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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Realization functors and Kan complexes

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Abstract

In this report we study a generalization of the adjoint to Quillens functor λ from the category of differentially graded Lie algebras to simplicial sets. We describe its construction and prove that its image is a Kan complex.

Acknowledgements

I would like to thank my supervisor Alexander Berglund for offering his guidance and experience during the writing period. Further I'm grateful for the insightful remarks made by the examiner Gregory Arone, and in particular for the suggestion of rephrasing that $\langle L \rangle$ is a Kan complex by considering a retraction of $\widehat{\mathbb{L}}(\Delta^n)$ into $\widehat{\mathbb{L}}(\Lambda^n_k)$.

Introduction

One aspect of mathematics is to classify objects and divide them into different categories. The methods are plentiful but mostly involve searching for properties that are invariant under certain operations. In topology, and for topological spaces X, two common invariants are the the homotopy groups $\pi_n(X)$ and the singular homology groups $H_n(X)$. These groups are invariant under homeomorphisms, but are too wide in the sense that non-homeomorphic spaces can have isomorphic homotopy/homology groups. Recall that two topological spaces X and Y are *homotopy equivalent* if there are two continuous maps $f : X \to Y$ and $g : Y \to X$ such that that $gf \simeq id_X$, and $fg \simeq id_Y$. In particular the homotopy groups $\pi_n(X)$ and $\pi_n(Y)$ are isomorphic, induced by f and g. Homotopy equivalence imposes an equivalence relation on spaces, and the study of spaces modulo the relation of homotopy equivalence is called homotopy theory. Calculating the homotopy groups is however complicated and methods to overcome this difficulty are of great importance. One way of simplification is by the means of *rational homotopy theory*. The theory is based on the observation that $\pi_0(X) = \pi_1(X) = 0$ for simply connected spaces X and further that $\pi_n(X) = \mathbb{Z}^r \oplus T$ for $n \geq 2$ if X is a CW-complex of finite type, and T denotes an abelian group generated by elements of finite order. The group T, also known as the torsion, is one component complicating the calculation of the homotopy groups. One way of simplification is to tensor the homotopy groups with $\mathbb{Q}$. This effectively removes the torsion, since elements of finte order vanish when tensored with $\mathbb{Q}$. What remains is a vector space $\pi_n(X) \otimes \mathbb{Q} = \mathbb{Q}^r$ over $\mathbb{Q}$. Serre [8] was the first to formalize the way of removing torsion, and his work lay the foundation for the rational homotopy theory. Later it was Quillen [7] who developed a theory on this freshly ploughed land. Quillen proved the existence of a differentially graded Lie algebra $\lambda(X)$ associated to a simply connected space X so that $H_*(\lambda(X)) \cong \pi_*(X) \otimes \mathbb{Q}$. The functor $\lambda : \mathbf{Top} \to \mathbf{DGL}$ showed that these categories were identical on the level of rational homotopy and homology respectively. Theoretically this construction was a succes, but as Hess [5] (p.768) puts it: "Performing actual calculations [...] was impossible in practice." Methods were developed to bridge this computability gap, with one of the pioneers being Sullivan [9] who's work is greatly influential in the theory today. Recently, the quartet Buijs, Félix, Murillo and Tanré [1] constructed a pair of functors that extends the functor of Quillen. In this report we will investigate further into the construction of one of their functors. But first we need to understand their approach to the subject at hand.

It turns out if you want to study topological spaces up to homotopy, you may as well study another object, namely simplicial sets. A simplicial set is an abstraction of a simplex, carrying some of the essential properties from simplices. The advantage is that we can induce this simplicial structure on all kind of mathematical objects such as groups, chain complexes and topological spaces. Since all these objects are based on sets, these are at the same time simplicial sets. The category of simplicial sets is denoted **sSet**. Its relation to topology is made explicit through the functors

$$S_\bullet : \mathbf{Top} \to \mathbf{sSet}$$
$$|\cdot| : \mathbf{sSet} \to \mathbf{Top},$$

where $S_\bullet$ is the singular simplicial set, and $|\cdot|$ the realization functor. We study these functors in greater detail in chapter 1. Just as we can associate the homotopy group to a topological space, there is a similar group structure we can associate to a simplicial set. Such simplicial sets are called Kan complexes and the associated group is, just as its topological counterpart, called the homotopy group and is denoted by $\pi_n(X)$. This similarity is no coincidence, since the theory of topological homotopy groups and simplicial homotopy groups are almost equivalent. In fact the task of calculating the homotopy group of a topological space can be translated to calculating the homotopy group of a simplicial set by using the functors $S_\bullet$ and $|\cdot|$. This cements the idea of studying simplicial sets instead of topological spaces.

In [1] the authors constructed the functors

$$\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$$
$$\mathcal{L} : \mathbf{sSet} \to \mathbf{DGL},$$

where $\langle\cdot\rangle$ is also referred to as the realization functor. The functor $\mathcal{L}$ generalize Quillens functor λ, and similarly $\langle\cdot\rangle$ generalize the adjoint of λ. These functors have the advantage of being much simpler than the functors created by Quillen. Further they satisfy

$$H_n(\mathcal{L}(K), \partial_a) \cong \pi_{n+1}(K) \otimes \mathbb{Q}$$

when K is a simply connected finite simplicial complex, and

$$H_n(L, \partial) \cong \pi_{n+1}(\langle L\rangle, 0)$$

when L is a complete DGL concentrated in finite degrees. In this report we will study the construction of the functor $\langle\cdot\rangle$, and some of its properties.

Writing this report I had two goals in mind. Firstly, to clarify some of the results presented in the original report. The main contribution is some minor clarification of their results which are presented in section 4 and 5, the most important being to show that $\langle L\rangle$ is a Kan complex. The second goal was to present the material in an approachable way to readers with no experience of simplicial homotopy theory. Therefore the necessary background is presented on a fairly simple level, plentiful of examples have been included, and most proofs are carried out in detail, even for basic concepts. Some prerequisites are topology, homological algebra and preferably some understanding of category theory. However most of the concepts are defined thoroughly and results presented within close range of the definitions.

Disposition: In the first section we give a brief introduction to simplicial sets, including standard examples and constructions. Further we define a special kind of simplicial sets, namely Kan complexes and show that we can associate the homotopy group to such simplicial sets. In section 2 we introduce the notion of differentially graded Lie algebras (DGLs) and some related basic definitions. The realization functor $\langle\cdot\rangle$ is defined using a collection of DGLs $\mathcal{L}_\bullet$, and the examples of this section are well connected to its construction and will be frequently referred to in later sections. Section 3 presents a fundamental result from simplicial homotopy theory, namely the the Dold-Kan correspondence. This correspondence plays two parts. On the one hand as an example that connects section 1 and 2. On the other hand it acts as a prelude to section 5 where we construct $\langle\cdot\rangle$ and see that the Dold-Kan correspondence serves a special case of $\langle\cdot\rangle$. In section 4 we define the cosimplicial DGL $\mathcal{L}_\bullet$, following the construction from [1] and show related results. In section 5 we present the definition of $\langle L\rangle$ and prove that it is a Kan complex when L is a complete DGL. We further show that $\pi_n\langle L\rangle \cong H_{n-1}(L)$ for L concentrated in positive degrees. We end the section with calculating the rational homotopy groups of the n-dimensional spheres.

Contents

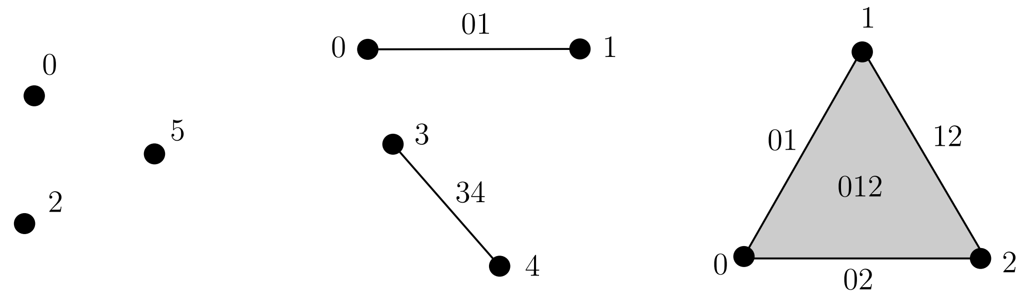
Chapter 1

Simplicial Theory

The idea behind simplicial theory is to study objects whose structure is similar to that of a simplex. Our understanding of simplices begins with its geometrical interpretation. A topological n-simplex $|\Delta|^n$ is the convex hull of $n+1$ points in a general position, usually described by

$$|\Delta^n| = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0 \text{ and } \sum t_i = 1\}.$$

Thus a 0-simplex $|\Delta^0|$ is a point, a 1-simplex $|\Delta^1|$ is a line, a 2-simplex $|\Delta^2|$ a triangle and so on. With this perspective, it is clear that $|\Delta^n|$ contain lower-dimensional simplices as faces. For example the triangle $|\Delta^2|$ contains three lines $|\Delta^1|$ represented by the subspaces $\{(0, t_1, t_2) \,|\, t_1 + t_2 = 1\}$, $\{(t_0, 0, t_2) \,|\, t_0 + t_2 = 1\}$ and $\{(t_0, t_1, 0) \,|\, t_0 + t_1 = 1\}$. Similarly $|\Delta^2|$ contains three points $|\Delta^0|$ represented by $\{(1, 0, 0)\}$, $\{(0, 1, 0)\}$ and $\{(0, 0, 1)\}$. A more general concept of simplices should preserve this structure. As we noted, $|\Delta^n|$ is the convex hull of $n+1$ points in a general position. With this convention, it becomes clear that the k-dimensional faces of $|\Delta^n|$ is in a bijective correspondence with subsets of $\{0, 1, ..., n\}$ of size k. That is, if we label the vertices of $|\Delta^n|$ with the integers $0, ..., n$, then any collection of k integers corresponds to a k-simplex of $|\Delta^n|$.



Interpretation of single numbers as 0-simplices, pair of numbers as 1-simplices and triples as 2-simplices.

Using this idea we define the abstract n-simplex Δ^n as the powerset of $\{0, ..., n\}$. The set of k-simplices Δ^n_k of Δ^n can then be interpreted as increasing k-tuples on $\{0, ...n\}$.

That is

$$\Delta^n_k = \{(v_0, ..., v_k) \,|\, 0 \leq v_i < v_{i+1} \leq n\}.$$

Thus the abstract 1-simplex Δ^1 becomes the set $\{(0), (1), (0, 1)\}$ and the abstract 2-simplex Δ^2 is the set $\{(0), (1), (2), (0, 1), (0, 2), (1, 2), (1, 2, 3)\}$. Later in this chapter we will also allow degenerate simplices such as $(0, 0)$ in the definition of Δ^n. This has the advantage that a k-simplex $(x_0, ..., x_k)$ in Δ^n can be interpreted as a monotone increasing map $\varphi : \{0, ..., k\} \to \{0, ..., n\}$ by defining $\varphi(i) = x_i$. This observation leads to the construction of a category $\mathbf{\Delta}$ where the objects are sets of the form $[n] = \{0, ..., n\}$ and the morphisms are monotone increasing maps between these sets, just as φ above. This will act as the foundation on which the simplicial theory lies upon.

We properly define $\mathbf{\Delta}$ together with other basic concepts of simplicial theory in the first part of this chapter. We include several examples, including the topological n-simplex $|\Delta^n|$ and the abstract n-simplex Δ^n. We also describe a method of creating functors from any category to the category of simplicial sets. One application of this is the construction of the functor $\langle \cdot \rangle : \mathbf{DGL} \to \mathbf{sSet}$ in section 4. In the second part we introduce the Kan condition of a simplicial set. Any simplicial set that satisfy the Kan-condition is called a Kan complex. We further define the homotopy group corresponding to a Kan complex and prove that this group is well defined and satisfies the group axioms. Lastly we provide an example from topology involving the singular simplicial set functor $S_\bullet : \mathbf{Top} \to \mathbf{sSet}$ and its adjoint, the realization functor $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$.

Definitions and examples of simplicial objects

Definition 1.1. Let $\mathbf{\Delta}$ be the category where

$$\text{Objects: Sets on the form } [n] = \{0, ..., n\} \text{ for } n \in \mathbb{N}.$$
$$\text{Morphisms: Monotone increasing maps } \varphi : [m] \to [n].$$

That is every morphism $\varphi : [m] \to [n]$ satisfies $\varphi(i) \leq \varphi(j)$ for $0 \leq i \leq j \leq m$.

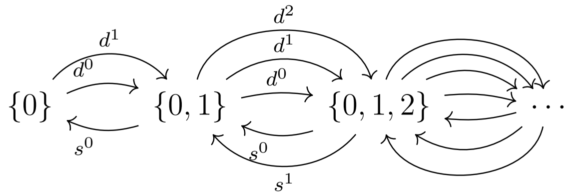
The category $\mathbf{\Delta}$ contains two families of morphisms, namely

$$d^i : [n-1] \to [n], \quad 0 \leq i \leq n,$$
$$s^i : [n+1] \to [n], \quad 0 \leq i \leq n,$$

where d^i is the unique injective function not containing i in its image, and s^i is the unique surjective function where i is hit twice. These maps have two fundamental properties. Firstly, these maps generate the category $\mathbf{\Delta}$ in the sense that every morphsim is a composition of d^i and s^i. Secondly, they satisfy the following list of relations

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad i < j \\ s^j d^i &= d^i s^{j-1}, \quad i < j \\ s^i d^i &= s^i d^{i+1} = 1 \\ s^j d^i &= d^{i-1} s^j, \quad i \geq j \\ s^j s^i &= s^i s^{j+1}, \quad i \leq j. \end{aligned}$$

This list is complete in the sense that every other relation between the d^i and s^i are derivable from these [4] (p.4). We can visualize this as



Both perspectives of the morphisms of $\mathbf{\Delta}$ will be used throughout this paper.

