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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Kuratowski's Theorem and properties of graphs embeddable on surfaces of higher genus, with respect to their Euler characteristic

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Abstract

This paper presents a proof of Kuratowski's Theorem and discuss properties of graphs embeddable on surfaces with respect to the embeddings Euler characteristic and genus of the surface.

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1 Introduction

Planar graphs, graphs that can be drawn on the plane with edges only intersecting at vertices, has properties in common with graphs embeddable on the sphere, some of which are Euler characteristic 2 and a common set of forbidden minors.
The purpose of this paper is to discuss these properties together with corresponding properties for graphs embeddable on other surfaces, with respect to their Euler characteristic, and give an extended proof of Kuratowski's Theorem, based on that of Diestel.

The paper assume some familiarity with graph theory and combinatorics.

2 Graph theory basics

The definitions in Chapter 2 are based on those of Diestel [1].

Definition 2.1. A graph G is a pair of sets $G = (V(G), E(G))$ such that $E(G)$ is a set of 2-element subsets of $V(G)$. The elements of $V(G)$ are called the vertices of G and the elements of $E(G)$ are its edges.

In this paper all graphs will be simple and connected. Graphs will be symbolically represented as a set of dots for vertices and lines between vertices represent edges. The notation used for an edge e that connect two vertices x and y will be $e = xy$

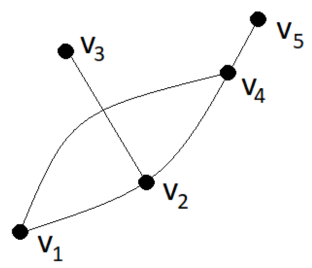

Figure 1: A graph G with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(G) = \{v_1v_2, v_1v_4, v_2v_3, v_2v_4, v_4v_5\}$

Definition 2.2. A vertex v is incident to an edge e if $v \in e$.

Definition 2.3. Two vertices $v, u \in V(G)$ are adjacent, or neighbours, if they are connected by an edge, that is there are an edge $e \in E(G)$ such that $u, v \in e$.

Hence the vertex v_1 of the graph in figure 1 is incident to the edges v_1v_2 and v_1v_4 as well as adjacent to the vertices v_2 and v_4.

Definition 2.4. A path is a sequence of vertices $v_1, ..., v_k$ such that v_i and v_{i+1} are adjacent for all $i = 1, ..., k-1$.

The graph in figure 1 has three possible paths between v_1 and v_4, $P_1 = v_1v_2v_4$, $P_2 = v_1v_2v_3v_2v_4$ and $P_3 = v_1v_4$.

Definition 2.5. A collection of paths are disjoint if no two paths in the collection has any vertices in common.

Definition 2.6. A cycle is a path where $v_1 = v_k$ and $v_i \neq v_j$ for all other i, j, $i \neq j$.

Definition 2.7. H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 2.8. H is a component of G if H is a maximal connected subgraph of G.

Definition 2.9. A graph is a complete graph if every pair of vertices are connected by an edge. Let K^n denote the complete graph with n vertices.

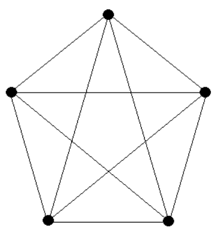

Figure 2: The complete graph with five vertices, K^5

Definition 2.10. A graph G is complete bipartite if $V(G)$ can be divided into two non empty subsets X and Y such that every $x \in X$ is adjacent to every $y \in Y$ and no vertex is adjacent to another vertex in the same part. Let $K_{m,n}$ denote the complete bipartite graph where X and Y contains m respectively n vertices.

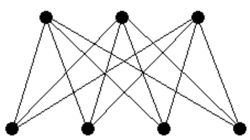

Figure 3: The complete bipartite graph $K_{3,4}$

Definition 2.11. Let G be a graph and assume there is an edge $e = xy$ in G. A subdivision of G is a graph obtained from G by removing an edge xy and adding a new vertex v and new edges xv and yv, instead of e.
Let $H = TG$ denote a graph H obtained as a subdivision of G.

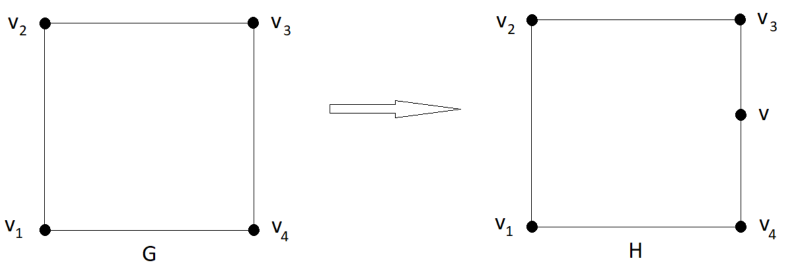

Figure 4: $H = TG$ obtained by subdividing the edge $e = v_3v_4$

Definition 2.12. The branch vertices of a graph $H = TG$ are the vertices in $V(G)$.

In figure 4 the branch vertices of H are V_1, V_2, V_3 and V_4.

Definition 2.13. Let $e = uv$ be an edge of a graph G. Let G/e, an edge contraction, be the graph obtained from G by removing the edge e and replace its incident vertices with a new vertex v_e, which is adjacent to the former neighbours of u and v.

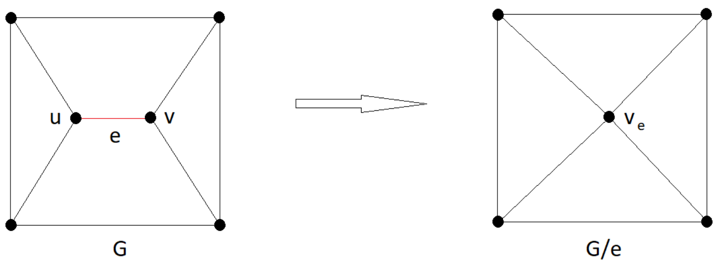

Figure 5: Contracting the edge $e = uv$ of G produces a new graph G/e

Definition 2.14. A graph H is a minor of a graph G if H can be obtained from G by edge contractions, removing edges and removing isolated vertices.

Hence if a graph $H = TG$ then G is also a minor of H.

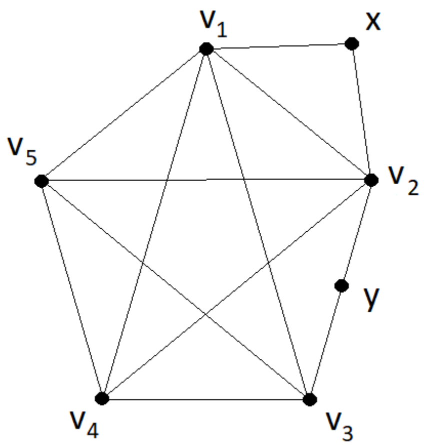

Figure 6: K^5 can be obtained by successive edge contractions or by deleting the vertex x and replace y and its incident edges with the edge v_2v_3

Definition 2.15. A graph H is a topological minor of a graph G if a subdivision of H is isomorphic to a subgraph of G.

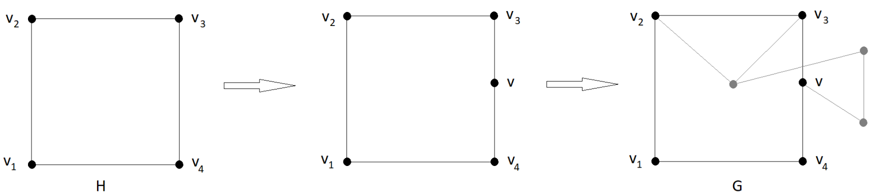

Figure 7: H is a topological minor of G

By the definition it follows that every topological minor is also an ordinary minor, the converse is however not necessarily true.

Definition 2.16. A graph is planar if it can be drawn in the plane such that any two edges intersects at a vertex or not at all.

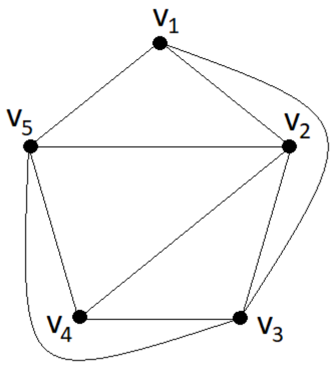

Figure 8: A planar graph

Definition 2.17. A vertex has degree n if it has exactly n incident edges. Let $deg(v)$ denote the degree of a vertex v.

