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## The Heat Equation

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**Abstract**

The aim of this thesis is to investigate various aspects of the heat equation. We first consider the derivation of the heat equation and relevant historical background. Thereafter, we explore the Fourier series solution to the heat equation in terms of the method of *separation of variables*. The analysis of the solutions to the heat equation are examined in light of two of their properties; that is to say, uniqueness and existence. Furthermore, the thesis treats two boundary conditions; namely, the homogeneous and inhomogeneous Neumann boundary condition and Dirichlet boundary condition for the homogeneous heat equation; with a focus on the latter. Finally, the thesis studies the finite or bounded domains in which we assume $a < x < b$ that is scaled to $0 < x < 2\pi$ in a one dimensional space where $x \in \mathbb{R}$ of an idealized, homogeneous rod that is infinitely thin.

**Keywords**: heat equation, Fourier series, partial differential equations, diffusion, separation of variables.

# Contents

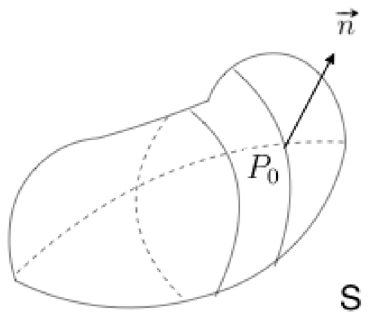
# 1 Introduction

## 1.1 Preliminaries and Notational Conventions

### 1.1.1 Heat Flux and Thermal Conductivity

**Definition 1.1.** Let $P_0 = (x_0, y_0, z_0)$ be a point on a body $\Omega$ (subset in $\mathbb{R}^n$) and assume its surface $S$ is a smooth surface through a point $P_0$ (see figure below):

Figure 1: A Body $\Omega$




Let $\vec{n}$ be a surface normal to $S$ at point $P_0$. At time $t$, the heat flux $\Phi = \Phi(P_0, t)$ along $S$ at point $P_0$ in the vector direction $\vec{n}$ is the amount of heat flow rate intensity in terms of energy per unit of time and unit of area that passes along $P_0$ in that direction. As such, the heat flux is measured in $\mathrm{J/m}^2 s$.

If $u(x, y, z, t)$ represents the temperature at the points $(x, y, z)$ of the body at time $t$ and if $n$ indicates the magnitude of the distance in direction $\vec{n}$; namely $n = |\vec{n}|$, then the heat flux $\Phi(x_0, y_0, z_0, t)$ is positive when the directional derivative $\frac{du}{dn}$ is negative at point $P_0$ and negative when $\frac{du}{dn}$ is positive at the same point. A fundamental postulate, known as Fourier's law within the mathematical theory of thermal conductivity, states that the magnitude of the flux $\Phi(x_0, y_0, z_0, t)$ is proportional to the directional derivative $\frac{du}{dn}$ (or temperature gradient $\nabla U$) at point $P_0$ and time $t$ so that the local heat flux density for an isotropic body $\vec{q}$; namely, one whose thermal and mechanical properties are identical in all directions, is given by

$$\vec{q} = -k\nabla U$$

where $k$ is a constant called the thermal conductivity of the material [W/mK][1].

**Definition 1.2.** Heat energy is the transmission of thermal energy from one system to another by kinetic energy due to a difference in temperature, flowing from a warmer (energy source) to a cooler object (energy receiver) with SI unit Joule (J). Thermal energy is the random kinetic energy of the moving particles in matter. However, when heated, objects expand and so the bonds that keep the atoms together stretch. This results in more elastic energy. Thus, thermal energy is the sum of kinetic and elastic energy of atoms and molecules. It is a type of internal energy since it is energy that is within the object [14, p. 128].

**Definition 1.3.** Entropy is described as the dispersal of energy. The larger the dispersal or spreading, the greater the entropy. Entropy was first introduced by Rudolf Clausius (1822-1888) in response to Carnot's use of the term *waste heat*. Clausius thus created an alternative version of the Second Law using the term *entropy*. To exemplify, suppose one adds ice to a glass of water: the water and ice are separate. The water has higher thermal energy compared to the ice, and so the system has low entropy. When the ice melts the water and ice can no longer be dissociated from one another. The thermal energy is dispersed throughout the system; thus, entropy has increased. However, thermal energy from the (warm) environment has been dispersed to the ice water (the system), and so the entropy of the environment has decreased. Whatsoever, calculations illustrate that the increase in the ice/water mixture is more significant than the decrease in the surrounding environment. Thus, an assertion of the Second Law is that the entropy of the system and the environment can never decrease. Maximum entropy is achieved when the temperature of the system and environment reaches equilibrium [14, p. 135].

[1]Square brackets denote the "dimension of"

# 2 The Heat Equation

## 2.1 Definition of the Heat Equation

A partial differential equation or PDE is any differential equation that consists of an unknown function of multiple independent variables and certain partial derivatives of that function. The distribution of thermal energy within a body $\Omega$ in $\mathbb{R}^n$ (where $\Omega \subset \mathbb{R}^n$ is open) can be described then, under appropriate premises, by the PDE

$$u_t = k\Delta u, \qquad x \in \Omega, \tag{1}$$

where $u(x,t)$ represents the temperature at a given point $x$, and time $t > 0$ . The Laplacian $\Delta$ is taken with regards to the spatial variables with arbitrary dimensions $x = (x_1, ..., x_n) : \Delta u = \Delta_x u = \sum_{i=1}^{n} u_{x_i x_i}$. Here, it is sufficient to assume that $k$ is a positive constant, known as the thermal diffusivity of the body. It governs the thermal conductivity of the medium, which by scaling $x$ allows us to fix it equal to 1 [6]. Otherwise, the coefficient $k$ is given by

$$k = \frac{\lambda}{\rho c}$$

where $\lambda$ is the thermal conductivity, $c$ is the specific heat capacity per unit mass $[c] = H/mT$, (basic units are $H$ = heat energy, $m$ = mass, $T$ = temperature) in other words the amount of heat energy required to increase the temperature of a material per unit mass [14], and $\rho$ is the density of the body (mass per unit volume). If not specified otherwise; then, $k$, $c$, and $\rho$ will be constant throughout the body. Under these assumptions, the derivatives of the spatial variables $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2})$ together with the temperature function are continuous throughout the entire interior of the corresponding body in which no heat is generated or destroyed. When

$$u_t - k\Delta u = 0$$

the equation is said to be homogeneous. The physical interpretation of this phenomenon can be such that the material lacks access to any heat sources. On the contrary, when

$$u_t - k\Delta u \neq 0$$

the partial differential equation is inhomogeneous and the opposite physical property applies. Physically, the heat equation describes the transmission of heat per unit volume over an infinitesimally small volume in a domain and obeys the second law of thermodynamics; namely, that all natural processes authorize heat to travel in the direction that prompts entropy to increase, i.e from bodies of higher temperature to bodies of lower temperature in efforts to reach equilibrium anew. This causes an irreversible process to transpire (entropy flows in the direction that prompts an increase in the entropy of the system and environment where the final entropy is greater than the initial entropy); after which, the entropy of the system plus the environment can remain constant for a reversible process to commence; namely where the initial entropy is equal to the final entropy in an equilibrium state [13]. In addition to this, the heat equation also describes how the density of some quantity varies in time, for instance the chemical concentration of a substance. An example is the rate of diffusion of gases and liquids. In a similar manner to conduction and the second law of thermodynamics, diffusion describes the procedure in which particles uniformly disperse from areas of higher concentration to areas of lower concentration. This partial differential equation is commonly stated together with an initial condition that designates the initial temperature distribution in a material, as well as a boundary condition (or lack thereof) that can take various forms and describes what occurs at the endpoints. For reasons of simplicity, we will from this point forward consider the one-dimensional case where $x \in \mathbb{R}$ and consider an idealized, homogeneous rod that is infinitely thin. Thus, the initial condition may be represented by

$$u(x,0) = g(x), \qquad 0 < x < a, \quad x \in \mathbb{R}, \tag{2}$$

where $g(x)$ is a distribution function of heat and is a function of $x$ only. By doing this, the initial temperature at every point on the material is specified; namely, the initial temperature distribution of the rod $u(x, 0)$ [10, p. 130]. Boundary conditions on the other hand, may vary. If the temperature at any end is kept constant, for example by the use of an ice water bath or heat; the conditions can be expressed as follows,

$$u(0,t) = T_0, \qquad u(a,t) = T_1 \qquad t > 0 \tag{3}$$

where $T_1$ and $T_0$ may be identical or different. Principally, the temperature at the boundary does not necessarily need to remain constant but merely be regulated or managed so as to be controlled. If we from this point onward consider $a$ and $b$ to be the endpoints (unless specified otherwise); namely, we assume the finite interval $a < x < b$, the boundary conditions become

$$u(a,t) = A(t), \quad u(b,t) = B(t) \tag{4}$$

where $A(t)$ and $B(t)$ are functions of time [10, p. 131]. This is known as the Dirichlet boundary condition or *condition of the first kind*. Overall, this can be expressed as,

$$\begin{cases} \partial_t u = u_{xx} + f(x,t) \\ u(x,0) = g(x), \quad a < x < b \\ u(a,t) = A(t), \quad u(b,t) = B(t) \end{cases}$$

where $f(x,t)$ is a function of space and time. Another boundary condition is known as the Neumann condition or *condition of the second kind* where the rate of flow of heat is regulated. In this case, a gradient of the type

$$u'_x(a,t) = A', \quad u'_x(b,t) = B' \tag{5}$$

is applied to each extremity. This is permitted due to the fact that Fourier's law of heat conduction stipulates that the heat flow rate is proportional to the magnitude of the negative gradient of the temperature [10]. In one dimension it is given by,

$$q = -ku'_x, \tag{6}$$

where $q$ is the heat flow rate in the positive direction, $k$ is the thermal conductivity, units $[k] = L^2/T$ ($L$ = length, $T$ = time) and $u'_x$ is the negative temperature gradient. Often, $A'$ which is a function of time is taken to be equal to zero,

$$u'_x(a,t) = 0$$

illustrating an insulated surface since there is no heat flow. In consideration of a finite body, the same can hold true for the surface at the other end. Specifically, the homogeneous Neumann boundary condition with respect to the inhomogeneous heat equation, stipulates the values of the derivative of the solution on the boundaries, which together with the initial conditions can be expressed in $\mathbb{R}$ as follows,

$$\begin{cases} \partial_t u = u_{xx} + f(x,t) \\ u(x,0) = g(x), \quad a < x < b \\ u_x(a,t) = A'(t), \quad u_x(b,t) = B'(t). \end{cases}$$

If $A' = B' = 0$, the rate of change in terms of transfer of heat in and out the boundary (flow rate) is null; thus, heat distribution is constant and controlled so the edge is insulated. The homogeneous Dirichlet condition on the other hand, maintains that the value (of the temperature) of the solution on the boundary is specified and equal to zero.

Furthermore, a union of the two can give rise to various different boundary conditions: one of which is known as the Robin boundary condition, named after French mathematical analyst Victor Gustave Robin, which describes the linear combination of the values of the derivative at the one boundary as well as the values of the function [10, p. 131]. This is a condition of the third kind and fulfills Newton's law of cooling [10, p. 131]. There are a myriad of properties the heat equation possesses; namely, stability, maximum principle, linearity, regularity, existence and uniqueness. This thesis however will be interested in the two latter for the analysis of the existence-solution; specifically, the Fourier series solution in terms of separation of variables. Furthermore, the thesis will treat the two of the aforementioned boundary conditions; namely, the homogeneous and inhomogeneous Neumann boundary condition for the homogeneous heat equation, as well as the homogeneous and inhomogeneous Dirichlet boundary condition for the homogeneous heat equation. The thesis will consider the finite case where we assume $a < x < b$ which is scaled to $0 < x < 2\pi$ in a one dimensional space.

The first chapter of this thesis is concerned with preliminaries, theorems, definitions and notational conventions deemed necessary.

The second chapter will deal with the derivation and definition of the heat equation, boundary conditions, related equations, and historical background where most information has been gathered from [16] and [10].

Chapter three in which we treat the 1D heat equation where the Fourier method is presented will begin with a section towards Fourier analysis which relies heavily on the theory presented in [15] and [18]. The method, i.e separation of variables uses elementary analysis.

## 2.2 Derivation of the Heat Equation

Consider a rod made of a specific heat-conductive material whose cylindrical surface is insulated, as such represented in Figure 1. The first problem that becomes evident in the quest to derive this PDE, is whether temperature can be expressed to account for all types of bodies: namely, those where temperature is uniformly distributed contra those where it is non-uniformly distributed [10]. To simplify matters, we intend to assume a uniform rod (made from a single material: where quantities such as volume, area or length of the material will be equal to the mass of any other equal quantity of the specific material; the specific heat capacity $c$, thermal conductivity $k$, density $\rho$ and cross-sectional area $A$ are constant e.t.c) and cross-section, where the temperature does not alter from one point to another on a section, in attempts to secure that the temperature depends solely on position $x$ and time $t$, as suggested in Figure 1 [9]. The fundamental concept when deriving this partial differential equation is to employ the first law of thermodynamics (a variant of the law of conservation of energy) to a cross-sectional strip with dimensions as such denoted in Figure. 2. The conservation of energy is a principle which maintains that no energy can be lost nor produced in an isolated or closed system and therefore the total energy remains constant. Energy can however change form, for instance a rock at the top of a cliff may possess potential energy but once it starts rolling downwards, that energy is transformed to kinetic energy. Specifically, this law can be formulated as,

$$\delta Q = dU + \delta W \tag{7}$$

which translates to: the amount of heat $\delta Q$ that is supplied to the region is equal to the change in internal energy $dU$ plus the amount of energy lost due to work done $\delta W$ in the system [10]. Furthermore, the law is also valid when in consideration of rates per unit time, rather than amounts. As such, we can re-define the law to equate: the rate at which heat enters a region plus what is produced inside is equal to the rate at which heat leaves the region plus the rate of heat storage. We now allow $q(x, t)$ where $[q] = H/tL^2$ ($H$ = heat energy, $t$ = time, $L$ = length) to signify the heat flux of the rod at point $x$ and time $t$, and $q(x + \Delta x, t)$ to signify the heat flux at point $x + \Delta x$ and time $t$ as illustrated in Figure. 2. As $q$ is a vector, it has direction and magnitude; thus, positive when the flow of heat is to the right. We let $A$ denote the area of a cross-section. Then, $Aq(x, t)$ and $Aq(x + \Delta x, t)$ define the rates at which heat enters and leaves the strip from the surfaces at $x$ and

$x + \Delta x$, respectively.

