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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Walrasian equilibrium and the Brouwer fixed-point theorem

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Abstract

This paper will state and prove the Brouwer fixed-point theorem using the no-differentiable retraction theorem. In the last section the Brouwer fixed point theorem will be applied to prove the existence of a Walrasian price equilibrium. This paper is divided into three sections. The First section states and proves the no-differentiable retraction theorem. The second section states and proves the Brouwer fixed-point theorem. The last section states and proves the existence of a Walrasian price equilibrium.

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1 Introduction

Proving the existence of solutions is an important task in many fields of mathematics. The endeavour to prove the existence of solutions in mathematics is often accomplished by utilizing fixed-point theorems. One of the most important fixed-point theorems is the Brouwer fixed-point theorem. The theorem states that for any continuous function mapping a compact convex set to itself has a fixed-point. This result has seemingly unlikely application in disciplines such as economics and game theory. Within economics in particular it has been applied in order to prove the existence of a general equilibrium in an economy, that is the existence of a price vector which equilibrates supply and demand of all markets in an economy.

The Brouwer fixed-point theorem can be proved using a multiple of approaches involving tools from homology and combinatorics. However, this paper will prove the theorem using the no-differentiable retraction theorem which states the impossibility of a differentiable retraction from the unit ball to its boundary. The main focus of this paper will be to prove the Brouwer fixed-point theorem, then apply it in the context of a simple general equilibrium model in order to prove the existence of an equilibrium price which clears all markets in an economy.

2 The No-differentiable retraction theorem

In this section I will state and prove the No-differentiable retraction theorem[5]. The No-differentiable retraction theorem will be crucial in the proof of the Brouwer fixed-point theorem, and by extension proving the existence theorem of Walrasian equilibrium.
In the first subsection I will present the definition of a retraction. I will also state the divergence theorem since it is applied in the proof of The No-differentiable retraction theorem. I will however not prove the divergence theorem since the focus of this section is to prove the no-differentiable retraction theorem.
In the second subsection I will state and prove the no-differentiable retraction theorem using the divergence theorem in the first subsection. I will also give an example of the theorem in the one-dimensional case.

Remark 2.1. I will throughout the paper denote vectors in bold and scalars in standard text whenever it is necessary. For example, $\boldsymbol{x}$ will denote a vector, and x_i will denote the ith component of a vector. If nothing else is stated x will just denote a scalar. This is to avoid confusion relating to dot-products and multiplication of scalars with scalars and scalars with vectors. Dot products will be denoted in the following way, for example $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$ and the norm is denoted as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$.

Remark 2.2. I will throughout this paper differentiate between open ball $\mathring{B}_k^n$ and closed balls with radius k with the notation B_k^n. Furthermore, I will denote the closed unit ball simply as B^n. In other words:

$$B_k^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq k\}$$

$$B^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$$

$$\mathring{B}^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < 1\}$$

$$\mathring{B}_k^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < k\}.$$

The boundary will be denoted as

$$\partial B_k^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = k\}.$$

Balls centered on a point $\boldsymbol{p}$, will be denoted $B_k^n(\boldsymbol{p})$.

2.1 Preliminary definitions, theorems and context

Definition 2.1 (Retraction)**.** Let $S \subseteq \mathbb{R}^n$ and $B \subseteq S$. A map $r : S \to B$ is said to be a retraction if it is continuous and $r(b) = b$ for all $b \in B$. The set B is called a retract of S.

Theorem 2.1 (The Divergence theorem)**.** *Let V be a compact subset of $\mathbb{R}^n$ which has a piecewise smooth boundary $\partial V = S$. Let $\boldsymbol{F} = (f_1, \ldots, f_n)$ be a continuously differentiable vector field, $\boldsymbol{N}$ is the outward pointing unit normal* *field of the boundary of* S *and* $\nabla \cdot \boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$ *is the divergence of* $\boldsymbol{F}$ *over* V*, then the following holds*

$$\iint \cdots \int_V \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \int \cdots \int_{\partial V} \boldsymbol{F} \cdot \boldsymbol{N} \, \mathrm{d}S.$$

For proof in the three-dimensional case see [10, p.468].
In the case of $n = 1$ the above formula reduces to the fundamental theorem of calculus.

2.2 The No-differentiable retraction theorem: Statement and proof

Let us begin this subsection with an easy example of the theorem in the one-dimensional case.

Example 2.1 (One-dimensional case of no-differentiable retraction)**.** We claim claim that there exists no differentiable $f : [-1, 1] \to \{-1, 1\}$, such that $f(1) = 1$ and $f(-1) = -1$.

Proof. Clearly $f'(x) = 0$ for all $x \in [-1, 1]$, since otherwise the range of f would contain an interval. But:

$$0 = \int_{-1}^{1} f'(x) \, \mathrm{d}x = f(1) - f(-1) = 1 - (-1) = 2 \neq 0$$

Which is a contradiction. □

The proof of the no-differentiable retraction theorem can be seen as a generalization of the proof above to dimension n. The idea behind the n-dimensional case is to replace f' by the Jacobian determinant and replacing the fundamental theorem of calculus by the n-dimensional divergence theorem. Since the divergence theorem can be seen as the analogue of the fundamental theorem of calculus.

Theorem 2.2. *There exists no twice differentiable map* $\boldsymbol{f} : B^n \to \partial B^n$*, such that* $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \partial B^n \subseteq \mathbb{R}^n$*, where* B^n *is the unit ball.*

Proof. Suppose by contradiction that $\boldsymbol{f}$ is such a retraction $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$. Let $J(\boldsymbol{x})$ be the Jacobian determinant of $\boldsymbol{f}$ at $\boldsymbol{x}$. Then expand the Jacobian determinant by the first column:

$$J(\boldsymbol{x}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = \begin{vmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{vmatrix}$$

$$J(\boldsymbol{x}) = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial f_1}{\partial x_i} E_i(\boldsymbol{x}).$$

$E_i(\boldsymbol{x})$ is the determinant from the matrix $M(\boldsymbol{x})$ below by omitting the ith row.

$$M(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

The Jacobian determinant vanishes on B^n since the following relation $f_1^2(\boldsymbol{x}) + \cdots + f_n^2(\boldsymbol{x}) = 1$ hold on B^n. If one take the gradient of the relation we get $2f_1\nabla f_1 + \cdots + 2f_n\nabla f_n = \mathbf{0}$ for all $\boldsymbol{x} \in B^n$, which means the vectors are linearly dependent since all $f_1, \ldots, f_n$ can not all be zero (because of the sum of the squares equal one). Hence the Jacobian determinant is zero. Integrating $J(\boldsymbol{x})$ over B^n (the integral over B^n is equal to zero since the Jacobian determinant vanishes on B^n). We find by using the product rule of differentiation that:

$$0 = \int_{B^n} \cdots \int J(\boldsymbol{x})\, \mathrm{d}x_1 \cdots\, \mathrm{d}x_n =$$

$$\int_{B^n} \cdots \int \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial}{\partial x_i}(f_1 E_i)\, \mathrm{d}x_1 \cdots\, \mathrm{d}x_n + \int_{B^n} \cdots \int \sum_{i=1}^{n} (-1)^i f_1 \frac{\partial E_i}{\partial x_i}\, \mathrm{d}x_1 \cdots\, \mathrm{d}x_n.$$

