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A Brief History of Elliptic Functions

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Abstract

This thesis explores the history of elliptic functions, beginning with their origins in elliptic integrals studied by Jacob Bernoulli and Euler, and ending up with the torus surface after trespassing into the complex domain. Along the way we encounter parallels between elliptic functions and trigonometric functions, and learn about the discovery of inverting elliptic integrals due to Abel, Jacobi and Gauss.

Contents

1 Introduction

Historically, elliptic functions were defined as inverse functions of elliptic integrals. As such, before we can speak of a history of these functions, we must start from the beginning, with the ellipse.

As their name suggests, *elliptic integrals*, arose from the study of arc lengths of ellipses. The discovery of Kepler's second law which states that an elliptical orbit sweeps out equal areas over equal times led mathematicians to pursue problems involving the rectification of ellipses, which led to elliptic integrals.

The first account of elliptic integrals was in 1655 when John Wallis studied the arc length of an ellipse. However, many mathematicians would soon realize that these integrals seemed impossible to solve by way of Leibniz's "closed form" solutions, or solutions by elementary functions. All efforts to express elliptic integrals in terms of functions composed from algebraic, circular, and exponential functions and their inverses failed, and in 1694 Jacob Bernoulli conjectured that the task was impossible. It was not until 1833 however, that the conjecture was eventually confirmed by Liouville.

In the meantime, while the impossibility of closed form solutions remained a mystery, a few notable mathematicians continued to tackle these problems. In so doing, they uncovered many interesting properties that elliptic integrals had in common, and they were soon able to be classified and systematized into its own subject that eventually gave rise to the theory of elliptic functions – a crucial step on the way to elliptic curves and the proof of Fermat's Last Theorem.

In the present work, we will use the following definition for elliptic integrals.

Definition 1. *Elliptic integrals* are functions of the form

$$f(x) = \int_c^x R(t, \sqrt{p(t)})dt,$$

where R is a rational function of its two arguments and p is a polynomial in one variable of degree 3 or 4 without repeated roots, and c is a constant.

1.1 Outline and reference material

Sections 2-4 constitute the first half of this thesis, covering bits and pieces of the classic theory of elliptic integrals.

Section 2 covers the rectification of the lemniscate to find its arc length, which led to the discovery of the important *lemniscatic integral*. Section 3 covers addition formulas for elliptic integrals due to Fagnano and Euler. Section 4 provides some terminology and a classification of three kinds of elliptic integrals based on the work of Legendre. Stillwell (2010), Siegel (1969) and Tkachev (2010) were consulted for the proofs given in Sections 2-3, with historical notes found in Bottazzini and Gray (2013) and Gray (2015).

Sections 5-6 constitutes the second half, where we learn about elliptic functions and their historical applications in geometry.

Section 5 covers the foundational works of Abel, Jacobi and Gauss who are each credited with independent discovery of elliptic functions. We learn about the idea of inverting elliptic integrals and the remarkable property of double periodicity from three different perspectives. In closing, Section 6 offers a less detailed exposition of some later geometric ideas tied to elliptic functions. For Section 5.1, we've done a close reading of Abel (2007) in translation together with some notes in Houzel (2004). Kolmogorov and Yushkevich (1996) and Hancock (1910) were consulted for Section 5.2, and Stillwell (2010) was used as reference for Section 5.3. Weil (1976), Rice and Brown (2013) and Stillwell (2010) were primarily consulted for Section 6.

2 Lemniscate of Bernoulli

The key that unlocked the properties of elliptic integrals was a *lemniscate* first discovered by Jacob Bernoulli in 1694. He described it as a modification of the ellipse. Recall that

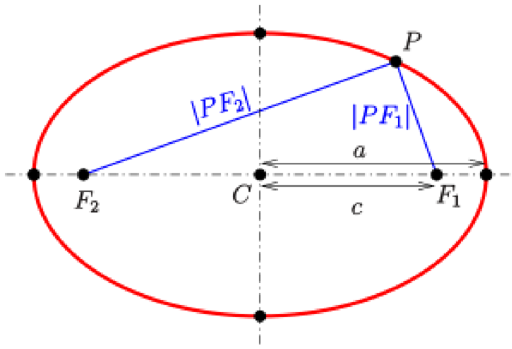

Figure 1: The ellipse

an ellipse can be defined using two fixed points, F_1 and F_2, called the *foci*, such that the sum of distances from any point P on the curve to the two foci is constant, usually denoted by the constant $2a$ given by the equation

$$|PF_1| + |PF_2| = 2a.$$

Jacob Bernoulli proposed instead to consider the curve given by the set of points for which the *product* of these distances is constant. Letting the two foci F_1 and F_2 be at distance $2a$ from each other, he set this product to be

$$|PF_1| \cdot |PF_2| = a^2.$$

This modified curve as defined by the constant product of the distances PF_1 and PF_2

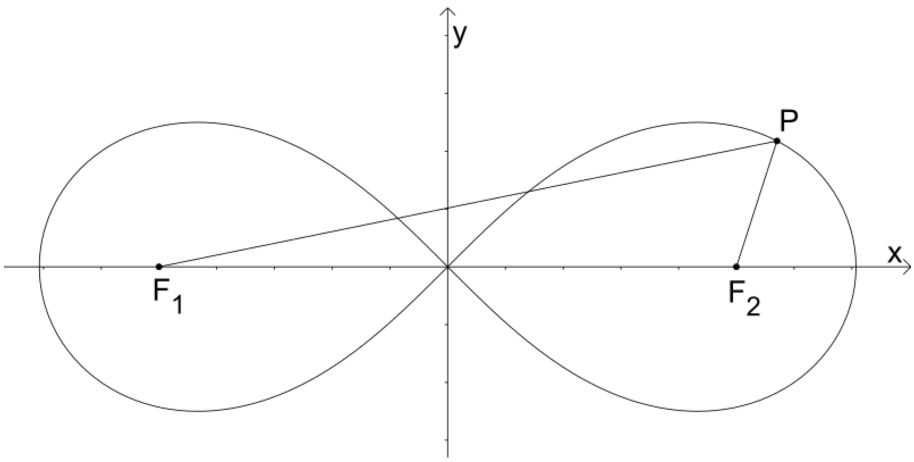

Figure 2: The lemniscate of Bernoulli

came to be known as the *lemniscate of Bernoulli*, and has the Cartesian equation

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$

He described it as "the form of a figure 8 on its side, as of a band folded into a knot, or of a lemniscus, or of a knot of a French ribbon."[1]

In polar form, it is written as $r^2 = 2a^2 \cos 2\theta$. Each of the two "loops" of the lemniscate as they appear in Figure 2 are commonly referred to as *lobes.* As the parameter a varies, the lobes are magnified but the shape of the curve remains the same. In what follows, we shall fix $a = 1/\sqrt{2}$ for convenience so that the polar equation can be written as

$$r^2 = \cos 2\theta. \tag{1}$$

Jacob Bernoulli then discovered the following result about the lemniscate, which is shown using elementary techniques from calculus.

Theorem 1. *The total length L of the curve with polar equation $r^2 = \cos 2\theta$, is given by the integral*

$$L = 4 \int_0^1 \frac{dr}{\sqrt{1 - r^4}}.$$

Proof. Recall from calculus the formula for the line element in polar coordinates

$$ds = \sqrt{(r\, d\theta)^2 + dr^2},$$

from which we obtain

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{ds}{d\theta}\right)^2}.$$

Now, half of one lobe of the lemniscate is traced out as θ goes from 0 to $\pi/4$. Then, since the curve is symmetric, we may restrict our consideration to the first quadrant. Recalling the formula for arc length from calculus, the total length L can thus be written as

$$L = 4 \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

From the polar equation (1), we obtain

$$2r \frac{dr}{d\theta} = -2 \sin 2\theta, \tag{2}$$

and hence

$$\frac{dr}{d\theta} = -\frac{\sin 2\theta}{r}, \quad \text{or} \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2}.$$

Manipulating the expression in the integrand, we get

$$L = 4 \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4 \int_0^{\pi/4} \sqrt{r^2 + \frac{\sin^2 2\theta}{r^2}}\, d\theta = 4 \int_0^{\pi/4} \sqrt{\frac{r^4 + \sin^2 2\theta}{r^2}}\, d\theta.$$

But $r^4 = \cos^2 2\theta$, and so $r^4 + \sin^2 2\theta = \cos^2 2\theta + \sin^2 2\theta = 1$, thus we can simplify the integrand to

$$4 \int_0^{\pi/4} \sqrt{\frac{r^4 + \sin^2 2\theta}{r^2}}\, d\theta = 4 \int_0^{\pi/4} \frac{d\theta}{r}.$$

[1] Tkachev (2014), p. 11

Finally, to obtain the desired result, we change the variable of integration to r. From (2), we get

$$d\theta = \frac{r}{\sin 2\theta} \cdot dr,$$

and since $\sin 2\theta = \sqrt{1 - \cos^2 2\theta} = \sqrt{1 - r^4}$, we thus obtain the desired expression in the integrand

$$\frac{d\theta}{r} = \frac{dr}{\sqrt{1 - r^4}}.$$

Observing that as θ varies over the interval $0 \leq \theta \leq \pi/4$ on the curve $r^2 = \cos 2\theta$, the radius r varies over the corresponding interval $0 \leq r \leq 1$, and we thus get the total length

$$L = 4 \int_0^1 \frac{dr}{\sqrt{1 - r^4}}.$$

$\square$

This result, while simple, came to be studied extensively for the remarkable properties of the *lemniscatic integral* derived from it, namely the function

$$f(x) = \int_0^x \frac{dt}{\sqrt{1 - t^4}}.$$

All efforts to express this integral in terms of functions composed from algebraic, circular, and exponential functions and their inverses failed, and Jacob Bernoulli conjectured that finding a "closed form" solution to this integral was impossible. As it turns out, the lemniscatic integral belongs to a class of functions called *elliptic integrals*, which all share the lack of closed form solutions as was later shown by Liouville in 1833.

