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The Platonic solids and finite rotation groups

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The Platonic solids and finite rotation groups

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Abstract

The Platonic solids have fascinated humanity for more than 2000 years. This thesis explores polygons and polyhedra in order to find the regular polyhedra. It turns out there are five of them; the regular tetrahedron, the cube, the regular octahedron, the regular icosahedron and the regular dodecahedron. They are together called the Platonic solids, named after Plato, who wrote about them in his dialogue "Timaeus". The thesis also examines the rotation groups of the Platonic solids, as well as the other two finite subgroups of the rotation group $SO(3)$.

Acknowledgements

I would like to thank my supervisor Gregory Arone, for his help with the theory and for the many examples he gave to help me understand. I also like to thank my referee Jörgen Backelin, for his comments and suggestions on how to improve my thesis.

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1.1 The history of the Platonic solids

1.1.1 Plato, Theaetetus and Euclid

Our story of the regular polyhedra will begin around 360 BC. At that time, Plato (429-347 BCE)[3] wrote about them in his dialogue "Timaeus", and consequently they are called Platonic solids [2]. In the dialogue he linked the regular solids to the elements fire, water, earth and air. The tetrahedron was fire, since it is the sharpest of the solids, it is also the smallest and therefore the driest. The cube was earth, because it is the most stable. The octahedron was air, since it can easily be spun between two fingers and is therefore the most unstable. The icosahedron was water, because of its many sides it flows easily. At last the dodecahedron was the universe, since it is the biggest of the solids and can enclose all the others.

The story continues with Theaetetus (417-369 BCE)[3], who was active at Plato's Academy in Athens. He studied the Platonic solids mathematically and realised that there are only five regular solids. He also gave a proof of this fact, which is probably the first proof of it.

The earliest preserved mathematical script that deals with the regular solids is called Elements and is written by Euclid (lived around 300 BCE) [3]. Elements consists of thirteen books, and in the last one Euclid constructs the regular solids and shows how they can be enclosed in a sphere. He also compares the edges of the solids to the radius of the sphere. The proof by Theaetetus, that there are only five regular solids, is included in the book.

1.1.2 Kepler and Euler

Throughout the history many scientists have investigated the Platonic solids. One that took a special interest in them was Kepler (1571-1630)[3]. He thought that it must be special that there are only five of them and tried to connect that fact to the rest of the world.

In that time only six planets in our solar system were known. These six planets had to be linked to the regular solids according to Kepler. First, he tried to correlate the orbits of the planets to polygons, but soon realized that it did not work. Then he made spheres of the orbits and placed the regular polyhedra inside the spheres. The first sphere was the orbit of Saturn, inside that sphere he inscribed a cube. Inside the cube, he inscribed another sphere that represented the orbit of Jupiter. He then continued with a regular tetrahedron, a sphere, a regular dodecahedron, a sphere, a regular icosahedron, a sphere, a regular octahedron and a sphere. In this way he argued that all the orbits of the planets were represented by the spheres separated by the regular solids. Since he knew that the orbits were not circular, he gave the spheres a thickness to fit better to the data presented by Copernicus (1473-1543). But even then, some parts of his model did not quite fit with the data. He solved this by simply stating that the data was wrong and that his model was right. He published the model in his first book "Mysterium Cosmographicum" in 1596. An illustration of the model from the book can be seen in Figure 1. Later in his life, Kepler used the same data to discover the laws of planetary motions.

The next big step for the Platonic solids where made by Euler (1707-1783)[3]. He wanted to classify the polyhedra by counting their features. He began with naming the different parts of the solids; the 0-dimensional components he called vertices, the 1-dimensional components he called edges and the 2-dimensional components he called faces. By counting the different components, he found the simple formula $V - E + F = 2$. To his own knowledge, he was the first to notice this relationship. It is surprising that it took this long for someone to notice this relationship, even though countless mathematicians have studied the Platonic solids for over 2000 years. But before Euler, the study of the polyhedra was focused on the properties that could be measured; the length, area, volume and angles. No one had explicitly referred to the edges before or tried to classify the polyhedra by the number of vertices, edges and face, and thus no one had counted them in order to compared them.

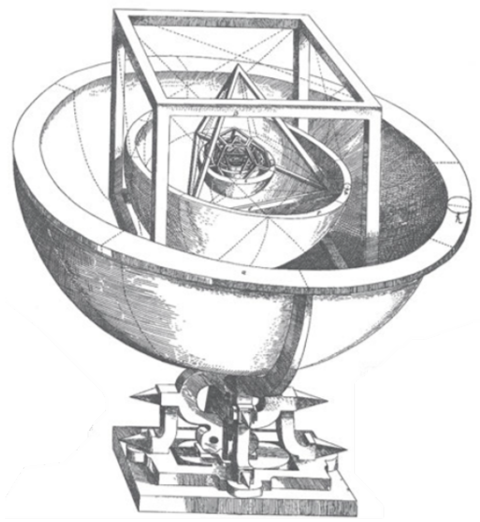


Figure 1: Kepler attempted to correlate the Platonic solids to the orbits of the six known planets (at the time). [12]

2 Polytopes

In order to understand what the Platonic solids are, this section will present some basic definitions about polygons and polyhedra. We start with some definitions about sets.

Definition 2.1. A *set* is a collection of selected objects. An object x in a set A is called an *element* of A, written $x \in A$.

Definition 2.2. Let A and B be two sets. If every element of A is also an element of B, then A is a *subset* of B, written $A \subseteq B$.

Definition 2.3. A subset S in R^n is *convex* if for every two points $\bar{x}$ and $\bar{y}$ in S

$$(1-t)\bar{x} + t\bar{y} \in S, \text{for all } 0 \leq t \leq 1.$$

Theorem 2.1. *Given any collection of convex sets, their intersection is itself a convex set.*

Proof. The intersection can either be empty, consist of a single point or consist of more than a single point. For an intersection that is empty or consists of a single point, the theorem is true by definition. For an intersection that contains more than a single point, choose two points A and B in the intersection. The line $\overline{AB}$ between the points must lie in each convex set, and thereby also in the intersection. □

Definition 2.4. The *convex hull* of a subset M of R^n is the intersection of all convex sets containing M. Since the intersection of any collection of convex sets is convex, it follows that the convex hull of M is itself convex, and it is the smallest convex set containing M.

Definition 2.5. A *hyperplane* in R^n is the set of solutions of a linear equation of the form

$$a_1x_1 + \ldots + a_nx_n = b$$

where the numbers $a_1, \ldots, a_n$ are not all zero.

Definition 2.6. Let P be the set of solutions of a finite collection of linear inequalities of the following form

$$b_{k,1}x_1 + \ldots + b_{k,n}x_n \leq b_{k,0}$$

where say $1 \leq k \leq N$. Assume that the inequalities are not redundant, and the set of solutions is non-empty and bounded. Then P is a *polytope* in R^n, and N is the number of hyperplanes that bound the polytope. If the interior of P is not empty, then we will say that P is a non-degenerate polytope. [1]

2.1 Polygons

The first step towards the Platonic solids is the polygon. A polygon is a two-dimensional polytope that has edges and vertices. Here follow some definitions of different polygons.

The following definition of a polygon can be shown to be equivalent to that of a non-degenerate polytope in R^2.

Definition 2.7. A *polygon* consists of a circuit of p line-segments $\mathbf{A_1A_2}$, $\mathbf{A_2A_3}$, ..., $\mathbf{A_pA_1}$ that are not allowed to intersect except at endpoints of adjacent segments. The line-segments are joined in consecutive pairs of p points $\mathbf{A_1}$, $\mathbf{A_2}$, ..., $\mathbf{A_p}$. The line-segments are called edges, the points are called vertices. The name of the polygon is determined by the number of edges. [1]

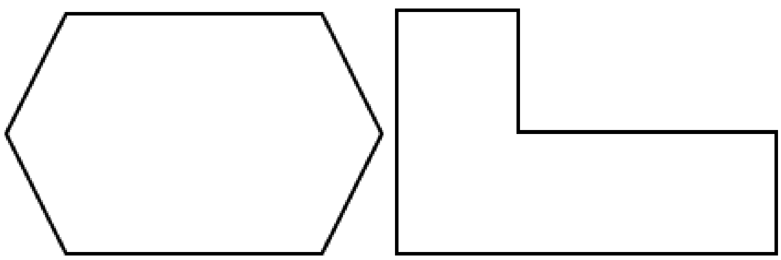
Definition 2.8. Let A be a set of points in R^3. If it exists a plane that contains all the points in A, then the points are called *coplanar*.