Figure 2: A heat-conductive rod.

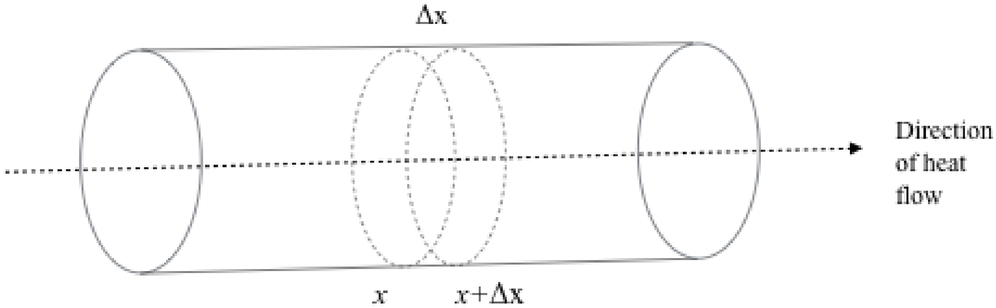
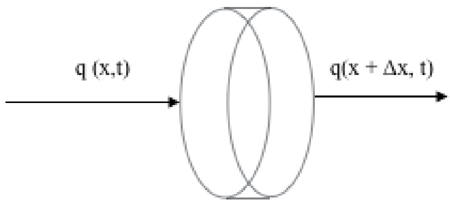



Figure 3: Cross-sectional strip.




The rate of change of temperature in the strip of the rod is proportional to the rate of heat storage. We assume that only the ends are exposed as the remaining surfaces of rod are insulated. Additionally, no source of heat may be found inside the rod. Therefore, if $c$ is the specific heat capacity per unit mass $[c] = H/mT$, and $\rho$ is the density of the body (mass per unit volume), we may estimate the rate of heat storage in the strip by (here, we use an alternative notation for the partial derivatives for reasons of simplicity)

$$\text{Heat energy storage of strip} = \rho c A \Delta x \frac{\partial u}{\partial t}(x, t)$$

where $u(x,t)$ is the temperature throughout the strip as it is arbitrarily thin. This can be deduced from the more general heat storage equation which asserts that $\Delta E = mc\Delta T$. Mass is equal to the density multiplied by the volume of a body which is equivalent to $\rho A \Delta x$, thereby resulting in the above formula by the following substitution:

we have

$$\rho c A \Delta x u(x,t)$$

and at a later time

$$\rho c A \Delta x u(x,t+h)$$

the change is:

$$\rho c A \Delta x (u(x,t+h) - u(x,t))$$

for the rate of change, we divide by the time increment

$$\frac{u(x,t+h) - u(x,t)}{t+h-t}$$

$$\frac{u(x,t+h) - u(x,t)}{h}$$

$$\lim_{h \to 0} \frac{u(x,t+h) - u(x,t)}{h} = \frac{\partial u}{\partial t}(x,t)$$

The mean value then:

$$u(x,t+h) - u(x,t)$$

$$= (t+h-t) \cdot \frac{\partial u}{\partial t}(x,t_1)$$

$$\frac{u(x,t+h) - u(x,t)}{h} = \frac{\partial u}{\partial t}(x,t_1), \quad t_1 \in [t, t+h]$$

where $t_1$ is close to $t$ if $h$ small.

Thus, the rate of heat energy storage in the strip is given by

$$\rho c A \Delta x \frac{\partial u}{\partial t}(x,t_1).$$

There is a multitude of ways in which energy may flow in (and out) of the strip; namely, through radiation, convection, chemical reaction and so forth. We shall account for all these different ways of heat entering and leaving, into what is called a 'generation rate.' We let $g$ be the rate of generation per unit volume $[g] = H/tL^3$, then the rate in which heat is generated inside the strip is given by $A\Delta xg$. We can now apply the law of conservation of energy on the strip, as all factors have been identified. We then get the expression

$$Aq(x,t) + A\Delta xg = Aq(x+\Delta x,t) + A\Delta x\rho c\frac{\partial u}{\partial t}.$$

We subtract $Aq(x+\Delta x,t)$ and $A\Delta xg$ from both sides and then divide by $\Delta x$

$$\frac{Aq(x,t) - Aq(x+\Delta x,t)}{\Delta x} = \frac{A\Delta x\rho c\frac{\partial u}{\partial t} - A\Delta xg}{\Delta x}$$

.

We factor out $A$ and divide both sides

$$\frac{q(x,t) - q(x+\Delta x,t)}{\Delta x} = \rho c\frac{\partial u}{\partial t} - g.$$

We recognize that the ratio

$$\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x}$$

describes by definition the partial derivative of $q$ with respect to $x$ in the limit $\Delta x \to 0$. If we take the limit of this difference quotient when $\Delta x$ tends to zero, we obtain

$$\lim_{\Delta x\to 0}\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} = \frac{\partial q}{\partial x}.$$

This limit therefore yields the form

$$-\frac{\partial q}{\partial x} = \rho c\frac{\partial u}{\partial t} - g \tag{8}$$

for the law of conservation of energy on the strip.

Given the dependent variables $q$ and $u$ we require another equation to connect the two.

Fourier's law of heat conduction stipulates that the heat flow rate is proportional to the magnitude of the negative gradient of the temperature [10]. In one dimension it is given by,

$$q = -\lambda\frac{\partial u}{\partial x} \tag{9}$$

where $q$ is the heat flow rate in the positive direction, $\lambda$ is the thermal conductivity and $\frac{\partial u}{\partial x}$ is the negative temperature gradient. If the body is not uniform, $\lambda$ may depend on $x$, as well as the temperature. For our purpose, it is equally valid to assume $\lambda$ to be a constant since we are dealing with a homogeneous body. Fourier's law substituted in the equation for the law of conservation of energy in (8) gives,

$$\frac{\partial}{\partial x}(\lambda \frac{\partial u}{\partial x}) = \rho c \frac{\partial u}{\partial t} - g \tag{10}$$

Thus,

$$\lambda \frac{\partial^2 u}{\partial x^2} = \rho c \frac{\partial u}{\partial t} - g \tag{11}$$

We assume that $\rho$, $c$ and $\lambda$ are constants. We multiply both sides with $\frac{1}{\rho c}$, and the heat balance equation is expressed as

$$\frac{\lambda}{\rho c} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{g}{\rho c}. \tag{12}$$

This yields, given that the thermal diffusivity $k$ is given by

$$k = \frac{\lambda}{\rho c}$$

that the equation can be written as,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{g}{\rho c}. \tag{13}$$

From this, we derive the heat equation

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = \frac{g}{\rho c} \tag{14}$$

for $0 < x < a$, and $t > 0$. When $g = 0$ we have the homogeneous case of the heat equation. When $g \neq 0$ then we have the alternate case where the equation is said to be inhomogeneous.

### 2.2.1 Boundary Conditions

Boundary conditions may take a variety of forms and are essential when solving a boundary value problem: they need to be imposed to get uniqueness when solving a differential equation whose domain is provided. Boundary conditions set requirements on the value of the function in the boundaries to the area in which the equation is to be solved. Since we consider an idealized body or rod, we assume $L$ to be a line segment $[0, a]$.

(i) The temperature $u$ is given on the line segment $L$. The boundary condition is

$$\begin{cases} u(0,t) = u_0(t) \\ u(a,t) = u_a(t) \end{cases}$$

where $u_0(t)$ and $u_a(t)$ is the temperature of the surrounding medium at the two endpoints.

(ii) Along $L$ an exchange of heat occurs with the surroundings, in such a way that it per unit of area and time, passes through the line segment $L$ and the surrounding medium's temperature is $u_0(t)$ in one boundary and $u_a(t)$ at the other. The boundary condition then takes the form

$$u'_x(0,t) + h(u(0,t) - u_0(t)) = 0 \tag{15}$$

where $h > 0$ is the heat exchange coefficient.

(iii) If the initial temperature is prescribed at time $t = 0$, another initial condition may arise; namely,

$$u(x,0) = f(x), x \in [0, a]$$

where $f(x)$ is the temperature distribution of the line segment at time $t = 0$.

### 2.2.2 Related Equations

If the heat flux is in 2D, there is no variation in the z-axis; for example,$u = u(x, y, t)$ and the heat equation can be written as

$$u'_t = k(u''_{xx} + u''_{yy}) + r \tag{16}$$

where $r$ is a given function. The 1D heat equation is a special case of the above equation:

$$u'_t = k(u''_{xx}) + r \tag{17}$$

where $u = u(x,t)$. If the temperatures remain constant in time, then $u'_t$ is eliminated (as it is no longer dependent on time) and set equal to zero; thus, we obtain Laplace's equation,

$$\nabla^2 u = u''_{xx} + u''_{yy} = 0, \tag{18}$$

where we assume $r = 0$ (this is a special case of Poisson's equation). The PDE, $\nabla^2 u = -r$ of elliptic type, (second order linear PDE where solutions to such equations do not have discontinuous derivatives; thereby, discontinuities are smoothed out) is named after the French mathematician and physicist Siméon Denis Poisson, as Poisson's equation [3].

The solutions to the 2D and respective 3D variations of the heat equation are known as harmonic functions which are characterised as being twice continuously differentiable functions $f : U \to \boldsymbol{R}$ where $U$ is an open subset of $\boldsymbol{R^n}$ that satisfies Laplace's equation.

## 2.3 Historical Background

During the 18th century, mathematicians did not seem to be concerned about the numerous disparities in their mathematical formulations and train of thought. The important aspect seemed to be that the methods worked (or seemed to do so); however, in the beginning of the 19th century a reconsideration of the fundamentals was deemed vital. Subsequently, some of the problems that began to arise concerned series; particularly, series of functions. In combination with the disquisition of some differential equations that pertain in physics, the Frenchman Jean Baptiste Joseph Fourier (1768-1830) studied series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

(19)

and suggested that each function in the interval $0 < x < 2\pi$ can be developed in a so-called Fourier series. Fourier's argument for this was regarded as non-convincing and in the discussion that followed, the question of whether a series of functions with continuous terms may have a discontinuous sum was debated, among other things. Both Leonhard Euler (1707- 1783) and Daniel Bernoulli (1700-1782), prior to this at around 1750, were also involved with the development of a theory regarding solutions in terms of trigonometric series or the present-day Fourier series [3, p. 101]. Even, Joseph-Louis Lagrange who with the use of vibrating string computed the coefficients of a trigonometric series and Jean-Baptiste le Rond d'Alembert who undertook preliminary investigations on the field, believed that the solutions seemed obscure.

Fourier was nonetheless the first to systematically study heat conduction theory. Fourier consequently became renowned due to his work that helped facilitate the solutions and analysis of heat conduction/transfer in solids and proved to be an effective mechanism for the analysis of the dynamic motion of heat. In addition, the equation has helped solve a myriad of diffusion-type problems ranging from the biological sciences and earth sciences to the social sciences. Fourier accomplished this with the help of trigonometrical series since he was intrigued by solutions in general and saw it as an unsolved problem of his time. He therefore solved a plethora of specific examples of the heat equation by separation of variables and expansions of the Fourier series, amongst others. From 1802 to 1807 he conducted his researches on not only heat diffusion but also Egyptology whenever he found spare time from his administrative position as Prefect (Governor) appointed by Napoleon for the Department of Isère in Grenoble [4]. It was later that the German mathematicians Bernhard Riemann (1826-1866) and Karl Theodor Wilhelm Weierstrass (1815-1897) laid solid grounds for mathematical analysis and thereby showed that even discontinuous functions can be expressed as trigonometric series. In fact, there are continuous functions that are not differentiable at any single point and one of the most famous examples was derived by Karl Weierstrass:

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

(20)

where $0 < a < 1$ , $ab > 1 + \frac{3\pi}{2}$ and $b$ is an odd integer greater than 1. The graph of $W(x)$ is, with modern terminology, a fractal curve. Another known continuous curve without a tangent at any point and constructed by geometry is von Koch's snowflake, named after the Swedish mathematician Helge von Koch (1870-1924). Principally, the one that is primarily associated with the amelioration of mathematical analysis is Weierstrass. It was he who first gave the familiar $\epsilon - \delta$ definitions for limits, continuity e.t.c. With Weierstrass, geometric arguments did not seem sufficient in mathematical analysis. He considered that instead of geometric intuition to be the building blocks of mathematical analysis, real numbers should be the fundamental basis. Weierstrass's project is commonly called "arithmetic analysis", and it was performed successfully during the 19th century [17]. In the end, a concern about the basis of the real numbers themselves emerged. The two most common ways of constructing them from the rational numbers are through the Cauchy sequences and Dedekind cuts. Furthermore, Bernhard Riemann, known for the notion of integral that was named after him in the field of mathematics recognized as real analysis. He expressed the integral as a limit value - previously, it had been perceived as *an infinite sum of infinitely small terms* [17]. Research on the Fourier series and their convergence (where and towards what do they converge?) led to the necessity to expand the concept of a function. Therefore, Fourier's lack of clarity and formality when defining the concept of an integral and function was salvaged by Riemann.