Definition 1.2. Let $\mathcal{C}$ be a category. A *simplicial object* C in $\mathcal{C}$ is a covariant functor

$$C : \mathbf{\Delta}^{op} \to \mathcal{C}.$$

We use the notation $C_n = C([n])$ and $\varphi^* = C(\varphi) : C_n \to C_m$ when $\varphi : [m] \to [n]$. The simplicial objects of a category $\mathcal{C}$ is itself a category, denoted $s\mathcal{C}$. The objects are simplicial objects of $\mathcal{C}$ and the morphisms are natural transformations. More explicitly if X, Y are two objects in $s\mathcal{C}$, a morphism from X to Y is a collection of $\mathcal{C}$-morphisms $\psi_i : X_i \to Y_i$ so that

$$\begin{array}{ccc} X_n & \xrightarrow{\psi_n} & Y_n \\ \downarrow{\scriptstyle X(\varphi)} & & \downarrow{\scriptstyle Y(\varphi)} \\ X_m & \xrightarrow{\psi_m} & Y_m \end{array}$$

commutes for every $\varphi : [m] \to [n]$. By the properties of d^i and s^i, a simplicial object C in $\mathcal{C}$ is equivalent to a sequence $\{C_n\}_{n\geq 0}$ together with morphisms corresponding to d^i and s^j. This fact leads to an equivalent definition of simplicial objects. A simplicial object is a sequence of objects $\{C_n\}_{n\geq 0}$ in $\mathcal{C}$ together with morphisms

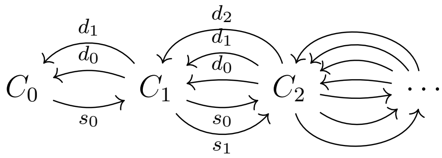
$$d_i : C_n \to C_{n-1}, \quad 0 \leq i \leq n$$

$$s_i : C_n \to C_{n+1}, \quad 0 \leq i \leq n$$

satisfying the relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \quad i < j \\ d_i s_j &= s_{j-1} d_i, \quad i < j \\ d_i s_i &= d_{i+1} s_i = 1 \\ d_i s_j &= s_j d_{i-1}, \quad i \geq j \\ s_i s_j &= s_{j+1} s_i, \quad i \leq j. \end{aligned}$$

Writing this as a diagram we have



The maps d_i and s_i will be referred to the i:th *face map* and i:th *degeneracy map* respectively. The elements of the set C_n are called the n-simplices of C. Any n-simplex in the image of a degeneracy map s^i is called *degenerate*, and *non-degenerete* otherwise.

Remark 1.3. Any simplicial object we consider is naturally included into the category of simplicial sets **sSet** using the forgetful functor.

Example 1.4. The *standard n-simplex* Δ^n is the simplicial set where the k-simplices is the set of morphisms in $\mathbf{\Delta}$ from $[k]$ to $[n]$. That is $\Delta^n_k = \mathbf{Hom}_{\mathbf{\Delta}}([k],[n])$. Naturally this is a covariant functor $\mathbf{Hom}_{\mathbf{\Delta}}(-,[n]) : \mathbf{\Delta}^{op} \to \mathbf{Set}$ since $f \in \Delta^n_k$ and $\varphi : [l] \to [k]$ imply that $\varphi^*(f) \in \Delta^n_l$ since $\varphi^*(f) = f \circ \varphi : [l] \to [n]$. Each map $f \in \Delta^n_k$ may be naturally identified with a k-tuple of elements from the set $[n]$ so that the sequence is monotone increasing. Thus equivalently we may identify Δ^n_k with the set $\{(v_0, ..., v_k) \in [n]^k \,|\, v_i \leq v_{i+1}\}$. In this context the face and degeneracy maps are defined as

$$d_i : \Delta^n_k \to \Delta^n_{k-1}, \qquad (v_0, ..., v_k) \mapsto (v_0, ..., \hat{v}_i, ..., v_k)$$

$$s_i : \Delta^n_k \to \Delta^n_{k+1}, \qquad (v_0, ..., v_k) \mapsto (v_0, ..., v_i, v_i, ..., v_k).$$

The notation $(v_0, ..., \hat{v}_i, ..., v_n)$ denotes removing the element v_i from the $n+1$-tuple. In other words, $(v_0, ..., \hat{v}_i, ..., v_n) := (v_0, ..., v_{i-1}, v_{i+1}, ..., v_n)$. For simplicity we write $(v_0, ..., v_n)$ as $v_0...v_n$.

The *boundary* $\dot{\Delta}^n$ of Δ^n is the simplicial subset generated by all k-simplices for $0 \leq k \leq n$ except the n-simplex $01...n$. The *p-horn* Λ^n_p of Δ^n is the simplicial subset generated by all k-simplices $0 \leq k \leq n$ except $0...n$ and $d_p(0...n)$. As the name suggests, there is a natural way of associating the non-degenerate vertices of the standard n-simplex Δ^n to the picture of a (surprise) n-simplex. We illustrate this below for the standard 2-simplex Δ^2, its boundary $\dot{\Delta}^n$ and the 0-horn Λ^n_i respectively.

Δ^2		
Simplices	Non-degenerate	Degenerate
0	0, 1, 2	-
1	01, 02, 12	00, 11, 22
2	012	000, 001, 002, 011, 022, 111, 112, 122, 222
3	-	0000, 0001, 0002,...
$\vdots$	$\vdots$	$\vdots$

Table 1: Elements of the standard 2-simplex Δ^2

$\dot{\Delta}^2$		
Simplices	Non-degenerate	Degenerate
0	0, 1, 2	-
1	01, 02, 12	00, 11, 22
2	-	000, 001, 002, 011, 022, 111, 112, 122, 222
3	-	0000, 0001, 0002,...
⋮	⋮	⋮

Table 2: Elements of the boundary $\dot{\Delta}^2$ of the 2-simplex.

$\dot{\Lambda}_0^2$		
Simplices	Non-degenerate	Degenerate
0	0, 1, 2	-
1	01, 02	00, 11, 22
2	-	000, 001, 002, 011, 022, 111, 222
3	-	0000, 0001, 0002,...
⋮	⋮	⋮

Table 3: Elements of the 0-horn Λ_0^2 of the 2-simplex.

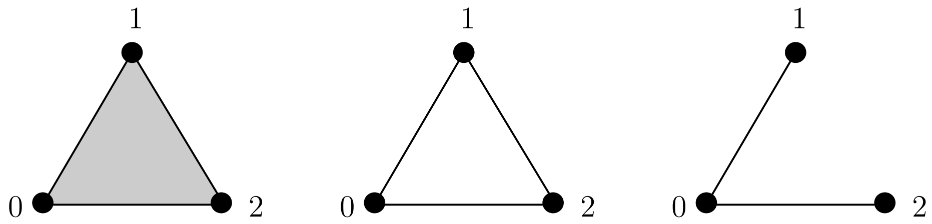


Diagram of the 2-simplex Δ^2, the 2-boundary $\dot{\Delta}^2$ and the 0-horn Λ_0^2.

Note that not all 3-simplices are in the boundary $\dot{\Delta}^2$. For example 0012 is not, since it is only generated by 012. Similarly 112 and 122 are not contained among the 2-vertices of the 0-horn Λ_0^2 since they are generated by $d_0(012) = 12$.

Example 1.5. Let G be a group. The *nerve* $N_\bullet G$ of G is the simplicial set with $N_n G = \{(g_1, ..., g_n) \,|\, g_i \in G\}$ and face and degeneracy maps defined as

$$d_i(g_1, ..., g_i, g_{i+1}, ..., g_n) = (g_1, ..., g_i \cdot g_{i+1}, ..., g_n)$$
$$s_i(g_1, ..., g_i, g_{i+1}, ..., g_n) = (g_1, ..., g_i, e, g_{i+1}, ..., g_n)$$

with the exception that $d_0(g_1, ...g_n) = (g_2, ..., g_n)$ and $d_n(g_1, ..., g_n) = (g_1, ..., g_{n-1})$. Equivalently the n-simplices are the unique compositions of n morphisms on G

$$G \xrightarrow{g_1} G \xrightarrow{g_2} G \xrightarrow{g_3} \cdots \xrightarrow{g_{n-2}} G \xrightarrow{g_{n-1}} G \xrightarrow{g_n} G$$

corresponding to multiplication by the elements $(g_1, ..., g_n)$. The i:th face map corresponds to compose the morphisms g_i and g_{i+1}, and the i:th degeneracy map corresponds to inserting the identity-morphism in the i:th position.

Definition 1.6. Let $\mathcal{D}$ be a category. A *cosimplicial object* D in $\mathcal{D}$ is a covariant functor

$$D : \mathbf{\Delta} \to \mathcal{D}.$$

We use the notation $D([n]) = D^n$ and $\varphi_* = D(\varphi) : D^n \to D^m$ when $\varphi : [n] \to [m]$. Equivalently a cosimplicial object is a sequence of objects $\{D^n\}_{n\geq 0}$ in $\mathcal{D}$ with morphisms $d^i : D^{n-1} \to D^n$ and $s^i : D^{n-1} \to D^n$ satisfying the relations of the morhpisms of $\mathbf{\Delta}$. The maps d^i and s^i are usually referred to the i:th *coface* and i:th *codegeneracy* maps.

Remark 1.7. Notice the the difference in notation with simplicial objects, where the k-simplices of a simplicial object is denoted by C_k and the k-simplices of a cosimplicial object is denoted by C^k.

Example 1.8. The collection of standard simplices $\Delta^\bullet = \{\Delta^n\}_{n\in\mathbb{N}}$ is a cosimplicial object. The coface maps $d^k : \Delta^n \to \Delta^{n+1}$ are defined by

$$d^k(i_0...i_p) = (j_0...j_p) \text{ where } j_l = \begin{cases} i_l & \text{if } l < k \\ i_l + 1 & \text{if } l \geq k \end{cases} \qquad \begin{matrix} 0 \leq k \leq n+1 \\ 0 \leq p \leq n. \end{matrix}$$

The codegeneracy maps $s^k : \Delta^n \to \Delta^{n-1}$ are defined as

$$s^k(i_0...i_p) = (j_0...j_p) \text{ where } j_l = \begin{cases} i_l & \text{if } l \leq k \\ i_l - 1 & \text{if } l > k \end{cases} \qquad 0 \leq k, p \leq n.$$

Unless explicitly stated, Δ^n will denote the standard n-simplex.

Example 1.9. The standard topological n-simplex is the topological space

$$|\Delta^n| = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0 \text{ and } \sum t_i = 1\}.$$

The collection $\{|\Delta^n|\}_{n\geq 0}$ is a cosimplicial topological space. The simplicial map $\varphi^* : |\Delta^m| \to |\Delta^n|$ induced by $\varphi : [m] \to [n]$ is defined by

$$\varphi^*(t_0, ..., t_m) = (s_0, ..., s_n), \text{ where } s_k = \sum_{i\in\varphi^{-1}(k)} t_i.$$

Due to their functorial nature, simplicial and cosimplicial objects can themselves generate a multitude of simplicial and cosimplicial sets.

Example 1.10. Let X be a cosimplicial object in the category $\mathcal{C}$. The composition of the functors $X : \mathbf{\Delta} \to \mathcal{C}$ and $\mathbf{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ give the functor

$$\mathbf{Hom}_{\mathcal{C}}(X(\,\cdot\,), -) : \mathbf{\Delta}^{op} \times \mathcal{C} \to \mathbf{Set}.$$

This can be viewed a functor from $\mathcal{C}$ to **sSet**. The object $\mathbf{Hom}_{\mathcal{C}}(X(\,\cdot\,)\,,C)$ is a simplicial set for any object C in $\mathcal{C}$ with the set of n-vertices being the morphisms

$$\mathbf{Hom}_{\mathcal{C}}(X(n)\,,C) = \mathbf{Hom}_{\mathcal{C}}(X_n\,,C).$$

The map $\varphi_* : \mathbf{Hom}_{\mathcal{C}}(X_m\,,C) \to \mathbf{Hom}_{\mathcal{C}}(X_n\,,C)$ induced by $\varphi^* : X_n \to X_m$ is defined on $f \in \mathbf{Hom}_{\mathcal{C}}(X_m\,,C)$ by $\varphi_* f = f \circ \varphi^*$. Similarly a simplicial object Y induces a functor

$$\mathbf{Hom}_{\mathcal{C}}(-\,,Y(\,\cdot\,)) : \mathcal{C} \to \mathbf{sSet}^{op}.$$

from $\mathcal{C}$ to cosimplicial sets.

Remark 1.11. In section 5 we will construct the functor $\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$ in this manner. That is we construct a cosimplicial DGL $\mathcal{L}_\bullet$ and define the simplicial set $\langle L\rangle = \mathbf{Hom}_{\mathbf{DGL}}(\mathcal{L}_\bullet, L)$ for $L \in \mathbf{DGL}$.

Example 1.12. The *singular simplicial set* $S_\bullet(T)$ of a topological space T is the simplicial set $S_\bullet(T) = \mathbf{Hom}_{\mathbf{Top}}(\{|\Delta^n|\}_{n\geq 0}\,,T)$.

Example 1.13. Let $\mathbf{\Delta}^\bullet$ be the cosimplicial set from example 1.8. This defines the functor

$$\mathbf{Hom}_{\mathbf{sSet}}(\mathbf{\Delta}^\bullet\,,-) : \mathbf{sSet} \to \mathbf{sSet}.$$

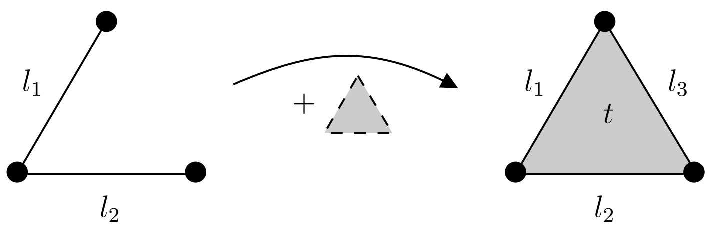
If X is a simplicial set, then $\mathbf{Hom}_{\mathbf{sSet}}(\mathbf{\Delta}^\bullet\,,X) \cong X$ by Yoneda's lemma. In particular the isomorphism of the n-simplices

$$\mathbf{Hom}_{\mathbf{sSet}}(\Delta^n\,,X_\bullet) \cong X_n$$

means that a simplicial map $f : \Delta^n \to X$ is uniquely determined by where it maps $(0...n)$.

The homotopy group $\pi_n(X, x_0)$ of a simplicial set

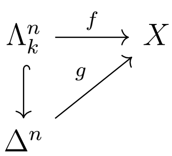
Next up we will define a group π_n associated so simplicial sets, called the n:th homotopy group. This group can however only be associated to simplicial sets that satisfy the Kan condition.The Kan condition has a straightforward geometric interpretation. Suppose that we have a horn of the 2-simplex. That is two 1-simplices l_1, l_2 that have a 0-simplex in common. We might *fill* this horn by finding a 2-simplex t containing l_1 and l_2.



Filling the 2-horn.

More generally, a simplicial set is a Kan complex if each horn Λ^n_k can be filled by an n-simplex. Given a Kan complex X, one can define the n:th homotopy group $\pi_n(X, x_0)$. The elements $\pi_n(X, x_0)$ will be a subset of the n-simplices of X having the $n-1$-simplex x_0 as their only face, modulo some equivalence relation. The group operation is best visualized for the 1:st homotopy group $\pi_1(X)$. Let l_1 and l_2 be two 1-simplices that have a 0-simplex in common. Together they become a horn of a 2-simplex. Since X is a Kan complex, one can fill the horn with some 2-simplex t having l_1 and l_2 as faces. One defines the third face l_3 of t to be their product. We note here the necessity of X being a Kan complex to guarantee the existence of l_3. This operation is however only well defined under the equivalence relation we impose. This section is devoted to define these concepts and verify that it is well defined.

Definition 1.14. Let X be a simplicial object. We say that X is a *Kan complex* if for each simplicial map $f : \Lambda^n_k \to X$, there is a simplicial map $g : \Delta^n \to X$ so that the following diagram commutes.