Definition 2.9. A polygon is called *plane* if all the vertices are coplanar. If not, the polygon is called *skew*.

A plane polygon divides its plane into two regions. The region that is finite is called the interior (or the inside) of the polygon. The other region is called the exterior (or the outside) of the polygon.

In this paper, polygons will always refer to plane polygons.

Example 2.1. Figure 2 shows two polygons. Although they look different, since they both have six edges, both are called hexagons. The left one is a convex hexagon and the right one is a non-convex hexagon.



(a) A convex polygon (b) Not a convex polygon

Figure 2: Two hexagons, one convex and one not convex.

Definition 2.10. A polygon is called *equilateral* if all its edges are the same length.

Definition 2.11. A polygon is called *equiangular* if all its interior angles are equal.

Definition 2.12. A polygon is *regular* if it is both equilateral and equiangular.

What do all these types of polygons look like? In Figure 3, a sample of different polygons is collected. All the polygons in the figure are convex, except the left equilateral polygon. A regular polygon is always convex, it is not possible for a polygon to be both equiangular and equilateral if it is not convex.

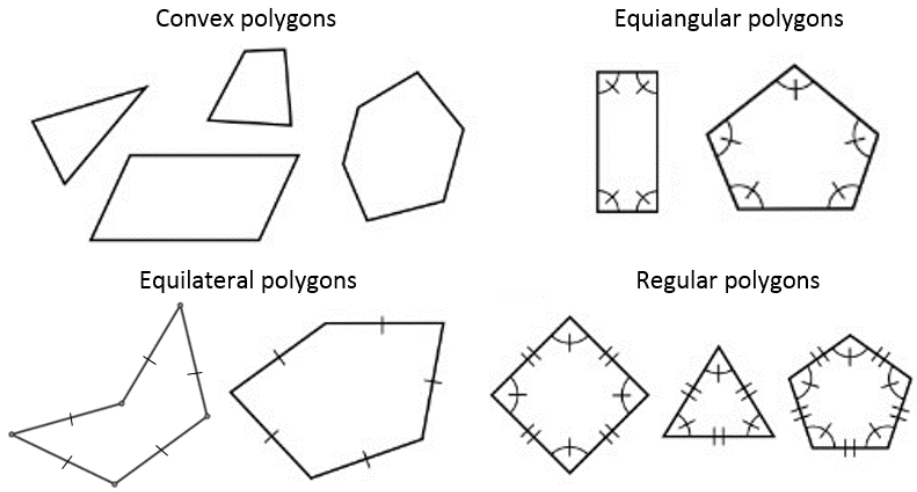


Figure 3: Different polygons. [7] [8]

2.2 Polyhedra

In three dimensions the polytopes are called polyhedra. They can have different characteristics and therefore look very different. The Platonic solids are one type of polyhedra, called regular. One of them, the cube, is shown in Figure 4.

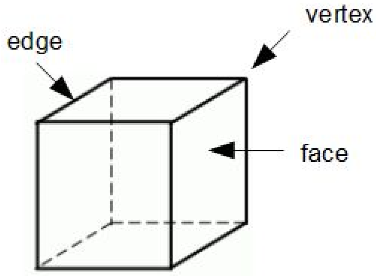


Figure 4: The faces, edges and vertices of a cube. [9]

The following definition of a polyhedron can be shown to be equivalent to that of a non-degenerate polytope in R^3 from Definition 2.6.

Definition 2.13. A *polyhedron* is a finite, connected set of plane polygons, such that every side of each polygon belongs to just one other polygon, provided that in each point of intersection between more than two polygons, the surrounding polygons form a single circuit. The polygonal surfaces of a polyhedron are called faces. The lines of intersection between two faces are called edges. The points of intersection between more than two faces are called vertices. Two faces only intersect on an edge, at a vertex or not at all. [1]

The Platonic solids are regular polyhedra. For a polyhedron to be regular, it should satisfy a few conditions [4]:

1. The polyhedron is convex.
2. Every face of the polyhedron is a regular polygon.
3. All faces are identical.
4. Every vertex is surrounded by the same number of faces.

In this paper, a definition based on flags and automorphisms will be used. We want to be able to define a regular polyhedron in a way that is related to rotations, which will be explored in Section 4.

Definition 2.14. A *flag* is a sequence of subsets $S_0, S_1, \ldots, S_k$ of a subset S of R^n, for which

$$S_0 \subset S_1 \subset \ldots \subset S_k.$$

A flag is *complete* if $\dim S_i = i$ and $k + 1 = \dim S$.

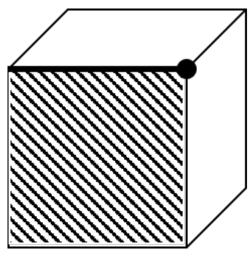


Figure 5: A complete flag of a cube (a vertex, an edge and a face).

For a polyhedron P, the subsets in a flag are represented by the vertices, edges and faces. The vertices are the 0-dimensional points P_0, the edges are 1-dimensional lines P_1 and the faces are 2-dimensional polygons P_2. A flag is the sequence of one vertex, one edge and one face that satisfy

$$P_0 \subset P_1 \subset P_2.$$

This forms a complete flag since $\dim P_i = i$ and $k+1 = \dim P$. An example of a flag is shown in Figure 5.

To be able to define an automorphisms of a polyhedron, a definition of Euclidean motions is needed.

Let $u = (x_1, \ldots, x_n)$ and $v = (y_1, \ldots, y_n)$ be vectors in R^n. Recall that that the inner product in R^n is defined by the formula $\langle u, v\rangle = x_1y_1 + \ldots + x_ny_n$. The norm of u is $|u| = \sqrt{\langle u, u\rangle}$. More generally, the inner product has a geometric interpretation $\langle u, v\rangle = |u||v| \cos(u, v)$.

Definition 2.15. In a Euclidean n-space R^n with the inner product $\langle -, -\rangle$ and the norm $|v| = \sqrt{\langle v, v\rangle}$, a *Euclidean motion* is a function $f : R^n \to R^n$ for which $|f(x) - f(y)| = |x - y|$ for any two $x, y \in R^n$. [2]

Before we move on to automorphisms and regular polyhedra, we will investigate some facts about Euclidean motions.

Lemma 2.1. Let $T : R^n \to R^n$ be a Euclidean motion that satisfies $T(\bar{0}) = \bar{0}$. Then T preserves norms, which means that $|T(\bar{u})| = |\bar{u}|$ for all vectors $\bar{u} \in R^n$.

Proof. Since T is a Euclidean motion, and $T(\bar{0}) = \bar{0}$, we can write

$$|T(\bar{u})| = |T(\bar{u}) - T(\bar{0})|.$$

Since

$$|T(\bar{u}) - T(\bar{0})| = |\bar{u} - \bar{0}| = |\bar{u}|.$$

we get that $|T(\bar{u})| = |\bar{u}|$. □

In Definition 2.15, a Euclidean motion is defined as a transformation that preserves distances. We shall now see that they also preserve inner products, which suggests that the Euclidean motions preserve angles. We begin with proving it for Euclidean motions that preserve the zero vector.

Lemma 2.2. Let $T : R^n \to R^n$ be a Euclidean motion that satisfies $T(\bar{0}) = \bar{0}$. Then T preserves inner products, which means that $\langle T(u), T(v)\rangle = \langle u, v\rangle$ for any $\bar{u}, \bar{v} \in R^n$.

Proof. Since T is a Euclidean motion, we know that

$$|T(\bar{u}) - T(\bar{v})| = |\bar{u} - \bar{v}|.$$

If we square the left side and expand it, we get

$$\begin{aligned}
|T(\bar{u}) - T(\bar{v})|^2 &= \langle T(\bar{u}) - T(\bar{v}), T(\bar{u}) - T(\bar{v})\rangle \\
&= |T(\bar{u})|^2 - 2\langle \bar{u}, \bar{v}\rangle + |T(\bar{v})|^2 \\
&= |\bar{u}|^2 - 2\langle \bar{u}, \bar{v}\rangle + |\bar{v}|^2.
\end{aligned}$$

Similar calculations give us

$$|\bar{u}-\bar{v}|^2 = |\bar{u}|^2 - 2\langle \bar{u}, \bar{v}\rangle + |\bar{v}|^2.$$

If we compare the two equations, we see that

$$\langle T(\bar{u}), T(\bar{v})\rangle = \langle \bar{u}, \bar{v}\rangle.$$

□

Lemma 2.3. Let $T : R^n \to R^n$ be a Euclidean motion that satisfies $T(\bar{0}) = \bar{0}$. Then the inner product $\langle T(u), T(v)\rangle$ preserves angles.

