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Koszul algebras and formality

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Abstract

In this thesis we study formal differential graded algebras and coalgebras. As our tools we use theory of Koszul algebras and some homotopical algebra. We also give some examples of Koszul algebras, formal differential graded algebras and non-formal differential graded algebras from algebraic topology.

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Contents

1 Introduction and notation

The homotopy groups $\pi_n(\mathrm{X})$ of a topological space X play an important role in algebraic topology. They are however often very hard to compute. A standard example which illustrates this is the fact not even for spheres are there complete descriptions of the homotopy groups. In rational homotopy theory the groups $\pi_n(X) \otimes \mathbb{Q}$ are studied for simply connected spaces X. The idea is to forget about the torsion in order to get a more computable theory. This was made precise by Serre in [14] and it was in some sense a success. For example there are complete descriptions of the rational homotopy groups of spheres. Later, Quillen and Sullivan would come up with ways to model rational homotopy theory of simply connected spaces using differential graded Lie algebras and commutative differential graded algebras respectiely [15], [16]. But even the rational homotopy groups can be hard to compute which is why formal topological spaces are interesting. Formal topological spaces are topological spaces whose rational homotopy type is determined by the rational cohomology ring $H^\bullet(X;\mathbb{Q})$. There are some equivalent definitions of formal spaces but one is that the differential graded cochain algebra $C^\bullet(X)$ is connected to its cohomology by a zig zag of quasi isomorphisms. There is a similar notion of coformal spaces which are defined by the property that the algebra $C_\bullet(\Omega_b X)$ be connected to its homology by a zig zag of quasi isomorphisms (here $\Omega_b X$ denotes the based loops space of X and the algebra structure on $C_\bullet(\Omega_b X)$ comes from the monoidal structure on $\Omega_b X$). For this reason, a differential graded algebra with the property that it is connected to its (co)homology by a zig zag of quasi isomorphisms is called formal. These will be some of our main objects of study. More precisely we will examine how the notions of formality and koszulity are related in algebra and also in topology. This thesis consists of five chapters. In chapter one we introduce some terminology and theory that will be used in later chapters. The most important objects for us will be differential graded algebras and differential graded coalgebras (abbreviated dga algebras and dga coalgebras respectively). In chapter two we summarise some theory necessary for us to state and prove the main theorem of this thesis. Most of the theory in chapter two concerns how dga coalgebras and dga algebras are related; we construct the bar and cobar functor, we give the space of linear maps from a dga coalgebra to a dga algebra the structure of a dga algebra and we introduce the twisted tensor product of a dga coalgebra and a dga algebra. In chapter three we prove the following special case of theorem 2.9 in [1].

Theorem 1. Let $\kappa : C \to A$ be a Koszul twisting morphism where A is a connected dga algebra and C is a connected dga coalgebra. The following are equivalent:

(1) C and A are formal.

(2) A is formal and $H_*(A)$ is Koszul.

(3) C is formal and $H_*(C)$ is Koszul.

We do so without having to introduce operads or A_∞-algebras. Instead of introducing A_∞-algebras we spend section 3.2 establishing certain factorisation and lifting properties in the category of dga algebras to prove the following preliminary result.

Proposition 2. Let $\kappa : C \to A$ be a Koszul twisting morphism where A is a connected dga algebra and C is a connected dga coalgebra. If $C \sim C'$ and $A \sim A'$ then there is a Koszul twisting morphism $\kappa' : C' \to A'$.

In chapter four we explain how our main theorem connects to algebraic topology. We introduce the notion of Koszul spaces and give two examples of such, namely spheres and euclidean configuration spaces (where we assume the number of points to be less than or equal to the dimension). We follow a proof from [2] for the intrinsic rational formality of Euclidean configuration spaces but as a warm up example we show how the same ideas can be used to prove intrinsic rational formality of spheres.

In chapter five we give an example of a topological space whose cohomology ring is Koszul but which is not formal over $\mathbb{Z}_2$. The example is $F_4(\mathbb{R}^2)$ and the proof of its non-formality comes from [2].

1.1 Preliminary definitions

The aim of this section is to establish the notation which we will use throughout the thesis. First of all we will denote by $\mathbb{K}$ a field and the vector spaces, algebras and coalgebras will be over $\mathbb{K}$. Almost all objects we are interested in will be graded vector spaces with extra structure. So let us first make precise what we mean by a graded vector space.

Definition 3. A *graded vector space* is a vector space V with a direct sum decomposition

$$V = \bigoplus_{j \in \mathbb{Z}} V_j.$$

For a homogenous element $v \in V$ we denote by $|v|$ its degree.

Most of the graded vector spaces that occur in this thesis come with what we call a differential structure. Here is the precise definition.

Definition 4. A *differential graded vector space* (V, d), also called a *chain complex* is a graded vector space together with a differential $d : V \to V$ such that $d(V_i) \subset V_{i-1}$ for all i and $d^2 = 0$. A *morphism of chain complexes* $(V, d_V) \to (W, d_W)$, also called a *chain map*, is a linear map, homogenous of degree 0, which commutes with the differentials.

We recall that any chain complex gives rise to a new graded vector space which is smaller in some sense.

Definition 5. The *homology* of a chain complex is by definition the graded vector space $H_\bullet(V, d_V) := \oplus_{n\in\mathbb{Z}} ker(d : V_n \to v_{n-1})/im(d : V_{n+1} \to V_n)$. Any chain map induces a linear map in homology. A chain map which induces an isomorphism in homology is called a *quasi isomorphisms.*

We will think of $\mathbb{K}$ as a chain complex with trivial differential which is zero in all degrees except 0 where it is $\mathbb{K}$. (We say that it is concentrated in degree 0.) A differential graded vector space (V, d_V) is called *augmented* if there is a chain map from $(V, d_V) \to (\mathbb{K}, 0)$. An augmented vector space is called *acyclic* if the augmentation map is a quasi isomorphism. This means we might call a chain complex acyclic if either their homology vanishes or their homology consists of just one copy of $\mathbb{K}$ in degree zero (it should be clear from context which is meant).

Sometimes we will encounter chain complexes (V, d_V) with differentials of degree $+1$ rather than -1. We call them *cochain complexes* and we indicate their grading with superscript as in $V = \oplus_{j\in\mathbb{Z}} V^j$. We say that a chain complex is *homologically graded* and a cochain complex is *cohomologically graded*. The following convention will however allow us to restrict ourselves to the study of chain complexes.

Convention 6. We think of a cochain complex (V, d_V) as a chain complex which is homologically graded by $V_n := V^{-n}$. We note that with this convention $d_V : V_n = V^{-n} \to V^{-n+1} = V_{n-1}$ is of degree -1 with respect to the homological degree.

Now we move on to the notion of algebras and coalgebras.

Definition 7. An *associative algebra* is a vector space A equipped with a linear map $\mu : A \otimes A \to A$ such that the following diagram commutes

$$\begin{CD} A\otimes A\otimes A @>{1\otimes\mu}>> A\otimes A \\ @V{\mu\otimes 1}VV @VV{\mu}V \\ A\otimes A @>{\mu}>> A \end{CD}\ .$$

The algebra A is called *unital* if comes equipped with a linear map $u : \mathbb{K} \to A$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{K}\otimes A & \xrightarrow{u\otimes 1} & A\otimes A & \xleftarrow{1\otimes u} & A\otimes\mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle\mu} & \swarrow_{\cong} & \\ & & A & & \end{array}\ .$$

Notice that a unital algebra has an identity element $u(1_{\mathbb{K}})$ which we usually denote 1_A. The last diagram in the previous definitions shows that 1_A is indeed a two-sided identity for the multiplication.

Together with the notion of algebras comes a notion of structure preserving maps between algebras.

Definition 8. Let (A, μ) and (A', μ') be associative algebras. A linear map $f : A \to A'$ is called an *algebra morphism* if it respects the multiplication. In other words if the following diagram commutes

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ \downarrow \mu & & \downarrow \mu' \\ A & \xrightarrow{f} & A' \end{array} .$$

If A and A' are unital we also require that $f(1_A) = 1_{A'}$

We note that $\mathbb{K}$ is itself a unital associative algebra with the usual multiplication and the unit being the identity map $\mathbb{K} \to \mathbb{K}$.

Definition 9. An algebra A is called *augmented* if there is an algebra morphism $\epsilon : A \to \mathbb{K}$. One often denotes by $\bar{A}$ the kernel of the augmentation map ϵ.

A lot of the algebras we will be dealing with come with an extra graded structure. Here is the precise definition.

Definition 10. We say that the algebra A is *graded* if it has a vector space decomposition

$$A = \bigoplus_{j \in \mathbb{Z}} A_j$$

such that the multiplication μ respects this decomposition meaning $\mu(A_i \otimes A_j) \subset A_{i+j}$. A *morphism of graded algebras* $f : A \to A'$ is a morphism of algebras which respects the grading, meaning $f(A_j) \subset A'_j$.

Example 11. The polynomial ring $A = \mathbb{K}[x_1, ..., x_m]$ is an example of a graded algebra. The degree n part is the linear span of all monomials of degree n: $A_n = \langle x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k} | n_1 + n_2 + ... + n_k = n \rangle$.

When working in the graded setting one often has to deal with a lot of minus signs which can make computations much harder to follow. There is however a convention which can make things somewhat simpler called the Koszul sign convention:

Convention 12. Throughout this thesis we will, unless otherwise stated, define the tensor product of two linear maps $f : V \to V'$, $g : W \to W'$ by the rule

$$f \otimes g(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

We now define the type of algebras that we will mostly be interested in, namely algebras which are also chain complexes.

Definition 13. A *differential graded associative algebra* (*dga algebra* for short) (A, d) is a graded associative algebra (A, μ) together with a differential $d : A \to A$ which is a derivation for the product. In other words d is a linear map of degree minus one which satisfies

$$d^2 = 0, \text{ and } d \circ \mu = \mu \circ (d \otimes 1 + 1 \otimes d).$$

A *morphism of dga algebras* $f : A \to A'$ is a morphism of graded algebras which commutes with the differentials, meaning $f \circ d_A = d_{A'} \circ f$. A dga algebra A is called *augmented* if there is a morphism of dga algebras $A \to \mathbb{K}$ where $\mathbb{K}$ is thought of as a dga algebra concentrated in degree 0.

Because a dga algebra (A, d) is a chain complex if we just forget about the algebra structure we can of course take homology of a dga algebra. Because the differential is a derivation for the product the homology of a dga algebra inherits the structure of a graded algebra.

As with chain complexes, any morphism of dga algebras $f : A \to A'$ induces an algebra morphism $f_\bullet : H_\bullet(A) \to H_\bullet(A')$. We are particularly interested in those maps $f : A \to A'$ that induce isomorphisms on homology. As for chain complexes we will call such maps *quasi isomorphisms.*

Example 14. Consider the unital algebra $A = \mathbb{K}[x, y]/I$ where $I = (x^2, y^2)$. We can force a grading on it by specifying the degree of the generators. Setting $|x| = 0$ and $|y| = 1$ for example then forces $|xy| = |x| + |y| = 1$. Also the degree of 1 in any unital graded algebra must be zero because $|1_A| = |1_A \cdot 1_A| = |1_A| + |1_A|$. We see that with this grading we get $A_0 = \langle 1, x\rangle$, $A_1 = \langle y, xy\rangle$ and $A_i = 0$ for all other i. We can define a differential by specifying what it does on the generators, $d : y \mapsto x$ and $d : x \mapsto 0$, and then use the Leibniz rule from definition 13 to extend this to any product of the generators (one also has to check that $d(I) \subset I$). For example $d(xy) = d(x)y + xd(y) = 0 + x^2 = 0$. This makes A into a dga algebra. We saw that A is concentrated in degrees 0 and 1 and as a chain complex it looks like

$$... \xleftarrow[d]{} 0 \xleftarrow[d]{} \mathbb{K}1_A \oplus \mathbb{K}x \xleftarrow[d]{} \mathbb{K}y \oplus \mathbb{K}xy \xleftarrow[d]{} 0 \xleftarrow[d]{} ...$$

and it is not so hard to compute its homology, $H_0(A) = \mathbb{K}1_A$, $H_1(A) = \mathbb{K}xy$. As a graded algebra $H_\bullet(A)$ is the trivial algebra on one generator xy of degree 1.

Sometimes a graded algebra A comes equipped with an extra grading which we will call weight. Such algebras we will call *weight graded.* The multiplication of A must respect both the original grading and this extra weight grading. We will require weight gradings to be concentrated in non negative weight. For an element $a \in A$ of a weight graded algebra we will denote by $|a|$ the degree and by $w(a)$ the weight of a. A weight graded dga algebra is often abbreviated wdga algebra.

An other type of algebraic objects that will occur frequently in this thesis are coalgebras. They are in a sense dual to algebras.

Definition 15. A *coalgebra* is a vector space C equipped with a linear map $\Delta : C \to C \otimes C$, called the coproduct, we call C an *coassociative coalgebra* if the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes 1} & C \otimes C \otimes C \end{array}$$

and we say it is counital if it is equipped with a linear map $\epsilon : C \to \mathbb{K}$ making the following diagram commute.

$$\begin{array}{ccccc} & & C & & \\ & \swarrow \cong & \downarrow \Delta & \searrow \cong & \\ C \otimes \mathbb{K} & \xleftarrow[1\otimes\epsilon]{} & C \otimes C & \xrightarrow{\epsilon\otimes 1} & \mathbb{K} \otimes C \end{array}.$$

A *morphism of coalgebras* $f : C \to C'$ is a linear map which commute with the coproduct, meaning $\Delta_{C'} \circ f = (f \otimes f) \circ \Delta_C$. If the coalgebras C and C' are counital with counits $u : C \to \mathbb{K}$ and $u' : C' \to \mathbb{K}$ we also require that f commutes with the counits, meaning $u' \circ f = u$.

We note that $\mathbb{K}$ is itself a counital associative coalgebra with multiplication defined by $1 \mapsto 1 \otimes 1$ and the counit being the identity $\mathbb{K} \to \mathbb{K}$.

Definition 16. A counital coalgebra C is called *coaugmented* if there is a coalgebra morphism $u : \mathbb{K} \to C$. We denote by 1_C the image of 1 under u.

We note that if C is coaugmented we must have $\epsilon \circ u = id_{\mathbb{K}}$ because u is a coalgebra morphism which means that it commutes with the counits but the counit of $\mathbb{K}$ is $id_{\mathbb{K}}$. Moreover there is a natural way to define a coalgebra structure on $ker(\epsilon)$ namely by $\bar{\Delta}(x) = \Delta(x) - 1_C \otimes x - x \otimes 1_C$. If Δ is coassociative then $\bar{\Delta}$ is too.

Definition 17. We say that a coaugmented coalgebra C is conilpotent if for all $c \in \bar{C}$ there is an integer n such that $\bar{\Delta}^n(c) = 0$ where $\bar{\Delta}^n$ is defined inductively as $\bar{\Delta}^n(c) := (\bar{\Delta} \otimes 1^{\otimes n-1}) \circ \bar{\Delta}^{n-1}(c)$.

As with algebras we will often encounter coalgebras with a graded structure. Here is the precise definition.

Definition 18. A *graded coalgebra* is a coalgebra C which has a vector space decomposition

$$C = \bigoplus_{j \in \mathbb{Z}} C_j$$

such that the coproduct respects the grading, meaning

$$\Delta(C_n) \subset \bigoplus_{i+j=n} C_i \otimes C_j.$$

A *morphism ofgraded coalgebras* $f : C \to C'$ is a morphism of coalgebras which respects the grading, meaning $f(C_i) \subset f(C_i')$ for all i.

Definition 19. A *differential graded associative coalgebra* (*dga coalgebra* for short) (C, d) is a graded associative coalgebra (C, Δ) together with a differential $d : C \to C$ which is a coderivation for the coproduct. In other words d is a linear map of degree -1 which satisfies

$$d^2 = 0 \text{ and } \Delta \otimes d = (d \otimes 1 + 1 \otimes d) \circ \Delta.$$

Example 20. If we take the linear dual of the algebra A from example 14 we get a coalgebra $C = A^*$ whose coproduct is given by the composite

$$A^* \xrightarrow{\mu_A^*} (A \otimes A)^* \cong A^* \otimes A^* \ .$$

(The isomorphism above exists since $dim(A) < \infty$.) We can give C a grading by $C_0 = \langle 1_A^*, x^* \rangle$, $C_{-1} = \langle y^*, (xy)^* \rangle$ where we have fixed the basis of A^* dual to the one in example 14. Then we can compute for example the coproduct of $(xy)^*$ by applying $\mu^*((xy)^*)$ to the basis elements of $A \otimes A$ obtained by taking tensor product of basis elements of A. We get

$$\mu_A^*((xy)^*)(1_A \otimes xy) = (xy)^*(\mu_A(1 \otimes xy)) = (xy)^*(xy) = 1,$$

$$\mu_A^*((xy)^*)(x \otimes y) = (xy)^*(\mu_A(x \otimes y)) = (xy)^*(xy) = 1,$$

$$\mu_A^*((xy)^*)(xy \otimes 1_A) = (xy)^*(\mu_A(xy \otimes 1_A)) = (xy)^*(xy) = 1,$$

and then the rest is zero because no other product of basis elements of A contain xy as a term so we get $\mu_A^*((xy)^*) = 1_A^* \otimes (xy)^* + x^* \otimes y^* + (xy)^* \otimes 1_A^*$. We also get a differential on C by taking the dual of the differential d in example 12. Explicitly it is given by

$$d^* : 1_A^* \mapsto 0,\ x^* \mapsto y^*,\ y^* \mapsto 0,\ (xy)^* \mapsto 0.$$

Using that μ_A is associative and that d is a derivation for μ_A one can check that μ_A^* is coassociative and that d^* is a coderivation for μ_A^* so (C, d^*) is a dga coalgebra.

Finally a *connected* (co)algebra is a non-negatively graded (co)algebra such that $A_0 = \mathbb{K}$ ($C_0 = \mathbb{K}$). A weight graded (co)algebra is connected with respect to weight if $A(0) = \mathbb{K}$ ($C(0) = \mathbb{K}$).

Next we introduce two functors that assign to any vector space V an algebra and a coalgebra respectively. The algebra is usually denoted $T(V)$ and called the tensor algebra and the coalgebra is denoted $T^c(V)$ and called the tensor coalgebra. Each of these satisfy a universal property that will come in handy in later chapters. For the proofs of the properties of the tensor (co) algebra we refer to [3].

Definition 21. Given a vector space V the *tensor algebra* of V is an algebra whose underlying vector space is

$$T(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

The elements of $T(V)$ are sums of elements of the form $v_1 \otimes v_2 \otimes ... \otimes v_n$. It is however customary to omit the tensor sign and denote this element by $v_1 \cdots v_n$. The multiplication is given by concatenating words meaning

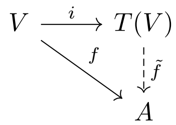
$$v_1 \cdots v_n \otimes u_1 \cdots u_m \mapsto v_1 \cdots v_n u_1 \cdots u_m.$$

As we mentioned the tensor algebra is a functor from the category of vector spaces to the category of augmented associative algebras; it assigns to any linear map $f : V \to W$ the algebra morphism

$$\oplus_{n \geq 0} f^{\otimes n} : T(V) \to T(W).$$

It is not so hard to check that this is indeed an algebra morphism and that this assignment respects the identity and the composition of maps. Now we move on to the universal property of the tensor algebra.

Proposition 22. Let V be a vector space. The tensor algebra $T(V)$ of V satisfies the following universal property: For any unital associative algebra A and linear map $f : V \to A$ there is an algebra morphism $\tilde{f} : T(V) \to A$ making the following diagram commute



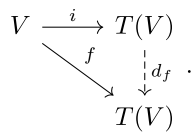
where i is the inclusion $V \hookrightarrow T(V)$.

We recall that if V and W are two graded vector spaces then $V \otimes W$ is graded too by

$$(V \otimes W)_n = \oplus_{i+j=n} V_i \otimes W_j.$$

This way when V is a graded vector space then $T(V)$ is graded too and moreover the multiplication on $T(V)$ respects this grading so in this case $T(V)$ is a graded algebra with the grading induced from V. Here is a result which allows us to uniquely extend any linear map $V \to T(V)$ to a derivation $T(V) \to T(V)$. We state the graded version here.

Proposition 23. Let V be a graded vector space. For any linear map $f : V \to T(V)$ of degree -1 there is unique derivation $d_f : T(V) \to T(V)$ which makes the following diagram commute



The corresponding construction for coalgebras goes as follows.

Definition 24. Given a vector space V the tensor coalgebra of V is the coalgebra whose underlying vector space is

$$T^c(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

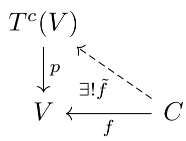
with comultiplication defined by deconcatenation of words meaning

$$v_1 \cdots v_n \mapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n \in T^c(V) \otimes T^c(V)$$

.

The tensor coalgebra is a functor from the category of vector spaces to the category of coaugmented associative coalgebras. It assigns to any linear map $f : V \to W$ the map $\oplus_{n\geq 0} f^{\otimes n} : T^c(V) \to T^c(W)$. Again it is not so hard to check that this is a coalgebra morphism and that this assignment respects the identity and the composition of maps. The tensor coalgebra also satisfies a universal property. It is in fact dual to the one the tensor algebra satisfied in the sense that all the arrows are just reversed.

Proposition 25. Let V be a vector space. The tensor coalgebra $T^c(V)$ satisfies the following universal property: For any conilpotent coalgebra C, and linear map $f : C \to V$ with $f(1_C) = 0$ there is a unique morphism of coaugmented coalgebras $\tilde{f} : C \to T^c(V)$ making the following diagram commute



where p is the projection $T^c(V) \twoheadrightarrow V$.

1.2 Simplicial sets

We end this first chapter with a short introduction to simplicial sets. This will mostly be used in chapter five but since simplicial sets give rise to natural examples of dga algebras we include it here. Denote by Δ the category whose objects are the sets $[n] := 0, 1, ..., n$ for $n \geq 0$ and whose morphisms are functions $f : [n] \to [m]$ such that $i \leq j \implies f(i) \leq f(j)$.

Definition 26. A *simplicial set* is a contravariant functor $X : \Delta \to Sets$. The elements of $X([n])$ are called *n-simplices*

Consider the following morphisms in Δ

$$d^j : [n-1] \to [n],\ d^j(i) = \begin{cases} i, \text{ if } i < j \\ i+1, \text{ if } i \geq j \end{cases}$$

$$s^j : [n+1] \to [n],\ s^j(i) = \begin{cases} i \text{ if } i \leq j \\ i-1, \text{ if } i > j \end{cases} .$$

Given a simplicial set X we will denote by d_j and s_j the set-functions $X(d^j)$ and $X(s^j)$ respectively. These functions are called face and degeneracy maps respectively. An element $x \in X([n])$ is called degenerate if $x = s_j(y)$ for some j and some $y \in X([n-1])$.