Theorem 2.18. A graph G contains K^5 or $K_{3,3}$ as a minor if and only if it contains K^5 or $K_{3,3}$ as a topological minor.

Proof. Based on [1, p. 103].
As every topological minor of G is also a minor of G, if G contains K^5 or $K_{3,3}$ as a topological minor then G also contains K^5 or $K_{3,3}$ respectively as a minor.

If G contains $K_{3,3}$ as a minor then $K_{3,3}$ is obtained from G by first removing edges and vertices, giving the connected graph G', and then contracting edges to obtain $K_{3,3}$. Where the edge contractions are not equivalent with removing edges and vertices.

Denote the parts of $K_{3,3}$ by $\{v_{i,1}, v_{i,2}, v_{i,3}\}$, $i \in \{1,2\}$ and let $U_{i,j}$, $j \in \{1,2,3\}$, denote the subgraph of G' which was contracted into $v_{i,j}$.
As $K_{3,3}$ is obtained from G' each pair of subgraphs $U_{i,j}$ and $U_{3-i,k}$, $k \in \{1,2,3\}$, are connected in G'.

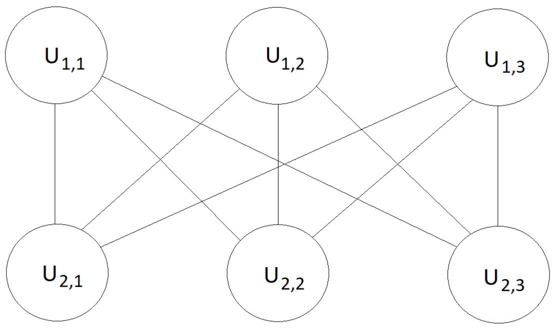

Figure 9: $U_{i,j}$ and $U_{3-i,k}$ are connected in G'

Let uv be a contracted edge, if $deg(u) = 1$ or $deg(v) = 1$ the edge contraction operation is equivalent with the operation of removing the vertex with degree 1 and its incident edge, hence $deg(u), deg(v) \geq 2$.
As the vertices of $K_{3,3}$ has degree 3 it follows that $deg(u), deg(v) \leq 3$ as the contraction of an edge $e = xy$ can not produce a new vertex v_e of lesser degree than x or y, unless x or y is of degree 1. Furthermore if $deg(u) = deg(v) = 3$ then the contracted vertex would be of degree 4, a contradiction as $K_{3,3}$ is obtained from G' by contracting edges.

Hence $deg(u) = deg(v) = 2$ or $deg(u) = 2$ and $deg(v) = 3$, but then the contraction of uv is equivalent to a topological minor.

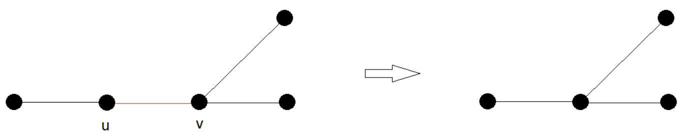

Figure 10: Contracting uv is equivalent to a topological minor

It follows that if G contains $K_{3,3}$ as a minor then it also contains a subgraph that is a subdivision of $K_{3,3}$, hence G contains $K_{3,3}$ as a topological minor.

If G is minimal with the property of containing K^5 as a minor then K^5 is obtained from G by first removing edges and vertices, giving the minimal connected graph G' and then contracting edges in G' to obtain K^5.

Denote the vertices of K^5 by v_i, $i \in \{1, 2, 3, 4, 5\}$, and let U_i denote the subgraph of G' which was contracted into v_i.
By the minimality of G' each U_i is minimally connected, and hence a tree, and there are exactly one edge between any pair of subgraphs U_i and U_j, $i \neq j$.

Let T_i denote U_i together with the four edges that join it to the other subgraphs U_j.
If each T_i is a $TK_{1,4}$ then G' is a subdivision of K^5.
If not then at least one T_i has two vertices, x, y, of degree 3. This yields a $K_{3,3}$ minor in G' by contracting T_i into the two vertices x and y, contracting every other T_j into a single vertex and then remove two edges.

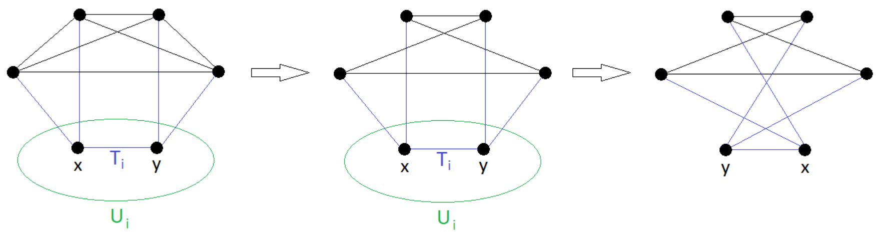

Figure 11: $K_{3,3}$ minor in G'

But as a $K_{3,3}$ minor is also a topological $K_{3,3}$ minor, if G contains K^5 as a minor then it contains a subgraph that is a subdivision of K^5 or a subgraph that is a subdivision of $K_{3,3}$. □

Definition 2.19. Let $G = (V, E)$ be a graph. If $S \subset V$ and $G - S$ is disconnected then S is a separator of G.

Hence if a graph $G = (V, E)$ can be divided into two nonempty subsets $A, B \subseteq V$ such that $S \subset V$ separates A and B in G, then S is a separator of G. Moreover any path containing vertices in both A and B must also contain at least one vertex in S.

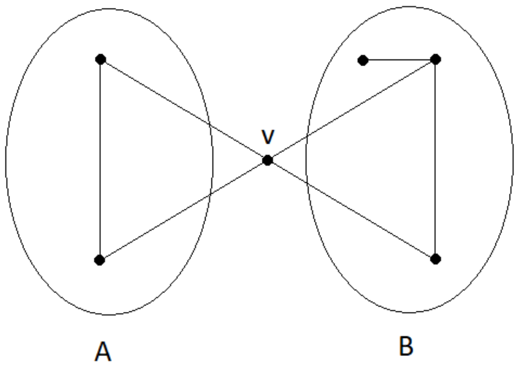

Figure 12: $S = \{v\}$ separates the graph $G = (A \cup B \cup \{v\}, E)$, whence any path between A and B must also contain v

Definition 2.20. A graph G is n-connected if no two vertices in G are separated by fewer than n other vertices in G.

By definition 2.20 the graph in figure 12 is 1-connected as it has a separator of one vertex, $S = \{v\}$.

Theorem 2.21. Menger's theorem. Let $G = (V, E)$ and $A, B \subseteq V$. The minimum number of vertices separating A from B in G is equal to the maximum number of disjoint $A - B$ paths in G.

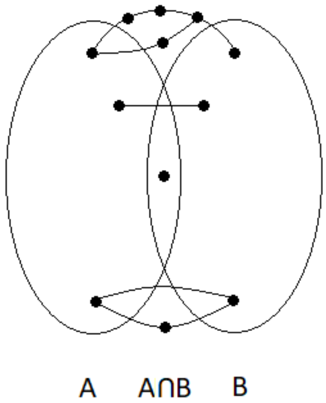

Figure 13: A and B are separated by a minimum of four vertices and there are exactly four disjoint $A - B$ paths

Proof. Based on [1, p. 67]
Let G be a graph and let n denote the minimum number of vertices separating A from B in G. Then the maximum number of disjoint $A-B$ paths in G is, by the pigeonhole principle, less than or equal to n.

Assume that, for all graphs with fewer edges than G, there are n disjoint $A-B$ paths in G and apply induction on the number of edges.

If $E = \emptyset$, then $|A \cap B| = n$ and there are n disjoint paths from A to B in G, as any vertex in $|A \cap B|$ is an $A - B$ path.