Proof. If we use the geometric interpretation of the inner product

$$|T(\bar{u})| \cdot |T(\bar{v})| \cdot \cos(T(\bar{u}), T(\bar{v})) = |\bar{u}| \cdot |\bar{v}| \cdot \cos(\bar{u}, \bar{v})$$

we see that

$$\cos(T(\bar{u}), T(\bar{v})) = \cos(\bar{u}, \bar{v})$$

which means that the inner product preserves angles. □

Lemma 2.4. Let $T : R^n \to R^n$ be a Euclidean motion that satisfies $T(\bar{0}) = \bar{0}$. Then T preserves angles.

Proof. Let $\bar{T}(\bar{u}) = T(\bar{u}) - T(\bar{0})$. Then $\bar{T}$ is a Euclidean motion and

$$\bar{T}(\bar{0}) = T(\bar{0}) - T(\bar{0}) = \bar{0}.$$

This means that $\bar{T}$ is a linear Euclidean motion that preserves inner products. Since $T(\bar{u}) = \bar{T}(\bar{u}) + T(\bar{0})$, and $T(\bar{0})$ is a translation, then $T(\bar{u})$ preserves angles. Which means that since a Euclidean motion preserves norms, it also preserves angles. □

Definition 2.16. An *automorphism* of a polyhedron is a Euclidean motion which transforms a convex polyhedron into itself.

Definition 2.17. If any two complete flags in a convex polyhedron P can be transformed into one another by an automorphism of P, then the polyhedron is called *regular*.

A polyhedron P that satisfies the definition of a regular polyhedron, also satisfies the conditions listed on page 10.

Every complete flag can be transformed into any other complete flag. Which means that in P, every face can be transformed into any other face, every edge can be transformed into any other edge and every vertex can be transformed into any other vertex. In order to have an automorphism of P that performs the transformations, all the faces need to be identical and all vertices need to be surrounded by the same number of faces.

An automorphism preserves distances. Which means that the distance between a vertex and the centre of a face is the same in every complete flag of P, the same apply to a vertex and a mid-edge point and to a mid-edge point and the centre of a face. This means that every polygonal face needs to be regular.

Since a regular polyhedron is convex according to the definition, it satisfies the conditions.

2.3 Schläfli symbol

In order to denote the regular polytopes, the Schläfli symbol can be used. It contains the information of the structure of the polytope. The symbol is named after the Swiss mathematician Ludwig Schläfli (1814-1895).

A regular polygon is denoted $\{p\}$, where p is the number of edges on the polygon. An equilateral triangle has the Schläfli symbol $\{3\}$ because of its three edges.

A regular polyhedron is denoted $\{p, q\}$, where p is the type of polygon its faces are made of and q is the number of edges that meet in each vertex. A cube has the Schläfli symbol $\{4, 3\}$ since the faces are squares and three edges meet at each vertex.

The symbol can also be used in higher dimensions. The d-dimensional regular polytope P has the Schläfli symbol $\{p_1, p_2, \dots, p_{d-1}\}$, where p_1 is the type of polygon and $\{p_2, \dots, p_{d-1}\}$ are its excellent vertex figures (defined in Definition 3.4).

2.4 Graphs

Instead of seeing a polyhedron as a three-dimensional body, we can forget that the faces should be polygons and convert the polyhedron into a graph, which often is called its skeleton, formed by its vertices and edges. With graphs, a specific formula called Euler's formula (Theorem 2.2) can be proven.

Definition 2.18. A *graph* $G = (V, E)$ consists of two finite sets V and E, such that the elements in E are unordered pairs of elements from V. The elements in V are called *vertices* and the elements in E are called *edges*. Two vertices u, v are connected by an edge if $\{u, v\}$ is an element in E.

If an element $\{u, v\}$ only is allowed once in E, then the graph is called a *simple graph*. If an element $\{u, v\}$ are allowed several times in E and elements of the type $\{v, v\}$ are allowed in E, then the graph is called a *multigraph*. In a multigraph, there can be several edges between u and v, and edges that go from v to v, called *loops*.

Unless stated otherwise, graphs will refer to simple graphs.

Definition 2.19. A *walk* is a sequence of vertices $v_1, v_2, \ldots, v_n$ in a graph, such that v_i and v_{i+1} is connected by an edge for $1 \leq i \leq n-1$. [5]

Definition 2.20. Let v be a vertex in a graph G. If there exists a walk from v to all the other vertices in G, the graph is called *connected*.

Definition 2.21. If a graph G can be drawn on a plane with no edges crossing each other, G is called a *planar graph*.

In planar graphs, the connected components of the complement of the graph are called *faces*.

Definition 2.22. Let a planar graph G have V vertices, E edges and F faces. Create a new planar graph G' that has a vertex for every face in G. For every edge in G, let there be an edge between the two vertices in G' that correspond to the two faces in G separated by the edge. For every vertex in G, there now exists a corresponding face in G'. The graph G' is called the *dual graph* of G with F vertices, E edges and V faces.

An example of how a dual graph of a planar graph can be created is demonstrated in Figure 6.

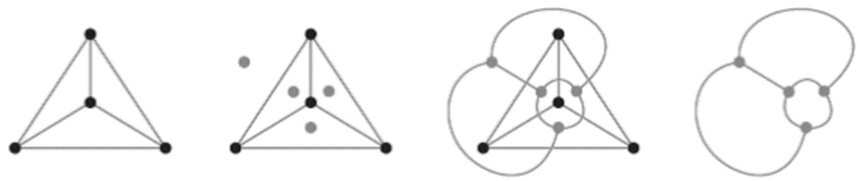


Figure 6: The steps for creating a dual graph. [10]

2.5 Dual polyhedra

Every convex polyhedron has a special relationship to another convex polyhedron, called the dual polyhedron. For a convex polyhedron with F faces, E edges and V vertices, the dual polyhedron has V faces, E edges and F vertices.

Definition 2.23. The *dual polyhedron* of a convex polyhedron is the convex hull of the centres of its faces. [2]

To find a dual polyhedron, begin with marking a point in the centre of every face of a convex polyhedron. If the polyhedron has F faces, these points will be the F vertices of the dual.

Then draw lines between the points that are on adjacent faces of the polyhedron. These lines form polygons that create the faces of the dual. This makes one polygon for every vertex in the polyhedron, and if the polyhedron has V vertices, the dual has V faces.

The dual now has V faces and F vertices, but does it have E edges? It does, because for every edge between two vertices in the polyhedron, there is an edge separating two faces in the dual and since the polyhedron has V vertices and the dual has V faces, the edges remain the same.

We have now formed a dual polyhedron with V faces, E edges and F vertices. This result is the same dual as taking the convex hull of the centres of the polygons faces, as in Definition 2.23, because the centres of the faces are chosen as the vertices, and the lines drawn, together with the faces that were created, is the same as the convex hull.

An example of a dual polyhedron can be seen in Figure 7. The dual of a dual polyhedron is the original polyhedron, only smaller.

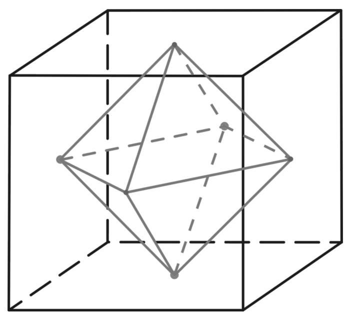


Figure 7: The octahedron is the dual of the cube. [11]

Definition 2.22 defined the dual of a planar graph. This can also be used to find the dual of a polyhedron P.

First, convert P into a planar graph G with the same number of faces, edges and vertices as P. The dual of G is G', which describes the dual polyhedron P' as well, since P' will have the same number of faces, edges and vertices as G'. The dual polyhedron P' can be created from the dual graph G' by connecting the vertices in P' according to how the vertices in G' are connected by the edges. This will create the same number of faces in P' as in G'. All that is left is to adjust the edges in order to make the faces regular polygons and the dual polyhedron is found.