We claim that:

$$\sum_{i=1}^{n} (-1)^i \frac{\partial E_i}{\partial x_i} \equiv 0$$

Note if $n = 2$ the expression reduces to the equality of mixed derivatives.
In order to prove the claim above, let $c_{i,j}(\boldsymbol{x})$ denote the determinant of the matrix obtained from $M(\boldsymbol{x})$ by omitting the ith row and replacing the jth row:

$$\left(\frac{\partial f_2}{\partial x_j}, \cdots, \frac{\partial f_n}{\partial x_j} \right)$$

by:

$$\left(\frac{\partial^2 f_2}{\partial x_i \partial x_j}, \cdots, \frac{\partial^2 f_n}{\partial x_i \partial x_j} \right).$$

If we use the rule of differentiating determinants we can see that.

$$\frac{\partial E_i}{\partial x_i} = \begin{vmatrix} \frac{\partial^2 f_2}{\partial x_1 \partial x_i} & \cdots & \frac{\partial^2 f_n}{\partial x_1 \partial x_i} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_2}{\partial x_j} & \cdots & \frac{\partial f_n}{\partial x_j} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x} \end{vmatrix} + \cdots + \begin{vmatrix} \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 f_2}{\partial x_j \partial x_i} & \cdots & \frac{\partial^2 f_n}{\partial x_j \partial x_i} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x} \end{vmatrix} + \cdots + \begin{vmatrix} \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_2}{\partial x_j} & \cdots & \frac{\partial f_n}{\partial x_j} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 f_2}{\partial x_n \partial x_i} & \cdots & \frac{\partial^2 f_n}{\partial x_n \partial x_i} \end{vmatrix}$$

We can see that $\partial E_i/\partial x_i = \sum_{j\neq i} c_{i,j}$.
The equality of mixed derivatives implies that $c_{j,i} = (-1)^{j-i-1}c_{i,j}$, as the row of the second derivative gets shifted $j-i-1$ rows when one passes from $c_{i,j}$ to $c_{j,i}$ if $i<j$ and $i-j-1$ rows otherwise. Hence:

$$\sum_{i=1}^{n}(-1)^i\frac{\partial E_i}{\partial x_i} = \sum_{i=1}^{n}(-1)^i\left(\sum_{j<i}c_{i,j}+\sum_{j>i}c_{i,j}\right) =$$

$$\sum_{j<i}(-1)^i c_{i,j} + \sum_{j>i}(-1)^i(-1)^{j-i-1}c_{j,i} = 0.$$

By substituting $\sum_{i=1}^{n}(-1)^i\frac{\partial E_i}{\partial x_i}$ into $\int_{B^n}\cdots\int J(\boldsymbol{x})\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n$, then we get the contradiction once we prove that:

$$I = \int_{B^n}\cdots\int\sum_{i=1}^{n}(-1)^{i+1}\frac{\partial}{\partial x_i}(f_1E_i)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n \neq 0$$

We now use the divergence theorem for the above integral I, applied to the vector field whose ith component is $(-1)^{i+1}f_1(\boldsymbol{x})E_i(\boldsymbol{x})$. We denote by $\mathrm{d}\sigma$ the surface element on the unit sphere ∂B^n. We also utilize the fact that the unit normal of ∂B^n coincides with $\boldsymbol{x} = (x_1,\cdots,x_n)$. Hence:

$$I = \int_{\partial B^n}\cdots\int f_1(\boldsymbol{x})\sum_{i=1}^{n}(-1)^{i+1}x_iE_i(\boldsymbol{x})\,\mathrm{d}\sigma$$

In order to calculate I, observe that $f_i(\boldsymbol{x}) \equiv x_i$ on ∂B^n. This means $f_i(\boldsymbol{x}) - x_i$ is constant on ∂B^n. Hence $\nabla f_i - \nabla x_i$ is orthogonal to ∂B^n on ∂B^n there. Thus there exists scalars λ_i(depending on $\boldsymbol{x}$) such that $\nabla f_i(\boldsymbol{x}) = \nabla x_i + \lambda_i x$ and the matrix M can be written as:

$$\begin{pmatrix} \lambda_2x_1 & \cdots & \lambda_nx_1 \\ 1+\lambda_2x_2 & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \lambda_2x_n & \cdots & 1+\lambda_nx_n \end{pmatrix}$$

The sum $\sum_{i=1}^{n}(-1)^{i+1}x_iE_i(\boldsymbol{x})$ is equal to the determinant:

$$\begin{vmatrix} x_1 & \lambda_2x_1 & \cdots & \lambda_nx_1 \\ x_2 & 1+\lambda_2x_2 & \cdots & \lambda_nx_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & \lambda_2x_n & \cdots & 1+\lambda_nx_n \end{vmatrix} =$$

$$\begin{vmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 1 \end{vmatrix} = x_1.$$

Moreover $f_1(\boldsymbol{x}) = x_1$ on ∂B. If we insert this result in:

$$I = \int_{\partial B^n} \cdots \int f_1(\boldsymbol{x}) \sum_{i=1}^{n} (-1)^{i+1} x_i E_i(\boldsymbol{x}) \, \mathrm{d}\sigma.$$

We get the result that $I = \int_{\partial B^n} \cdots \int x_1^2 \, \mathrm{d}\sigma > 0$, which contradicts:

$$\int_{B^n} \cdots \int J(\boldsymbol{x}) \, \mathrm{d}x_1 \ldots \, \mathrm{d}x_n = 0.$$

This concludes the proof.

□

For further discussions relating to theorem 2.2 , see [5, p. 265-268].

3 The Brouwer fixed-point theorem

In this section I will state and prove the Brouwer fixed-point theorem using the no-differentiable retraction theorem which I proved in the previous section.
In the first subsection I will introduce necessary definitions, lemmas and theorems which will be necessary in order to prove the Brouwer fixed-point theorem. I will also provide a short history of the Brouwer fixed point theorem.
In the second subsection I will state two versions of the Brouwer fixed-point theorem: one which relates to the closed unit ball in $\mathbb{R}^n$, and another slightly more general version which relates to any compact convex subset in $\mathbb{R}^n$. I will prove both versions, since the second depends on the first. I will also provide an example of theorem in the one-dimensional case.