The set-functions d_j, s_i satisfy the following identities [5]

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases}.$$

Given a simplicial set X we can define a chain complex $C_\bullet(X)$ over any field $\mathbb{K}$ as follows. In degree n we have $\hat{C}_n(X) = \mathbb{K}X([n])$ is the free $\mathbb{K}$-vector space on the set $X([n])$. The differential is given by

$$\partial : \hat{C}_n(X) \to \hat{C}_{n-1}(X),\ x \mapsto \sum_{i=0}^{n} (-1)^i d_i(x).$$

Using the identities above one can see that $\partial^2 = 0$. Also using the identities above one can show that the subspaces $D_n^X \subset \hat{C}_n^X$ spanned by the degenerate elements of $X([n])$ form a sub complex and we define the *normalised chain complex* of X to be

$$C_\bullet(X) = \hat{C}_\bullet(X)/D_\bullet(X).$$

We will however be more interested in the dual cochain complex $C^\bullet(X) := (\oplus_n Hom(C_n(X), \mathbb{K}), \partial^*)$. The cochain complex $C^\bullet(X)$ has a product, called the cup product. Let $f \in C^p(X),\ g \in C^q(X),\ x \in X([p+q])$ and let

$$\iota : [p] \to [p+q],\ i \mapsto i$$

and

$$\eta : [q] \to [p+q],\ i \mapsto p+i.$$

Then the cup product can be defined by the following formula

$$f \cup g(x) = (-1)^{pq}(f \circ X(\iota)(x)) \cdot (g \circ X(\eta)(x)).$$

Directly from these formulas one can show the cup product is associative and that the following formula holds

$$\partial^*(f \cup g) = \partial^*(f) \cup g + (-1)^{|f|} f \cup \partial^*(g).$$

In other words $(C^\bullet(X), \cup, \partial^*)$ is dga algebra.

There is another multiplication that we will use which is of degree -1 $\cup_1 C^p(X) \otimes C^q(X) \to C^{p+q-1}(X)$. If f, g are as before and $y \in X([p+q-1])$ and if we for $j \in \{0, 1, ..., p-1\}$ define

$$\iota^j : [p] \to [p+q-1],\ i \mapsto \begin{cases} i \text{ if } i \leq j \\ i+q-1 \text{ if } i > j \end{cases}$$

$$\eta^j : [q] \to [p+q-1],\ i \mapsto i+j$$

then $\cup_1$ can be defined by

$$f \cup_1 g(y) = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} (f \circ X(\iota^j)(y)) \dot{(} g \circ Y(\eta^j)(y)).$$

The definition goes back to Steenrod [13] and he also proved that the following formula hold.

Proposition 27.

$$\partial^*(f \cup_1 g) = \partial^*(f) \cup g + (-1)^p f \cup \partial^*(g) + (-1)^{p+q-1} f \cup g + (-1)^{pq+p+q} g \cup f.$$

Example 28. An example of a simplicial set is the set of singular simplices of a topological space T. It is defined on objects by

$$S(T)([n]) = \{\sigma : \Delta^n \to T : \sigma \text{ is continuous}\}.$$

It is defined on morphisms by
$S(T)(h : [n] \to [m])(\sigma) = \sigma \circ \langle x_{h(0)}, x_{h(1)}, ..., x_{(h(m))} \rangle$ where $\langle x_{h(0)}, x_{h(1)}, ..., x_{(h(m))} \rangle : \Delta_n \to \Delta_m$ is the map sending a point $(t_0, ..., t_n) \in \Delta_n$ to $\sum_{i=0}^n t_i e_{h(i)}$ and $\{e_i\}$ is the standard basis of $\mathbb{R}^m$. The face are given explicitly by

$$d_i : S_n(X) \to S_{n-1}(X),\ \sigma \mapsto \sigma \circ \langle x_0, ..., \hat{x_i}, ..., x_n \rangle$$

where $\hat{x_i}$ means we omit x_i. The degeneracy maps are given by

$$s_i : S_n(X) \to S_{n+1},\ \sigma \mapsto \sigma \circ \langle x_0, ..., x_i, x_i, ..., x_n \rangle.$$

We can then define the normalised singular chain complex of T with coefficients in some field $\mathbb{K}$ as we did for a general simplicial set above $C_\bullet(T) := C_\bullet(S(T))$. It gives rise to the homology of T

$$H_\bullet(T; \mathbb{K}) := H_\bullet(C_\bullet(S(T))).$$

We can further define the cochain algebra of the topological space T with coefficients in $\mathbb{K}$ by

$$C^\bullet(T; \mathbb{K}) := C^\bullet(S(T))$$

which gives rise to the cohomology ring of T

$$H^\bullet(T; \mathbb{K}) := H^\bullet(C^\bullet(S(T))).$$

A *map between simplicial sets* is a natural transformation $X \to Y$. One can show that this is equivalent to a family of maps $X([n]) \to Y([n])$ that commute with the face and degeneracy maps. A sub simplicial set $Z \subset X$ is a simplicial set such that $Z([n]) \subset X([n])$ for all n and $Z(f : [n] \to [m]) = X(f : [n] \to [m])|_{Z([m])}$. Given a map of simplicial sets $\phi = \{\phi_n : X([n]) \to Y([n])\}$ and a sub simplicial set $Z \subset Y$ the inverse image $\phi^{-1}(Z)$ is a sub simplicial set of X in a natural way. Also the intersection of two sub simplicial sets $Z \subset X$ and $Y \subset X$ is a simplicial set in a natural way by $Z \cap Y([n]) := Z([n]) \cap Y([n])$ and $Z \cap Y(f : [n] \to [m]) = X(f : [n] \to [m])|_{Z([m]) \cap Y([m])}$.

2 Bar construction, cobar construction & twisting morphisms

In this chapter we study the relation between dga algebras and dga coalgebras further. We introduce the convolution algebra and twisted tensor products. We also study the bar and cobar adjunction. The theory in this chapter can be found in chapter two of [3]. I have decided to only go into detail on those proofs that I find especially interesting or that are only sketched in [3]. In this chapter the algebras (coalgebras) will assumed to be augmented (coaugmented) and concentrated in non-negative degrees.

2.1 Bar and cobar construction

There are functors going from from the category of dga algebras to the category of dga coalgebras and vice versa. These functors will occur a lot in this thesis so let us give explicit descriptions of them and prove some properties that they enjoy.

Starting with a dga algebra (A, μ, d_A) the bar construction of A is the coalgebra $T^c(\mathbb{K}s \otimes \bar{A})$ where $\mathbb{K}s$ is a one dimensional graded vector space concentrated in degree 1. We use the notation $s\bar{A} := \mathbb{K}s \otimes \bar{A}$. We will define two differentials d_1 and d_2 on A and show that their sum $d_1 + d_2$ is a differential as well.

The first differential d_1 comes from d_A. Indeed, we can define a differential $d_1^{(1)}$ on $s\bar{A}$ by $sa \mapsto -sd_A(a)$. We can then take the tensor product of this to get differentials

$$d_1^{(n)} = \sum_i 1 \otimes ... \otimes d_1^{(1)} \otimes ... \otimes 1 : (s\bar{A})^{\otimes n} \to (s\bar{A})^{\otimes n}.$$

We can then define d_1 to be the direct sum of all the $d_1^{(i)}$'s ($d^{(0)} = 0$) which is then a differential on $T^c(V)$.

The second differential d_2 is induced by the product in A. In formulas we have

$$d_2(sa_1 \otimes ... \otimes sa_n) = \sum_i (-1)^{i-1+|a_1|+|a_2|+...+|a_n|} sa_1 \otimes ... \otimes s\mu(a_i, a_{i+1}) \otimes ... \otimes sa_n.$$

Proposition 29. d_1 and d_2 are indeed differentials on the graded coalgebra $T^c(s\bar{A})$. Moreover, they anti commute so that their sum $d_1 + d_2$ is again a differential.

Proof. To prove that d_1 and d_2 are coderivations for the coproduct one just has to write out the formulas. The fact that d_1 squares to zero follows from d_A being a differential. To see that $d_2^2 = 0$ one has to write out the formulas, keeping close attention to the signs and use the associativity of μ. Finally, proving that $d_1 \circ d_2 + d_2 \circ d_1 = 0$ also just comes down to writing out the formulas and using the fact that d_A is a derivation for μ. ☐

We can now conclude that $BA = (T^c(s\bar{A}), d = d_1 + d_2)$ is a differential graded coalgebra. To see that B is in fact a functor we have to say what it does on morphisms. If $f : A \to A'$ is a morphism of augmented dga algebras then $f(\bar{A}) \subset \bar{A}'$. Then we can define a linear map $f_s : s\bar{A} \to s\bar{A}'$ by $sa \mapsto sf(a)$. But then we can take tensor powers of f_s to get linear maps $f_s^{\otimes n} : (s\bar{A})^{\otimes n} \to (s\bar{A}')^{\otimes n}$. Finally we take the direct sum of of these maps to get Bf :

$$Bf := \bigoplus_{n \geq 0} f_s^{\otimes n} : T^c(s\bar{A}) \to T^c(s\bar{A}').$$

It is not so hard to check that Bf is a coalgebra morphism that commutes with the differential and that $B1_A = B1_{BA}$ and $B(f \circ g) = Bf \circ Bg$ when f and g are composable.

Now we move on to the cobar functor Ω. So let (C, Δ, d_C) be a coaugmented dga coalgebra. As an associative graded algebra we have

$$\Omega C := T(s^{-1}\bar{C})$$

where $s^{-1}\bar{C}$ is short for $\mathbb{K}s^{-1} \otimes \bar{C}$. $\mathbb{K}s^{-1}$ being the one dimensional vector space concentrated in degree -1. As with B we have two differentials on Ω, let us call them δ_1 and δ_2. The first one comes from the original differential on C. First we get a differential $\delta_1^{(1)} : s^{-1}\bar{C} \to s^{-1}\bar{C}$ by $sc \mapsto -sd_C(c)$. Then we get differentials

$$\delta_1^{(n)} = \sum 1 \otimes ... \otimes \delta_1^{(1)} \otimes ... \otimes 1 : (s^{-1}\bar{C})^{\otimes n} \to (s^{-1}\bar{C})^{\otimes n}$$

for all n. Finally we can take the direct sum of all of these to get

$$\delta_1 = \bigoplus_{i \geq 0} \delta_1^{(i)} : \Omega C \to \Omega C.$$

The other differential δ_2 is induced by the coproduct on C. More precisely, we can define it as follows. Let $\Delta_s : \mathbb{K}s^{-1} \to \mathbb{K}s^{-1}$ be the map defined by $s^{-1} \mapsto -s^{-1} \otimes s^{-1}$. Let $\tau : \mathbb{K}s^{-1} \otimes \bar{C} \to \bar{C} \otimes \mathbb{K}s^{-1}$ be the map defined by $s^{-1} \otimes c \mapsto (-1)^{|c|} c \otimes s^{-1}$ and let $\bar{\Delta}$ be the reduced coproduct in C. Then we consider the following composition

$$\begin{array}{ccc} \mathbb{K}s^{-1} \otimes \bar{C} & \xrightarrow{\Delta_s \otimes \bar{\Delta}} & \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1} \otimes \bar{C} \otimes \bar{C} \\ & \swarrow_{1 \otimes \tau \otimes 1} & \\ \mathbb{K}s^{-1} \otimes \bar{C} \otimes \mathbb{K}s^{-1} \otimes \bar{C} & \hookrightarrow & T(s^{-1}\bar{C}) \end{array}.$$

This composition is a linear map $\mathbb{K}s^{-1}\bar{C} \to T(\mathbb{K}s^{-1}\bar{C})$. It has degree -1 because Δ_s does and all the other maps have degree 0. Hence by proposition 23 it extends uniquely to a derivation $\delta_2 : T(\mathbb{K}s^{-1}\bar{C}) \to T(\mathbb{K}s^{-1}\bar{C})$.

Proposition 30. δ_1 and δ_2 are indeed differentials on the graded algebra $T(s^{-1}\bar{C})$. Moreover, the two differential δ_1 and δ_2 anti commute so that their sum $\delta_1 + \delta_2$ is again a differential.

Proof. To see that δ_1 is a derivation one just has to write out the formulas. After showing this, the fact that δ_1 squares to zero follows from the fact that d_C does. We know already that δ_2 is derivation so let us prove that it squares to zero. Let $s^{-1}c$ be an element of $s^{-1}\bar{C}$ and let

$$\bar{\Delta}(c) = \sum_i a_1^i \otimes a_2^i.$$

We then note that the linear map $s^{-1}\bar{C} \to T(s^{-1}\bar{C})$ we used to define δ_2 is given explicitly by

$$s^{-1}\bar{C} \ni s^{-1}c \mapsto -\sum_i (-1)^{|a_1^i|} s^{-1}a_1^i \otimes s^{-1}a_2^i.$$

Now let us call $\bar{\Delta}(a_1^i) = \sum_j b_1^{ij} \otimes b_2^{ij}$ and $\bar{\Delta}(a_2^i) = \sum_j f_1^{ij} \otimes f_2^{ij}$. Then, using the formula for $\delta_2(s^{-1}c)$ from above, the fact that δ_2 is a derivation and the fact that $\bar{\Delta}$ respects the grading of $\bar{C}$ we get

$$\delta_2^2(s^{-1}c) =$$

$$\sum_{ij} (-1)^{|b_2^{ij}|} s^{-1}b_1^{ij} \otimes s^{-1}b_2^{ij} \otimes s^{-1}a_2^i + \sum_{ij} (-1)^{|f_1^{ij}|-1} s^{-1}a_1^i \otimes s^{-1}f_1^{ij} \otimes s^{-1}f_2^{ij}.$$

But since $\bar{\Delta}$ is coassociative we know $\sum_{ij} s^{-1}b_1^{ij} \otimes s^{-1}b_2^{ij} \otimes s^{-1}a_2^i = \sum_{ij} s^{-2}a_1^i \otimes s^{-1}f_1^{ij} \otimes s^{-1}f_2^{ij}$ and since

$$(s^{-1}\bar{C})^{\otimes 3} = \bigoplus_{n,r,t\in\mathbb{Z}} s^{-1}\bar{C}_n \otimes s^{-1}\bar{C}_r \otimes s^{-1}\bar{C}_t$$

is a direct sum we see that the components, on each side of the equality, belonging to $\bar{C}_n \otimes \bar{C}_r \otimes \bar{C}_t$, must equal for all $n, r, t \in \mathbb{Z}$. But then, if we fix n, r and t and only consider the terms in the two sums

$$\sum_{ij} (-1)^{|b_2^{ij}|} s^{-1}b_1^{ij} \otimes s^{-1}b_2^{ij} \otimes s^{-1}a_2^i + \sum_{ij} (-1)^{|f_1^{ij}|-1} s^{-1}a_1^i \otimes s^{-1}f_1^{ij} \otimes s^{-1}f_2^{ij}$$

belonging to $\bar{C}_n \otimes \bar{C}_r \otimes \bar{C}_t$ we have $|b_2^{ij}| = |f_1^{ij}| = r$ so if we still only consider the terms belonging to $\bar{C}_n \otimes \bar{C}_r \otimes \bar{C}_t$ we get

$$\sum_{ij} (-1)^r s^{-1}b_1^{ij} \otimes s^{-1}b_2^{ij} \otimes s^{-1}a_2^i + \sum_{ij} (-1)^{r-1} s^{-1}a_1^i \otimes s^{-1}f_1^{ij} \otimes s^{-1}f_2^{ij} =$$

$$(-1)^r \sum_{ij} s^{-1}b_1^{ij} \otimes s^{-1}b_2^{ij} \otimes s^{-1}a_2^i + (-1)^{r-1} \sum_{ij} s^{-1}a_1^i \otimes s^{-1}f_1^{ij} \otimes s^{-1}f_2^{ij} = 0.$$

But since the same argument holds for all n, r and t we see that $\delta_2^2(s^{-1}c)$ is indeed 0. Because δ_2^2 is zero on the generators of ΩC it follows that $\delta_2^2 = 0$

It remains to show that the two differentials anti commute. First we show that $\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1$ is zero on $s^{-1}\bar{C}$. To do this we recall that we have

$$\delta_1|_{s^{-1}\bar{C}} = 1 \otimes d_C : \mathbb{K}s^{-1} \otimes \bar{C} \to \mathbb{K}s^{-1} \otimes \bar{C},$$

$$\delta_1|_{(s^{-1}\bar{C})^{\otimes 2}} = 1 \otimes d_C \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes d_C : (\mathbb{K}s^{-1} \otimes \bar{C})^{\otimes 2} \to (\mathbb{K}s^{-1} \otimes \bar{C})^{\otimes 2}$$

and

$$\delta_2|_{s^{-1}\bar{C}} = (1 \otimes \tau \otimes 1) \circ (\Delta_s \otimes \bar{\Delta}) : \mathbb{K}s^{-1} \otimes \bar{C} \to (\mathbb{K}s^{-1} \otimes \bar{C})^{\otimes 2}.$$

Then we check

$$\delta_2 \circ \delta_1|_{s^{-1}\bar{C}} = (1 \otimes \tau \otimes 1) \circ (\Delta_s \otimes \bar{\Delta}) \circ (1 \otimes d_C) = (1 \otimes \tau \otimes 1) \circ (\Delta_s \otimes (\bar{\Delta} \circ d_C)).$$

Using that d_C is a coderivation we get

$$(1 \otimes \tau \otimes 1) \circ (\Delta_s \otimes ((1 \otimes d_C + d_C \otimes 1) \circ \bar{\Delta})) =$$
$$-(1 \otimes \tau \otimes 1) \circ (1 \otimes 1 \otimes 1 \otimes d_C + 1 \otimes 1 \otimes d_C \otimes 1) \circ (\Delta_s \otimes \bar{\Delta}) =$$
$$-(1 \otimes 1 \otimes 1 \otimes d_C + 1 \otimes d_C \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\Delta_s \otimes \bar{\Delta}) =$$
$$-\delta_1 \circ \delta_2|_{s^{-1}\bar{C}}$$

where the minus sign appears because the map $(1 \otimes d_C + d_C \otimes 1)$ which is of degree -1 jumps over the map Δ_s which is also of degree -1. The above computations shows that $\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1$ is zero on $s^{-1}\bar{C}$. Now because $\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1$ is zero on the generators of ΩC an inductive argument show that it is in fact zero on all of ΩC.

□

We have seen that the cobar construction of a coaugmented dga coalgebra is an augmented dga algebra (the augmentation is given by the projection $T(s^{-1}\bar{C}) \to \mathbb{K}$). But the cobar construction is in fact a functor from the category of conilpotent dga coalgebras to the category of augmented dga algebras. Indeed, it assigns to any morphism of conilpotent dga coalgebras $f : C \to C'$ the map

$$1 \oplus \Big(\bigoplus_{n \geq 0} (1 \otimes f|_{\bar{C}})^{\otimes n} \Big) : T(s^{-1}\bar{C}) \to T(s^{-1}\bar{C}').$$

It is not so hard to check that this map is indeed a morphism of dga algebras and that this assignment respects the identity and composition of maps

2.2 The convolution algebra

In this section we give the set of linear maps from a dga coalgebra to a dga algebra $Hom(C, A)$ the structure of a dga algebra. We identify, in this algebra, certain elements called twisting morphisms. These twisting morphisms give rise to a chain complex structure on $C \otimes A$ which we call twisted tensor products.

We will denote by $Hom(C, A)$ the set of all linear maps from C to A. We give it a graded structure by $Hom(C, A)_r := \{f : C \to A | f(C_i) \subset A_{i+r}\}$. Define a multiplicaiton in $Hom(C, A)$ by

$$f \star g := \mu \circ (f \otimes g) \circ \Delta.$$

Since Δ and μ have degree zero and $f \otimes g$ has degree $|f|+|g|$ we see that $\star$ respects the grading on $Hom(C, A)$. Finally we define a differential on $Hom(C, A)$.

Proposition 31. The linear map $\partial : Hom(C, A) \to Hom(C, A)$ defined by

$$\partial(f) = d_A \circ f - (-1)^{|f|} f \circ d_C$$

makes $(Hom(C, A), \star, \partial)$ into a dga algebra.

Proof. To see that $\star$ is associative one just has to write out the formulas. Let us check that ∂ is a differential. First we note that since d_C and d_A both have degree -1 the map $\partial(f)$ has degree $|f| - 1$ which means ∂ is of degree -1. Next we check that ∂ is a derivation for $\star$:

$$\partial(f \star g) = d_A \circ (f \star g) - (-1)^{|f|+|g|}(f \star g) \circ d_C =$$

$$d_A \circ \mu \circ (f \otimes g) \circ \Delta - (-1)^{|f|+|g|} \mu \circ (f \otimes g) \circ \Delta \circ d_C.$$

But using that d_A is a derivation and d_C is a coderivation we can move things around until we get to

$$\begin{aligned} \mu \circ \big((d_A \circ f - (-1)^{|f|} f \circ d_C) \otimes g\big) \circ \Delta + \\ (-1)^{|f|} \mu \circ \big(f \otimes (d_A \circ g - (-1)^{|g|} g \circ d_C)\big) = \\ \partial(f) \star g + (-1)^{|f|} f \star \partial(g). \end{aligned}$$

Checking that $\partial^2 = 0$ again just comes down to writing out the formulas.

□

We are interested in certain special elements in this dga. These will play an important roll in adjunction between the bar functor and the cobar functor.

Definition 32. An element α of the dga algebra $Hom(C, A)$ is called a *twisting morphism* if it has degree -1, satisfies the *Mauer-Cartan equation* $\alpha \star \alpha + \partial(\alpha) = 0$ and if it is zero when composed with the augmentation map of A or with the coaugmentation map of C. The subset consisting of all twisting morphisms of $Hom(C, A)$ is denoted $Tw(C, A)$.

Starting with two chain complexes (V, d_V) and (W, d_W) there is a natural way to define a chain complex structure on their tensor product $V \otimes W$. Indeed, we have already used that $d_V \otimes 1 + 1 \otimes d_W$ is a differential on the graded vector space $V \otimes W$. We will now introduce a different chain complex structure in the case $V = C$ is a dga coalgebra and $W = A$ is a dga algebra and there is a twisting morphism $\alpha : C \to A$. For this purpose we note that any linear map $\alpha : C \to A$ gives rise to a linear map $d_\alpha : C \otimes A \to C \otimes A$, namely the composite

$$C \otimes A \xrightarrow{\Delta \otimes 1} C \otimes C \otimes A \xrightarrow{1 \otimes \alpha \otimes 1} C \otimes A \otimes A \xrightarrow{1 \otimes \mu} C \otimes A \ .$$

Lemma 33. For α and β in $Hom(C, A)$ we have the following relation

$$d_\alpha \circ d_\beta = d_{\alpha \star \beta}$$

Proof. We have

$$d_\alpha \circ d_\beta = (1 \otimes \mu) \circ (1 \otimes \alpha \otimes 1) \circ (\Delta \otimes 1) \circ (1 \otimes \mu) \circ (1 \otimes \beta \otimes 1) \circ (\Delta \otimes 1).$$

Using that μ is associative and Δ is coassociative we can move things around until we reach

$$(1 \otimes \mu) \circ (1 \otimes (\mu \circ (\alpha \otimes \beta) \circ \Delta) \otimes 1) \circ (\Delta \otimes 1) = d_{\alpha \star \beta}.$$

□

Proposition 34. If $\alpha : C \to A$ is a twisting morphism then the map $d'_\alpha = d_\alpha + d_C \otimes 1 + 1 \otimes d_A$ is a differential on $C \otimes A$. The chain complex $(C \otimes A, d'_\alpha)$ is referred to as the twisted tensor product of C and A and denoted $C \otimes_\alpha A$.