Despite Bernoulli's conjecture, or perhaps instigated by it, the lemniscatic integral was nonetheless investigated by many subsequent mathematicians who then found universal properties that could be extended to more general elliptic integrals. Most notably, investigations of addition theorems for these integrals would later play an important part in the development of *elliptic functions*. In the next section we shall look at two such addition theorems.

3 Algebraic Addition Theorems

A familiar example from calculus of a function which admits an algebraic addition theorem is

$$f(x) = e^x.$$

The addition theorem then states that

$$e^{u+v} = e^u \cdot e^v,$$

or

$$f(u+v) = f(u) \cdot f(v).$$

In what follows, we will use the following definition.

Definition 2. Let $f(u)$ be an analytic function. If there exists an algebraic equation of the form

$$f(u+v) = G(f(u), f(v)),$$

where G is a polynomial in $f(u)$ and $f(v)$ with coefficients that do not depend on u and v, we say that $f(u)$ admits an *algebraic addition theorem.*

3.1 Fagnano's doubling formula

Following the ideas of Jacob Bernoulli, one early mathematician to study the properties of the lemniscate was Giulio Carlo Fagnano (1682-1766). Fagnano's research was published in the period 1714-1720 in an obscure Venetian journal and was not widely known. One result, referred to as Fagnano's Theorem, relates the sum of appropriately chosen arcs of an ellipse to the coordinates of the points involved.

We shall concern ourselves with another result, namely his discovery of a formula for doubling the arc of the lemniscate, which was the first step to a general addition theorem for elliptic integrals. Perhaps the best way to understand Fagnano's doubling formula is to compare it to an analogous case, the more familiar integral from calculus that bears striking similarities to the lemniscatic integral $\int_0^x \frac{dt}{\sqrt{1-t^4}}$.

Example 1. Consider the integral

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

If we let

$$u = \sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}},$$

then

$$2u = 2\int_0^x \frac{dt}{\sqrt{1-t^2}}. \tag{3}$$

Now, in order to formalize a doubling formula for the arcsine integral, let $x = f(u)$ be the solution to

$$u = \int_0^{f(u)} \frac{dt}{\sqrt{1-t^2}}.$$

Then, by recognizing that $x = f(u) = \sin u$, being the inverse function of $u = \sin^{-1} x$, we see that the addition formula for $f(u+u)$ here is simply a special case of the sine addition theorem:

$$\sin 2u = 2 \sin u \sqrt{1 - \sin^2 u}.$$

In other words we have

$$f(2u) = \sin 2u = 2f(u)\sqrt{1 - f(u)^2}. \tag{4}$$

The relation in (4) has a corresponding relation in the upper limits of integration in (3), thus we obtain a doubling formula for the arcsine integral, namely

$$2\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}.$$

Perhaps inspired by the analogous trigonometric case in the example above, Fagnano arrived at his formula for the lemniscate by attempting to rationalize the integrand with a substitution similar to a natural substitution for the arcsine integral.

Theorem 2 (Fagnano's Doubling Theorem)**.** *Let $f(u)$ be defined as the solution to*

$$u = \int_0^{f(u)} \frac{dt}{\sqrt{1-t^4}}.$$

Then,

$$f(2u) = f(u+u) = \frac{2f(u)\sqrt{1-f(u)^4}}{1+f(u)^4}.$$

Proof. We follow here a reconstruction of Fagnano's proof given by Siegel (1969), filling in some details along the way. In the analogous case of the arcsine integral, we recall from calculus that the integrand $dt/\sqrt{1-t^2}$ may be rationalized via the substitution $t = 2v/(1+v^2)$ to make the new integrand $2dv/(1+v^2)$. Fagnano employs two similar such substitutions in succession to create a mapping that doubles the arc length of the lemniscate.

Since the lemniscate is symmetric with respect to both coordinate axes, we can restrict ourselves to the portion of the arc length that lies in the first quadrant. Keeping in mind that u is the arc length, we may regard t as an independent variable which varies over the interval $0 \le t \le 1$.

For the first substitution, let

$$t^2 = \frac{2v^2}{1+v^4} \tag{5}$$

Taking the derivative on both sides and dividing by 2 yields

$$t\,\frac{dt}{dv} = \frac{2v(1-v^4)}{(1+v^4)^2},$$

and we also observe from (5) that

$$t = \frac{\sqrt{2}\,v}{\sqrt{1+v^4}}.$$

Hence,

$$\frac{dt}{dv} = \frac{1}{t} \cdot \frac{2v(1-v^4)}{(1+v^4)^2} = \frac{2v\sqrt{1+v^4}}{\sqrt{2}\,v} \cdot \frac{(1-v^4)}{(1+v^4)^2} = \frac{\sqrt{2}}{\sqrt{1+v^4}} \cdot \frac{1-v^4}{1+v^4}. \tag{6}$$

Now, to carry out the substitution, we can rewrite the radical expression in the integrand of $\int dt/\sqrt{1-t^4}$ as

$$\sqrt{1-t^4} = \sqrt{1 - \frac{4v^4}{(1+v^4)^2}} = \sqrt{\frac{(1-v^4)^2}{(1+v^4)^2}} = \frac{1-v^4}{1+v^4}. \tag{7}$$

In view of (6) and (7), we see then that

$$\frac{dt}{dv} = \frac{\sqrt{2}}{\sqrt{1+v^4}} \cdot \sqrt{1-t^4},$$

and hence

$$\frac{dt}{\sqrt{1-t^4}} = \sqrt{2}\frac{dv}{\sqrt{1+v^4}}. \tag{8}$$

While this does not succeed at rationalizing the integrand, the relation in (8) gives a monotonic mapping of the interval $0 \le v \le 1$ onto $0 \le t \le 1$. It follows that

$$\int_0^t \frac{dt'}{\sqrt{1-t'^4}} = \sqrt{2}\int_0^v \frac{dv'}{\sqrt{1+v'^4}} \quad (0 \le t \le 1), \tag{9}$$

where v is the solution of (5) which lies in the interval $0 \le v \le 1$.

But since

$$t = \sqrt{2}\frac{v}{\sqrt{1+v^4}},$$

we may rewrite the relation in (9) as

$$\sqrt{2}\int_0^v \frac{dv'}{\sqrt{1+v'^4}} = \int_0^{\sqrt{2}v/\sqrt{1+v^4}} \frac{dt'}{\sqrt{1-t'^4}}.$$

Fagnano then makes a second substitution

$$v^2 = \frac{2w^2}{1-w^4}, \tag{10}$$

such that when employed in succession to the first substitution in (5) we obtain

$$t^2 = \frac{2v^2}{1+v^4} = \frac{4w^2(1-w^4)}{(1+w^4)^2}. \tag{11}$$

Using the same method as following the first substitution to obtain the relation in (9), one can check that the second substitution (10) yields the relation

$$\frac{dv}{\sqrt{1+v^4}} = \sqrt{2}\frac{dw}{\sqrt{1-w^4}},$$

and the corresponding integrals

$$\int_0^v \frac{dv'}{\sqrt{1+v'^4}} = \sqrt{2}\int_0^w \frac{dw'}{\sqrt{1-w'^4}}.$$

Now, in view of (11), we have

$$t = \frac{2w\sqrt{1-w^4}}{1+w^4}, \tag{12}$$

and the corresponding relation between integrals is

$$\int_0^t \frac{dt'}{\sqrt{1-t'^4}} = 2\int_0^w \frac{dw'}{\sqrt{1-w'^4}}. \tag{13}$$

But we have here in (12) a simple algebraic relation between the upper limits t and w of the lemniscatic integrals. In other words, if $f(u)$ is the solution to

$$u = \int_0^{f(u)} \frac{dt}{\sqrt{1-t^4}},$$

then in view of (12) and (13),

$$f(2u) = \frac{2f(u)\sqrt{1-f(u)^4}}{1+f(u)^4},$$

which is what we wanted to prove. □

3.2 Euler

Fagnano's research on the lemniscate was given to Leonhard Euler for review on December 23, 1751; Euler was requested by the Berlin Academy of Sciences to examine Fagnano's recently published book and draft a suitable letter of thanks. Less than five weeks later, on January 27, 1752, Euler read to the Academy the first of a series of papers giving new derivations for Fagnano's results on elliptic integrals. According to Jacobi, the theory of elliptic functions was born in this span of five weeks.[2] By 1753, Euler had a general addition theorem for lemniscatic integrals, which he was able to extend to more general elliptic integrals five years later.

It is worth mentioning that Euler's addition theorems do not cover all elliptic integrals. However, as we shall see in Section 4, the general form of the elliptic integral $\int R(t, \sqrt{p(t)})dt$, where R is a rational function and $p(t)$ is a polynomial of degree 3 or 4, reduces to just three kinds, of which Euler was able to find an addition theorem for the first kind. We shall provide here his addition theorems for lemniscatic integrals and for elliptic integrals of the first kind, known as Euler's Addition Theorem.