2.6 Euler's formula

The 14:th of November in 1750 was a special day for the polyhedra. On this day Euler sent a letter to his friend Christian Goldbach. The letter contained the definition of an edge of a polyhedron and a relationship between the number of vertices, edges and faces in a convex polyhedron. The relationship is known as Euler's formula.

Theorem 2.2. *A convex polyhedron with V vertices, E edges and F faces satisfies*

$$V - E + F = 2. \tag{1}$$

Proof. To prove Euler's formula, begin with projecting a convex polyhedron into a plane and make it a planar connected multigraph. We can imagine that we remove one of the faces of the polyhedron and then unfold it and place it on a plane. This creates a planar graph. The edges and vertices of the polyhedron are represented by edges and vertices in the graph. The faces of the polyhedron are represented by the area enclosed by the edges and vertices, the face that we imagine that we removed is represented by the area outside the graph. Since the graph is a planar graph, no edges are crossing each other, and every face is enclosed by at least three vertices and three edges in the beginning (since every face on a polyhedron is enclosed by at least three vertices and three edges).

Now we want to remove edges and vertices until there is only one vertex and no edges left. Begin with choosing an edge. If it is enclosed between two vertices, shrink the edge until it disappears, and the two vertices join and become one vertex. Now the number of vertices has decreased by one, and the number of edges has decreased by one. The number of faces is conserved. The expression $V - E + F$ remains the same.

If the edge chosen is a loop, remove it (a loop can be created when we eliminate an edge $\{u, v\}$ and a vertex v if there are two edges between v and u, which is allowed in a multigraph). Then the number of edges decreases with one and the number of faces decreases with one. The number of vertices is conserved. Even now the expression $V - E + F$ remains the same.

Continue to shrink and remove edges until no edges remain. All that do remain are one vertex and one face (the outside), see Figure 8. Now

$$V - E + F = 1 - 0 + 1 = 2.$$

Since the expression $V - E + F$ have not changed while removing and shrinking edges, the equation

$$V - E + F = 2$$

is true for the graph we began with, and therefore true for the polyhedron. $\square$

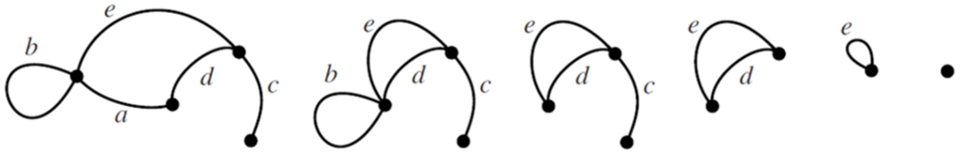


Figure 8: By removing edges, the multigraph can be reduced. [12]

3 The Platonic solids

Finally, we have reached the Platonic solids after studying the theory behind them. In this section we will learn more about the regular polyhedra and investigate why there are only five of them. We will take a closer look on each of the solids to see what makes them so special.

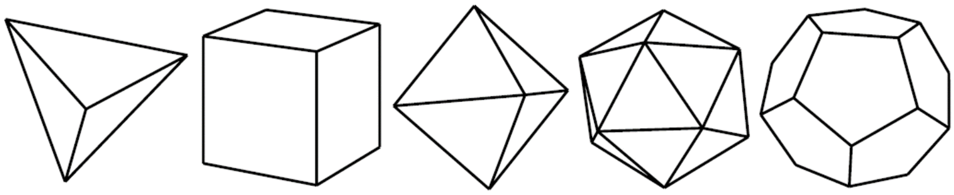


Figure 9: The five Platonic solids. [12]

Definition 3.1. The *Platonic solids* are:

- The regular tetrahedron
- The regular hexahedron (cube)
- The regular octahedron
- The regular icosahedron
- The regular dodecahedron

Polyhedron	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Icosahedron	12	30	20
Dodecahedron	20	30	12

Table 1: The number of vertices, edges and faces of the Platonic solids.

3.1 Tetrahedron

The tetrahedron is made of four triangles. It has four vertices, six edges and four faces, as can be seen in Table 1. The regular tetrahedron is made of four equilateral triangles. It is the first polyhedron from the left in Figure 9, and has the Schläfli symbol $\{3, 3\}$.

To form a regular tetrahedron, begin with an equilateral triangle as a base. Through the centre of the triangle, draw a line perpendicular to the base. Choose a point on the line. From that point, draw lines to the three vertices of the triangle. Each such line is the same length, which means that three isosceles triangles are formed. Adjust the distance to the point on the line to make the isosceles triangle into equilateral triangles. These four equilateral triangles form the regular tetrahedron.

Example 3.1. The tetrahedron can also be formed by the convex hull of four vertices in a cube. For it to be a regular tetrahedron, all the edges of the convex hull must be the same length. For a cube with vertices $(\pm 1, \pm 1, \pm 1)$, a regular tetrahedron is formed by the vertices

$$(1, -1, 1), (1, 1, -1), (-1, -1, -1), (-1, 1, 1).$$

The dual of a tetrahedron is another tetrahedron. It is self-dual. A convex hull of the centres of its faces creates triangles, and because the solid has both four vertices and four faces, the result is a tetrahedron upside down as seen in Figure 10.

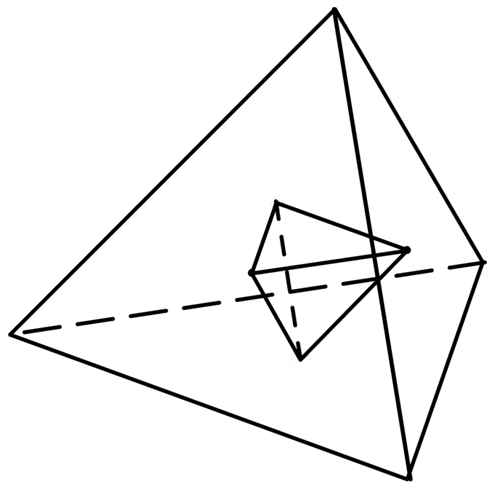


Figure 10: The dual of a tetrahedron is another tetrahedron. [13]

3.2 Cube

The regular hexahedron is called a cube. It has eight vertices, twelve edges and six faces. It is the second polyhedron from the left in Figure 9. The cube has the Schläfli symbol $\{4, 3\}$.

In order to form a cube, begin with a square as a base. Draw lines perpendicular to the base through each of the four vertices of the square.

On a chosen side of the square, make one point on each of the lines at the same distance from the square. The convex hull of those four points creates a square, fill in those new lines. Adjust the distance to the points from the base to make the rectangles into squares. These six squares form the cube.

3.3 Octahedron

The third polyhedron from the left in Figure 9 is called a regular octahedron. It consists of eight equilateral triangles and has six vertices, twelve edges and eight faces. The octahedron has the Schläfli symbol $\{3, 4\}$. The cube and octahedron are duals, as can be seen in Figure 11.

In order to form a regular octahedron, make two pyramids with squares as bottoms and triangles as sides. Adjust the edges of the bottom and the height of the pyramids to make the triangles equilateral. Then place the pyramids base to base and remove the two squares. We have now constructed an octahedron of eight equilateral triangles.

Example 3.2. Since the octahedron is the dual of the cube, the octahedron can also be created by the convex hull of the centres of the faces of the cube. If the cube has the vertices $(\pm1, \pm1, \pm1)$, then the vertices of the octahedron is formed by the points $(\pm1, 0, 0)$, $(0, \pm1, 0)$ and $(0, 0, \pm1)$.

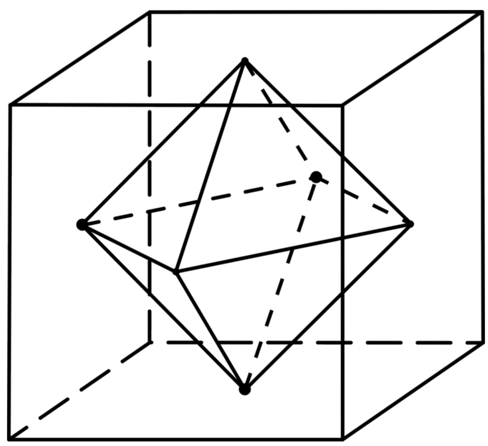


Figure 11: The dual of the cube is the octahedron. [14]

3.4 Icosahedron

The icosahedron is the Platonic solid that has the largest number of faces. Its twenty faces are made of equilateral triangles. The solid has twelve vertices and thirty edges. The icosahedron is the second polyhedron from the right in Figure 9.