Note that a simplicial map $f : \Lambda^n_k \to X$ is uniquely determined where it maps its nondegenerate $n-1$-simplices. Thus defining f is equivalent of choosing a collection of $n-1$ vertices $x_0, ..., \widehat{x_k}, ..., x_n$ in X so that

$$d_i x_j = d_{j-1} x_i \text{ for } i < j \text{ and } i, j \neq k. \tag{1.0.1}$$

Note that g is uniquely defined by some n-vertice x in X by example 1.13. Thus equivalently, X is a Kan complex if for each collection $x_0, ..., \hat{x_k}, ..., x_n \in X_{n-1}$ satisfying (1.0.1), there is an $x \in X_n$ so that $d_i x = x_i$ for $i \neq k$. This condition will be referred to as the *Kan-condition.*

Example 1.15. A general simplicial set is not a Kan complex. Here follows three standard examples.

- The standard n-simplex Δ^n is not a Kan complex. As an example, consider the standard 1-simplex Δ^1 and the vertices $y_0 = 00$ and $y_2 = 01$. But the 2-simplices 000 and 011 are the only ones that satisfy $d_0(000) = y_0$ and $d_2(011) = y_2$ respectively. Since they are not equal, the Kan condition is not satisfied.

- Every simplicial group $G_\bullet$ is a Kan complex. Since the simplicial maps are group homomorphism, they preserve a rich enough structure so that a lift is possible. For example see [4].

- The singular simplicial set $S_\bullet(T)$ of a topological space T is a Kan complex. For details see example 1.21.

Let X be a simplicial set and $x \in X_n$. Let δx denote the $n+1$-tuple containing the images of x under the face maps d_i. That is

$$\delta x = (d_0 x, d_1 x, ..., d_n x).$$

Let $x_0 \in X_0$. A *base-point* x_0 is the collection of degenerate vertices that can be obtained from x_0. Due to the simplicial relations, any such element will be on the form $s_0^n x_0$. For brevity we let x_0 denote $s_0^n x_0$.

Definition 1.16. Let X be a Kan complex and $x_0 \in X_0$ a base point. Let $n \geq 1$ and consider the set $\tau_n(X, x_0)$ of n-simplices $x \in X_n$ so that $\delta x = (x_0, ..., x_0)$. That is

$$\tau_n(X, x_0) = \{x \in X_n \mid \delta x = (x_0, ..., x_0)\}.$$

Define a relation $\sim$ on $\tau_n(X, x_0)$ by

$$x \sim y \text{ if and only if } \exists \omega \in X_{n+1} \text{ so that } \delta\omega = (x, y, x_0, ..., x_0).$$

We will show that $\sim$ is an equivalence relation, and we set $\pi_n(X, x_0)$ to be the set of equivalence classes under this relation. That is $\pi_n(X, x_0) = \tau_n(X, x_0)/\sim$.

Proposition 1.17. *The relation $\sim$ is an equivalence relation.*

Proof. **Reflexivity:** Let $x \in \tau_n(X, x_0)$. The simplex $s_0 x$ gives

$$\begin{aligned}\delta s_0 x &= (d_0 s_0 x, d_1 s_0 x, d_2 s_0 x, ..., d_{n+1} s_0 x) \\ &= (x, x, s_0 d_1 x, ..., s_0 d_n x) \\ &= (x, x, x_0, ..., x_0).\end{aligned}$$

By definition this means that $x \sim x$.
Symmetry: Let $x, y \in \tau_n(X, x_0)$ so that $x \sim y$. Let $\omega \in X_{n+1}$ where

$$\delta\omega = (x, y, x_0, ..., x_0).$$

Consider the collection of $n+1$ simplices

$$(\hat{y_0}, y_1, y_2, y_3, ..., y_{n+1}) := (\,\cdot\,, s_0 y, \omega, x_0, ..., x_0).$$

These vertices satisfy the Kan condition (1.0.1) and so we find $\chi \in X_{n+2}$ so that $d_i \chi = y_i$. In particular if we set $y_0 = d_0 \chi$, then

$$\delta y_0 = (y, x, x_0, ..., x_0)$$

due to the relation $d_j y_0 = d_0 y_{j+1}$ for $0 \leq j \leq n$. Hence $y \sim x$.
Transitivity: Suppose that $x \sim y$ and $y \sim z$. Since we have already shown symmetry, we have $z \sim y$. Thus there are $\omega, \chi \in X_{n+1}$ so that

$$\begin{aligned}\delta\omega &= (x, y, x_0, ..., x_0) \\ \delta\chi &= (z, y, x_0, ..., x_0).\end{aligned}$$

The collection of $n+1$ simplices

$$(\hat{y_0}, y_1, y_2, y_3, ..., y_{n+1}) = (\,\cdot\,, \omega, \chi, x_0, ..., x_0)$$

satisfy the Kan condition. Let $\theta \in X_{n+2}$ so that $d_i\theta = y_i$ for $i \geq 1$. Set $y_0 = d_0\theta$ and from the simplicial identities we gather

$$\delta y_0 = (x, z, x_0, ..., x_0).$$

Hence $x \sim z$, and we have shown that $\sim$ is an equivalence relation on $\pi(X, x_0)$. □

Remark 1.18. Let $\sim'$ be a relation on $\tau_n(X, x_0)$ defined as

$$x \sim' y \text{ if and only if } \delta\omega = (x_0, ..., x_0, x, y, x_0, ...x_0).$$

It turns out that this is also an equivalence relation, and that is equivalent to $\sim$. That is $x \sim y$ if and only if $x \sim' y$. For example see lemma 1.22 [2].

So far we have shown that $\tau_n(X, x_0)/\sim$ is a collection of equivalence classes. Next up we want to define a group operation on $\pi_n(X, x_0)$ making it into a group. We note that if $x, y \in \tau_n(X, x_0)$, then the collection

$$(y_0, \hat{y_1}, y_2, y_3, ..., y_{n+1}) = (y, \,\cdot\,, x, x_0, ..., x_0)$$

of n-vertices satisfies the Kan-condition. Let ω be the $n+1$-simplex which fills this horn. In other words

$$\delta\omega = (y, d_1\omega, x, x_0, ..., x_0).$$

We use this construction to define a group operation on $\pi_n(X, x_0)$.

Proposition 1.19. *The assignment $x \cdot y = d_1\omega$ is a well defined group operation on the equivalence classes of $\pi_n(X_n, x_0)$.*

Definition 1.20. The group $\pi_n(X, x_0)$ is called the *n:th homotopy group* at the base point x_0.

Proof. First we show that the product is well defined on the equivalence classes, and thereafter we prove the group axioms.
Well defined: We will show that the operation is well defined in two steps. First that $x \sim x'$ implies that $x \cdot y \sim x' \cdot y$, and then second that $y \sim y'$ implies that $x \cdot y \sim x \cdot y'$. It then follows that the operator as a whole is well defined. Suppose that $x \sim x'$ with the $n+1$ vertices ω_x, χ, χ' such that

$$\begin{aligned} \delta\omega_x &= (x', x, x_0, x_0, ..., x_0) \\ \delta\chi &= (y, (x \cdot y), x, x_0, ..., x_0) \\ \delta\chi' &= (y, (x' \cdot y), x', x_0, ..., x_0). \end{aligned}$$

The collection

$$(y_0, y_1, \hat{y_2}, y_3, y_4, ..., y_{n+1}) = (\chi', \chi, \,\cdot\,, \omega_x, x_0, ..., x_0)$$

satisfy the Kan-condition. Let θ be the corresponding $n+2$ vertex and set $y_2 = d_2\theta$. Then y_2 satisfies

$$\delta y_2 = \big((x' \cdot y), (x \cdot y), x_0, ..., x_0\big).$$

Hence $x \cdot y \sim x' \cdot y$. Assuming $y \sim y'$, then a similar argument can be made to show $x \cdot y \sim x \cdot y'$. Let ω_y, χ, χ' be the $n+1$ vertices satisfying

$$\begin{aligned} \delta\omega_y &= (x_0, y, y', x_0, ..., x_0) \\ \delta\chi &= (y, (x \cdot y), x, x_0, ..., x_0) \\ \delta\chi' &= (y', (x \cdot y'), x, x_0, ..., x_0). \end{aligned}$$

Note the use of remark 1.18 for ω_y. Then the collection

$$(y_0, \hat{y_1}, y_2, y_3, y_4, ..., y_{n+1}) = (\omega_y, \,\cdot\,, \chi, \chi', x_0, ..., x_0)$$

satisfies the Kan-condition, and gives the $n+1$ vertex y_1 showing $x \cdot y \sim x \cdot y'$.
Identity: The vertex x_0 is the identity. This follows from remark 1.18 by

$$\begin{aligned} x \sim x &\iff \delta\omega = (x, x, x_0, ..., x_0) \text{ some } \omega \in X_{n+1} \\ &\iff \delta\chi = (x_0, x, x, x_0, ..., x_0) \text{ some } \chi \in X_{n+1}. \end{aligned}$$

Hence $[x_0][x] = [x][x_0] = [x]$.
Inverse: We note that there always exist an inverse of x by observing that the n-vertices

$$(y_0, y_1, \hat{y_2}, y_3 ..., y_n) = (x, x_0, \,\cdot\,, x_0, ..., x_0)$$

satisfies the Kan-condition. So there is a $n+1$ vertex ω so that

$$\delta\omega = (x, x_0, y, x_0, ..., x_0).$$

That is $[y][x] = [x_0]$ and so $[x]^{-1} = [y]$.
Associativity: Let $x, y, z \in \pi_n(X, x_0)$ and consider the $n+1$ simplices $\omega_0, \omega_1, \omega_3$ that correspond to $y \cdot z$, $(x \cdot y) \cdot z$ and $x \cdot y$ respectively. That is

$$\begin{aligned} \delta\omega_0 &= \big(z, (y \cdot z), y, x_0, ..., x_0\big) \\ \delta\omega_1 &= \big(z, \big((x \cdot y) \cdot z\big), (x \cdot y), x_0, ..., x_0\big) \\ \delta\omega_3 &= \big(y, (x \cdot y), x, x_0, ..., x_0\big). \end{aligned}$$

The collection $(\omega_0, \omega_1, \,\cdot\,, \omega_3, x_0, ..., x_0)$ satisfies the Kan-condition. Let θ be the corresponding $n+2$ vertex and set $\omega_2 = d_2\theta$. Then by the simplicial identities we have

$$\begin{aligned} d_0\omega_2 &= d_1\omega_0 = yz \\ d_1\omega_2 &= d_1\omega_1 = (xy)z \\ d_2\omega_2 &= d_2\omega_3 = x. \end{aligned}$$

However note that $d_0\omega_2 = yz$ and $d_2\omega_2 = x$, and so by definition $d_1\omega_2 = x(yz)$. This together with the statement above gives $\big([x][y]\big)[z] = [x]\big([y][z]\big)$. □

The usefullness of the homotopy group comes into play when considering functors between **sSet** and other categories that have a similar structure.

Example 1.21. As mentioned in example 1.15, the singular simplicial set is a Kan complex. One way of showing this is through finding a functor $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ that is left adjoint to $S_\bullet(\cdot) : \mathbf{Top} \to \mathbf{sSet}$. Let X be a simplicial set and Δ^n the topological n-simplex. The *geometric realization* $|X|$ of X is the topological space defined by the quotient

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim .$$

The relation $\sim$ is induced by all $\varphi : [m] \to [n]$ by

$$(\varphi^*(x), \mathbf{t}) \sim (x, \varphi_*(\mathbf{t}))$$

for all $x \in X_n$ and all $\mathbf{t} \in \Delta^m$. The realization functor is left adjoint to the singular simplicial set functor. That is, there is a bijection of the set of morphisms $\mathbf{Top}(|X|, T) \cong \mathbf{sSet}(X, S_\bullet(T))$. What we can note is that the geometric realization of the standard n-simplex Δ^n is homeomorphic to the standard topological simplex $|\Delta^n|$, explaining the notation. Similarly the realization of the k-horn Λ^n_k is homeomorphic to the topological k-horn. We use this to show that $S_\bullet(T)$ is a Kan complex. Note the relation between the geometric n-simplex and geometric k-horn

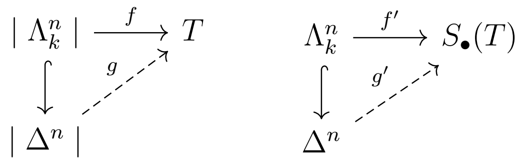
$$|\Delta^n| = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0\}$$

$$|\Lambda^n_k| = \{(t_0, ..., t_n) \in |\Delta^n| \mid t_k = 0\} \subset |\Delta^n|.$$

In particular we can define a strong deformation retract $H : |\Delta^n| \times [0,1] \to |\Delta^n|$ of $|\Delta^n|$ into $|\Lambda^n_k|$ by

$$H\big((t_0, ..., t_n), s\big) = (t_0 + s\frac{t_k}{n}, ..., (1-s) \cdot t_k, ..., t_n + s\frac{t_k}{n}).$$

Thus if T is a topologcial space and $f : |\Lambda^n_k| \to T$ a continuous map, then one can define $g : |\Delta^n| \to T$ using the deformation retract. Adjointness then gives a lift $g' : \Delta^n \to T$ of $f' : \Lambda^n_k \to S_\bullet(T)$ since $\mathbf{Top}(|\Lambda^n_k|, T) \cong \mathbf{sSet}(\Lambda^n_k, S_\bullet(T))$.



Thus $S_\bullet(T)$ is a Kan complex.

Remark 1.22. Simplicial theory was first developed as a tool to study the topological homotopy groups from a combinatorial perspective. The adjointness of $S_\bullet$ and $|\cdot|$ is one of the fundamental results showing that these structures are similar. In particular, if $X \in \mathbf{sSet}$ is a Kan complex, then

$$\pi_n(|X|, x) = \pi_n(X, x).$$

Similarly for a topological space T we have that

$$\pi_n(T, t) \cong \pi_n(S_\bullet(T), t).$$

Thus if one want to study topological spaces up to homotopy equivalence, then it is sufficient to study homotopy in **sSet**.

Chapter 2

Lie Theory and Chain complexes

The aim of this report is to describe the functor $\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$ constructed in [1]. In the first chapter we introduced the target category **sSet** of $\langle\cdot\rangle$. In this chapter we instead focus on the domain of this functor, namely differentially graded Lie algebras, ord DGLs in short. DGLs are often used in deformation theory and rational homotopy theory, but we do not study any such connections in this report. In short a DGL is a Lie algebra with the additional structure of a chain complex. Recall that a Lie algebra L is a vector space together with a bilinear product

$$[-,-] : L \times L \to L$$

called a *Lie bracket*. The chain complex structure is given on L by a decomposition $L = \bigoplus_{p\in\mathbb{Z}} L_p$ together with a differential $\partial_p : L_p \to L_{p-1}$. That is a linear map so that $\partial^2 = 0$. The decomposition gives a grading of the elements of L which the Lie bracket preserve in the sense that if $x \in L_p$ and $y \in L_q$, then $[x, y] \in L_{p+q}$. The chain complex structure also imply that we can study L by means of homology

$$H_n(L, \partial) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

We will construct $\langle\cdot\rangle$ as in example 1.10 by finding a cosimplicial DGL $\mathcal{L}^\bullet$ and the purpose of this chapter is to define the necessary tools to achieve this.