Suppose G has an edge $e = xy$, let v_e be the contracted vertex of G/e and let

$$A' := \begin{cases} A, & \text{if } A \cap \{x, y\} = \emptyset \\ A - \{x, y\} + \{v_e\}, & \text{otherwise} \end{cases}$$
$$B' := \begin{cases} B, & \text{if } B \cap \{x, y\} = \emptyset \\ B - \{x, y\} + \{v_e\}, & \text{otherwise.} \end{cases}$$

Then $A' - B'$ paths in G/e that does not contain v_e are also $A - B$ paths in G, as the vertices in the path remain unchanged by the edge contraction.

Consider an undirected $A' - B'$ path in G/e, P', which contain v_e.
Let $P' = v_0 ... v_i v_e v_{i+1} ... v_n$, $v_e \notin A' \cap B'$, then P' induces one of three possible undirected paths, P, in G.

- $P = v_0 ... v_i x y v_{i+1} ... v_n$
- $P = v_0 ... v_i x v_{i+1} ... v_n$
- $P = v_0 ... v_i y v_{i+1} ... v_n$

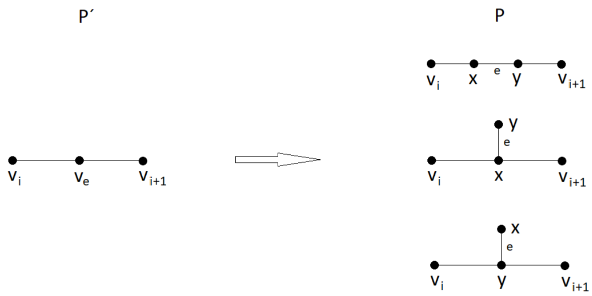

Figure 14: P' induces one of three possible undirected paths, P, in G

Furthermore, if $v_e \in A' \cap B'$ then at least one of the following cases occur:

- $x \in A \cap B$, $y \notin A \cap B$
- $y \in A \cap B$, $x \notin A \cap B$
- $x \in A$, $y \in B$
- $y \in A$, $x \in B$
- or $x, y \in A \cap B$

where the first four cases induce a corresponding path P as before and the last case correspond to two possible disjoint paths in G, $P = x$ and $P = y$.

Hence each collection of disjoint $A' - B'$ paths in G/e has a corresponding collection of disjoint $A - B$ paths in G.

As $|E(G/e)| < |E(G)|$ the assertion hold for G/e, hence there is a $C \subseteq V[G/e]$ which separates A' from B' in G/e. If C has exactly n vertices the proof is done, hence it remains to consider the case where C has fewer than n vertices.

If $v_e \notin C$ then C would also separate A and B in G, contradicting $|C| < n$. Hence $v_e \in C$.

As C separates A' and B' in G/e, any path in G between A and B must contain vertices either in $C - v_e$ or $\{x, y\}$.

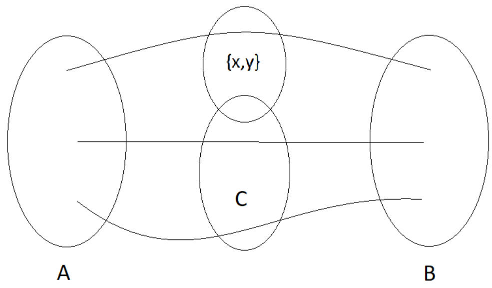

Figure 15: $(C - v_e) \cup \{x, y\}$ separates A and B in G

Hence $D := (C - v_e) \cup \{x, y\}$ separates A from B in G, and by the induction hypothesis, $|D| \geq n$.
Since $|D| = |C| - 1 + 2 = |C| + 1$ and $|C| < n$, it follows that $|D| = n$.

Consider $G - e$ and let S separate A from D in $G - e$.
Furthermore, since D is an $A - B$ separator in G, it must also separate A from B in $G - e$. If not there would exist an $A - B$ path in $G - e$ not going through D, however this path would then also be an $A - B$ path in G not passing through D. This gives a contradiction as D is an $A - B$ separator in G.

As every $A - D$ path in $G - e$ must pass through S, and D is an $A - B$ separator in $G - e$, S is also an $A - B$ separator in $G - e$ and $|S| \geq n$.

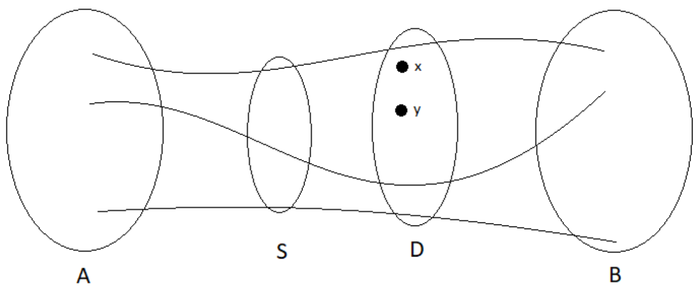

Figure 16: S and D are $A - B$ separators in $G - xy$

Since $|E(G-e)| < |E(G)|$ the assertion hold for $G-e$, hence $G-e$ has n disjoint $A-D$ paths.
Similarly there are n disjoint $B-D$ paths in $G-e$.

If an $A-D$ path meet any $B-D$ path outside of D they would form an $A-B$ path not going through D, but as D separates A and B this gives a contradiction. Hence $A-D$ and $B-D$ paths can only meet in D.
As each collection of $A-D$ and $B-D$ paths are disjoint each path end at a distinct vertex in D. Hence, since $|D| = n$, the n disjoint $A-D$ paths and the n disjoint $B-D$ paths can be combined to n disjoint $A-B$ paths in $G-e$.

However, as the edge $e = xy$ connects two vertices in D, no path contain both the vertices x and y. Therefore the n disjoint $A-B$ paths of $G-e$ are also the n disjoint $A-B$ paths of G. □

Definition 2.22. Let G be a planar graph, then the regions of G enclosed by edges are the faces of G.

Hence the planar graph in figure 8 has six faces, one outer face and five faces enclosed by the cycles $v_1v_2v_3v_1$, $v_1v_2v_5v_1$, $v_2v_3v_4v_2$, $v_2v_4v_5v_2$ and $v_3v_4v_5v_3$.

Theorem 2.23. Euler's formula. Let $G = (V, E)$ be a simple, connected planar graph with v vertices, e edges and f faces, then $v - e + f = 2$.

Proof. By induction on f.

If $f = 1$ then G does not contain any cycles and $e = v - 1$, hence $v - e + f = v - (v-1) + 1 = 2$

Assume the assertion hold for $f \leq n$ and consider the case where $f = n + 1$.
As $f > 1$ there exists an edge xy which separates two faces of G. Then $G - xy$ is connected and has v vertices, $e - 1$ edges and $f - 1$ faces. Hence, by the induction hypothesis, $v - (e-1) + (f-1) = v - e + f = 2$. □

Corollary 2.24. The graphs K^5 and $K_{3,3}$ are non planar.

Proof. Assume $G = K^5$ with $v = 5$ vertices and $e = 10$ edges is planar. By Theorem 2.23, $v - e + f = 2$, hence $f = 2 - v + e = 7$.

Consider a face of G.
As any cycle of G has a minimum of three edges, and as each edge is the boundary of two faces, the minimum number of edges in G is $e = \frac{3f}{2} = \frac{3 \cdot 7}{2}$. But as $\frac{21}{2} > 10$ this gives a contradiction, hence G is non planar.

Assume $G = K_{3,3}$ with $v = 6$ vertices and $e = 9$ edges is planar. By Theorem 2.23, $v - e + f = 2$, hence $f = 2 - v + e = 5$.

Consider a face of G.
As G is bipartite any cycle of G has a minimum of four edges, and as each edge is the boundary of two faces, the minimum number of edges in G is $e = \frac{4f}{2} = 10$. But as $10 > 9$ this gives a contradiction, hence G is non planar. □

3 Kuratowski's Theorem

Lemma 3.1. If G is 3-connected and $|G| \neq 4$ then G has an edge e such that G/e is 3-connected.

Proof. Based on [1, p. 64].
Assume there is no such edge. Then, for each edge xy, the graph G/xy obtained by contracting the edge xy can be divided into two subgraphs that are separated by a set S containing at most two vertices.
Since every vertex in G is at least three connected, the contracted vertex v_{xy} of G/xy must lie in S and, as G/xy is 2-connected, $|S| = 2$. Hence G has a vertex $z \notin \{x, y\}$ such that $S = \{v_{xy}, z\}$ separates G/xy.

