SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

An Introduction to Poincare's model of Hyperbolic Geometry

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2019 - No K14

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An Introduction to Poincare's model of Hyperbolic Geometry

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Rikard Bögvad

2019

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May 2019

It should be known that geometry enlightens the intellect and sets one's mind right. All of its proofs are very clear and orderly. It is hardly possible for errors to enter into geometrical reasoning, because it is well arranged and orderly. Thus, the mind that constantly applies itself to geometry is not likely to fall into error. In this convenient way, the person who knows geometry acquires intelligence

Ibn Khaldun (1332-1406) [1]

[1] Carl C. Gaither, Alma E. Cavazos-Gaither. Gaither's Dictionary of Scientific Quotations. Springer Science Business Media, Dey 15, 1390 AP - Science - 2867 pages

Abstract

Just in recent centuries this indisputable belief, that Euclidean geometry is the absolute and invariable truth which completely justifies the physical space, came to inadequacy. Geometricians of nineteenth century demonstrated that there could be another possible form of geometry. In the following lines we concentrate on non-Euclidean geometry and basically hyperbolic geometry introducing hyperbolic distance, geodesics, hyperbolic triangles and their interesting differences with Euclidean triangles and hyperbolic area.
Key words: non-Euclidean geometry, hyperbolic geometry, geodesic.

I want to thank my mentor Rikard Bøgvad.

Contents

1 Introduction

For about 2000 years Euclidean geometry had been thought to be absolute truth. Euclidean geometry is based on five obvious truths which are called Euclidean Postulates:

The Euclidean Postulates:

1. Postulate I: To draw a straight line from any point to any point. (That through any two distinct points there exists a unique line).
2. Postulate II: To produce a finite line continuously in a straight line. (That any segment may be extended without limit).
3. Postulate III: To describe a circle with any center and distance. (Meaning of course, radius)."Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center."[2]
4. Postulate IV: All right angles are equal to one another. (Where two angles that are congruent and supplementary are said to be right angles).
5. Postulate V: If a straight line falling upon two straight lines makes the interior angles on the same side less than two right angles (in sum) then the two straight lines, if produced indefinitely, meet on that side on which are the two angles less than the two right angles.[3]

Fifth postulate can be rephrased. In the fifth century, the philosopher Proclus re-stated Euclid's fifth postulate in the following form, which has become known as the parallel postulate: Exactly one line parallel to a given line can be drawn through any point not on the given line.

First four postulates are easy to believe but the fifth one was somehow controversial and perhaps Euclid himself knew it because he avoided using it until he had proven the first twenty eight theorems of the Elements (the famous book of Euclid in geometry). During 2000 years many subtle thinkers doubted the fifth postulate and the result of their delicacy was the promotion of geometry to a higher level. Consider postulate V as this question: from any given point P out of a given line L how many lines parallel to L can be drawn?

[2] mathworld.wolfram.com/EuclidsPostulates.html

[3] Saul Stahl. The Poincaré Half-plane: A Gateway to Modern Geometry. Jones and Bartlett Learning, 1993 - Mathematics

The way one answers this question, determines the kind of geometry that one is dealing with.

1. There is one and only one line parallel to L passing P. This is the parallel postulate and is Euclid's 5th Axiom[4] rephrased. This approach gives the standard **Euclidean geometry**.
2. There is no parallel line to L passing P. Line in the spherical geometry is a huge circle that calls geodesic. In this kind of geometry, there is not only one line passing through two points,such that north and south poles, but also there are infinitely lines passing through them, which it does not hold the Euclidean first postulate. In this kind of geometry just axiom 2 and 4 satisfied. Triangles in the spherical geometry makes by circles and sum of there's angles are more than π. This approach is seen in the **spherical geometry**.
3. There is not parallel line to L passing P. In this kind of geometry any 2 lines intersect 2 points. This phenomenon occurs in **elliptic geometry.** In the elliptic geometry the IV Euclidean postulate does not hold.
4. More than one line parallel to L pass from P. This branch leads to **hyperbolic geometry**.

Then there are two kinds of geometry: **Euclidean** and **non-Euclidean**. Non-Euclidean geometry itself is divided in three subdivisions:

- **spherical geometry**
- **elliptic geometry**
- **hyperbolic geometry**

Spherical geometry
Spherical geometry is the geometry of the two-dimensional surface of a sphere. It is an example of a geometry that is not Euclidean. Two practical applications of the principles of spherical geometry are navigation and astronomy.

[4] A postulate is not quite the same as an axiom. Axioms are general statements that can apply to different contexts, whereas postulates are applicable only in one context, geometry in this case.

Hyperbolic geometry
Hyperbolic geometry(also called Bolyai–Lobachevsky geometry or Lobachevskian geometry) is a non-Euclidean geometry. The parallel postulate of Euclidean geometry is replaced with: For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R. A modern use of hyperbolic geometry is in the theory of special relativity.[5]

We will present one model of hyperbolic geometry: the Poincare's half plane. Through an interpretation of "lines" as shortest distance between points with respect to a distance function that is different from the usual Euclidean distance, we will see the first Euclidean axiom holds and the parallel axiom is not. Our main focus in this paper is on hyperbolic geometry.

[5] https://en.wikipedia.org/wiki/Hyperbolic-geometry.

2 Hyperbolic geometry

2.1 Hyperbolic distance

To introduce hyperbolic distance, and to explain its difference with Euclidean distance,we want to use the method that Sal Stahl used in his book ‘’Poincaré half plane” which is applying the concepts of velocity and temperature.We assume that the X axis is infinitely cold and by getting close to it, everything then contracts, for example moving from A to B takes less time than C to D and from C to D less time than E to F, because the more one gets close to X axis the smaller everything would be including our ruler and the moving object. This less velocity looks as distances are longer. More precisely, we assume that every distance in hyperbolic geometry is reversely proportional to its distance to the X axis so it can be assumed that distances in hyperbolic plane are equal to their amount in an Euclidean plane, divided by their (distance to X axis):

$$Hyperbolic \quad distance = \frac{Euclidean \quad distance}{y}.$$

It means moving from A to B takes half time of moving from C to D and one tenth of moving from E to F.