Today, Fourier analysis is a highly developed and technically challenging field of mathematics: the study of approximating and presenting functions as the sum of trigonometric functions. The general 3D heat equation has, through time, been solved via different complicated methods that have transpired due to the help of modern computer engineering. Fourier's findings have influenced a number of other fields over the past two centuries; namely, electricity, molecular diffusion, flow in porous materials and stochastic diffusion. Georg Simon Ohm of Germany (1787–1854) who was curious of the nature of electricity and its relation to magnetism became aware of the analogy with heat conduction and regarded that the flow of electricity is precisely analogous to the flow of heat. To describe this relation he formulated the equation,

$$\gamma u'_t = \chi(u''_{xx}) - \frac{bc}{\omega}u, \tag{21}$$

where $\gamma$ is analogous to heat capacity, $\chi$ is electrical conductivity, $u$ is the electrostatic force, $b$ is a transfer coefficient, $c$ is the circumference, and $\omega$ is the area of cross section. Ohm was not entirely correct in his formulations however, which led to another scientist James Clerk Maxwell (1831–1879) in the field of mathematical physics to experimentally derive the equation but in another context. Consequently, a major progress in terms of terrestrial heat flow studies saw the development of a probe that measures temperature gradients in the bottom of the oceans by Edward Crisp Bullard (1907–1980) in 1949. In terms of molecular diffusion, mathematician and medical practitioner Adolf Fick (1829–1901) helped Thomas Graham (1805–1869) to see the analogy between heat conduction in solids and diffusion of solutes in liquids and expressed this in a parabolic partial differential equation in 1855. The analogy of Fourier's heat conduction model did not only apply to the diffusion of liquids but also gases and solids. Soon after, in consideration of flow in porous materials, engineers Jules-Juvenal Dupuit (1804–1866) and Philipp Forchheimer (1852–1933) published the theoretical foundation which considered how the heat equation was applicable in the analysis of water flow in groundwater and the ciculation of water to wells, in 1863 and 1886 respectively. Forchheimer [1886] further illustrated how the stable drainage of water can be expressed by the use of the Laplace equation and used complex variable theory to solve 2D problems in the volume in which the flow takes place that may occur in dams. Lastly, in the first half of the nineteenth century, processes such as the flow of electric current; diffusion in the three states of matter (liquids, solids, and gases); and the movement of solutions in porous mediums were all directly affected by Fourier's heat conduction model. In such evaluations, Fourier's model was used in an empirical manner, to decode experimental data from observable systems. Contrary to an empirical use of the heat diffusion equation, the second half of the nineteenth century observed a more theoretical approach to the problems concerned with the heat equation: stochastic processes coined by Langevin. It marked the beginning of an expansion towards issues of a more theoretical character, concerning the general manifestation of random processes [16, p. 165]. The birth of stochastic differential equations remained somewhat implied in the findings of four distinct scientists: the theory of sound by Lord Rayleigh [1880] which ultimately showed that the calculation to find the amplitude and intensity of $n$ vibrations of undetermined phase satisfies Fourier's heat conduction equation; the law of error by Edgeworth [1883] where the differential equation he derived described the nature of compound error; the theory of speculation by Bachelier [1900] where due to the randomness of stock prices where a comparison between stock option prices and the diffusion equation could be made; and lastly the theory of Brownian motion by Einstein [1905] where particles suspended in a fluid collide resulting in random fluctuations or motion.

# 3 Solving the Heat Equation

## 3.1 The Fourier Series

**Definition 3.1.** A function $f$ is said to be *even* if for every $x \in D_f$ it is true that $-x \in D_f$ and $f(-x) = f(x)$. In other words, the graph is symmetric about the $y$-axis.

A function $g$ is said to be *odd* if for every $x \in D_f$ it is true that $-x \in D_f$ and $g(-x) = -g(x)$. In other words, the graph has rotational symmetry with respect to the origin.

**Definition 3.2.** Let $f \in C^\infty$ (in other words $f$ has derivatives $f^{(n)}$ for all $n$, and $f^{(n)}$ is continuous) at a point $a$ and suppose that

$$f(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{22}$$

has a positive radius of convergence (see Definition 3.9 for convergence): the series converges for some $r > 0$ so that $|x - a| < r$, and the biggest $r$ so that this holds true is called radius of convergence. A function $f$ is analytic if there exists an open interval $I$ such that $I \subseteq \mathbb{R}$ and its Taylor series about any point, say $x$ where $x \in I$, converges to the function in some neighborhood for every point in its domain. See [12 p. 232].

In this chapter we will treat functions that are not so smooth as the aforementioned $f$, functions that perhaps have a finite number of derivatives at some points while being discontinuous in other (points). In such cases they will not have a power series expansion of the type in (28). To attain representations of non-smooth functions, we turn to expansions in terms of trigonometric functions such as

$$1, \cos x, \cos 2x, \cos 3x, ..., \cos nx, ...,$$

$$\sin x, \sin 2x, \sin 3x, ..., \sin nx, ...$$

A trigonometric series looks as follows,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{23}$$

where $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ are independent of $x$ whilst dependent of $n$.
For convenience, take $I = [-\pi, \pi]$ and $f : I \to \mathbb{R}$. The coefficients $a_n$ and $b_n$, $n = 0, 1, 2, ...$, can be determined so that $f$ can be represented by (23). To accomplish this, we will utilize the orthogonality relationships of the trigonometric functions listed in Theorem 1. However, let us first consider the following Lemma:

**Lemma 3.1.** *Integrals of Even and Odd Functions*

*If* $f : [-c, c] \to \mathbb{R}$ *is even,* $\Longrightarrow \int_{-c}^{c} f(x)dx = 2\int_0^c f(x)dx$.

*If* $g : [-d, d] \to \mathbb{R}$ *is odd,* $\Longrightarrow \int_{-d}^{d} g(x)dx = 0$ *. The proof is left to the reader.*

*An example of an even function is* $x \mapsto \cos x$ *and an example of an odd function is* $x \mapsto \sin x$.

*Note: Let* $f_1 : I_1 \to \mathbb{R}$ *be odd and* $f_2 : I_2 \to \mathbb{R}$ *be even; for* $x \in I_1 \cap I_2$ *we have* $f_1 \cdot f_2(x)$ *odd and* $f_1^2(x)$, $f_2^2(x)$ *even. This is utilized in determining* $a_n$ *and* $b_n$.

**Theorem 1.** For $m$,$n$ integers, we have

$$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx dx = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx = \pi \cdot \delta_{mn} \tag{24}$$

and

$$\int_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = 0, \quad \text{for} \quad m, n = 1, 2, ..., \tag{25}$$

where $\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ otherwise, is known as Kronecker's delta. Furthermore, these relations can be verified with single variable calculus.

*Proof:*

To prove (24) we begin with the case when $m \neq n$. We get,

$$\int_{-\pi}^{\pi} \cos(mx) \cdot \cos(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos(mx)\cos(nx)dx =$$

We then use the product-to-sum identity: $2\cos\theta\cos\phi = \cos(\theta - \phi) + \cos(\theta + \phi)$. We obtain,

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos(mx - nx) + \cos(mx + nx)dx =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)dx + \int_{-\pi}^{\pi} \cos((m+n)x)dx =$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n}\right]_{-\pi}^{\pi} =$$

where $m-n$ in the denominator is defined since $m \neq n$,

$$= \frac{1}{2}\left(\frac{\sin((m-n)\pi)}{m-n} - \frac{\sin(-(m-n)\pi)}{m-n}\right) + \frac{1}{2}\left(\frac{\sin((m+n)\pi)}{m+n} - \frac{\sin(-(m+n)\pi)}{m+n}\right),$$

and $m-n \neq 0$ implies that it is a whole number. So if we let $m-n=k$, then $\sin(k\pi)=0$ and

$$\int_{-\pi}^{\pi} \cos(mx)\cdot\cos(nx)dx = 0.$$

Thus, $\delta_{mn} = 0$ for $m \neq n$.

The proof for $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx$ is similar and left to the reader.

We continue with the second case; namely, when $m = n$.

$$\int_{-\pi}^{\pi} \cos(nx)\cdot\cos(nx)dx = \int_{-\pi}^{\pi} \cos^2(nx)dx =$$

We apply Pythagorean identities and the sum rule:

$$\int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos(2nx)}{2}\ dx = \int_{-\pi}^{\pi} \frac{1}{2} + \int_{-\pi}^{\pi} \frac{\cos(2nx)}{2} dx =$$

$$= \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(2nx)}{4n}\right]_{-\pi}^{\pi} = \frac{1}{2}\pi + \frac{1}{2}\pi + \frac{\sin(2n\pi)}{4n} - \frac{\sin(-2n\pi)}{4n} = \pi + 0 = \pi.$$

Thus, $\delta_{mn} = 1$ for $m = n$. The analysis for $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx$ is similar and left to the reader.

To prove (25), we can consider the rules of odd and even functions of integrals as such in Lemma 3.1. We know that the product of $\cos mx \cdot \sin nx$ is odd since $\cos mx$ is even and $\sin nx$ is odd which means that the integral will be zero.

**Definition 3.3.** (Uniform convergence) Asssume that $f$ is defined $f : [-\pi, \pi] \rightarrow \mathbb{R}$, then the series $\sum_{n=1}^{\infty} f_n(x)$ is said to converge uniformly to $f(x)$ in $-\pi \leqslant x \leqslant \pi$ if

$$\sup_{-\pi \leqslant x \leqslant \pi} |f(x) - S_N(f)(x)| \rightarrow 0$$

as $N \rightarrow \infty$, where $S_N(x)$ is the $N$th partial sum defined by $S_N(x) = \sum_{k=1}^{N} f_k(x)$.

[15, p. 173]

**Theorem 2.**

Let $I = [-\pi, \pi]$ and $f \in C(I)$. Suppose that the series

$$s = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \text{cos} nx + b_n \text{sin} nx), \tag{26}$$

converges uniformly towards $f, \forall x \in I$ (written $s_N = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \text{cos} nx + b_n \text{sin} nx) \rightarrow f(x)$ when $N \rightarrow \infty$ ). Then,

$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\text{cos} nx dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\text{sin} nx dx \end{cases} \tag{27}$$

and $n = 1, 2, 3...$

*Proof:* First, we define the partial sums

$$s_k(x) = \frac{a_0}{2} + \sum_{m=1}^{k}(a_m \text{cos} mx + b_m \text{sin} mx), \tag{28}$$

$s_k(x) \rightarrow f(x)$ implies that $s_k(x)\text{cos} nx \rightarrow f(x)\text{cos} nx$, when $k \rightarrow \infty$, for every fixed $n$. We recognize directly that,

$$|s_k(x)\text{cos} nx - f(x)\text{cos} nx| = |s_k(x) - f(x)| \cdot |\text{cos} nx| \leq |s_k(x) - f(x)| \rightarrow 0$$

$$\Rightarrow s_k(x)\text{cos} nx \rightrightarrows f(x)\text{cos} nx.$$

(29)

In a similar fashion, we have

$$s_k(x)\text{sin}nx \to f(x)\text{sin}nx,$$

for every fixed $n$. We then acquire for every fixed $n$,

$$f(x)\text{cos}nx = \frac{a_0}{2}\text{cos}nx + \sum_{m=1}^{\infty}(a_m\text{cos}mx\text{cos}nx + b_m\text{sin}mx\text{sin}nx).$$

This uniformly-convergent series can be integrated one term at a time, between $-\pi$ and $\pi$, due to its uniform convergence since we can change the summation order of two convergent series (the series and the Riemann-sum that is its integral). This leads to

$$\int_{-\pi}^{\pi} f(x)\text{cos}nx dx = \pi \cdot a_n. \tag{30}$$

Similarly, for $f(x)\text{sin}nx$ we attain the latter formula in (34).

**Definition 3.4.** The coefficients $a_n$ and $b_n$ are known as the Fourier coefficients of $f$ and are often written as $a_n(f)$ and $b_n(f)$, respectively. When $a_0$, $a_n$ and $b_n$ are given in the form exhibited in (27), then (26) represents the Fourier series of a function $f(x)$. □

Now let $f : [-\pi, \pi] \to \mathbb{R}$ be an integrable function. The coefficients can be determined in accordance to (27). However, this is no guarantee that the series in (23) converges towards $f(x)$; in general we have,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{cos}nx + b_n\text{sin}nx) \tag{31}$$

to highlight that the series may or may not converge towards $f$. One of the most fundamental problems in Fourier analysis is to identify the classes of functions where $=$ replaces $\sim$.

**Definition 3.5.** Let $f : D_f \to \mathbb{R}$ be a function and let $k$ be a cluster point of $D_f$. Then the left-hand limit of $f$ at $k$ is written as $\lim_{x\to k^-} f(x) = A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left.\begin{array}{r} k \leqslant x < k+\delta \\ x \in D_f \end{array}\right\} \Rightarrow |f(x)-A| < \epsilon$$

Similarly, the right-hand side limit of $f$ at $k$ is written as $\lim_{x\to k^+} f(x) = B$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left.\begin{array}{r} k-\delta < x < k \\ x \in D_f \end{array}\right\} \Rightarrow |f(x)-B| < \epsilon.$$

When both the right-hand side limit and the left-hand side limit exist and are equal, then the limit of $f(x)$ when $x \to k$ exists and is equal to that value.