In this chapter we first present the axioms of a DGL. Further we define related concepts needed in the construction of $\mathcal{L}^\bullet$, such as completeness and the free Lie algebra $\mathbb{L}(V)$ generated by V. Lastly we include examples and results of DGLs linked to $\mathcal{L}^\bullet$.

Basic definitions

Definition 2.1. A *differential graded Lie algebra* L consists of a triple $(L, [-,-], \partial)$ where L is a vector space over $\mathbb{Q}$, $[-,-] : L \times L \to L$ is a bilinear map and $\partial : L \to L$ is a linear map satisfying the following properties

- L is a *graded vector space*. That is
  - $L = \bigoplus_{p\in\mathbb{Z}} L_p$ where L_p are vector spaces. If $x \in L_p$ for some p, then say that x is *homogeneous of degree* p. We denote this by $|x| = p$.

- The map $[-,-]$ is a *Lie bracket*. That is if x, y, z are homogeneous elements, then
  - $[x, y] = -(-1)^{|x||y|}[y, x]$
    (Graded antisymmmtetry)
  - $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$
    (Graded Jacobi identity)
  - $|[x, y]| = |x| + |y|$
- The linear map ∂ is a *differential*. That is
  - $\partial^2 = 0$

  and for homogeneous elements x, y it satisfies
  - $|\partial x| = |x| - 1$
  - $\partial[x, y] = [\partial x, y] + (-1)^{|x|}[x, \partial y]$. (Graded Leibniz rule)

We will usually refer to a differentially graded Lie algebra by DGL-algebra. Further a DGL without differential is a *Graded Lie algebra* and a *Lie algebra* when there also is no grading.
A DGL subalgebra $I \subset L$ is a *Lie ideal* if $[L, I] \subset I$. The grading of L together with the differential ∂ defines a natural *chain complex* on the homogeneous components of L.

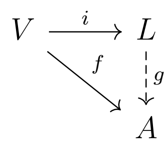
$$\cdots \longrightarrow L_{n+1} \xrightarrow{\partial^{n+1}} L_n \xrightarrow{\partial^n} L_{n-1} \longrightarrow \ldots$$

The *n:th homology group* $H_n(L, \partial)$ is the quotient $\ker \partial^n / \operatorname{im} \partial^{n+1}$. We say that L is *concentrated in positive degrees* or *positively graded* if $L = \bigoplus_{p \geq 0} L_p$ An element $a \in L_{-1}$ is called a *Maurer-Cartan* element if

$$\partial a = -\frac{1}{2}[a, a].$$

Denote the set of Maurer-Cartan elements by $MC(L)$. If $\alpha \in L$, then $\mathrm{ad}_\alpha : L \to L$ is called the *adjoint map* defined by $\mathrm{ad}_\alpha(x) = [\alpha, x]$. If $a \in MC(L)$ and ∂ a differential on L, then we set $\partial_a(x) = \mathrm{ad}_a(x) + \partial(x)$. In particular $\partial_a(x)$ is a differential on L so that $|\partial_a(x)| = |x| - 1$.

Definition 2.2. Let V be a graded vector space, L a graded Lie algebra and $i : V \to L$ a morphism of graded vector spaces. If every morphism of graded vector spaces $f : V \to A$ factors uniquely through i for every graded Lie algebra A, then L is *free on* V. In other words, for every morphism of graded vector spaces $f : V \to A$, there is a unique Lie algebra morphism $g : L \to A$ so that the diagram commutes.



A DGL-algebra is free if it free as a graded Lie algebra, and we denote such an algebra by $\mathbb{L}(V)$. We will also say that $\mathbb{L}(V)$ is the *free Lie algebra generated by* V. We may also extend the definition to free Lie algebras generated by a collection of elements $\{a_i\}_{i\in I}$ of given degrees. We denote this by $\mathbb{L}(\{a_i\}_{i\in I})$, and interpret it as the free Lie algebra generated by the graded vector space V spanned by $\{a_i\}_{i\in I}$. Similarly as other free structures, any morphism of DLGs $f : \mathbb{L}(V) \to L$ is completely determined where it maps the generators.

For every graded vector space V, there is a free Lie algebra $\mathbb{L}(V)$ generated by V. This algebra is unique up to isomorphism.

Example 2.3. Construction of $\mathbb{L}(V)$: Let V be a graded vector space over $\mathbb{Q}$. Define the tensor algebra $T(V)$ as

$$T(V) = \bigoplus_{i\geq 1} V^{i\otimes} = V \oplus (V \otimes V) \oplus ...$$

This is a graded associative algebra, with multiplication defined as $x \cdot y = x \otimes y$, and with degrees for pure tensors given by $|x \otimes y| = |x| + |y|$. From this we can define a graded Lie algebra $T(V)^{Lie}$ on the same underlying set by letting the bracket be defined as

$$[x, y] = x \otimes y - (-1)^{|x||y|} y \otimes x.$$

A routine check shows that this bracket preserves grading, satisfies graded antisymmetry and the graded Jacobi identity. Next define the sequence of graded vector spaces $\Gamma^n V$ inductively, where $\Gamma^1 V = V$ and $\Gamma^{n+1} V = [V, \Gamma^n V]$ for $n \geq 1$. The $\Gamma^n V$s are disjoint except at 0 and satisfy $[\Gamma^n V, \Gamma^m V] \subset \Gamma^{m+n} V$. Further $\bigoplus_{i=0}^{\infty} \Gamma^i V \subset T(V)^{Lie}$ is a Lie subalgebra, and it is free on V. Thus we may set $\mathbb{L}(V) = \bigoplus_{i=0}^{\infty} \Gamma^i V$.

Remark 2.4. Note that there are two gradings of a free graded Lie algebra. One that is given by the degree of the elements as described in definition 2.1, and one that is given by the composition $\bigoplus_{i=0}^{\infty} \Gamma_i V$ described in example 2.3 The latter corresponds to how many brackets each term is composed of.

Definition 2.5. Let $\mathbb{L}(V)$ be the free graded Lie algebra generated by V and consider the sequence $\Gamma^n V$ from example 2.3. If $x \in \Gamma^n V$ for some n, then say that x is *homogeneous of length n*. We denote this by $|x|_l = n$. We will only say that x is homogeneous if it is clear from context that we refer to degree or length. Let $\varphi : \mathbb{L}(V) \to \mathbb{L}(W)$ be a graded Lie algebra morphism. We write

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + ...$$

where φ_i denotes the DGL-morphism that satisfy

$$|\varphi_i(x)|_l = |x|_l + i - 1$$

for x homogeneous by length. That is, φ_i is the component of φ that increases the length of elements by $i - 1$. Say that φ is of length i.

Example 2.6. Let V be a graded vector space with basis $\{x, y\}$ each of degree 0. Then $\mathbb{L}(V) = \Gamma^1 V \oplus \Gamma^2 V \oplus \Gamma^3 \oplus \cdots$ where the first components are spanned by the elements below.

$$\begin{array}{ll} \Gamma^1 V : & x, y \\ \Gamma^2 V : & [x, y] \\ \Gamma^3 V : & [x, [x, y], \ [y, [x, y]] \end{array}$$

Example 2.7. If (L, ∂) is a free DGL, then we may decompose the differential ∂ by length as

$$\partial = \partial_1 + \partial_2 + \partial_3 + \cdots$$

where $|\partial_i(x)|_l = |x|_l + i - 1$. Note in particular that this decomposition gives that

$$\partial^2 = (\partial_1\partial_1) + (\partial_1\partial_2 + \partial_2\partial_1) + (\partial_1\partial_3 + \partial_2\partial_2 + \partial_3\partial_1) + ...$$

where each component $\sum_{i=1}^{k} \partial_i \partial_{k+1-i}$ is of length k. Hence $\partial^2 = 0$ implies that each component $\sum_{i=1}^{k} \partial_i \partial_{k+1-i} = 0$. In particular $\partial_1^2 = 0$, and so $\partial_1 : L \to L$ is a differential on L. We call ∂_1 the *linear part* of ∂.

Definition 2.8. Let L be a Lie algebra and let $\{\Gamma^n L\}_{n \geq 1}$ be *the lower central series* of L. That is a sequence of Lie ideals defined by

$$\Gamma^1 L = L, \qquad \Gamma^n L = [L, \Gamma^{n-1} L] \text{ for } n \geq 2.$$

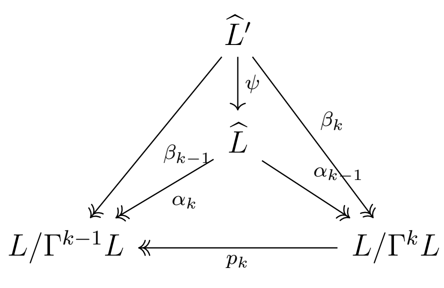
The quotients $L/\Gamma^n L$ are Lie algebras, and the projections on the form $p_n : L/\Gamma^n L \to L/\Gamma^{n-1} L$ are Lie morphism since $\Gamma^{n+1} L \subset \Gamma^n L$. Thus we gather the tower of Lie algebras

$$L/\Gamma^1 L \xleftarrow{p_2} L/\Gamma^2 L \xleftarrow{p_3} L/\Gamma^3 L \longleftarrow \cdots$$

Note in particular that $L/\Gamma^1 L = 0$. The *completion of* L is a Lie algebra $\widehat{L}$ together with morphisms $\alpha_i : \widehat{L} \to L/\Gamma^i L$ so that

i) $\alpha_{k-1} = p_k \circ \alpha_k$ for every $p_k : L/\Gamma^k L \to L/\Gamma^{k-1} L$.

ii) For any other such $\widehat{L}'$ with maps $\beta_i : \widehat{L}' \to L/\Gamma^i L$ satisfying i), there is a map $\psi : \widehat{L}' \to \widehat{L}$ so that $\beta_k = \alpha_k \circ \psi$.



Every Lie algebra have a completion and we say that L is complete if it isomorphic to its completion. If V is a graded vector space and $\mathbb{L}(V)$ the free Lie algebra generated by V, then we denote its completion by $\widehat{\mathbb{L}}(V)$.

Remark 2.9. If L is a complete free Lie algebra, then any $\alpha \in L$ can be described as a possibly infinite sum

$$\alpha = \sum_{i=0}^{\infty} \alpha_i$$

where $|\alpha_i|_l = i$. In fact the completion make such sums well defined as long as $\alpha_i \in \Gamma^i L$. This convergence can also be verified from a topological viewpoint. The lower central series defines a neighborhood basis of the identity element, which by addition can be extended to a neighborhood basis of every point, and in particular define a topology. Any such series will converge in this topology.

The first components of $\mathcal{L}_\bullet$

With the newly defined concepts in mind we are able to construct the the first DGLs of the cosimplicial DGL $\mathcal{L}_\bullet$. The first example corresponds to the 0-simplex, and the LS-interval corresponds to the 1-simplex. Lastly we include how two LS-intervals may be glued together to a third interval using the BCH product. This will set the framework on which $\mathcal{L}_\bullet$ will be constructed.

Example 2.10. Define $(\mathcal{L}_0, \partial)$ to be the complete free DGL $(\widehat{\mathbb{L}}(a), \partial)$ where a is a Maurer-Cartan element. Note that the differential is uniquely defined by this since $\partial a = -\frac{1}{2}[a, a]$. We see that $\mathcal{L}_0$ is spanned by a in degree -1 and $[a, a]$ in degree -2. This is so since $[a, [a, a]] = 0$. Note that in this case $(\mathbb{L}(a_0), \partial) = (\widehat{\mathbb{L}}(a_0), \partial)$.

Definition 2.11. The *Lawrence-Sullivan model of the interval* is the complete free DGL-algebra $(\widehat{\mathbb{L}}(a, b, x), \partial)$ where a, b are Maurer-Cartan element and x is of degree zero. The differential in defined on x by

$$\partial x = \mathrm{ad}_x(b) + \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_x^i(b - a)$$

where B_i are the Bernoulli-numbers. For more details of this construction see [6].

Remark 2.12. Note that if $(\widehat{\mathbb{L}}(a, b, x), \partial)$ is a LS-interval, then $(\widehat{\mathbb{L}}(b, a, -x), \partial)$ is an LS-interval as well. One do this by showing

$$\partial(-x) = [-x, a] + \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_{-x}^i(a - b).$$

Linearity of the differential gives that

$$\partial(-x) = -\partial x = -[x, b] - \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_x^i(b - a).$$

Now $B_i = 0$ for odd i except $i = 1$ since then $B_1 = -\frac{1}{2}$. Further $\mathrm{ad}_x^i(c) = \mathrm{ad}_{-x}^i(c)$ for even i. Thus

$$-[x,b] - \sum_{i=0}^{\infty} \frac{B_i}{i!}\mathrm{ad}_x^i(b-a) = -[x,b] + \frac{1}{2}[x,b-a] - \sum_{\substack{i=0\\ i\neq 1}}^{\infty} \frac{B_i}{i!}\mathrm{ad}_{-x}^i(b-a)$$

One then easily notes that

$$-[x,b] + \frac{1}{2}[x,b-a] = [-x,a] - \frac{1}{2}[-x,a-b]$$

and so the claim follows since

$$\begin{aligned}\partial(-x) &= -[x,b] + \frac{1}{2}[x,b-a] - \sum_{\substack{i=0\\ i\neq 1}}^{\infty} \frac{B_i}{i!}\mathrm{ad}_{-x}^i(b-a)\\ &= [-x,a] - \frac{1}{2}[-x,a-b] + \sum_{\substack{i=0\\ i\neq 1}}^{\infty} \frac{B_i}{i!}\mathrm{ad}_{-x}^i(a-b)\\ &= [-x,a] + \sum_{i=0}^{\infty} \frac{B_i}{i!}\mathrm{ad}_{-x}^i(a-b).\end{aligned}$$

Definition 2.13. Let L be a complete Lie algebra. Then we define the *Baker-Campbell-Hausdorff product* $*$ on L for $x, y \in L$ as the formal power series expansion

$$x * y = \log(e^x e^y).$$

We have the explicit formula given by

$$x * y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \cdots$$

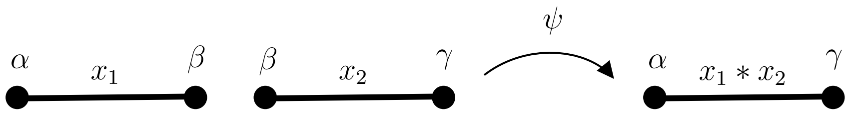
The product is associative, and $-x$ is an inverse for $x \in L$, i.e $x * (-x) = 0$. Note in particular that the BCH product is closed on the subspace L_0 of degree 0.

There is a natural way of adding two LS-intervals by means of the BCH-formula.