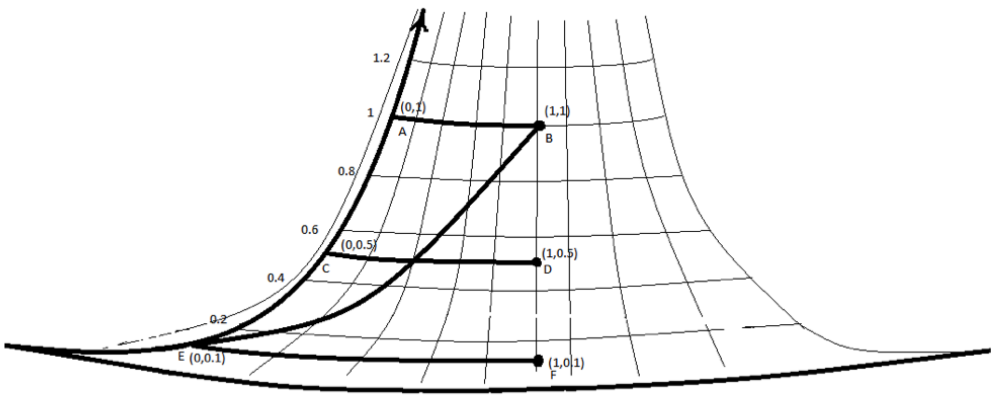

Figure 1: Hyperbolic plane

But this is not a very accurate definition. It only treats distances between points with the same y-coordinates.To be a little more accurate consider the triangle ABC with $A(x, y)$, $B(x + dx, y)$ and $C(x + dx, y + dy)$. The Euclidean length of AC would be:

$$dx^2 + dy^2.$$

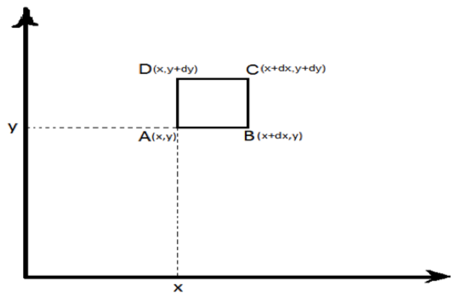

Figure 2: Euclidean length of AD in rectangle $ABCD$ is $\sqrt{dx^2 + dy^2}$

The approximation of hyperbolic length of AC would be:

$$\frac{\sqrt{dx^2 + dy^2}}{y}.$$

Definition 2.1. The hyperbolic length of any given curve

The hyperbolic length of any given curve K is given by summing small rectangles and letting their size go to 0 and is then:

$$\int_K \frac{\sqrt{dx^2 + dy^2}}{y}.$$

This should be compared to the usual definition:

Definition 2.2. Length of an arc in the Euclidean plane of any given curve is :

$$\int_K \sqrt{dx^2 + dy^2}.$$

We recall the motivation for this. To accurately measure the length of an arc consider the arc K and an arbitrary point "$r(t)$" on it:

$$r(t) = (x(t), y(t)).$$

Length of the arc between points $r(t)$ and $r(t+dt)$ is more and more close to the size of vector subtraction of $r(t)$ and $r(t+dt)$ vectors,that is $|r(t+dt)-r(t)|$. In the figure above :

$$r(t+dt) = (x+dx, y+dy)$$

$$r(t) = (x, y)$$

and

$$\Big|r(t+dt) - r(t)\Big| = \sqrt{(dx)^2 + (dy)^2} \approx \Big|r'(t)dt\Big|.$$

Since

$$dx = x'(t)dt, \quad dy = y'(t)dt,$$

the size of $r'(t)$ is :

$$\Big|r'(t)\Big|dt = \sqrt{x'^2(t) + y'^2(t)}dt.$$

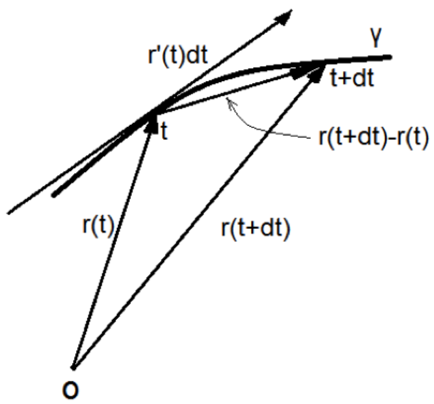

Figure 3: Length of a curve

After integration the length of curve K will be:

$$L = \int_a^b \Big|r'(t)\Big|dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt.$$

This may also be expressed as:

$$L = \int_K ds.$$

Where

$$ds = \sqrt{dx^2 + dy^2}.$$

The important question here is: what is the shortest way between two given points? Is the straight line the shortest way that links two point together as in Euclidean plane? Let's find out, in an example which we have taken from [3].

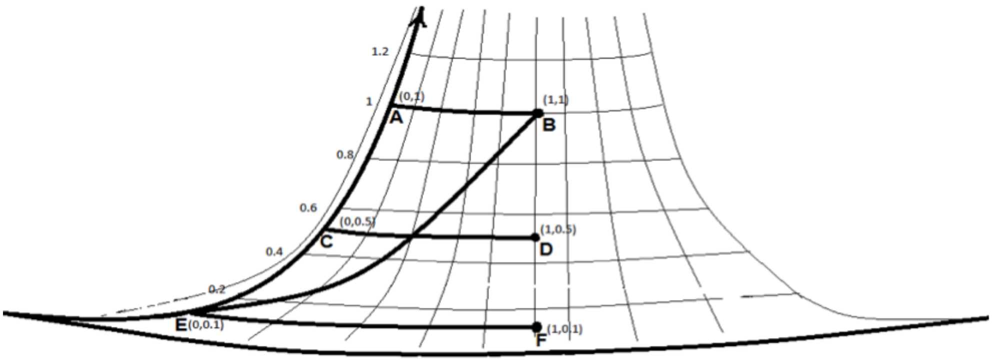

Figure 4: Comparing hyperbolic lengths of EB and $EA + AB$ [3]

We compare EB and $EA + AB$ in the picture above using the arc length:

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{y}.$$

The equation of line EB can be acquired from:

$$Y - y_0 = \frac{(y_1 - y_0)}{(x_1 - x_0)}(X - x_0).$$

$$E = (0, 0.1)$$

$$B = (1, 1)$$

Then:

$$Y - 0.1 = \frac{(1 - 0.1)}{(1 - 0)}(X - 0) \Rightarrow Y = 0.9X + 0.1.$$

The differential is:

$$dy = 0.9dx.$$

Then the hyperbolic length of EB is:

$$EB = \int_0^1 \frac{\sqrt{(dx)^2 + (0.9dx)^2}}{0.9x + 0.1} = \Big[\sqrt{1.81}\ln(0.9x{+}0.1)\Big]_0^1 = -\sqrt{1.81}\ln 0.1 \approx 3.442.$$

For $EA : dx = 0$, then hyperbolic length of

$$EA = \int_{0.1}^1 \frac{\sqrt{(dy)^2}}{y} = \int_{0.1}^1 \frac{dy}{y} = \Big[\ln y\Big]_{(0.1)}^1 = -\ln 0.1 \approx 2.303.$$

And for AB:

$$AB = \frac{\sqrt{(1)^2 + (0dy)^2}}{1} = 1.$$

hyperbolic length of $AB = 1$.

$$EA + AB = 2.303 + 1 = 3.303 \text{ but } EB = 3.442.$$

$$EA + AB < EB.$$

We knew in Euclidean geometry that the straight line between two points is the shortest possible way but as we saw above there was a typical route consisting of two perpendicular straight lines (EA and AB) which was shorter than the straight way (EB). In hyperbolic plane the straight line between two points, is not the shortest way. This shortest way is called a geodesic segment and one of our aims is to determine it.