In the previous subsection, we used the analogy of the sine addition theorem to find a doubling formula for the arc length of the lemniscate. Euler's generalizations of Fagnano's formula draws upon the same analogy. In this section, we take a look at a reconstruction of Euler's train of thought given by Siegel (1969), pp. 7-10, beginning again with the analogy of trigonometric functions.

3.2.1 General Addition Formula for the Lemniscate

Recall once more the sine addition theorem from Example 1, except this time in its more general form:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

[2] Weil (1983), p. 1.

or

$$\sin(x+y) = \sin x\sqrt{1-\sin^2 y} + \sin y\sqrt{1-\sin^2 x}. \tag{14}$$

Similarly to the argument used in Example 1 of the previous subsection above, if we substitute

$$u = \sin x, \quad v = \sin y$$

in (14), we get

$$\sin(x+y) = u\sqrt{1-v^2} + v\sqrt{1-u^2}. \tag{15}$$

Taking the inverse functions in the arcsine integral, we obtain the relation

$$\int_0^u \frac{dt}{1-t^2} + \int_0^v \frac{dt}{1-t^2} = \int_0^{\phi(u,v)} \frac{dt}{1-t^2},$$

with $\phi(u,v) = u\sqrt{1-v^2} + v\sqrt{1-u^2}$.

Now, in order to obtain a corresponding version of the addition theorem for the lemniscatic integral, compare the general sine addition theorem (15) with its special case from Example 1, $\sin(x+x) = 2u\sqrt{1-u^2}$. Intuitively, we may try to replace the numerator $2u\sqrt{1-u^4}$ in Fagnano's formula by the expression $u\sqrt{1-v^4}+v\sqrt{1-u^4}$. In the denominator, we choose the simplest symmetric function of u and v which becomes $1+u^4$ for $u=v$, namely, $1+u^2v^2$.

The corresponding substitution for the lemniscatic integral then becomes

$$\phi(u,v) = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}. \tag{16}$$

To show that this yields the desired addition theorem for arc lengths of the lemniscate, we consider the curve $\phi(u,v) = a$, for some constant a, and let v be an independent variable, with u as dependent variable, and then find the differential equation which the curve satisfies. Thus, when $v=0$, we have

$$a = \phi(u,0) = \frac{u\sqrt{1-0} + 0\cdot\sqrt{1-u^4}}{1+u^2\cdot 0} = u,$$

and so $u=a$ when $v=0$.

With the above in mind, we can find the differential equation in the following way. We introduce the abbreviations $U = \sqrt{1-u^4}, V = \sqrt{1-v^4}$ and differentiate (16):

$$d\phi = \phi_u \cdot du + \phi_v \cdot dv = 0, \tag{17}$$

with

$$\phi_u = \frac{(UV - 2vu^3)(1+u^2v^2) - 2uv^2(uUV + v - vu^4)}{U(1+u^2v^2)^2};$$

$$\phi_v = \frac{(VU - 2uv^3)(1+u^2v^2) - 2vu^2(vVU + u - uv^4)}{V(1+u^2v^2)^2}.$$

The numerators in ϕ_u and ϕ_v both further simplify to $UV - u^2v^2UV - 2uv^3 - 2vu^3$, which becomes $\sqrt{1-u^4}$ or $\sqrt{1-a^4}$ when $v=0$, which is nonzero for sufficiently small a. Hence, u as a function of v satisfies the differential equation obtained from (17):

$$\frac{du}{dv} = -\frac{\phi_u}{\phi_v}.$$

For clarity, we may denote the common factors in ϕ_u and ϕ_v by

$$\sigma(u,v) = \frac{UV - u^2v^2UV - 2uv^3 - 2vu^3}{(1+u^2v^2)^2},$$

and write the equation in (17) as

$$\frac{\sigma(u,v)\cdot du}{\sqrt{1-u^4}} + \frac{\sigma(u,v)\cdot dv}{\sqrt{1-v^4}} = 0$$

Since $\sigma(u,v)$ is locally nonzero for $v = 0$, we may eliminate the common factors and rewrite this as

$$\frac{du}{\sqrt{1-u^4}} + \frac{dv}{\sqrt{1-v^4}} = 0.$$

Integrating with respect to v, with lower limit 0, we obtain

$$\int_{\phi(u,v)}^{u} \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = 0,$$

or

$$\int_0^u \frac{dt}{\sqrt{1-t^4}} - \int_0^{\phi(u,v)} \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = 0,$$

and so,

$$\int_0^u \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = \int_0^{\phi(u,v)} \frac{dt}{\sqrt{1-t^4}},$$

with

$$\phi(u,v) = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2},$$

which is the desired addition formula.

3.2.2 Euler's Addition Theorem

Shortly after this discovery, Euler took the last essential step towards a complete addition theorem for elliptic integrals. By replacing the expression $1-t^4$ under the radical in the lemniscatic integral with the polynomial $P(x) = 1 + cx^2 - x^4$, where $0 < c < 1$, he could extend his addition formula to *elliptic integrals of the first kind.*

Theorem 3 (Euler's Addition Theorem). *Let c be a constant such that $0 < c < 1$, and let $P(x)$ be the polynomial*

$$P(x) = 1 + cx^2 - x^4,$$

Then,

$$\int_0^u \frac{dt}{\sqrt{P(t)}} + \int_0^v \frac{dt}{\sqrt{P(t)}} = \int_0^{\phi(u,v)} \frac{dt}{\sqrt{P(t)}},$$

where

$$\phi(u,v) = \frac{u\sqrt{P(v)} + v\sqrt{P(u)}}{1-u^2v^2}.$$

Proof. The argument here follows along the same lines as the particular case in the previous subsection 3.2.1, where $P(t) = 1-t^4$. Again, we fix $\phi(u,v) = a$, for some constant a, and differentiate:

$$d\phi = \phi_u \cdot du + \phi_v \cdot dv = 0,$$

with

$$\phi_u = \frac{(\sqrt{P(u)P(v)} + cuv - 2vu^3)(1+u^2v^2) - 2uv^2(u\sqrt{P(u)P(v)} + v + cvu^2 - vu^4)}{\sqrt{P(u)}(1+u^2v^2)^2};$$

$$\phi_v = \frac{(\sqrt{P(u)P(v)} + cuv - 2uv^3)(1+u^2v^2) - 2vu^2(v\sqrt{P(u)P(v)} + u + cuv^2 - uv^4)}{\sqrt{P(v)}(1+u^2v^2)^2}.$$

with both numerators simplifying to $(\sqrt{P(u)P(v)} + cuv)(1-u^2v^2) - 2uv(u^2+v^2)$. Using the same argument as in the lemniscatic case, it follows that

$$\frac{du}{\sqrt{P(u)}} + \frac{dv}{\sqrt{P(v)}} = 0,$$

and hence,

$$\int_0^u \frac{dt}{\sqrt{P(t)}} + \int_0^v \frac{dt}{\sqrt{P(t)}} = \int_0^{\phi(u,v)} \frac{dt}{\sqrt{P(t)}},$$

with

$$\phi(u,v) = \frac{u\sqrt{P(v)} + v\sqrt{P(u)}}{1+u^2v^2}, \quad P(t) = 1 + ct^2 - t^4.$$

□

4 General Elliptic Integrals

Euler himself realized that his results were restricted to polynomials $p(t)$ of degree 4 in $\int dt/\sqrt{p(t)}$, and did not touch on other cases of the general form of elliptic integrals $\int R(t, \sqrt{p(t)})dt$. However, as it turns out, there is no real difference if p is a cubic or a quartic, and the general form is reducible to three kinds.

To see why degree 3 and 4 are essentially the same, consider the following example.

Example 2. Suppose we have the elliptic integral

$$\int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}},$$

where the polynomial under the radical is of degree 3. We now show that the substitution $x = \frac{1}{y}$ transforms the expression under the radical into a polynomial of degree 4.

First, we may assume that the roots of $p(x) = (x-a)(x-b)(x-c)$ are distinct. Were this not the case, we could simply pull the repeated factor out of the radical and be left with a polynomial $p_1(x)$ of degree 1 or 2 under the radical sign and express the integral in terms of inverse trigonometric functions.

Now, letting $x = \frac{1}{y}$, we get

$$\frac{dx}{dy} = -\frac{1}{y^2}, \quad \text{and} \quad \sqrt{(x-a)(x-b)(x-c)} = \sqrt{(\frac{1}{y}-a)(\frac{1}{y}-b)(\frac{1}{y}-c)},$$

and so

$$\frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{-dy}{\sqrt{y^4(\frac{1}{y}-a)(\frac{1}{y}-b)(\frac{1}{y}-c)}} = $$

$$= \frac{-dy}{\sqrt{y(1-ya)(1-yb)(1-yc)}}$$

where $\tilde{p}(y) = y(1-ya)(1-yb)(1-yc)$ is a polynomial of degree 4.

More generally, the same idea can be applied to any integral of the form

$$\int R(x, \sqrt{p(x)})dx$$

where $\deg(p) = 3$. The general idea is to make a change of variables $z = (ax+b)/(cx+d)$, and by suitable choice of coefficients a, b, c, d the polynomial $p(x)$ can be transformed to a new polynomial $\tilde{p}(z)$, such that $\deg \tilde{p} = 4$. The reverse reduction is also possible, transforming a quartic into a cubic. We refer to Bateman and Erdelyi (1953), for how these reductions can be carried out.