Proposition 2.14. *Define the LS-intervals L, L_1 and L_2 as*

$$\begin{aligned}L &= (\widehat{\mathbb{L}}(a,b,x),\partial)\\ L_1 &= (\widehat{\mathbb{L}}(\alpha,\beta,x_1),\partial)\\ L_2 &= (\widehat{\mathbb{L}}(\beta,\gamma,x_2),\partial).\end{aligned}$$

Set $L_3 = (\widehat{\mathbb{L}}(\alpha,\beta,\gamma,x_1,x_2),\partial)$ to be the free complete DGL with generators and relations from L_1 and L_2. Then the map $\psi : L \to L_3$ defined by $\psi(a) = \alpha$, $\psi(b) = \gamma$ and $\psi(x) = x_1 * x_2$ is a DGL-morphism.



Gluing two LS-intervals together with the BCH-formula.

In particular the image of ψ is an embedded LS-interval in L_3. Further it is a sub-DGL

$$\big(\widehat{\mathbb{L}}(\alpha, \gamma, (x_1 * x_2)), \partial\big) \subset \big(\widehat{\mathbb{L}}(\alpha, \beta, \gamma, x_1, x_2), \partial\big).$$

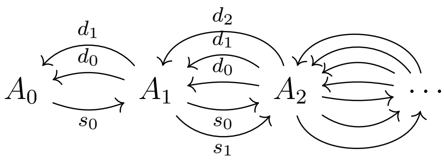
Proof. See Theorem 2 in [6]. □

Chapter 3

Interlude: The Dold-Kan correspondence

In this chapter we present a fundamental theorem of simplicial homotopy theory, called the Dold-Kan correspondence. Not only is it an important result, it does also serve as a special case of the realization functor $\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$ in section 5. Essentially the Dold-Kan is an equivalence between the category of simplicial abelian groups $\mathbf{sAb}$ and the category of positively graded chain complexes $\mathbf{Ch}_+$. Furthermore this equivalence preserve homology and homotopy in their respective categories. We present the functors of this equivalence, so that a meaningful comparison of $\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$ can be made in section 4 and 5.

Consider the category of simplicial abelian groups $\mathbf{sAb}$. That is, objects are sequences $A = \{A_n\}_{n\geq 0}$ of abelian groups together with face and degeneracy maps d_i and s_i which are groups homomorphisms.



Note in particular that each object of $\mathbf{sAb}$ is a Kan-complex by example 1.15. Let $\mathbf{Ch}_+$ be the category of positively graded chain complexes. The objects of $\mathbf{Ch}_+$ are sequences of $\mathbb{Z}$-modules $\{C_n\}_{n\geq 0}$ together with a differential $\partial_n : C_n \to C_{n-1}$,

$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \longleftarrow \ldots$$

The morphisms of $\mathbf{Ch}_+$ are chain maps. The structure of these categories have some similarities. Not only do their objects consist of sequences of abelian groups with homomorphisms between them, there are also associated groups to each of the objects respectively. Namely the homotopy group $\pi_n(A, a_0)$ to a simplicial abelian group, and the homology group $H_n(C, \partial)$ to a positively graded chain complex. This similarity is confirmed by the Dold-Kan correspondence.

Theorem 3.1. *(The Dold-Kan correspondence) There exists two functors*

$$N : s\mathbf{Ab} \to \mathbf{Ch}_+ \text{ and } K : \mathbf{Ch}_+ \to s\mathbf{Ab}.$$

which together form an equivalence. Further if A is a simplicial abelian group, then

$$\pi_n(A, 0) \cong H_n(NA, \partial).$$

Knowing what we are aiming for, we present two ways of imposing a chain complex structure on a simplicial abelian group. One of which is the normalization functor N in the Dold-Kan correspondence.

Definition 3.2. Let A be a simplicial abelian group. *The Moore complex* A^{Mo} is the chain complex with

$$A_n^{Mo} = A_n.$$

The differential $\partial : A_n^{Mo} \to A_{n-1}^{Mo}$ is defined using the face maps $d_i : A_n \to A_{n-1}$ by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

A simple verification using the simplicial identities shows that $\partial^2 = 0$.

The normalized chain complex NA is the chain complex with

$$NA_n = \bigcap_{i=0}^{n-1} \ker d_i.$$

The differential $\partial : NA_n \to NA_{n-1}$ is defined by

$$\partial = (-1)^n d_n.$$

Similarly one verifies that $\partial^2 = 0$ by the simplicial identities ($d_{n-1}d_n = d_{n-1}d_{n-1}$).

The construction of the normalized chain complex defines the functor $N : s\mathbf{Ab} \to \mathbf{Ch}_+$, and is the functor given in the Dold-Kan correspondence. Note however that the normalized chain complex is not very different from the Moore complex. For one there is a natural inclusion of chain complexes $i : NA \to A^{Mo}$. But it turns out that they have more in common.

Let $A \in \mathbf{sAb}$, and define $D_n \subset A_n$ to be the subgroup generated by the degenerate simplices of A_n. That is

$$D_n = \{a \in A_n \,|\, a = s_i b \text{ for some } b \in A_{n-1}\}$$

where we interpret A_{-1} as the trivial group. If ∂ is the differential on the Moore-complex, then $\partial(D_n) \subset D_n$ due to the simplicial identites. In particular the quotient A_n^{Mo}/D_n, i.e A^{Mo} modulo degeneracies, is a chain complex with the induced differential ∂ from A^{Mo}. Denote the latter chain complex by A^{Mo}/D.

Proposition 3.3. *Let A be a simplicial abelian group, $i : NA \to A^{Mo}$ the natural inclusion of chain complexes and $p : A^{Mo} \to A^{Mo}/D$ the projection of chain complexes. Then the composite $p \circ i : NA \to A^{Mo}/D$ is an isomorphism of chain complexes.*

Further the inclusion map i is a chain homotopy equivalence. That is

$$H_n(NA, \partial) \cong H_n(A^{Mo}, \partial).$$

Proof. See theorem 2.1 in [4]. □

We now turn our focus to the other functor of the Dold-Kan correspondence. Recall from example 1.10 that a cosimplicial object $C_\bullet$ in a category $\mathcal{C}$ defines a simplicial set $\mathbf{Hom}_{\mathcal{C}}(C_\bullet, X)$ for every object $X \in \mathcal{C}$. In other words $C_\bullet$ defines a functor $\mathbf{Hom}_{\mathcal{C}}(C_\bullet, -)) : \mathcal{C} \to s\mathbf{Set}$. We will use this idea to construct the functor $K_\bullet$, and so the first step is to find a cosimplicial chain complex $C_\bullet$.

Recall the standard n-simplex Δ^n from example 1.4. It is in itself a simplicial set, but it also a component of the cosimplicial set $\mathbf{\Delta}^\bullet$ from example 1.8. Let $\mathbb{F}_{\mathbb{Z}}(\Delta^n)$ be the free abelian group generated by Δ^n. The simplicial maps $\varphi : \Delta^n_i \to \Delta^n_j$ are extended to homomorphisms $\varphi_* : \mathbb{F}_{\mathbb{Z}}(\Delta^n_i) \to \mathbb{F}_{\mathbb{Z}}(\Delta^n_j)$ by the universal property of free abelian groups. Similar to the cosimplicial set $\mathbf{\Delta}^\bullet$ having Δ^n as components, we set $\mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet)$ to be the simplicial set whose components are the abelian groups $\mathbb{F}_{\mathbb{Z}}(\Delta^n)$. The cosimplicial maps $\varphi : \Delta^n \to \Delta^m$ naturally extends to group homomorphisms $\varphi^* : \mathbb{F}_{\mathbb{Z}}(\Delta^n) \to \mathbb{F}_{\mathbb{Z}}(\Delta^m)$. Now $\mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet)$ is a cosimplicial abelian group, but we want a cosimplicial chain complex. By applying the normalization functor N on each component $\mathbb{F}_{\mathbb{Z}}(\Delta^n)$ of $\mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet)$, we gather a sequence of chain complexes $NF_{\mathbb{Z}}(\Delta^n)$. Denote this sequence by $C_*(\mathbf{\Delta}^\bullet)$, that is $C_*(\mathbf{\Delta}^\bullet) = \{N\mathbb{F}_{\mathbb{Z}}(\Delta^n)\}_{n \geq 0}$. The cosimplicial maps of $\mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet)$ induce chain maps on $C_*(\mathbf{\Delta}^\bullet)$ making it into the desired cosimplicial chain complex.

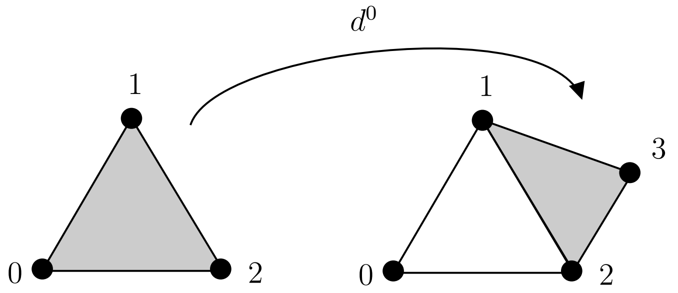
$$\begin{array}{ccc}
\mathbf{\Delta}^\bullet & \mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet) & C_*(\mathbf{\Delta}^\bullet) \\
\\
\Delta^{n+1} & \mathbb{F}_{\mathbb{Z}}(\Delta^{n+1}) & N\mathbb{F}_{\mathbb{Z}}(\Delta^{n+1}) \\
d^i \uparrow \; \downarrow s^i & d^i \uparrow \; \downarrow s^i & d^i \uparrow \; \downarrow s^i \\
\Delta^n & \mathbb{F}_{\mathbb{Z}}(\Delta^n) & N\mathbb{F}_{\mathbb{Z}}(\Delta^n)
\end{array}$$

Diagram of the cosimplicial structures of $\mathbf{\Delta}^\bullet$, $\mathbb{F}_{\mathbb{Z}}(\mathbf{\Delta}^\bullet)$ and $C_*(\mathbf{\Delta}^\bullet)$.

Example 3.4. (The cosimplicial chain complex $C_*(\Delta^\bullet)$) Due to proposition 3.3 the chain complexes $N\mathbb{F}_{\mathbb{Z}}(\Delta^n)$ are isomorphic to the Moore-complex $\mathbb{F}_{\mathbb{Z}}(\Delta^n)^{Mo}$ modulo degeneracies. In particular $\mathbb{F}_{\mathbb{Z}}(\Delta^n)^{Mo}$ modulo degeneracies is isomorphic to the free abelian group generated by the non-degenerate simplices of Δ^n due to the groups being free. So $N\mathbb{F}_{\mathbb{Z}}(\Delta^2)$ would correspond to the chain complex

$$0 \xleftarrow{d_0} \mathbb{F}_{\mathbb{Z}}(0,1,2) \xleftarrow{d_0 - d_1} \mathbb{F}_{\mathbb{Z}}(01,02,12) \xleftarrow{d_0 - d_1 + d_2} \mathbb{F}_{\mathbb{Z}}(012) \longleftarrow 0.$$

The coface maps d^i of $C_*(\Delta^\bullet)$ act just as in $\boldsymbol{\Delta}^\bullet$, meaning that they represent the different inclusions of the n-simplex into the $n+1$-simplex. For example we may include the 2-simplex 012 of Δ^2 into the 2-simplex 123 of Δ^3 by $d^0(012) = 123$. Note that the faces of 012 map to the correct faces of $d^0(012)$ since $d^0(01) = 12$, $d^0(12) = 23$ and $d^0(02) = 13$.



Inclusion of 2-simplex into 3-simplex by 0:th face map

The codegeneracy maps s^i act similarly, but map any element to zero if its image is degenerate.

We use the cosimplicial chain complex $C_*(\boldsymbol{\Delta}^\bullet)$ to define the functor K.

Definition 3.5. The *Dold-Kan functor* $K_\bullet : \mathbf{Ch}_+ \to \mathbf{sAb}$ is defined by mapping $D \in \mathbf{Ch}_+$ to the simplicial abelian group $K_\bullet(D)$ whose set of n-simplices is

$$K_n(D) = \mathbf{Ch}_+(C_*(\Delta^n), D).$$

A straightforward verification shows that $K_n(D)$ is in fact an abelian group, and that the face and degeneracy maps defined as in example 1.10 are group homomorphisms.

The functors N and K do not only form an equivalence, they preserve the homotopy group and homology group of **sSet** and $\mathbf{Ch}_+$ respectively. The normalization functor N preserve homology.

Theorem 3.6. *If A is a simplicial abelian group, then*

$$\pi_n(A, 0) \cong H_n(NA, \partial).$$

Similarly if L is a positively graded chain complex, then

$$\pi_n(K(L), 0) \cong H_n(L, \partial).$$

Proof. See corollary 2.7 in [4]. □

Chapter 4

The cosimplicial DGL $\mathcal{L}_\bullet$

In this section we follow the outline of [1] and construct the cosimplicial DGL $\mathcal{L}_\bullet$. The idea starts out simple. We first define a sequence of DGLs $(\mathcal{L}_n, \partial)$ which are free and complete and whose generators correspond to the non-degenerate simplices of the standard n-simplex Δ^n. The grading of 0-simplices are -1, and increase for higher-dimensional simplices. So the 1-simplices have degree 0, the 2-simplices have degree 1 and so on. We have already seen the definition of $\mathcal{L}_0$ and $\mathcal{L}_1$ in section 2, and will define $\mathcal{L}_n$ inductively using them as the base case. At the same stage we define the cosimplicial maps $d^i : \mathcal{L}_n \to \mathcal{L}_{n+1}$ to be the inclusion of the i:th n-face into the $n+1$-simplex similar to the cosimplicial topological space in example 1.9. Further we impose an additional property of $\mathcal{L}_\bullet$, namely that it is inductive. Essentially it means that the image of an n-simplex under the differential is expressible by its faces. More precisely it maps the n-simplex to an element generated by simplices of smaller dimension than n for $n \geq 2$. This property will prove useful in section 5 where we show that the simplicial set $\langle L \rangle$ is a Kan complex. Further we show that the twisted homology of $\mathcal{L}_n$ with a Maurer-Cartan element is zero, that is $H(\mathcal{L}_n, \partial_{a_0}) = 0$ and finish of with proving the existence of the cosimplicial DGL $\mathcal{L}_\bullet$. Our presentation of $\mathcal{L}_\bullet$ differs from [1] in three main ways. Firstly we include a more detailed formulation of proposition 4.11, explicating both the formulation and the proof. Secondly the proof of corollary 4.12 was only hinted at in [1], but is here carried out in full detail. Lastly proposition 4.10 better clarifies the use of proposition 2.14 but is otherwise similar. The remaining results are carried out in a similar fashion for the sake of completion, except for the construction of the degeneracy maps which have been left out. Their existence will only be of theoretical use in this report.