**Definition 3.6.** A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous at $[a, b]$ iff i) there exists a partition

$$a = x_0 < x_1 < x_2 < ... < x_n = b \tag{32}$$

such that $f \in C(x_{k-1}, x_k)$ and ii) with every $x_k$ there exists both $f(x_k-)$ and $f(x_k+)$ where $f(x_k+)$ denotes the right-hand limit of $f$ at $x_k$ and $f(x_k-)$ denotes the left-hand limit of $f$ at $x_k$, as per definition (3.5).

A piecewise continuous function has a finite number of discontinuities at $x_0, x_1, ..., x_n$ and at every such point there exists $\lim_{x\to x_k-} f(x)$ and $\lim_{x\to x_k+} f(x)$. The magnitude of $f(x_k+)-f(x_k-)$ represents the jump at $x_k$, whereas $a_n(f)$ and $b_n(f)$ are not affected when and if the value of the function changes at a finite number of points in $[a, b]$. One can show that, two functions $f_1$ and $f_2$ that are identical ($f_1 = f_2$) except at a finite number of points, have $a_n(f_1) = a_n(f_2)$ and $b_n(f_1) = b_n(f_2)$; in other words, $f_1$ and $f_2$ have the same Fourier series.

We say that a piecewise continuous function is standardized at a discontinuous point $x_i$ if $f(x_i) = \frac{1}{2}(f(x_i+) + f(x_i-))$. With standardizing the Fourier series is not altered; therefore in the continuation, all functions are standardized.

**Definition 3.7.** A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise smooth if i) $f$ is piecewise continuous and ii) $f'$ is piecewise continuous in every subinterval $]x_{k-1}, x_k[$, $k = 1, 2, ..., n$. □
[10, p. 68].

Let $f : [-\pi, \pi) \to \mathbb{R}$ be piecewise continuous. The periodic enlargement $\widetilde{f}$ of $f$ is given by the formula,

$$\widetilde{f}(x) = \begin{cases} f(x), & \pi \leq x < \pi \\ \widetilde{f}(x-2\pi), & x \in \mathbb{R} \end{cases} \tag{33}$$

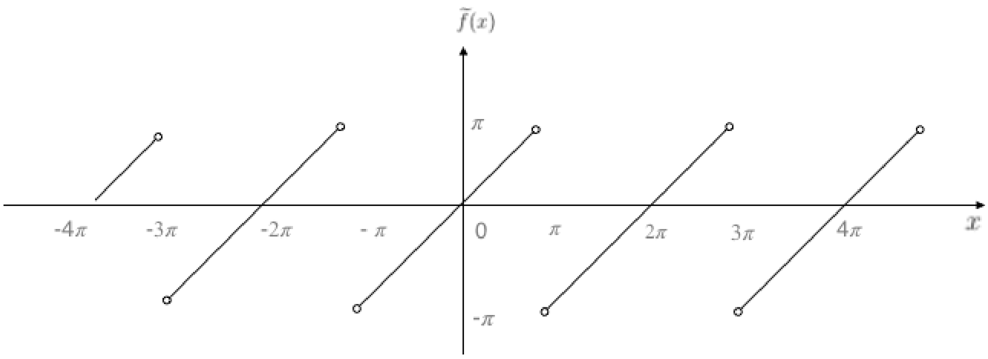
We then standardize $\widetilde{f}$ at $-\pi$ and $\pi$ as well as all other discontinuous points so that the domain of the function (set of independent variables or input for which the function is defined) $D_{\widetilde{f}} = \mathbb{R}$.

*Example 1.* We want to derive the Fourier series of the function $f(x)$ defined by

$$f(x) = x, \qquad x \in [-\pi, \pi].$$

The periodical enlargement $x \mapsto \widetilde{f}(x), x \in \mathbb{R}$ is standardized as depicted in Figure 4. We obtain,

Figure 4: Periodical enlargement of $f$ in *Example 1.*




$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0$$

due to the fact that $x\cos nx$ is an odd function, since $x$ is an odd function and $\cos nx$ is an even function in the interval $x \in [-\pi, \pi]$. Furthermore, the integral of an odd function will always be zero. To compute $b_n$ we begin with integration by parts,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = \frac{1}{\pi}\left[\frac{x(-\cos(nx))}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1(-\cos(nx))}{n}dx = \tag{34}$$

$$= \frac{1}{\pi}\left(\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(-n\pi)}{n}\right) + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx$$

$$= \frac{-\cos(n\pi)}{n} - \frac{\cos(-n\pi)}{n} + \frac{1}{\pi}\left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{-\cos(n\pi)}{n} - \frac{\cos(-n\pi)}{n} + \frac{1}{\pi}\left(\frac{\sin(n\pi)}{n^2} - \frac{\sin(-n\pi)}{n^2}\right)_{-\pi}^{\pi}.$$

Here, $\frac{\sin(n\pi)}{n^2}$ and $\frac{\sin(-n\pi)}{n^2}$ are equal to zero and $\cos(n\pi) = (-1)^n$. We obtain,

$$b_n = \frac{-\cos(n\pi)}{n} - \frac{\cos(-n\pi)}{n}\frac{-(-1)^n}{n} - \frac{(-1)^n}{n}$$

$$= \frac{-2(-1)^n}{n} = -1\frac{2(-1)^n}{n} = \frac{2(-1)^{n-1}}{n}.$$

So,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = 2\frac{(-1)^{n-1}}{n},$$

where $f(x) \sim 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...)$, $x \in [-\pi, \pi]$ and $b_n$ represents $\widetilde{f}(x)$.

### 3.1.1 Sine and Cosine Series

**Definition 3.8.** (Orthogonal system) A set of orthogonal functions $\{\phi_1, ..., \phi_n, ...\}_1^\infty$ is complete if $\forall\epsilon$, where $\epsilon > 0$, there exist scalars $a_1, a_2, ...$ so that

$$\left\|f - \sum_{k=1}^{\infty} a_k\phi_k\right\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x) - \sum_{k=1}^{\infty} a_k\phi_k(x)\right|^2 dx < \epsilon$$

where $\|f\|$ is the L2-norm.

Suppose that we wish to find the Fourier series of a function $f : [0, \pi] \to \mathbb{R}$. Since the Fourier coefficients $a_n(f)$ and $b_n(f)$ are given in terms of integrals from $-\pi$ to $\pi$ we have to somehow alter $D_f$ to $[-\pi, \pi]$. We can do this easiest by defining $f$ arbitrarily in the subinterval $[-\pi, 0)$. Since we are interested in $f:[0,\pi]\to\mathbb{R}$, the convergence properties in $[-\pi,0)$ are of no significant interest. However, we have two choices: it is useful to expand $f$ in $[-\pi,0)$ either as an even function, namely $f(-x)=f(x)$, $-\pi\leqslant x<0$, or as an odd function $f(-x)=-f(x)$, $-\pi\leqslant x<0$. We therefore have,

$$f_j(x)=\begin{cases} f(x), 0\leqslant x\leqslant\pi \\ f(-x), -\pi\leqslant x<0\end{cases} \tag{35}$$

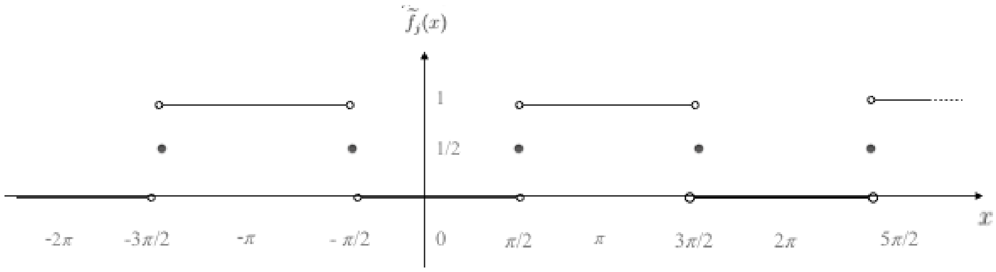
$$f_u(x)=\begin{cases} f(x), 0\leqslant x\leqslant\pi \\ -f(x), -\pi\leqslant x<0\end{cases} \tag{36}$$

where $f_j(x)\sim$ cosine series and $f_u(x)\sim$ sine series and form a complete orthogonal system in the interval $[-\pi,\pi]$ (where one can project an arbitrary square-integrable function on a complete base in a infinite dimensional function space); the expansions of which go by the name half-range expansions.

*Example 2.* Given a function $f(x)$ defined by,

$$f(x)=\begin{cases} 0, \text{for} \quad 0\leqslant x<\frac{\pi}{2} \\ 1, \text{for} \quad \frac{\pi}{2}\leqslant x\leqslant\pi\end{cases} \tag{37}$$

we are to determine its Fourier series. We first extend $f$ to an even periodic function, as represented in the figure below, and then is standardized so that $\widetilde{f}(\frac{\pi}{2})=\widetilde{f}(\frac{3\pi}{2})=\widetilde{f}(-\frac{\pi}{2})=\frac{1}{2}$.

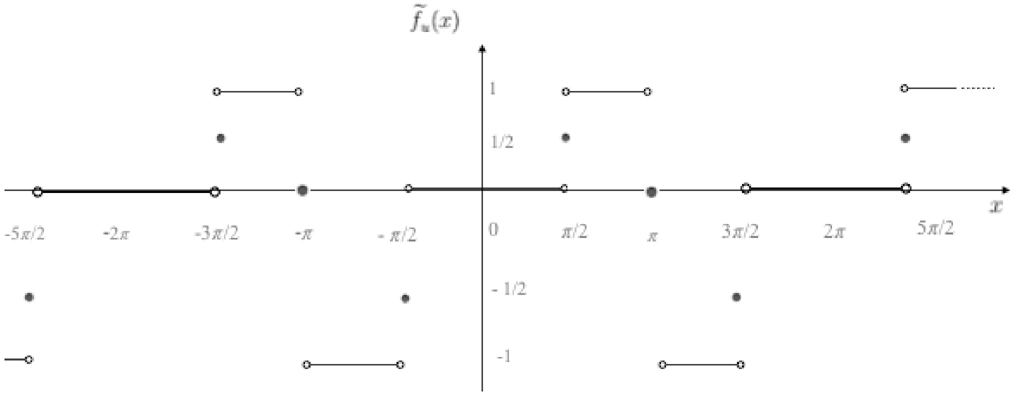



If $\widetilde{f}_j(x)$ is even, in accordance to the formulas in (34), we obtain $b_n=0\Rightarrow \mathrm{a}_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{f}_j(x)\mathrm{cos}nxdx=\frac{2}{\pi}\int_0^{\pi}\widetilde{f}_j(x)\mathrm{cos}nxdx=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\mathrm{cos}nxdx, n=0,1,2,...$

$\Rightarrow \mathrm{a}_0=1$ and $a_n=\frac{2}{\pi}\left[\frac{\mathrm{sin}nx}{n}\right]_{\frac{\pi}{2}}^{\pi}=\frac{2}{n}(-\mathrm{sin}\frac{n\pi}{2}), n\geqslant 1$ and by applying (33) and Definition 3.2 we acquire,

$\Rightarrow \widetilde{f}_j(x) \sim \frac{1}{2} - \frac{2}{\pi}(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - ...)$ for $-\pi \leqslant x \leqslant \pi$.

We now extend $f$ to an odd periodic function $\widetilde{f}_u(\frac{3\pi}{2})$, as shown in the figure below,



The standardized function has $\widetilde{f}_u(x) = (-\frac{3\pi}{2}) = \widetilde{f}_u(\frac{\pi}{2}) = ... = \frac{1}{2}, \widetilde{f}_u(-\pi) = \widetilde{f}_u(\pi) = 0$ and $\widetilde{f}_u(-\frac{\pi}{2}) = \widetilde{f}_u(\frac{3\pi}{2}) = ... = -\frac{1}{2}$.With the same reasoning used above for the case of the even function, we obtain $a_n = 0$, for $n = 0, 1, 2, ...$, and $b_n = \frac{2}{\pi}\int_0^{\pi} \widetilde{f}_u(x)\text{sin}nxdx = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \text{sin}nxdx = \frac{2}{n\pi}(\text{cos}\frac{n\pi}{2} - \text{cos}n\pi)$ or more precisely,

$$b_n = \begin{cases} \frac{2}{n\pi}, n \quad \text{odd} \\ \frac{2}{k\pi}((-1)^k - 1), n = 2k, \qquad k \geqslant 1. \end{cases} \tag{38}$$

$\Rightarrow \widetilde{f}_u(x) \sim \frac{2}{\pi}(\frac{\text{sin}x}{1} - 2\frac{\text{sin}2x}{2} + \frac{\text{sin}3x}{3} + \frac{\text{sin}5x}{5} - 2\frac{\text{sin}6x}{6} + \frac{\text{sin}7x}{7} - ...)$, for $-\pi \leqslant x \leqslant \pi$.

Note: If $f$ is piecewise smooth in the interval $[c-\pi, c+\pi]$, we can construct an expansion of a periodic function to $f$ and determine the Fourier coefficients $a_n$ and $b_n$ in accordance to formula (27). However, since the trigonometric functions sine and cosine are periodic with period $2\pi$, these coefficients are given by

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{cos}nxdx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{sin}nxdx$.