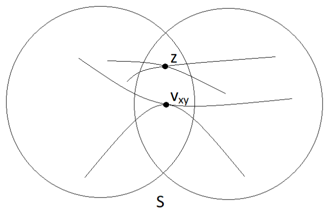

Figure 17: G/xy has a separating set S of exactly two vertices

Let $T := \{x, y, z\}$, then any two vertices separated by S in G/xy are separated by T in G. As T separates G every vertex in T has a neighbour in every maximal connected subgraph C of $G - T$.

Choose xy, z and C such that $|C|$ is as small as possible and let u be a neighbour of z in C. By assumption G/zu is again not 3-connected, hence there exists a vertex v such that $U = \{z, u, v\}$ separates G, and every vertex in U has a neighbour in every component of $G - U$.

As x and y are adjacent and U separates G, $G - U$ has a component D such that $D \cap \{x, y\} = \emptyset$.

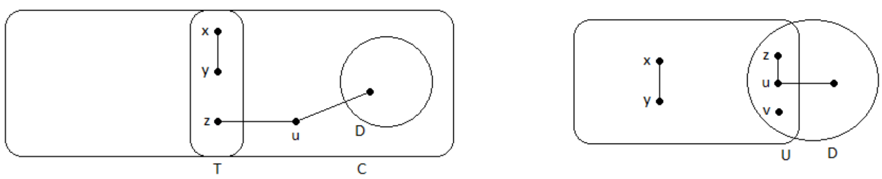

Figure 18

Then, as $u \in C$, every neighbour of u in D is also in C and $D \cap C \neq \emptyset$, whence $D \subset C$. This gives a contradiction as xy, z and C were chosen such that $|C|$ was as small as possible. □

Lemma 3.2. Every 3-connected graph G without a K^5 or $K_{3,3}$ minor is planar.

Proof. Based on [1, p. 104].
By induction on $|G|$.
Let $|G| = 4$, as G is 3-connected it follows that $G = K^4$ and the assertion hold.

Let $|G| > 4$ and assume the assertion hold for smaller graphs. By Lemma 3.1 G has an edge xy such that G/xy is 3-connected, and as the minor relation is transitive G/xy has no K^5 or $K_{3,3}$ minor either.

Hence, by the induction hypothesis, G/xy has an embedding G' in the plane. Let v_{xy} be the contracted vertex of G/xy and f the face of $G' - v_{xy}$ which contained v_{xy}, with boundary C.

Let $X := N_G(x)\backslash\{y\}$ and $Y := N_G(y)\backslash\{x\}$.
As $v_{xy} \in f$, $G'' := G' - \{v_{xy}v | v \in Y\backslash X\}$ is equivalent with an embedding of $G - y$, in which x is represented by v_{xy}.

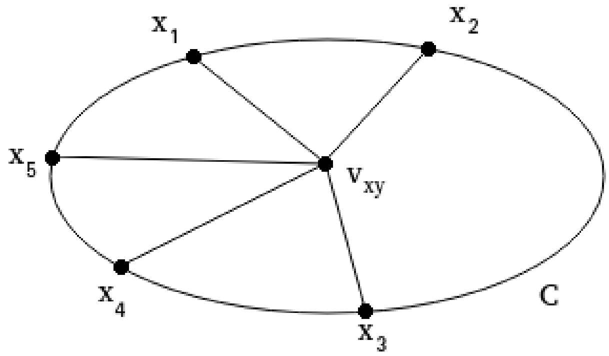

Figure 19: G''

An embedding of G can then be constructed by adding y to G''.

Since G' is 3-connected $G' - v_{xy}$ is 2-connected, and hence C is a cycle.
Let $x_1, ..., x_k$ be the vertices of X on C and let $P_i = x_i...x_{i+1}$, $(i = 1, ..., k; k+1 = 1)$, be the paths on C between them.

Suppose $Y \nsubseteq V(P_i)$ for some i and consider the three possible cases.
If y has a neighbour $y' \in P_i$ for some i, it has another neighbour $y'' \in C - P_i$ separated in C by $x' := x_i$ and $x'' := x_{i+1}$. Then x, y', y'' and y, x', x'' form the branch vertices of a $TK_{3,3}$ in G, a contradiction.

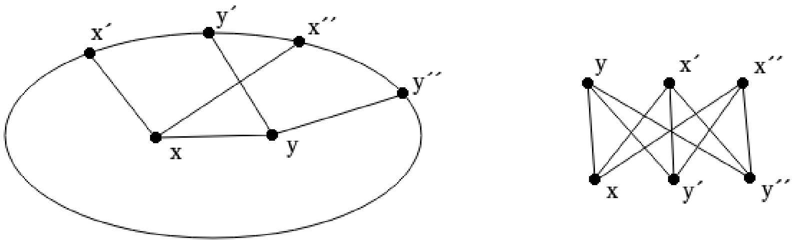

Figure 20: Branch vertices of a $TK_{3,3}$ in G

If $Y \subseteq X$ and $Y \cap X \leq 2$, then y has exactly two neighbours, y' and y'', on C, separated by two vertices x' and x''. As in the first case these form the branch vertices of a $TK_{3,3}$ in G, a contradiction.
If y and x has three common neighbours on C, v_1, v_2, v_3, then these form the branch vertices of a TK^5 in G, a contradiction.

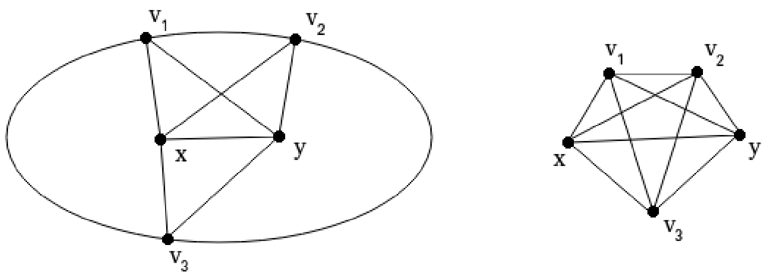

Figure 21: Branch vertices of a TK^5 in G

Hence $Y \subseteq V(P_i)$ for some i.

Fix i so that $Y \subseteq P_i$. The set $C \backslash P_i$ is contained in one of the two faces of the cycle $C_i := xP_ix$, let the other face of C_i be f_i. Since f_i contains points of f, close to x, but no points of its boundary C, $f_i \subseteq f$.
The plane edges xx_j, $j \notin \{i, i+1\}$, meet C_i only in x and end outside f_i in $C \backslash P_i$, whence f_i meet none of those edges. Therefore f_i is contained in, and equal to a face of G'', and y and its incident edges can be placed in f_i $\square$

Lemma 3.3. Let Υ be a set of 3-connected graphs. Let G be a graph with a proper separation $\{V_1, V_2\}$ of order $\kappa(G) \leq 2$. If G is edge maximal without a topological minor in Υ, then so are $G_1 := G[V_1]$ and $G_2 := G[V_2]$, and $G_1 \cap G_2 = K^2$

Proof. Based on [1, p. 105].
As G is maximal with the property of having no topological minor in Υ, every edge e added to G must lie in a $TK \subseteq G + e$, $K \in \Upsilon$.
Let $S := V_1 \cap V_2$ and $v \in S$.
If, for some v, v is not connected to a neighbour in every component of $G_i - S$, $i \in \{1, 2\}$, then $S \backslash \{v\}$ would separate G, contradicting $|S| = \kappa(G)$.
Hence every v must have a neighbour in every component of $G_i - S$, $i \in \{1, 2\}$.

If $S = \emptyset$, let e join a vertex, v_1, in V_1 to a vertex, v_2, in V_2. By Theorem 2.21 a 3-connected TK must have its branch vertices in a single V_i, $i \in \{1, 2\}$. Since the arising TK must contain e and there can only be one path, containing e, between G_1 and G_2 a contradiction is reached.