First, a description of how a regular icosahedron can be constructed will be given, but since this description is quite informal, a formal proof will follow.

To construct a regular icosahedron, begin with a regular octahedron. Colour the faces alternately black and white. Create a direction on each edge so that the face on the left side of the edge is black and the face on the right side is white when following the edge in the chosen direction, see Figure 12a. Choose a point on each line in any given ratio a:b in the chosen direction. Since the octahedron has twelve edges, we get twelve points. The convex hull of these points forms the icosahedron, but it will not automatically be regular.

In order to make the icosahedron regular, the ratio $a\colon b$ need to be specified. Begin with noticing that eight of the triangles on the icosahedron lie on the faces of the octahedron. If the edges on the regular octahedron is $a+b$, the eight triangles are equilateral with edges $\sqrt{a^2+b^2-ab}$. The rest of the triangles on the icosahedron are isosceles with two edges that are $\sqrt{a^2+b^2-ab}$ and one that is $\sqrt{2}a$.

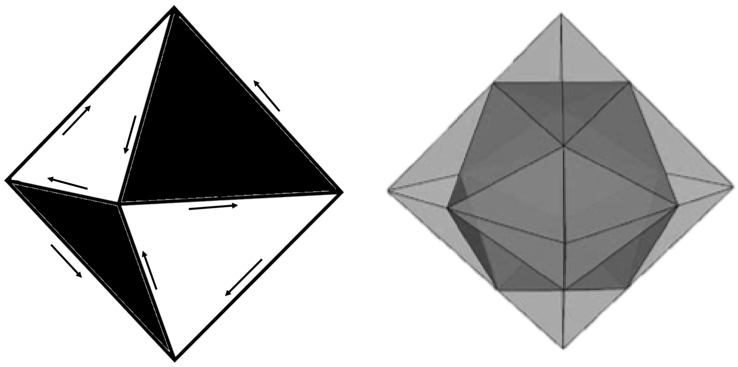
We want to choose the ratio $a\colon b$ to make all the triangles equilateral, so

$$\sqrt{2}a = \sqrt{a^2+b^2-ab}$$

which give

$$\frac{a}{b} = \frac{\sqrt{5}-1}{2} = \frac{1}{\tau}.$$

To conclude, we divide the twelve edges on a regular octahedron according to the golden ratio τ to find the vertices of a regular icosahedron, see Figure 12b.



(a) [15] (b) [16]

Figure 12: By colouring the octahedron alternately black and white like in (a), the icosahedron can be found by dividing the edges of the octahedron according to the golden ratio, like in (b).

Example 3.3. For a regular octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$, the length of the edges will be $\sqrt{2}$. If $a + b = 1$, the vertices on the regular icosahedron will be

$$(0, \pm a, \pm b), (\pm b, 0, \pm a), (\pm a, \pm b, 0)$$

given that $a = 1 - \tau$ and $b = \tau$. The length of the edges on the regular icosahedron is then $2(1 - \tau)$.

Our next task is to prove that the icosahedron is a regular polyhedron. For this we need to use vertex figures and a regularity criterion.

A vertex figure of a polyhedron can be formed by choosing a vertex and then mark one point on each of the edges connected to the vertex. Draw straight lines between those points to form a polygon. This polygon is called a vertex figure of the polyhedron. An example of how it can look like is given in Figure 13. A more explicit definition uses a closed half space.

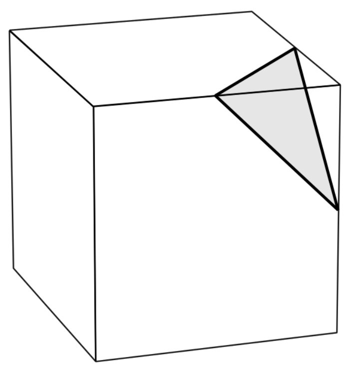


Figure 13: A vertex figure of a cube. [17]

Definition 3.2. Let $h : R^n \to R$ be a linear function. Let γ be a real number. Then H is a *closed half space* of R^n if

$$H = \{x \in R^n : h(x) \leq \gamma\}.$$

The *boundary* of H is

$$\partial H = \{x \in H : h(x) = \gamma\}.$$

Definition 3.3. Let v be a vertex of a polyhedron P. A *vertex figure* Q of P is the intersection between P and the boundary of a closed half space H that contain all vertices of P except for v:

$$Q = P \cap \partial H.$$

A special kind of vertex figures is the one that goes through all the vertices that is connected to the vertex v by an edge. This figure is called an excellent vertex figure and is shown in Figure 14. The excellent vertex figure of a regular polyhedron is a regular polygon.

Definition 3.4. Let v be a vertex in a regular polyhedron P. Let a closed half space H contain all vertices of P except for v, and let the boundary of the closed half space ∂H go through the vertices that is connected to v by one edge. The intersection between P and ∂H is called an *excellent vertex figure.*

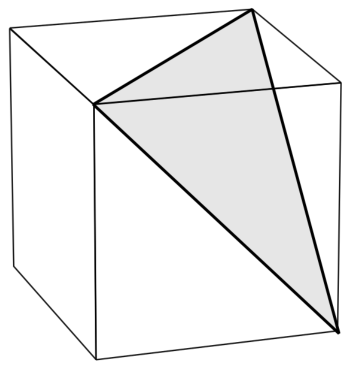


Figure 14: An excellent vertex figure of a cube. [18]

Lemma 3.1. *Regularity criterion.* A convex polyhedron P is *regular* if it has a vertex v with the following properties: [2]

(i) For every vertex w of P ther is an automorphism of P that transforms v to w.

(ii) There is a regular vertex figure Q of P at v.

(iii) For every automorphism γ of Q there is an automorphism g of P with $g(v) = v$ and $g|Q = \gamma$.

We want to prove that if a polyhedron satisfies the regularity criterion, then it is regular according to Definition 2.17.

Proof. Let P be a polyhedron that satisfies the criterion of Lemma 3.1.

First, let (v, e_1, f_1) and (v, e_2, f_2) be two complete flags of P containing the vertex v. We want to show that there exists an automorphism g of P such that

$$g(v) = v$$

$$g(e_1) = e_2$$
$$g(f_1) = f_2.$$

For $i = 1, 2$ let
$$x_i = Q \cap e_i$$
$$w_i = Q \cap f_i$$

Then x_i is a vertex of Q and w_i is an edge of Q. Which means that (x_1, w_1) and (x_2, w_2) are complete flags of Q.

According to ii), there exists an automorphism γ of Q such that
$$\gamma(x_1) = x_2$$
$$\gamma(w_1) = w_2.$$

According to iii), there exists an automorphism g of P such that
$$g(v) = v$$
$$g(x_1) = x_2$$
$$g(w_1) = w_2.$$

Since $x_i \in e_i$ and $w_i \in f_i$ then
$$g(e_1) = e_2$$
$$g(f_1) = f_2.$$

This proves that any two complete flags, containing the vertex v, can be transformed into each other by an automorphism of P.

Now let (v, e_1, f_1) and (w, e_2, f_2) be any two complete flags of P. According to i), there is an automorphism h of P such that
$$h(v) = w.$$

We can write
$$h^{-1}(w, e_2, f_2) = (v, e', f').$$

By the first part of the proof we know that there is an automorphism g of P such that $g(v) = v$, $g(e_1) = e'$ and $g(f_1) = f'$. Then
$$hg(v) = w$$
$$hg(e_1) = e_2$$
$$hg(f_1) = f_2.$$

This proves that any two complete flags of P can be transformed into each other by an automorphism of P. Therefore, a polyhedron that satisfies the regularity criterion also satisfies the definition of a regular polyhedron.

□

Theorem 3.1. *The icosahedron is the regular solid* $\{3, 5\}$.