The standard form of the Fourier expansion has hitherto been considered in the interval $-\pi \leq x < \pi$. In other cases, it is required to build the Fourier series of $f(x)$ that is defined in an interval $-L \leq x < L$, where $L$ is a positive number $\neq \pi$. This is achieved by change of variables; we therefore introduce a new variable $t$ that ranges from $-\pi$ to $\pi$ when $x$ varies between $-L$ and $L$:

$$\frac{t}{\pi} = \frac{x}{L} \iff t = \frac{\pi x}{L} \iff x = \frac{Lt}{\pi}$$

The function $f : [-L, L] \to \mathbb{R}$ is transformed thereby to $g : [-\pi, \pi] \to \mathbb{R}$, $g(t) = f(\frac{Lt}{\pi}) = f(x)$ and if we suppose that $f(x)$ fulfills Dirichlet's conditions, so does $g(t)$. We expand therefore $g(t)$ to a Fourier series in the usual form

$$g(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where the Fourier coefficients are again the usual

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\text{cos}ntdt, \quad n = 0, 1, 2, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\text{sin}ntdt, \quad n = 1, 2, ...$$

We can then come back to $f$:

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\right) \tag{39}$$

with the Fourier coefficients,

$$a_n(f) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, ...$$

$$b_n(f) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, ...$$

*Example 3* Let us expand in a Fourier series, the function $f : [-2, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{for} \quad -2 \leqslant x < 0 \\ 1, & \text{for} \quad 0 \leqslant x \leqslant 2 \end{cases} \tag{40}$$

Then,
$a_0 = 1$,
$a_n(f) = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{0}^{2}\cos\frac{n\pi x}{2}dx$
$\forall n \geq 1 : a_n = \frac{1}{2}\int_0^2 \cos\frac{n\pi x}{2}dx = 0;$
$\forall n \geq 1 : b_n = \frac{1}{2}\int_0^2 \sin\frac{n\pi x}{2}dx = \frac{1}{2}[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}]_0^2 = \frac{1-(-1)^n}{n\pi};$
Note: $\cos n\pi = (-1)^n$.
We obtain the expansion (when $k = 2n - 1$),

$$f(x) \sim \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1-(-1)^k}{n\pi}\sin\frac{k\pi x}{2} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2}.$$

This converges to a 4-periodic $\widetilde{f}(x)$ with $f(0) = f(\pm 2) = f(\pm 4) = ... = \frac{1}{2}$.

We will now consider the case of the Fourier transform for functions defined on an interval of the type: $[a, b)$, with $a, b$ being two arbitrary real numbers with $a < b$. To do so, we will expand $f(x)$ to a periodic function within the domain; namely, with period $T = b - a$. We let $L$ be half the distance of the interval, that is $L = \frac{T}{2}$, and $c = \frac{a+b}{2}$. We represent the extended function as $F(x)$, in other words,

$$F(x) = f(x) \quad \text{for} \quad a \leqslant x \leqslant b, \quad F(x+2L) = F(x). \tag{41}$$

We introduce variable $s$ that ranges between $-\pi$ to $\pi$ when $x$ varies from $a$ to $b$, as follows

$$s = \frac{\pi}{L}(x-c) \tag{42}$$

which results from first centering the interval $[a,b)$ at 0 and then scaling the interval since the length of $[a,b)$ is $b-a$ and the length of $[-\pi,\pi]$ is $2\pi$ so the scaling factor is $\frac{2\pi}{b-a} = \frac{\pi}{L}$. (Thus, we attain a Fourier transform on a general interval between $a$ and $b$ by translating the interval so that it's centered at 0). We denote

$$H(s) := F(x) = F\Bigg(\frac{L}{\pi}(s + c\frac{\pi}{L})\Bigg). \tag{43}$$

We then obtain,

$$H(s+2\pi) = F\Bigg(\frac{L}{\pi}(s+2\pi+c\frac{\pi}{L})\Bigg) = F\Bigg(\frac{L}{\pi}(s+c\frac{\pi}{L})+2L\Bigg) = F(x+2L) = F(x) = H(s). \tag{44}$$

Thus, $H(s)$ is $2\pi$ periodic, and its Fourier series can be expressed as

$$H(s) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(\widetilde{a_n}\cos(ns) + \widetilde{b_n}\sin(ns)), \tag{45}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} H(s)ds = \frac{1}{\pi}\int_a^b F(x)\frac{\pi}{L}dx = \frac{1}{L}\int_a^b F(x)dx, \tag{46}$$

$$\begin{aligned}\widetilde{a_n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} H(s)\cos(ns)ds = \frac{1}{\pi}\int_a^b F(x)\cos\Big(\frac{n\pi}{L}(x-c)\Big)\frac{\pi}{L}dx \\ &= \frac{1}{L}\int_a^b F(x)\cos\Big(\frac{n\pi}{L}(x-c)\Big)dx,\end{aligned}$$

and

$$\widetilde{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} H(s)\sin(ns)ds = \frac{1}{\pi}\int_a^b F(x)\sin\Big(\frac{n\pi}{L}(x-c)\Big)\frac{\pi}{L}dx$$

$$= \frac{1}{L}\int_a^b F(x)\sin\Big(\frac{n\pi}{L}(x-c)\Big)dx.$$

The Fourier series is therefore given by,

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\widetilde{a_n}\cos\Big(\frac{n\pi}{L}(x-c)\Big) + \widetilde{b_n}\sin\Big(\frac{n\pi}{L}(x-c)\Big)\right). \tag{47}$$

We observe that in (47), there is a "movement" in the trigonometric terms, in other words, $x$ is shifted to the right by $c$ units. Whatsoever, by the use of the orthogonality conditions of functions $\sin\big(\frac{n\pi}{L}x\big)$ and $\cos\big(\frac{n\pi}{L}x\big)$ in the interval $[a,b]$, the Fourier series can be acquired without this shift. To begin with, we note the orthogonality conditions of the aforementioned trigonometric functions (sine and cosine) in the interval $[a,b)$. Since $b = a + 2L$, for any integer $n \neq 0$, we obtain,

$$\int_a^b \sin\Big(\frac{n\pi}{L}x\Big)dx = \int_a^b \cos\Big(\frac{n\pi}{L}x\Big)dx = 0. \tag{48}$$

For any integers $n$ and $m$,

$$\int_a^b \sin\Big(\frac{n\pi}{L}x\Big)\cos\Big(\frac{m\pi}{L}x\Big)dx = 0. \tag{49}$$

For any integers $n > 0$ and $m > 0$,

$$\int_a^b \sin\Big(\frac{n\pi}{L}x\Big)\sin\Big(\frac{m\pi}{L}x\Big)dx = \int_a^b \cos\Big(\frac{n\pi}{L}x\Big)\cos\Big(\frac{m\pi}{L}x\Big)dx = \delta_{mn}L, \tag{50}$$

in which

$$\delta_{mn} = \begin{cases} 1, m = n \\ 0, m \neq n \end{cases} \tag{51}$$

The orthogonal conditions allow us to write the Fourier series of $F(x)$ as

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\right), \tag{52}$$

where $a_0$ is the same as in (46) and

$$a_n = \frac{1}{L}\int_a^b F(x)\cos(\frac{n\pi x}{L})dx, \quad b_n = \frac{1}{L}\int_a^b F(x)\sin(\frac{n\pi x}{L})dx, \tag{53}$$

and in accordance to a lemma, the Fourier series (47) of function $F(x)$ is equivalent to the Fourier series (52). For the proof of this Lemma, readers are advised to see [19, p. 13]. Thus, the Fourier series representation for an arbitrary interval [a,b) is the given by the same formula as that with interval $-L \leq x < L$ shown in (39), but where $L = \frac{T}{2}$. [19, p. 10-12].

### 3.1.2 Convergence Theorems

The criteria to follow is used to decipher when Fourier Series can be differentiated and integrated termwise.

**Definition 3.9.** If the sequence $(s_n)_{n=0}^{\infty}$ of the partial sums to a series $\sum_{k=0}^{\infty} a_k$ tends to a limit, the series is called convergent. If the partial sums do not have a limit, the series is called divergent.

**Theorem 3.** (Pointwise Convergence)$f$ is smooth implies

$$\lim_{n\to\infty} |f(x) - S_N(f)(x)| = 0.$$

[15, p. 173]

**Theorem 4.** Let $f$ be $2\pi$ periodic and piecewise smooth. Then $S_N(x) \to \tilde{f}(x)$ pointwise on $\mathbb{R}$, where

$$\tilde{f}(x) := \frac{(f(x_+) + f(x_-))}{2}.$$

[1, p. 4].

**Lemma 3.2.** Let $f$ be a $2\pi$ periodic function that is piecewise smooth. Then the Fourier coefficients $a_n(f)$ and $b_n(f)$, $n \geq 1$, fulfill the following inequalities:

$$|a_n| \leq \frac{c}{n}, \quad |b_n| \leq \frac{c}{n}, \quad n = 1, 2, ...$$

where $c$ is dependent only on $f$.

*Proof:* Suppose that the jump occurs in

$$-\pi = x_0 < x_1 < x_2 < ... < x_r = \pi.$$

Then we have for $a_n(f), n = 1, 2, ...,$ $a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt = \frac{1}{\pi}\sum_{k=1}^{r}\int_{x_{k-1}}^{x_k} f(t)\cos ntdt$. Integration by parts gives

$$a_n(f) = \frac{1}{\pi}\sum_{k=1}^{r}\left[\frac{f(t)\sin nt}{n}\right]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}\frac{1}{n}f'(t)\sin ntdt.$$

Since $f$ and $f'$ are bounded, we obtain an estimation of $a_n$ by means of the triangle inequality. We obtain,

$$|a_n(f)| = \left|\frac{1}{n}\left\{\frac{1}{\pi}\sum_{k=1}^{r}\left[f(t)\sin nt\right]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}f'(t)\sin ntdt\right\}\right|$$

$$= \frac{1}{n}\left|\left\{\frac{1}{\pi}\sum_{k=1}^{r}\left[f(t)\sin nt\right]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}f'(t)\sin ntdt\right\}\right|$$

$$= \frac{1}{n}\left|\left[\frac{1}{\pi}\sum_{k=1}^{r}\left[f(x_k)\sin nx_k - f(x_{k-1})\sin nx_{k-1}\right] - \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}f'(t)\sin ntdt\right|\right.$$

$$\leq \frac{1}{n}\left\{\frac{1}{\pi}\sum_{k=1}^{r}|f(x_k)\sin nx_k| + \frac{1}{\pi}\sum_{k=1}^{r}|f(x_{k-1})\sin nx_{k-1}| + \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}|f'(t)||\sin nt|dt\right\}$$

Since $|\sin x| \leq 1$ for any $x$,

$$\leq \frac{1}{n}\left\{\frac{1}{\pi}\sum_{k=1}^{r}|f(x_k)| + \frac{1}{\pi}\sum_{k=1}^{r}|f(x_{k-1})| + \int_{x_{k-1}}^{x_k}\frac{1}{\pi}\sum_{k=1}^{r}|f'(t)|dt\right\}$$

and since $|f(x)| \leq \max|f| < c$ due to the fact that $|f'(x)| \leq \max|f'| < c'$ for any $x$ and constant $c$, since a continuous function on a closed interval maintains a maximum in that interval,

$$\leq \frac{1}{n}\left\{\frac{1}{\pi}c\cdot r + \frac{1}{\pi}c\cdot r + \frac{1}{\pi}c'\cdot r\right\} = \frac{1}{\pi}\left(\frac{K\cdot r}{n}\right) = \frac{K\cdot r}{\pi n}. \qquad (54)$$

since $f(x) \leqslant c$ so each term is less than or equal to a constant and so the whole sum is less than or equal to a constant multiplied by the number of terms $r$, namely $c \cdot r$.

The estimation of $b_n$ is acquired similarly.

**Corollary** Suppose that $f$ and its derivative of order $p$ are $2\pi$ periodic and piecewise smooth. Then the Fourier coefficients inequalities are fulfilled, namely

$$|a_n| \leq \frac{c}{n^p}, \quad |b_n| \leq \frac{c}{n^p}, \quad n = 1, 2, 3, ...$$

where $c$ does not depend on $n$.

*Proof:* We apply integration by parts $p$ times, in a similar manner as in lemma 3.2. The corollary states that the more derivatives $f$ has, the faster its Fourier coefficients tend to zero as $n \to \infty$.

**Proposition 1** Assume that $f$ is $2\pi$ periodic, continuous, and piecewise smooth. Abbreviate by $a_n, b_n$ the Fourier coefficients of $f$ and by $A_n, B_n$ the Fourier coefficients of $f'$. Then,

$$A_n = nb_n \quad B_n = -na_n.$$

*Proof:* Integration by parts gives,

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx dx = \frac{1}{\pi}\Big[f(x)\cos nx\Big]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = nb_n$$

since $\cos\pi$ and $f$ are both periodic. Similarly, $B_n = -na_n$. [9]

As a concequence, we get:

**Theorem 5.** (Term by term differentiation of the Fourier series)

Let $f$ be continuous everywhere and be $2\pi$ periodic. Let $f'$ be piecewise smooth and standardised where the function $f$ satisfies the conditions $f(-\pi) = f(\pi), f'(-\pi) = f'(\pi)$. Then,

(i) The Fourier series of $f$ uniformly converges towards $f$ on $[-\pi, \pi]$ for all $x$.

(ii) The series acquired after the termwise differentiation of the Fourier series of $f$ converges in every point to $f'$.