3 The minimal hyperbolic distance between two points

It has been shown that the geodesic segment in a hyperbolic plane is in one of these two forms below:

- A vertical line perpendicular to the X axis.

- An arc of a circle with a center on the X axis.

Theorem 1. *The hyperbolic length of a line segment parallel to the Y axis from point A to B is:*

$$\ln \frac{y_2}{y_1}.$$

Proof. First we calculate the hyperbolic length of these geodesics then we can prove that these are the true shortest ways. Vertical line: for any given line perpendicular to the X axis at point "a" the equation is $x = a$.

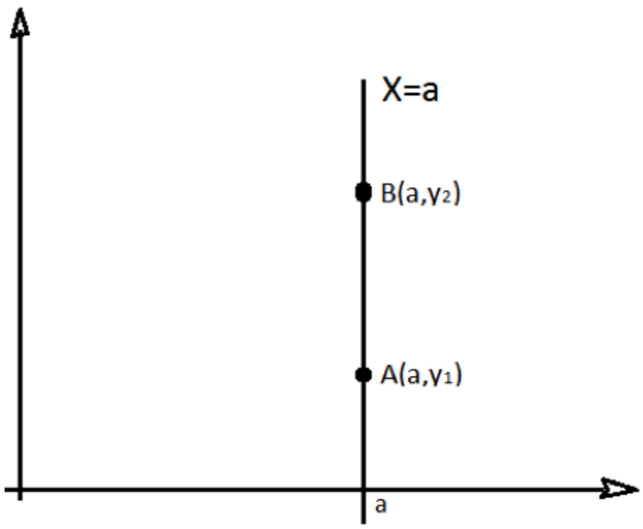

Figure 5: vertical line

Therefore $dx = 0$ and the length of segment AB with $A(a, y_1)$ and $B(a, y_2)$ on this line would be:

$$\int_c \frac{\sqrt{(dx^2 + dy^2)}}{y} = \int_{y_1}^{y_2} \frac{dy}{y} = lny_2 - \ln y_1 = \ln \frac{y_2}{y_1}.$$

In this case $x_1 = x_2$, assume that equation of "G" is:

$$x = f(y).$$

$$\frac{dx}{dy} = f'.$$

Again this is piece wise true since $y_1 \neq y_2$.

$$G = \int_{y_1}^{y_2} \frac{\sqrt{(f'^2 dy^2 + dy^2)}}{y} = \int_{y_1}^{y_2} \frac{\sqrt{(f'^2+1)}}{y} \geq dy$$

$$\int_{y_1}^{y_2} \frac{dy}{y} = ln\frac{y_2}{y_1}.$$

Theorem 2. *The hyperbolic length of a circle segment an arc of a circle between P and Q with a center on the X axis is:*

$$\ln\frac{|\csc\beta - \cot\beta|}{|\csc\alpha - cot\alpha|}.$$

Proof. For any point P on the arc AB of any circle with center C and orthogonal to the X axis, $x = c + r\cos(\theta)$ and $y = r\sin(\theta)$.

Then $dx = -r\sin\theta$ and $dy = r\cos\theta$. And the hyperbolic length of $arcAB$ would be:

$$\int_{\alpha}^{\beta} \frac{\sqrt{(dx^2 + dy^2)}}{y} =$$

$$\int_{\alpha}^{\beta} \frac{\sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}}{r\sin\theta} d\theta =$$

$$\int_{\alpha}^{\beta} \frac{rd\theta}{r\sin\theta} = \int_{\alpha}^{\beta} \frac{d\theta}{\sin\theta} = \int_{\alpha}^{\beta} \frac{\sin\theta}{\sin^2\theta} d\theta =$$

$$\int_{\alpha}^{\beta} \frac{\sin\theta}{1-\cos^2\theta} d\theta = \left[u = \cos\theta, du = -\sin\theta dx\right]$$

$$= -\int_{\alpha}^{\beta} \frac{du}{1-u^2} = -\frac{1}{2}\int_{\alpha}^{\beta} (\frac{1}{1-u} + \frac{1}{1+u}) du$$

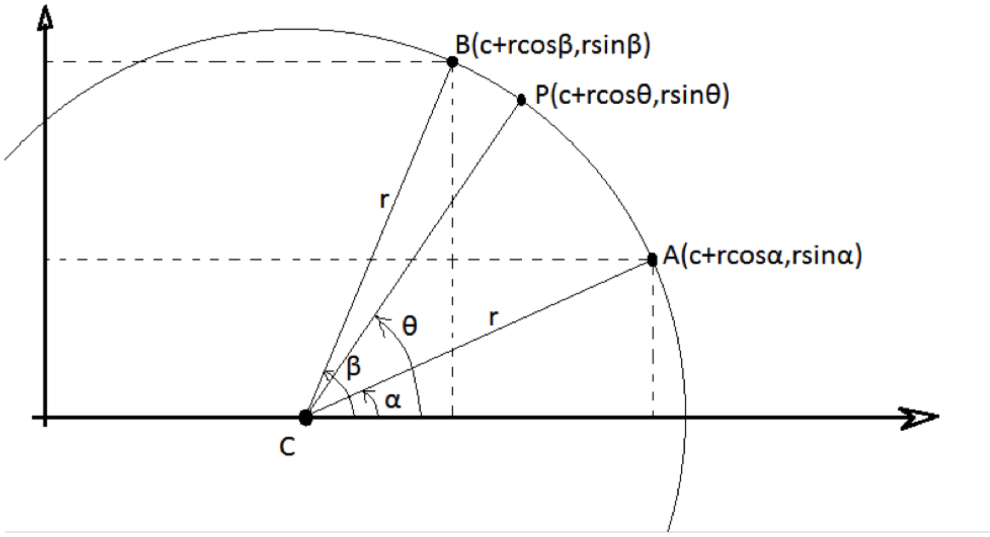

Figure 6: Circle with center C on Xaxis

$$= -\frac{1}{2}(\ln\left|k+1\right| - \ln\left|k-1\right|) + C = \frac{1}{2}\ln\left|\frac{k-1}{k+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + C = \frac{1}{2}\ln\left|\frac{(\cos\theta-1)}{(\cos\theta+1)}\frac{(\cos\theta-1)}{(\cos\theta-1)}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(\cos\theta-1)^2}{\sin^2\theta}\right| + C$$

$$= \ln\left|\frac{\cos\theta-1}{\sin\theta}\right| + C$$

$$= \ln\left|csc\theta - cot\theta\right| + C$$

$$\ln\left|\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right|.$$

And it remains be proved that these two amounts are the minimums or the geodesics.But first let us state what we proved.