3.1 Preliminary definitions, theorems and context

The Brouwer fixed-point theorem is one of the most well-known and important existence principles in mathematics. It has proved to be a useful tool in order to prove the existence of solutions in several areas of pure and applied mathematics as well as in fields such as economics and game theory.
The theorem is named after the Dutch mathematician and philosopher, Luitzen Egbertus Jan Brouwer(1881-1966), who made important contributions to fields such as topology, set-theory and complex analysis. He is also the founder of the mathematical philosophy of intuitionism. However, several proofs of specific cases of the theorem were already proved before Brouwer proved it for any finite dimensional case in 1910[13].
The theorem is said to have originated from Brouwer's observation of a cup of coffee. If one stirs the liquid in a cup of coffee it would appear as if at least one point in the liquid does not move, where one can conceive the coffee cup as a compact convex set and the stirring or the moving of the liquid as the transformation[13].
In order to proceed we need to state necessary definitions, theorems and lemmas in order to prove the theorems in the second subsection [1, p.28-33].

Definition 3.1 (Bounded set)**.** A set $S \subset \mathbb{R}^n$ is said to be bounded if it can be contained within a ball of finite radius.

Definition 3.2 (Closed set)**.** A set $S \subset \mathbb{R}^n$ is said to be closed if and only if it contains all of its limit-points. Alternatively a set $S \subset \mathbb{R}^n$ is closed if for any sequence of points $\{X_n\}_{n\in\mathbb{N}}$ in S such that the limit $\lim_{x\to\infty} X_n$ exists it holds that $\lim_{x\to\infty} X_n \in S$.

Definition 3.3 (Compact set)**.** A set $S \subset \mathbb{R}^n$ is said to be compact if and only if it is both closed and bounded.

Definition 3.4 (Fixed-point)**.** For a continuous map $f : X \to X$, a point x is said to be a fixed-point if $f(x) = x$.

Definition 3.5 (Fixed-point property)**.** A space X has the fixed-point property if every map from X to itself has a fixed point.

Definition 3.6 (Convex set)**.** A set $S \subseteq \mathbb{R}^n$ is said to be convex if for every $x, y \in S$, $(1-t)x + ty \in S$ for $t \in [0,1] \subset \mathbb{R}$.

Definition 3.7 (Bijection)**.** A function $f : X \to Y$ is said to be bijective if it is both injective and surjective, that is f is a bijection if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$, and $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. A bijective function will have a well-defined inverse.

Definition 3.8 (Topological space)**.** A topological space is a set X together with a collection of open subsets T of X that satisfies four conditions:

1. The empty set is in T.

2. X is in T .

3. The intersection of a finite number of subsets in T also belong to T.

4. The union of an arbitrary number of sets in T is also in T.

T is called the topology of X.

Definition 3.9 (Continuous function)**.** If X and Y are topological spaces, a map $f : X \to Y$ is said to be continuous if for every open subset $U \subset Y$ its preimage $f^{-1}(U)$ is open in X.

Definition 3.10 (Homeomorphism)**.** Two topological spaces X, Y are homeomorphic if there exists a bijective function $f : X \to Y$ such that both f and f^{-1} are continuous.

Definition 3.11 (Neighborhood)**.** If X is a topological space and p is a point in X, a neighborhood of p is a subset V of X that includes an open set U containing p. That is, $p \in U \subseteq V$.

Definition 3.12 (Hausdorff space)**.** A topological space X is said to be a Hausdorff space if given any pair of points $p_1, p_2 \in X$, there exists neighborhoods U_1 of p_1 and U_2 of p_2 with $U_1 \cap U_2 = \emptyset$.

Theorem 3.1. *If X has the fixed-point property and X is homeomorphic to Y. Then Y have the fixed point property.*

Proof. Let $h : X \mapsto Y$ be a homeomorphism and suppose that $g : Y \to Y$ is continuous. We must show that g has a fixed-point in Y. Notice that

$$h^{-1} \circ g \circ h : X \mapsto X$$

is continuous. Since X has the fixed-point property there exists $x^* \in X$ with

$$h^{-1} \circ g \circ h(x^*) = x^*.$$

Hence $g(y^*) = y^*$ where $h(x^*) = y^*$. This ends the proof. $\square$

Lemma 3.1 (Closed map lemma)**.** Suppose F is a continuous map from a compact space to a Hausdorff space. Then if F is bijective it is a homeomorphism.

The proof of the closed map lemma is omitted since it is beyond the scope of this paper. The proof can be found in [6, p.102-103].

Lemma 3.2 (Any metric space is Hausdorff)**.** Let $M = (X, d)$ be a metric space then M is a Hausdorff space.

Proof. Let $x, y \in X$ and $x \neq y$, then $d(x, y) > 0$. Let $\varepsilon = \frac{d(x,y)}{2}$. Consider the open balls $\mathring{B}^n_\varepsilon(x)$ and $\mathring{B}^n_\varepsilon(y)$. We claim that they are disjoint.
Suppose $\mathring{B}^n_\varepsilon(x)$ and $\mathring{B}^n_\varepsilon(y)$ are not disjoint. Then there exists $z \in M$ such that $z \in \mathring{B}^n_\varepsilon(x)$ and $z \in \mathring{B}^n_\varepsilon(y)$. Then $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. Hence $d(x, z) + d(z, y) < 2\varepsilon = d(x, y)$. This contradicts the definition of a metric. Hence the open balls must be disjoint.
The balls $\mathring{B}^n_\varepsilon(x)$ and $\mathring{B}^n_\varepsilon(y)$ are disjoint open sets. Then by the definition of Hausdorff space the lemma follows. □

Theorem 3.2. *Any compact and convex subset D of $\mathbb{R}^n$ with a non-empty interior is homeomorphic to the closed unit ball $B^n \subset \mathbb{R}^n$.*

Proof. Let $\boldsymbol{p}$ be an interior point of D. Without loss of generality let this interior be $\mathbf{0} = \boldsymbol{p} \in int(D)$. Then there exists a ε such that the open ball $\mathring{B}^n_\varepsilon(\mathbf{0})$ is contained in D using the dilation $\boldsymbol{x} \mapsto \boldsymbol{x}/\varepsilon$ we can assume that the open ball $\mathring{B}_1(\mathbf{0}) \subseteq D$. We claim that each closed ray starting at the origin intersects ∂D in exactly one point. Let R be such a closed ray. Because D is compact, its intersection with R is compact. Therefore there is a point $\boldsymbol{x}_0$ in this intersection at which the distance to the origin assumes its maximum. It follows that $\boldsymbol{x}_0$ lies on the boundary ∂D. To demonstrate that there can be only one such point, we need to show that the line segment from $\mathbf{0}$ to $\boldsymbol{x}_0$ consists entirely of interior points of D except for $\boldsymbol{x}_0$ itself. Any point on this segment other than $\boldsymbol{x}_0$ can be written in form $t\boldsymbol{x}_0$ for $0 \leq t < 1$. Suppose $z \in \mathring{B}^n{}_{1-t}(t\boldsymbol{x}_0)$, and let $\boldsymbol{y} = (\boldsymbol{z} - \boldsymbol{x}_0)/(1-t)$. Since $(1-t)\|\boldsymbol{y}\| = \|\boldsymbol{z} - \boldsymbol{x}_0\| \leq 1-t$, it follows that $\|\boldsymbol{y}\| < 1$, so $\boldsymbol{y} \in \mathring{B}_1(\mathbf{0}) \subseteq D$. Since $\boldsymbol{y}$ and $\boldsymbol{x}_0$ are both in D and $\boldsymbol{z} = t\boldsymbol{x}_0 + (1-t)\boldsymbol{y}$, it follows from convexity that $\boldsymbol{z} \in D$. Thus, the open ball $\mathring{B}^n{}_{1-t}(t\boldsymbol{x}_0)$ is contained in D, which implies that $t\boldsymbol{x}_0$ is an interior point. Now we define a map $\boldsymbol{f} : \partial D \to \partial B^n$:

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

In other words, $\boldsymbol{f}(\boldsymbol{x})$ is the point where the line segment from the origin to $\boldsymbol{x}$ intersects the unit sphere. Since $\boldsymbol{f}$ is the restriction of a continuous map, it is continuous, and the discussion above shows that it is bijective. Since ∂D is compact, $\boldsymbol{f}$ is a homeomorphism by the closed map lemma. Let us define $\boldsymbol{F} : B^n \to D$:

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\| \boldsymbol{f}^{-1}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) & \boldsymbol{x} \neq \mathbf{0} \\ 0 & \boldsymbol{x} = \mathbf{0} \end{cases}$$

Then $\boldsymbol{F}$ is continuous away from the origin because $\boldsymbol{f}^{-1}$ is, and at the origin because boundedness of $\boldsymbol{f}^{-1}$ implies $\boldsymbol{F}(\boldsymbol{x}) \to \mathbf{0}$ as $\boldsymbol{x} \to \mathbf{0}$. Geometrically, $\boldsymbol{F}$ maps each radial line segment connecting $\mathbf{0}$ with a point $\boldsymbol{\omega} \in \partial B^n$ linearly onto the radial segment from $\mathbf{0}$ to the point $\boldsymbol{f}^{-1}(\boldsymbol{\omega})$. By convexity $\boldsymbol{F}$ takes its values in D. The map $\boldsymbol{F}$ is injective since points on distinct rays are mapped to distinct rays, and each radial segment is mapped linearly to its image. It is surjective because each point $\boldsymbol{y} \in D$ is on some ray from $\mathbf{0}$. By the closed map lemma $\boldsymbol{F}$ is a homeomorphism. This concludes the proof. □

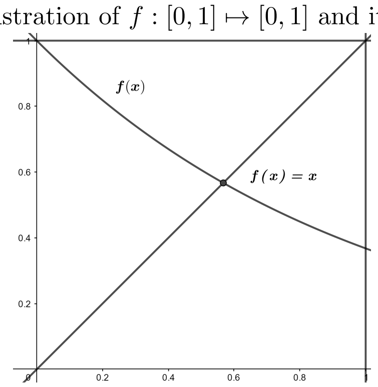
For further discussion relating to theorem 3.4, see [6, p.128-129].

3.2 Brouwer fixed-point theorem: Statement and proof

We are now ready to prove the Brouwer fixed-point theorem. I will prove two equivalent versions of the theorem, where the second is a slightly more general version of the first, since it generalizes the result from the first to any compact and convex subset of $\mathbb{R}^n$.
In the proof of the first version of the Brouwer fixed-point theorem, that is in the case of the closed unit ball, I will apply the no-differentiable retraction theorem from section one. However, this will not be sufficient for any continuous function, just in the case of a twice differentiable function, therefore I will need to complement the proof with an approximation argument.
Let us consider a one-dimensional example of the Brouwer fixed-point theorem.

Example 3.1 (One-dimensional case of Brouwer fixed-point theorem)**.** Consider a continuous function $f : [a, b] \to [a, b]$ then there exists c such that $f(c) = c$. This is a direct application of Brouwers theorem since f is a continuous function from a compact convex set to itself. However, this statement is also an immediate consequence of the well-known intermediate-value theorem. The following figure gives and illustration of this. Any curve going from the left side to the right side must intersect the line segment between 0 and 1.

Figure 1: Illustration of $f : [0, 1] \mapsto [0, 1]$ and its fixed-point.

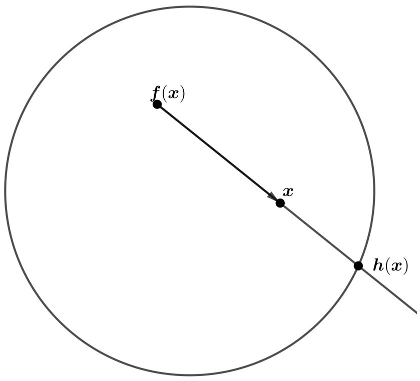


Theorem 3.3. *Let $B^n \subset \mathbb{R}^n$ be the closed unit ball. Let $\boldsymbol{f} : B^n \to B^n$ be a continuous function. Then $\boldsymbol{f}$ has a fixed point. In other words, B^n has the fixed point property.*

Proof. Assume that the function $\boldsymbol{f} : B^n \to B^n$does not have any fixed points. Assume also that $\boldsymbol{f}$ is C^2.
Since $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{x}$, there is a unique line passing through $\boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{x})$. Let $\boldsymbol{h}(\boldsymbol{x})$ be intersection point of this line and the unit sphere that is closer to $\boldsymbol{x}$ than to $\boldsymbol{f}(\boldsymbol{x})$. See Figure 2. If $\boldsymbol{x} \in \partial B^n$, then $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}$, that is $\boldsymbol{h}(\boldsymbol{x})$ restrict to the identity on ∂B^n. Since x is on the line segment between $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$, one can write the vector $\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})$ as a multiple t times the vector $\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})$, where $t \geq 1$, this is illustrated in (figure 2). In other words

$$\boldsymbol{h}(\boldsymbol{x}) = t\boldsymbol{x} + (1-t)\boldsymbol{f}(\boldsymbol{x}).$$

Figure 2: Illustration of the retraction $\boldsymbol{h}(\boldsymbol{x})$.