4.1 Legendre normal form

Following the works of Euler and Fagnano, much of the classic theory of elliptic integrals was systematized by Legendre. For 40 years he published papers and books on the subject, including their various addition and transformation theorems, tables of values computed with the addition theorems, as well as their classification into three kinds. However, his work did not attract the interest of his peers to the degree that he had hoped until in 1827 Abel and Jacobi took it up and completely transformed the subject, unlocking an entirely new direction of mathematics by studying the exotic new properties of the inverses of elliptic integrals – an idea that had not occurred to Legendre. To quote Legendre:[3]

> "Scarcely had my work seen the light of day, scarcely could its title have become known to scientists abroad, when I learned with as much astonishment and satisfaction that two young geometers, MM. Jacobi of Königsberg and Abel of Christiania, had succeeded in their own individual work in considerably improving the theory of elliptic functions at its highest points."

Legendre showed in 1792 how any integral of the form $\int R(x, \sqrt{p(x)})dx$, where R is a rational function and $p(x)$ is a quartic polynomial with real coefficients and without repeated factors, could be reduced to the form

$$\int \frac{Qdt}{\sqrt{(1-t^2)(1-c^2t^2)}},$$

where Q is a rational function of t. Applying the substitution $t = \sin\phi$, he further reduced this to

$$\int \frac{Qd\phi}{\sqrt{1-c^2\sin^2\phi}}.$$

Legendre also introduced some terminology, calling the variable ϕ the *amplitude* of the elliptic integral, the real parameter c the *modulus* (with $0 < c < 1$), and the quantity $b := \sqrt{1-c^2}$ the *complementary modulus*. Writing $\Delta(\phi)$ for the radical expression $\sqrt{1-c^2\sin\phi}$, Legendre then showed using partial fraction decomposition that the integral $\int Qd\phi/\Delta$ is equal to an elementary function, plus an elliptic integral of the form

$$\int (A + B\sin^2\phi)\frac{d\phi}{\Delta},$$

where A and B are constants. He thus confines his attention to this last integral, which in turn reduces to one of three distinct kinds, a classification known today as *Legendre normal form.*

The three kinds, denoted by the functions $F(\phi)$, $E(\phi)$ and $\Pi(\phi)$ respectively, are as follows:

$$F(\phi) = \int_0^\phi \frac{d\psi}{\Delta(\psi)}; \qquad \text{(\textit{elliptic integral of the first kind})}$$

$$E(\phi) = \int_0^\phi \Delta(\psi)d\psi; \qquad \text{(\textit{elliptic integral of the second kind})}$$

$$\Pi(\phi) = \int_0^\phi \frac{d\psi}{(1+n^2\sin^2\psi)\Delta(\psi)}, \qquad \text{(\textit{elliptic integral of the third kind})}$$

where n may be real or complex.

[3] Bottazzini and Gray (2013), p. 15.

5 Elliptic Functions

Consider again the familiar "circular integral" from Example 1 in Section 3.1:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$$

Experience has taught us that the inverse of this function, namely $\sin x$, is much easier to deal with. So it is with other cases of circular integrals – inverting them leads to trigonometric functions $g(x)$, most of which are periodic, in the sense that

$$g(x + 2\pi) = g(x).$$

Similarly, it is convenient to replace certain elliptic integrals by their inverses, which came to be known as *elliptic functions*. The idea of inverting elliptic integrals to obtain elliptic functions was originally due to Abel, Jacobi and Gauss. Abel had the first publication, Gauss had priority on the idea some 30 years earlier but did not publish, and Jacobi published his own inversion two years after Abel. While the independence of Jacobi's discovery is unclear, he is still credited with originality due to having fundamental new ideas of his own to contribute to the new theory of elliptic functions.

5.1 Abel's Recherches

In January 1827, Niels Henrik Abel (1802-1829) published his first major paper on elliptic integrals in Crelle's *Journal*, presenting the first account of a remarkable new direction for the subject. By studying the inverses of elliptic integrals, the theory of *elliptic functions* finally came to life.

The paper, titled *Recherches sur les fonctions elliptiques*, has a lucid style of exposition and provides a long introduction to elliptic functions. In it, he provides several addition theorems for these functions, shows that they are doubly periodic, gives them formulae for multiplication and division and expresses his functions as infinite series and products. He also proves many other interesting results, such as that the circumference of the lemniscate can be divided by ruler and compass alone, and a general transformation theorem for transforming one elliptic integral into another. We shall in this subsection content ourselves with a brief survey of the first part of his large memoir and end with the discovery of double periodicity.

Abel begins his memoir by recalling Legendre's classification of the general elliptic integral of the first kind, slightly reformulating it and introducing an additional parameter e in the integrand

$$\alpha = \int_0^x \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}},$$

of which he writes:

> "M. Legendre takes c^2 to be positive, but I have observed that the formulae become simpler, if we take c^2 to be negative, equal to $-e^2$. In the same way, to have more symmetry, I write $1 - c^2x^2$ for $1 - x^2$ [...]"

He then proposes to consider the inverse function, namely $x = \phi(\alpha)$, writing that this function satisfies

$$\frac{d\phi}{d\alpha} = \sqrt{(1-c^2x^2)(1+e^2x^2)}. \tag{18}$$

Further, he introduced the auxiliary functions

$$f(\alpha) = \sqrt{1 - c^2\phi^2(\alpha)}, \quad F(\alpha) = \sqrt{1 + e^2\phi^2(\alpha)},$$

and defined

$$\frac{\omega}{2} = \int_0^{1/c} \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}}.$$

The function ϕ is positive in the range $0 < \alpha < \frac{\omega}{2}$, with $\phi(0) = 0$ and $\phi(\frac{\omega}{2}) = \frac{1}{c}$. Since α is an odd function of x, we have

$$\phi(-\alpha) = -\phi(\alpha).$$

Introducing complex variables, he replaced α formally by $i\beta$, such that $ix = \phi(i\beta)$, and noted that

$$\beta = \int_0^{x} \frac{dt}{\sqrt{(1+c^2t^2)(1-e^2t^2)}}$$

is real and positive on the interval $0 < x < 1/e$. Inverting the β integral, Abel defined

$$\frac{\tilde{\omega}}{2} = \int_0^{1/ie} \frac{dt}{\sqrt{(1+c^2t^2)(1-e^2t^2)}},$$

where x is positive in the range $0 < \beta < \frac{\tilde{\omega}}{2}$.

Continuing, Abel notes that the values of $\phi(\alpha)$ are known for every real value of α on the interval $-\frac{\omega}{2} < \alpha < \frac{\omega}{2}$, and similarly for every imaginary value of $i\beta$ on the interval $-\frac{\tilde{\omega}}{2} < i\beta < \frac{\tilde{\omega}}{2}$.

Recalling Euler's addition theorem for elliptic integrals, Abel extended the definition of his functions to the entire complex domain with similar addition formulae:

$$\begin{cases} \phi(\alpha+\beta) = \dfrac{\phi(\alpha)\cdot f(\beta)\cdot F(\beta) + \phi(\beta)\cdot f(\alpha)\cdot F(\alpha)}{1+e^2c^2\phi^2(\alpha)\cdot\phi^2(\beta)}, \\ f(\alpha+\beta) = \dfrac{f(\alpha)\cdot f(\beta) - c^2\phi(\alpha)\cdot\phi(\beta)\cdot F(\alpha)\cdot F(\beta)}{1+e^2c^2\phi(\alpha)\cdot\phi^2(\beta)}, \\ F(\alpha+\beta) = \dfrac{F(\alpha)\cdot F(\beta) + e^2\phi(\alpha)\cdot\phi(\beta)\cdot f(\alpha)\cdot f(\beta)}{1+e^2c^2\phi^2(\alpha)\cdot\phi^2(\beta)}. \end{cases} \tag{19}$$

(19) Abel notes that these formulae can be deduced from other known properties of elliptic functions found in Legendre, but verifies them in his own way by means of differential equations in the following manner.

First, he squares the auxiliary functions

$$\begin{cases} f^2(\alpha) = 1 - c^2\phi^2(\alpha), \\ F^2(\alpha) = 1 + e^2\phi^2(\alpha). \end{cases}$$

Differentiating yields

$$\begin{cases} f(\alpha)\cdot\dfrac{df}{d\alpha} = -c^2\phi(\alpha)\cdot\dfrac{d\phi}{d\alpha}, \\ F(\alpha)\cdot\dfrac{dF}{d\alpha} = e^2\phi(\alpha)\cdot\dfrac{d\phi}{d\alpha}. \end{cases}$$

But from the observation in (18) we have

$$\frac{d\phi}{d\alpha} = \sqrt{\Big(1 - c^2\phi^2(\alpha)\Big)\Big(1 + e^2\phi^2(\alpha)\Big)},$$

which by definition of the auxiliary functions is precisely equal to $f(\alpha) \cdot F(\alpha)$. Thus, the functions ϕ, f, F are related by the following equations

$$\begin{cases} \dfrac{d\phi}{d\alpha} = f(\alpha) \cdot F(\alpha), \\ \dfrac{df}{d\alpha} = -c^2\phi(\alpha) \cdot F(\alpha), \\ \dfrac{dF}{d\alpha} = e^2\phi(\alpha) \cdot f(\alpha). \end{cases} \tag{20}$$

Abel then shows the first equality in (19). Let r denote the right-hand side of the first equation, i.e.

$$r = \frac{\phi(\alpha) \cdot f(\beta) \cdot F(\beta) + \phi(\beta) \cdot f(\alpha) \cdot F(\alpha)}{1 + e^2c^2\phi^2(\alpha) \cdot \phi^2(\beta)}.$$