In this case despite the former case $x_1 \neq x_2$ the perpendicular bisector of the segment connecting P and Q is not parallel to X axis and intersects it in

$C(c,0)$. Consider a polar coordination centered on C and horizontal axis coincident on X axis. The curve "G" is assume to be parametrised by:

$$r = f(\theta).$$

And the coordinates of P and Q would be:

$$P(r_P,\alpha), Q(r_Q,\beta).$$

Clearly G has to be at least piece wise of this form since $x_1 \neq x_2$.
And the hyperbolic length of "G" is:

$$\int_G \frac{\sqrt{(dx^2+dy^2)}}{y}.$$

Relating equations of Cartesian and polar coordination systems are:

$$x = c + r\cos\theta, y = r\sin\theta.$$

Then:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r\frac{d\cos\theta}{d\theta} = r'\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\frac{d\sin\theta}{d\theta} = r'\sin\theta + r\cos\theta$$

$$\Rightarrow dx^2 + dy^2 = (r'\cos\theta - rsin\theta)^2 d\theta^2 + (r'\sin\theta + r\cos\theta)^2 d\theta^2 =$$

$$\left[r'^2(\cos^2\theta + \sin^2\theta) + 2rr'(\cos\theta + sin\theta) + r^2(\cos^2\theta + sin^2\theta)\right]d\theta^2 = (r'^2 + r^2)d\theta^2$$

$$\Rightarrow \int_\alpha^\beta \frac{\sqrt{(dx^2+dy^2)}}{y} = \int_\alpha^\beta \frac{\sqrt{(r'^2+r^2)}}{r\sin\theta}d\theta \geq \int_\alpha^\beta \frac{\sqrt{(r^2)}}{r\sin\theta}d\theta =$$

$$\int_\alpha^\beta \frac{d\theta}{\sin\theta} = \int_\alpha^\beta \csc\theta = \ln(\csc\theta - \cot\theta)\Big|_\alpha^\beta = \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - cot\alpha}.$$

The last is the hyperbolic distance of an arc connecting P and Q which is part of a circle centered on X axis or our known geodesic. Lastly we can see that out of an integral we have a formula by which we can calculate the length of arc AB using central angles that points A and B make with the X- axis. With this inequality we see that any form of connecting lines between any two given points is more extended than the arc.

This is the hyperbolic distance of a vertical line connecting P and Q or our known geodesic.With this inequality we see that any form of curve between any two given points is more extended than the vertical line.[6]

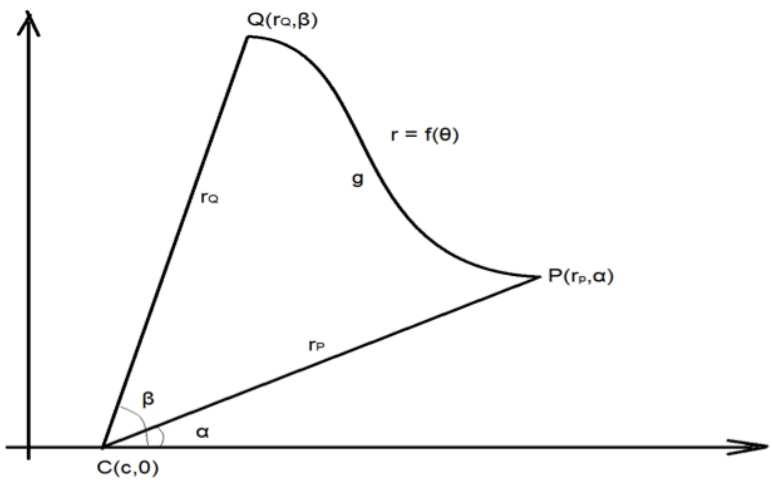

Figure 7: We assume that the geodesic connecting P and Q has an arbitrary form

[6]Stahl. The Poincaré Half-plane: A Gateway to Modern Geometry. Jones Bartlett Learning, 1993 - Mathematics

3.1 An example

As a numerical example consider the figure 8. We calculate the length of different geodesics on the line $x = 1$ between points $(1,1), (1,2)$ and $(1,3)$ and arcs PQ, QR and PR on the circle C.

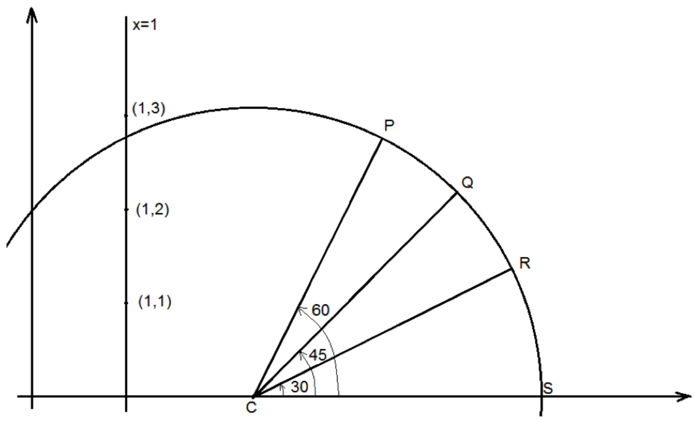

Figure 8: Numerical example for geodesics

Between $(1,1)$ and $(1,2)$ the hyperbolic length is :

$$g_1 = \ln \frac{y_2}{y_1} = \ln \frac{2}{1} = \ln 2 = 0.6931.$$

Between $(1,2)$ and $(1,3)$ the hyperbolic length is :

$$g_2 = \ln \frac{3}{2} = \ln 1.5 = 0.4056.$$

Between $(1,1)$ and $(1,3)$the hyperbolic length is :