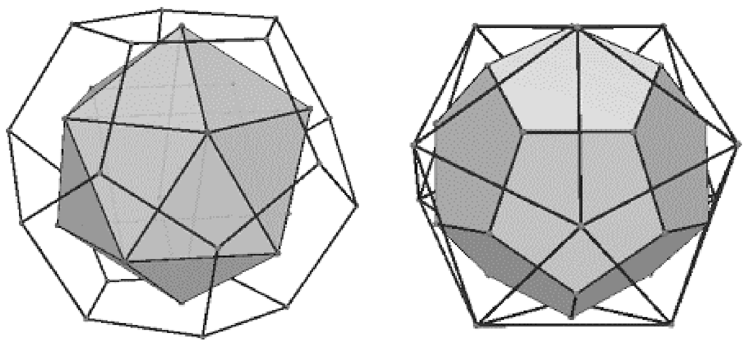
Proof. We start with our black and white coloured octahedron. There exists an automorphism of the octahedron that does not change the colouring. Every such automorphism is also an automorphism of the icosahedron that is created by dividing the edges of the octahedron according to the golden ratio. The automorphisms act transitively on the edges of the octahedron, which means that they also act transitively on the vertices of the icosahedron.

The icosahedron can also be seen as an antiprism and two pentagonal pyramids, with the bases of the pyramids placed on the top and bottom of the antiprism. We can choose a vertex v and see it as the top of one of the pyramids. Then the pentagonal base of that pyramid is an excellent vertex figure of the icosahedron at v. The automorphism of the excellent vertex figure is an automorphism of the icosahedron with v as a fixed point.

These descriptions of the automorphisms of the icosahedron result in that the icosahedron satisfy the criteria for a regular polyhedron written in Lemma 3.1. Therefore the icosahedron is the regular polyhedron $\{3, 5\}$. $\square$

3.5 Dodecahedron

The last of the regular polyhedra is the dodecahedron. It has twenty vertices, thirty edges and twelve pentagonal faces. It is the dual of the icosahedron, see Figure 15, and can be found by forming the convex hull of the centres of the faces of the icosahedron.



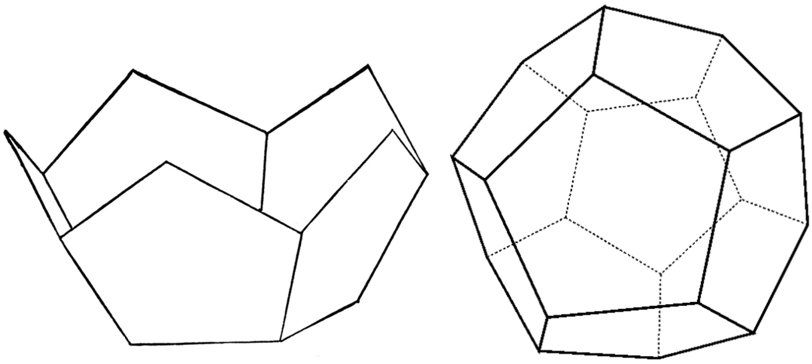
(a) An icosahedron inside a dodecahedron. [19]

(b) A dodecahedron inside an icosahedron. [20]

Figure 15: The icosahedron and the dodecahedron are duals.

Since we proved that the icosahedron exists, the dodecahedron also exists. It is the first polyhedron from the right in Figure 9. It has the Schläfli symbol $\{5, 3\}$.

To understand the construction of the dodecahedron, take a regular pentagon and surround it by five regular pentagons, one on each edge of the first pentagon, and fold it to something that look like a bowl. Two such bowls can be put together to form the dodecahedron, see Figure 16.



(a) Six pentagons create a bowl [21] (b) A regular dodecahedron [22]

Figure 16: The dodecahedron can be seen as two bowls that are put together.

Example 3.4. For a dodecahedron centred at the origin, with edge length $\frac{2}{\tau} = \sqrt{5} - 1$, the vertices are located at the following coordinates:

$$(\pm 1, \pm 1, \pm 1), (0, \pm\tau, \pm 1/\tau), (\pm 1/\tau, 0, \pm\tau), (\pm\tau, \pm 1/\tau, 0)$$

where $\tau = \frac{\sqrt{5}+1}{2}$.

3.6 The vertices of regular polyhedra

Now we have gone through the five Platonic solids, but how do we know that there are not more of them? A rigorous proof will be given, but first an informal argument. Can it be seen from how the vertices are constructed that there are no more than five Platonic solids?

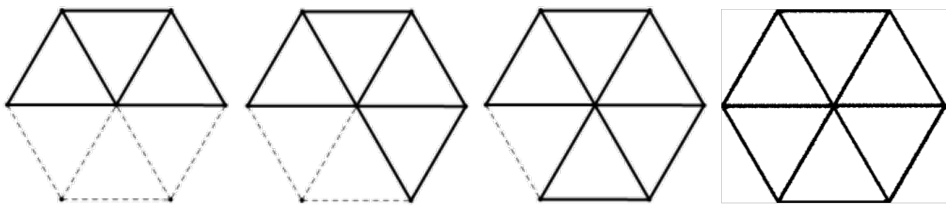


Figure 17: A vertex can be created with three, four or five equilateral triangles, but not with six, since six equilateral triangles create a flat construction. [23][24]

Each vertex in a regular polyhedron is made by at least three polygonal faces whose vertices meet in one point.

We will start with equilateral triangles. Create a vertex of three triangles, this is the vertex of a regular tetrahedron. If another triangle is added, the vertex of a regular octahedron is found. With five triangles, we get the vertex of a regular icosahedron. What will happen if a sixth triangle added? Since the inner angle of an equilateral triangle is $60°$, the total angle of a vertex with six triangles is $360°$. This means that a vertex can not be created of six equilateral triangles since it is a flat construction, see Figure 17. With more than six triangles, the vertex will not be convex and therefore a polyhedron made with these vertices will not be convex, which is a criterion for the regular solids. Only three different regular solids can be constructed with triangles.

The next regular polygon is the square. Three squares create the vertex of a cube. Four squares create a flat construction, and thereby a vertex can not be constructed of four squares. With more than four squares the vertex will not be convex, so only one regular solid can be constructed with squares.

Next we will use regular pentagons. Three pentagons construct the vertex of a regular dodecahedron. The inner angle of a regular pentagon is $108°$. Three pentagons can create a vertex since they together are $324°$, but a vertex of four or more pentagons will together be more than $360°$ and therefore can not construct a convex vertex. Only a regular dodecahedron can be constructed with regular pentagons.

What about the polyhedra made of regular polygons with more edges than five? Three regular hexagons, with inner angle $120°$, will together make a flat construction. More than three will not create a convex vertex. For regular polygons with seven edges or more, not even three of them can create a convex vertex since the total angle of these vertices are greater than $360°$. This means that only five different vertices, that could be a part of a regular solid, can be created. They are shown is Figure 18.

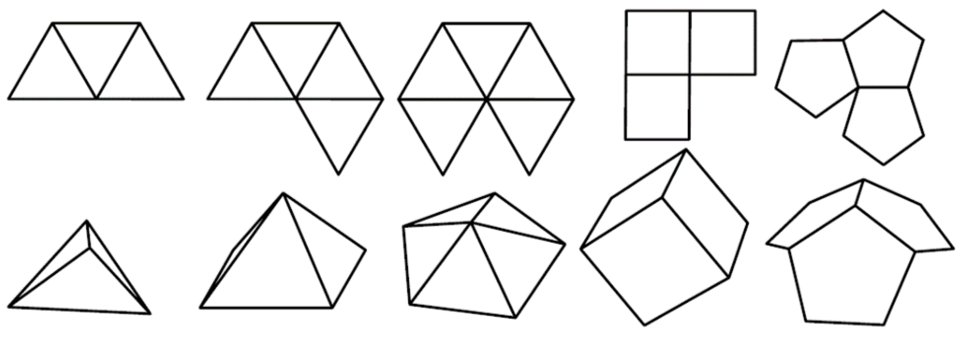


Figure 18: Five possible vertices of the Platonic solids, laid flat at the top and folded into a vertex at the bottom. [12]

3.7 Five regular polyhedra

After examining the vertices, it is now time to formally prove that there are no more than five Platonic solids.