*Proof:*

To show that the Fourier Series of $f$ converges uniformly to $f$, we use the fact that $f'(x)$ is piecewise smooth and consequently that $f''(x)$ is piecewise continuous. We assume $f$ is smooth or piecewise smooth which implies, $f(x) = \lim_{N\to\infty} \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ given that $f$ is piecewise smooth then it satisfies the assumption of Theorem 4 and uniformly converges. We also assume Theorem 4. Since, from Theorem 4, we know that $S_N(x) \to f(x)$ pointwise (where $S_N(x)$ is the partial sum of the Fourier series of $f$) , we may write for each $x \in \mathbb{R}$:

$$f(x) - S_N(f)(x) = \left(\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nx + b_n \sin nx)\right) - \left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx)\right)$$

$$= \sum_{n=N+1}^{\infty} a_n \cos nx + b_n \sin nx$$

and

$$|f(x) - S_N(f)(x)| = \left| \sum_{n=N+1}^{\infty} a_n \cos nx + b_n \sin nx \right| \leq \sum_{N+1}^{\infty} (|a_n| + |b_n|).$$

We abbreviate by $a_n, b_n$ the Fourier coefficients of $f$ and by $A_n, B_n$ the Fourier coefficients of $f'$. To prove the absolute convergence it is enough to show that $|a_n|, |b_n| \leq \frac{M}{n^2}$ for some constant $M$ independent of $n$. Then, $\sum_{n\geq 1} |a_n|$ and $\sum_{n\geq 1} |b_n|$ converge by the comparison test which implies the absolute convergence of the Fourier series. Additionally, uniform convergence is implied from the Weierstrass M-test. From proposition 1, $a_n = \frac{-B_n}{n}$. This means,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx.$$

We have assumed that $f'$ is piecewise smooth; therefore, $f''$ is continuous excluding a finite number of points where it has a jump. Suppose that $f''$ is continuous in the interval $a < x < b$, then

$$\frac{1}{n\pi} \int_a^b f'(x) \sin nx dx = \frac{1}{n^2\pi} \Big[ f'(x) \cos nx \Big]_a^b - \frac{1}{n^2\pi} \int_a^b f''(x) \cos nx dx$$

$$= \frac{1}{n^2\pi}\Big[f(b^-)\cos nx - f(a^+)\cos nx\Big] + \frac{1}{n^2\pi}\int_a^b f''(x)\cos nx dx.$$

Given that $|f'(x)|, |f''(x)| \leq K$ for a constant $K$ and all $x$ (at points $x$ where $f'$ and $f''$ have a jump discontinuity, the left-and right-hand side derivatives for $f'$ have to be taken.) So,

$$\frac{1}{n\pi}\left|\int_a^b f'(x)\sin nx dx\right| \leq \frac{K(2+(b-a))}{n^2\pi} \leq \frac{2K(1+\pi)}{n^2\pi} \leq \frac{4K}{n^2}.$$

Given that $f''$ has a finite number of discontinuities, the integral $\int_{-\pi}^{\pi} f'(x)\sin nx dx$ can be expressed as a finite number, say $p$, of integrals $\int_a^b f'(x)\sin nx dx$ over integrals where $f''$ is continuous. So,

$$|a_n| = \frac{1}{n\pi}\left|\int_{-\pi}^{\pi} f'(x)\sin nx dx\right| \leq \frac{4Kp}{n^2}.$$

Accordingly, $|b_n| \leq \frac{4Kp}{n^2}$.

From $|a_n| \leq \frac{4K}{n^2}$ we can deduce,

$$|f(x) - S_N(f)(x)| \leqslant \sum_{N+1}^{\infty}(|a_n| + |b_n|) \leq \sum_{N+1}^{\infty}\frac{c}{n^2},$$

independent of $x$. We then recall from calculus that the tail of a convergent series tends to 0; namely, as $\sum_1^\infty \frac{1}{n^2}$ converges implies that $\sum_{N+1}^\infty \frac{1}{n^2} \to 0$ when $N \to \infty$.

See [8] and [1] for more detail.

To show the convergence of the differentiated series to $f'$, let the dip of $f'$ occur in

$$-\pi = x_0 < x_1 < x_2 < ... < x_r = \pi$$

We define

$$g(x) = \int_{-\pi}^{x} f'(t)dt$$

and note that $g$ is continuous because $f'$ is continuous. In accordance to the fundamental theorem of calculus, $g' - f' \equiv 0$ for $x_{i-1} < x < x_i, i = 1, 2, ..., r$

so $g - f = k$, where $k$ is a constant in all subintervals. When both $g$ and $f$ are continuous, so is $g - f \equiv$ a constant.

The termwise differentiation of the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

gives the Fourier series of $f'(x)$, since

$$\frac{d}{dn}(a_n \cos nx + b_n \sin nx)$$

$$= -na_n \sin nx + nb_n \cos nx)$$

Where we are allowed to differentiate under the summation symbol due to the fact that the series is convergent. Finally, we attain

$$f'(x) = \sum_{n=1}^{\infty}(nb_n \cos nx - na_n \sin nx).$$

As by Proposition 1. [8].

**Theorem 6.** (Term by term integration of the Fourier series) Let $f$ be $2\pi$ periodic and piecewise smooth. Suppose that the Fourier coefficient $a_0 = 0$ and define,

$$F(x) = \int_{-\pi}^{x} f(t)dt.$$

Then the Fourier series for $F(x)$ is obtained by termwise integration of the Fourier series of $f$, except the constant term $A_0$ that is given by

$$A_0 = -\frac{1}{\pi}\int_{-\pi}^{\pi} xf(x)dx.$$

*Proof:*

The condition $a_0 = 0$ is required so that $F$ can be $2\pi$ periodic. The relation between the Fourier coefficients of $F$ and the Fourier coefficients of $f$ follows from Theorem 5. To find $A_0$ we observe that,

$$
\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\int_{-\pi}^{x} f(t)dt\Big)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big(\int_{t}^{\pi} dx\Big)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - t)f(t)dt \\
&= -\frac{1}{\pi}\int_{-\pi}^{\pi} tf(t)dt.
\end{aligned}
$$

Note:

(i) If $f$ is such that $a_0 \neq 0$, we define $f(x) - \frac{1}{2}a_0 = g(x)$ and use Theorem 6.

(ii) Theorem 6 does not require uniform convergence for the derivative $F'(x) = f(x)$. [11, p. 66].

### 3.1.3 Fourier Method

The vector spaces that interest us are function spaces.

**Definition 3.10.** A linear space of functions or a *function space* is a class of functions with a fixed domain and range together with addition and multiplication by scalars. The elements of the space are functions (between two sets).

**Definition 3.11.** *A linear operator* $\mathscr{L}$ on a given function space transforms every real-valued function $u$ to a function $\mathscr{L}u$, which does not necessarily have to belong to the same space, and possesses the property of preserving the linear operations mentioned in the aforementioned Definition 3.10. These two operations can be summarized in the following relation:

$$\mathscr{L}(c_1u_1 + c_2u_2) = c_1\mathscr{L}u_1 + c_2\mathscr{L}u_2, \quad c_1, c_2 \in \mathbb{R}.$$

where $c_1$ and $c_2$ are scalars. For $c_1 = c_2 = 1$ we attain addition and for $c_1 = 1, c_2 = 0$ scalar multiplication. This can be expanded to a finite number of functions. If $(u_i)_{i=1}^n$ and $(c_i)_{i=1}^n$, from above it follows that

$$\mathscr{L}\sum_{i=1}^{n}(c_iu_i) = \sum_{i=1}^{n} c_i\mathscr{L}u_i. \quad (n \in \mathbb{N})$$

is called linear if $\mathscr{L}$ is a linear operator. If $\sum_{i=1}^{n} c_i \mathscr{L} u_i = 0$, it is identified as a homogeneous linear equation. Alternatively, if the right-hand side is not equal to zero it is an inhomogeneous linear equation. Thus, every linear homogeneous PDE has the form $\mathscr{L}u = 0$. Furthermore, $L^2$ is the vector space of all functions so that the integral of the square of the absolute value is finite in some interval i.e $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$. An example of a linear $\mathscr{L}$ on $L^2$ space is $\mathscr{L} = x^2 \frac{\partial^2}{\partial x^2} - 3x \frac{\partial}{\partial x} + 4$. The one-dimensional heat equation in $L^2$ takes the form,

$$\mathscr{L}u = (\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2})u = 0 \iff u'_t - ku''_{xx} = 0.$$

The above equation is homogeneous since the right hand side $\equiv 0$. Accordingly, a PDE $\mathscr{L}u = f$, with $f \neq 0$ is an inhomogeneous PDE. The most common linear operations in this thesis are integration and differentiation.

A boundary value problem consists of a PDE and the corresponding conditions. These conditions can be either homogeneous or inhomogeneous in nature. Consider an idealized, uniform a rod with an isolated surface that occupies the interval $[\,a, b]$ as mentioned in (4) and has homogeneous boundary conditions, such as

$$u(a,t) = 0, \quad u'_x(a,t) = 0 \quad \text{or} \quad u'_x(a,t) = hu(a,t) \tag{55}$$

where $h > 0$ is a constant.

All can be included in the form

$$\cos\alpha \cdot u_x(a,t) - \sin\alpha \cdot u(a,t) = 0$$

where $0 \leq \alpha \leq \pi$; when $\alpha = \frac{\pi}{2}$ we have $u(a,t) = 0$ and when $\alpha = 0$ we have $u'_x(a,t) = 0$. When $\tan\alpha = h$, we have the third condition $u'_x(a,t) = hu(a,t)$. In the same manner, the general condition at $x = a$ can be written,

$$\cos\beta \cdot u'_x(b,t) + \sin\beta \cdot u(b,t) = 0.$$

The constant $\beta$ is not related to $\alpha$. Note that we can write

$$L_\alpha = \cos\alpha\frac{\partial}{\partial x} - \sin\alpha, \quad L_\beta = \cos\beta\frac{\partial}{\partial x} + \sin\beta$$

similarly, the conditions are expressed

$$(L_\alpha u)(a,t) = 0, \quad (L_\beta u)(b,t) = 0.$$

The superposition principle is fundamental when solving linear boundary value problems with the Fourier method. It is the basic approach that states that the Fourier expression of the general solution can be expressed as the sums of simple solutions. In other words, the principle of superposition states:

**Principle of Superposition.** If $u_1, u_2, ...$ are solutions of the same linear homogeneous PDE $\mathscr{L}u = 0$, then so is

$$u = c_1 u_1 + c_2 u_2 + ...$$

where $c_1, c_2, ...$ are constants (real or complex).

It includes series of the type

$$u = \sum_{i=1}^{\infty} c_i u_i$$

(provided it converges) where $c_i$ are constants and $u_i$ specified functions. [11, p. 3]

**Theorem 7.** Suppose that $f_n : [a,b] \to \mathbb{R}$, for each $k = 1, 2, ...$, is integrable on $[a,b]$ and that $f_n(x) \to f(x)$ uniformly on $[a,b]$ as $n \to \infty$.

Then the limit function $f(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$. This also holds for series.

**Theorem 8.** Suppose that every smooth function $u_i$ in an infinite sequence $u_1, u_2, u_3, ...$ has a continuous derivative on $[a,b]$ and satisfies a linear homogeneous PDE $\mathscr{L}u_i = 0$ or the homogeneous boundary conditions $Lu_i = 0$. Suppose further that:
i) The series $\sum_{n=1}^{\infty} u_n(x_0)$ converges at some point on $x_0 \in [a,b]$, and the series of derivatives $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[a,b]$, to $f(x) = \sum_{n=1}^{\infty} u_n'(x)$ say, and is differentiable with regard to all the derivatives that are included in $\mathscr{L}F$ or $LF$ and
ii) there is a given continuity condition at the boundary that is fulfilled by $\mathscr{L}F$ (since it has to be at least twice differentiable*)* Then even

$$F(x) = \sum_{n=1}^{\infty} c_n u_n(x) \tag{56}$$

satisfies the same PDE or the same boundary conditions, where the series $\sum_{n=1}^{\infty} u_n(x)$ converges at every $x$, and the sum $F(x) = \sum_{n=1}^{\infty} c_n u_n(x)$ is differentiable with $F'(x) = f(x)$ (so $f(x)$ is the derivative of $F(x) = \sum_{n=1}^{\infty} c_n u_n(x)$).

*Proof:* To prove this theorem, for reasons of simplicity, we will suppose that $\mathscr{L} = \frac{d}{dx}$.

Since each $u_n'$ is continuous and $\sum_{n=1}^{\infty} u_n'$ is uniformly convergent, $f$ is continuous on $[a, b]$ since the limit of a uniformly convergent sequence of continuous functions is continuous. Thus, $u_n'$ and $f$ are integrable on $[a, b]$. Employing Theorem 7 to $u_n'$ and $f$ on $[x_0, x]$ we get

$$\sum_{n=1}^{\infty} \int_{x_0}^{x} u_n'(x)dx = \int_{x_0}^{x} f(x)dx. \tag{57}$$

The LHS of (57) can be expressed as

$$\sum_{n=1}^{\infty} \int_{x_0}^{x} u_n'(x)dx = \lim_{m\to\infty} \sum_{n=1}^{m} \int_{x_0}^{x} u_n'(x)dx =$$

$\lim_{m\to\infty} \sum_{n=1}^{m}(u_n(x) - u_n(x_0)) = \lim_{m\to\infty}(\sum_{n=1}^{m} u_n(x) - \sum_{n=1}^{m} u_n(x_0))$
(58)

The limit $\lim_{n\to\infty} \sum_{n=1}^{m} u_n(x_0)$ exists due to the hypothesis that (56) converges, therefore the series $F(x) = \lim_{m\to\infty} \sum_{n=1}^{m} u_n(x)$ converges for each $x \in [a, b]$. Therefore, a function $F : [a, b] \to \mathbb{R}$ is well-defined and from (57) and (58) we acquire

$$F(x) - F(x_0) = \int_{x_0}^{x} f(x)dx.$$

Differentiation in $x$ gives $F'(x) = f(x)$ for $x \in [a, b]$ since $f$ is continuous.