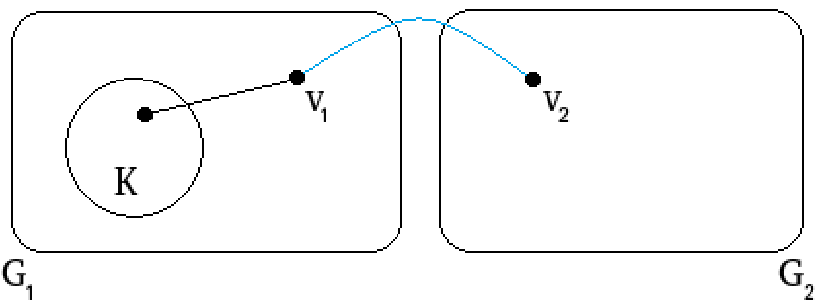

Figure 22: $S = \emptyset$

If $S = \{v\}$, let e join a neighbour of v in $V_1 \backslash S$, v_1, to a neighbour of v in $V_2 \backslash S$, v_2. As in the previous case, a 3-connected TK must have all its branch vertices in a single V_i, $i \in \{1, 2\}$, and can, at most, meet V_{3-i} in a path P containing v and e. But as $v_i v_j P v$ can be replaced with $v_i v$ a TK is produced in $G_i \in G$, a contradiction is reached.

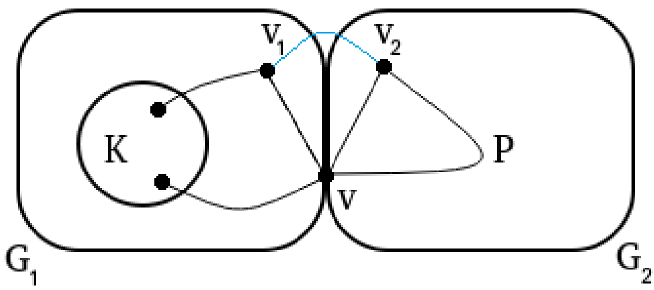

Figure 23: $S = \{v\}$

Let $S = \{x, y\}$ and assume that $S \neq K^2$. Let $e = xy$ be an additional edge for G, then there must be a $TK \subseteq G + e$ with e in TK. As in the previous cases a 3-connected TK must have its branch vertices in a single V_i, $i \in \{1, 2\}$, and e must be an edge in the arising TK. But then a contradiction is reached as e can be replaced with an xPy path, as x and y are connected to every component of $G_i - S$, which yields a TK in G. Hence $S = K^2$.

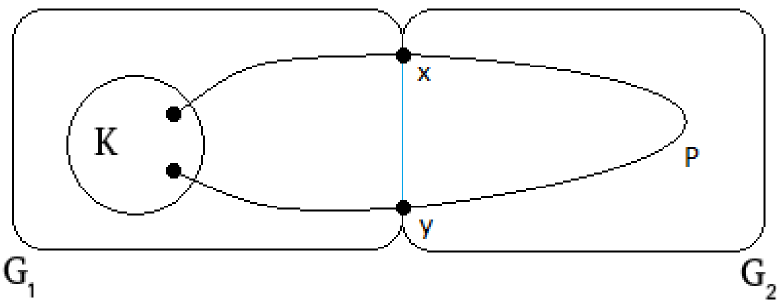

Figure 24: $S = \{x, y\}$, $e \neq xy$

It remains to show that G_1 and G_2 are edge maximal with the property of not having a topological minor in Υ.

Let P be a path as above and e be an additional edge for G_i, $i \in (1, 2)$, replacing xPy with xy if necessary. This yields a TK either in $G_i + e$ or G_{i-3}.
If the TK lies in G_i it shows edge maximality of G_i. If it lies in G_{i-3} a contradiction is reached as $G_{i-3} \subseteq G$. □

Lemma 3.4. If $|G| \geq 4$ and G is edge maximal without a $TK_{3,3}$ or TK^5 then G is 3-connected

Proof. Based on [1, p. 106].
By induction on $|G|$.
For $|G| = 4$, $G = K^4$ and the assertion hold.

For $|G| > 4$, let G be edge maximal with the property of not having a $TK_{3,3}$ or TK^5. Let G have a proper separation $\{V_1, V_2\}$, $G_1 := G[V_1]$ and $G_2 := G[V_2]$, and suppose $\kappa(G) \leq 2$. As the forbidden TK^5 and $TK_{3,3}$ are 3 connected we have, by lemma 3.3, that since G is edge maximal without a TK^5 or $TK_{3,3}$ then so are G_1 and G_2, and $G_1 \cap G_2 = K^2$.
Hence, by the induction hypothesis, G_1 and G_2 are either a triangle or 3-connected, and, by lemma 3.2, planar as they cannot contain a TK^5 or $TK_{3,3}$.

Let $G_1 \cap G_2 = \{x, y\}$ and choose a drawing of G_i, $i \in \{1, 2\}$, with a face f_i containing xy and a vertex $z_i \neq x, y$ on its boundary. Let z_1z_2 be an additional edge for G and, as G is edge maximal, let K denote the arising TK^5 or $TK_{3,3}$ in $G + z_1z_2$.

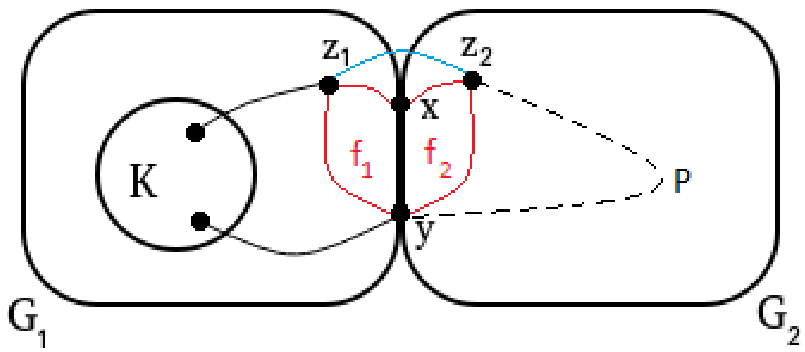

Figure 25: $G + z_1z_2$

If the branch vertices of K lies in the same G_i of $G + z_1z_2$ then z_1z_2 can be replaced by a path containing z_iPx or z_iPy. But then either $G_i + xz_i$ or $G_i + yz_i$, which are planar by the choice of z_i, contains a K, a contradiction.

If K is a TK^5 there must, by Theorem 2.21, be at least four independent paths between its branch vertices. But $G + z_1 z_2$ contains three independent paths between $G_1 - G_2$ and $G_2 - G_1$, a contradiction.

If K is a $TK_{3,3}$ with one branch vertex, v, in $G_1 - G_2$ or $G_2 - G_1$, assume K lies in G_1 with v in $G_2 - G_1$. Then $G_1 + v + \{vx, vy, vz_1\}$, which is planar by the choice of z_1, can be drawn with v in f_1, a contradiction.

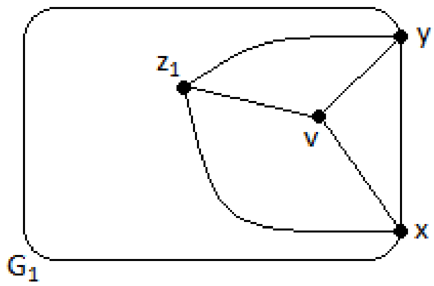

Figure 26: $G_1 + v + \{vx, vy, vz_1\}$

If K is a $TK_{3,3}$ with more than one branch vertex in $G_1 - G_2$ or $G_2 - G_1$ at least four indepent paths is required between $G_1 - G_2$ and $G_2 - G_1$, by Theorem 2.21. As $G + z_1 z_2$ has at most three independent paths between $G_1 - G_2$ and $G_2 - G_1$, a contradiction is reached. □

Theorem 3.5. kuratowski's theorem. A graph G is planar if and only if it contains neither K^5 nor $K_{3,3}$ as a minor.

Proof. Let $G = (V, E)$ be a graph and $\{TK^5, TK_{3,3}\} \nsubseteq G$. Let S be a set of additional edges such that $G' = (V, E + S)$ is edge maximal with the property of having no topological minor in $\{K^5, K_{3,3}\}$.
By lemma 3.4 G' is 3-connected and hence, by lemma 3.2, G' is planar.

As G' is planar it has an embedding in the plane, but then, by removing the edges in S, so does G, and, by Theorem 2.18, G contains no minor in $\{K^5, K_{3,3}\}$. □

4 Surfaces

The definitions in Chapter 4 are, unless otherwise stated, based on those of P.A. Firby and C.F. Gardiner [2].