$$g_3 = \ln \frac{3}{1} = \ln 3 = 1.0987.$$

$$g_1 + g_2 = 0.6931 + 0.4056 = 1.0987 = g_3.$$

$$PQ : g_4 = \ln \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} =$$

$$\ln \frac{\csc 60 - \cot 60}{\csc 45 - \cot 45} = \ln \frac{1.154 - 0.57735}{1.4142 - 1} =$$

$$\ln \frac{0.5773}{0.414} = \ln 1.3939 = 0.3321.$$

$$QR : g_5 = \ln \frac{\csc 45 - cot45}{\csc 30 - cot30} = \ln \frac{1.4142 - 1}{2 - 1.73}$$

$$= \ln \frac{0.4142}{0.268} = \ln 1.5455 = 0.43536.$$

$$PR : g_6 = \ln \frac{\csc 6 - \cot 60}{\csc 30 - \cot 30} = \ln \frac{1.1547 - 0.57735}{2 - 1.73}$$

$$= \ln \frac{0.5773}{0.26} = \ln 2.1543 = 0.76746.$$

$$PQ + QR = 0.3321 + 0.43536 = 0.76746 = PR.$$

In this example we demonstrated that hyperbolic distance is different from their respective values in Euclidean plane as for distances equal to unique value in Euclidean plane which have an intercept larger than $y = 1$, they are smaller but when the intercept is smaller than $y = 1$, the distance would be larger.

Proposition 1. *From two given points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$,just and only one circle can be drawn makes a right angle with x-axis.*

Proof. We should draw linking line between P and Q and then its perpendicular bisector which intersects X axis at point C. CPQ is an isosceles triangle. Its equal sides are CP and CQ and its base is PQ which its corresponding altitude is the same mentioned perpendicular bisector.

Now by sides CP or CQ as radius and point C as center, a circle can be drawn. This is a unique circle which pass through those two points and has centered on X axis. Since C is located on X axis, the diameter of circle is coincided on X axis and since diameter (radius) is perpendicular on circumference of circle at intersecting point, therefore it can be said that the circle is perpendicular on X axis at intersecting points.

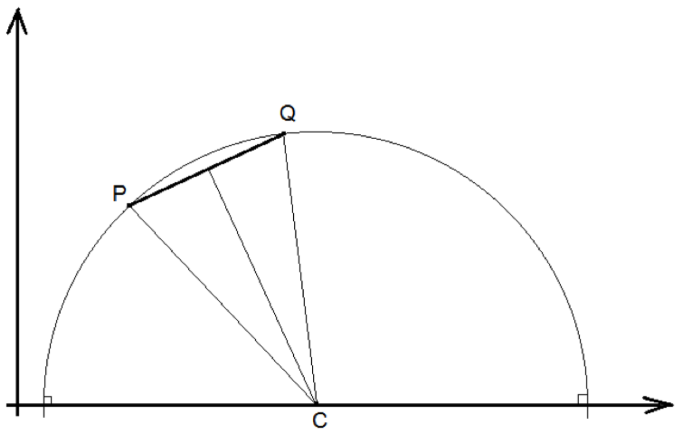

Figure 9: Through two given points there is an unique circle centered on a given line.

Identifying "lines" with geodesics, we have shown that there is a unique "line" between two points in the Poincare' half-space. This corresponds to axiom I of Euclidean geometry. In fact axioms I-IV will be true, and so we can assume any statement from Euclidean geometry ,which is proved without use of the parallel axiom. In what follows we will assume that axioms I-IV hold in the hyperbolic plane, and will use propositions that are derived from these axioms.

4 Hyperbolic triangle

A hyperbolic triangle is made from three points which are not located on one single geodesic and its sides are three geodesics that cut each other at those three points. The main goal here is to demonstrate that in every hyperbolic triangle the sum of inner angles is less than π. First we review the angles made in cross points of geodesics and introduce a way to know their amounts: look at the picture below:

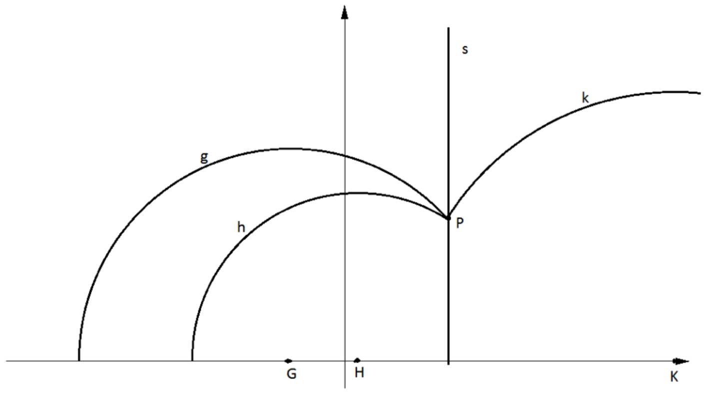

Figure 10: Angles made in an intersection point of geodesics

4.1 Sum of angles in hyperbolic triangle

Theorem 3. *The sum of the angles in a hyperbolic triangle is less than π.*

We have a straight geodesic s perpendicular to X axis and bowed geodesics g, h, and k with centers G, H, and K respectively which are all intersect at point P. We will describe the angles between these geodesics.

(see figure 11)We denote the angles between two geodesics as $\angle$ (geodesic 1,geodesic 2) and the angle with head A and whose sides end at B and C as $\angle BAC$ or $\angle CAB$. We denote the tangents at the intersection points to the geodesic g by g', etc. After drawing radiuses and tangent lines of geodesics g, h, and k entering to P it can be shown that:

$$\angle(s, g) = \angle PGS.$$

$$\angle(g, h) = \angle GPH.$$

$$\angle(g,k) = \pi - \angle GPK.$$

.