Taking the dot product on both sides of the formula above we get

$$\boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{h}(\boldsymbol{x}) = (t\boldsymbol{x} + (1-t)\boldsymbol{f}(\boldsymbol{x})) \cdot (t\boldsymbol{x} + (1-t)\boldsymbol{f}(\boldsymbol{x})).$$

Which is equivalent to

$$t^2\|\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\|^2 + 2t\boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})) + \|\boldsymbol{f}(\boldsymbol{x})\|^2 - \|\boldsymbol{h}(\boldsymbol{x})\|^2 = 0,$$

where $\|\boldsymbol{h}(\boldsymbol{x})\| = 1$. Solving this equation for t gives us

$$t = \frac{-\boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})) \pm \sqrt{\|\boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}))\|^2 + \|\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\|^2(1 - \|\boldsymbol{f}(\boldsymbol{x})\|^2)}}{\|\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\|^2}.$$

There are always two distinct real roots. In order to demonstrate this, we need to show that the expression in the square root is always a positive real number.

Since $\|\boldsymbol{f}(\boldsymbol{x})\| \leq 1$ the expression inside the square root must be non-negative, so it remains to show that it is non-zero. If it is zero, then since $\|\boldsymbol{f}(\boldsymbol{x}) - x\| > 0$, it follows that $(\boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}))) = 0$ and $1 - \|\boldsymbol{f}(\boldsymbol{x})\| = 0$, which means that $\|\boldsymbol{f}(\boldsymbol{x})\| = 1$ and $\boldsymbol{x} \cdot \boldsymbol{f}(\boldsymbol{x}) = \|\boldsymbol{f}(\boldsymbol{x})\| = 1$. From the Cauchy-Schwarz Inequality and $\|\boldsymbol{x}\| \leq 1$, it follows that $\|\boldsymbol{x}\| = 1$ which implies that $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$ which is a contradiction. This means that the expression inside the square root is positive, so it has two roots.
There are no roots such that $0 < t < 1$, the triangle inequality implies that

$$\|(1-t)\boldsymbol{f}(\boldsymbol{x}) + t\boldsymbol{x}\| \leq (1-t)\|\boldsymbol{f}(\boldsymbol{x})\| + t\|\boldsymbol{x}\| \leq 1.$$

So the function

$$q(t) = t^2\|\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})\|^2 + 2t\boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x})) + \|\boldsymbol{f}(\boldsymbol{x})\|^2 - 1$$

lies on the interval $[-1, 1]$ if $0 < t < 1$. Suppose the value is zero for some t_0. Since there are two distinct roots for the above polynomial it follows that the latter does not attain its maximum at t_0, hence there is some t_1 such that $0 < t_1 < 1$ and the value of the function at t_1 is positive. This leads to a contradiction of our observation of the behavior of the function that a root such as t_0 must be false.
There is one root of $q(t)$ such that $t \leq 0$ and a second root such that $t \geq 1$. We know that $q(0) \leq 0$ and that $\lim\limits_{t \to -\infty} q(t) = +\infty$. By continuity there must be some t_1 such that $q(t_1) = 0$. In a similar way we know that $q(1) \leq 0$ and $\lim\limits_{t \to +\infty} q(t) = +\infty$, so again by continuity there must be a $t_2 > 1$ such that $q(t_2) = 0$.
The unique root satisfying $t \geq 1$ is then a C^2 function of $\boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{x})$. It therefore follows that $\boldsymbol{h}(\boldsymbol{x})$ for this t is a C^2 function since $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{x}$[11].
We can therefore conclude that $\boldsymbol{h}(\boldsymbol{x})$ is a twice differentiable retraction from B^n to ∂B^n, contradicting the No-differentiable retraction theorem [5, p.268].
It now remains to show that this result implies there exists no continuous retraction. We can demonstrate this by an approximation argument.
Let us begin by supposing that a continuous $\boldsymbol{f} : B^n \to B^n$ is without fixed point, then $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}\| > 0$ for all $\boldsymbol{x} \in B^n$, this implies that there exists a $\varepsilon > 0$ such that $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}\| > \varepsilon$ for all $\boldsymbol{x} \in B^n$.
It follows from the fact that smooth functions are dense among continuous functions[4, p. 47] that there exists a smooth function $\tilde{\boldsymbol{f}} : B^n \to B^n$ such that $\|\boldsymbol{f}(\boldsymbol{x}) - \tilde{\boldsymbol{f}}(\boldsymbol{x})\| < \frac{\varepsilon}{2}$ for all $\boldsymbol{x} \in B^n$.
We now claim that $\tilde{\boldsymbol{f}}$ does not have a fixed point. Indeed for all $x \in B^n$

$$\varepsilon < \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}\| = \|\boldsymbol{f}(\boldsymbol{x}) - \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \tilde{\boldsymbol{f}}(\boldsymbol{x}) - \boldsymbol{x}\| \leq \|\boldsymbol{f}(\boldsymbol{x}) - \tilde{\boldsymbol{f}}(\boldsymbol{x})\| + \|\tilde{\boldsymbol{f}}(\boldsymbol{x}) - \boldsymbol{x}\| <$$

$$\frac{\varepsilon}{2} + \|\tilde{\boldsymbol{f}}(\boldsymbol{x}) - \boldsymbol{x}\|.$$

Then

$$0 < \frac{\varepsilon}{2} < \|\tilde{\boldsymbol{f}}(\boldsymbol{x}) - \boldsymbol{x}\|.$$

Therefore $\tilde{\boldsymbol{f}}$ does not have any fixed point, which demonstrates that the above result hold in the case of a general continuous function. This concludes the proof. □

I will now state and prove the Brouwer fixed-point theorem in the case of any compact convex subset of $\mathbb{R}^n$. This will heavily rely on theorems from the previous subsection.

Theorem 3.4 (Brouwer fixed-point theorem)**.** *If S is a compact, convex subset of $\mathbb{R}^n$ with an non-empty interior, and $\boldsymbol{f}$ is a continuous map $\boldsymbol{f} : S \to S$ then there exists a $\boldsymbol{x} \in S$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.*

Proof. Since S is a compact convex subset of $\mathbb{R}^n$ with an non-empty interior, therefore S is homeomorphic to the closed unit ball B^n by theorem 3.2. We know from theorem 3.3 that B^n has the fixed-point property. Since B^n has the fixed-point property and B^n homeomorphic to S, it follows from theorem 3.1 that S has the fixed-point property. This proves the theorem.

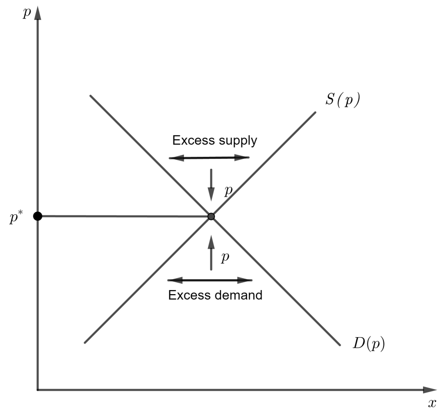
4 Walrasian equilibrium

In this section I will prove the existence of a Walrasian price equilibrium using the second version of the Brouwer fixed-point theorem presented in the previous section.
In the first subsection I will present a basic introduction of general equilibrium theory. This will include some history of general equilibrium theory as well as definitions and propositions.
In the second subsection I will state and prove the existence of a general price equilibrium or as it is sometimes called a Walrasian price equilibrium [12, p.31-37].