The above discussion can be applied for linear homogeneous boundary condition $Lu = 0$. In this case we may require that the function $Lu$ satisfies a continuity condition at the points on the boundary so that their value there represent the limit when these points approach the domain from within. [2, p. 1]

## 3.2 Method: Separation of Variables

In this section, we will begin with a description of the most common method used for solving the heat equation; namely, the separation of variables. In other words, we search for particular solutions in the form

$$u(x,t) = X(x)T(t) \tag{59}$$

and attempt to obtain ordinary differential equations for $X(x)$ and $T(t)$. The aforementioned equations will contain a parameter called the *separation constant*. The function $u(x,t)$ in (59) is called a *separated solution*. Subsequently, we can employ the superposition principle to acquire more general solutions of a linear homogeneous PDE as sums of separated solutions. A second order homogeneous PDE in two variables, say $x$ and $t$, can always be expressed in the canonical form

$$a(x,t)u''_{xx} + c(x,t)u''_{tt} + d(x,t)u'_x + e(x,y)u'_t + f(x,y)u = 0.$$

If $a = 0$ or if $c = 0$ the equation is parabolic, and such is the one-dimensional homogeneous heat equation $u''_{xx} - ku'_t = 0$, where $a = e = 1$ and the rest zero. Furthermore, a boundary value problem or an initial value problem are said to be well-posed if they satisfy the following three conditions:

i) Existence: there is at least one solution,

ii) Uniqueness: there is at most one solution

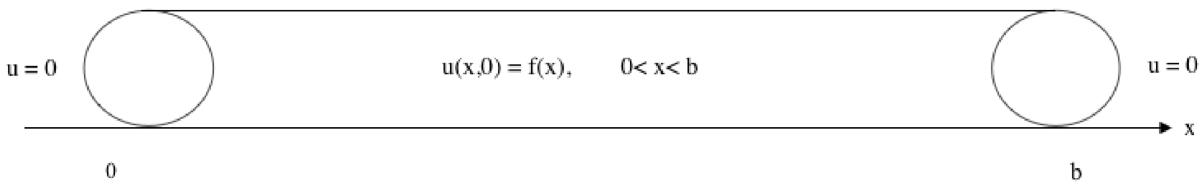
iii) The solution depends continuously on data and parameters; which in turn regulate the behaviour of the functions. [11, p. 9]

*Example 4* The linear boundary value problem, namely

$$\begin{cases} u'_t = ku''_{xx}, & 0 < x < b, \qquad t > 0 \\ u(0,t) = 0, & t \geqslant 0 \\ u(b,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x), & 0 \leqslant x \leqslant b \end{cases} \tag{60}$$

is a problem for temperature $u(x,t)$ for an idealized, heat conducting homogeneous rod that is infinitely thin of length $b$. It is also assumed that $f$ is piecewise smooth and that the surface is completely insulated. The cross-sectional area is circular due to the cylindrical shape of the rod and the diameter $d \ll b$. It is assumed that $k$ is a positive constant and that the rod lacks a heat source. To determine the non-trivial functions that satisfy the conditions $u'_t - ku''_{xx} = 0$ and $u(0,t) = u(b,t) = 0$ we let the coordinate system be represented as shown in Figure 5 above and use the method of the separation of variables of variables $x$ and $t$.

Figure 5: Idealized, homogeneous rod




Let, $u(x,t) = X(x) \cdot T(t)$.
The imposition of the boundary conditions yields:

$$u(0,t) = u(b,t) = 0 \iff X(0)T(t) = X(b)T(t) = 0 \iff X(0) = X(b) = 0 \tag{61}$$

for arbitrary T(t). We differentiate the function $u(x,t) = X(x) \cdot T(t)$ with respect to $t$ as well as with respect to $x$ and then put it back into our PDE in (60) to obtain,

$$X(x) \cdot T'(t) = k \cdot X''(x) \cdot T(t) \iff \frac{T'(t)}{k \cdot T(t)} = \frac{X''(x)}{X(x)} = -\alpha^2$$

$$\iff \begin{cases} X''(x) + \alpha^2 X(x) = 0 \\ T'(t) + k\alpha^2 T(t) = 0 \end{cases} \qquad (62)$$

where $\alpha$ is a positive constant since the left-hand side of (62) only depends on the variable $t$ and the right-hand side only depends on the variable $x$ and since the two of them are equal, we conclude that there must exist a constant $\alpha$. We know that the constant is equal to $-\alpha^2$ since if we consider the other two cases; namely, when the constant is equal to $\alpha^2$ or 0 we obtain the following:

For a constant $\alpha^2$ (as opposed to $-\alpha^2$), $\quad \alpha \in \mathbb{R}$ :

$$X''(x) - \alpha^2 X(x) = 0$$

where the solution is

$$X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} \tag{63}$$

We apply the boundary conditions,
$X(0) = 0$ which substituted into (63) gives $0 = C_1 + C_2 \Rightarrow \mathrm{C}_1 = -C_2$

$$X(b) = 0 \Rightarrow C_1 e^{\alpha b} - C_1 e^{-\alpha b} = 0 \Rightarrow C_1(e^{2\alpha b} - 1) = 0, \alpha > 0, b > 0$$

$$\Rightarrow C_1 = 0, C_2 = 0$$

where we recover a trivial solution.
For $\alpha^2 = 0$

$$X''(x) = 0$$

We integrate twice,

$$X'(x) = C_1 \tag{64}$$

$$X(x) = C_1 x + C_2$$

For some constants $C_1, C_2$.
We apply the boundary conditions,

$$X(0) = 0 \Rightarrow 0 = C_2$$
$$\Rightarrow X(x) = C_1 x$$

$$X(b) = 0 \Rightarrow 0 = C_1 b, \quad b > 0$$

$$\Rightarrow C_1 = 0$$

which is a trivial solution. Thus, (62) is equal to $-\alpha^2$ as the other two cases recover trivial solutions. It now becomes an issue of solving the so-called Sturm-Liouville-problem [11, p. 84-85].

$$X''(x) + \alpha^2 X(x) = 0, \quad X(0) = X(b) = 0. \tag{65}$$

$$X'' + \alpha^2 X = 0 \iff X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$X(0) = 0 \Rightarrow C_1 = 0 \Rightarrow X(x) = C_2 \sin \alpha x$$

$$X(b) = 0 \Rightarrow C_2 \sin \alpha b = 0 \Rightarrow \sin \alpha b = 0 \iff \alpha b = n\pi$$

$$\iff \alpha = \frac{n\pi}{b}, \quad n = 1, 2, 3, ... \tag{66}$$

All these numbers constitute the spectrum (eigenvalues) of the operator $-\frac{d^2}{dx^2}$. The corresponding eigenfunctions are

$$X_n(x) = B_n \cdot \sin \frac{n\pi x}{b}, \quad n = 1, 2, 3...$$

(67)

Let us return to the T-equation in (62). We insert the value of $\alpha^2$ and the equation is transformed to

$$T'(t) + k(\frac{n^2\pi^2}{b^2})T(t) = 0$$

and has solutions

$$T_n(t) = C_n e^{-\frac{kn^2\pi^2 t}{b^2}} \tag{68}$$

The non-trivial solutions to the homogeneous equation that fulfill the boundary conditions $u(0, t) = u(b, t) = 0$ is given by

$$U_n(x, t) = X_n(t)T_n(t) = a_n \cdot e^{-\frac{kn^2\pi^2 t}{b^2}} \cdot \sin \frac{n\pi x}{b}$$

where $a_n = B_n C_n, n = 1, 2, ...$ We formally (i.e glossing over some details such as convergence) build a series since the principle of superposition states that the sum of solutions of a linear homogeneous equation is also a solution:

$$u(x, t) = \sum_{n=1}^{\infty} a_n \cdot e^{-\frac{kn^2\pi^2 t}{b^2}} \cdot \sin \frac{n\pi x}{b}.$$

(69)

Note: we need to worry whether this makes sense as a function later. Now: in accordance to the initial condition, we substitute $t = 0$ in (69) and obtain the condition:

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{b}$$

(70)

We now need to determine the constants $a_n$ from this expression; therefore, we use an orthogonality relationship. Note that from earlier computations; namely, equation (50):

$$\int_0^b \sin \frac{n\pi x}{b} \cdot \sin \frac{m\pi x}{b} dx = \begin{cases} 0, & n \neq m \\ \frac{b}{2}, & n = m \end{cases}$$

so if we go back to 70 and apply the above, we begin by multiplying $\sin \frac{m\pi x}{b}$ to both sides and then integrating over $[0, b]$ and obtain:

$$\int_0^b f(x) \sin\Big(\frac{m\pi x}{b}\Big) dx = \int_0^b \sum_{n=1}^{\infty} a_n \sin\Big(\frac{n\pi x}{b}\Big) \sin\Big(\frac{m\pi x}{b}\Big) dx \qquad (71)$$

$$\int_0^b f(x) \sin\Big(\frac{m\pi x}{b}\Big) dx = \sum_{n=1}^{\infty} \int_0^b a_n \sin\Big(\frac{n\pi x}{b}\Big) \sin\Big(\frac{m\pi x}{b}\Big) dx. \qquad (72)$$

(above, we ignore the question about interchanging $\sum_{n=1}^{\infty}$ and $\int_0^b$.) On the right hand side we will be integrating an infinite number of terms we assume that $n = m$ since we have an infinite number of terms from $n = 1$ to $\infty$ where the integral will be zero for $n \neq m$ but one of them will correspond to $n = m$ and in that case we get $\frac{1}{2}$ but for every other term we will be adding zero:

$$\int_0^b f(x) \sin\Big(\frac{m\pi x}{b}\Big) dx = a_m\Big(\frac{b}{2} + 0 + 0 + 0 + ...\Big) \qquad (73)$$

$$\int_0^b f(x)\sin\Big(\frac{m\pi x}{b}\Big)dx = \frac{a_m b}{2} \tag{74}$$

Thus $f$ can be expanded in a Fourier Series with $a_n$ given by:

$$\forall n \geqslant 1 : a_n = \frac{2}{b}\int_0^b f(x)\sin\Big(\frac{n\pi x}{b}\Big)dx \tag{75}$$

Finally, the series

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\frac{2}{b}\int_0^b f(s)\sin\frac{n\pi s}{b}ds\Big)e^{-\frac{kn^2\pi^2 t}{b^2}}\sin\frac{n\pi x}{b},$$

(76)

is a formal solution to the heat equation problem 60. A formal solution is an object that, on the face of its appearance, solves a certain problem, but where it is not clear whether the object is well-defined as a function or similar.

*Example 5* We consider Example 1 from chapter 3 where we have computed the Fourier coefficients of $f(x) = x$ for $x \in [-\pi, \pi]$. Hence, our problem is

$$\begin{cases} u'_t = ku''_{xx}, & -\pi < x < \pi, \qquad t > 0 \\ u(-\pi,t) = 0, & t \geqslant 0 \\ u(\pi,t) = 0, & t \geqslant 0 \\ u(x,0) = f(x) = x, & -\pi \leqslant x \leqslant \pi \end{cases}$$

In Example 1 we obtained,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = 2\frac{(-1)^{n-1}}{n},$$

The same method is used as in Example 4 and from (76) we obtain,

$$u(x,t) = \sum_{n=1}^{\infty} 2\frac{(-1)^{n-1}}{n}e^{-\frac{kn^2\pi^2 t}{b^2}}\sin(nx).$$

*Example 6* (Continuation of Example 4) Suppose that the initial temperature distribution is given by $f(x) = x(b-x)$ where $a = 0$, $b = 1$ and $k = \frac{1}{10}$.

Let us recall that $\alpha_n = \frac{n\pi}{b}$ which in our case is reduced to $n\pi$. Also, note that the following analysis for functions defined on $[0,1]$ is permitted due to the fact that the Fourier transform for functions defined on an interval of the type: $[a,b)$, with $a,b$ being two arbitrary real numbers with $a < b$ is given by (52).

From (75), we aim to compute the Fourier coefficients

$$a_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx$$

Integration by parts yields

$$a_n = 2\int_0^1 x(1-x)\Big(\frac{\cos(n\pi x)}{n\pi}\Big)' dx$$

$$= \frac{2}{n\pi}\left[\Big[-x(1-x)\frac{\cos(n\pi x)}{n\pi}\Big]_0^1 + \int_0^1 (1-2x)\frac{\cos(n\pi x)}{\alpha_n}dx\right]$$

$$= \frac{2}{n\pi}\int_0^1 (1-2x)\left(\frac{\sin(n\pi x)}{n\pi}\right)' dx$$

$$= \frac{2}{n\pi}\left[\Big[(1-2x)\frac{\sin(n\pi x)}{n\pi}\Big]_0^1 - \int_0^1 (-2)\frac{\sin(n\pi x)}{n\pi}dx\right]$$

$$= \frac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)dx = \frac{4}{(n\pi)^2}\left[-\Big[\frac{\cos(n\pi x)}{n\pi}\Big]_0^1\right] = \frac{4[1-(-1)^n]}{(n\pi)^3}$$

.

So the solution is

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}e^{-\frac{n^2\pi^2 t}{10}}\sin(n\pi x).$$

For more detail concerning this example, readers are advised to consult [7, p. 9].

*Example 7* (Continuation of Ex.4) In this example we will consider another case which differs from the previous two due to the fact that one boundary condition is inhomogeneous. The method to be used will reduce the inhomogeneous problem to a homogeneous problem.