Definition 4.1. A surface is a connected compact Hausdorff topological space locally homeomorphic to a unit disc in the plane.
All surfaces considered in this paper will be compact surfaces.

Definition 4.2. A surface is orientable if a two dimensional figure following any closed loop on the surface can not return to its starting point as a mirror image of itself, otherwise the surface is non orientable.

Definition 4.3. A plane model is a polygonal representation, showing certain pairs of edges as identified, of a surface.
Figure 27 shows a plane model of a torus constructed by making two loop cuts, which allow the torus to be opened up and form a plane model. Direction of edges are indicated by arrows on the edge.

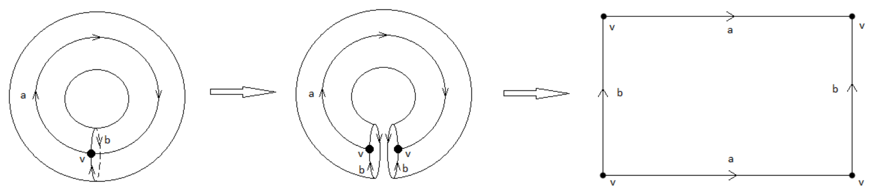

Figure 27: Acquiring a plane model of a torus

By definition 4.2 the Möbius band is a non orientable surface as any figure traveling a lap around the Möbius band will return to its starting point as a mirror image of itself.

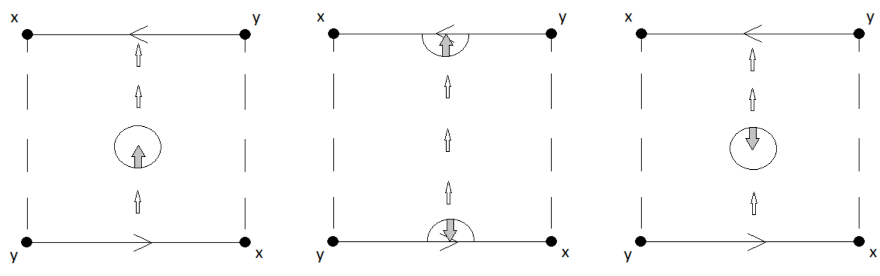

Figure 28: Two dimensional figure traveling along the plane model of a Möbius band

Definition 4.4. A crosscap is obtained by making a hole in a sphere and attach the boundary of a Möbius band to it [10].

As the Möbius band, and hence a crosscap, is a non orientable surface a sphere with crosscaps is also non orientable.

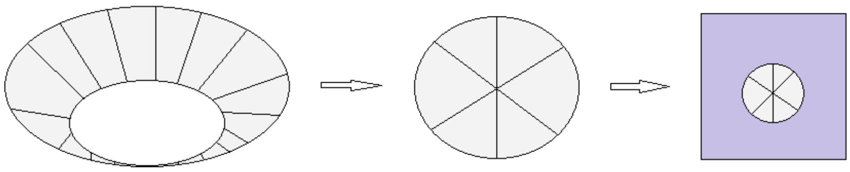

Figure 29: Attaching a Möbius band to a hole in a surface

Definition 4.5. The genus of a surface is the number of handles attached to a sphere if the surface is orientable, or the number of crosscap attached to a sphere if the surface is non orientable.
Let M_k denote a orientable surface M of genus k and let N_k denote a non orientable surface N of genus k.

Figure 29 and 30 visualizes the process of attaching a Möbius band and a handle respectively to a surface. It follows that a surface is orientable precisely if it does not contain a crosscap, and non orientable otherwise.

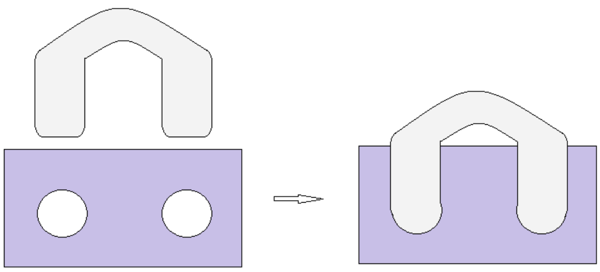

Figure 30: Attaching a handle to a surface

Definition 4.6. Let a graph triangulate a surface such that it has an embedding on the surface where any two triangles are either disjoint or meet at a common vertex or along a complete common edge. This gives a triangulation of the surface.

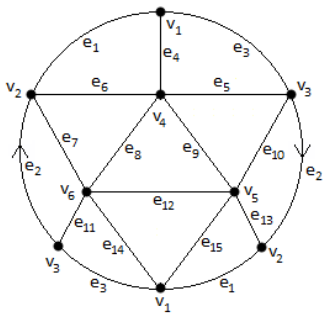

Figure 31: Triangulation of the surface N_1

Definition 4.7. The Euler Characteristic χ of a closed surface S, with a triangulation consisting of v vertices, e edges and f faces is given as $\chi(S) = v - e + f$.

Definition 4.8. For a closed surface S with Euler characteristic $\chi(S)$ its genus g is given by
$g = 1 - \frac{\chi(S)}{2}$ if S is orientable.
$g = 2 - \chi(S)$ if S is non orientable.

Definition 4.9. Two surfaces are homeomorphic if one of the surfaces can be stretched, bent or squashed to look like the other, without tearing or gluing points together.

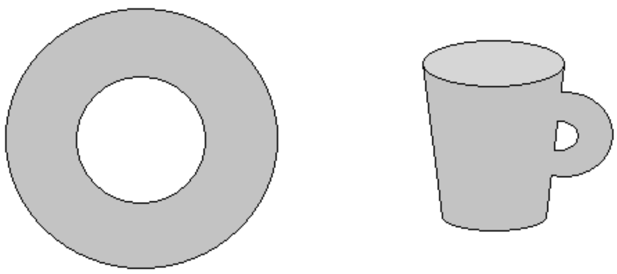

Figure 32: Two homeomorphic surfaces M_1.

Theorem 4.10. Two compact surfaces S_1 and S_2 are homeomorphic if and only if they are both either orientable or non-orientable and $\chi(S_1) = \chi(S_2)$.

Proof. See [3, pp. 393-399] □

4.1 Special surfaces

As the genus of a surfaces increase, by the addition of extra handles for orientable surfaces and crosscap for non orientable surfaces, its Euler characteristic decrease, by 4.8. Hence, by 4.10, a new surface is obtained that is not homeomorphic to the previous surface.

4.1.1 Sphere

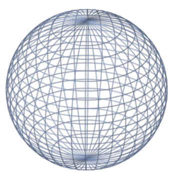

Figure 33: Sphere. Image by Geek3
https://commons.wikimedia.org/wiki/File:Sphere_wireframe_10deg_10r.svg

The sphere is the basic orientable surface of genus 0, with Euler characteristic $\chi(S) = 2 - 2g = 2$. Let S or M_0 denote the sphere.

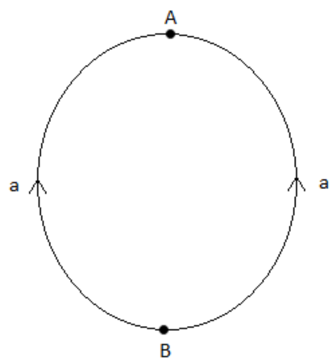

Figure 34: Plane model of the sphere

4.1.2 Torus

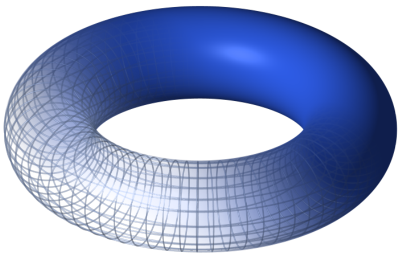

Figure 35: Torus. Image by LucasVB
https://commons.wikimedia.org/wiki/File:Torus.png

The torus is the orientable surface formed by adding a handle to a sphere, whence the torus has genus 1 and Euler characteristic $\chi(T) = 2 - 2g = 0$. Let T or M_1 denote the torus.