4.1 Preliminary definitions, lemmas and context

Economics have always been concerned with the relationship between supply and demand, and how equilibrium between these occurs. The study of this is called equilibrium analysis. One of the central questions in equilibrium analysis is if there exist prices which equilibrate supply and demand.
The study of market equilibrium within the field of economics can be divided into two different types of models. The first is called partial equilibrium theory which studies equilibrium in a single market. The second is called general equilibrium theory, which studies the simultaneous equilibrium of all markets in an economy. There is a well-known illustration of this in the partial equilibrium case. The diagram has the independent variable on the y-axis and the dependent variable on the x-axis since this is the convention in economics.

Figure 3: Supply and demand with equilibrium price p^* .



One of the first attempts at studying general equilibrium was made by the French economist Léon Walras[14]. For Léon Walras the price mechanism was crucial for general equilibrium, that is the process which leads an economy to an equilibrium price which clears all markets. Léon Walras imagined a scenario of

a price adjustment process where the agents meet on a public square where a so called 'Walrasian auctioneer' calls out prices. After the auctioneer has done this the agents calls out their demands at those prices. The auctioneer then adjust the prices and calls out a new price, this process repeats itself until demand equals supply, that is the general equilibrium or Walrasian equilibrium.
From the study of general equilibrium two central questions arise:

1. Whether a general equilibrium even exists?

2. And if it exists, what properties does it have?

I will in this paper focus on the first question. The second question relates to questions about uniqueness of equilibrium and welfare or efficiency properties of equilibrium. The question relating to existence was proved using fixed point theory by Gérard Debreu and Kenneth Arrow in their 1954 article on the existence of competitive equilibrium [2]. For further discussion about welfare properties of equilibrium such as the fundamental theorems of welfare economics, see [7, p.3-24], [12, p.141-151], [9, p.545-627].

I will now consider a simple Walrasian or general equilibrium model of an exchange economy.
The Walrasian model allows us to consider an exchange economy with I agents $i \in \mathcal{A} = \{1, 2, \ldots, I\}$ and n commodities or goods $l \in \mathcal{C} = \{1, 2, \ldots, n\}$, a bundle of commodities or goods are described as a non-zero vector $\boldsymbol{x} \in \mathbb{R}^n_+$, and a non-zero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$. Each agent i has an non-zero endowment $\boldsymbol{e}^i \in \mathbb{R}^n_+$ and a utility function $u^i : \mathbb{R}^n_+ \to \mathbb{R}$. We therefore define the economy $\mathcal{E} = \big((\boldsymbol{e}^i, u^i)_{i \in \mathcal{A}}\big)$. Each agent i is faced with an optimization problem of maximizing his utility function $u^i(\boldsymbol{x})$ subject to his budget constraint $\mathcal{B}_i$. The budget constraint can be interpreted as stating that the agent cannot consume for a higher value then the value of his endowments. One can construct the following optimization problem:

$$\max u^i(\boldsymbol{x})$$

subject to:

$$\boldsymbol{x} \in \mathcal{B}_i = \{\boldsymbol{x} \cdot \boldsymbol{p} \leq \boldsymbol{e}^i \cdot \boldsymbol{p}\}$$

Given an initial endowment, the solution to this optimization problem yields the demand function $\boldsymbol{x}^i = \boldsymbol{D}^i(\boldsymbol{p})$ for agent i.

Definition 4.1 (Walrasian equilibrium)**.** A Walrasian equilibrium or general equilibrium for an economy $\mathcal{E}$ is defined as a vector $\big(\boldsymbol{p}^*, (\boldsymbol{x}^i)_{i \in \mathcal{A}}\big)$ that satisfies two conditions:

1. For all $i \in \mathcal{A}$

$$\boldsymbol{x}^i \in \arg\max_{\boldsymbol{x} \in \mathcal{B}_i} u^i(\boldsymbol{x})$$

2. Markets clear for all $l \in \mathcal{C}$

$$\sum_{i \in \mathcal{A}} D_l^i(\boldsymbol{p}) = \sum_{i \in \mathcal{A}} e_l^i$$

Remark 4.1. Arguments of the maxima or "arg max" are the points of the domain of some function at which the function values are maximized.

In general equilibrium theory, only relative prices matter not their numerical values. Therefore we can define our price space to the 'price simplex'.

Definition 4.2 (Price simplex)**.** The 'Price simplex' is defined in the same way as the unit simplex.

$$P = \left\{ \boldsymbol{p} \in \mathbb{R}^n : \boldsymbol{p} \geq \boldsymbol{0}, \sum_{l=1}^{n} p_i = 1 \right\} \subset \mathbb{R}^n$$

Lemma 4.1. The price simplex P is convex and compact.

Proof. This follow from the fact that P is just the unit simplex which has the property of being compact and convex. To show that P is convex let $\boldsymbol{x}, \boldsymbol{y} \in P$, for $t \in [0,1]$. We can see that since$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k = 1$:

$$\sum_{k=1}^{n} (tx_k + (1-t)y_k) = t \sum_{k=1}^{n} x_k + (1-t) \sum_{k=1}^{n} y_k = 1$$

Hence $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in P$, in other words P is convex.
We know that P is bounded since for any $\boldsymbol{p} \in P$,

$$\|\boldsymbol{p}\| = \sum_{k=1}^{n} p_k^2 \leq \left(\sum_{k=1}^{n} p_k \right)^2 = 1.$$

This proves it is bounded.
Since P is the intersection of two closed sets

$$\{\boldsymbol{p} \in \mathbb{R}^n : \boldsymbol{p} \geq \boldsymbol{0}\}$$

and

$$\left\{ \boldsymbol{p} \in \mathbb{R}^n : \sum_{k=1}^{n} p_k = 1 \right\}$$

The intersection of two closed sets is closed. Therefore, P is closed. This proves the lemma.

□

Definition 4.3 (The excess demand function)**.** The excess demand function $\boldsymbol{Z} : P \to \mathbb{R}^n$, is defined as:

$$\boldsymbol{Z}(\boldsymbol{p}) = \sum_{i \in \mathcal{A}} \boldsymbol{D}^i(\boldsymbol{p}) - \sum_{i \in \mathcal{A}} \boldsymbol{e}^i.$$

Where $\boldsymbol{D}^i : P \to \mathbb{R}^n$ is the demand function for agent i.
The excess demand function $\boldsymbol{Z}$ is assumed to have two properties:

1. Walras' Law: For $\boldsymbol{p} \in P$

$$\boldsymbol{p} \cdot \boldsymbol{Z}(\boldsymbol{p}) = \sum_{l=1}^{n} p_l Z_l(\boldsymbol{p}) = 0$$

2. Continuity: $\boldsymbol{Z}(\boldsymbol{p})$ is a continuous function.

In the case of the excess demand function we say there is an excess demand if $Z_l > 0$ for some good l and there is an excess supply of $Z_l < 0$, for some good l. Walras' law can be thought of as a scarcity assumption, that is, it says that if there exists an excess demand it must be matched with an excess supply in another market.