Proof. We need to show that $d'^2_\alpha = 0$. Using that $(1 \otimes d_A + d_C \otimes 1)$ squares to zero we get

$$(d_\alpha + 1 \otimes d_A + d_C \otimes 1)^2 = d_\alpha^2 + d_\alpha \circ (1 \otimes d_A + d_C \otimes 1) + (1 \otimes d_A + d_C \otimes 1) \circ d_\alpha.$$

If we expand the second two terms we get

$$d_\alpha \circ (1 \otimes d_A + d_C \otimes 1) + (1 \otimes d_A + d_C \otimes 1) \circ d_\alpha =$$
$$d_\alpha \circ (1 \otimes d_A) + d_\alpha \circ (d_C \otimes 1) + (1 \otimes d_A) \circ d_\alpha + (d_C \otimes 1) \circ d_\alpha.$$

Using the definition of d_α we see that the first and the third term above give

$$(1 \otimes \mu) \circ (1 \otimes \alpha \otimes 1) \circ (\Delta \otimes 1) \circ (1 \otimes d_A) + (1 \otimes d_A) \circ (1 \otimes \mu) \circ (1 \otimes \alpha \otimes 1) \circ (\Delta \otimes 1).$$

Using the fact that d_A is a derivation for μ we can move things around until we reach

$$(1 \otimes \mu) \circ (1 \otimes (d_A \circ \alpha) \otimes 1) \circ (\Delta \otimes 1) = d_{d_A \circ \alpha}.$$

Similarly one can show that $d_\alpha \circ (d_C \otimes 1) + (d_C \otimes 1) \circ d_\alpha = d_{\alpha \circ d_C}$. Then if we put all of this together and use the previous lemma and the fact that composition and tensor products of linear maps are additive operations, we get

$$(d_\alpha + 1 \otimes d_A + d_C \otimes 1)^2 = d_\alpha^2 + d_{d_A \circ \alpha} + d_{\alpha \circ d_C} = d_{\alpha \star \alpha + \partial(\alpha)}.$$

But this last expression is zero since α satisfies the Mauer-Cartan equation.

□

As we showed in the previous proposition any twisting morphism $\alpha : C \to A$ gives rise to a chain complex $(C \otimes_\alpha A)$. We are particularly interested in those that give rise to an acyclic chain complex.

Definition 35. A *Koszul twisting morphism* is a twisting morphism $\alpha : C \to A$ for which the twisted tensor product $C \otimes_\alpha A$ is acyclic.

2.3 Bar-cobar adjunction and the fundamental theorem of twisting morphisms

Twisting morphisms are closely related to the bar and cobar functors that we introduced in the previous section. In fact, as we will see the bar and cobar functors form a pair of adjoint functors and the easiest way to describe the adjunction is through $Tw(C, A)$.

Proposition 36. Let C be a conilpotent dga coalgebra and let A be an augmented dga algebra. There are bijections

$$Hom_{dga-alg}(\Omega C, A) \cong Tw(C, A) \cong Hom_{conil.\ dga-coalg.}(C, BA).$$

Proof. The first bijection goes as follows. To a morphism $f : \Omega C \to A$ of dga algebras we assign the map $\tilde{f} = f \circ \iota$ where $\iota : C \to \Omega C$ is defined by $1_C \mapsto 0$, $\bar{C} \ni c \mapsto s^{-1}c$. Then $\tilde{f}$ has degree -1, vanishes on $\mathbb{K} \subset C$ and maps $\bar{C}$ into $\bar{A}$. To see that $\tilde{f}$ satisfies the Mauer-Cartan equation we note that

$$0 = d_A \circ f(s^{-1}c) - f \circ \delta_1(s^{-1}c) - f \circ \delta_2(s^{-1}c) =$$

$$d_A \circ \tilde{f}(c) + \tilde{f} \circ d_C(c) + \tilde{f} \star \tilde{f} = \partial(\tilde{f})(c) + \tilde{f} \star \tilde{f}(c)$$

where the first equality follows from f being a chain map. On the other hand if $\alpha \in Tw(C, A)$ we can define a degree zero map $\alpha' : s^{-1}\bar{C} \to A$ by $s^{-1}c \mapsto \alpha(c)$. By the universal property of the tensor algebra we get an algebra morphism $f_\alpha : \Omega C \to A$. To see that f_α is a chain map we let $\bar{\Delta}(x) = \sum_i x_i \otimes y_i$ which gives

$$d_A(f_\alpha(s^{-1}c)) - f_\alpha(d_{\Omega C}(s^{-1}c)) = d_A(f_\alpha(s^{-1}c)) - f_\alpha(\delta_1(s^{-1}c)) - f_\alpha(\delta_2(s^{-1}c)) =$$

$$d_A(f_\alpha(s^{-1}c)) + f_\alpha(s^{-1}d_C(c)) - f_\alpha(\sum_i (-1)^{|x_i|} s^{-1}x_i \otimes s^{-1}y_i)) =$$

$$d_A(f_\alpha(s^{-1}c)) + f_\alpha(s^{-1}d_C(c)) - \sum_i (-1)^{|x_i|} f_\alpha(s^{-1}x_i) f_\alpha(s^{-1}y_i) =$$

$$d_A(f_\alpha(s^{-1}c)) + f_\alpha(s^{-1}d_C(c)) - \sum_i (-1)^{|x_i|} \alpha(x_i)\alpha(y_i) = \partial(\alpha)(c) + \alpha \star \alpha(c) = 0.$$

Finally we check that these assignments are inverses of each other. First we assign to $f \in Hom_{dga-alg}(\Omega C, A)$ the twisting morphism $\tilde{f} = f \circ \iota$. Then we assign to the twisting morphism $\tilde{f}$ the unique dga algebra morphism $F : \Omega C \to$

A which satisfies $F(s^{-1}c) = \tilde{f}(c)$. But $f(s^{-1}c) = f \circ \iota(c) = \tilde{f}(c)$ so $F = f$. On the other hand, if we assign to the twisting morphism $\alpha : C \to A$ the unique dga algebra morphism $f_\alpha : \Omega C \to A$ which satisfies $f_\alpha(s^{-1}c) = \alpha(c)$. To f_α we then assign the twisting morphism $\tilde{f_\alpha} = f_\alpha \circ \iota$ which is precisely α since $f_\alpha(s^{-1}c) = \alpha(c)$. This proves the first bijection.

For the second bijection let $g : C \to BA$ be a morphism of conilpotent dga coalgebras. Let $\tilde{g}$ be the composition $\pi \circ g$ where $\pi : BA \to A$ is zero everywhere except on $s\bar{A}$ on which it is defined as $sa \mapsto a$. Then $\tilde{g}$ is of degree -1 it vanishes on $\mathbb{K} \subset C$ and it maps $\bar{C}$ into $\bar{A}$. Next we show that $\tilde{g}$ satisfies the Mauer-Cartan equation. Let $c \in C$. Since g is a chain map we have

$$\pi \circ g \circ d_C(C) = \pi \circ d_{BA} \circ g(c). \qquad (\triangle)$$

The left hand side is $\tilde{g} \circ d_C(c)$. The right hand side we can rewrite as

$$\pi \circ d_1 \circ g(c) + \pi \circ d_2 \circ g(c).$$

Let us first study the first term. For this purpose let $g(c) = sc_1 + M$ where M consists of terms of word length other than one. Since d_1 fixes word length and π vanishes on everything of word length other than one we get $\pi \circ d_1 \circ g(c) = \pi \circ d_1(sc_1) = \pi(-sd_A(c_1)) = -d_A(c_1) = -d_A \circ \pi \circ g(c) = -d_A \circ \tilde{g}(c)$.
Now we study the second term which I claim is precisely $-\tilde{g} \star \tilde{g}$. To see this we will show that the following diagram commutes

$$\begin{array}{ccccccc} C & \xrightarrow{g} & BA & \xrightarrow{d_2} & BA & \xrightarrow{\pi} & A \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_{BA}} & & & \nearrow{\scriptstyle -\mu} & \\ C \otimes C & \xrightarrow{g \otimes g} & BA \otimes BA & \xrightarrow{\pi \otimes \pi} & A \otimes A & & \end{array} .$$

This would prove the claim because composing the arrows on top gives us precisely $\pi \circ d_2 \circ g$ and going down, right, right and then up is precisely $-\tilde{g} \star \tilde{g}$. The first square however commutes because g is a morphism of coalgebras so it remains to check that the pentagon to the right commutes. Because $\pi \circ d_2$ and $(\pi \otimes \pi) \circ \Delta_{BA}$ both vanish on elements of BA of word lenght other than two it remains to check that the pentagon commutes for elements of the form $\sum_i sa_i \otimes sb_i$. Let us use small and large tensor symbols ($\otimes$ and $\bigotimes$) to distinguish between elements of BA and $BA \otimes BA$. We have

$$-\mu \circ (\pi \otimes \pi) \circ \Delta_{BA}\big(\sum_i sa_i \otimes sb_i\big) =$$

$$-\mu \circ (\pi \otimes \pi)\big(\sum_i (1 \bigotimes sa_i \otimes sb_i + sa_i \bigotimes sb_i + sa_i \otimes sb_i \bigotimes 1)\big) =$$

$$-\mu\big(\sum_i (-1)^{|a_i|+1} a_i \otimes b_i\big) = \sum_i (-1)^{|a_i|} \mu(a_i \otimes b_i).$$

On the other hand we have

$$\pi \circ d_2\big(\sum_i sa_i \otimes sb_i\big) = \pi\big(\sum_i (-1)^{|a_i|} s\mu(a_i \otimes b_i)\big) = \sum_i (-1)^{|a_i|} \mu(a_i \otimes b_i)$$

so the diagram does indeed commute. But then we can rewrite the equation ($\triangle$) as

$$\tilde{g} \circ d_C = -d_A \circ \tilde{g} - \tilde{g} \star \tilde{g}$$

or if we bring everything to the left hand side we get precisely

$$\partial(\tilde{g})(c) + \tilde{g} \star \tilde{g}(c) = 0$$

and since c was arbitrary we see that $\tilde{g}$ satisfies the Mauer-Cartan equation so it is a twisting morphism.

On the other hand if we start with a twisting morphism $\psi : C \to A$ we can define a linear map $\hat{\psi}$ as the composite

$$C \xrightarrow{\psi} \bar{A} \xrightarrow{S} s\bar{A} \hookrightarrow BA$$

where the map S is defined by $1_A \mapsto 0$ and $a \mapsto sa$ for $a \in \bar{A}$. We see that $\hat{\psi}$ has degree zero. Since ψ is a twisting morphism $\psi(1_C) = 0$ and since BA is cofree on the vector space $s\bar{A}$ proposition 25 tells us that there is a unique coalgebra morphism $\Psi : C \to BA$ which lifts $\hat{\psi}$. To see that $\hat{\psi}$ is in fact a morphism of dga coalgebras we must show that

$$\Psi \circ d_C(c) = d_{BA} \circ \Psi(c)$$

for any $c \in C$. To do this we recall from the proof of proposition 1.2.1 in [3] that Ψ can be defined by the following formula

$$\Psi(c) = \sum_{n \geq 1} \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1}(c).$$

Then we have

$$\Psi \circ d_C(c) = \sum_{n \geq 1} \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1} \circ d_C(c) = \sum_{n \geq 1} \sum_{i=0}^{n-1} \hat{\psi}^{\otimes n} \circ (1^{\otimes i} \otimes d_C \otimes 1^{\otimes n-i}) \circ \bar{\Delta}^{n-1}(c) =$$

$$\sum_{n \geq 1} \sum_{i=0}^{n-1} (\hat{\psi}^{\otimes i} \otimes (\hat{\psi} \circ d_C) \otimes \hat{\psi}^{n-i-1}) \circ \bar{\Delta}^{n-1}(c).$$

We also have

$$d_B A \circ \Psi(c) = d_1 \circ \Psi(c) + d_2 \circ \Psi(c).$$

To give a more explicit description of the first term let d'_A be the map defined by $sa \mapsto -sd_A(a)$. Then we get

$$d_1 \circ \Psi(c) = d_1 \left(\sum_{n \geq 1} \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1}(c) \right) = \sum_{n \geq 1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes d'_A \otimes 1^{\otimes n-i-1}) \circ \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1}(c) =$$

$$\sum_{n \geq 1} \sum_{i=0}^{n-1} (\hat{\psi}^{\otimes i} \otimes (d'_A \circ \hat{\psi}) \otimes 1^{\otimes n-i-1}) \circ \bar{\Delta}^{n-1}(c).$$

To give a more explicit description of the second term $d_2 \circ \Psi(c)$ let us define a map $S^{-1} : s\bar{A} \to A$ by $sa \mapsto a$. Then we note that the differential d_2 has the following formula when restricted to $(s\bar{A})^{\otimes n}$

$$d_2|_{(s\bar{A})^{\otimes n}} = -\sum_{i=0}^{n-2}(1^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1})) \otimes 1^{n-i-2}).$$

This gives

$$d_2 \circ \Psi(c) = d_2\Bigg(\sum_{n\geq 1} \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1}(c)\Bigg) =$$

$$-\sum_{n\geq 1}\sum_{i=0}^{n-2}(1^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1})) \otimes 1^{n-i-2}) \circ \hat{\psi}^{\otimes n} \circ \bar{\Delta}^{n-1}(c) =$$

$$-\sum_{n\geq 1}\sum_{i=0}^{n-2}(\hat{\psi}^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1}) \circ \hat{\psi}^{\otimes 2}) \otimes \hat{\psi}^{n-i-2}) \circ \bar{\Delta}^{n-1}(c).$$

Because $BA = \bigoplus_{n\geq 0}(s\bar{A})^{\otimes n}$ is a direct sum it is enough to check that the component of $\Psi \circ d_C(c)$ that lies in $(s\bar{A})^{\otimes n}$ equals the component of $d_{BA} \circ \Psi(c)$ that lies in $(s\bar{A})^{\otimes n}$ for each n. Using the explicit expressions we have found this amounts to showing that

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i} \otimes (\hat{\psi} \circ d_C) \otimes \hat{\psi}^{n-i-1}) \circ \bar{\Delta}^{n-1}(c) =$$

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i} \otimes (d'_A \circ \hat{\psi}) \otimes 1^{\otimes n-i-1}) \circ \bar{\Delta}^{n-1}(c)-$$

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1}) \circ \hat{\psi}^{\otimes 2}) \otimes \hat{\psi}^{n-i-1}) \circ \bar{\Delta}^{n}(c).$$

If we rewrite the last sum in the following way

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1}) \circ \hat{\psi}^{\otimes 2}) \otimes \hat{\psi}^{n-i-1}) \circ \bar{\Delta}^{n}(c) =$$

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i} \otimes (S \circ \mu \circ (S^{-1} \otimes S^{-1}) \circ \hat{\psi}^{\otimes 2} \circ \bar{\Delta}) \otimes \hat{\psi}^{n-i-1}) \circ \bar{\Delta}^{n-1}(c)$$

and collect all three sums on the left side and use bilinearity of the tensor product we are left with showing that

$$\sum_{i=0}^{n-1}(\hat{\psi}^{\otimes i}\otimes\Bigg(\hat{\psi}\circ d_C - d'_A\circ\hat{\psi} + (S\circ\mu\circ(S^{-1}\otimes S^{-1})\circ\hat{\psi}^{\otimes 2}\circ\bar{\Delta})\Bigg)\otimes\hat{\psi}^{n-i-1})\circ\bar{\Delta}^{n-1}(c) = 0.$$

But using that $\hat{\psi}(x) = s\psi(x)$ and $\psi(1_C) = 0$ the last expression can be written as

$$s(\partial(\psi)(x) + \psi \star \psi(x))$$

which is zero since ψ is a twisting morphism. So Ψ is indeed a morphism of dga coalgebras.

To see that these assignments are inverses to each other let $g : C \to BA$ be a morphism of conilpotent dga coalgebras. Then we assign to it the twisting morphism $\tilde{g} = \pi \circ g$. Then to $\tilde{g}$ we assign the unique morphism of conilpotent dga coalgebras $C \to BA$ that lifts $S \circ \tilde{g}$. But if $p : BA \to s\bar{A}$ we see that $p \circ g = S \circ \pi \circ g = S \circ \tilde{g}$ so g lifts $S \circ \tilde{g}$. If we on the other hand start with a twisting morphism ψ, we assign to it the unique morphism of conilpotent dga coalgebras Ψ that lifts $S \circ \psi$. Then we assign to Ψ the twisting morphism $\pi \circ \Psi = S^{-1} \circ p \circ \Psi = S^{-1} \circ S \circ \psi = \psi$. We have thus proved the second bijection. $\square$

When discussing these bijections it is helpful to have the following diagram in mind

$$\begin{array}{ccc} C & \xrightarrow{f_\alpha} & BA \\ \downarrow{\scriptstyle \iota} & \searrow^{\alpha} & \downarrow{\scriptstyle \pi} \\ \Omega C & \xrightarrow{g_\alpha} & A \end{array}$$

where ι, α and π are linear maps between graded vector spaces, f_α is a morphism of conilpotent dga coalgebras and g_α is a morphism of augmented dga algebras. When we refer to the (co)algebra morphisms corresponding to a twisting morphism α we will usually denote them by f_α and g_α as in this diagram.

The following fundamental theorem on twisting morphisms gives a close relation between quasi isomorphisms $\Omega C \to A$, Koszul twisting morphisms $C \to A$ and quasi isomorphisms $C \to BA$ through the bijections from the proposition above. We only sketch the proof and refer the reader to [3] for details.

Theorem 37. For any twisting morphism $\alpha : C \to A$, where C is a connected wdga coalgebra and A is a connected wdga algebra which is connected with respect to weight and homological degree the following are equivalent:
1) The twisted tensor product $C \otimes_\alpha A$ is acyclic.
2) The map $g_\alpha : \Omega C \to A$ corresponding to α through the bijections of the previous proposition is a quasi isomorphism.
3) The map $f_\alpha : C \to BA$ corresponding to α through the bijections of the previous proposition is a quasi isomorphism.

Proof. The proof depends on the following facts:

1) $\pi : BA \to A$ is a Koszul twisting morhpism,

2) $\iota : C \to \Omega C$ is a Koszul twisting morphism,

3) given a map $f : C \to C'$ between wdga coalgebra, a map $g : A \to A'$ between wdga algebras and twisting morphisms $\alpha : C \to A$ and $\alpha' : C' \to A'$ such that $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ is a chain map we have that if two out of the maps f, g and $f \otimes g$ are quasi isomorphisms then so is the third.

Then to prove the equivalence 1) $\iff$ 2) we consider the map $id_C \otimes g_\alpha : C \otimes_\iota \Omega C \to C \otimes_\alpha A$. By the second fact $C \otimes_\iota \Omega C$ is acyclic. Also one checks that $id_C \otimes g_\alpha$ is a chain map which means, using fact 3), that α is a Koszul twisting morphism if and only if g_α is a quasi isomorphism. Similarly one proves the equivalence 1) $\iff$ 3) by considering the map $f_\alpha \otimes id_A : C \otimes_\alpha A \to BA \otimes_\pi A$. □

Remark 38. The theory introduced in chapter 2 before this last theorem works just as well over an arbitrary commutative ring rather than a field. The proof of this last theorem however, and in particular the proof of fact 3, cannot be modified in any obvious way to work for arbitrary modules over some commutative ring. The problem arises in the comparison lemma in [3] (which is what we called fact 3 in the proof above). The proof of this lemma begins by filtering the weight n part of the twisted tensor product by

$$F_s(C \otimes A)^{(n)} := \bigoplus_{d+m \leq s} C_d^{(m)} \otimes A^{(n-m)}.$$

Then one looks at the associated spectral sequence and notes that first differential is $d_C \otimes 1_A$ and the second differential is $1_C \otimes d_A$. Working over a field one can then conlcude that the second page of the spectral sequence look like $E^2_{pq} = \oplus_{m=0}^n H_{p-m}(C_\bullet^{(M)}) \otimes H_{(t+m)}(A_\bullet^{(n-m)})$. But for this to be true when working over some commutative ring we would need the extra requirement that $A_k^{(m)}$ and $H_{(k)}(C_\bullet^{(m)})$ are flat modules.

3 Koszulity, formality and how they are related

The goal of this chapter is to give a detailed proof of theorem 2.9 of [1] restricted to the case of associative algebras. By restricting ourselves to the associative case we do not need to introduce operads. To prove proposition 2.5 of [1] we devote one section to prove certain factorization and lifting properties in the category of dga algebras.

3.1 Main theorem

Given a Koszul twisting morphism $C \to A$ we will in this section investigate some equivalent conditions for when C and A are both formal. These equivalent conditions use the notion of Koszul algebras and Koszul coalgebras. We begin by giving some definitions

Definition 39. A *weak equivalence* of dga algebras A and A' is just a quasi isomorphism. We say that two dga algebras A and A' are weakly equivalent if there is a zig-zag diagram of weak equivalences connecting them

$$A \xrightarrow{\sim} B_0 \xleftarrow{\sim} \cdots \xrightarrow{\sim} B_k \xleftarrow{\sim} A'.$$

A *weak equivalence* between dga coalgebras C and C' is a morphism of dga coalgebras f such that $\Omega f : \Omega C \to \Omega C'$ is a quasi isomorphism. We say that two coalgebras C and C' are *weakly equivalent* if there is a zig-zag diagram of weak equivalences connecting them

$$C \xrightarrow{\sim} D_0 \xleftarrow{\sim} \cdots \xrightarrow{\sim} D_k \xleftarrow{\sim} C'.$$

Finally we say that a dga algebra (dga coalgebra) is *formal* if it is weakly equivalent to its homology.

Now let A be a weight graded dga algebra with $A(0) = \mathbb{K}$ (recall that we require weight gradings to be concentrated in non negative weight). Then A is augmented with the augmentation map being the projection onto $A(0)$. This weight grading carries over to the bar construction. Because the bar construction comes with another natural grading, namely word length, the coalgebra BA is in fact bigraded. Because $\bar{A}$ is concentrated in positive weight BA will have a lower triangular structure which is depicted in table 1. Recall from the previous chapter that the differential on BA is a sum $d_1 + d_2$ where d_2 decreases the word length by one. Let us denote by $\mathscr{D}$ the diagonal $\bigoplus_{i \geq 0}(sA(1)^{\otimes i})$ of BA. Let us further denote the intersection $\mathscr{D} \cap ker(d_2)$ by $A^{¡}$.

Definition 40. We say that the dga algebra A is Koszul if there is a weight grading on A such that $A(0) = \mathbb{K}$ and the inclusion $A^{¡} \hookrightarrow BA$ is a quasi isomorphism.

We can do something similar with coalgebras. Assume C is a weight graded dga coalgebra and assume $C(0) = \mathbb{K}$. C is then counital with counit given by the projection $C \to C(0) = \mathbb{K}$. Also it is conilpotent because

$$\bar{\Delta} : C(n) \to \bigoplus_{k+l=n,\ k,l>0} C(k) \otimes C(l)$$

lowers weight in each factor. The weight grading on C gives a weight grading on ΩC. ΩC is then bigraded, by weight and word length. Because $\bar{C}$ is concentrated in positive degree ΩC will have a lower triangular structure as depicted in table 2. The differential on ΩC is a sum $\delta_1 + \delta_2$ where δ_2 increases the word length by one. Let us denote by $\mathscr{D}_C$ the diagonal $\bigoplus_n (s^{-1}C(1))^{\otimes n}$ of ΩC. Let us further denote by $C^{¡}$ the quotient $\mathscr{D}_C/(im\delta_2 \cap \mathscr{D}_C)$.

Definition 41. We say that the dga coalgebra C is Koszul if there is a weight grading on C such that $C(0) = \mathbb{K}$ and the projection $\Omega C \to C^{¡}$ is a quasi isomorphism.

Proposition 42. Let $\kappa : C \to A$ be a Koszul twisting morphism where C is a coaugmented connected dga coalgebra and A is an augmented connected dga algebra. If C and A both have trivial differentials then C and A are Koszul. Moreover $C \cong A^{¡}$ and $A \cong C^{¡}$.