Differentiating with respect to α yields

$$\begin{aligned} \frac{dr}{d\alpha} = &\frac{\phi'(\alpha)f(\beta)F(\beta) + \phi(\beta)F(\alpha)f'(\alpha) + \phi(\beta)f(\alpha)F'(\alpha)}{1 + e^2c^2\phi^2(\alpha)\phi^2(\beta)} \\ &- \frac{\Big(\phi(\alpha)f(\beta)F(\beta) + \phi(\beta)f(\alpha)F(\alpha)\Big)2e^2c^2\phi(\alpha)\phi^2(\beta)\phi'(\alpha)}{\Big(1 + e^2c^2\phi^2(\alpha)\phi^2(\beta)\Big)^2}. \end{aligned}$$

Writing $\phi\alpha, f\alpha, F\alpha$ instead of $\phi(\alpha), f(\alpha), F(\alpha)$ to shorten the notation and save space in the margins, and substituting in the values for $\phi'\alpha, f'\alpha, F'\alpha$ from (20), we get

$$\begin{aligned} \frac{dr}{d\alpha} = &\frac{f\alpha.F\alpha.f\beta.F\beta}{1 + e^2c^2\phi^2\alpha.\phi^2\beta} - \frac{2e^2c^2\phi^2\alpha.\phi^2\beta.f\alpha.f\beta.F\alpha.F\beta}{\Big(1 + e^2c^2\phi^2\alpha.\phi^2\beta\Big)^2} \\ &+ \frac{\phi\alpha.\phi\beta\Big(1 + e^2c^2\phi^2\alpha.\phi^2\beta\Big)\Big(-c^2F^2\alpha + e^2f^2\alpha\Big) - 2e^2c^2\phi\alpha.\phi\beta.\phi^2\beta.f^2\alpha.F^2\alpha}{\Big(1 + e^2c^2\phi^2\alpha.\phi^2\beta\Big)^2}. \end{aligned}$$

Substituting the values $1 - c^2\phi^2\alpha$ and $1 + e^2\phi^2\alpha$ for $f^2\alpha$ and $F^2\alpha$, and simplifying, we obtain

$$\frac{dr}{d\alpha} = \frac{\Big(1 - e^2c^2\phi^2\alpha.\phi^2\beta\Big)\Big((e^2 - c^2)\phi\alpha.\phi\beta + f\alpha.f\beta.F\alpha.F\beta\Big) - 2e^2c^2\phi\alpha.\phi\beta\Big(\phi^2\alpha + \phi^2\beta\Big)}{\Big(1 + e^2c^2\phi^2\alpha.\phi^2\beta\Big)^2}.$$

Now, clearly

$$\frac{dr}{d\alpha} = \frac{dr}{d\beta}, \tag{21}$$

since α and β enter symmetrically into the expression for r; and permuting α and β in the expression for $dr/d\alpha$ does not change its value.

Here, Abel writes

"This equation in partial differentials shows us that r is a function of $\alpha+\beta$; so we will have

$$r=\psi(\alpha+\beta).$$

The form of the function ψ will be found by giving a suitably chosen value to β."

However, he does not explain how to see this. Continuing the argument, suppose $\beta=0$. Since $\phi(0)=0$, $f(0)=1$, $F(0)=1$, we get

$$\begin{aligned} r &= \frac{\phi(\alpha)\cdot f(0)\cdot F(0)+\phi(0)\cdot f(\alpha)\cdot F(\alpha}{1+e^2c^2\phi^2(\alpha)\phi^2(0)} \\ &= \frac{\phi(\alpha)\cdot 1\cdot 1+0\cdot f(\alpha)\cdot F(\alpha)}{1+e^2c^2\phi^2(\alpha)\cdot 0} \\ &= \phi(\alpha). \end{aligned}$$

Hence, the desired function ψ is such that $\psi(\alpha)=\phi(\alpha)$, and so

$$r=\psi(\alpha+\beta)=\phi(\alpha+\beta),$$

and the first addition formula holds.

The other two formulae can be verified in a similar fashion.

From these, Abel deduced other formulae, such as

$$\phi(\alpha+\beta)+\phi(\alpha-\beta)=\frac{2\phi(\alpha)\cdot f(\beta)\cdot F(\beta)}{R},\quad \phi(\alpha+\beta)-\phi(\alpha-\beta)=\frac{2\phi(\beta)\cdot f(\alpha)\cdot F(\alpha)}{R},$$

and

$$\phi(\alpha+\beta)\phi(\alpha-\beta)=\frac{\phi^2(\alpha)-\phi^2(\beta)}{R},$$

where $R=1+e^2c^2\phi^2(\alpha)\phi^2(\beta)$; with similar such formulae for f and F.

He also found that

$$\phi(\alpha\pm\frac{\omega}{2})=\pm\frac{1}{c}\frac{f(\alpha)}{F(\alpha)},\quad \phi(\alpha\pm\frac{\tilde{\omega}}{2}i)=\pm\frac{i}{e}\frac{F(\alpha)}{f(\alpha)},$$

and hence

$$\phi(\frac{\omega}{2}+\alpha)=\phi(\frac{\omega}{2}-\alpha),\quad \phi(\frac{\tilde{\omega}}{2}i+\alpha)=\phi(\frac{\tilde{\omega}}{2}i-\alpha).$$

From these relations, he obtained

$$\phi(\alpha+\omega)=-\phi(\alpha)=\phi(\alpha+\tilde{\omega}i),$$

and

$$\phi(2\omega+\alpha)=\phi(\alpha)=\phi(2\tilde{\omega}i+\alpha)=\phi(\omega+\tilde{\omega}i+\alpha).$$

Again, similar such relations were found for the functions f and F, and thus the functions ϕ, f, F are *periodic*, such as in the case for ϕ:

$$\phi(m\omega+n\tilde{\omega}i\pm\alpha)=\pm(-1)^{m+n}\phi(\alpha).$$

In particular, we get

$$\phi(\alpha+2\omega)=\phi(\alpha),\quad \text{and} \quad \phi(\alpha+2i\tilde{\omega})=\phi(\alpha).$$

And so, one and the same function was found to have two distinct periods, unlike the circular functions which have one period such as $\sin(x+2\pi)=\sin(x)$. This fact allowed Abel to deduce many other interesting results about these elliptic functions, but double periodicity is one of their key properties.

5.2 Jacobi

In the same year of 1827 as Abel published his *Recherches*, an academic rivalry was born as Carl Gustav Jacob Jacobi entered the scene with his own ideas on the theory of elliptic integrals. He took a different approach from Abel's into the subject however, staying closer to Legendre in notation and direction. Jacobi's admiration for Legendre's work is showcased in a letter he wrote to Legendre dated to August 5th, 1827:[4]

> Sir,
>
> A young geometer dares to present to you some discoveries he has made in the theory of elliptic functions, to which he has been led by assiduous study of your great writings. It is to you, Sir, that this brilliant part of analysis owes the high degree of perfection to which it has been brought, and it is only by following in the footsteps of so great a master, that geometers will be able to go beyond the boundaries by which they had formerly been restricted. It is then to you that I must offer what follows as a sign of admiration and acknowledgement.

The letter proceeds with some of his early discoveries on transformations of elliptic integrals, which we shall not go over in this paper, focusing rather on some of his results in a later foundational work that would come to be the definitive account of the theory of elliptic functions for at least a generation, titled *Fundamenta nova functionum ellipticarum* (New foundations of elliptic functions), published in 1829. Given that the original work is written in Latin, we shall instead follow here brief summaries of the *Fundamenta nova* given by Kolmogorov and Yushkevich (1996), and Hancock (1917).

5.2.1 Fundamenta nova

Jacobi chose as his starting point the elliptic integral of the first kind, just as Abel had done, except writing it in the Legendre normal form, as

$$u = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.$$

The parameter k is called the *modulus* ($0 < k < 1$), and the variable ϕ the *amplitude*. Jacobi often used the abbreviation $F(k, \phi) := \int_0^\phi \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$ to denote the elliptic integral of the first kind, and by $F_1(k) := F(k, \frac{\pi}{2})$ denoted the case when $\phi = \frac{\pi}{2}$. The integral $F_1(k)$ is called a *complete* integral. And again, as Abel had done, Jacobi proposed to study the upper limit ϕ as a function of u, using the notation $\phi = am\, u$ for *amplitude of* u.

Since the substitution $x = \sin \phi$ yields

$$u = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

we obtain $x = \sin am\, u$ for x as a function of u. Along with this function, Jacobi introduced two other elliptic functions: $\cos \phi = \cos amu$ and $\Delta\phi = \Delta amu = \sqrt{1 - k^2 \sin^2 \phi}$.

[4]C. G. J. Jacobi to A. M. Legendre, August 5, 1827, in *Abel on Analysis: Papers on abelian and elliptic functions and the theory of series*, translated and edited by Philip Horowitz (Heber City: Kendrick Press, 2007), p. 528.