Definition 4.4 (Equilibrium price)**.** $\boldsymbol{p}^* \in P$ is said to be an equilibrium price vector if $\boldsymbol{Z}(\boldsymbol{p}^*) \leq \boldsymbol{0}$(the inequality holds coordinatewise) with $p_l^* = 0$ such that $Z_l(\boldsymbol{p}^*) < 0$. That is, $\boldsymbol{p}^*$ is an equilibrium price vector if demand $\boldsymbol{D}^i(\boldsymbol{p}^*)$ equals the endowments with possible excess supply of free commodities.

Definition 4.5 (The price adjustment function)**.** The price adjustment function $\boldsymbol{T} : P \to P$ is defined for market l as:

$$T_l(\boldsymbol{p}) = \frac{\max[0, p_l + Z_l(\boldsymbol{p})]}{\sum_{k=1}^{n} \max[0, p_k + Z_k(\boldsymbol{p})]}.$$

This price adjustment function can be thought of as the "Walrasian auctioneer" adjusting prices such that demand equal supply. If there is an excess demand auctioneer increases the price, if there is an excess supply the auctioneer lowers the price until equilibrium is attained.

Lemma 4.2. If Walras law is fulfilled and $\boldsymbol{Z}(\boldsymbol{p})$ is continuous then the price adjustment function $\boldsymbol{T}(\boldsymbol{p})$ is also continuous.

Proof. This follows from the fact that the operations max, sum and division with by an nonzero continuous function maintain continuity. The denominator $\sum_{k=1}^{n} \max[0, p_k + Z_k(\boldsymbol{p})]$ in $T_l(\boldsymbol{p})$ can never be zero because this would mean that all goods have excess supply simultaneously, which violates the scarcity assumption of Walras law. □

4.2 The existence of Walrasian price equilibrium: Statement and proof

I will now proceed to state the theorem of existence of a general equilibrium or an Walrasian equilibrium in an exchange economy[12].
I will use the results and definitions from the previous subsection in order to prove the theorem. In particular lemma 4.1 and lemma 4.2 will be crucial in order to apply the Brouwer fixed-point theorem. This is because P is a compact convex set by lemma 4.1 and $\boldsymbol{T} : P \to P$ is a continuous function by lemma 4.2.

Theorem 4.1. *Let Walras law and the continuity assumptions be fulfilled, then there exists a* $\boldsymbol{p}^* \in P$ *such that* $\boldsymbol{p}^*$ *is the equilibrium price.*

Proof. This proof relies on the lemma that the price simplex is compact and convex and the lemma which states that the price adjustment function is continuous. Since P is compact and convex and $\boldsymbol{T} : P \to P$ is a continuous function it follows from the Brouwer fixed-point theorem that there exists a fixed point $\boldsymbol{p}^* \in P$ such that $T(\boldsymbol{p}^*) = \boldsymbol{p}^*$, that is the price in which the auctioneer stops adjusting. We need to show that this break in doing adjustments is the right thing to do for the auctioneer, that is, we need to show that this $\boldsymbol{p}^*$ constitute a general equilibrium price. That is, all markets clear with the exception of potential oversupply of free good. We have to divide this in two cases.
If we consider the case when $\boldsymbol{T}(\boldsymbol{p}^*) = \boldsymbol{p}^*$ for each good k, then $T_k(\boldsymbol{p}^*) = p_k^*$. Looking at the numerator we can distinguish two cases.
Case 1:

$$p_k^* = 0$$

Case 2:

$$p_k^* = \frac{\max[0, p_k^* + Z_k(\boldsymbol{p}^*)]}{\sum_{k=1}^{n} \max[0, p_k^* + Z_k(\boldsymbol{p}^*)]} > 0.$$

In **Case 1** we have by Brouwer the fixed-point p_k^*. We can then show from the price adjustment function that the following equality must hold

$$p_k^* = 0 = max[0, p_k^* + Z_k(\boldsymbol{p}^*)].$$

Hence $0 \geq p_k^* + Z_k(\boldsymbol{p}^*) = Z_k(\boldsymbol{p}^*)$ and $Z_k(\boldsymbol{p}^*) \leq 0$. This is the case of free goods.
In **Case 2**: Let

$$\lambda = \frac{1}{\sum_{k=1}^{n} \max[0, p_k^* + Z_k(\boldsymbol{p}^*)]}.$$

So that $T_k(\boldsymbol{p^*}) = \lambda(p_k^* + Z_k(\boldsymbol{p}^*))$. Since $\boldsymbol{p^*}$ is the fixed point of $\boldsymbol{T}$ we have $p_k^* = \lambda(p_k^* + Z_k(\boldsymbol{p^*})) > 0$. This expression holds true for all k with $p_k^* > 0$ and λ is the same for all k. Let us consider

$$(1 - \lambda)p_k^* = \lambda Z_k(\boldsymbol{p}^*),$$

Then multiply both sides by $Z_k(\boldsymbol{p}^*)$ to get

$$(1-\lambda)p_k^* Z_k(\boldsymbol{p}^*) = \lambda Z_k(\boldsymbol{p}^*)^2$$

Now sum over all k in **Case 2**

$$(1-\lambda)\sum_{k\in Case2} p_k^* Z_k(\boldsymbol{p}^*) = \lambda \sum_{k\in Case2} Z_k(\boldsymbol{p}^*)^2$$

Walras law then says

$$0 = \sum_{k=1}^{n} p_k^* Z_k(\boldsymbol{p}^*) = \sum_{k\in Case1} p_k^* Z_k(\boldsymbol{p}^*) + \sum_{k\in Case2} p_k^* Z_k(\boldsymbol{p}^*).$$

For $k \in Case1$, $p_k^* Z_k(\boldsymbol{p}^*) = 0$ so

$$0 = \sum_{k\in Case1} p_k^* Z_k(\boldsymbol{p}^*)$$

Therefore,

$$\sum_{k\in Case2} p_k^* Z_k(\boldsymbol{p}^*) = 0.$$

Hence,

$$0 = (1-\lambda)\sum_{k\in Case2} p_k^* Z_k(\boldsymbol{p}^*) = \lambda \sum_{k\in Case2} Z_k(\boldsymbol{p}^*)^2.$$

Using Walras' Law we established that the left-hand side equals 0, but the right-hand side only equals 0 if $Z_k(\boldsymbol{p}^*) = 0$ for all k such that $p_k^* > 0$ (in **Case 2**). Thus $\boldsymbol{p}^*$ is an equilibrium price. This concludes the proof. □

5 Concluding remarks

It is rather remarkable that one can construct a mathematical narrative starting with the divergence theorem and end with a proof of the existence of price equilibrium in an economy. Since the divergence theorem is usually applied in the field of physics and engineering, one can then consider it rather strange that is has something to do with an economic model. Furthermore, the Brouwer fixed-point theorem can initially seem somewhat divorced from any practical application, least of all an economic application.