Theorem 3.2. *There are at most five regular polyhedra.*

Proof. Assume that we have a regular polyhedron. Define n and m as

n= the number of edges on each face

m= the number of edges that meet at each vertex

The polyhedron has F faces, and on every face there are n edges. By counting all edges on every face, all the edges have been counted twice, since every edge belongs to two faces. This gives

$$E = \frac{F \cdot n}{2}. \tag{2}$$

Every face has n vertices (the same as the number of edges on each face). When counting all the vertices on all the faces, the vertices are counted m times too much since m faces meet at every vertex (the number of faces that meet at each vertex is the same as the number of edges that meet at each vertex). This gives

$$V = \frac{F \cdot n}{m}. \tag{3}$$

Euler's formula states that

$$V - E + F = 2.$$

Insert Equation 2 and 3 in Euler's formula:

$$\frac{F \cdot n}{m} - \frac{F \cdot n}{2} + F = 2.$$

Solving for F gives:

$$F = \frac{4m}{2n - mn + 2m}.$$

Because m and the number of faces are positive

$$2n - mn + 2m > 0. \tag{4}$$

In order to create a polyhedron, we need at least three edges on each face and at least three edges must meet at each vertex:

$$m, n \geq 3. \tag{5}$$

The inequalities in Equation 4 and 5 gives us

$$2(n + m) > mn \tag{6}$$

$$
\begin{aligned}
2n &> m(n-2) \\
\frac{2n}{n-2} &> m \geq 3 \\
2n &> 3n-6 \\
n &< 6.
\end{aligned}
$$

Because of the symmetry in Equation 6 we also get that

$$m < 6.$$

In total we have:

$$
\begin{cases}
3 \leq n < 6 \\
3 \leq m < 6 \\
2n - mn + 2m > 0.
\end{cases}
\tag{7}
$$

The only solutions the system in Equation 7 are

$$(n, m) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$$

which represent the five Platonic solids. □

We have now proved that the five Platonic solids are:

- The regular tetrahedron $\{3, 3\}$
- The regular hexahedron (cube) $\{4, 3\}$
- The regular octahedron $\{3, 4\}$
- The regular icosahedron $\{3, 5\}$
- The regular dodecahedron $\{5, 3\}$

4 Rotation groups

In this section we will investigate the rotation groups of the Platonic solids. The question to answer is: *"In how many ways can a regular polyhedron be rotated and still look the same?"*. The answer is different depending on which regular polyhedron that is considered. Dual polyhedra have the same rotation group, so the cube and octahedron will share one group. The same applies to the icosahedron and dodecahedron. The tetrahedron will have its own group.

We will also take a quick look at the other finite subgroups of the rotation group $SO(3)$.

First, some definitions about groups.

Definition 4.1. A *group* consists of a set G, together with binary operation $*$ defined on G which satisfies the following axioms: [5]

1. *(Closure)* For all x and y in G:
$$x * y \in G.$$
2. *(Associativity)* For all x, y and z in G:
$$(x * y) * z = x * (y * z).$$
3. *(Identity)* There is an element e, called the identity, in G such that
$$e * x = x * e = x$$
for all x in G.
4. *(Inverse)* For all x in G, there is an element x', called the inverse, in G such that
$$x * x' = x' * x = e.$$

Definition 4.2. Let G be a group, and let $|G|$ be finite. Then $|G|$ is called the *order* of G.

Definition 4.3. Let G be a group and let H be a non-empty subset of G. Then H is called a *subgroup* of G if for every $x, y \in H$:

$$xy \in H$$

$$x^{-1} \in H.$$

If G is finite, then $xy \in H$ is enough for H to be a subgroup of G.

Definition 4.4. Let G be a group. If all the elements in G are a power of an element $x \in G$, then G is called a *cyclic* group. The element x is called the *generator* of G.

4.1 Rotations of the tetrahedron

The rotation group of the regular tetrahedron has the order twelve, which can be seen in Table 2. It is called the *tetrahedral group*.

4 axes through vertices and faces	2 elements each	8
3 axes through mid-edge points	1 element each	3
The identity		1
In total:		12 elements

Table 2: The order of the tetrahedral group is twelve.

A rotation has an axis and an angle. For a regular polyhedron with Schläfli symbol $\{p, q\}$, there are three different kinds of rotations:

(i) q-fold: The axis passes through a vertex. The angle is a multiple of $2\pi/q$.

(ii) twofold: The axis passes through the midpoint of an edge. The angle is π.

(iii) p-fold: The axis passes through the centre of a face. The angle is a multiple of $2\pi/p$.

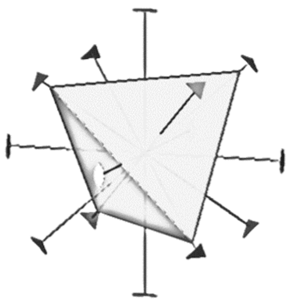


Figure 19: The rotation axes through a tetrahedron. [27]

For the regular tetrahedron, the q-fold and the p-fold rotations will be the same, since an axis through a vertex will go through the face on the opposite side of the tetrahedron. The axis can be rotated by an angle of $2\pi/3$ two times, and since the tetrahedron has four vertices, the four axes contribute with eight rotations.

The twofold rotations will have an axis that goes through two edges, because the edges in the tetrahedron occur in antipodal pairs. The tetrahedron's six edges will contribute with three axes and therefore three rotations.

The last element of the tetrahedral group is the identity. All the seven axes of rotation can be seen in Figure 19.

4.2 Rotations of the cube and octahedron

In a regular octahedron, the vertices, edges and faces occur in antipodal pairs. Which means that axes through half of the vertices, edges and faces cover all the rotations. In an octahedron with F faces, E edges and V vertices, there are $V/2$ q-fold axes through vertices, $E/2$ twofold axes through edges and $F/2$ q-fold axes through centre of faces. All the axes can be seen in Figure 20.

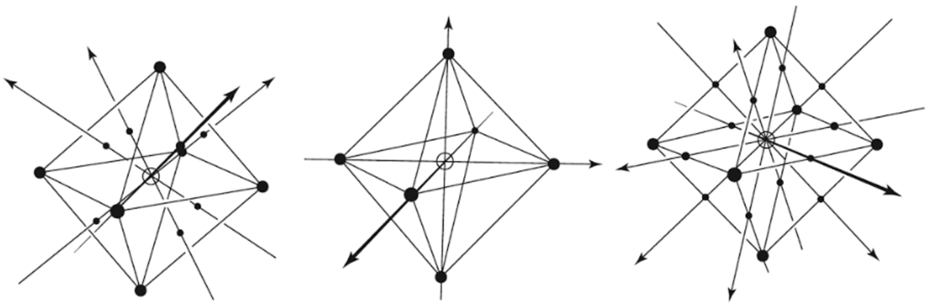


Figure 20: The rotation axes through an octahedron. [28]

The axis for a q-fold rotation can be rotated three times by an angle of $2\pi/4$. The three antipodal pairs of vertices contribute to nine rotations.

The octahedron has six antipodal pairs of edges. An axis through the edges can be rotated once by an angle of π. This contributes with six rotations.

The last axis is through the faces. The four antipodal pairs of faces create four axes that can be rotated by an angle of $2\pi/3$ two times and together contribute to eight rotations.

At last is the identity. In total the *octahedral group* has 24 elements, which can be seen in Table 3.

3 axes through vertices	3 elements each	9
6 axes through mid-edge points	1 element each	6
4 axes through face centres	2 element each	8
The identity		1
In total:		24 elements

Table 3: The order of the octahedral group is 24.

The octahedron and the cube are duals. This means that they have the same rotation group. The rotation around an axis through a vertex in the octahedron corresponds to the rotation around an axis through a centre of a face in the cube. The axis through a mid-edge point in the octahedron corresponds to an axis through a mid-edge point on the cube. The axis through the centre of a face in the octahedron corresponds to an axis through a vertex in the cube. An example with three axes can be seen in Figure 21.

This means that since the octahedron has nine q-fold rotations, six twofold rotations and eight p-fold rotations, the cube has eight q-fold rotations, six twofold rotations and nine p-fold rotations.