Definition 4.1. Define $D^{-1}\Delta$ to be the standard n-simplex without degeneracies. Set $D^{-1}\Delta^\bullet$ to be the sequence $\{D^{-1}\Delta^n\}_{n\geq 0}$. Now $D^{-1}\Delta^\bullet$ is naturally included in the cosimplicial set $\Delta^\bullet$ from example 1.8, and the coface maps $d^k\Delta^n \to \Delta^{n+1}$ defined by

$$d^k(i_0...i_p) = (j_0...j_p) \text{ where } j_l = \begin{cases} i_l & \text{if } l < k \qquad 0 \leq k \leq n+1 \\ i_l + 1 & \text{if } l \geq k \qquad 0 \leq p \leq n. \end{cases}$$

are well defined when restricted to $D^{-1}\Delta^n$. Thus $D^{-1}\Delta^\bullet$ have the same cosimplicial structure as $\Delta^\bullet$ excluding degenerate elements. Similarly we set $D^{-1}\dot{\Delta}^n$ and $D^{-1}\Lambda^n_k$ to be the boundary and the k-horn excluding degenerate elements. We induce a grading on $D^{-1}\Delta^n$ by letting the p-vertices $D^{-1}\Delta^n_p$ being of degree $p-1$. Let $\widehat{\mathbb{L}}(\Delta^n)$ be the free and complete graded Lie algebra generated by the set $D^{-1}\Delta^n$. Similarly we let $\widehat{\mathbb{L}}(\dot{\Delta}^n)$ and $\widehat{\mathbb{L}}(\Lambda^n_k)$ be the free and complete graded Lie algebras generated by $D^{-1}\dot{\Delta}^n$ and $D^{-1}\Lambda^n_k$ respectively. The generator corresponding to $(i_0, ..., i_p)$ will be denoted $a_{i_0 \dots i_p}$.

Note that with this grading the 0-simplices $a_0, ..., a_n$ of $\widehat{\mathbb{L}}(\Delta^n)$ have degree -1. We define a differential d_Δ on $\widehat{\mathbb{L}}(\Delta^n)$ by

$$d_\Delta a_{i_0 \dots i_p} = \sum_{k=0}^{p} (-1)^k a_{i_0 \dots \hat{i}_k \dots i_p}. \tag{4.0.1}$$

Definition 4.2. A *sequence of compatible models of* $\mathbf{\Delta}$ is a sequence of DGLs $\{(\mathcal{L}_n, \partial)\}_{n \geq 0}$ so that

- $\mathcal{L}_n$ is the free complete graded Lie algebra generated by the set $D^{-1}\Delta^n$. That is $\mathcal{L}_n = \widehat{\mathbb{L}}(\Delta^n)$.
- The linear part of ∂ satisfies $\partial_1 = d_\Delta$ as in (4.0.1).
- The generators $a_0, ..., a_n$ of degree -1 in $\mathcal{L}_n$ are Maurer-Cartan elements. That is $\partial a_i = -\frac{1}{2}[a_i, a_i]$.
- The coface maps $d^i : \mathcal{L}_{n-1} \to \mathcal{L}_n$ are induced by the coface maps on $\Delta^\bullet$ and are DGL morphisms.

We refer to the elements of the sequence $\{\mathcal{L}_n, \partial\}_{n \geq 0}$ as models of $\mathbf{\Delta}$.

Remark 4.3. If $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ is a model of $\mathbf{\Delta}$, then the sub Lie algebras $\widehat{\mathbb{L}}(\dot{\Delta}^n)$ and $\widehat{\mathbb{L}}(\Lambda^n_k)$ are DGLs by restricting the differential ∂ from $\widehat{\mathbb{L}}(\Delta^n)$.

Remark 4.4. Note the similarity of the DGL $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ and the chain complex $N\mathbb{F}(\Delta^n)$ from the Dold-Kan chapter. As mentioned in example 3.4, $N\mathbb{F}(\Delta^n)$ is naturally isomorphic to the chain complex where the k:th component in the chain is the free abelian group generated by the non-degenerate k-simplices of Δ^n. Further d_Δ coincide exactly with the differential of $N\mathbb{F}(\Delta^n)$. Thus these constructions are equivalent if we restrict ourselves to elements of length one in $(\widehat{\mathbb{L}}(\Delta^n), \partial)$.

Definition 4.5. A sequence $\{(\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ of compatible models is *inductive* if

$$\partial_{a_0} a_{0 \dots n} \in \widehat{\mathbb{L}}(\dot{\Delta}^n).$$

for $n \geq 2$.

Example 4.6. The model of a point $(\mathcal{L}_0, \partial)$ is the complete DGL from example 2.10. That is

$$(\mathcal{L}_0, \partial) = (\widehat{\mathbb{L}}(a_0), \partial)$$

where a_0 is a Maurer-Cartan element. Note that the linear part of ∂ satisfies the second condition. That is $\partial_1 a_0 = d_\Delta a_0 = 0$.
A model of the interval $(\mathcal{L}_1, \partial)$ is given by the LS-interval from example 2.11 with $a = a_0$, $b = a_1$ and $x = a_{01}$. That is

$$(\mathcal{L}_1, \partial) = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial).$$

We directly verify that $\partial_1 = d_\Delta$.

Remark 4.7. Note that these models are vacuously inductive, since the condition stated in the definition only applies for models $\mathcal{L}_n$ with $n \geq 2$.

Example 4.8. The model of the triangle $(\mathcal{L}_2, \partial) = (\widehat{\mathbb{L}}(\Delta^2), \partial)$ is given by

$$\mathcal{L}_2 = \widehat{\mathbb{L}}(\Delta^2) = \widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}).$$

Let the coface maps $d^i : (\mathcal{L}_1, \partial) \to (\mathcal{L}_2, \partial)$ define how the differential acts on all generators except a_{012}. That is ∂ is defined on 0 and 1-simplices when they are seen as elements in the LS-interval structure as seen below.

$$\begin{aligned} d^0(\mathcal{L}_1) &= d^0(\widehat{\mathbb{L}}(a_0, a_1, a_{01})) = \widehat{\mathbb{L}}(a_1, a_2, a_{12}) \\ d^1(\mathcal{L}_1) &= d^1(\widehat{\mathbb{L}}(a_0, a_1, a_{01})) = \widehat{\mathbb{L}}(a_0, a_2, a_{02}) \\ d^2(\mathcal{L}_1) &= d^2(\widehat{\mathbb{L}}(a_0, a_1, a_{01})) = \widehat{\mathbb{L}}(a_0, a_1, a_{01}) \end{aligned}$$

Taking $\widehat{\mathbb{L}}(a_1, a_2, a_{12})$ as an example, we see that a_1 and a_2 are Maurer Cartan elements and that

$$\partial a_{12} = \mathrm{ad}_{a_{12}}(a_2) + \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_x^i (a_2 - a_1).$$

Using the BCH-formula, define the differential of a_{012} as

$$\partial_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1}.$$

Remark 4.9. The differential is implicitly defined since

$$\partial a_{012} = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}].$$

Further we directly verify that $\partial_1 = d_\Delta$ expanding the first terms of the BCH-formula.

Proposition 4.10. *The Lie algebra $(\mathcal{L}_2, \partial)$ as defined in example 4.8 is a DGL algebra. Furthermore, it is an inductive model of the triangle.*

Proof. We only need to verify that $\partial^2 a_{012} = 0$ in order to show that $(\mathcal{L}_2, \partial)$ is a DGL. First define the DGL morphism

$$\psi : (\widehat{\mathbb{L}}(a, b, x), \partial) \to (\widehat{\mathbb{L}}(\alpha, \beta, \gamma, x_1, x_2), \partial)$$

following the construction in proposition 2.14 with

$$\begin{aligned} L &= (\widehat{\mathbb{L}}(a, b, x), \partial) \text{ A general LS-interval} \\ L_1 &= (\widehat{\mathbb{L}}(a_0, a_2, (a_{01} * a_{12})), \partial) \\ L_2 &= (\widehat{\mathbb{L}}(a_2, a_0, -a_{02}), \partial). \end{aligned}$$

We note that L_1 and L_2 are LS-intervals by proposition 2.14 and remark 2.12 respectively. Hence ψ becomes

$$\psi : (\widehat{\mathbb{L}}(a, b, x), \partial) \to (\widehat{\mathbb{L}}(a_0, a_2, a_0, (a_{01} * a_{12}), -a_{02}), \partial)$$

where $\psi(a) = \psi(b) = a_0$ and $\psi(x) = a_{01} * a_{12} * a_{02}^{-1}$. Using that ψ is a DGL-morphism gives that

$$\begin{aligned} \partial(a_{01} * a_{12} * a_{02}^{-1}) &= \partial(\psi x) = \psi(\partial x) \\ &= \psi\Big(\mathrm{ad}_x(b) + \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_x^i (b - a)\Big) \\ &= \Big(\mathrm{ad}_{\psi(x)}(\psi(b)) + \sum_{i=0}^{\infty} \frac{B_i}{i!} \mathrm{ad}_{\psi(x)}^i (\psi(b) - \psi(a))\Big) \\ &= \mathrm{ad}_{\psi(x)}(\psi(b)) = \mathrm{ad}_{a_{01} * a_{12} * a_{02}^{-1}}(a_0). \end{aligned}$$

In particular $\mathrm{ad}_{a_{01} * a_{12} * a_{02}^{-1}}(a_0) = -[a_0, (a_{01} * a_{12} * a_{02}^{-1})]$ implying that $\partial_{a_0}^2 a_{012} = 0$. Equivalently $\partial^2 a_{012} = 0$. Lastly note that the model is inductive since $\partial_{a_0} a_{012} \in \widehat{\mathbb{L}}(\dot{\Delta}^2)$ by direct verification of the definition. □

Next up we will compute the homology group of each model $(\widehat{\mathbb{L}}(\Delta^n), \partial)$, but we first need a proposition. Let V and W be graded vector spaces of finite dimension and $(\widehat{\mathbb{L}}(V \oplus W), \partial)$ a free complete DGL generated by $V \oplus W$. Let $J \subset (\widehat{\mathbb{L}}(V \oplus W), \partial)$ be the ideal generated by W. That is

$$J = \{x \in \widehat{\mathbb{L}}(V \oplus W) \,|\, x \in W \text{ or } x \in [W, \widehat{\mathbb{L}}(V \oplus W)]\}.$$

The quotient $\widehat{\mathbb{L}}(V \oplus W)/J$ is a complete graded Lie algebra which is isomorphic to $\widehat{\mathbb{L}}(V)$. Note that J is only an ideal of the underlying graded Lie algebra, and need thus not be closed under the differential. However if J is closed under the differential, i.e $\partial(J) \subset J$, then ∂ induces a differential on $\widehat{\mathbb{L}}(V)$. Further the projection map

$$p : (\widehat{\mathbb{L}}(V \oplus W), \partial) \to (\widehat{\mathbb{L}}(V), \partial)$$

becomes a DGL morphism. Trivially (J, ∂) also becomes a DGL when $\partial(J) \subset J$. In particular the graded vector space W becomes a chain complex with the linear part ∂_1 of ∂ acting as the differential since $\partial_1 W \subset W$. Lastly it is seen that the linear subspace W of J becomes a chain complex (W, ∂_1). To summarize, if $\partial(J) \subset J$, then

- (W, ∂_1) is a chain complex
- $(J, \partial) \subset (\widehat{\mathbb{L}}(V \oplus W), \partial)$ is a free and complete DGL.
- $(\widehat{\mathbb{L}}(V), \partial) := (\widehat{\mathbb{L}}(V \oplus W)/J, \partial)$ is a free and complete DGL.

Proposition 4.11. *If $\partial(J) \subset J$ and $H_k(W, \partial_1) = 0$ for all k, then*

- *$H_k(J, \partial) = 0$ for all k*
- *The projection $p : (\widehat{\mathbb{L}}(V \oplus W), \partial) \to (\widehat{\mathbb{L}}(V), \partial)$ is a quasi-isomorphism. That is, $H_k(\widehat{\mathbb{L}}(V \oplus W), \partial) \cong H_k(\widehat{\mathbb{L}}(V), \partial)$ for all k.*

Proof. We proceed in two steps. Consider the projection

$$p : (\mathbb{L}(V \oplus W), \partial_1) \to (\mathbb{L}(V), \partial_1)$$

of the free but *not* complete DGLs and let K be its kernel. That is $K = J \cap (\mathbb{L}(V \oplus W)$. We have that $H(K, \partial_1) = 0$ according to [1].
Next following [1] we show that $H(J, \partial) = 0$. Let $x \in J$ be a cycle. Decomposing x by length, we have that $x = x_n + y_{n+1}$ where x_n is homogeneous of length n and y_{n+1} is composed of homogeneous terms of length greater than n. Note that $x_n \in K$ and thus $\partial_1 x_n = 0$. Hence $H_k(K, \partial_1) = 0$ gives that there is some $z_n \in K$ such that $\partial_1 z_n = x_n$. Next consider $x - \partial z$, which now has length strictly larger than n. Again we find the term of lowest length in $x - \partial z$, note that it is in K and find its boundary. By continuing this process we gather a sequence $\{z_n\}_{n \geq 0}$ so that $z_n \in K$ and $x = \partial(\sum_{n \geq 0} z_n)$. Hence $H(J, \partial) = 0$ follows.
We finally show that p is a quasi-isomorphism. Note that we have the short exact sequence

$$0 \longrightarrow (J, \partial) \hookrightarrow (\widehat{\mathbb{L}}(V \oplus W), \partial) \xrightarrow{p} (\widehat{\mathbb{L}}(V), \partial) \longrightarrow 0$$

of chain maps. By the zig-zag lemma we get the induced long exact sequence of homology groups

$$\cdots \longrightarrow H_k(J, \partial) \longrightarrow H_k(\widehat{\mathbb{L}}(V \oplus W), \partial) \xrightarrow{p^*} H_k(\widehat{\mathbb{L}}(V), \partial) \longrightarrow H_{k-1}(J, \partial) \longrightarrow \cdots$$

But since $H(J, \partial) = 0$ exactness implies that $H_k(\widehat{\mathbb{L}}(V \oplus W), \partial) \cong H_k(\widehat{\mathbb{L}}(V), \partial)$. This isomorphism is induced by the projection p, which is what we wanted to show. ☐

Corollary 4.12. *If $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ is an inductive model, then*

- $H(\widehat{\mathbb{L}}(\Delta^n), \partial_{a_0}) = 0$
- $H(\widehat{\mathbb{L}}(\Lambda^n_i), \partial_{a_0}) = 0$ *for all* $i = 0, ..., n$.

Proof. Using the notation of proposition 4.11, set

$$V = \{a_0\} \text{ and } W = \{a_i - a_0 \,|1 \leq i \leq n\} \cup \{a_{i_0 \dots i_p} \in \Delta^n \,|\, 1 \leq p \leq n\}.$$

Then clearly $\widehat{\mathbb{L}}(V \oplus W) = \widehat{\mathbb{L}}(\Delta^n)$. The proof then boils down to show that V and W satisfies the hypotheses of proposition 4.11, namely

- $H(V, \partial_{a_0}) = 0$
- $\partial_{a_0}(J) \subset J$
- $H(W, \partial_1) = 0$.