In 1838, C. Gudermann (teacher of Weierstrass) gave the simpler notation $sn\, u$, $cn\, u$, $dn\ u$ for the three elliptic functions, such that

$$x = \sin\phi = sn\ u,$$

$$\sqrt{1-x^2} = \cos\phi = cn\ u,$$
$$\sqrt{1-k^2x^2} = \Delta\phi = dn\ u.$$

From the above definitions, it follows at once that

$$sn^2\ u + cn^2\ u = 1, \quad \text{and} \quad dn^2\ u + k^2\ sn^2\ u = 1.$$

Jacobi then defines a real positive quantity K such that

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = F\Big(k, \frac{\pi}{2}\Big).$$

Jacobi also defined the *complementary modulus* $k' = \sqrt{1-k^2}$, and a corresponding quantity K' which is the same function of the complementary modulus k' as K is of the modulus k, i.e.

$$K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}} = F\Big(k', \frac{\pi}{2}\Big)$$

5.2.2 Double periodicity of sn(u), cn(u) and dn(u)

Jacobi then establishes the addition theorems and double periodicities of the functions $sn\ u$, $cn\ u$, and $dn\ u$ following the same route as taken by Abel. In another letter to Legendre, dated to January 12, 1828, he gives an outline of Abel's demonstration of double periodicity, but in his own notation. He writes:[5]

> Since my last letter, some researches of the greatest importance on the elliptic functions have been published by a young geometer, who perhaps is personally known to you. It is the first part of a memoir of M. Abel, of Christiania [...]. As I suppose that this memoir has not yet reached you, I shall tell you of the more interesting details. But, for greater convenience, I shall employ the notation I ordinarily use.

The argument that follows in the letter is very brief and leaves out many details, and while Jacobi gives a similar demonstration in the *Fundamenta nova*[6], the proof is unannotated and difficult to follow. A clever reconstruction of Jacobi's train of thought was given by Hancock[7], which we transcribe here using Gudermann's simplified notation.

First, to find the real periods of $sn\ u$, $cn\ u$, $dn\ u$ consider the integral

$$u = \int_{n\pi-\pi/2}^{n\pi} \frac{d\phi}{\Delta\phi},$$

[5]C. G. J. Jacobi to A. M. Legendre, January 12, 1828, in *ibid.*, p. 538.
[6]Jacobi (1829), §19.
[7]Hancock (1917), pp. 29-31.

where $\Delta\phi = \sqrt{1 - k^2 \sin^2 \phi}$, and n is a positive integer. Substituting $\phi = n\pi - \theta$, we get $d\phi = -d\theta$, with $0 < \theta < \pi/2$. After changing sign, we obtain the integral

$$u = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta\theta} = K.$$

Similarly, putting $\phi = n\pi + \theta$ in the integral

$$u' = \int_{n\pi}^{n\pi+\pi/2} \frac{d\phi}{\Delta\phi}$$

gives

$$u' = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta\theta} = K.$$

It follows that we can write

$$\int_0^{\frac{n\pi}{2}} \frac{d\phi}{\Delta\phi} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\phi}{\Delta\phi} + ... + \int_{(n-1)\frac{\pi}{2}}^{\frac{n\pi}{2}} \frac{d\phi}{\Delta\phi} = nK,$$

since each of the n pieces is equal to K. Thus,

$$am\, nK = \frac{n\pi}{2},$$

and since $am\ K = \frac{\pi}{2}$, we have

$$am\, nK = n\ am\ K.$$

Now, any angular distance α may be expressed in the form $\alpha = n\pi \pm \beta$, where $0 \leq \beta \leq \frac{\pi}{2}$, and we note that

$$\int_0^{n\pi+\beta} \frac{d\phi}{\Delta\phi} = \int_0^{n\pi} \frac{d\phi}{\Delta\phi} + \int_{n\pi}^{n\pi+\beta} \frac{d\phi}{\Delta\phi} = 2nK + u,$$

where $u = \int_{n\pi}^{n\pi+\beta} \frac{d\phi}{\Delta\phi}$. Thus for any angular distance α, we can write

$$\alpha = n\pi \pm \beta = am(2nK \pm u),$$

and

$$2n\ am\ K \pm am\ u = am(2nK \pm u).$$

Using this, we can now derive some formulas for the elliptic functions $sn\ u$, $cn\ u$, $dn\ u$.

$$sn(u \pm 2K) = \sin am(u \pm 2K) = \sin(amu \pm 2amK) = \sin(amu \pm \pi) = -\sin amu = -snu,$$

and

$$sn(u \pm 4K) = \sin am(u \pm 4K) = \sin(am\, u \pm 4am\ K) = \sin(am\, u \pm 2\pi) = sn\, u. \quad (22)$$

The formulas for $cn\ u$ and $dn\ u$ are shown in similar fashion to be

$$cn(u \pm 2K) = -cn\ u, \qquad cn(u \pm 4K) = cn\ u, \quad (23)$$

$$dn(u \pm 2K) = dn\, u, \qquad dn(u \pm 4K) = dn\, u.$$

We see then that $4K$ is a period of $sn\, u$, $cn\, u$ and $dn\, u$, with $2K$ also being a period of $dn\, u$.

Having found the real periods, we turn to the imaginary periods. First, Jacobi supposes the following relations

$$\sin\phi = i\tan\psi, \quad \cos\phi = \frac{1}{\cos\psi}, \quad \Delta(\phi, k) = \frac{\Delta(\psi, k')}{\cos\psi}, \tag{24}$$

where $\Delta(\phi, k) = \sqrt{1 - k^2\sin^2\phi}$. Then, $d\phi = i\frac{d\psi}{\cos\psi}$, from which he obtains

$$\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \frac{i\, d\psi}{\sqrt{1 - k'^2\sin^2\psi}}.$$

Then,

$$\int_0^\phi \frac{dt}{\sqrt{1 - k^2\sin^2 t}} = i\int_0^\psi \frac{dt}{\sqrt{1 - k'^2\sin^2 t}}$$

Letting

$$u = \int_0^\psi \frac{dt}{\sqrt{1 - k'^2\sin^2 t}},$$

such that $\psi = am(u, k')$, he gets

$$iu = \int_0^\phi \frac{dt}{\sqrt{1 - k^2\sin^2 t}},$$

and so $\phi = am(iu, k)$. Substituting $\phi = am(iu, k)$ and $\psi = am(u, k')$ into the relations in (24), he obtains the formulae

$$sn(iu, k) = i\, tn(u, k'),$$

$$cn(iu, k) = \frac{1}{cn(u, k')},$$

$$dn(iu, k) = \frac{dn(u, k')}{cn(u, k')},$$

where $tn(u, k') = \tan\, am\,(u, k')$ in our notation.

As a last step, we replace u by $u + 4K'$ in the above formulae, and use our findings in (22) and (23) to obtain

$$sn[i(u{+}4K'), k] = itn(u{+}4K', k') = i\frac{\sin am(u + 4K', k')}{\cos am(u + 4K', k')} = i\frac{sn(u, k')}{cn(u, k')} = itn(u, k').$$

But $i\, tn(u, k') = sn(iu, k)$, and so

$$sn(iu + 4iK', k) = sn(iu, k).$$

In similar fashion, we find that

$$cn(iu + 4iK', k) = cn(iu, k), \quad \text{and} \quad dn(iu + 4iK', k) = dn(iu, k).$$

Substituting iu for u, we get

$$sn(u \pm 4iK', k) = sn(u, k),$$

$$cn(u \pm 4iK', k) = cn(u, k),$$

$$dn(u \pm 4iK', k) = dn(u, k).$$

Furthermore, we also find that

$$\begin{aligned} sn(iu \pm 2iK', k) &= i\, tn(u + 2K', k') \\ &= i\, \frac{\sin am(u + 2K', k')}{\cos am(u + 2K', k')} \\ &= i\, \frac{-sn(u, k')}{-cn(u, k')} \\ &= i\, tn(u, k') \\ &= sn(iu, k). \end{aligned}$$

Again, changing iu to u and omitting the modulus k from our notation we get

$$sn(u \pm 2iK') = sn\, u.$$

From this and from (22), it follows that

$$sn(u \pm 4K \pm 2iK') = sn\, u.$$

Thus we find that $sn\, u$ is doubly periodic with the periods

$$4K = 4 \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad \text{and} \quad 2iK' = 2i \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}}.$$

In a similar fashion, one will find the periods for $cn\, u$ and $dn\, u$, and we can write more generally that

$$sn(u + 2mK + 2n\, iK') = (-1)^m sn\, u,$$

$$cn(u + 2mK + 2n\, iK') = (-1)^{m+n} cn\, u,$$

$$dn(u + 2mK + 2n\, iK') = (-1)^n dn\, u.$$

5.2.3 Jacobi Theta functions

Another important development found in the *Fundamenta Nova* is the *Jacobi theta functions.* If elliptic functions can be considered as analogs of circular functions, then theta functions are the elliptic analogs of the exponential functions. Whittaker and Watson (1990) note that while theta functions had appeared before Jacobi's time–the first such function to appear being the partition function $\Pi_{n=1}^{\infty}(1 - x^n z)^{-1}$ of Euler in *Introductio in Analysin Infinitorum, I* (Lausanne, 1748)–Jacobi was the first to study them systematically.

In the present work, we will not go into detail on the subject of these functions, as its domain is of such magnitude and extent that even a partial mapping of it would require its own proper introduction, including some preliminary background in complex analysis and Fourier series. We shall content ourselves with merely a brief glance as to how it relates to elliptic functions.

The first appearance of the theta function $\Theta(u)$ in *Fundamenta Nova* is in §52, where it is given by the formula

$$\Theta(u) = \Theta(0) \exp \int_0^u Z(t) dt,$$

where $Z(u)$ is defined in terms of complete and incomplete elliptic integrals of the first and second kinds as

$$Z(u) = \frac{F_1 E(\phi) - E_1 F(\phi)}{F_1 \Delta(\phi)},$$

where we recall that $F(\phi)$ is the elliptic integral of the first kind with modulus k omitted from the notation, $F_1 = F(\pi/2)$ is its complete integral, $E(\phi)$ is the elliptic integral of the second kind (which we saw very briefly in Section 4) and E_1 is its complete integral. The quantity $\Theta(0)$ here is an indeterminate constant.