Proof. Because κ is a twisting morphism we get a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f_\kappa} & BA \\ \Big\downarrow{\scriptstyle \iota} & \searrow^{\kappa} & \Big\downarrow{\scriptstyle \pi} \\ \Omega C & \xrightarrow{g_\kappa} & A \end{array}$$

where f_κ is a morphism of dga coalgebras and g_κ is an morphism of dga algebras. Moreover, since κ is a Koszul twisting morphism it follows from the fundamental theorem of twisting morphisms that f_κ and g_κ are quasi isomorphisms. Note that A and C are weight graded by homological grading because they both have trivial differentials so the fundamental theorem of twisting morphisms does indeed apply. Now we define a weight grading on C by

$$C(p) = f_\kappa^{-1}\big((s\bar{A})^{\otimes p}\big).$$

Because f_κ is a morphism of augmented coalgebras $C(0) = f_\kappa^{-1}(\mathbb{K}) = \mathbb{K}$. Next we will define a weight grading on A. First we note that since the differential on C is trivial the differential on ΩC increases word length by precisely 1. From this it follows that the homology of the cobar construction admits a weight grading

$$H_\bullet(\Omega C) = \bigoplus_{p\geq 0} \frac{ker(d_{\Omega C} : (s^{-1}C)^{\otimes p} \to s^{-1}C)^{\otimes p-1})}{im(d_{\Omega C} : (s^{-1}C)^{\otimes p+1} \to s^{-1}C)^{\otimes p})}.$$

But then we can transport this weight grading to A via

$$H_\bullet(\Omega C) \cong H_\bullet(A) = A$$

where the last equality follows from the fact that $d_A = 0$.

Now we show that $A^{¡} \hookrightarrow BA$ is a quasi isomorphism. To do this I claim that κ vanishes outside weight 1. Indeed this follows from that fact that π vanishes outside word length 1. Also I claim that $im(\kappa) \subset A(1)$. Indeed, Since κ vanishes outside $C(1)$ it is enough to check that $\kappa(C(1)) \subset A(1)$. Let $c \in C(1)$. Then $\iota(c) = s^{-1}c \in s^{-1}C(1)$ lies on the diagonal of ΩC. Because $d_{\Omega C}$ increases word length by precisely 1 any element on the diagonal is a cycle. So $[s^{-1}c] \in H_*(\Omega C)^{(1)}$ and $g_\kappa(s^{-1}c) = (g_\kappa)_*[s^{-1}c] \in A(1)$. Since $\kappa = g_\kappa \circ \iota$ it follows that κ lands in $A(1)$. But since $im(\kappa) \subset A(1)$ it follows from how the second bijection of proposition 36 is defined that $f_\kappa(C) \subset \mathscr{D}_A$. Also since C has trivial differential $f_\kappa(C) \subset ker(d_{BA})$ so $f_\kappa(C) \subset A^{¡}$. I claim that this inclusion is in fact an equality. To see this let $x \in A^{¡}$. Because f_κ is a quasi isomorphism there is a cycle $z \in C$ such that $f_\kappa(z) = x + b$ for some boundary b. Because f_κ lands on the diagonal $\mathscr{D}_A$ b is a boundary in the diagonal but $\mathscr{D} \cap imd_{BA} = 0$ so $f_\kappa(z) = x$ and $f_\kappa(C) = A^{¡}$ and since f_κ is a quasi isomorphism and $d_C = 0$ f_κ must be injective and then $C \cong A^{¡}$. But this implies A is Koszul because we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad\sim\quad} & BA \\ & {\scriptstyle\cong}\searrow \quad \nearrow & \\ & A^{¡} & \end{array}.$$

Next we show that also $p : \Omega C \twoheadrightarrow C^{¡}$ is a quasi isomorphism. To do this I claim that $ker(p) = ker(g_\kappa)$. For the inclusion ($\subset$) we note that since κ vanishes outside $C(1)$ it follows from how the first bijection of proposition 36 is defined that g_κ vanishes outside the diagonal $\mathscr{D}_C$. Also since g_κ is a chain map it takes boundaries to boundaries but A has trivial differential so g_κ vanishes on $imd_{\Omega C}$ aswell. For the inclusion ($\supset$) suppose $x \in ker(g_\kappa)$. Then since both p and g_κ vanish outside $\mathscr{D}_C$ we can assume $x \in \mathscr{D}_C$. But $d_{\Omega C}$ is trivial on $\mathscr{D}_C$ so then x is a cycle and $(g_\kappa)_*[x] = 0$. But g_κ is quasi isomorphism so x must be a boundary and then $p(x) = 0$. This means g_κ descends to an injective map $\hat{g_\kappa} : C^{¡} \to A$ and since A has trivial differential and g_κ is a quasi isomorphism g_κ is surjective and then g_κ is too. So $C^{¡} \cong A$. But then C is Koszul because we have a commutative diagram

$$\begin{array}{ccc} \Omega C & \xrightarrow{\sim} & A \\ & \searrow \quad \nearrow_{\cong} & \\ & C^{¡} & \end{array} .$$

□

Now we are ready to state the main theorem. We will however postpone the proof of it until the end of this chapter.

Theorem 43. Let $\kappa : C \to A$ be a Koszul twisting morphism where A is a connected dga algebra and C is a connected dga coalgebra. The following are equivalent:

(1) C and A are formal.

(2) A is formal and $H_\bullet(A)$ is Koszul.

(3) C is formal and $H_\bullet(C)$ is Koszul.

3.2 Factorisation- and lifting properties in the category of dga algebras

In this section we show that the morphisms of augmented dga algebras have certain nice factorization and lifting properties. The main purpose of this section is to prove the following result.

Proposition 44. Let $\kappa : C \to A$ be a Koszul twisting morphism where A is a connected dga algebra and C is a connected dga coalgebra. If $C \sim C'$ and $A \sim A'$ then there is a Koszul twisting morphism $\kappa' : C' \to A'$.

But the theory developed in this chapter has a broader interest and is part of what is often called homotopical algebra.

First we recall that given a chain complex (V, d_V) the tensor algebra $T(V)$ is an augmented dga algebra. Indeed by proposition 23 the composition

$$V \xrightarrow{d_V} V \hookrightarrow T(V)$$

induces a unique derivation on $T(V)$ and because $d^2 = 0$ on the generators it follows that $d^2 = 0$.

Next, we show that the category of augmented dga algebras has pushouts. Consider the following diagram of augmented dga algebras and morphisms of such

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ \downarrow{\scriptstyle f} & & \\ A & & \end{array} \quad . \qquad (*)$$

We construct its pushout as follows. Think of $(\bar{A} \oplus \bar{B}, d = d_A \oplus d_B)$ as a chain complex. Then we saw that the tensor algebra $T(\bar{A} \oplus \bar{B})$ is an augmented dga algebra. In the following proposition we show that a pushout of the diagram above can be defined as

$$A *_{f,g} B := T(\bar{A} \oplus \bar{B})/I$$

where I is the two sided ideal generated by all elements of the following form

$$a_1 \otimes a_2 - \mu_A(a_1 \otimes a_2),\ b_1 \otimes b_2 - \mu_B(b_1 \otimes b_2),\ f(c) - g(c)$$

for $a_1, a_2 \in \bar{A}$ $b_1, b_2 \in \bar{B}$ and $c \in \bar{C}$.

Proposition 45. $A *_{f,g} B$ is an augmented dga algebra. Moreover $A *_{f,g} B$ is a pushout of the diagram $(*)$

Proof. To prove this we first note that I is generated by homogenous elements (μ_A, μ_B, f g are all linear maps of degree zero) so I is a graded ideal and then $T(\bar{A} \oplus \bar{B})/I$ inherits a grading from $T(\bar{A} \oplus \bar{B})$. Next we show that $d(I) \subset I$ which would then imply d descends to a differential on $A *_{f,g} B$. It is enough that $d(\alpha) \in I$ on the generators α. We have

$$d(a_1 \otimes a_2 - \mu_A(a_1 \otimes a_2)) =$$
$$d_A(a_1) \otimes a_2 - \mu_A(d_A(a_1) \otimes a_2) + (-1)^{|a_1|} a_1 \otimes d_A(a_2) - (-1)^{|a_1|} \mu_A(a_1 \otimes d_A(a_2))$$

which is in I. Similarly $d(b_1 \otimes b_2 - \mu_B(b_1 \otimes b_2)) \in I$. Finally we check that

$$d(f(c) - g(c)) = d_A(f(c)) - d_B(g(c)) = f(d_C(c)) - g(d_C(c)) \in I.$$

For the second statement suppose we have a commutative diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle k} \\ A & \xrightarrow{h} & D \end{array} .$$

Then we claim there is a unique morphism of augmented algebras $\phi : A *_{f,g} B \to D$ such that $\phi \circ i_A = h$ and $\phi \circ i_B = k$ where $i_A : A \to A *_{f,g} B$ is defined by $1_A \mapsto 1$ and $\bar{A} \ni a \mapsto a$ and similarly for $i_B : B \to A *_{f,g} B$. But since we have a linear map $h \oplus k : \bar{A} \oplus \bar{B} \to D$ the universal property of the tensor algebra gives us a unique morphism of algebras $\tilde{\phi} : T(\bar{A} \oplus \bar{B}) \to D$. Let us check that $\tilde{\phi}(I) = 0$. We have

$$\tilde{\phi}(a_1 \otimes a_2 - \mu_A(a_1 \otimes a_2)) = h(a_1)h(a_2) - h(\mu_A(a_1 \otimes a_2)) = h(a_1)h(a_2) - h(a_1)h(a_2) = 0$$

and similarly $\tilde{\phi}(b_1 \otimes b_2 - \mu_B(b_1 \otimes b_2)) = 0$. Also $\tilde{\phi}(f(c) - g(c)) = h \circ f(c) - k \circ g(c) = 0$. So we get a well defined map $\phi : A *_{f,g} B \to D$ which makes everything commute. Also this map ϕ is unique because if there was some other map ϕ' then ϕ and ϕ' agrees on $i_A(A)$ and $i_B(B)$ which generate $A *_{f,g} B$ so they agree everywhere. □

We will now use pushouts to construct new algebras from old ones by adding generators. This construction reminds us of adding cells to topological spaces because it can be used to turn cycles of the original algebra into boundaries. Let A be a dga algebra. Let V be a chain complex with zero differential and let $CV = V \oplus sV$ be the chain complex with differential defined by

$$v \mapsto 0, sv \mapsto v.$$

The composition $V \hookrightarrow CV \hookrightarrow T(CV)$ induces an algebra morphism $T(V) \to T(CV)$. Now if we choose a basis $\{e_i\}$ for V and to each basis element a cycle of A a_i we can define a morphism of dga algebras $T(V) \to A$ by specifying $e_i \mapsto a_i$. In this scenario we say that a dga algebra B is obtained from A by *adding cells* if B fits into a pushout diagram

$$\begin{array}{ccc} T(V) & \longrightarrow & T(CV) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} .$$

Moreover, we say that a morphism of augmented dga algebras $f : A \to B$ is a *relative cell algebra inclusion* if there is a sequence

$$A = A_0 \to A_1 \to A_2 \to ...$$

such that A_i is obtained from A_{i-1} by adding cells and such that B is the colimit of this sequence.

Proposition 46. Let $f : A \to B$ be a relative cell algebra inclusion. Then f has the following lifting property. For any surjective quasi isomorphism $p : X \twoheadrightarrow Y$ and commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow f & & \downarrow p \\ B & \xrightarrow{k} & Y \end{array}$$

there exists $H : B \to X$ such that $H \circ f = h$ and $p \circ H = k$.

Proof. We prove this in four steps.

Step 1: We show that for any vector space V the inclusion $V \hookrightarrow CV$ has the corresponding lifting property in the category of chain complexes. Indeed, consider the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ CV & \xrightarrow{k} & Y \end{array}$$

where p is a surjective quasi isomorphism. Then I claim that for each $v \in V$ there exists a $x \in X$ such that $d_X(x) = h(v)$. To see this, note that since v is a cycle in V and h is a chain map $h(v)$ is a cycle in X. But then $[ph(v)]$ is an element of $H(Y)$. But

$$[ph(v)] = [k(v)] = [kd_{CV}(sv)] = [d_Y(k(sv))] = 0$$

and since $p_\bullet : H(X) \to H(Y)$ is an isomorphism $[h(v)] = 0$. In other words $h(v)$ is a boundary so there is an $x \in X$ such that $h(v) = d_X(x)$. Moreover, I claim that we can choose x such that $p(x) = k(sv)$. To see this let x' be such that $d_X(x') = h(v)$. Then since $d_Y k(sv) = k(v) = ph(v) = pd_X(x') = d_Y p(x')$ we see that $p(x') - k(sv) \in ker(d_Y)$. But then since $p_\bullet : H(X) \cong H(Y)$ there is a cycle $w \in X$ such that $[p(w)] = [p(x') - k(sv)]$ or in other words $p(w) = p(x') - k(sv) + d_Y(a)$ for some $a \in Y$. But since p is surjective there is a $c \in X$ such that $p(c) = a$. If we then set $x = x' - w + d_X(c)$ we have $d_X(x) = d_X(x') = h(v)$ and

$$p(x) = p(x') - p(w) + pd_X(c) = p(x') - p(x') + k(sv) + d_Y(a) - d_Y p(c) = \\ k(sv) + d_Y(a) - d_Y(a) = k(sv).$$

Now we can define $H : CV = V \oplus sV \to X$ as follows. Let $\{e_i\}$ be a basis for V. Then $\{e_i\} \cup \{se_i\}$ is a basis for CV. We define H by $e_i \mapsto h(e_i)$ and $se_i \mapsto x_i$ where $x_i \in X$ is an element such that $p(x_i) = k(se_i)$ and $d_X(x_i) = h(e_i)$ (such x_i exist by the argument above). Then H is a chain map by construction and $H \circ i = h$ and $p \circ H = k$.

Step 2: We show that the induced morphism of augmented dga algebras $T(V) \to T(CV)$ has the lifting property. For this purpose we consider the following commutative diagram of dga algebras

$$\begin{array}{ccc} T(V) & \xrightarrow{h} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ T(CV) & \xrightarrow{k} & Y \end{array} .$$

By only considering the underlying chain complexes we get a commutative diagram of chain complexes

$$\begin{array}{ccccc} V & \hookrightarrow & T(V) & \xrightarrow{h} & X \\ \downarrow & & \downarrow i & & \downarrow p \\ CV & \hookrightarrow & T(CV) & \xrightarrow{k} & Y \end{array}.$$

From the first step of this proof we know there is a linear map $H : CV \to X$ which turns the commutative (big) rectangle into two commutative triangles. By the universal property of the tensor algebra there is a unique algebra morphism $\tilde{H} : T(CV) \to X$ such that

$$\begin{array}{ccc} & & X \\ & \overset{H}{\nearrow} & \uparrow \tilde{H} \\ CV & \hookrightarrow & T(CV) \end{array}$$

commutes. But then the linear map $V \hookrightarrow T(V) \to X$ extends to two algebra morphisms $h : T(V) \to X$ and $H \circ i : T(V) \to X$ so by the universal property of $T(V)$ we get $h = \tilde{H} \circ i$. Also the linear map $CV \hookrightarrow T(CV) \to Y$ extends to two algebra morphisms $k : T(CV) \to Y$ and $p \circ \tilde{H} : T(CV) \to Y$ so we must have $p \circ \tilde{H} = k$. The fact that $\tilde{H}$ is a chain map follows from H being a chain map.

Step 3: We show that if A_{i+1} is obtained from A_i by adding cells then the map $A_i \to A_{i+1}$ has the lifting property. Again we consider a commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{h} & X \\ \downarrow j & & \downarrow p \\ A_{i+1} & \xrightarrow{k} & Y \end{array}$$

where p is a surjective quasi isomorphism. Because A_{i+1} is obtained from A_i by adding cells we get a commutative diagram

$$\begin{array}{ccccc} T(V) & \xrightarrow{s} & A_i & \xrightarrow{h} & X \\ \downarrow & & \downarrow j & & \downarrow p \\ T(CV) & \xrightarrow{t} & A_{i+1} & \xrightarrow{k} & Y \end{array}$$

where the left square is a pushout diagram. In step 2 we proved there is a map $\tilde{H} : T(CV) \to X$ such that the big rectangle above splits into two commutative triangles. But then since the left square is a pushout diagram we get a map $H : A_{i+1} \to X$ such that $H \circ t = \tilde{H}$ and $H \circ j = h$. It remains to check that $p \circ H = k$. This too follows from the fact that the left square is a pushout because the maps $p \circ h : A_i \to Y$ and $k \circ t : T(CV) \to Y$ induces a unique map $A_{i+1} \to Y$ but both k and $p \circ H$ fit.

step 4: We show that the relative cell algebra inclusion $f : A \to B$ has the lifting property in the statement of the proposition. Since f is a cell algebra inclusion there is a diagram

$$A = A_0 \to A_1 \to A_2 \to ...$$

and B is the colimit of this diagram. This means we have maps $f_i : A_i \to B$ such that

$$\begin{array}{ccccccccc} ... & \longrightarrow & A_{i-1} & \longrightarrow & A_i & \longrightarrow & A_{i+1} & \longrightarrow & ... \\ & & & \searrow^{f_{i-1}} & \downarrow f_i & \swarrow_{f_{i+1}} & & & \\ & & & & B & & & & \end{array}$$

commutes for all $i > 1$. Starting with the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow f & & \downarrow p \\ B & \xrightarrow{k} & Y \end{array}$$

we can replace f and get a new commutative diagram

$$\begin{array}{ccccc} A & & \xrightarrow{h} & & X \\ \downarrow & & & & \downarrow p \\ A_1 & \xrightarrow{f_1} & B & \xrightarrow{k} & Y \end{array}$$

and since A_1 is obtained from A by adding cells we saw in step 3 of this proof that there exists $H_1 : A_1 \to X$ such that

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow & \nearrow_{H_1} & \\ A_1 & & \end{array}$$

commutes and $k \circ f_1 = p \circ H_1$. But then we get a commutative diagram

$$\begin{array}{ccccc} A_1 & & \xrightarrow{H_1} & & X \\ \downarrow & & & & \downarrow p \\ A_2 & \xrightarrow{f_2} & B & \xrightarrow{k} & Y \end{array}$$

and we get a map H_2 such that

$$\begin{array}{ccc} A_1 & \xrightarrow{H_1} & X \\ \downarrow & \nearrow_{H_2} & \\ A_2 & & \end{array}$$

commutes and $k \circ f_2 = p \circ H_2$. Continuing like this we get maps $H_i : A_i \to X$ such that

$$\begin{array}{ccc} A_i & \xrightarrow{H_i} & X \\ \downarrow & \nearrow_{H_{i+1}} & \\ A_{i+1} & & \end{array}$$

commutes and $k \circ f_i = p \circ H_i$ for all i. But since $B = Colim A_i$ we get a map $H : B \to X$ such that $H_i = H \circ f_i$ for all i. In particular $h = H_0 = f_0 \circ H = f \circ H$. It remains to check that $p \circ H = k$. This however follows from the universal property of a colimit because we get maps $k \circ f_i : A_i \to Y$ such that

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \Big\downarrow{\scriptstyle k \circ f_i} & \swarrow_{k \circ f_{i+1}} & \\ Y & & \end{array}$$

commutes for all i. But then there is a unique map $F : B \to Y$ such that $k \circ f_i = F \circ f_i$ but both k and $p \circ H$ fits as F so they must equal.

□

Next we will prove an extremely useful factorization property that morphisms in the category of augmented dga algebras enjoy.

Proposition 47. Any morphism of augmented dga algebras $f : A \to B$ factors as $f = p \circ j$ where j is a relative cell algebra inclusion and a quasi isomorphism and p is surjective. Moreover j has the following lifting property: for any surjective morphism $q : X \to Y$ and commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle q} \\ B & \xrightarrow{k} & X \end{array}$$

there exists a map $H : B \to X$ such that $H \circ j = h$ and $q \circ H = k$.

Proof. Let V be the graded vector space with basis $\{s^{-1}e_b\}_{b \in B}$ where $|e_b| = |b| - 1$. Let $A[e_b]_{b \in B}$ be the algebra we get from the pushout

$$\begin{array}{ccc} T(V) & \longrightarrow & T(CV) \\ \Big\downarrow{\scriptstyle u_A \circ \epsilon_{T(V)}} & & \Big\downarrow \\ A & \longrightarrow & A[e_b]_{b \in B} \end{array} \qquad (*)$$

where the leftmost map is the composite of the augmentation of $T(V)$ and the unit of A. Then let W be the graded vector space with basis $\{s^{-1}x_b\}_{b \in B}$ where

$|x_b| = |b|$. Then we define $A[e_b, x_b, dx_b = e_b]_{b\in B}$ as the pushout

$$\begin{array}{ccc} T(W) & \longrightarrow & T(CW) \\ \Big\downarrow{\scriptstyle s^{-1}x_b \mapsto e_b} & & \Big\downarrow \\ A[e_b]_{b\in B} & \longrightarrow & A[e_b, x_b, dx_b = e_b]_{b\in B} \end{array} \qquad . \qquad (**)$$

Then let $j : A \to A[e_b, x_b, dx_b = e_b]_{b\in B}$ be the inclusion. Further let $f' : A[e_b]_{b\in B} \to B$ be the map induced by $f : A \to B$ and $T(CV) \to \mathbb{K} \to B$. Then we let $p : A[e_b, x_b, dx_b = e_b]_{b\in B} \to B$ be the map induced by $f' : A[e_b]_{b\in B} \to B$ and $T(CW) \to B,\ s^{-1}x_b \mapsto db,\ x_b \mapsto b$. Now I claim that $p \circ j = f$. Indeed if we denote by $i : A \to A[e_b]_{b\in B}$ and $i' : A[e_b]_{b\in B} \to A[e_b, x_b, dx_b = e_b]_{b\in B}$ we see that $f = f' \circ i = p \circ i' \circ i = p \circ j$.

The fact that j is a cell algebra algebra inclusion follows from the fact that $A[e_b, x_b, dx_b = e_b]_{b\in B}$ was obtained from A in two steps by adding cells. Also, p is surjective by construction.

It remains to check that j has the lifting property.

We first show that the linear map $0 \to CW$ has the corresponding lifting property in the category of chain complexes. Indeed, consider the following commutative diagram of chain complexes

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ CW & \longrightarrow & Y \end{array}$$

where p is surjective. We can then define $H : CW \to X$ by $x_b \mapsto c_b$ where c_b is any element in the preimage $p^{-1}(k(x_b))$ and $s^{-1}x_b \mapsto d_X(c_b)$.

As in the proof of the previous proposition it then follows that the morphism of dga algebras $T(0) = \mathbb{K} \to T(CW)$ has the desired lifting property.