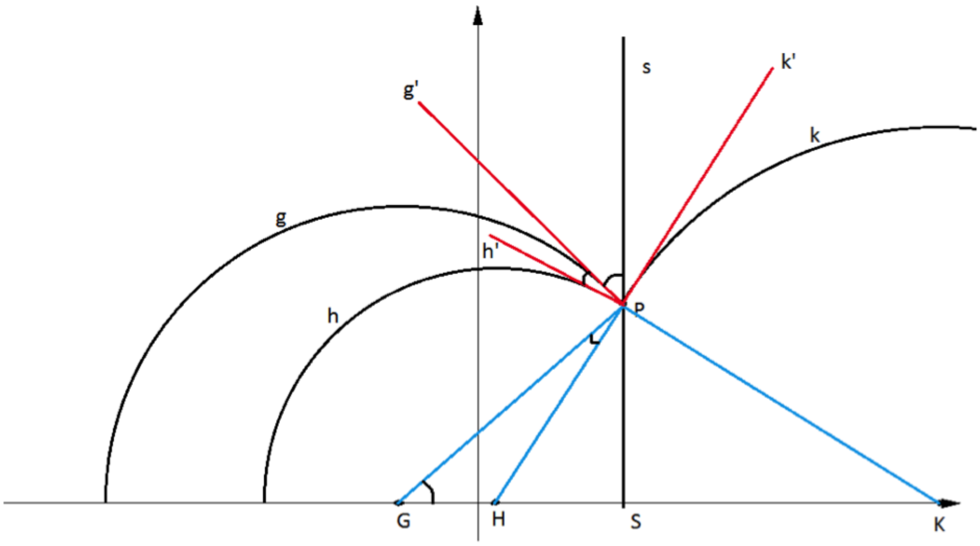

Figure 11: Angles of tangent lines of circles at intersection point

For the first of these equalities, we use the following argument. we know that tangent line Pg' is perpendicular at radius GP at intersection point.

$$\angle(s,g) = \angle sPg' =$$

$$\pi - \angle g'PG - \angle GPS =$$

As we know that angle $g'PG$ is a right angle, then we have:

$$\angle(s,g) = \pi - \frac{\pi}{2} - \angle GPS =$$

This last equality follows:

$$\frac{\pi}{2} - \angle GPS = \angle PGS.$$

Since in the triangle GPS the angle GSP is a right angle, and then we have:

$$\angle GPS + \angle PGS = \frac{\pi}{2}.$$

Now we will give the argument determining $\angle(g, h)$. By the definition we get the first equality below:

$$\angle(g, h) = \angle g'Ph' =$$

$$\angle g'PG - \angle h'PG =$$

As we know that $\angle g'Pg$ is a right angle.

$$\angle(g, h) = \frac{\pi}{2} - \angle h'PG$$

again we have that $\angle h'PH$ s a right angle, then:

$$\angle h'PH - \angle h'PG = \angle GPH.$$

This proves the second equality at the beginning of this section. To see the third,consider the geodesics g and k. Both $g'P$ and $k'P$ are tangent lines of those geodesics:

$$\angle(g, k) = \angle g'Pk' =$$

$$\angle g'Ps + \angle sPk' =$$

We already found that $\angle g'Ps = \angle PGS$ and semilarity $\angle SPk' = PKS$. Now we have:

$$\angle(g, h) = \angle PGS + \angle PKS = \pi - \angle GPK.$$

This proves all equalities on page 21.

Now to prove that the sum of inner angles in a hyperbolic triangle is less than π it is simpler to start with right triangles. We will use the preceding way to calculate angles.

Consider the triangle ABC with heads $A(0, k), B(s, t)$, and $C(0, 1)$ as above. The head C is a right angle, the side BC is part of a geodesic centered at origin, the side AB is part of a geodesic centered at $D(-d, 0)$, and the side AC is part of positive region of Y axis. Now we know by what we have shown above, that(see figure 12):

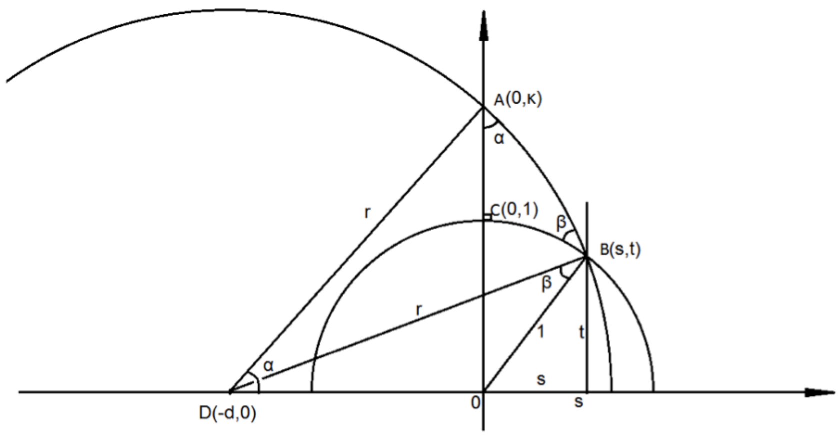

Figure 12: Angles of hyperbolic triangle ABC

$$\alpha = \angle CAB = \angle ADO.$$

and

$$\beta = \angle ABC = \angle DBO.$$

Since $\angle ACB = \dfrac{\pi}{2}$ then we will show that:

$$\alpha + \beta < \frac{\pi}{2}$$

or

$$\alpha < \frac{\pi}{2} - \beta.$$

Because α and β both are acute angles, this is equivalent to showing that .

$$\sin \alpha < \sin(\frac{\pi}{2} - \beta)$$

$$\Leftrightarrow \sin \alpha < \cos \beta.$$

According to rule of cosines in triangle DBO :

$$r^2 + 1^2 - 2(1)(r)\cos \beta = d^2$$

$$\Leftrightarrow \cos\beta = \frac{(r^2 + 1^2 - d^2)}{2r}.$$

Then

$$\sin\alpha < \cos\beta \Leftrightarrow \frac{k}{r} < \frac{(r^2 + 1^2 - d^2)}{2r}.$$

In the right-angle triangle AOD:

$$r^2 = d^2 + k^2$$

Then $r^2 + 1 - d^2 = k^2 + 1$, so the sought inequlity becomes:

$$\frac{k}{r} < (\frac{k^2 + 1}{2r})$$

$$\Leftrightarrow (\frac{k}{r})r < (\frac{k^2 + 1}{2r})r$$

$$\Leftrightarrow 2k < k^2 + 1$$

$$\Leftrightarrow 0 < k^2 + 1 - 2k$$

$$= (k-1)^2.$$

Because we reached to a true inequality then our first step ($\alpha + \beta < \frac{\pi}{2}$) has been true.

We have proved so far that **the sum of inner angles in a hyperbolic right triangle is less than π.** To prove generally, first we claim that in any triangle at least there is one altitude inside the triangle using Euclid's $17th$ proposition that says **"In any triangle, two angles taken together in any manner are less than two right angles"**. Also according to Euclid's $16th$ proposition when an altitude is external, one of the adjacent angles of external altitude is obtuse. Therefore there must be two other acute angles a triangle with n external altitude and consequently the altitude of the third angle (the obtuse one) would be internal.