The fundamental theta functions, $\Theta(u)$ and $\mathrm{H}(u)$, can be represented by the everywhere convergent trigonometric series

$$\Theta(u) = 1 - 2q \cos 2v + 2q^4 \cos 4v - 2q^9 \cos 6v + ...,$$

$$\mathrm{H}(u) = 2q^{1/4} \sin v - 2q^{9/4} \sin 3v + 2q^{25/4} \sin 5v - ...,$$

where $v = \pi u/2K$ and $q = e^{-\pi K'/K}$. Both functions are periodic, for instance Jacobi shows that:

$$\Theta(u + 2K) = \Theta(u),$$

however, they are not doubly periodic. Instead of an imaginary period $2iK'$, we get

$$\Theta(u + 2iK') = -\exp\left[\frac{\pi(K' - iu)}{K}\right]\Theta(u).$$

On the other hand, as a remarkable fact, the quotients

$$\frac{\mathrm{H}(u)}{\Theta(u)}, \quad \frac{\mathrm{H}(u+K)}{\Theta(u)}, \quad \frac{\Theta(u+K)}{\Theta(u)}$$

turn out to be doubly periodic functions! In fact, Jacobi shows that the elliptic functions $sn\ u$, $cn\ u$, $dn\ u$ can be expressed in terms of the fundamental theta functions by the formulas

$$sn\ u = \frac{1}{\sqrt{k}} \frac{\mathrm{H}(u)}{\Theta(u)}, \quad cn\ u = \sqrt{\frac{k'}{k}} \frac{\mathrm{H}(u+K)}{\Theta(u)}, \quad dn\ u = \sqrt{k'} \frac{\Theta(u+K)}{\Theta(u)}.$$

As noted by Kolmogorov and Yushkevich (1996), due to this relation the theta functions are valuable for obtaining numerical results in problems involving elliptic functions, the reason being that their trigonometric series representation converge very rapidly when u and k are real and k satisfies the condition $0 < k < 1$.

5.3 Gauss

Of course, as with any history of mathematics, it can not end without Gauss having left his footprints somewhere down the line. While Abel was the first to publish the idea of inverting elliptic integrals to obtain elliptic functions in 1827, with Jacobi publishing his own inversion two years later, Gauss had the idea in the late 1790s but did not publish it.

Gauss studied the lemniscatic integral in 1797 and defined the "lemniscatic sine function" $x = sl(u)$ as its inverse by

$$u = \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

He found that the function $sl(u)$ was periodic, like the sine, with period

$$2\bar{\omega} = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

Gauss also studied complex arguments of $sl(u)$, since it follows from $i^2 = -1$ that

$$\frac{d(it)}{\sqrt{1-(it)^4}} = i\frac{dt}{\sqrt{1-t^4}},$$

and hence $sl(iu) = isl(u)$ and the lemniscatic sine function has a second period $2i\bar{\omega}$.

Another discovery that Gauss cherished was a remarkable relationship between the *arithmetic-geometric mean function* (agM) and the period of the lemniscatic sine. The arithmetic-geometric mean of two positive real numbers x and y, written agM(x, y), is usually defined as follows.

Start with

$$\begin{cases} x = a_0, \\ y = g_0; \end{cases}$$

then iterate the sequences a_n and g_n defined by

$$\begin{cases} a_{n+1} = \frac{1}{2}(a_n + g_n), \\ g_{n+1} = \sqrt{a_n \cdot g_n}. \end{cases}$$

These two sequences then converge to the same number, which is the arithmetic-geometric mean of x and y, denoted by agM(x, y). The discovery that Gauss had made was that this seemingly unrelated function satisfied the following relation:

$$\text{agM}(1, \sqrt{2}) = \frac{\pi}{\bar{\omega}},$$

where $\bar{\omega}$ is a period of the elliptic function $sl(u)$!

Unfortunately, Gauss never published any these findings, choosing to remain silent until Abel's results appeared in 1827 – at which point he wrote to Bessel in 1828:[8]

> I shall most likely not soon prepare my investigations on the transcendental functions which I have had for many years – since 1798. [...] Herr Abel has now, as I see, anticipated me and relieved me of the burden in regard to one third of these matters.

[8] Stillwell (2010), p. 236.

6 Elliptic Curves

Jacobi's *Fundamenta nova* attracted great interest among brilliant young minds in the field of elliptic functions. In 1836, just seven years after its publication, the book came into the hands of young Karl Weierstrass, who was teaching himself mathematics. The book turned out to be too difficult for him, but perhaps it piqued his curiosity, as he enrolled in Münster Academy shortly after to be the only member of the audience for a series of lectures on elliptic functions held by Gudermann.

Another few years later, a young Gotthold Eisenstein picked up an interest in elliptic functions, and would soon teach classes on the subject at the University of Berlin, where among his students one would find Bernhard Riemann.

These men would soon take the theory of elliptic functions deeper into the realm of complex numbers, developing it alongside the complex analysis, and unlock remarkable geometric properties of these functions that would lay the groundwork for modern day algebraic geometry.

6.1 Eisenstein

Throughout the history of elliptic functions, we have seen that the trigonometric functions were a valuable source of inspiration. No less so for Eisenstein. He considered a formula given by Euler[9], namely

$$\frac{1}{\sin^2 z} = \sum_{m=-\infty}^{\infty} \frac{1}{(z+m\pi)^2},$$

where one may note that replacing z by $z + 2\pi$ on both sides does not change the sum. Eisenstein then replaced the single period $m\pi$ by the periods $m\omega_1$, $n\omega_2$ to construct the analogous series

$$\sum_{m,n=-\infty}^{\infty} \frac{1}{(z+m\omega_1+n\omega_2)^2},$$

where $\omega_1, \omega_2 \in \mathbb{C}$ and $\omega_1/\omega_2 \notin \mathbb{R}$. Eisenstein then argued that the series converges by a process now called *Eisenstein summation.* As noted by Weil[10], however, he was unaware of the concept of uniform convergence, and therefore assumed tacitly that the series could be differentiated term by term.

We may again note that this convergent series remains unchanged when z is replaced by $z + \omega_1$ or $z + \omega_2$. Indeed, as it turns out, the function defined by Eisenstein's series is an elliptic function, and as expected of an elliptic function is doubly periodic.

6.2 Weierstrass

While studying at Münster Academy to become a teacher, Weierstrass attended a course held by Christoph Gudermann in 1839-1840 on elliptic functions, the first such course to be taught at any institute. From that point on, elliptic functions became one of his lifelong interests, and he went on to make many discoveries that greatly enriched the subject, particularly with a view to applications in geometry.

[9]Shown to be a consequence of Euler's reflection formula in Andrews, et al. (1999).
[10]Weil (1976), p. 5

6.2.1 Weierstrass elliptic function

In 1863, Karl Weierstrass drew on Eisenstein's work to define perhaps the most famous elliptic function, the *Weierstrass $\wp$-function*:

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}.$$

The $\wp$-function, and all of its derivatives ($\wp'(z), \wp''(z)$, and so on) are doubly periodic with periods ω_1, ω_2. Furthermore, Weierstrass showed that any elliptic function with periods ω_1, ω_2 can be written as a rational function in terms of $\wp(z)$ and $\wp'(z)$.

By carrying out a Taylor series expansion about 0, Weierstrass was able to show the important identity concerning the $\wp$-function:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{25}$$

where

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4}$$

and

$$g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6}.$$

For a detailed proof, see for example Kirwan (1992), pp. 118-119. In view of (25), it is not hard to see that the point $(\wp(z), \wp'(z))$ lies on the cubic curve

$$y^2 = 4x^3 - g_2x - g_3,$$

where these constants g_2 and g_3 depend only on ω_1 and ω_2 and are thus determined by the elliptic function.

6.2.2 Weierstrass normal form

Using the $\wp$-function, Weierstrass was able to systematize the theory of elliptic integrals and their inverses in an entirely new way, beginning with his own classification of the three kinds of elliptic integrals, called *Weierstrass normal form*:

$$I_1 = \int \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}; \qquad \text{(first kind)}$$

$$I_2 = \int \frac{tdt}{\sqrt{4t^3 - g_2t - g_3}}; \qquad \text{(second kind)}$$

$$I_3 = \int \frac{dt}{(t-c)\sqrt{4t^3 - g_2t - g_3}}. \qquad \text{(third kind)}$$

He then went on to prove addition theorems, which for the elliptic integral of the first kind would be:

$$\int_0^{x_1} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} + \int_0^{x_2} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} = \int_0^{x_3} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}},$$

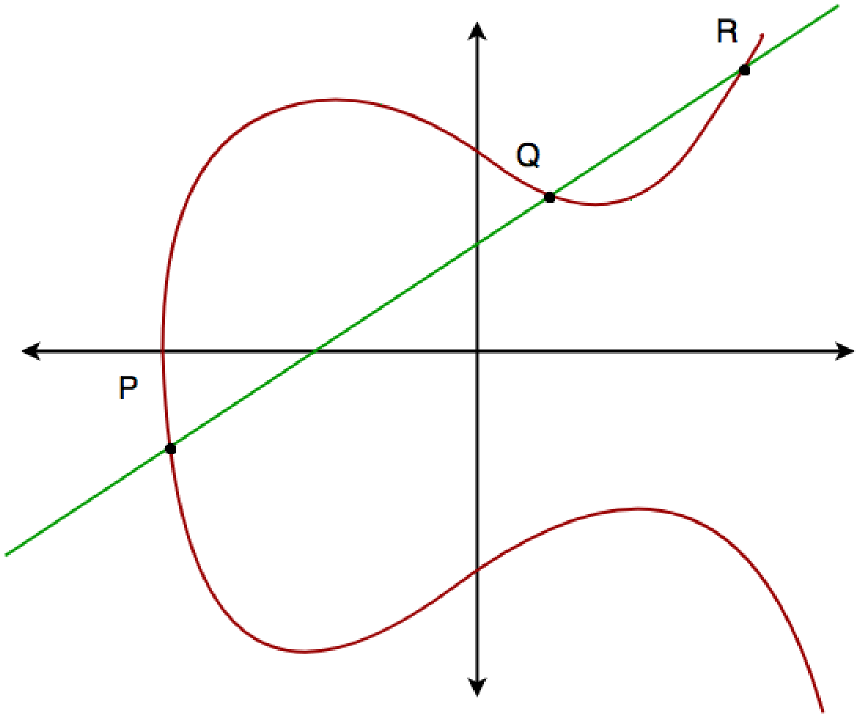

Figure 3: Collinear points on cubic curve $y^2 = 4x^3 - g_2x - g_3$

where x_3 is the x-coordinate of the point $R = (x_3, y_3)$ on the plane cubic

$$y^2 = 4x^3 - g_2x - g_3$$

of the straight line through $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, as shown in Figure 3.