But using the fact that $(*)$ and $(**)$ are pushout diagrams one can show that there is a pushout diagram

$$\begin{array}{ccc} \mathbb{K} & \longrightarrow & A \\ \Big\downarrow{\scriptstyle l} & & \Big\downarrow{\scriptstyle j} \\ T(CW) & \longrightarrow & A[e_b, x_b, dx_b = e_b]_{b\in B} \end{array} \qquad . \qquad (***)$$

and then we are done because we have seen that $\mathbb{K} \to T(CW)$ has the desired lifting property and as in the proof of the previous proposition we get that j has it too. Finally we show that j is a quasi isomorphism. To do this we introduce the dga algebra

$$A[CW] = \mathbb{K} \oplus \bar{A} \oplus (\bar{A} \otimes CW \otimes \bar{A}) \oplus (\bar{A} \otimes CW \otimes \bar{A} \otimes CW \otimes \bar{A}) \oplus ...$$

with multiplication given by

$$a_1 \otimes s^{-1}x_b \otimes ... \otimes x_{b'} \otimes a_n \bigotimes a'_1 \otimes x_{b''} \otimes ... \otimes a'_m \mapsto$$

$$a_1 \otimes s^{-1}x_b \otimes ... \otimes x_{b'} \otimes a_n a'_1 \otimes x_{b''} \otimes ... \otimes a'_m$$

and where the differential is obtained from d_A and d_{CW} by taking tensor products and direct sums. Now I claim that $A[CW]$ also fits into the pushout diagram $(***)$ where $T(CW)$ is identified with the subalgebra

$$\mathbb{K} \oplus (\mathbb{K}1_A \otimes CW \otimes \mathbb{K}1_A) \oplus (\mathbb{K}1_A \otimes CW \otimes \mathbb{K}1_A \otimes CW \otimes \mathbb{K}1_A) \oplus ... \subset A[CW]$$

and A is identified with the subalgerba $\mathbb{K} \oplus \bar{A} \subset A[CW]$. We have to show that given a commutative diagram

$$\begin{array}{ccc} \mathbb{K} & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \phi_1} \\ T(CW) & \xrightarrow{\phi_2} & D \end{array}$$

there is a unique morphism $\phi : A[CW] \to D$ such that

$$\begin{array}{ccccc} T(CW) & \hookrightarrow & A[CW] & \hookleftarrow & A \\ & {\scriptstyle \phi_2}\searrow & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \phi_1} & \\ & & D & & \end{array}$$

commutes. If ϕ exists it is unique because A and $T(CW)$ generate $A[CW]$. For existence we define $\phi(a_1 \otimes s^{-1}x_b \otimes ... \otimes x'_b \otimes a_n) = \phi_1(a_1)\phi_2(s^{-1}x) \cdots \phi_2(x'_b)\phi_1(a_n)$. This is well defined because on each direct summand $\bar{A} \otimes CW \otimes \bar{A} \otimes CW \otimes ... \otimes \bar{A}$ it is the composition of $\phi_1 \otimes \phi_2 \otimes \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_1$ and the product in D. The fact that ϕ is a chainmap and an algebra morphism follows from the fact that ϕ_1 and ϕ_2 are. But then since $A[CW]$ and $A[e_b, x_b, dx_b = e_b]_{b \in B}$ both fit in the pushout diagram $(***)$ there is an isomorphism $A[CW] \cong A[e_b, x_b, dx_b = e_b]$ which fixes A. Because CW is acyclic it follows from Kunneth's formula [9] that $H(A[CW]) = \mathbb{K} \oplus H(\bar{A}) = H(A)$ and the inclusion $A \hookrightarrow A[CW]$ is a quasi isomorphism but then so is $j : A \hookrightarrow A[e_b, x_b, dx_b = e_x]_{b \in B}$ because there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & A[e_b, x_b, dx_b = e_b]_{b \in B} \\ \downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle \cong} & \\ A[CW] & & \end{array} .$$

□

We prove one final lifting property. In the language of homotopical algebra this shows that shows that ΩC is cofibrant.

Proposition 48. Let C be a coaugmented dga coalgebra which is connected. Then the the unit map $\mathbb{K} \to \Omega C$ has the following lifting property: For any surjective quasi isomorphism of dga algebras $p : A \to B$ and dga algebra morphism

$f : \Omega C \to B$ we can find a lift H as in the diagram below

$$\begin{array}{ccc} & & A \\ & \overset{H}{\nearrow} & \downarrow p \\ \Omega C & \xrightarrow{f} & B \end{array} .$$

Proof. We define maps $H_n : T(s^{-1}(\oplus_{i=0}^{n}\bar{C}_i)) \to A$ inductively. Set $H_{-1} : T(0) = \mathbb{K} \to A$ be the unit map for A. Assume by induction that we have constructed a morphism of augmented dga algebras H_{n-1} such that

$$\begin{array}{ccccc} & & & & A \\ & & \overset{H_{n-1}}{\nearrow} & & \downarrow p \\ T(\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i) & \hookrightarrow & \Omega C & \xrightarrow{f} & B \end{array}$$

commutes. This gives us a commutative diagram of vector spaces

$$\begin{array}{ccccc} & & & & A \\ & & \overset{\phi_{n-1}}{\nearrow} & & \downarrow p \\ \oplus_{i=0}^{n-1} s^{-1}\bar{C}_i & \hookrightarrow & \Omega C & \xrightarrow{f} & B \end{array}$$

where ϕ_{n-1} is the composition $\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i \hookrightarrow T(\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i) \xrightarrow{H_{n-1}} A$. We will extend it to a map $\phi_n : \oplus_{i=0}^{n} s^{-1}\bar{C}_i \to A$. Let $\{e_\alpha^n\}$ be a basis for $\bar{C}_n$. Let $z_\alpha = H_{n-1}(d_{\Omega C}(s^{-1}e_\alpha^n))$. Note that H_{n-1} is defined on $d_{\Omega C}(e_\alpha^n)$ because $d_{\Omega C} = d_1 + d_2$ where d_1 is induced by d_C which maps e_α^n into $\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i$ and d_2 which is induced by the coproduct maps e_α^n into

$$\bigoplus_{i+j=n} s^{-1}\bar{C}_i \otimes s^{-1}\bar{C}_j \subset T(\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i).$$

Because $d_{\Omega C}(s^{-1}e_\alpha^n)$ is a cycle in $T(\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i)$ and H_{n-1} is a chain map z_α is a cycle too. Also $d_{\Omega C}(s^{-1}e_\alpha^n)$ is a boundary in ΩC so $p(z_\alpha) = f(d_{\Omega C}(s^{-1}e_\alpha^n))$ is a boundary in B. But since p is a quasi isomorphism this implies z_α is a boundary in A. Let $a_\alpha \in A$ be such that $d_A(a_\alpha) = z_\alpha$. Then since

$$d_B(p(a_\alpha)) = p(H_{n-1}(d_{\Omega C}(s^{-1}e_\alpha^n))) = f(d_{\Omega C}(s^{-1}e_\alpha^n)) = d_B(f(s^{-1}e_\alpha^n))$$

we see that $p(a_\alpha) - f(s^{-1}e_\alpha^n)$ is a cycle in B. But the fact that p is surjective and that p is a quasi isomorphism implies that there is a cycle $a'_\alpha \in A$ such that $p(a'_\alpha) = p(a_\alpha) - f(s^{-1}e_\alpha^n)$. Then $d_A(a_\alpha - a'_\alpha) = d_A(a_\alpha) = H_{n-1}(d_{\Omega C}(s^{-1}e_\alpha^n))$ and $p(a_\alpha - a'_\alpha) = f(s^{-1}e_\alpha^n)$. We can now define a linear map $\phi_n : \oplus_{i=0}^{n} s^{-1}\bar{C}_i \to A$ by $\phi_n = \phi_{n-1}$ on $\oplus_{i=0}^{n-1} s^{-1}\bar{C}_i$ and $\phi_n(e_\alpha^n) = a_\alpha - a'_\alpha$. This induces $H_n : T(\oplus_{i=0}^{n} s^{-1}\bar{C}_i) \to A$ and H_n is a morphism of dga algebras because is commutes with the differential on all the generators by construction. Moreover we have $f|_{T(\oplus_{i=0}^{n} s^{-1}\bar{C}_i)} = p \circ H_n$ by construction. □

Now we are ready to prove proposition 45.

Proof. We will prove four special cases from which the proposition will then follow. As the first case let's assume there is a weak equivalence $C' \to C$ and $A = A'$. Then since $\kappa : C \to A$ is a Koszul twisting morphism there is a quasi isomorphism $\Omega C \to A$ (fundamental theorem of twisting morphisms). But since there is a weak equivalence $C \to C'$ we get a quasi isomorphism $\Omega C' \to \Omega C$ and since composition of quasi isomorphisms is again a quasi isomorphism we get a quasi isomorphism $\Omega C' \to A$ and then there is a Koszul twisting morphism $\kappa' : C \to A$ (fundamental theorem of twisting morphisms).

The second case where we assume $C = C'$ and assume there is a weak equivalence $A \to A'$ is very similar to the first case.

As our third case we assume there is a weak equivalence $C \to C'$ and $A = A'$. Then there is a quasi isomorphism $f : \Omega C \to \Omega C'$. I claim that there is one in the opposite direction as well. Indeed, we know that f admits a factorization $f = p \circ j$ where p is surjective and j is a quasi isomorphism which has the lifting property described in proposition 48. But since f and j are both quasi isomorphisms it follows that p must be too. We get a commutative diagram

$$\begin{array}{ccccc} \mathbb{K} & \longrightarrow & \Omega C & \xrightarrow{j} & B \\ \downarrow & & & & \downarrow p \\ \Omega C' & & \xrightarrow{=} & & \Omega C' \end{array} .$$

By proposition 49 we get a map H

$$\begin{array}{ccccc} \mathbb{K} & \longrightarrow & \Omega C & \xrightarrow{j} & B \\ \downarrow & & \overset{H}{\nearrow} & & \downarrow p \\ \Omega C' & & \xrightarrow{=} & & \Omega C' \end{array}$$

such that everything commutes. Also, because p is a quasi isomorphism so is H. Next consider the commutative diagram

$$\begin{array}{ccc} \Omega C & \xrightarrow{=} & \Omega C \\ \downarrow j & & \downarrow \\ B & \longrightarrow & \mathbb{K} \end{array} .$$

Because of the lifting property that j enjoys we get a map h as in the diagram

$$\begin{array}{ccc} \Omega C & \xrightarrow{=} & \Omega C \\ \downarrow j & \overset{h}{\nearrow} & \downarrow \\ B & \longrightarrow & \mathbb{K} \end{array}$$

and beacuse j is a quasi isomorphism h is too. But then $h \circ H : \Omega C' \to \Omega C$ is a quasi isomorphism and composing with the quasi isomorphism $\Omega C \to A$ we get a quasi isomorphism $\Omega C' \to A$ and then there is a Koszul twisting morphism $C' \to A$.

As our last case we consider $C = C'$ and there is a quasi isomorphism $g : A' \to A$. Factorize g as $q \circ i$ where p is surjective and i is a quasi isomorphism with the lifting property described in proposition 48. By proposition 49 we get the dashed arrow ϕ in the diagram below

$$\begin{array}{ccc} A' & \xrightarrow{i} & D \\ & \overset{\phi}{\nearrow} & \downarrow q \\ \Omega C & \longrightarrow & A \end{array} .$$

Because i and $g = q \circ i$ are quasi isomorphisms q is too. But then since the bottom arrow is a quasi isomorphism ϕ must be as well. Now by the lifting property that i enjoys we get the dashed arrow k in the diagram below.

$$\begin{array}{ccc} A' & \xrightarrow{=} & A' \\ \downarrow i & \overset{k}{\nearrow} & \downarrow \\ D & \longrightarrow & \mathbb{K} \end{array} .$$

Because i is a quasi isomorphism k is too. But then $k \circ \phi : \Omega C \to A'$ is a quasi isomorphism and then there is a Koszul twisting morphism $\kappa' : C \to A'$ (by the fundamental theorem of twisting morphisms).

□

Before moving on to the proof of the main theorem we state a proposition very similar to proposition 49 which we will use later in chapter 4.

Proposition 49. Let W be a graded coalgebra which as a vector space has an extra weight grading $W = \oplus_{n \geq 0} W_{(n)}$. If $(T(W), d)$ is a dga algebra such that d is homogenous of degree -1 with respect to weight then $(T(W), d)$ has the same lifting property as ΩC in proposition 49.

Proof. The proof is very similar to that of proposition 49 but instead of using induction on the homological degree we use induction on the weight. □

3.3 Proof of main theorem

The following lemma is the final thing needed to prove the main theorem.

Lemma 50. Suppose there is a quasi isomorphism of dga coalgebras $\phi : C \to BA$. Then it is a weak equivalence, i.e. $\Omega C \to \Omega BA$ is a quasi isomorphism.

Proof. There are commutative diagrams,

$$\begin{array}{ccc} BA & \xrightarrow{=} & BA \\ \downarrow & \searrow & \downarrow \pi \\ \Omega BA & \xrightarrow{\epsilon} & A \end{array} , \quad \begin{array}{ccc} C & \xrightarrow{\phi} & BA \\ \downarrow & \overset{\kappa}{\searrow} & \downarrow \pi \\ \Omega C & \xrightarrow{\psi} & A \end{array}$$

where all the vertical maps are quasi isomorphisms. Now I claim that we have an equality of maps $\epsilon \circ \Omega\phi = \psi$. To see this let $s^{-1}c_1 \otimes ... \otimes s^{-1}c_k$ be an element of ΩC. Then from how the functor Ω is defined on morphisms we see that

$$\Omega\phi(s^{-1}c_1 \otimes ... \otimes s^{-1}c_k) = \phi(c_1) \otimes ... \otimes \phi(c_k).$$

But then, since ϵ is the algebra morphism corresponding to the twisting morphism π through the bijections of proposition 36 we see that

$$\epsilon \circ \Omega\phi(s^{-1}c_1 \otimes ... \otimes s^{-1}c_k) = \pi \circ \phi(c_1) \cdots \pi \circ \phi(c_k).$$

But then by commutativity of the diagram above we see that the last expression equals $\kappa(c_1) \cdots \kappa(c_k)$ which is precisely $\psi(s^{-1}c_1 \otimes ... \otimes s^{-1}c_k)$. Now since ϵ and $\epsilon \circ \Omega\phi = \psi$ are both quasi isomorphisms it follows that $\Omega\phi$ is too.

□

We are ready to prove the main theorem.

Proof. We begin by proving that (1) $\implies$ (2) and (3). Assume C and A are formal. By proposition 45 there is a Koszul twisting morphism $\kappa' : H_\bullet(C) \to H_\bullet(A)$. Since these have trivial differential it follows from proposition 43 that $H_\bullet(C)$ and $H_\bullet(A)$ are Koszul.

Next we prove (2) $\implies$ (1). Since $A \sim H_\bullet(A) =: A'$ proposition 45 tells us there is a Koszul twisting morphism $\kappa' : C \to A'$. By the fundamental theorem of twisting morphisms there is a quasi isomorphism $C \to BA'$ and by the previous lemma this is in fact a weak equivalence. But A' is Koszul so the inclusion $A'^{\text{¡}} \hookrightarrow BA'$ is a quasi isomorphism. The previous lemma implies that this is in fact a weak equivalence. But $A'^{\text{¡}}$ has trivial differential so $H_\bullet(A'^{\text{¡}}) = A'^{\text{¡}}$. From what we just said we know that $H_\bullet(A'^{\text{¡}}) \cong H_\bullet(BA')$ and since we had a quasi isomorphism $C \to BA'$ we have an isomorphism $H_\bullet(C) \cong H_\bullet(BA')$. Putting all of this together gives a zig-zag diagram

$$C \xrightarrow{\sim} BA' \xleftarrow{\sim} A'^{\text{¡}} = H_\bullet(A'^{\text{¡}}) \cong H_\bullet(BA'^{\text{¡}}) \cong H_\bullet(C)$$

which proves that C is formal.

Finally we prove (3) $\implies$ (1). Since C is now formal we know $C \sim H_\bullet(C) =: C'$. By proposition 45 there is a Koszul twisting morphism $\kappa' : C' \to A$. But then there is a quasi isomorphism $\Omega C' \to A$. Also, since C' is Koszul there is a quasi isomorphism $\Omega C' \to C'^{\text{¡}}$. But $C'^{\text{¡}}$ has trivial differential so $H_\bullet(C'^{\text{¡}}) = C'^{\text{¡}}$. Since we already said there are quasi isomorphisms $\Omega C' \to A$ and $\Omega C' \to C'^{\text{¡}}$ we get isomorphisms $H_\bullet(A) \cong H_\bullet(\Omega C') \cong H_\bullet(C'^{\text{¡}})$. Since quasi isomorphisms between dga algebras are by definition weak equivalences we get a zig-zag diagram

$$A \xleftarrow{\sim} \Omega C' \xrightarrow{\sim} C'^{\text{¡}} = H_\bullet(C'^{\text{¡}}) \cong H_\bullet(\Omega C') \cong H(A)$$

which proves that A is formal.

□

3.4 Alternative definition of Koszul algebras

The definition of Koszul algebras that we gave in section 3.1 is somewhat different from the classical notion of Koszul algebras in two ways. The more classical way refers to some quadratic data of the algebra when defining the Koszul property, where as the definition we use is not a priori limited to quadratic algebras. The other difference is that the definition we use allows for the algebra to come with a differential. We will in this section show that for algebras with trivial differential a Koszul algebra using our definition is equivalent to the more classical. Let us briefly recall the classical definition.

Definition 51. A *quadratic data* (V, R) is a graded vector space V and a subspace $R \subset V \otimes V$. A *quadratic algebra* with quadratic data (V, R) is a graded algebra $A(V, R)$ isomorphic to $T(V)/(R)$. A *quadratic coalgebra* with quadratic data (V, R) is a graded coalgebra C(V,R) isomorphic to the following subcoalgebra of the tensor coalgebra

$$\mathbb{K} \oplus V \oplus R \oplus \big(R \otimes V \cap V \otimes R\big) \oplus ... \oplus \Big(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\Big) \oplus ...$$

One can show [3] that given quadratic data (V, R) there is a twisting morphism

$$\alpha : C(sV, s^2R) \to A(V, R)$$

which is zero everywhere except on the weight one part sV where it is defined by $sV \ni sv \mapsto v \in V$. The more classical definition of Koszul algebras is as follows.

Definition 52. The quadratic algebra (V, R) is Koszul in the classical sense if the twisting morphism $\alpha : (sV, s^2V) \to (V, R)$ is a Koszul twisting morphism.

Proposition 53. If a connected weight graded algebra A is Koszul as in section 3.1 then it is quadratic and Koszul in the classical sense.

Proof. A being Koszul means by definition that $A^{¡} \hookrightarrow BA$ is a quasi isomorphism. By the fundamental theorem of twisting morphisms we have a commutative diagram

$$\begin{array}{ccccc} & & BA & & \\ & \overset{\sim}{\nearrow} & & \searrow & \\ A^{¡} & & \longrightarrow & & A \\ & \searrow & & \overset{\sim}{\nearrow} & \\ & & \Omega A^{¡} & & \end{array}$$

where the arrows marked with $\sim$ are quasi isomorphisms. Almost by definition $A^{¡}$ is the weight graded coalgebra

$$A^{¡} = \mathbb{K} \oplus sA(1) \oplus (sA(1))^{\otimes 2} \cap ker(d_2 : (sA(1))^{\otimes 2} \to sA(2)) \oplus ...$$

$$\oplus \bigcap_{i+2+j=n} \Big((sA(1))^{\otimes i} \otimes (sA(1))^{\otimes 2} \cap ker(d_2 : (sA(1))^{\otimes 2} \to sA(2)) \otimes (sA(1))^{\otimes j} \Big) \oplus ...$$

Then $\Omega A^{¡}$ looks as in table 2 with C replaced by $A^{¡}$ and we see that the diagonal is just the tensor algebra of $A(1)$, $\mathscr{D}_{\Omega A^{¡}} = T(s^{-1}A^{¡}(1)) = T(A(1))$. As in the proof of proposition 43 one can show that the quasi isomorphism $g : \Omega A^{¡} \to A$ in the diagram above vanishes outside the diagonal. Moreover, since A has trivial differential, so does $A^{¡}$ and then the only differential on $\Omega A^{¡}$ is the one induce by the coproduct δ_2. But then, since the diagonal of $\Omega A^{¡}$ is contained in the kernel of δ_2 and the map g is a quasi isomorphism we see that A is isomorphic to a quotient of the diagonal $A \cong T(A(1))/(im(\delta_2) \cap T(A(1)))$. But $(im(\delta_2) \cap T(A(1))$ is precisely the two sided ideal in $T(A(1))$ generated by $im\big(\delta_2 : s^{-1}A^{¡}(2) \to A(1) \otimes A(1)\big) \subset A(1) \otimes A(1)$. This proves that A is quadratic with quadratic data $(V, R) = (A(1), im(\delta_2 : s^{-1}A^{¡}(2) \to A(1) \otimes A(1)))$.

To see that A is Koszul in the classical sense we just have to note that under the identification $A \cong T(V)/(R)$ we have $ker\big(d_2 : sA(1) \otimes sA(1) \to sA(2)\big) = ker\big(d_2 : sV \otimes sV \to s(V \otimes V/R)\big) = s^2R$ because d_2 is induced by the product of A and under the identification $A \cong T(V)/(R)$ this is just concatenating tensors and then taking the quotient. But then we see that $A^{¡} = C(sV, s^2R)$ and the Koszul twisting morphism in the diagram above is precisely the map α from definition 20.

□

4 Connection to topology and some examples

Much of the algebra developed in the previous chapters is motivated by problems in algebraic topology. For instance the cobar construction Ω is closely related to Moore's loop space construction of a topological space (denoted Ω_b). Adams' theorem states that $H_\bullet(\Omega C_\bullet(X)) \cong H_\bullet(\Omega_b X)$ for a simply connected topological space X. Also the theory of chapter 3.2 is related to topology; the category of topological spaces is one of the first examples of model categories. The main theorem in this thesis has a topological counterpart too. We will state, but not prove it, in the following section.

4.1 Coformal and formal topological spaces

The notion of formality originally comes from algebraic topology.

Definition 54. Let $\mathbb{K}$ be a field. A topological space X is $\mathbb{K}$-*formal* if $C^\bullet(X; \mathbb{K})$ is formal as a dga algebra.

For topological spaces we also have the notion of coformality. Before giving the precise definition we recall what Moore's loop space of a topological space X is and we quickly describe the algebra structure on it. Moore's loop space of a pointed topological space (X, x_0) is the set of pairs (f, t_0) where f is continuous function $[0, \infty) \to X$ and $t_0 \in [0, \infty)$ such that $f(0) = x_0$ and $f(t) = x_0$ for all

$t \geq t_0$. It inherits a topology from the product $X^{[0,\infty)} \times [0,\infty)$. Moreover it has a natural monoid structure defined by

$$(f, t_0)\dot{(}g, t_1) = (f * g, t_0 + t_1)$$

where $f * g : [0, \infty) \to X$ is the function defined by

$$f * g(t) = \begin{cases} f(t), \ 0 \leq t \leq t_0 \\ g(t), \ t_0 \leq t \leq t_0 + t_1. \end{cases}$$

This multiplication is continuous and hence induces a multiplication on $C_\bullet(X)$. One can show that this makes $C_\bullet(X)$ into a dga algebra.

Definition 55. A topological space is $\mathbb{K}$-coformal if $C_\bullet(\Omega_b X; \mathbb{K})$ is formal as a dga algebra.

Spaces that are both formal and coformal have the special property that their loop space homology can be computed from their cohomology algebra. Here is the topological counterpart to the main theorem in the previous chapter.

Theorem 56. Let X be a simply connected topological space of finite $\mathbb{K}$-type then conditons 1-3 below are equivalent and they imply condition 4.
1) X is both formal and coformal,
2) $C^\bullet(X; \mathbb{K})$ is formal as a dga coalgebra and $H^\bullet(X)$ is Koszul.
3) $C_\bullet(\Omega X : \mathbb{K})$ is formal as a dga algebra and $H_\bullet(\Omega X)$ is Koszul.
4) $H_\bullet(\Omega_b X : \mathbb{K})$ and $H^\bullet(X)$ are both Koszul and they are Koszul dual

$$H_\bullet(\Omega_b X : \mathbb{K}) \cong H^\bullet(X : \mathbb{K})^!.$$

4.2 Bigraded- and filtered models

In what follows we will work with cochain algebras of topological spaces. These are concentrated in non-negative cohomological degree. In this section we develop some tools that will aid us in our study of these cochain algebras and in particular formality of those. More precisely we define the bigraded model of a graded algebra A. Then we describe how the differential of the bigraded model can be perturbed to get a filtered model for any cochain algebra whose cohomology is A. Halperin and Stasheff first used these ideas in their article [7] when studying obstructions to homotopy equivalences and Haouari later generalised their ideas to the non-commutative case.