Now note that Pro.17 in Elementa is proved without using the parallell axiom, and so will be true for hyperbolic triangle ABC with an internal altitude AH (picture below). We will have two right triangles, AHB and AHC and as we had for right triangles:(see figure 14)

$$\angle A_1 + \angle B < \frac{\pi}{2}.$$

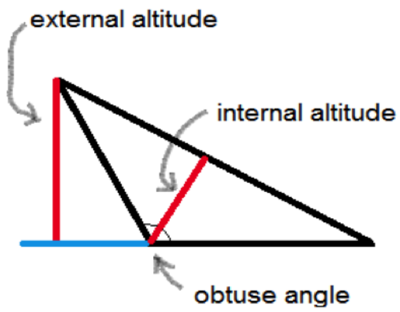

Figure 13: There is at least one internal altitude in any triangle

$$\angle A_2 + \angle C < \frac{\pi}{2}.$$

In triangle ABC, sum of angles is equal to:

$$\angle A + \angle B + \angle C.$$

Since

$$\angle A = \angle A_1 + \angle A_2,$$

we have that

$$\angle A + \angle B + \angle C = \angle A_1 + \angle A_2 + \angle B + \angle C =$$

$$\angle A_1 + \angle B + \angle A_2 + \angle C < \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\angle A + \angle B + \angle C < \pi.$$

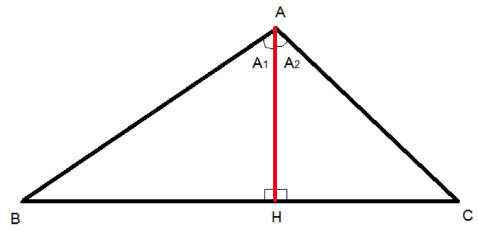

Figure 14: An internal altitude bisects any triangle to two right triangles

5 Hyperbolic area

In Euclidean plane the area of any given region (shape) R with either straight or curved borders is calculable in this manner:

$$Area(R) = \int\int_R dx\,dy.$$

Then as we defined before:

$$Hyperbolic \quad distance = \frac{Euclidean \quad distance}{y},$$

therefore in hyperbolic plane it can be deduced that hyperbolic area would be:

$$ha(R) := \int\int_R \frac{dx}{y}\frac{dy}{y} = \int\int_R \frac{dxdy}{y^2}.$$

Consider figure below:

As it is obvious, the area of regions A and B in Euclidean plane is 1 and C is infinite but it is very different in hyperbolic plane. Let us explain that with some examples:

Example 1:
Hyperbolic area of A

$$A = ha(A) = \int_0^1\int_0^1 \frac{dxdy}{y^2} = \int_0^1 \frac{dy}{y^2} = \infty.$$

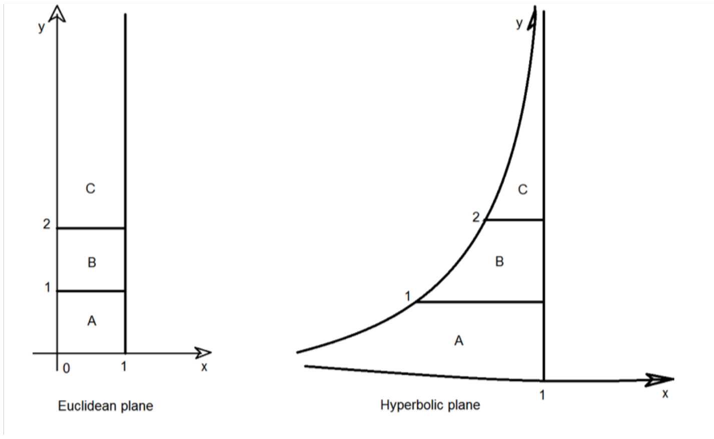

Figure 15: Comparing areas in Euclidean and hyperbolic planes

Example 2:
Hyperbolic area of B

$$B = ha(B) = \int_1^2 \int_0^1 \frac{dxdy}{y^2} = \int_1^2 \frac{dy}{y^2} =$$

$$\left[\frac{-1}{y}\right]_1^2 = \frac{-1}{2} - \frac{-1}{1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Example 3:
Hyperbolic area of C

$$C = ha(C) = \int_2^\infty \int_0^1 \frac{dxdy}{y^2} =$$

$$\int_2^\infty \frac{dy}{y^2} = \left[\frac{-1}{y}\right]_2^\infty = 0 - \frac{-1}{2} = \frac{1}{2}.$$

Proposition 2. *The hyperbolic area of a hyperbolic triangle is $\pi - (\alpha + \beta + \gamma)$, if the angles of the triangle are α, β, γ.*

To calculate the area of a hyperbolic triangle we first need to know the area R restricted between two straight geodesics e and d perpendicular to the X-axis, and a curved geodesic "DE" between these two. (see figure 16).

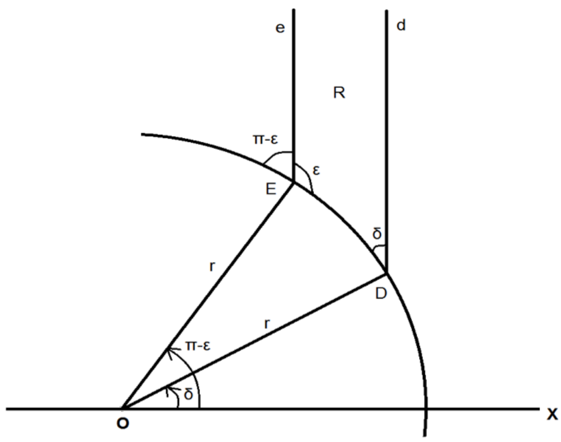

Figure 16: Area of the hyperbolic triangle

"O" is the center of DE and

$$\angle XOD = \delta.$$

and

$$\angle XOE = \pi - \epsilon.$$

Euclidean equation of the circle DE is:

$$x^2 + y^2 = r^2.$$

or

$$y = \sqrt{r^2 - x^2}, \text{ if } y \geq 0.$$

Then the hyperbolic area of R is:

$$ha(R) = \int_{r\cos(\pi-\epsilon)}^{r\cos\delta} \Big(\int_{\sqrt{r^2-x^2}}^{\infty} \frac{dy}{y^2} \Big) dx =$$

$$\int_{r\cos(\pi-\epsilon)}^{r\cos\delta}\int_{\sqrt{r^2-x^2}}^{\infty}\frac{dy}{y^2}dx = \int_{r\cos(\pi-\epsilon)}^{\cos\delta} y^{-1}dx$$