The first condition is easily verified by inspecting example 2.10.
Next we show that $\partial_{a_0}(J) \subset J$, by first proving that $\partial_{a_0}(W) \subset J$. We decompose ∂_{a_0} by length by

$$\partial_{a_0} = \partial_1 + \partial_2 + \partial_3 + ...$$

Let $w \in W$, then $\partial_1 w \in J$ since $\partial_1 w \in W$. It remains to show that the non-linear parts of ∂ maps w to J, that is $\partial_{\geq 2} w \subset J$. This reduces to three cases. First if w is of degree -1, that is $w = a_i - a_0$, then $\partial_{a_0}(a_i - a_0) \in J$ since

$$\begin{aligned}
\partial_{a_0}(a_i - a_0) &= \partial(a_i - a_0) + [a_i - a_0, a_0] \\
&= -\frac{1}{2}[a_i, a_i] + \frac{1}{2}[a_0, a_0] + [a_i, a_0] - [a_i, a_i] \\
&= \frac{1}{2}([a_i, a_0] - [a_i, a_i] + [a_i, a_0] - [a_0, a_0]) \\
&= \frac{1}{2}([a_i, a_i - a_0] + [a_i - a_0, a_0]).
\end{aligned}$$

If $|w| = 0$, then $w = a_{i_0 i_1}$. In particular there is an explicit formula for $\partial_{a_0} w$ by definition 2.11. Inspection gives that $\partial_{a_0} a_{i_0 i_1} \in J$.
Lastly if $|w| \geq 1$ then without loss of generality we may assume that $w = a_{01 \dots k}$ with $k \geq 2$. Since $\widehat{\mathbb{L}}(\Delta^n)$ an inductive model, it follows that $\partial_{\geq 2} a_{01 \dots k} \in \widehat{\mathbb{L}}(\dot{\Delta}^k)$. Let $i \geq 2$ and suppose that a term of length i in $\partial_i a_{01 \dots k}$ contain the factors $b_1, ..., b_i \in \dot{\Delta}^n$. Then counting degrees we gather

$$k - 2 = |\partial^i_{a_0} a_{01 \dots k}| = |b_1| + ... + |b_i|, \text{where } -1 \leq |b_j| \leq k - 2.$$

It is not possible that all $|b_j| = -1$ since $k - 2 \geq 0$, and so at least one element is of degree 0 or higher. Hence the term will be in W, showing that $\partial_{a_0} W \in J$. Now proving $\partial_{a_0} x \in J$ for any $x \in J$ is done by using induction over the length of x. The inclusion $\partial_{a_0} W \subset J$ acts as the base case, while the induction step is easy to work out

and therefore left out. The result $\partial_{a_0} J \subset J$ then follows.
Lastly we show that $H(W, \partial_1) = 0$. Let $W_i \subset W$ denote the subspace of W of degree i and $|U|$ denote the dimension of a vector space U. Further let $\partial_1^k : W_k \to W_{k-1}$ denote ∂_1 from W_k to W_{k-1}. The following is a chain complex

$$0 \longleftarrow W_{-1} \xleftarrow{\partial_1^0} W_0 \xleftarrow{\partial_1^1} W_1 \longleftarrow \cdots \longleftarrow W_{n-2} \xleftarrow{\partial_1^{n-1}} W_{n-1} \longleftarrow 0$$

Note that $|W_{-1}| = n$ and $|W_k| = \binom{n+1}{k+2}$ for $k \geq 0$. The maps $\partial_1^k : W_k \to W_{k-1}$ are linear, so

$$|\operatorname{im} \partial_1^k| + |\ker \partial_1^k| = |W_k|. \tag{4.0.2}$$

Claim: $|\operatorname{im} \partial_1^k| = \binom{n}{k+1}$ for $k \geq 0$.
Given the claim, we can solve for $|\ker \partial_1^k|$ in (4.0.2). Thus

$$\begin{aligned} |\operatorname{im} \partial_1^k| + |\ker \partial_1^k| &= |W_k| \\ \binom{n}{k+1} + |\ker \partial_1^k| &= \binom{n+1}{k+2} \\ |\ker \partial_1^k| &= \binom{n+1}{k+2} - \binom{n}{k+1} = \binom{n}{k+2}. \end{aligned}$$

where the last step is only valid for $k < n-1$. However $k = n-1$ gives

$$|\ker \partial_1^{n-1}| = \binom{n+1}{n+1} - \binom{n}{n} = 0.$$

Now we see that $|\operatorname{im} \partial_1^k| = |\ker \partial_1^{k-1}|$ and so $H(W, \partial_1) = 0$ follows.
Proof of claim: Let $k \geq 0$ and consider all elements in W_k on the form $a_{0i_1 \dots i_{k+1}}$ for increasing sequences $0 < i_1 < i_2 < \dots < i_k$. There are $\binom{n}{k+1}$ such elements and the claim follows if the collection $\partial_1^k a_{0i_1 \dots i_{k+1}}$ is a basis of $\operatorname{im} \partial_1^k$. We note that these elements are linearly independent since the term $a_{i_1 \dots i_k}$ is unique in the sum $\partial_1^k a_{0i_1 \dots i_k}$. Now suppose that $a_{i_0 \dots i_{k+1}} \in W_k$ so that $i_0 \neq 0$. We will show that

$$\partial_1 a_{i_0 \dots i_{k+1}} = \sum_{p=0}^{k+1} (-1)^p \partial_1 a_{0 i_0 \dots \widehat{i_p} \dots i_{k+1}}. \tag{4.0.3}$$

Note that $\partial_1 a_{0 i_0 \dots \widehat{i_p} \dots i_{k+1}}$ are elements from the basis-set. Set $A_p = a_{0 i_0 \dots \hat{i}_p \dots i_{k+1}}$ for $0 \leq p \leq k+1$. Then

$$\partial_1 A_p = \sum_{q=0}^{k+1} (-1)^q B_{(p,q)}$$

where we set

$$B_{(p,q)} = \begin{cases} a_{i_0 \dots \widehat{i_p} \dots i_{k+1}} & \text{if } q = 0 \\ a_{0 i_0 \dots \widehat{i_{q-1}} \dots \widehat{i_p} \dots i_{k+1}} & \text{if } 0 < q \leq p \\ a_{0 i_0 \dots \widehat{i_p} \dots \widehat{i_q} \dots i_{k+1}} & \text{if } q > p \end{cases}.$$

Thus (4.0.3) is equivalent to

$$\partial_1 a_{i_0 \dots i_{k+1}} = \sum_{0 \leq p,q \leq k+1} (-1)^{p+q} B_{(p,q)}.$$

By a simple observation we note that

$$\partial_1 a_{i_0 \dots i_{k+1}} = \sum_{p=0}^{k+1} (-1)^p a_{i_0 \dots \widehat{i_p} \dots i_{k+1}} = \sum_{0 \leq p \leq k+1} (-1)^p B_{(p,0)}.$$

Hence it remains to show that

$$\sum_{\substack{0 \leq p \leq k+1 \\ 1 \leq q \leq k+1}} (-1)^{p+q} B_{(p,q)} = 0. \tag{4.0.4}$$

Note that $B_{(p,q)}$ satisfies the relation

$$B_{(p,q)} = \begin{cases} B_{(q,p+1)} & \text{if } q > p \\ B_{(q-1,p)} & \text{if } q \leq p \end{cases}.$$

Using this relation, one easily shows that (4.0.4) holds. Hence we have shown that the elements $\partial_1 a_{0 i_1 \dots i_{k+1}}$ is a basis of $\operatorname{im} \partial_1^{k-1}$, and so $|\operatorname{im} \partial_1^k| = \binom{n}{k+1}$ for $k \geq 0$.
The proof of $H(\Lambda_k^n, \partial_{a_0}) = 0$ follows using the same proof as above with some slight modifications. At first let V and W be as before, only removing $a_{0 \dots n}$ and $a_{0 \dots \hat{\imath} \dots n}$ from W. Then note that $\partial_1^{k-2} : W_{k-2} \to W_{k-3}$ is injective. The result then follows. □

Theorem 4.13. *There exists an inductive sequence of compatible models $\{\mathcal{L}_n, \partial\}_{n \geq 0}$ of $\mathbf{\Delta}$ as in definition 4.2.*

Proof. Define the first three models as in the examples 2.10, 2.11 and 4.8 respectively. That is

$$\begin{aligned} (\mathcal{L}_0, \partial) &= (\widehat{\mathbb{L}}(a_0), \partial) \\ (\mathcal{L}_1, \partial) &= (\widehat{\mathbb{L}}(a_{01}, a_0, a_1), \partial) \\ (\mathcal{L}_2, \partial) &= (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), \partial). \end{aligned}$$

We have already seen that these are inductive. Next suppose that there are inductive models $(\widehat{\mathbb{L}}(\Delta^k), \partial)$ for $2 \leq k < n$. Following [1] , define $(\mathcal{L}_n, \partial)$ by letting $\mathcal{L}_n = \widehat{\mathbb{L}}(\Delta^n)$

and where the differential is defined on simplices $a_{i_0\dots i_p}$ for $p < n$ using the coface maps $d^i : \widehat{\mathbb{L}}(\Delta^{n-1}) \to \widehat{\mathbb{L}}(\Delta^n)$. Thus it remains to define $\partial a_{0\dots n}$. If we consider the element $a_{01\dots n-1}$, then $\partial_{a_0} a_{01\dots n-1} \in \widehat{\mathbb{L}}(\Lambda^n_n)$ by inductivity of the model. Since $\partial^2_{a_0} a_{01\dots n-1} = 0$ and $H(\mathbb{L}(\Lambda^n_n), \partial_{a_0}) = 0$, there is some $\Gamma \in \mathbb{L}(\Lambda^n_n)$ so that $|\Gamma| = n-2$ and $\partial_{a_0} a_{01\dots n-1} = \partial_{a_0}\Gamma$. We set

$$\partial_{a_0} a_{01\dots n} = (-1)^n (a_{01\dots n-1} - \Gamma).$$

Clearly $\partial^2_{a_0} a_{01\dots n} = 0$ and so ∂ is a differential. Furthermore the model is clearly inductive and satisfies condition i) and iii) from Definition 4.2. It remains to show that condition ii) is satisfied, that is

$$\partial_1 a_{0\dots n} = \sum_{i=0}^{n} (-1)^i a_{0\dots \hat{i} \dots n}. \tag{4.0.5}$$

Let Γ_1 and $\partial_1 a_{0\dots n}$ be the linear part of Γ and $\partial_{a_0} a_{0\dots n}$ respectively. Hence

$$\partial_1 a_{0\dots n} = (-1)^n (a_{0\dots n-1} - \Gamma_1).$$

Now $\partial_1^2 a_{0\dots n} = 0$ and so

$$\partial_1 \Gamma_1 = \partial_1 a_{0\dots n-1}.$$

If we set $\omega = (-1)^{n-1}\Gamma_1 - \sum_{i=0}^{n-1} (-1)^i a_{0\dots \hat{i} \dots n}$, then $|\omega| = n-2$ and $\omega \in \widehat{\mathbb{L}}(\Lambda^n_n)$. In particular

$$\begin{aligned} \partial_1 \omega &= (-1)^{n-1} \partial_1 \Gamma_1 - \sum_{i=0}^{n-1} \partial_1 a_{0\dots \hat{i} \dots n} \\ &= (-1)^{n-1} \partial_1 a_{0\dots n-1} + (-1)^n \partial_1 a_{0\dots n-1} = 0. \end{aligned}$$

By $H(\widehat{\mathbb{L}}(\Lambda^n_n)) = 0$, there is some $\gamma \in \widehat{\mathbb{L}}(\Lambda^n_n)$ so that $\omega = \partial_1 \gamma$ where $|\gamma| = n-1$. But there is no generator of degree $n-1$ in $\widehat{\mathbb{L}}(\Lambda^n_n)$, so $\gamma = 0$. And in particular $\omega = 0$ implies that

$$\Gamma_1 = \sum_{i=0}^{n-1} (-1)^i a_{0\dots \hat{i} \dots n}.$$

Hence equation 4.0.5 is satisfied. This completes the proof. □

Note that the we so far only have defined the coface maps $d^i : \mathcal{L}_n \to \mathcal{L}_{n+1}$ and so it remains to define the codegeneracy maps $s^i : \mathcal{L}_n \to \mathcal{L}_{n-1}$ for us to have the finalized cosimplicial DGL $\mathcal{L}_\bullet$. However we do here only satisfy to state that such maps can be defined on the inductive sequence $\{\mathcal{L}_n\}_{n\geq 0}$.

Theorem 4.14. *Any inductive sequence $\{\mathcal{L}_n\}_{n\geq 0}$ of $\boldsymbol{\Delta}$ admits a cosimplicial DGL $\mathcal{L}_\bullet$. The n-simplices of $\mathcal{L}_\bullet$ is $(\mathcal{L}_n, \partial)$ and the coface maps are induced from the model of $\boldsymbol{\Delta}$.*

Proof. See theorem 3.4 in [1]. □

Chapter 5

Homotopy Theory

In this section we present the adjoint functors $\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}$ and $\mathcal{L} : \mathbf{sSet} \to \mathbf{DGL}$ developed in [1], where **DGL** here denotes the category of complete DGLs. The construction of $\langle\cdot\rangle$ follows the methods of example 1.10 and the Dold-Kan functor in definition 3.5. Our main interest will be in the functor $\langle\cdot\rangle$. We show that $\langle L\rangle$ is a Kan complex when L is a complete DGL, and further that

$$\pi_n(\langle L\rangle, 0) \cong H_{n-1}(L, \partial),\ n \geq 0$$

for L concentrated in positive degrees. Lastly as an application we calculate $\pi_k(S^n) \otimes \mathbb{Q}$ using $\mathcal{L}$. The main contribution of this section is showing that $\langle L\rangle$ is a Kan complex, which was not shown in [1]. The construction of the isomorphism above follows the ideas of [1], but is carried out in greater detail.

The realization functor $\langle\cdot\rangle$

Definition 5.1. Let **DGL** be the category of complete DGLs. Let $L \in \mathbf{DGL}$ and define the simplicial set $\langle L\rangle = \mathbf{DGL}(\mathcal{L}_\bullet, L)$. We call $\langle L\rangle$ the *realization of* L. Note that $\langle L\rangle$ is a simplicial set according to example 1.10, with $\langle L\rangle_n = \mathbf{DGL}(\mathcal{L}_n, L)$. If $\varphi : [m] \to [n]$ is a simplicial map and

$$f \in \langle L\rangle_n, \quad f : \mathcal{L}_n \to L$$

then $\varphi^* f = f \circ \varphi_*$ where $\varphi_* : \mathcal{L}_m \to \mathcal{L}_n$. Thus we have a functor

$$\langle\cdot\rangle : \mathbf{DGL} \to \mathbf{sSet}.$$

Remark 5.2. The construction of $\langle\cdot\rangle$ is a generalization of the Dold-Kan functor $K_\bullet : \mathbf{Ch}_+ \to \mathbf{sAb}$. A DGL is abelian if the Lie bracket is trivial, i.e $[x, y] = 0$ for all x, y. So when L is an abelian DGL, then it is a chain complex. We also noted in remark 4.4 that $N\mathbb{F}(\Delta^n)$ is naturally included in $\widehat{\mathbb{L}}(\Delta^n)$. Now if $f \in K_n(L)$, i.e $f : N\mathbb{F}(\Delta^n) \to L$, then this map naturally extends to a DGL morphism $f : \widehat{\mathbb{L}}(\Delta^n) \to L$, and vice verca. In particular $\langle L\rangle$ becomes a simplicial abelain group, with the addition defined pointwise on the functions.