Definition 57. Let A be a graded algebra concentrated in non-negative cohomological degrees. A *bigraded model* for A is a quasi isomorphism $\rho : (T(V), d) \to (A, 0)$ such that
i) $V = \oplus_{k,n \geq 0} V_n^k$ is a bigraded vector space, making $T(V)$ a bigraded algebra, and $d : T(V)_n^k \to T(V)_{n-1}^{k+1}$,
ii)$H_n(T(V), d) = 0$ for $n > 0$. We call the upper grading on V and $T(V)$ cohomological degree and the lower grading weight.

If A is for example the cohomology algebra of a dga algebra (Y, d_Y) then a bigraded model for A together with the following theorem can be of use when examining whether (Y, d_Y) is formal or not.

Theorem 58. Let C be a dga algebra concentrated in positive cohomological degrees and assume the cohomology $H(C) = A$ is connected. If $\rho : (T(Z), d) \to (A, 0)$ is a bigraded model for A then there is a differential D on $T(Z)$ and a morphism of dga algebras $\pi : (T(Z), D) \to (C, d_C)$ such that
$i)(D - d) : Z_n \to \bigoplus_{i=0}^{n-2} T(Z)_i$
$ii)$if $z \in T(Z_0)$ then $\pi(z)$ is a cycle in C and $[\pi z] = \rho z$
$iii)\pi$ is a quasi isomorphism.
Moreover, if D' is some other differential such that $(T(V), D')$ also has all the properties above, then there is an isomorphism $\psi : (T(W), D) \to (T(W), D')$ such that $\phi - id$ lowers filtration level.

The proof is from an article by Stephen Halperin and James Stasheff [7]. They prove it for graded commutative dga algebras but the same arguments work for non-commutative dga algebras. We include the proof of the existence since we will be referring to the construction performed in the proof. For the last statement concerning uniqueness however we refer to [7] and [10] for the non-commutative case. The proof of the theorem depends on the following lemma.

Lemma 59. If we have a differential D' on $T(Z_{\leq n})$ such that $(D' - d) : Z_l \to \oplus_{i=0}^{l-2} T(Z)_i$ for $0 \leq l \leq n$. Then if $u \in \oplus_{i=0}^{n-1} T(Z)_i$ is in $ker(D')$ there are elements $v \in \oplus_{i=0}^{n} T(Z)_i$ and $\alpha \in A$ such that $u = D'(v) + \eta(\alpha)$ where $\eta : A \to T(Z_0)$ is a linear map such that $\rho \circ \eta = id_A$.

Proof. The lemma is proved using induction on n. For $n = 1$ we have $u \in T(Z)_0$. Set $\alpha = \rho(u)$. Then since u and $\eta(\rho(u))$ both are in $T(Z_0)$ they are both in $ker(d)$. But $\rho(u - \eta(\rho(u))) = 0$ and since ρ is a quasi isomorphism there is a $v \in T(Z)_1$ such that $dv = u - \eta(\alpha)$. Finally since $D' - d$ lowers the weight by 2 D' and d agree on $T(Z)_0$ and we get

$$u = D'(v) + \eta(\alpha).$$

Now assume the claim is proved up to $n - 1$ and we will prove it holds for n aswell. Now $u \in \oplus_{i=0}^{n-1} T(Z)_i$ and we can write

$$u = \sum_{i=0}^{n-1} u_i,\ u_i \in T(Z)_i.$$

By assumption $(D' - d)(u) \in \oplus_{i=0}^{n-3} T(Z)_i$ and since $D'(u) = 0$ and d lowers the weight by precisely 1 we get $d(u_{n-1}) = 0$. Because the cohomology $H(T(Z), d)$ vanishes in positive weight there is a $v' \in T(Z)_n$ such that $u_{n-1} = d(v')$. But then $u - D'(v') = -(D' - d)(v') \in \oplus_{i=0}^{n-2} T(Z)_i$ and since $D'(u - D'(v')) = 0$ the induction hypothesis tells us there are $\alpha \in A$ and $v'' \in \oplus_{i=0}^{n-1} T(Z)_i$ such that $u - D'(v') = \eta(\alpha) + D'(v'')$ and if we set $v = v' + v''$ we get $u = \eta(\alpha) + D'(v)$ as desired. $\square$

Now we are ready to prove the theorem.

Proof. We construct the maps D and π inductively. We first set $D = d$ on $T(Z_{\leq 1})$. Define $\pi|_{Z_0} : Z_0 \to C$ on a basis by sending a basis element z to any representative of $\rho(z) \in A = H(C)$. This extends to a unique morphism of algebras $T(Z_0) \to C$ and we have $\pi \circ D = d_C \circ \pi = 0$ on $T(Z_0)$ so it is in fact a morphism of dga algebras. Next we define π on an basis of Z_1. Let $x \in Z_1$ be a basis element. Then I claim that $\pi(d(x))$ is a boundary in C. Indeed this follows from how π was defined on $T(Z_0)$ because

$$[\pi(d(x))] = \rho(d(x)) = 0.$$

Then let $c \in C$ be an element such that $d_C(c) = \pi(d(x))$ and define π on Z_1 by $x \mapsto c$. This extends to a unique morphism of dga algebras $T(Z_{\leq 1}, D) \to (C, d_C)$. Now we extend π and D to $T(Z_{\leq 2})$ as follows. We fix a basis for Z_2 and consider a basis element z. Then $d(z) \in T(Z)_1 \subset T(Z_{\leq 1})$ and because $D = d$ on $T(Z_{\leq 1})$ we have $d(z) \in ker(D)$. But since $\pi : (T(Z_{\leq 1}), D) \to (C, d_C)$ is a morphism of dga algebas $\pi(d(z))$ is a cycle in C so $[\pi(d(z))]$ is an element of A. D is then defined on Z_2 by

$$z \mapsto dz - \eta([\pi(d(z))]).$$

Note that since d is of degree 1 and η and π are both of degree 0 (upper degree) D is of degree 1 and it then extends to a derivation on $T(Z_{\leq 2})$. We saw already that $D^2 = 0$ on Z_0 and Z_1 and if $z \in Z_2$ then $D^2(z) = dD(z) = d(d(z) - \eta[\pi(d(z))]) = 0$ because η lands in $T(Z_0) \subset ker(d)$. So D extends to a differential on $T(Z_{\leq 2})$. Now we extend π to Z_2. We saw already that $\pi(d(z))$ is a cycle in C and since η lands in $T(Z_0) \subset ker(D)$ and $\pi : (T(Z_{\leq 1}), D) \to (C, d_C)$ is a morphism of dga algebras $\pi(\eta([\pi(d(z))]))$ is also a cycle of C. Then $\pi(D(z))$ is a cycle too and we have

$$[\pi(D(z))] = [\pi(d(z))] - [\pi(\eta([\pi(d(z))]))] = [\pi(d(z))] - \rho(\eta([\pi(d(z))])) = [\pi(d(z))] - [\pi(d(z))] = 0$$

which means $\pi(D(z))$ is in fact a boundary in A. We then define π on a basis for Z_2 by $z \mapsto c$ where c is such that $d_C(c) = \pi(D(z))$. This extends π to $T(Z_{\leq 2})$. Defined in this way D satisfies *i)* because $(D - d) = 0$ on Z_0 and Z_1 and for $z \in Z_2$ we have $(D - d)(z) = \eta([\pi(d(z))]) \in T(Z_0)$.

Now for the induction step assume a differential D is defined on $T(Z_{\leq n-1})$ such that *i)* holds and suppose further $\pi : (T(Z_{n-1}, D)) \to (C, d_C)$ has been defined. We then extend D to Z_n. Fix a basis for Z_n and let z be a basis element. Then $D(d(z)) = (D - d)(d(z)) \in \oplus_{i=0}^{n-3} T(Z)_i \subset T(Z_{\leq n-3})$ so by the previous lemma there are elements $\alpha \in A$ and $v \in \oplus_{i=0}^{n-2} T(Z)_i$ such that $D(d(z)) = \eta(\alpha) + D(v)$. Now I claim that $\alpha = 0$. Indeed we have $\pi(\eta(\alpha)) = \pi(D(d(z))) - \pi(D(v)) = d_A(\pi(d(z) - v))$ which means $\pi(\eta(\alpha))$ is boundary so $0 = [\pi(\eta(\alpha))] = \rho(\eta(\alpha)) = \alpha$. This means $d(z) - v \in ker(D)$. Then if $(d(z))^m$

and v^m denote the components of $d(z)$ and v of cohomological degree m we have $d(z)^m - v^m \in ker(D)$ for all m. Let $w = \sum_m \lambda_m v^m$ where

$$\lambda_m = \begin{cases} 0 \text{ if } (d(z))^m = 0, \\ 1 \text{ if } (d(z))^m \neq 0 \end{cases} .$$

Then $d(z) - w \in ker(D)$ aswell and we can extend D to Z_n by

$$z \mapsto d(z) - w - \eta([\pi(d(z) - w)]).$$

$D|_{Z_n}$ is then homogenous of degree 1 and $D^2(z) = 0$. Also,

$$(D - d)(z) = -w - \eta([\pi(d(z) - w)])) \in \oplus_{i=0}^{n-2} T(Z)_i$$

so we can extend D to $T(Z_{\leq n})$ such that $i)$ is satisfied. But then we note that for $z \in Z_n$ we have

$$[\pi(D(z))] = [\pi(d(z) - w - \eta([\pi(d(z) - w)]))] = [\pi(d(z) - w)] - [\pi(\eta([\pi(d(z) - w)]))] =$$

$$[\pi(d(z) - w)] - [\pi(d(z) - w)] = 0$$

and we can define π on a basis of Z_n by $z \mapsto c$ where $d_C(c) = \pi \circ D(z)$ and then we get a morphism of dga algebras $\pi : (T(Z_{\leq n}), D) \to (C, d_C)$. This concludes the induction step and so we have proved existence of D and π. It remains to prove that π is a quasi isomorphism. If we think of A as a dga algebra with trivial differential then since $im\eta \subset T(Z_0) \subset ker(D)$ we have $\eta : (A, 0) \to (T(Z), D)$ is a morphism of cochain complexes. We know that $\pi_* \circ \eta_* = id_A$ so η_* is injective. Also it follows from the previous lemma that η_* is surjective so η_* is an isomorphism and then π_* must be its inverse.

□

Now we explain how to get a bigraded model for a graded algebra A in the case A is weight graded and Koszul. To do this introduce the following convention.

Convention 60. The vector space $\mathbb{K}s$ is concentrated in homological degree 1 and cohomological degree -1 and the vector space $\mathbb{K}s^{-1}$ is concentrated in homological degree -1 and cohomological degree 1. This is

Theorem 61. *If A is a non-negatively graded algebra which is Koszul then a bigraded model for A is given by $\Omega A^{¡} \twoheadrightarrow A$.*

Proof. For the proof of this we do not use convention 6. The reason for this is that we want to use the fact that a positively graded algebra can be thought of as a dga algebra concentrated in non-negative homological degrees with trivial differential but it can also be though of as a dga algebra concentrated in non-negative cohomological degrees with trivial differential. We will however use convention 60.

We call the grading on A homological grading and denote it by subscript. We equip A with another grading called the cohomological grading, denoted by superscript, which coincides with the homological grading: $A_n = A^n$ for all n. Recall that A being Koszul means that it also has an extra weight grading such that $A^{¡} \hookrightarrow BA$ is a quasi isomorphism. Then we have a commutative diagram

$$\begin{array}{ccc} A^{¡} & \xrightarrow{f} & BA \\ \downarrow & \overset{\kappa}{\searrow} & \downarrow \\ \Omega A^{¡} & \xrightarrow{g} & A \end{array}$$

where the horizontal maps are quasi isomorphisms. Now $A^{¡} \subset T^c(sA(1))$ which has a homological grading and a cohomological grading that come from the homological grading and cohomological grading of A and $\mathbb{K}s$ (see the convention above) and because d_2 is homogenous with respect to both homological and cohomological grading these gradings passes over to $A^{¡} = ker(d_2) \cap T(sA(1))$. We also have a weight grading on $A^{¡}$ namely $A^{¡}(k) = A^{¡} \cap (sA(1))^{\otimes k}$. Now we introduce a weight grading on $\bar{A}^{¡}$ which we will call syzygy degree. Set $\bar{A}^{¡}[k] := A^{¡} \cap (sA(1))^{\otimes k+1}$ which induces a syzygy degree on $\Omega A^{¡}$ too. We have that $\Omega A^{¡}$ has a triangular form with respect to weight and word length (see table 2 with $C = A^{¡}$). As in the proof of proposition 43 one can show that $g : \Omega A^{¡} \to A$ vanishes outside the diagonal. But $\Omega A^{¡}$ also has a triangular form with respect to syzygy degree and weight (see table 3) and we see that the column of syzygy degree 0 is precisely the diagonal in table 2 so g vanishes outside $(\Omega A^{¡})[0]$. Now we note that the differential δ_2 of $\Omega A^{¡}$ is homogenous of degree 1 with respect to cohomological degree. Also, $\Omega A^{¡}$ has a basis consisting of elements which are homogenous with respect to homological and cohomolgical grading and whose homological degree differ from their cohomological degree by a multiple of two, and from this it follows that δ_2 satisfies the Leibniz rule with respect to the cohomological grading. I claim that $\Omega A^{¡}$ with the cohomological grading and the syzygy grading is a bigraded model for A. Indeed the differential δ_2 is homogenous of degree -1 with respect to syzygy degree. The cohomology $H^{\bullet}(\Omega A^{¡})$ vanishes in positive syzygy degree. The quasi isomorphism g respects the cohomological grading because it vanishes outside syzygy degree 0 and in syzygy degree 0 the cohomological degree agrees with the homological degree (note that $s^{-1}A^{¡}[0] = s^{-1}sA(1) = A(1)$. □

4.3 Spheres

Spheres are examples of Koszul spaces. We will here show that the rational cohomology ring of spheres is Koszul. Then we use this fact to show that spheres are formal.

We know from [6] that

$$H^m(S^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } m \in \{0, n\} \\ 0 \text{ else} \end{cases} .$$

The ring structure is then completely determined; for degree reasons we must have $H^\bullet(S^n : \mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$ where x has cohomological degree n. Let's prove that it is Koszul.

Proposition 62. $\mathbb{Q}[x]/(x^2)$ is Koszul.

Proof. The algebra $A := \mathbb{Q}[x]/(x^2)$ is weight graded with $A(0) = \mathbb{Q}1_A$ and $A(1) = \mathbb{Q}x$ and zero in every other weight. With this weight grading we have $A^¡ = T^c(sx)$ is the tensor coalgebra on one generator sx of degree $n+1$. We will show that $\alpha : A^¡ \hookrightarrow BA \twoheadrightarrow A$ is a Koszul twisting morphism. The kernel K of the augmentation map $A^¡ \otimes_\alpha A \twoheadrightarrow \mathbb{Q}$ has a basis $\{(sx)^n \otimes 1_A |\ n \geq 1\} \cup \{(sx)^n \otimes x | n \geq 0\}$. Because A and $A^¡$ both have trivial differential the differential on the twisted tensor product is given by

$$d_\alpha((sx)^n \otimes x^k) = \begin{cases} 0 \text{ if } n = 0 \text{ or } k = 1 \\ (sx)^{n-1} \otimes x^{k+1} \text{ else} \end{cases} .$$

Then if $\omega = \sum_{n,k} c_{nk}(sx)^n \otimes x^k$ is in $K \cap ker(d_\alpha)$ we have $c_{nk} = 0$ whenever $n \neq 0$ and $k \neq 1$. But then ω is in fact of the form

$$\omega = \sum_{n \neq 0} c_{n1}(sx)^n \otimes x + c_{01} 1 \otimes x =$$

$$d_\alpha(\sum_{n \neq 0} c_{n1}(sx)^{n+1} \otimes 1_A + c_{01}(sx) \otimes 1_A).$$

This means α is a Koszul twisting morphism and by the fundamental theorem of twisting morphisms $A^¡ \hookrightarrow BA$ is a quasi isomorphism so A is Koszul. □

Next we will show that spheres are formal over $\mathbb{Q}$.

Let $A = T(\mathbb{Q}x)/(x^2)$ be the cohomology ring of S^n. We construct its bigraded model as in the proof of theorem 59. We have

$$A^¡ = \mathbb{Q} \oplus \mathbb{Q}sx \oplus (\mathbb{Q}sx)^{\otimes 2} \oplus ...$$

where sx has cohomological degree $n-1$. Since A is Koszul a bigraded model for A is given by $\Omega A^¡$ where $A^¡$ is bigraded with cohomological degree (denoted by superscript) and syzygy degree (denoted by subscript).

Theorem 63. S^n is intrinsically formal over $\mathbb{Q}$ for $n \geq 2$.

Proof. Let Y be a topological space with $H^\bullet(Y;\mathbb{Q}) = H(S^n;\mathbb{Q}) = A$. Then by theorem 4.2 there is a differential D on $T(s^{-1}\bar{A}^¡)$ such that $(D - \delta_2)$ lowers the syzygy degree by 2 and a quasi isomorphism $\pi : (T(s^{-1}\bar{A}^¡), D) \to (C^\bullet(Y), \partial)$. We know that $D = \delta_2$ on $(s^{-1}\bar{A}^¡)_{\leq 1}$. And if $z \in (s^{-1}\bar{A}^¡)_{\geq 2}$ has weight at least 2 then it has cohomological degree atleast $1 + 3(n-1) = 3n - 2$. But if $n \geq 2$ then $3n - 2 > n + 1$. We see that $D = \delta_2$ in cohomological degree $(s^{-1}\bar{A}^¡)^{\leq n+1}$. Then we can define $\phi : (s^{-1}\bar{A}^¡) \to A$ by $\phi|_{(s^{-1}\bar{A}^¡)_{\leq 1}} = \rho$ and $\phi|_{(s^{-1}\bar{A}^¡)_{>1}} = 0$ where ρ is the quasi isomorphism $T((s^{-1}\bar{A}^¡), \delta_2) \to (A, 0)$. Then we extend ϕ to an algebra morphism $T(s^{-1}\bar{A}^{¡}) \to A$. I claim that this is in fact a quasi isomorphism $(T(s^{-1}\bar{A}^{¡}), D) \to (A, d_A = 0)$. First we note that it is a morphism of graded algebras because it respects the cohomological degree of the generators. Next we check that it commutes with the differential. Because $d_A = 0$ this just comes down to checking that ϕ vanishes on boundaries. For $z \in T(s^{-1}\bar{A}^{¡})^{\leq n+1} \subset T((s^{-1}\bar{A}^{¡})_{\leq 1})$ we have

$$\phi(D(z)) = \phi(\delta_2(z)) = \rho(\delta_2(z)) = 0$$

since ρ vanishes on δ_2-boundaries. Also, if $z \in T(s^{-1}\bar{A}^{¡})^{\geq n+2}$ then $D(z) \in T(s^{-1}\bar{A}^{¡})^{\geq n+2}$ and then $\phi(D(z)) \in A^{\geq n+2} = 0$.

The fact that ϕ is a quasi isomorphism follows from the fact that $H^\bullet(T(s^{-1}\bar{A}^{¡}), D) = A$ vanishes in cohomological degree $\geq n+1$ and $T(s^{-1}\bar{A}^{¡})^{<n+1} \subset T((s^{-1}\bar{A}^{¡})_{\leq 1})$ so $\phi^\bullet = \rho^\bullet$ is an isomorphism. But then we have a zig-zag

$$(C^\bullet(Y), \partial) \xleftarrow[\sim]{} (T(s^{-1}\bar{A}^{¡}), D) \xrightarrow{\sim} (A, 0)$$

proving that Y is formal.

By theorem 57 it then follows that spheres are both formal and coformal.

□

4.4 Euclidean configuration spaces

In this section we will see another example of a Koszul space, namely configuration spaces. We prove that $F_k(\mathbb{R}^n)$ is intrinsically formal over $\mathbb{Q}$, for $k \leq n$, following an article [2] by Paolo Salvatore. We will not explain how to compute the cohomology ring of $F_k(\mathbb{R}^n)$ but refer to [1].

Proposition 64. The cohomology ring $H^\bullet(F_k(\mathbb{R}^n); \mathbb{Q})$ admits a quadratic presentation (V, R) where V is a free graded vector space concentrated in degree $n-1$ with basis

$$\{A_{ij} | 1 \leq i < j \leq k\}$$

and $R \subset V \otimes V$ is spanned by

$$\{A_{ij}^2\} \cup \{A_{ij}A_{jk} + (-1)^n A_{jk}A_{ik} + (-1)^n A_{ik}A_{ij} | i < j < k\} \cup$$

$$\{A_{ij}A_{kl} - (-1)^{(n-1)^2} A_{kl}A_{ij}\}.$$

Proposition 65. $H^\bullet(F_k(\mathbb{R}^n); \mathbb{Q})$ has a graded basis provided by the set

$$\bigcup_{l \geq 0} \{A_{i_1 j_1} \cdots A_{i_l j_l} | j_1 < \cdots < j_l, i_t < j_t\}$$

Remark 66. The generators A_{ij} are in fact pullbacks of the standard generator of $H^\bullet(S^{n-1})$ under the maps $\pi_{ij} : F_k(\mathbb{R}^n) \to S^{n-1}$ which maps $(x_0, ..., x_k) \in F_k(\mathbb{R}^n)$ to $\frac{x_i - x_j}{|x_i - x_j|} \in S^{n-1}$.

Let $A = H^\bullet(F_k(\mathbb{R}^n); \mathbb{Q})$. The basis in the previous proposition is a so called PBW-basis and as a result of this A is a Koszul algebra [8]. Its Koszul dual coalgebra is given by

$$A^{¡} := \mathbb{Q} \oplus sV \oplus s^2 R \oplus ... \oplus \big(\bigcap_{i+2+j=n} (sV)^{\otimes i} \otimes (s^2 R) \otimes (sV)^{\otimes j} \big) \oplus ...$$

We note that $A^{¡}$ is bigraded. The upper grading is the cohomological degree and it is induced by the cohomological grading on V (recall it is concentrated in cohomological degree $n-1$ so sV is concentrated in degree $n-2$) and the lower grading is the syzygy degree introduced in the end of section 4.2. Since A is Koszul $\Omega A^{¡} \twoheadrightarrow A$ is a bigraded model for A.

Theorem 67. $F_k(\mathbb{R}^n)$ is intrinsically formal over $\mathbb{Q}$ for $n \geq k$.