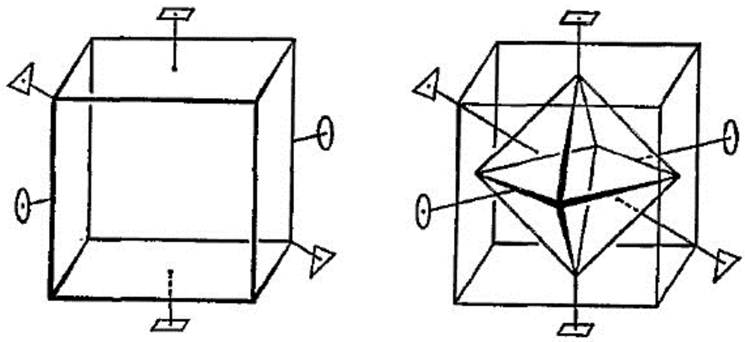


Figure 21: The axes of rotation in the cube and octahedron correspond to each other. [29]

4.3 Rotations of the icosahedron and dodecahedron

The group of rotations of the icosahedron and the dodecahedron is called the *icosahedral group*. The elements of the group can be seen in Table 4.

6 axes through vertices	4 elements each	24
15 axes through mid-edge points	1 element each	15
10 axes through face centres	2 element each	20
The identity		1
In total:		60 elements

Table 4: The order of the icosahedral group is 60.

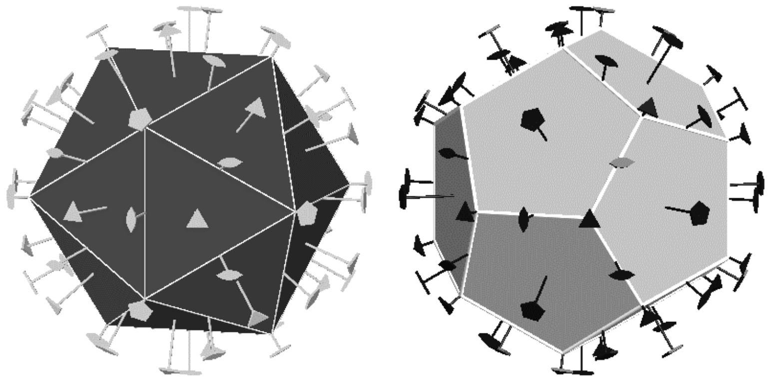
In a regular icosahedron the vertices, mid-edge points and the centres of faces occur in antipodal pairs.

An axis through two antipodal vertices will result in q-fold rotations. The axis can be rotated four times by an angle of $2\pi/5$. Since the icosahedron has twelve vertices, the axes will contribute with 24 rotations.

The twofold rotations are created by rotation around fifteen axes through antipodal edges. These will contribute with 15 rotations.

The axis through an antipodal pair of centres of faces can be rotated two times with an angle of $2\pi/3$. These ten axes will contribute with 20 p-fold rotations.

At last we also have the identity for the icosahedron. These 60 rotations create the icosahedral group. The axes of rotation can be seen in Figure 22.



(a) The rotation axes through an icosahedron. [25]

(b) The rotation axes through a dodecahedron. [26]

Figure 22: The axes of rotation through an icosahedron and a dodecahedron are the same.

4.4 Permutation groups

The tetrahedral, octahedral and icosahedral groups can be identified with permutation groups.

Definition 4.5. Let X be a non-empty finite set. A *permutation* of X is a bijection from X to X.

Definition 4.6. A permutation that exchanges the place of two elements in a non-empty finite set, and leaves the rest unaltered, is called a *transposition.*

Definition 4.7. Let X be a non-empty finite set. Every permutation σ of X can be written as a product of transpositions. If the number of transpositions is even, then σ is an *even* permutation. If the number of transpositions is odd, then σ is an *odd* permutation.

Definition 4.8. Let X be a non-empty finite set. Let G be a set of permutations of X. If G is a group, then G is called a *permutation group* of X.

Definition 4.9. Let $X_n = \{1, 2, \dots, n\}$ be a finite set with n elements. Let S_n be a permutation group that contains all the permutations of X_n. Then S_n is called a *symmetric group*. The order of S_n is $n!$.

Definition 4.10. Let S_n be a symmetric group. A subgroup A_n, consisting of all even permutations in S_n, is called an *alternating group*. The order of A_n is $\frac{n!}{2}$.

Definition 4.11. Let G be a set of all automorphism of a polyhedron P. Then G is called an *automorphism group* of P.

4.4.1 The tetrahedral group

The tetrahedral group can be identified as the alternating group A_4. The symmetric group S_4, that consists of all permutations of four vertices, is the same as the full automorphism group of the tetrahedron. The even permutation are rotations, and these create A_4.

4.4.2 The octahedral group

The octahedral group can be identified as the symmetric group S_4. The octahedron has four pairs of parallel faces, which the elements of the octahedral group permute. No element that is not equal to the identity gives the identity. All the 24 possible permutations occur, and these create S_4.

4.4.3 The icosahedral group

The icosahedral group can be identified as the alternating group A_5. To show this, divide the edges of the icosahedron into five classes with six edges in each. Let two edges belong to the same class if and only if they are perpendicular or parallel to each other. In this way, the five edges connected to the same vertex will belong to five different classes, see Figure 23, as well as the five edges that belong to two faces that share an edge.

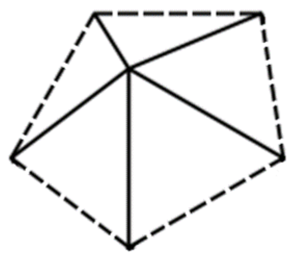


Figure 23: The five edges connected to the same vertex in the icosahedron will belong to five different classes. [30]

How are the five classes permuted by the icosahedral group? The q-fold rotations will give cyclic permutations of order five, since there are five edges connected to one vertex. The twofold rotations will give two transpositions of the five classes, since there are five edges of two faces that share an edge. The p-fold rotations will give cyclic permutations of three classes. The two classes that do not contain an edge that belong to the face that axis goes through, can not be permuted by these rotations.

This shows that the icosahedral group is a subgroup of S_5. Since only the 60 even permutations occur, the icosahedral group is the alternating group A_5.

4.5 Finite subgroups of the rotation group SO(3)

The tetrahedral, octahedral and icosahedral groups are subgroups of the rotation group $SO(3)$, which contains all the rotations around the origin in R^3. There are two other infinite series of finite subgroups of $SO(3)$. They are called the cyclic subgroup and the p-dihedral subgroup.

Definition 4.12. Let a group G consist of all transformations of the Euclidean n-space R^n that preserve the distance between any two points in R^n. If all the transformations in G fix a given point, G is called an *orthogonal group* $O(n)$. [6]

Definition 4.13. Let $O(n)$ be an orthogonal group. Let a subgroup $SO(n)$ contain all transformations in $O(n)$ that fix the origin. Then $SO(n)$ is called a *special orthogonal group.* In three dimensions, $SO(n)$ is called the *rotation group* $SO(3)$. [7]

Up to conjugacy the finite subgroups of $SO(3)$ are: [2]

- the cyclic groups of order $p = 2, 3, \ldots$
- the dihedral groups of order $2p$ ($p = 2, 3, \ldots$)
- the tetrahedral group
- the octahedral group
- the icosahedral group.

A cyclic subgroup consists of all rotations around a fixed axis. The subgroups are of any order $p < \inf$ and the angles of rotation are $2\pi j/p$, where $j = 1, 2, \ldots, p$.

Let three lines be perpendicular to each other and go through the origin. The identity and the rotations with an angle of $180°$ around each line form a subgroup. This group is of order four and is called the 2-dihedral group. Any two 2-dihedral groups are conjugate in $SO(3)$.

Let a polygon have p edges, with $p \geq 3$, and be centred at the origin. A line through the origin that is perpendicular to the polygon is an axis that create p-fold rotations. The axes that are parallel to the polygon are twofold and passes through the vertices and mid-edge points. There are p such axes. These rotations form the p-dihedral group of order $2p$. Any two p-dihedral groups are conjugate in $SO(3)$.

5 Summary

The Platonic solids continue to fascinate mathematicians to this day. It can easily be understood why so many have studied them throughout history. There is much to learn and several different ways to investigate them. They can be seen as 3-dimensional objects or as graphs, their rotations can be seen as permutations or as groups. In this thesis we have concluded that the Platonic solids are

- the regular tetrahedron
- the regular hexahedron (cube)
- the regular octahedron
- the regular icosahedron
- the regular dodecahedron.

We have found that the rotations of the solids can be divided into three groups; the tetrahedral group of order twelve, octahedral group of order 24 and icosahedral group of order 60. We also took a quick look at the two other finite subgroups of the rotation group $SO(3)$, the cyclic groups of order $p = 2, 3 \ldots$ and the dihedral groups of order $2p$ ($p = 2, 3, \ldots$).

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