Suppose that the temperature at one of the endpoints is $u(b,t) = u_0, t \geqslant 0$. Then the problem becomes

$$\begin{cases} u'_t = ku''_{xx}, \quad (0 < x < b, \quad t > 0) \\ u(0,t) = 0, \quad u(b,t) = u_0, \quad (t \geqslant 0) \\ u(x,0) = f(x), \quad (0 \leqslant x \leqslant b) \\ (77) \end{cases}$$

We now need to build an appropriate auxiliary function that satisfies the boundary conditions, say $j(x)$. If, $g(t)$ represents the boundary condition on the end of the rod and $h(t)$ represents the boundary condition at the beginning of the rod, then the construction of an auxiliary function is given by,

$$j(x,t) = \frac{g(t) - h(t)}{b} x + h(t)$$

$$j(x,t) = \frac{u_0 - 0}{b} x + 0 = \frac{u_0}{b} x.$$

We now let $u(x,t) = v(x,t) + j(x)$ so we obtain, $u(x,t) = v(x,t) + \frac{u_0}{b}x$. Substituted in (77) (firstly, we wish to find $v(0,t)$ so putting $x = 0$ we attain $u(0,t) = v(0,t) + \frac{u_0}{b} \cdot 0$ which gives that $v(0,t) = 0$; secondly, we want $v(b,t)$, so if we substitute $x = b$, we get $u_0 = v(b,t) + \frac{u_0}{b} b$ and obtain that $v(b,t) = 0$; thirdly, we want $v(x,0)$ so if $t = 0$ then, $u(x,0) = v(x,0) + \frac{u_0}{b}x$, and we obtain that $f(x) = v(x,0) + \frac{u_0}{b}x)$ produces the homogeneous problem

$$\begin{cases} v'_t(x,t) = kv''_{xx}(x,t), \quad (0 \leqslant x \leqslant b, \quad t > 0) \\ v(0,t) = 0, \quad v(b,t) = 0, \quad (t \geqslant 0) \\ v(x,0) = f(x) - \frac{u_0 x}{b}, \quad (0 \leqslant x \leqslant b) \end{cases}$$

The solution can be written directly from 76,

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{b} \int_0^b (f(s) - \frac{u_0 s}{b}) \sin \frac{n\pi s}{b} ds\right) \cdot e^{-\frac{kn^2\pi^2 t}{b^2}} \sin \frac{n\pi x}{b}.$$

[18, p. 28].

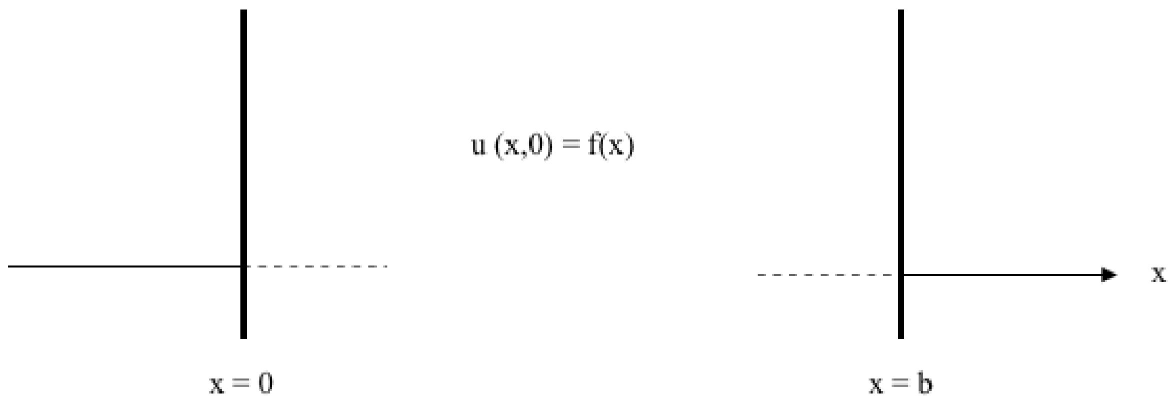
*Example 8*

In this example, we will treat another case; namely, a Neumann boundary condition. The linear boundary value problem

$$\begin{cases} u'_t(x,t) = ku''_{xx}(x,t), \quad (0 < x < b, \quad t > 0) \\ u'_x(0,t) = 0, \quad u'_x(b,t) = 0, \quad (t \geqslant 0) \\ u(x,0) = f(x), \quad (0 \leqslant x \leqslant b) \end{cases} \tag{78}$$

is a problem for the temperature $u(x,t)$ of a heat conducting flat disk (which we consider as an interval), bounded by $x = 0$ and $x = b$, but otherwise not bounded in the y- and z- axes. Its sides are insulated so no heat comes in or out. It is assumed that a heat source is absent here.

Figure 6: Heat conducting flat disk




Separation of variables $u(x,t) = X(x)T(t)$ gives

$$\Rightarrow \begin{cases} X''(x) + \alpha^2 X(x) = 0, \quad X'(0) = X'(b) = 0 \\ T'(t) + \alpha^2 T(t) = 0 \end{cases} \tag{79}$$

Equation (78) with the boundary conditions constitute of an eigenvalue problem (Sturm-Liouville's problem) with eigenvalues

$$\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi^2}{2}\right), \quad n = 1, 2, 3...$$

and eigenfunctions

$$X_0(x) = 1, \quad X_n(x) = \cos\frac{n\pi x}{b}, \quad n = 1, 2, 3...$$

The T-equation from (79) has the corresponding solutions

$$T_0(t) = 1, \quad T_n(t) = e^{-(\frac{n\pi}{b})^2 kt}$$

The product of the two constitutes the solution to (78):

$$u_n(x,t) = X_n(x)T_n(t) = e^{-(\frac{n\pi}{b})^2 kt} \cos\frac{n\pi x}{b} \quad n \geqslant 1.$$

In accordance to the superposition principle, the generalized linear combination becomes,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{b})^2 kt} \cos\frac{n\pi x}{b}. \tag{80}$$

which is the solution to

$$u'_t = ku''_{xx}, \quad u'_x(0,t) = u'_x(b,t) = 0.$$

To find $A_0$ and $A_n$, we take into account the initial condition,

$$u(x,0) = f(x) \Leftrightarrow 2\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{b} = f(x) \Leftrightarrow$$

$$\Leftrightarrow A_0 = \frac{2}{b}\int_0^b f(x)dx$$

and

$$A_n = \frac{1}{b}\int_0^b f(x)\cos\frac{n\pi x}{b}dx.$$

We substitute the values of $A_0$ and $A_n$ back to our solution (80). Thus, the problem is well-posed and as such entirely solved.

### 3.2.1 Existence and Uniqueness of Solutions to the Boundary Value Problem of the Heat Equation

We have hitherto encountered formal solutions in which a concern whether $u(x,t)$ makes sense as a function arises, since if we start with a nice function $f$ as the initial condition, then the series we build not only "solves" the problem but actually produces a function of $x$ and $t$. However, one could imagine that if $f$ is very wild, for example not piecewise continuous, then the function series we might obtain may not give a function that is differentiable. This is the distinction between formal solution and solution in the sense of pointwise convergence satisfying the heat equation as a function. Thus, we are to now examine two properties of solutions which will help clarify any uncertainties and illustrate how the heat equation satisfies both properties.

I) Existence

In the preceding section, for the boundary-value problem (60) in Example 4, we found that (76) is the formal solution of the heat conduction problem, where $a_n$ is given by (75). We shall illustrate that series (76) (or (69)) is the formal solution to the problem, if $f(x)$ is continuous in $[0, b]$, that $f(0) = f(b) = 0$ and that $f'$ is piecewise continuous in $[0, b]$. Since $f(x)$ is bounded, we have

$$|a_n| = \frac{2}{b}\left|\int_0^b f(x)\sin\left(\frac{n\pi x}{b}\right)dx\right| \leqslant \frac{2}{b}\int_0^b |f(x)|dx \leqslant C, \qquad (81)$$

where $C$ is a positive constant. So, for some finite $t_0 > 0$, we have

$$\left|a_n e^{-\frac{kn^2\pi^2 t}{b^2}} \sin\frac{n\pi x}{b}\right| \leqslant Ce^{-\frac{kn^2\pi^2 t_0}{b^2}}, \quad t \geq t_0.$$

The ratio test for numerical series illustrates how the series of terms $\sum_{n=1}^{\infty} e^{-\frac{kn^2\pi^2 t_0}{b^2}}$ converges. Therefore, in accordance to the Weierstrass M-test, series (76) converges absolutely with respect to $x$ and $t$ when $t \geq t_0$ and $0 \leqslant x \leqslant b$. Hence $u(x, t)$ makes sense as a function.

Since the series is convergent and differentiating a linear operator, term by term differentiation of (76) with respect to $t$ gives

$$u_t' = -\sum_{n=1}^{\infty} a\left(\frac{n\pi}{b}\right)^2 ke^{-\frac{kn^2\pi^2 t}{b^2}} \sin\frac{n\pi x}{b}.$$

(82)

We note that for $t \geq t_0$ we have,

$$\left|-a_n K\left(\frac{n\pi}{b}\right)^2 e^{-\frac{kn^2\pi^2 t}{b^2}} \sin\frac{n\pi x}{b}\right| \leqslant C\left(\frac{n\pi}{b}\right)^2 Ke^{-\frac{kn^2\pi^2 t_0}{b^2}}$$

and the series of terms

$$C\left(\frac{n\pi}{b}\right)^2 \cdot Ke^{-(\frac{n\pi}{b})^2 kt_0}$$

converges in accordance to the ratio test. It follows that $u_t'$ converges uniformly from the Weierstrass M-test when $t \geq t_0$ and $0 \leqslant x \leqslant b$. In the same way, if series (69) is differentiated two times with respect to $x$, we obtain

$$u''_{xx} = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{b}\right)^2 e^{-\frac{kn^2\pi^2 t}{b^2}} \sin\frac{n\pi x}{b}. \tag{83}$$

From (82) and (83),

$$u_t' = ku_{xx}''.$$

Thus, equation (69) is the solution to the 1D heat equation and boundary-value problem in (60) in the region $0 \leqslant x \leqslant b$, $t \geq t_0$.

Subsequently, we shall show that the boundary conditions are fulfilled. We note that series (69) which represents $u(x,t)$ uniformly converges in the interval $0 \leqslant x \leqslant b$, $t \geq t_0$. Given that a function defined by a uniformly convergent series of continuous functions is itself continuous, in turn means that $u(x,t)$ is continuous at $x = 0$ and $x = b$ from which it follows that when $x = 0$ and $x = b$, series (69) satisfies

$$u(0,t) = u(b,t) = 0, \quad \forall t > 0.$$

It remains to show that $u(x,t)$ satisfies the initial condition

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant b.$$

As of previous assumptions, $f(x)$ given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{b}$$

is uniformly and absolutely convergent. So, by Abel's test of convergence (which states that if $\sum_1^{\infty} b_n(x)$ converges uniformly on $P$ and that $\{a_n(x)\}$ is a monotone uniformly bounded sequence, then $\sum_1^{\infty} a_n(x)b_n(x)$ converges uniformly on $P$) [5, p. 434], the product of the terms of a uniformly convergent series (which we know is uniformly convergent from Theorem 5)

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{b}$$

and a uniformly bounded and monotone sequence $e^{-(\frac{n\pi}{b})^2 kt}$ is uniformly convergent with respect to $t$. Thus, the series

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cdot e^{-\frac{kn^2\pi^2 t}{b^2}} \cdot \sin \frac{n\pi x}{b}$$

converges uniformly for $0 \leqslant x \leqslant b, t \geqslant 0$, and in the same manner, $u(x,t)$ is continuous in $0 \leqslant x \leqslant b, t \geqslant 0$. The initial condition

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant b$$

is therefore fulfilled. The existence of solution is thus confirmed. [15, p. 251].

II) Uniqueness

Uniqueness is a fundamental feature of the heat equation. It demonstrates how any solution is decided by the corresponding initial and boundary conditions. There are different ways to prove this feature; however, we will focus on the *energy method.*

**Theorem 9.** Let $u(x,t)$ be a continuous differentiable function. If $u(x,t)$ satisfies the PDE

$$u'_t = ku''_{xx}, \quad 0 \leqslant x \leqslant b, \quad t > 0$$

with the boundary conditions

$$u(0,t) = 0, u(b,t) = 0, \qquad t \geqslant 0$$

and initial condition

$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant b$$

then there exists at most one solution $u$.

*Proof* Suppose that there exist two solutions $u_1$ and $u_2$ to the heat equation on $(0,b)$. Let $v(x,t) = u_1(x,t) - u_2(x,t)$. Then,

$$\begin{cases} v'_t = kv''_{xx}, \quad 0 \leqslant x \leqslant b, \quad t > 0 \\ v(0,t) = v(b,t) = 0, \qquad t \geqslant 0 \\ v(x,0) = 0, \quad 0 \leqslant x \leqslant b \end{cases}$$

(84)

We consider the "energy" which is defined by the integral of the function (where $v = v(x,t)$),

$$J(t) = \frac{1}{2k}\int_0^b v^2 dx.$$

$J(t)$ is differentiated with respect to $t$ and we obtain,

$$J'(t) = \frac{1}{k}\int_0^b v v'_t dx$$

$$= \int_0^b v v''_{xx} dx.$$

by virtue of the second equation in (84). Integration by parts gives

$$= \int_0^b v v''_{xx} dx = [v v'_x]_0^b - \int_0^b {v'_x}^2 dx.$$

Since $v(0,t) = v(b,t) = 0$,

$$J'(t) = -\int_0^b {v'_x}^2 dx \leqslant 0.$$

From the initial condition $v(x,0) = 0$ we have that $J(0) = 0$. This condition and $J'(t) \leqslant 0$ indicate that $J(t)$ is a decreasing function of $t$. Thus,

$$J(t) \leqslant 0.$$

But per definition of $J(t)$

$$J(t) \geqslant 0.$$

Hence,

$$J(t) = 0, \quad t \geqslant 0.$$

Since $v(x,t)$ is continuous $J(t) = 0$ implies

$$v(x,t) = 0,$$

in $0 \leqslant x \leqslant b, t \geqslant 0$. From which it follows that $u_1 = u_2$ and the solution (if it exists) is unique. [15, p. 254]. One could apply this method for Newmann's boundary conditions since the $[vv'_x]$ part from the integration by parts that disappeared because $v$ is zero at the boundary points, (homogenenous Dirichlet conditions) disappears even if it is $v'_x$ that is zero on the boundary points (homogeneous Neumann boundary conditions).

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