Proof. Let Y be a topological space with $H^\bullet(Y) \cong H^\bullet(F_k(\mathbb{R}^n)) = A$. Recall that we already constructed a bigraded model for A namely $\Omega A^{¡}$. Theorem 4.2 applied to the cochain algebra $C := C^\bullet(Y)$ gives a differential D on $T(s^{-1}\bar{A}^{¡})$ such that

$$(D - d_2) : s^{-1}\bar{A}^{¡}{}_l \to \oplus_{i=0}^{l-2} T(s^{-1}\bar{A}^{¡})_i, \text{ for all } l.$$

I claim that $D = d_2$ on $(s^{-1}\bar{A}^{¡})_{<n}$. To see this let $x \in (s^{-1}\bar{A}^{¡})_{<n}$ be a homogenous element of syzygy degree i. Since the syzygy degree is the word length minus 1 and the elements of sV are of cohomological degree $n-2$ (see proposition 65 and convention 61) we see that x must have cohomological degree $1 + (i+1)(n-2)$. But then $(D-d)(x)$ has cohomological degree $2 + (i+1)(n-2)$. To get a contradiction assume $(D-d)(x) \neq 0$. Then there is some non-zero homogenous monomial $m \in T(s^{-1}\bar{A}^{¡})$ that occurs in $(D-d)(x)$. We note that m must lie in $(s^{-1}\bar{A}^{¡})^{\otimes l}$ where $l \equiv 2$ modulo $n-2$. This follows from the fact that the cohomological degree of $(D-d)(x)$ is congruent to 2 modulo $n-2$ and the homogenous elements of $s^{-1}\bar{A}^{¡}$ are all congruent to 1 modulo $n-2$. I claim that m cannot be in $s^{-1}\bar{A}^{¡} \otimes s^{-1}\bar{A}^{¡}$. Indeed, since m was assumed homogenous this would mean $m = v_a \otimes v_b$ for some $v_a \in (s^{-1}\bar{A}^{¡})_a$ and $v_b \in (s\bar{A}^{¡})_b$ which means $v_a \otimes v_b$ have cohomological degree $1 + (a+1)(n-2) + 1 + (b+1)(n-2) = 2 + (a+b+2)(n-2)$ which would mean $a+b+2 = i+1$. This would however imply $v_a \otimes v_b$ is have syzygy degree $a+b = i-1$ which contradicts the fact that $(D-d)$ decreases syzygy degree by atleast 2. But then $m \in (s^{-1}\bar{A}^{¡})^{\otimes l}$ where l is atleast $2 + (n-2) = n$. But then the cohomological degree of m is atelast $n(1 + (n-2)) = 2 + (n+1)(n-2)$ which would mean i is at least n which contradicts the fact that $x \in (s^{-1}\bar{A}^{¡})_{<n}$. So $(D-d)(x) = 0$ as claimed.

Then we can define a map $\phi : T(s^{-1}\bar{A}^{¡}) \to A$ by $\phi|_{(s^{-1}\bar{A}^{¡})_{<n}} = \rho$ and $\phi|_{(s^{-1}\bar{A}^{¡})_{\geq n}} = 0$ where ρ is the quasi isomorphism $(T(s^{-1}\bar{A}^{¡}), d) \to (A, 0)$. I claim that ϕ is a quasi isomorphism $(T(s^{-1}\bar{A}^{¡}), D) \to (A, 0)$. First we note that it is a morphism of graded algebras because it respects the cohomological grading on the generators. To see that it commutes with the differentials it is enough to see that it vanishes on coboundaries. But if $z \in T(s^{-1}\bar{A}^{¡})^{\leq (k-1)(n-1)}$

then $z \in T((s^{-1}\bar{A}^{¡})_{<n})$ and we have

$$\phi(D(z)) = \phi(d(z)) = \rho(d(z)) = 0$$

because ρ vanishes on $d-$boundaries. If $z \in T(s^{-1}\bar{A}^{¡})^{>(k-1)(n-1)}$ then Dz has cohomological degree at least $(k-1)(n-1)+1$ but then $\phi(D(z)) \in A^{1+(k-1)(n-1)} = 0$. Finally to see that ϕ is a quasi isomorphism we note that because $H^{\bullet}(C,D) = A$ only lives in degrees less $(k-1)(n-1)$ where ϕ and ρ agree we must have $\phi^{\bullet} = \rho^{\bullet}$ which is an isomorphism.

□

By theorem 57 it then follows that $F_k(\mathbb{R}^n)$ (with $n \geq k$) is both formal and coformal.

5 Non-formality of planar configuration space with four points over characteristic two

In the previous two examples we used koszulness of the cohomology rings of some topological spaces, together with the theory of bigraded and filtered models to prove formality of these spaces. In this section we will see that these ideas can also be used to prove non-formality. We explain how Salvatore [2] proves the non-formality of the planar configuration space with four points over characteristic two. In the first section we introduce the Barratt-Eccles simplicial set WS_k where S_k denotes the symmetric group. Then we define a filtration on it $\mathcal{F}_t(WS_k)$ called the Smith's filtration. This filtration has the property that in level 2, its geometric realisation is homotopy equivalent to $F_k(\mathbb{R}^2)$ which allows us to study the formality of $F_4(\mathbb{R}^2)$ by studying the formality of the cochain algebra of the simplicial set $\mathcal{F}_2(WS_4)$. To study the formality of this algebra we construct the filtered model of it up to filtration level 2 which allow us to define the obstruction class $[\alpha]$. We then explain why non-triviality of this obstruction class implies the non-formality of the cochain algebra of the simplicial set $\mathcal{F}_2(WS_4)$.

5.1 Barrat-Eccles simplicial set

Now we describe the simplicial set which has the property that its geometric realization is homotopy equivalent to $F_4(\mathbb{R}^2)$. Let us denote by S_k the symmetric group on $\{1, 2, ..., k\}$ and let us denote a permutation $\sigma \in S_k$ by $\sigma = (\sigma(1)\ \sigma(2)\ \cdots\ \sigma(k))$. By definition the k'th Barrat-Eccles simplicial set is the contravariant functor WS_k defined on objects by $WS_k([l]) = S_k^{l+1}$ and on morphisms by

$$WS_k(f : [n] \to [m]) : S_k^{m+1} \to S_k^{n+1}, (\sigma_0, ..., \sigma_m) \mapsto (\sigma_{f(0)}, ..., \sigma_{f(n)}).$$

With these definitions we can make the face and degeneracy maps explicit:

$$d_i(\sigma_0, ..., \sigma_n) = (\sigma_{d^i(0)}, ..., \sigma_{d^i(n-1)}) = (\sigma_0, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$$

$$s_i(\sigma_0, ..., \sigma_n) = (\sigma_{s^i(0)}, ..., \sigma_{s^i(n+1)}) = (\sigma_0, ..., \sigma_i, \sigma_i, ..., \sigma_n).$$

We define for all $t \geq 0$ a filtration on WS_2. More precisely we define a chain of sub simplicial sets

$$... \subset \mathcal{F}_t(WS_2) \subset \mathcal{F}_{t+1}(WS_2) \subset ...$$

where $\mathcal{F}_t(WS_2)$ is the sub simplicial set spanned by all non-degenerate simplices of degree at most $t-1$:

$$\mathcal{F}_t(WS_2)([n]) := \begin{cases} WS_2([n]), \text{ if } n \leq t-1 \\ \cup_i s_i(WS_2([n-1])) \text{ if } n > t-1 \end{cases}.$$

Now for all $1 \leq i \neq j \leq k$ consider the simplicial maps $\pi_{ij} : WS_k \to WS_2$ defined on $WS_k([n])$ by

$$\pi_{ij}(\sigma_0, ..., \sigma_n) = (\tau_0, ..., \tau_n)$$

where

$$\tau_l = \begin{cases} (12) \text{ if } \sigma_l^{-1}(i) < \sigma_l^{-1}(j) \\ (21) \text{ if } \sigma_l^{-1}(i) > \sigma_l^{-1}(j) \end{cases}.$$

Using these maps we can define filtrations on WS_k for all k by

$$\mathcal{F}_t(WS_k) = \cap_{i,j} \pi_{ij}^{-1} \mathcal{F}_t(WS_2).$$

We are particularly interested in $\mathcal{F}_2(WS_k)$.

The following proposition explains why we are interested in the simplicial set $\mathcal{F}_t(WS_k)$. The proof of this proposition requires more background on simplicial sets so we simply refer to theorem 6.2 and proposition 7.9 in [11].

Proposition 68. The geometric realization of $\mathcal{F}_t(WS_k)$ is homotopy equivalent to $F_k(\mathbb{R}^t)$. Moreover the realization of the maps of simplicial π_{ij} defined above corresponds, up to homotopy, to the projection $\pi_{ij} : F_k(\mathbb{R}^t) \to S^{t-1}$ defined by $(x_1, ..., x_k) \mapsto \frac{x_i - x_j}{|x_i - x_j|}$.

Let us denote the normalised cochain algebra (over $\mathbb{Z}_2$) of $\mathcal{F}_2(WS_k)$ by $\mathcal{E}_2^\bullet(k)$. We recall from chapter one that it is a dga algebra under the cup product.

5.2 Filtered model for $\mathcal{E}_2^\bullet(4)$

Let $A = H^\bullet(F_4(\mathbb{R}^2); \mathbb{Z}_2)$. It has the same presentation as in proposition 64 but over $\mathbb{Z}_2$ now. We recall that since A is Koszul there is a quasi isomorphism $\Omega A^¡ \twoheadrightarrow A$. In this section we will however think of $\Omega A^¡$ in a slightly different way. It follows from proposition 3.2.1 in [3] that there is an isomorphism of graded coalgebras $A^¡ \cong \oplus_i s^i(A^!(i))^*$ where $A^!$ is the quadratic dual of A and $A^!(i)$ is the image of $(V^*)^{\otimes i}$ under the projection $T(V^*) \twoheadrightarrow A^!$. From now on we identify $A^¡$ with the desuspended, graded dual of $A^!$ through this isomorphism. Under this identification we have $\Omega A^¡ = (T(W), d)$ where $W = s^{-1}\big(\oplus_{i\geq 1} s^i(A^!(i))^*\big)$ and d is now induced by the coproduct μ^* of $\oplus_{i\geq 1} s^i(A^!(i))^*$ dual to the multiplicaiton of $A^!$. The syzygy degree on $\bar{A}^¡$ described in the previous chapter corresponds to the following weight grading $\overline{(A^!)^*}(k) = (A^!(k+1))^*$. We have the following description of $A^!$

Proposition 69. $A^! \cong T(B)/(S)$ where B has a basis $\{B_{ij} | 1 \leq i < j \leq 4\}$ and $S \subset B \otimes B$ is spanned by

$$B_{ij}B_{jk} + B_{jk}B_{ij} + B_{jk}B_{ik} + B_{ik}B_{jk}, \ i, j, k \text{ distinct},$$

$$B_{ij}B_{st} + B_{st}B_{ij}, \quad \text{for } \{i, j\} \cap \{s, t\} = \emptyset$$

where we use the convention that $B_{ij} = B_{ji}$.

Proof. Set $B_{ij} = A^*_{ij}$ where A_{ij} are the generators for A. We then have to prove that $R^{\perp}$ is spanned by the relations above where R is as in proposition 65 except with coefficients in $\mathbb{Z}_2$. It is not so hard to see that the relations above all vanish on R so it remains to see that they span R. We know from proposition 66 the Hilbert series for A starts with $h_A(x) = 1 + 6x + 11x^2 + ...$ Then since A is Koszul theorem 3.5.1 in [3] tells us $h_{A^!}(-x)h_A(x) = 1$ which tells us $h_{A^!} = 1 + 6x + 25x^2$. So since $dim(B \otimes B) = 36$ we only have to show that the relations above span an 11 dimensional subspace but it is not so hard to find 11 linearly independent relations of the desired form. ☐

This is the so called Yang-Baxter algebra and in [12] the following basis is computed for it.

Proposition 70. The algebra $A^!(n)$ admits a basis $\{B_{i_1j_1} \cdots B_{i_nj_n} | j_1 \leq j_2 \leq ... \leq j_n, \ i_t < j_t \text{ for all } t\}$.

Under the identification $A^¡ \cong \oplus_i s^i(A^!(i))^*$ the the quasi isomorphism $\Omega A^¡ \twoheadrightarrow A$ is defined by $B^*_{ij} \mapsto A_{ij}$ and $(B_{i_1j_1} \cdots B_{i_nj_n})^* \mapsto 0$ for $n > 1$.

We know that $H^\bullet(\mathcal{E}_2^\bullet(4)) \cong A$ so theorem 4.2 tells us that there is a differential D on $T(W)$ and a quasi isomorphism $\phi : (T(W), D) \to \mathcal{E}_2^\bullet(4)$ such that $D - d$ decreases the weight by at least 2. In this section we will make the map ϕ and the differential D explicit up to $T(W_{\leq} 2)$ by following the proof of theorem. Before doing so we give names to certain elements of $\mathcal{E}_2^\bullet(k)$.

Definition 71. Set $\omega^k_{ij} := (\pi_{ij}|_{\mathcal{F}_2(WS_k)})^*\big((12), (21)\big) \in \mathcal{E}_2^1(k)$ for all k.

Definition 72. Set $Ar := \big((132), (312)\big)^* \in \mathcal{E}_2^1(3)$.

We want to transfer the element Ar to $\mathcal{E}_2^1(4)$. We do this using maps of simplicial sets $f_{ijk} : \mathcal{F}_2(WS_4) \to WS_3$, one for each triple $1 \leq i < j < k \leq 4$. These maps are defined as follows.

Definition 73. We set

$$f_{ijk}\big(\sigma_0, ..., \sigma_m\big) = \big(\tau_1, ..., \tau_m\big)$$

where τ_t is the permutation which only remembers how σ_t permutes i, j and k. More precisely let τ' be the unique permutation on the set $\{i, j, k\}$ such that $(\tau(i)\ \tau(j)\ \tau(k))$ is the order in which i, j and k occur in $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \sigma(4))$ then let $\tau \in S_3$ be the permutation corresponding to τ' through the bijection $i \leftrightarrow 1\ j \leftrightarrow 2\ k \leftrightarrow 3$.

Now we start building the filtered model for $\mathcal{E}_2^\bullet(4)$

Proposition 74. The algebra morphism $\phi : T(s^{-1}W_{\leq 1}) \to \mathcal{E}_2^\bullet(4)$ defined on generators by

$$B_{ij}^* \mapsto \omega_{ij}^4,\ 1 \leq i \leq j \leq 4,$$

$$s(s^{-1}B_{ij}s^{-1}B_{kl})^* \mapsto \begin{cases} 0, \text{ if } i = k \text{ and } j = l, \\ \omega_{ij}^4 \cup_1 \omega_{kl}^4, \text{ if } j \neq l, \\ f_{ikl}^*(Ar) + \omega_{il}^4 \cup_1 \omega_{kl}^4, \quad \text{if } j = l \text{ and } i < k \\ f_{kil}^*(Ar), \quad \text{if } j = l \text{ and } i > k \end{cases}$$

agrees with the construction in the proof of theorem 4.2. In other words $\phi : (T(s^{-1}W_{\leq 1}), d) \to (\mathcal{E}_2^\bullet(4), \partial)$ is a morphism of differential graded cochain algebras and it can be extended to a filtered model $(T(s^{-1}W), D) \to (\mathcal{E}_2^\bullet(4), \partial)$ as in theorem 4.2.

The proof is almost only computation. To perform these computations we need the following two lemmas.

Lemma 75. An element $(\sigma, \tau, \gamma) \in WS_3([2])$ is in $\mathcal{F}_2(WS_3)([2])$ if and only if the order in which any two indices $i \neq j \in \{1, 2, 3\}$ appear does not change more than once in

$$\big(\sigma, \tau, \gamma\big) = \big((\sigma(1)\ \sigma(2)\ \sigma(3)), (\tau(1)\ \tau(2)\ \tau(3)), (\gamma(1)\ \gamma(2)\ \gamma(3))\big).$$

Proof. ($\implies$) If the order of two indices $i \neq j \in \{1, 2, 3\}$ does change more than once in (σ, τ, γ) then $\pi_{ij}(\sigma, \tau, \gamma) \in WS_2$ is non-degenerate and therefore not in $\mathcal{F}_2(WS_2)$ and then $(\sigma, \tau, \gamma) \notin \mathcal{F}_2(WS_3)$.
($\Longleftarrow$) If the order in which i, j appear in σ, τ and γ only changes zero or one time then $\pi_{ij}((\sigma, \tau, \gamma))$ is degenerate and if this holds for all $i \neq j \in \{1, 2, 3\}$ then $(\sigma, \tau, \gamma) \in \bigcap_{ij} \pi_{ij}^{-1}(\mathcal{F}_2(WS_2([2]))) = \mathcal{F}_2(WS_3)([2])$. ☐

Lemma 76. The set $\mathcal{F}_2(WS_3)([2])$ contains the following 36 simplices

$$((321), (132), (123))\ ((321), (213), (123))\ ((321), (231), (123))$$
$$((321), (231), (213))\ ((321), (312), (123))\ ((321), (312), (132))$$
$$((231), (123), (132))\ ((231), (213), (123))\ ((231), (213), (132))$$
$$((231), (321), (132))\ ((231), (321), (312))\ ((231), (312), (132))$$
$$((312), (123), (213))\ ((312), (132), (123))\ ((312), (132), (213))$$

$$((312),(321),(213))\ ((312),(321),(231))\ ((312),(231),(213))$$
$$((123),(132),(321))\ ((123),(132),(312))\ ((123),(213),(321))$$
$$((123),(213),(231))\ ((123),(231),(321))\ ((123),(231),(321))$$
$$((123),(312),(321))\ ((132),(123),(213))\ ((132),(123),(231))$$
$$((132),(213),(231))\ ((132),(213),(231))\ ((132),(321),(231))$$
$$((132),(312),(321))\ ((132),(312),(231))\ ((213),(123),(132))$$
$$((213),(123),(312))\ ((213),(132),(312))\ ((213),(321),(312))$$
$$((213),(231),(321))\ ((213),(231),(312))$$

Proof. First we note that W is concentrated in cohomological degree 1 and $|\omega^4_{ij}| = |\omega^4_{ij} \cup_1 \omega^4_{kl}| = |f^*_{kil}(Ar)| = |f_{kil}(Ar)| = 1$ so ϕ is a degree zero map. Next we recall that in the proof of theorem 4.2 that when defining the map ϕ on the generators in weight zero all we require is that they get mapped to a cocycle which represents a the element in cohomology which that generator gets mapped to by the bigraded model. In our case we need that ω^4_{ij} is a cocycle representing A_{ij}. To see that it is a cocycle we check that $((12),(21))^* \in \mathcal{E}^1_2(WS_2)$ is a cocyle. This however follows from the fact that $\mathcal{F}_2(WS_2)([2])$ has only degenerate simplices so $\mathcal{E}^2_2(WS_2) = 0$. On the generators $s^{-1}(sB_{ij}sB_{kl})^*$ 1 of weight 1 all we require is that $\partial^*\phi(s^{-1}(sB_{ij}sB_{kl})^*) = \phi d(s^{-1}(sB_{ij}sB_{kl})^*)$. To prove this we have to make some computations.

Computation 1: For simplicity we drop the suspensions for a while. First we compute $\phi d((B_{ij}B_{kl})^*)$. First thing to note is that $d((B_{ij}B_{kl})^*) \in W_0 \otimes W_0$ which has a basis $\{B^*_{ij} \otimes B^*_{kl} | i < j, k < l\}$. Since we work over $\mathbb{Z}_2$ we only have to check which such basis elements occur in $d((B_{ij}B_{kl})^*)$. Since d is induced by the coproduct μ^* dual to the multiplication of $A^!$ we will compute $\mu^*(B_{ij}B_{kl})^*(B_{st} \otimes B_{qr})$ to see if $B^*_{st} \otimes B^*_{qr}$ occurs i the expression for $d(B_{ij}B_{kl})^*$. We have $\mu^*(B_{ij}B_{kl})^*(B_{st} \otimes B_{qr}) = (B_{ij}B_{kl})^*(B_{st}B_{qr})$. But to compute the last expression we must know how $B_{st}B_{qr}$ is expressed in the basis of proposition 71. Using the relations in proposition 70 we get

$$B_{st}B_{qr} = \begin{cases} B_{st}B_{qr} \text{ if } t \leq r \ , \\ B_{qr}B_{st} \text{ if } r < t \text{ and } s \neq q, r \ , \\ B_{sr}B_{st} + B_{rt}B_{st} + B_{st}B_{rt} \text{ if } r < t \text{ and } s = q \ , \\ B_{qr}B_{rt} + B_{qt}B_{rt} + B_{rt}B_{qt} \text{ if } r < t \text{ and } s = r \end{cases} .$$

Then we can compute $(B_{ij}B_{kl})^*(B_{st}B_{qr})$ by dividing into cases depending on i, j, k and l. First we have $(B_{ij}B_{ij})^*(B_{st}B_{qr}) = 1$ only when $(s,t) = (q,r) = (i,j)$ so $d(B_{ij}B_{ij})^* = B^*_{ij} \otimes B^*_{ij}$. Next we have for $j \neq l$

$$(B_{ij}B_{il})^*(B_{st}B_{qr}) =$$

$$\begin{cases} 1 \text{ if } (s,t) = (i,j) \text{ and } (q,r) = (i,l) \text{ (from the case } t \leq r) \ , \\ 1 \text{ if } (q,r) = (i,j) \text{ and } (s,t) = (i,l) \text{ (from the case } r < t,\ s \neq q, r) \end{cases}$$

so $d(B_{ij}B_{il})^* = B^*_{ij} \otimes B^*_{il} + B^*_{il} \otimes B^*_{ij}$. Similar computations give the other cases and all the computations can be summarised as

$$d(B_{ij}B_{kl}) = \begin{cases} B^*_{ij} \otimes B^*_{ij} \text{ if } (i,j) = (k,l), \\ B^*_{ij} \otimes B^*_{kl} + B^*_{kl} \otimes B^*_{ij} \text{ if } j \neq l, \\ B^*_{ij} \otimes B^*_{kl} + B^*_{ij} \otimes B^*_{ik} + B^*_{kj} \otimes B^*_{ik} \text{ if } j = l \text{ and } i < k, \\ B^*_{ij} \otimes B^*_{kl} + B^*_{kj} \otimes B^*_{ki} + B^*_{ij} \otimes B^*_{ki} \text{ if } j = l \text{ and } i > k \end{cases}.$$

Then finally we get

$$\phi(d(B_{ij}B_{kl})^*) = \begin{cases} \omega^4_{ij} \cup \omega^4_{ij} \text{ if } (i,j) = (k,l), \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{kl} \cup \omega^4_{ij} \text{ if } j \neq l, \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{ij} \cup \omega^4_{ik} + \omega^4_{kj} \cup \omega^4_{ik} \text{ if } j = l \text{ and } i < k, \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{kj} \cup \omega^4_{ki} + \omega^4_{ij} \cup \omega^4_{ki} \text{ if } j = l \text{ and } i > k \end{cases}.$$

Finally we note that $\omega^4_{ij} \cup \omega^4_{ij} = (\pi_{ij}|_{\mathcal{F}_2(WS_k)})^*(((12),(21))^* \cup ((12),(21))^*) = 0$ because $((12),(21))^* \cup ((12),(21))^* \in \mathcal{E}^2_2(2) = 0$.