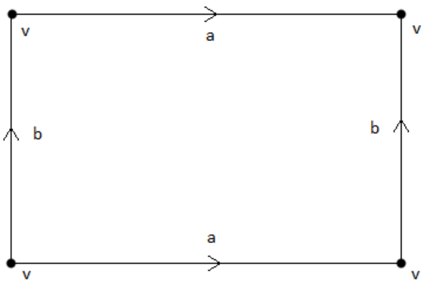

Figure 36: Plane model of the torus

4.1.3 Real Projective Plane

The real projective plane is the basic non orientable surface of genus 1, with Euler characteristic $\chi(P) = 2 - g = 1$, obtained by adding a crosscap to a sphere. Let P or N_1 denote the real projective plane.

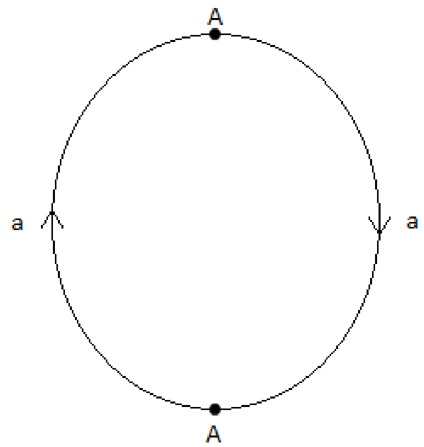

Figure 37: Plane model of the projective plane

4.1.4 Klein Bottle

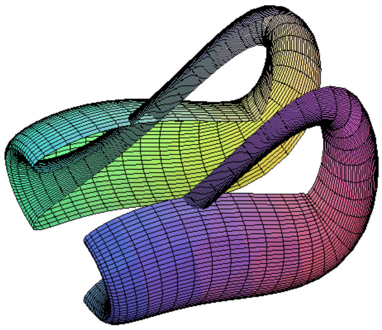

Figure 38: Klein bottle. Image by Theon
https://commons.wikimedia.org/wiki/File:Bouteille_Klein_2Mobius.png

The Klein bottle is the non orientable surface of genus 2, and Euler characteristic $\chi(K) = 2 - g = 0$, obtained by adding two crosscaps to a sphere. Let K or N_2 denote the Klein bottle.

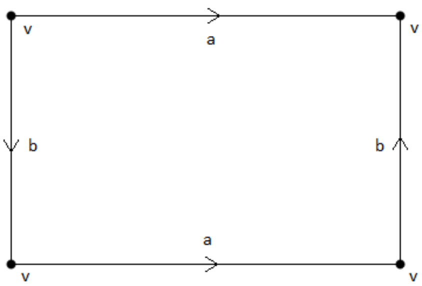

Figure 39: Plane model of the Klein bottle

5 Embedding graphs on surfaces

The definitions in Chapter 5 are based on those of P.A. Firby and C.F. Gardiner [2].

Definition 5.1. A graph is embedded on a surface if it has a fixed geometrical representation on the surface such that any two edges does not intersect, other than at a vertex.

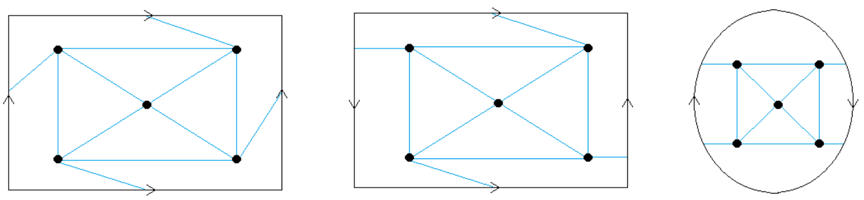

Figure 40: K^5 embedded on the torus, Klein bottle and projective plane respectively

Definition 5.2. The characteristic of a graph G embedded on a compact surface M is the maximum value of $\chi(M)$ for which G can be embedded. Let $\gamma(G)$ denote the characteristic of G.

Hence if a graph is not embeddable on a surface, there exist another surface, obtained by attaching handles or crosscaps to the original surface, for which the graph is embeddable.
Consider $G = K^5$. G is not embeddable on the sphere but has an embedding on the surface obtained by attaching a crosscap to the sphere, as seen in the third image of figure 40. Hence $\gamma(K^5) = \chi(N_1) = 2 - g = 1$.

Definition 5.3. A graph G is minimally embedded in a compact surface M if $\gamma(G) = \chi(M)$.

Definition 5.4. M is a minimal surface for G if G has a minimal embedding on M.

By definition 5.3 and 5.4 K^5 is minimally embedded in the projective plane and the projective plane is a minimal surface for K^5, as any other surface for which K^5 is embeddable has a lower Euler characteristic than the projective plane.

Definition 5.5. A 2-cell embedding is an embedding of a graph such that each of its faces is homeomorphic to an open disc.

The proof of Theorem 5.6 lies outside the scope of this paper [2, p. 108].
Theorem 5.6. If the connected graph G with v vertices and e edges is embedded on the compact surface M, and the embedding produces f faces, then
$v - e + f \geq \chi(M)$,
with equality if and only if the embedding is 2-cell.

Corollary 5.7. A graph is embeddable on the sphere if and only if it is planar.

Proof. The corollary follows from Theorem 2.23, Theorem 5.6 and the fact that the sphere has Euler characteristic $\chi(S) = 2 - 2g = 2$. □

Theorem 5.8. If the connected graph G with v vertices and e edges is 2-cell embedded in a surface M with Euler characteristic $\chi(M)$, then $\chi(M) \leq v - \frac{e}{3}$.

Proof. Let G be 2-cell embedded in a compact surface, producing f faces. Then, by Theorem 5.6, $\chi(M) = v - e + f$.
As each cycle in the pattern formed by the graph has a minimum of three edges, every edge must lie in exactly two faces, and every face must contain at least three edges, which gives the inequality $3f \leq 2e$.
Hence $\chi(M) \leq v - e + \frac{2e}{3} = v - \frac{e}{3}$. □

Corollary 5.9. If G is the complete graph K^n, $n \geq 3$, then 5.8 becomes $\chi(M) \leq \frac{7n-n^2}{6}$.

Proof. As the complete graph K^n, $n \geq 3$, has n vertices and $\binom{n}{2} = \frac{n(n-1)}{2}$ edges,
$v - \frac{e}{3} = n - \frac{n(n-1)}{6} = \frac{7n-n^2}{6}$. □

Equality in Corollary 5.9 for minimal embeddings on orientable surfaces, was proven by a series of mathematicians [2, p. 116], and a professor of French literature, Jean Mayer [6, p. 519]. The Klein bottle, non orientable surface with Euler characteristic $\chi(K) = 0$, is a special case for which K^7 does not embed, proven by Philip Franklin in 1934 [4].

Definition 5.10. A well-quasi-ordering is a binary relation that is transitive and reflexive such that any infinite sequence of elements contains a pair x_i and x_j such that $x_i \leq x_j$ where $i < j$.

Theorem 5.11. Robertson–Seymour theorem. Ordering of finite graphs by the minor relation provides well-quasi-orderings of them.

Theorem 5.11 was proven by Neil Robertson and Paul D. Seymour in a series of 20 papers between 1983 and 2004 [9], this proof lies outside the scope of this paper.

Definition 5.12. A family of graphs is minor closed if for any graph G in the family, any minor of G is also in the family.

Theorem 5.13. For any family of graphs that is minor closed there is a finite set of forbidden minors.

Proof. Let F be a minor closed family of graphs and let G be the complement of F. By Theorem 5.11 there is a finite set K of minimal elements in G, as there can be no infinite antichain. The forbidden minors for F are then precisely the graphs in K. $\square$

5.1 Forbidden minors

The family of graphs that can be embedded in a closed surface is minor closed and hence, by Theorem 5.13, has a finite set of forbidden minors. These are currently known for the sphere, $\{K^5, K_{3,3}\}$ [Theorem 3.5], and the projective plane, with 35 forbidden minors or 103 forbidden topological minors [7, p. 198], which were discovered by Glover H.H., Huneke J.P. and Wang C.S. [5, p. 49]. The complete set of forbidden minors for the torus, with more than 17'000 known forbidden minors [8], and the Klein bottle is an ongoing research topic.