As it turns out, Weierstrass' integral $\int dt/\sqrt{4t^3 - g_2t - g_3}$ with its remarkable geometric properties came to be the most convenient integral to use as a basis for the theory of elliptic functions. Its inverse, being the Weierstrass $\wp$-function, also parametrizes the curve $y^2 = 4x^3 - g_2x - g_3$.

6.2.3 Parameterization of cubic curves

As noted above, Weierstrass provides an identity in (25) that would suggest to us the idea that cubic curves of the form

$$y^2 = ax^3 + bx^2 + cx + d \tag{26}$$

could be parameterized as

$$x = f(z), \quad y = f'(z),$$

where f and its derivative f' are elliptic functions. This view was suggested by Jacobi (1834), and in 1864 German mathematician Alfred Clebsch introduced the idea of such a parameterization following the work of Eisenstein and Weierstrass. Finally, in 1901, Henri Poincaré tied these ideas together, effectively marking the birth of the study of *elliptic curves*, so named for requiring elliptic functions for their parameterization.

Now, suppose we have an elliptic curve such as (26) above, such that points (x, y) on the curve may be written as $f(z), f'(z)$, where f and its derivative f' are elliptic functions with periods ω_1, ω_2. This gives a map of the z plane $\mathbb{C}$ onto the curve (26) for which the preimage of a given point on the curve is a set of points in $\mathbb{C}$ of the form

$$z + \Lambda = \{z + m\omega_1 + n\omega_2 \ : \ m, n \in \mathbb{Z}\},$$

where

$$\Lambda = \{m\omega_1 + n\omega_2 \ : \ m, n \in \mathbb{Z}\}$$

is called a *lattice of periods of* f. This lattice can be seen as a partition of the complex plane into equivalence classes, generated by the periods ω_1 and ω_2, as in Figure 4 below.

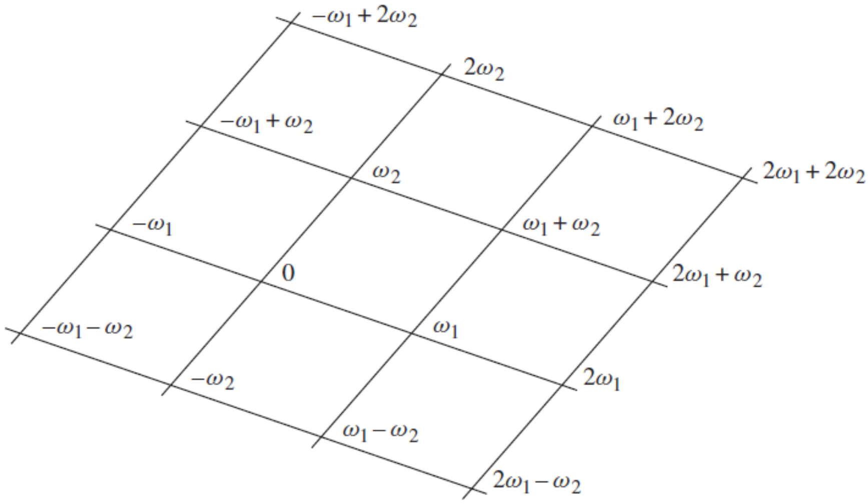

Figure 4: Lattice of periods

The parameterization

$$x = f(z), \quad y = f'(z)$$

is then a one-to-one correspondence between the points $(f(z), f'(z))$ on the elliptic curve and the equivalence classes $z + \Lambda$, in other words, the elliptic curve is *isomorphic* to the space $\mathbb{C}/\Lambda$. Topologically, this space is equivalent to a torus. To see this, we take one of the infinitely many parallellograms that make up the lattice, and mark its edges by A, B, C, and D as in Figure 5 below.

We then roll up the parallellogram into a cylinder, gluing together the edges A and C. Next, we stretch the cylinder and bend it in such a way that edges B and D can be glued together. The resulting surface is called a *torus*.

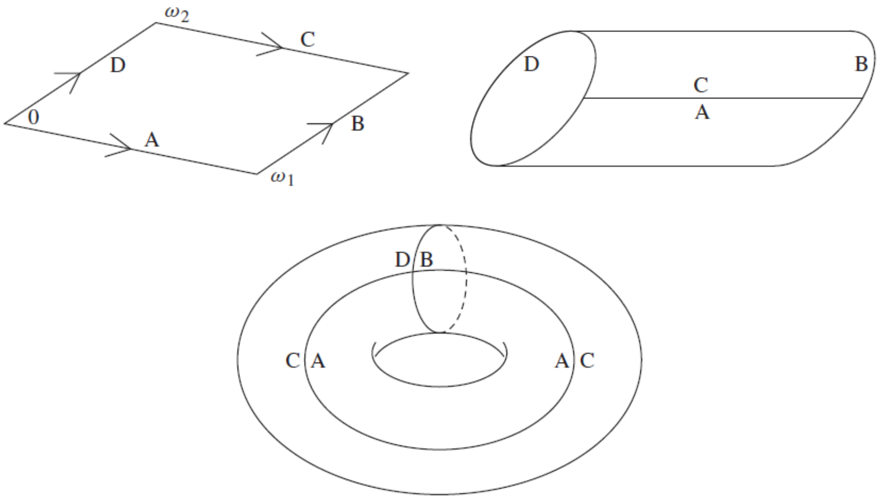

Figure 5: The topology of $\mathbb{C}/\Lambda$

7 Closing Thoughts

> Whoever wishes to foresee the future must consult the past; for human events ever resemble those of preceding times. This arises from the fact that they are produced by men who ever have been, and ever shall be, animated by the same passions, and thus they necessarily have the same results.
>
> *Machiavelli*

While the results in mathematics are clearly not the same across generations, we may observe a common trend after undertaking this journey from elliptic integrals to elliptic functions and the torus surface–namely, trigonometric functions as a source of inspiration. To us observers glancing back at the works of past mathematicians, a certain clarity becomes apparent that would not have been present in the mind of Jacob Bernoulli when he studied the lemniscate. We see now that the the idea inverting elliptic integrals to then use the elliptic functions to parameterize curves runs parallel with the idea of inverting the integral $\int dx/\sqrt{1-x^2}$ to obtain the trigonometric functions which can be used to conveniently parameterize the circle.

While Fagnano and Euler saw some similarities between the integrands $1/\sqrt{1-x^2}$ and $1/\sqrt{1-x^4}$, many other mathematicians at the time must have felt like the undergraduate student of today who would know how to integrate the former expression, but is told not to worry about the latter. Yet, despite the similarities, it did not seem to have occurred to Fagnano or Euler that inverting the lemniscatic integral would have been a good idea–or atleast, it must not have seemed significant to them.

It was not until Abel, Jacobi and Gauss that the link between trigonometric functions and elliptic functions was to be clearly seen. One can only guess at how far their insight reached without the techniques of complex analysis available to later mathematicians such as Eisenstein and Weierstrass, but it is clear that each of them understood that there were geometric ideas hiding in the elliptic functions as well.

Gauss, when he famously discovered a theorem on dividing the circle into n equal parts with ruler and compass, wrote in the *Disquisitiones arithmeticae*:[11]

> The principles of this theory which we are going to explain actually extend much further than we will indicate. For they can be applied not only to circular functions but just as well to other transcendental functions, e.g. to those which depend on the integral $\int(1/\sqrt{1-x^4})dx$.

Abel soon proved Gauss's suspicion when he discovered that the lemniscate can be divided into as many equal parts as the circle. In a letter to his teacher and friend Bernt Holmboe, he wrote:[12]

> Thus I have found that with ruler and compasses, one can divide the circumference of the lemniscate (polar equation $z=\sqrt{\sin 2\phi}$) into as many equal parts as can the circle, as Gauss has shown, e.g. into 17 parts.

One might pause to wonder, when a mathematician in today's world stumbles upon a curious coincidence or analogy that unites two mathematical problems, is there a deeper

[11]Gauss (1801), Art. 355.

[12]N. H. Abel to B. Holmboe, March 4, 1827, in Abel (2007), p. 504.

connection to be found beyond the coincidences? Are we as unaware of the underlying truths that are right in front of us as Euler and Fagnano were of inverting the lemniscatic integral?

> “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> *Joseph Fourier* ”

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