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6 Appendix

Here I will present an alternative way to prove the Brouwer fixed-point theorem.

Theorem 6.1. *Every non-empty closed, convex subset D of $\mathbb{R}^n$ is a retract of $\mathbb{R}^n$.*

Proof. Let us define $\boldsymbol{R}_D : \mathbb{R}^n \to D$. For any $\boldsymbol{x} \in \mathbb{R}^n$, there exists a unique $\boldsymbol{y} \in D$ with an minimum distance from $\boldsymbol{x}$, that is [3, p.50]:

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \inf\{\|\boldsymbol{x} - \boldsymbol{c}\| : \boldsymbol{c} \in D\}.$$

We define $\boldsymbol{R}_D(\boldsymbol{x}) = \boldsymbol{y}$ to be a function sending a point $\boldsymbol{x} \in \mathbb{R}^n$ to its nearest point in D. We need to show that it is continuous. It is enough to show that $\boldsymbol{R}_D$ is non-expansive. In other words we want to show that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$\|\boldsymbol{R}_D(\boldsymbol{x}) - \boldsymbol{R}_D(\boldsymbol{y})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Let us denote $\boldsymbol{R}_D(\boldsymbol{x})$ and $\boldsymbol{R}_D(\boldsymbol{y})$ by $\boldsymbol{x}'$ and $\boldsymbol{y}'$ respectively. Because of convexity we know that for all $\boldsymbol{x}', \boldsymbol{y}' \in D$ and for $t \in (0, 1)$

$$(1 - t)\boldsymbol{x}' + t\boldsymbol{y}' \in D.$$

By definition $\|\boldsymbol{x}' - \boldsymbol{x}\|$ is the minimum distance between $\boldsymbol{x}$ and any point in D. Therefore,

$$\|[(1 - 0)\boldsymbol{x}' + 0\boldsymbol{y}'] - \boldsymbol{x}\|^2 = \|\boldsymbol{x}' - \boldsymbol{x}\|^2 \leq \|[(1 - t)\boldsymbol{x}' + t\boldsymbol{y}'] - \boldsymbol{x}\|^2.$$

Therefore $\|[(1 - t)\boldsymbol{x}' + t\boldsymbol{y}'] - \boldsymbol{x}\|^2$ is increasing at $t = 0$. That is,

$$\frac{d}{dt}\|[(1 - t)\boldsymbol{x}' + t\boldsymbol{y}'] - \boldsymbol{x}\|^2 = 2((1 - t)\boldsymbol{x}' + t\boldsymbol{y}' - \boldsymbol{x}) \cdot (\boldsymbol{y}' - \boldsymbol{x}') \geq 0.$$

Then at $t = 0$, we know $(\boldsymbol{x}' - \boldsymbol{x}) \cdot (\boldsymbol{y}' - \boldsymbol{x}') \geq 0$. In the same way we know that at $t = 0$:

$$\frac{d}{dt}\|[(1 - t)\boldsymbol{y}' + t\boldsymbol{x}' - \boldsymbol{y}] - \boldsymbol{x}\|^2 = (\boldsymbol{y}' - \boldsymbol{y}) \cdot (\boldsymbol{x}' - \boldsymbol{y}') \geq 0$$

Now consider a function $d : \mathbb{R} \to \mathbb{R}^n$ which we define as

$$d(t) = \|\boldsymbol{x}' - \boldsymbol{y}' + t[\boldsymbol{x} - \boldsymbol{x}' - (\boldsymbol{y} - \boldsymbol{y}')]\|^2,$$

where we can see that

$$\frac{d}{dt}d(0) = 2(\boldsymbol{y}' - \boldsymbol{x}') \cdot (\boldsymbol{x}' - \boldsymbol{x}) + 2(\boldsymbol{x}' - \boldsymbol{y}') \cdot (\boldsymbol{y}' - \boldsymbol{y}) \geq 0.$$

Hence $d(t)$ is an upwards sloping parabola therefore $d(t)$ is non-decreasing on the interval $[0, \infty)$. Then

$$d(0) = \|\boldsymbol{x}' - \boldsymbol{y}'\|^2 \leq \|\boldsymbol{x} - \boldsymbol{y}\|^2 = d(1).$$

Hence $\boldsymbol{R}_D$ is non expansive and therefore continuous. Also, we know that for all $\boldsymbol{x} \in D$

$$\boldsymbol{R}_D(\boldsymbol{x}) = \boldsymbol{x}.$$

Therefore $\boldsymbol{R}_D$ is a continuous function and a retraction, this concludes the proof. □

Theorem 6.2. *If X has the fixed point property and $A \subseteq X$ is retract of X then A has the fixed-point property.*

Proof. Let $f : A \to A$ be a continuous function and $r : X \to A$ is a retraction. We must show that f has a fixed point in A. Notice that

$$f \circ r : X \to A \subseteq X$$

Since X has the fixed-point property there exists $x^* \in X$ with

$$f \circ r(x^*) = x^*$$

However $f(r(x^*)) \in A$ and therefore $x^* \in A$. Since $x^* \in A$ and $r : X \to A$ is a retraction, we have $r(x^*) = x^*$. As a result we have $f(x^*) = x^*$, $x^* \in A$. This ends the proof. □

Theorem 6.3 (Brouwer fixed-point theorem)**.** *If S is a compact, convex subset of $\mathbb{R}^n$, and $\boldsymbol{f}$ is a continuous map $\boldsymbol{f} : S \to S$ then there exists a $\boldsymbol{x} \in S$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.*

Proof. Since S is compact it is bounded. Therefore there exists a $K > 0$ such that for all $\boldsymbol{x} \in S$ $\|\boldsymbol{x}\| < K$. This implies that there exists a closed ball $B_K^n \subset \mathbb{R}^n$ with finite radius K such that $S \subset B_K^n$. Since B^n has the fixed-point property and B_K^n is homeomorphic to B^n then B_K^n also have the fixed-point property. Since S is a non-empty, closed and convex subset of $\mathbb{R}^n$ then S is a retract of $\mathbb{R}^n$. Since B_K^n has the fixed-point property and $S \subset B_K^n$ and S is a retract of B_K^n, then one can conclude that S has the fixed point property. This proves the theorem. □

A similar proof of theorem 6.3 using other methods to prove the fixed point property of the closed unit ball can be found in [8, p.13].