Computation 2: Now we want to compute $\partial^*(\phi(B_{ij}B_{kl}))^*$. We have of course $\partial^*(\phi(B_{ij}B_{ij})^*) = \partial^*(0) = 0$. Next, we want to compute $\partial^*(\phi(B_{ij}B_{kl})^*)$ for $j \neq l$. We have by definition of ϕ

$$\partial^*(\phi(B_{ij}B_{kl})^*) = \partial^*(\omega^4_{ij} \cup_1 \omega^4_{kl}).$$

Because ω^4_{ij} and ω^4_{kl} are coboundaries proposition 27 tells us

$$\partial^*(\phi(B_{ij}B_{kl})^*) = \partial^*(\omega^4_{ij} \cup_1 \omega^4_{kl}) = \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{kl} \cup \omega^4_{ij}.$$

Next we want to compute, for $j = l$ and $i > k$, $\partial^*(\phi(B_{ij}B_{kl})) = \partial^*(f^*_{ikj}(Ar)) = f^*_{kil}(\partial^* Ar)$. We recall that

$$Ar \in \mathcal{E}^1_2(3) = Hom(\mathbb{Z}_2\mathcal{F}_2(WS_3)([2]), \mathbb{Z}_2).$$

Let us express $\partial^* Ar$ in the basis dual to the one in lemma 77 Since

$$\partial^*(Ar)(\sigma_0, \sigma_2, \sigma_3) = Ar(\sigma_1, \sigma_2) + Ar(\sigma_0 + \sigma_2) + Ar(\sigma_0 + \sigma_1)$$

we see that out of the elements in lemma 1.3 dAr vanishes on all except those in which (132) and (312) occur (in that order). We get

$$\partial^*(Ar) = ((123),(132),(312))^* + ((132),(312),(321))^* +$$

$$((132),(312),(231))^* + ((213),(132),(312))^*.$$

Now I claim that this equals $\omega^3_{13} \cup \omega^3_{12} + \omega^3_{23} \cup \omega^3_{12} + \omega^3_{23} \cup \omega^3_{13}$. To prove this we will compute these three cup products in terms of the basis dual to the one in lemma 77. We first note that

$$\omega^3_{ij} \cup \omega^3_{kl}(\sigma_0, \sigma_1, \sigma_2) =$$

$$\omega^3_{ij}(\sigma_0,\sigma_1)\omega^3_{kl}(\sigma_1,\sigma_2) = ((12),(21))^*\big(\pi_{ij}(\sigma_0,\sigma_1)\big)((12),(21))^*\big(\pi_{kl}(\sigma_1,\sigma_2)\big).$$

The last expression however is 1 precisely when $(i\ j)$ is a subsequence of $(\sigma_0(1)\ \sigma_0(2)\ \sigma_0(2))$, $(j\ i)$ is a subsequence of $(\sigma_1(1)\ \sigma_1(2)\ \sigma_1(2))$, $(k\ l)$ is a subsequence of $(\sigma_1(1)\ \sigma_1(2)\ \sigma_1(2))$ and $(l\ k)$ is a subsequence of $(\sigma_2(1)\ \sigma_2(2)\ \sigma_2(2))$. Using this we see that out of the elements in lemma 77 $\omega^3_{13}\cup\omega^3_{12}$ vanish on all except $((123),(312),(321))$, $((132),(312),(321))$, $((132),(312),(231))$ which means

$$\omega^3_{13}\cup\omega^3_{12} = ((123),(312),(321))^* + ((132),(312),(321))^* + ((132),(312),(231)).$$

Similarly one computes

$$\omega^3_{23}\cup\omega^3_{12} = ((123),(132),(321))^* + ((123),(312),(321))^*,$$

$$\omega^3_{23}\cup\omega^3_{13} = ((123),(132),(321))^* + ((123),(132),(312))^* + ((312),(132),(312))^*$$

and adding these up gives

$$((123),(132),(312))^* + ((132),(312),(321))^* +$$

$$((132),(312),(231))^* + ((213),(132),(312))^*$$

which we said was $\partial^* Ar$. So we have

$$\partial^*(\phi(B_{ij}B_{kl})) = \partial^*(f^*_{ikj}(Ar)) =$$

$$f^*_{kil}(\partial^* Ar) = f^*_{kil}(\omega^3_{13}\cup\omega^3_{12} + \omega^3_{23}\cup\omega^3_{12} + \omega^3_{23}\cup\omega^3_{13}) =$$

$$f^*_{kil}(\omega^3_{13})\cup f^*_{kil}(\omega^3_{12}) + f^*_{kil}(\omega^3_{23})\cup f^*_{kil}(\omega^3_{12}) + f^*_{kil}(\omega^3_{23})\cup f^*_{kil}(\omega^3_{13}).$$

But I claim that $f^*_{kil}(\omega^3_{13}) = \omega^4_{kl}$. To see this we apply $f^*_{kil}(\omega^3_{13})$ to a non-degenerate simplex $(\sigma_0,\sigma_1)\in\mathcal{F}_2(WS_4)([1])$.

$$f^*_{kil}(\omega^3_{13}(\sigma_0,\sigma_1) = \omega^3_{13}\circ f_{kil}(\sigma_0,\sigma_1) =$$

$$\pi^*_{13}(((12),(21))^*)(f_{kil}(\sigma_0,\sigma_1)) = ((12),(21))^*(\pi_{13}(\tau_0,\tau_1))$$

where τ_0 and τ_1 are as in the definition of f_{kil}. This last expression however is 0 unless $(1\ 3)$ is a subsequence of τ_0 and $(3\ 1)$ is a subsequence of τ_1 in which case it is 1. But this happens precisely when $(k\ l)$ is a subsequence of σ_0 and $(l\ k)$ is a subsequence of σ_1 from which we conclude that

$$f^*_{kil}(\omega^3_{13}(\sigma_0,\sigma_1) = ((12),(21))^*(\pi_{kl}(\sigma_0,\sigma_1)) = \omega^4_{kl}(\sigma_0,\sigma_1).$$

Similarly one can show that $f^*_{kil}(\omega^3_{12}) = \omega^4_{ki}$ and $f^*_{kil}(\omega^3_{23}) = \omega^4_{il}$. We get

$$\partial^*(\phi(B_{ij}B_{kl})) = \omega^4_{kl}\cup\omega^4_{ki} + \omega^4_{il}\cup\omega^4_{ki} + \omega^4_{il}\cup\omega^4_{kl},\ \text{for } j=l,\ k<i.$$

Finally we will compute, for $j=l$ and $i<k$

$$\partial^*(\phi(B_{ij}B_{kl})) = \partial^*(f^*_{ikl}(Ar)) + \partial^*(\omega^4_{il}\cup_1\omega^4_{kl}).$$

The first term we just computed (but with the roles of i and k switched) and for the second term we use proposition 27. This gives for $j = l$ and $i < k$

$$\partial^*(\phi(B_{ij}B_{kl})) = \omega^4_{il} \cup \omega^4_{ik} + \omega^4_{kl} \cup \omega^4_{ik} + \omega^4_{kl} \cup \omega^4_{il} + \omega^4_{il} \cup \omega^4_{kl} + \omega^4_{kl} \cup \omega^4_{il} =$$

$$\omega^4_{il} \cup \omega^4_{ik} + \omega^4_{kl} \cup \omega^4_{ik} + \omega^4_{il} \cup \omega^4_{kl}$$

To summarise we have

$$\partial^*(\phi((B_{ij}B_{kl})^*)) = \begin{cases} 0 \text{ if } (i,j) = (k,l), \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{kl} \cup \omega^4_{ij} \text{ if } j \neq l, \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{ij} \cup \omega^4_{ik} + \omega^4_{kj} \cup \omega^4_{ik} \text{ if } j = l \text{ and } i < k, \\ \omega^4_{ij} \cup \omega^4_{kl} + \omega^4_{kj} \cup \omega^4_{ki} + \omega^4_{ij} \cup \omega^4_{ki} \text{ if } j = l \text{ and } i > k \end{cases}.$$

□

Now we will define a map $W_2 \to A(2)$ where. This map will in the next section prove to be an obstruction to formality for $\mathcal{E}_2^\bullet(4)$.

Definition 77. $\alpha(w) := [\phi(d(w))]$.

We note that this definitions makes sense since $dw \in T(W_{\leq 1})$ on which ϕ is defined and by the previous proposition we have

$$\partial^*(\phi(dw)) = \phi(d^2 w) = 0$$

so $\phi(dw)$ is indeed a cycle. So far we have constructed the filtered model of $\mathcal{E}_2^\bullet(4)$ up $T(W_{\leq 1})$. We will however need to construct D one step further, to $T(W_{\leq 2})$. Following the construction in the proof of theorem 4.3 we see that we need to fix a map linear $\eta : A \to T(W_0)$ such that $\rho \circ \eta = id_A$.

Definition 78. Define $\eta : A \to T(W_0)$ on the basis in proposition 65 by

$$A_{i_1 j_1} \cdots A_{i_t j_t} \mapsto B^*_{i_1 j_1} \otimes ... \otimes B^*_{i_t j_t} \in T(W_0).$$

We note that then $\rho(B^*_{i_1 j_1} \otimes ... \otimes B^*_{i_t j_t}) = \rho(B^*_{i_1 j_1}) \cdots \rho(B^*_{i_t j_t}) = A_{i_1 j_1} \cdots A_{i_t j_t}$ as we wanted so we can define D on on W_2 by $D(z) = d(z) - \eta(\alpha(z))$ as in the proof of theorem 4.2.

5.3 Hochshild homology

In this section we recall the twisted hom-space $Hom^\tau(W, A)$ and show that α as defined in the previous section is a cycle in this complex. Then we explain why non-triviality of the class of α is implies non-formality of $F_k(\mathbb{R}^2)$. Consider the map $\tau : W \to A$ which is zero everywhere except in weight 1 where it is given by $sB^*_{ij} \mapsto A_{ij}$.

Lemma 79. $\tau \star \tau = 0$

Proof. Because τ vanishes outside weight 1 and the coproduct of W respects weight $\tau \star \tau$ definitely vanishes outside $W(2)$. To see that it vanishes in weight 2 as well we use the computations done in the proof of proposition 75. Because d is induced by the coproduct we have (recalling the relations in A)

$$\tau \star \tau((B_{ij}B_{kl})^*) = \begin{cases} A_{ij}^2 = 0 \text{ if } (i,j) = (k,l), \\ A_{ij}A_{kl} + A_{kl}A_{ij} = 0 \text{ if } j \neq l, \\ A_{ij}A_{kj} + A_{ij}A_{ik} + A_{kj}A_{ik} = 0 \text{ if } j = l,\ i < k, \\ A_{ij}A_{kj} + A_{kj}A_{ki} + A_{ij}A_{ki} = 0 \text{ if } j = l,\ i > k \end{cases}.$$

□

This will allow us to put a twisted differential structure on $Hom(W, A)$. Let us define $\partial : Hom(W, A) \to Hom(W, A)$ by $f \mapsto f \star \tau + \tau \star f$.

Lemma 80. $(Hom(W, A), \partial)$ is a dga algebra.

Proof. Then ∂ is indeed a differential because $\partial^2(f) = \partial(f \star \tau + \tau \star f) = f \star \tau \star \tau + \tau \star f \star \tau + \tau \star f \star \tau + \tau \star \tau \star f = 0$. Also $\partial(f \star g) = f \star g \star \tau + \tau \star f \star g$ where as $\partial(f) \star g + f \star \partial(g) = f \star \tau \star g + \tau \star f \star g + f \star g \star g + f \star \tau \star g = f \star g \star \tau + \tau \star f \star g$. □

We now note that the map α defined in the end of the last section is an element of $Hom(W, A)$ and we will see that it is in fact a cycle with respect to the differential ∂.

Proposition 81. $\partial(\alpha) = 0$

Proof. We first note that since α vanishes outside W_2 and τ outside W_0 we see that $\alpha \star \tau + \tau \star \alpha$ vanish outside W_3. Let $X \in W_3$. Then we want to compute

$$\mu_A \circ (\alpha \otimes \tau) \circ \mu^{\bar{*}}_{A^!}(X) + \mu_A \circ (\tau \otimes \alpha) \circ \mu^{\bar{*}}_{A^!}(X).$$

But since the differential d on $T(W)$ is induced by the coproduct of W we have

$$\mu_A \circ (\alpha \otimes \tau) \circ \mu^{\bar{*}}_{A^!}(X) + \mu_A \circ (\tau \otimes \alpha) \circ \mu^{\bar{*}}_{A^!}(X) = \mu_A \circ (\alpha \otimes \tau) \circ d(X) + \mu_A \circ (\tau \otimes \alpha) \circ d(X).$$

Because the syzygy degree is weight minus one, and the coproduct of W respects the weight, it follows that $d(X) \in W_2 \otimes W_0 \oplus W_1 \otimes W_1 \oplus W_0 \otimes W_2$. Let us write $d(X) = \sum x_i \otimes y_i + \sum a_i \otimes b_i + \sum w_i \otimes z_i$ the first sum is in $W_2 \otimes W_0$, the second in $W_1 \otimes W_1$ and the third in $W_0 \otimes W_2$. Then we see that

$$\partial(\alpha)(X) = \mu_A \circ (\alpha \otimes \tau)\Big(\sum x_i \otimes y_i + \sum a_i \otimes b_i + \sum w_i \otimes z_i\Big) +$$

$$\mu_A \circ (\tau \otimes \alpha)\Big(\sum x_i \otimes y_i + \sum a_i \otimes b_i + \sum w_i \otimes z_i\Big) = \sum \alpha(x_i)\tau(y_i) + \sum \tau(w_i)\alpha(z_i).$$

Let us show that this is in fact zero in A. Since $d(X) \in W_2 \otimes W_0 \oplus W_1 \otimes W_1 \oplus W_0 \otimes W_2$ and since $D - d$ lowers syzygy degree by 2 (see theorem 4.2) we see that

$$D(dX) = (D - d)(dX) = (D - d)\Big(\sum x_i \otimes y_i + \sum a_i \otimes b_i + \sum w_i \otimes z_i\Big) =$$

$$\sum(D-d)(x_i)\otimes y_i+\sum w_i\otimes(D-d)(z_i)=\sum\eta(\alpha((x_i))\otimes y_i+\sum w_i\otimes\eta(\alpha(z_i))$$

where we used the definition of D on W_2 from the end of the previous chapter and the fact that we are working over $\mathbb{Z}_2$ for the last equality. But since ϕ can be extended to a filtered model $(T(W),D)\to(\mathcal{E}_2^\bullet(4),\partial)$ we see that $\phi(DdX)$ is a boundary in $\mathcal{E}_2^\bullet(4)$. So

$$0=[\phi(DdX)]=\Big[\sum\phi(\eta\alpha(x_i))\phi(y_i)+\sum\phi(w_i)\phi(\eta\alpha(z_i))\Big].$$

Because $\eta\alpha(x_i),y_i,w_i$ and $\eta\alpha(z_i)$ all land in W_0 they are all cycles and then the last expression equals

$$\sum[\phi(\eta\alpha(x_i))]\cdot[\phi(y_i)]+\sum[\phi(w_i)]\cdot[\phi(\eta\alpha(z_i))].$$

By property *ii)* of the filtered model (see theorem 4.2) we have $[\phi(v)]=\rho(v)$ for any $v\in W_0$ and since $\rho\eta=id_A$ we see that the last expression equals

$$\sum\alpha(x_i)\cdot\rho(y_i)+\sum\rho(w_i)\cdot\alpha(z_i).$$

However comparing the definitions of ρ and τ we see that they equal on W_0 meaning the last expression is precisely $\sum\alpha(x_i)\tau(y_i)+\sum\tau(w_i)\alpha(z_i)$ which we said was $\partial(\alpha)(X)$ proving that α is a cycle in $Hom(W,A)$. □

We have now identified the obstruction class $[\alpha]\in H(Hom(W,A))$. Proving that it is non-trivial however requires some more work and we refer to [2] for the details behind the following lemma.

Lemma 82. The class of α is non-trivial.

Let us however explain how the non-triviality of $[\alpha]$ implies non-formality of $F_k(\mathbb{R}^n)$. First we have the following lemma from Salvatore's article

Lemma 83. There is no isomorphism $\psi:(T(W),d)\to(T(W),D)$ such that $\psi-id$ lowers the syzygy degree.

Proof. To get a contradiction assume there is an isomorphism ψ as in the statement of the lemma. Since $\psi-id$ is supposed to lower the syzygy degree we must have $\psi=id$ on W_0. On W_1 ψ must be of the form $\psi=id+f$ where $f:W_1\to T(W_0)$ and on W_2 ψ must be of the form $\psi=id+f_1+f_0$ where $f_1:W_2\to T(W_{\le1})$ and $f_0:W_2\to T(W_0)$. Now let $x\in W_2$. By assumption we have $\psi\circ d(x)=D\circ\psi(x)$. Let us write $d(x)(x)=\sum a_i\otimes b_i+\sum c_i\otimes d_i\in W_1\otimes W_0\oplus W_0\otimes W_1$. Then

$$\psi(dx)=\psi(\sum a_i\otimes b_i+\sum c_i\otimes d_i)=\sum(id+f)(a_i)\otimes b_i+\sum c_i\otimes(id+f)(d_i)=$$

$$\sum a_i\otimes b_i+\sum c_i\otimes d_i+\sum f(a_i)\otimes b_i+\sum c_i\otimes f(d_i)=$$

$$dx+\sum f(a_i)\otimes b_i+\sum c_i\otimes f(d_i)$$

and

$$D \circ \psi(x) = D(x) + D \circ f_1(x) + D \circ f_0(x) = dx + \eta\alpha(x) + d \circ f_1(x).$$

Comparing these two expressions we see that

$$\sum f(a_i) \otimes b_i + \sum c_i \otimes f(d_i) = \eta\alpha(x) + d \circ f_1(x)$$

and applying ρ on both sides give

$$\sum \rho \circ f(a_i) \otimes \rho(b_i) + \sum \rho(c_i) \otimes \rho \circ f(d_i) = \alpha(x).$$

Finally we note that $\rho \circ f : W_1 \to A$ and $\rho = \tau$ on W_0 so the above equality becomes $\alpha(x) = \partial(\rho \circ f)$. Since x was arbitrary we see that $[\alpha]$ is trivial which contradicts the previous lemma.

□

Now we are ready to state Salvatore's main result in [2]

Theorem 84. The configuration space $F_4(\mathbb{R}^2)$ is not formal over $\mathbb{Z}_2$.

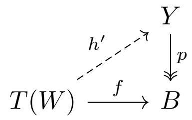
Proof. Recall from proposition 69 that the geometric realization of $\mathcal{F}_n(WS_k)$ is homotopy equivalent to $F_k(\mathbb{R}^n)$. This mean the following are quasi isomorphic $C^\bullet(F_4(\mathbb{R}^2)) \simeq C^\bullet(\mathcal{F}_2(WS_4)) = \mathcal{E}_2^\bullet(4)$ so formality of $F_k(\mathbb{R}^n)$ is equivalent to formality of the dga algebra $\mathcal{E}_2^\bullet(4)$. Now to get a contradiction let us assume $\mathcal{E}_2^\bullet(4)$ is formal. This means there is a zig-zag of dga algebras and quasi isomorphisms

$$\mathcal{E}_2^\bullet(4) \xleftarrow[\sim]{} A_0 \xrightarrow{\sim} \ldots \xleftarrow[\sim]{} A_n = A \ .$$

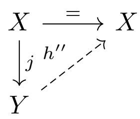
We may assume that this zig-zag induces the isomorphism $H^\bullet(\mathcal{E}_2^\bullet(4)) \cong A$ under which we have been working in this section (we can extend the zig zag by a suitable isomorphism $A \cong A$ if necessary). I claim that in this situation there is a quasi isomorphism $q : (T(W), d) \to \mathcal{E}_2^\bullet(4)$ such that $[q(w)] = \rho(w)$ for all $w \in T(W_0)$. Indeed, to prove this it suffices to show that $(T(W), d)$ has the following lifting property. Given quasi isomorphisms $f : (T(W), d) \to (B, d_B)$ and $g : (X, d_X) \to (B, d_B)$ there is a quasi isomorphism $h : (T(W), d) \to (X, d_X)$ such that the following diagram commutes on the level of homology

$$\begin{array}{ccc} & \overset{h}{\dashrightarrow} & X \\ & & \big\downarrow g \\ T(W) & \xrightarrow{f} & B \end{array} .$$

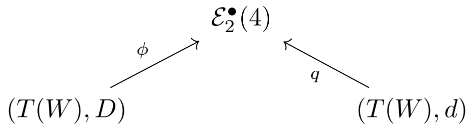
To prove this we first factor $g = p \circ j$ as in proposition 48 where $p : Y \to B$ is surjective and $j : X \to Y$ is a cell algebra inclusion and a quasi isomorphism and it has the lifting property in proposition 48. But since g and j are quasi isomorphisms p is too. Then by proposition 50 there is a map $h' : (T(W), d) \to (Y, d_Y)$ such that



commutes. Because f and p are both quasi isomorphisms h' is too. But then since j has the lifting property from proposition 48 there is a map h'' such that



commutes. Note that h'' is necessarily a quasi isomorphism. Then $h = h'' \circ h'$ is a quasi isomorphism with the desired property. But now we have two filtered models



which by theorem 4.2 means there is an isomorphism $\psi : (T(W), d) \to (T(W), D)$ which lowers the syzygy degree. This however cannot happen by the previous lemma so $F_4(\mathbb{R}^n)$ is not formal.

$\square$

So over $\mathbb{Z}_2$ $F_4(\mathbb{R}^2)$ gives an example of a space whose cohomology ring is Koszul but which is not formal.

	0	1	2	3	4	$\cdots$
0	$\mathbb{K}$	0	0	0	0	$\cdots$
1	0	$sA(1)$	0	0	0	$\cdots$
2	0	$sA(2)$	$sA(1) \otimes sA(1)$	0	0	$\cdots$
3	0	$sA(3)$	$sA(1) \otimes sA(2) \oplus sA(2) \otimes sA(1)$	$sA(1) \otimes sA(1) \otimes sA(1)$	0	$\cdots$
4	0	$sA(4)$	$\bigoplus_{i+j=4}(sA(i)) \otimes (sA(j))$	$\bigoplus_{i+j+k=4}(sA(i)) \otimes (sA(j)) \otimes (sA(k))$	$(sA(1))^{\otimes 4}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1: BA decomposed into weight and word length. The rows show the weight that come from the weight grading on A. The columns show word lenght.

	0	1	2	3	4	$\cdots$
0	$\mathbb{K}$	0	0	0	0	$\cdots$
1	0	$s^{-1}C(1)$	0	0	0	$\cdots$
2	0	$s^{-1}C(2)$	$s^{-1}C(1) \otimes s^{-1}C(1)$	0	0	$\cdots$
3	0	$s^{-1}C(3)$	$\bigoplus_{i+j=3} s^{-1}C(i) \otimes s^{-1}C(j)$	$s^{-1}C(1) \otimes s^{-1}C(1) \otimes s^{-1}C(1)$	0	$\cdots$
4	0	$s^{-1}C(4)$	$\bigoplus_{i+j=4} s^{-1}C(i) \otimes s^{-1}C(j)$	$\bigoplus_{i+j+k=4} s^{-1}C(i) \otimes s^{-1}C(j) \otimes s^{-1}C(k)$	$s^{-1}C(1)^{\otimes 4}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 2: The table shows ΩC decomposed by weight and word length. The rows show weight and the columns show word length.

$\cdots$	$\cdots$		$\cdots$		$\cdots$	(4)
0	$s^{-1}A^{¡}[2]$	$\to$	$(s^{-1}A^{¡}[0] \otimes s^{-1}A^{¡}[1]) \oplus (s^{-1}A^{¡}[1] \otimes s^{-1}A^{¡}[0])$	$\to$	$s^{-1}A^{¡}[0] \otimes s^{-1}A^{¡}[0] \otimes s^{-1}A^{¡}[0]$	(3)
0	0		$s^{-1}A^{¡}[1]$	$\to$	$s^{-1}A^{¡}[0] \otimes s^{-1}A^{¡}[0]$	(2)
0	0		0		$s^{-1}A^{¡}[0]$	(1)
0	0		0		$\mathbb{K}$	(0)
3	2		1		0	

Table 3: $\Omega A^{¡}$ decomposed by syzygy degree (indicated on the last row) and weight (indicated on the last column).

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