and then,since

$$y^{-1} = \frac{1}{\sqrt{r^2-x^2}}.$$

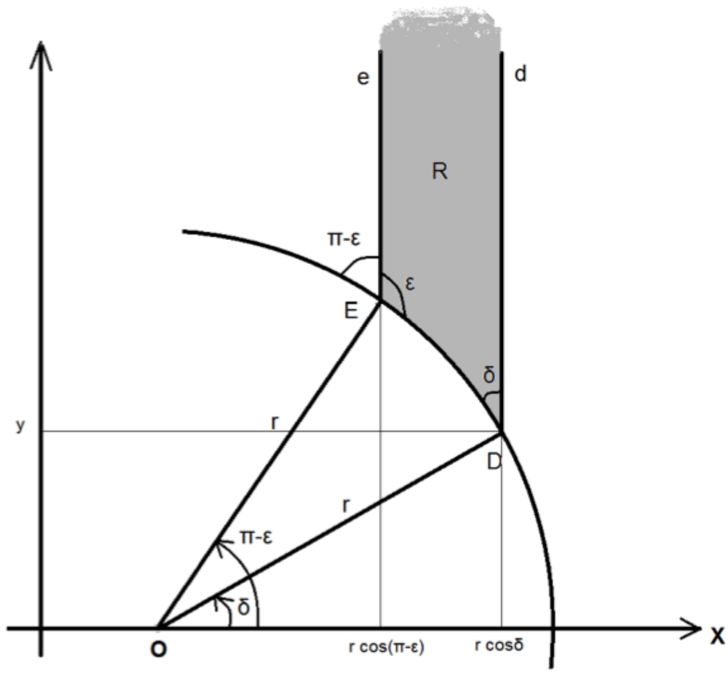

Figure 17: Area of R^7

$$ha(R) = \int_{r\cos(\pi-\epsilon)}^{\cos\delta)} y^{-1}dx$$

$$= \int_{r\cos(\pi-\epsilon)}^{(r\cos\delta)} \frac{dx}{\sqrt{r^2-x^2}}$$

$$= \arcsin\frac{x}{r}\Big|_{r\cos(\pi-\epsilon)}^{\cos\delta}.$$

We know that:

$$\cos(\pi - \epsilon) = -\cos\epsilon.$$

Then:

$$ha(R) = \arcsin(\cos\delta) – \arcsin(-\cos\epsilon) =$$

$$\arcsin(\cos\delta) + arc\sin(\cos(\epsilon)) =$$

$$\frac{\pi}{2} - \delta + \frac{\pi}{2} - \epsilon = \pi - \delta - \epsilon.$$

Now we use this result to calculate the area of a hyperbolic triangle limited by the $Yaxis$, geodesic AB and geodesic BC as in the figure below in which:

$$\theta = \beta + \varphi.$$

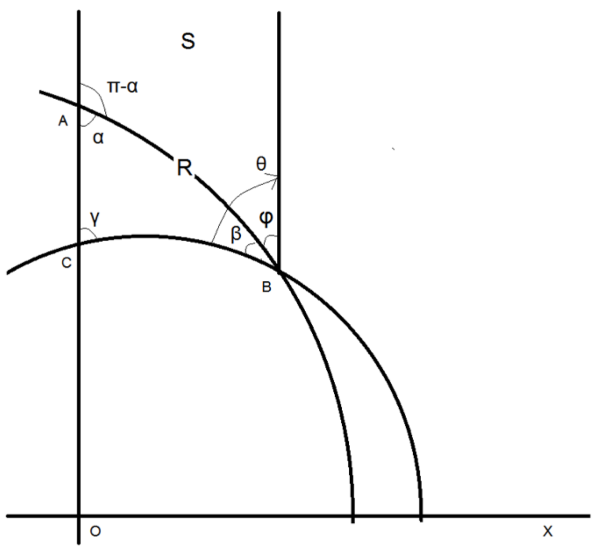

Figure 18: Area of triangle ABC

Consider figure 18. To find the area of triangle ABC we can subtract of the area R from the area S.Hence, by the previous result:

$$ha(ABC) = ha(R) – ha(S).$$

$$ha(ABC) = \pi - \gamma - \theta – (\pi - \varphi - (\pi - \alpha))$$

$$= \pi - \theta - \gamma + \varphi - \alpha$$

$$= \pi - \alpha - (\theta - \varphi) - \gamma$$

$$\Rightarrow ha(ABC) = \pi - \alpha - \beta - \gamma = \pi – (\alpha + \beta + \gamma).$$

Proposition 3. *The area of an hyperbolic triangle is* $\pi - (\alpha + \beta + \gamma)$.

This means that we can calculate the area of our hyperbolic triangle if we know its angles. It follows that the same result hold for an arbitrary hyperbolic triangle.(by subdividing the triangle, by a vertical line)

5.1 Some more differences between hyperbolic and Euclidean triangle

Then the area of a hyperbolic triangle is equal to two right angles minus the sum of its angles.The interesting subject here is the dependence of area of hyperbolic triangle on its angles despite of in Euclidean plane that there is no relation between area of a triangle and its angles as there is infinite number of triangles with the same angles with infinitely small areas to infinitely large areas. In the hyperbolic plane triangles with the same angles have the same area. Even triangles with equal sum of angles but different angles, also have equal areas.

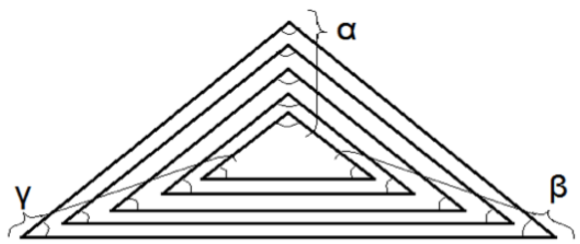

Figure 19: Triangles with same angles but different areas in Euclidean plane

References

[1] Carl C. Gaither, A. E.-G. (2012). Gaither's Dictionary of Scientific Quotations. Switzerland AG: Springer Science Business Media.

[2] Lars-Christer Böiers, A. P. (2010). Analys i en variabel. inbunden, Svenska: Studentlitteratur.

[3] Stahl, S. (1993). The Poincaré Half-plane: A Gateway to Modern Geometry . Burlington, Massachusetts, United States: Jones Bartlett Learning.

[4] https://en.wikipedia.org/wik/Spherical-geometry.

[5] https://en.wikipedia.org/wiki/Hyperbolic-geometry.

[6] mathworld.wolfram.com/EuclidsPostulates.html

-Figures are taken mostly from [3].