5.1.1 Forbidden minors of the projective plane

The 35 forbidden minors of the projective plane are, by [7, p. 198]

- The three possible disjoint unions of the graphs K^5 and $K_{3,3}$.
- The three graphs obtained by merging an edge in $2K_{3,3}$, $2K^5$ or $K_{3,3}$ and K^5.
- The six graphs that can be obtained by merging two edges of K^5 and $K_{3,3}$, and, if the merged edges are adjacent to each other, removing the edge connecting them.
- $K_{3,5}$ and $K_{4,4} - e$, where e is an edge.
- The graph obtained by taking the disjoint union of $2K_{2,3}$ and label the vertices in the two parts with three vertices v_{11}, v_{12}, v_{13} and v_{21}, v_{22}, v_{23} respectively. Then add the three edges $v_{11}v_{21}, v_{12}v_{22}, v_{13}v_{23}$.
- The graph obtained by, in the previous graph, contracting two of the edges $v_{11}v_{21}, v_{12}v_{22}, v_{13}v_{23}$ and subdivide the remaining edge to get a vertex v. Then add two edges such that v becomes adjacent to the two contracted vertices.
- The graph obtained by, in one of the original $K_{2,3}$ in the graph above, add an edge connecting the two vertices in one of the parts with two vertices, and, in the same $K_{2,3}$, contract the edge between v and the vertex which was subdivided.
- The graph obtained by taking the disjoint union of $2K^4$, and add edges to connect one vertex in K^4_i, $i \in 1, 2$, to two vertices in K^4_{3-i}. Then connect the rest of the vertices of K^4_i to one other vertex in K^4_{3-i}, such that one vertex in each K^4 has five neighbours and the rest has four neighbours.
- The graph obtained by taking the disjoint union of K^4 and $K_{2,3}$ and add a total of four new edges to connect one of the vertices in the part of $K_{2,3}$ with three vertices, H, to two vertices in K^4. And connect the remaining two vertices in H to each of the remaining two vertices in K^4.

- The graph obtained by taking the disjoint union of K^4 and $K_{2,3}$ and add a total of five edges to connect two of the vertices in the part of $K_{2,3}$ with three vertices to one vertex in K^4. And connect the three remaining vertices in $K_{2,3}$ to unique vertices of the three remaining vertices of K^4.

- The graph obtained by taking the disjoint union of $2K^4$ and add a total of four edges to connect each vertex in one of the K^4 to unique vertices in the other K^4.

- The graph obtained by taking the disjoint union of $2K^5$, select one edge and one vertex not incident to the selected edge in each of the K^5 and merge the selected edges and vertices.

- The graph obtained by taking the disjoint union of $K_{2,3}$ and K^4 and add a total of four vertices to K^4 such that exactly one vertex in K^4 is adjacent to a vertex in the part of $K_{2,3}$ with two vertices, and each vertex in the part with three vertices is adjacent to exactly one unique vertex in K^4, such that each vertex in K^4 has exactly four neighbours.

- The 12 graphs in figure 41.

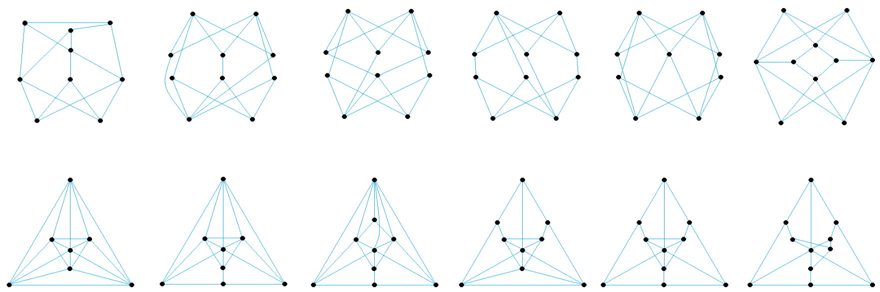

Figure 41

5.2 Special embeddings

5.2.1 Embeddings on surfaces of higher genus

For a graph to be 2-cell embeddable in a surface each of its faces must, by definition 5.5, be homeomorphic to an open disc. Hence

- No face may meet itself other than along edges or at vertices.
- Each face must have a cycle at its boundary containing at least three vertices.
- No two edges meet other than at vertices.

Therefore, as the genus of the surface increase, a graph must have one cycle for each handle to be 2-cell embeddable on a surface.

5.2.2 Embeddings of the complete graphs

Consider 2-cell embeddings of the complete graph K^n, $n \geq 3$, on a surface M. By corollary 5.9,

- K^3: $\chi(M) \leq \frac{7 \cdot 3 - 3^2}{6} = 2$.
- K^4: $\chi(M) \leq 2$.
- K^5: $\chi(M) \leq \frac{5}{3}$.
- K^6: $\chi(M) \leq 1$.
- K^7: $\chi(M) \leq 0$.
- K^8: $\chi(M) \leq -\frac{4}{3}$.

Hence, as the sphere has Euler characteristic $\chi(S) = 2$, no complete graph with five or more vertices are embeddable on the sphere. This result is consistent with Theorem 3.5 as K^n is a minor of K^{n+1}.

As the torus has Euler characteristic $\chi(T) = 0$ it has an embedding of K^7, as seen in figure 43, but no embeddings of K^n, $n \geq 8$

The projective plane, with Euler characteristic $\chi(P) = 1$, has an embedding of K^6 but no embeddings of K^n, $n \geq 7$.

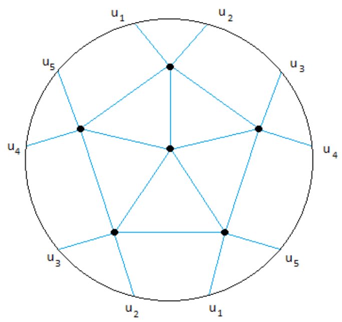

Figure 42: K^6 embedded on the plane model of the projective plane

The Klein bottle, with Euler characteristic $\chi(K) = 0$, is a special case and has no embedding of K^7 [4]. Hence the complete graph with six vertices is the largest complete graph embeddable on the Klein bottle.

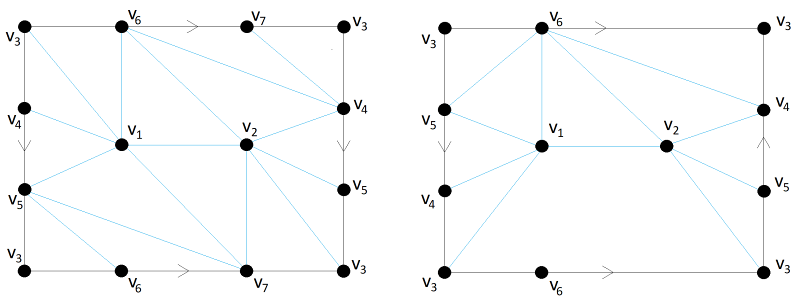

Figure 43: K^7 embedded on the torus and K^6 embedded on the Klein bottle

References

[1] Reinhard Diestel. *Graph Theory.* Springer, 2010.

[2] P.A. Firby and C.F. Gardiner. *Surface Topology.* Ellis Horwood, 1992.

[3] George K. Francis and Jeffrey R. Weeks. "Conway's ZIP Proof". In: *The American Mathematical Monthly* (May 1999). URL: https://www.jstor.org/stable/2589143 (visited on 03/15/2019).

[4] Philip Franklin. "A six color problem". In: *Studies in Applied Mathematics* (1934).

[5] Jonathan L. Gross and Thomas W. Tucker. *Topics in Topological Graph Theory.* Cambridge University Press, 2009.

[6] I.M. James. *History of Topology.* Elsevier, 1999.

[7] Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces.* Johns Hopkins University Press, 2001.

[8] Wendy Myrvold and Jennifer Woodcock. "A large set of torus obstructions and how they were discovered". In: *The Electronic Journal of Combinatorics* (Jan. 25, 2018). URL: https://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i1p16 (visited on 05/24/2019).

[9] Neil Robertson and P.D. Seymour. "Graph minors I–XX". In: *Journal of Combinatorial Theory, Series B* (1983–2004).

[10] Eric W. Weisstein. *"Cross-Cap." From MathWorld–A Wolfram Web Resource.* URL: http://mathworld.wolfram.com/Cross-Cap.html (visited on 05/23/2019).