The homotopy of $\langle L\rangle$

With the newly defined simplicial set $\langle L\rangle$ for a complete DGL L, we want show that $\pi_n(\langle L\rangle, 0) \cong H_{n-1}(L, \partial)$. But for the homotopy group $\pi_n(\langle L\rangle, 0)$ to be well defined we first need to verify that $\langle L\rangle$ is a Kan complex. Now $\langle L\rangle$ is a Kan complex if every simplicial map $f : \Lambda^n_k \to \langle L\rangle$ can be lifted to a simplicial map $g : \Delta^n \to \langle L\rangle$. In other words the diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & \langle L\rangle \\ \big\downarrow & \overset{g}{\nearrow} & \\ \Delta^n & & \end{array}$$

commutes. Note that f defines a DGL morphism $f : \widehat{\mathbb{L}}(\Lambda^n_k) \to L$, and similarly g defines $g : \widehat{\mathbb{L}}(\Delta^n) \to L$ so that g restricted to $\widehat{\mathbb{L}}(\Lambda^n_k)$ equals f. Thus the Kan condition is equivalent of finding a DGL morphism g for each f so that the diagram below commutes.

$$\begin{array}{ccc} \widehat{\mathbb{L}}(\Lambda^n_k) & \xrightarrow{f} & \langle L\rangle \\ \big\downarrow & \overset{g}{\nearrow} & \\ \widehat{\mathbb{L}}(\Delta^n) & & \end{array}$$

We proceed finding g in a similar fashion as in example 1.21. There we showed that the singular simplicial set $S_\bullet(T)$ is a Kan complex by finding a continuous map $H : |\Delta^n| \to |\Lambda^n_k|$ which mapped the inlcuded k-horn $|\Lambda^n_k| \subset |\Delta^n|$to itself. And so any map $h : |\Lambda^n_k| \to T$ could be lifted to $|\Delta^n|$ by the composition with H, i.e $h \circ H : |\Delta^n| \to T$.

Proposition 5.3. *Let $n \geq 1$ and set $\iota : \widehat{\mathbb{L}}(\Lambda^n_k) \to \widehat{\mathbb{L}}(\Delta^n)$ to be the inclusion. Then there is a DGL-morphism*

$$\psi : (\widehat{\mathbb{L}}(\Delta^n), \partial) \to (\widehat{\mathbb{L}}(\Lambda^n_k), \partial)$$

so that $\psi \circ \iota = id$.

Proof. We only need to define where ψ maps $a_{0\ldots n}$ and $a_{0\ldots\widehat{k}\ldots n}$ since ψ is already determined on the generators of $\widehat{\mathbb{L}}(\Lambda^n_k)$. In particular ψ needs to satisfy

$$\partial\psi(a_{0\ldots n}) = \psi(\partial a_{0\ldots n}) \tag{5.0.1}$$

$$\partial\psi(a_{0\ldots\widehat{k}\ldots n}) = \psi(\partial a_{0\ldots\widehat{k}\ldots n}). \tag{5.0.2}$$

Without loss of generality we assume fixed values for k. We deal with this in cases. If $n = 1$, then

$$\psi : (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial) \to (\widehat{\mathbb{L}}(a_0), \partial),$$

where $(\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$ is the LS-interval. In particular we have a formula for ∂a_{01}. Defining ψ by $\psi(a_1) = a_0$ and $\psi(a_{01}) = 0$, then (5.0.1) is satisfied since $\psi(a_1)$ is a Maurer-Cartan element and (5.0.2) is satisfied since

$$\psi(\partial a_{01}) = \psi\Big(\mathrm{ad}_{a_{01}}(a_1) + \sum_{i=0}^{\infty} \frac{B_i}{i!}\mathrm{ad}^i_{a_{01}}(a_1 - a_0)\Big) = 0 = \partial\psi(a_{01}).$$

If $n=2$, then consider the 1-horn $\widehat{\mathbb{L}}(\Lambda^2_1)$. Hence we want to define a DGL morphism

$$\psi : (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), \partial) \to (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}), \partial).$$

By proposition 4.11 the map $\varphi : (\widehat{\mathbb{L}}(a_0, a_2, a_{02}), \partial) \to (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}), \partial)$ defined by $\varphi(a_0) = a_0$, $\varphi(a_2) = a_2$ and $\varphi(a_{02}) = a_{01} * a_{12}$ is a DGL-morphism. Hence the definition $\psi(a_{02}) = \varphi(a_{02})$ is compatible with 5.0.2. So far we have defined ψ on $\widehat{\mathbb{L}}(\dot{\Delta}^2)$, and it remains to define $\psi(a_{012})$. By inductivity we have that $\partial_{a_0} a_{012} \in \widehat{\mathbb{L}}(\dot{\Delta}^2)$ and so $\psi(\partial_{a_0} a_{012})$ is defined. Further $\psi(\partial_{a_0} a_{012})$ is a ∂_{a_0}-cycle in $\widehat{\mathbb{L}}(\Lambda^2_1)$. Now $H(\Lambda^2_1, \partial_{a_0}) = 0$ by corollary 4.12, and so $\psi(\partial_{a_0} a_{012}) = \partial_{a_0}\beta$ for some $\alpha \in \widehat{\mathbb{L}}(\Lambda^2_1)$. If we define $\psi(a_{012}) = \alpha$, then 5.0.1 is satisfied since

$$\partial_{a_0}\psi(a_{012}) = \partial_{a_0}\alpha = \psi(\partial_{a_0} a_{012}).$$

Hence ψ is a DGL morphism. For the case $n \geq 3$ we proceed in a similar way. We first define $\psi : \widehat{\mathbb{L}}(\dot{\Delta}^n) \to \widehat{\mathbb{L}}(\Lambda^n_k)$ by defining $\psi(a_{0\ldots\widehat{k}\ldots n})$. Now $\partial_{a_0} a_{0\ldots\widehat{k}\ldots n} \in \widehat{\mathbb{L}}(\dot{\Delta}^{n-1}) \subset \widehat{\mathbb{L}}(\Lambda^n_k)$ by inductivity and so $\psi(\partial_{a_0} a_{0\ldots\widehat{k}\ldots n})$ is well defined and a ∂_{a_0}-cycle. Again by corollary 4.12 $H(\widehat{\mathbb{L}}(\Lambda^n_k), \partial_{a_0}) = 0$, and so $\psi(\partial_{a_0} a_{0\ldots\widehat{k}\ldots n}) = \partial_{a_0}\beta$ for some $\beta \in \widehat{\mathbb{L}}(\Lambda^n_k)$. Defining $\psi(a_{0\ldots\widehat{k}\ldots n}) = \beta$ makes ψ a well defined DGL morphism on the boundary $\widehat{\mathbb{L}}(\dot{\Delta}^n)$. Finally we define ψ on $\widehat{\mathbb{L}}(\Delta^n)$. Again $\partial_{a_0} a_{0\ldots n} \in \widehat{\mathbb{L}}(\dot{\Delta}^n)$ by inductivity. Thus $\psi(\partial_{a_0} a_{0\ldots n})$ is defined and it is a ∂_{a_0}-cycle. Thus $\psi(\partial_{a_0} a_{0\ldots n}) = \partial_{a_0}\gamma$ for some $\gamma \in \widehat{\mathbb{L}}(\Lambda^n_k)$. Defining $\psi(a_{0\ldots n}) = \gamma$ completes the definition of ψ and makes it into a DGL morphism. □

Corollary 5.4. *Let L be a complete DGL. Then the simplicial set $\langle L \rangle_\bullet$ is a Kan complex.*

Proof. As mentioned above, $\langle L \rangle$ is a Kan complex if each DGL morphism $f : \widehat{\mathbb{L}}(\Lambda^n_k) \to L$ has a lift $g : \widehat{\mathbb{L}}(\Delta^n) \to L$. In the case $n = 0$, then $\widehat{\mathbb{L}}(\Lambda^0_0) = \emptyset$, and so g only needs to map to some Maurer-Cartan element of L. This is always possible, since the identity is a Maurer-Cartan element. If $n \geq 1$, then we use ψ from proposition 5.3 and define $g = f \circ \psi$. Now g is naturally a DGL morphism with the desired properties, completing the proof. □

Let (L, ∂) be a complete and non negatively graded DGL. Recall that the n:th homotopy group $\pi_n(\langle L \rangle, 0)$ is described by

$$\pi_n(\langle L \rangle, 0) = \{f : \mathcal{L}_n \to L \,|\, \delta f = (0, ..., 0)\}/\sim$$

where $f \sim g$ if there is some $h \in \langle L \rangle_{n+1}$ so that

$$\delta(h) = (f, g, 0, ..., 0).$$

Alternatively $\pi_n(\langle L \rangle, 0) = \cap^n_{i=0} \ker d_i / \sim$ where $\ker d_i = \{f : \mathcal{L}_n \to L \,|\, d_i f = 0\}$. We define a map $\varphi : \pi_n(\langle L \rangle, 0) \to H_{n-1}(L, \partial)$ by evaluating f at $a_{0\ldots n}$. That is $\varphi(f) = [f(a_{0\ldots n})]$.We show that φ is well defined and a homomorphism for $n \geq 2$.

Remark 5.5. We let $\pi_n\langle L \rangle$ denote the n:th homotopy group $\pi_n(\langle L \rangle, 0)$.

Remark 5.6. It also holds that $\pi_1\langle L\rangle \cong H_0(L)$ when $H_0(L)$ is considered with the group structure induced by the BCH-formula. We do not however cover this case here.

Theorem 5.7. *Let $n \geq 2$ and L a complete and non negatively graded DGL. The map $\varphi : \pi_n\langle L\rangle \to H_{n-1}(L, \partial)$ is well defined and a group homomorphism.*

Proof. Recall that $\mathcal{L}_n$ is inductive, that is

$$\partial_{a_0} a_{0\dots n} \in \widehat{\mathbb{L}}(\dot{\Delta}^n) \text{ for } n \geq 2. \tag{5.0.3}$$

Also note that if $f \in \pi_n\langle L\rangle$, then

$$f(\alpha) = 0 \text{ for any } \alpha \in \widehat{\mathbb{L}}(\dot{\Delta}^n). \tag{5.0.4}$$

At first we need to verify $\varphi(f) \in \ker \partial^{n-1}$. Since $n \geq 2$ we note that $\partial_{f(a_0)} f(a_{0\dots n}) = \partial f(a_{0\dots n})$ since f vanishes on a_0 and so $f[a_0, a_{0\dots n}] = 0$. Now f being a DGL morphism together with (5.0.3) and (5.0.4) gives

$$\partial f(a_{0\dots n}) = \partial_{f(a_0)} f(a_{0\dots n}) = f(\partial_{a_0} a_{0\dots n}) = 0.$$

Well defined: Suppose that $f \sim g$ and $h \in \langle L\rangle_{n+1}$ so that $\delta h = (f, g, 0, ..., 0)$. We want to show that $[f(a_{0..n})] = [g(a_{0..n})]$, that is $f(a_{0..n}) - g(a_{0..n}) \in \operatorname{im} \partial^n$. Now

$$\begin{aligned} f(a_{0\dots n}) &= d_0 h(a_{0\dots n}) = h(a_{1\dots n+1}) \\ g(a_{0\dots n}) &= d_1 h(a_{0\dots n}) = h(a_{02\dots n+1}) \end{aligned}$$

since $d_0 h = f$ and $d_1 h = g$. In particular we have that h is zero on all generators except $a_{0\dots n}$, $a_{1\dots n}$ and $a_{02\dots n}$. Now gather

$$\begin{aligned} \partial h(a_{0..n+1}) &= h(\partial_1 a_{0..n+1}) + h(\partial_{>1} a_{0..n+1}) \\ &= \sum_{i=0}^{n+1} (-1)^i d_i h(a_{0..n}) + h(\partial_{>1} a_{0..n+1}) \\ &= f(a_{0..n}) - g(a_{0..n}) + h(\partial_{>1} a_{0..n+1}). \end{aligned}$$

The element $\partial_{>1} a_{0..n+1}$ have degree $n-1$, and all terms are of length greater than one. So any term must have a factor which is not $a_{0\dots n}$, $a_{1\dots n}$ or $a_{02\dots n}$. But since h is zero on any such factor, we gather $h(\partial_{>1} a_{0..n+1}) = 0$. The result then follows since $\partial h(a_{0\dots n+1}) = f(a_{0\dots n}) - g(a_{0\dots n})$.

Homomorphism: Suppose $f, g \in \pi_n\langle L\rangle$ with fg represented by $d_1 h$ for some $h \in \langle L\rangle_{n+1}$. That is $\delta h = (g, fg, f, 0, ..., 0)$. We want to show that $[\varphi(f)] + [\varphi(g)] = [\varphi(fg)]$ in $H_{n-1}(L, \partial)$. Equivalently, $f(a_{0\dots n}) + g(a_{0\dots n}) - fg(a_{0\dots n}) \in \operatorname{im} \partial^n$. Now note that

$$\begin{aligned} g(a_{0\dots n}) &= d_0 h(a_{0\dots n}) = h \circ d^0(a_{0\dots n}) \\ &= h(d^0 a_{0\dots n}) = h(a_{12\dots n+1}) \end{aligned}$$

and similarly

$$fg(a_{0\dots n}) = h(d^1 a_{0\dots n}) = h(a_{02\dots n+1})$$
$$f(a_{0\dots n}) = h(d^2 a_{0\dots n}) = h(a_{013\dots n+1}).$$

Hence

$$h\partial(a_{0\dots n+1}) = h\partial_1(a_{0\dots n+1}) + h\partial_{>1}(a_{0\dots n+1}) = (f - fg + g) + h\partial_{>1}(a_{0\dots n}).$$

The inductiveness of $\mathcal{L}_{n+1}$ together with (5.0.4) gives $\partial_{>1}(a_{0\dots n+1}) = 0$, proving the claim. □

Theorem 5.8. *The map $\varphi : \pi_n\langle L\rangle \to H_{n-1}(L)$ is an isomorphism of groups.*

Proof. We have already shown that φ is well defined and a homomorphism. Thus it suffices to show that it is a bijection. First consider the case when $n = 1$. Let $[f] \in \pi_1\langle L\rangle$ with a representative $f : \mathcal{L}_1 \to L$ so that $\varphi[f] = [0]$. Equivalently $f(a_{01}) = x$ for some $x \in \operatorname{im}\partial_1$. Suppose $y \in L$ so that $x = \partial y$. Define $h : \mathcal{L}_2 \to L$ by $h(a_{012}) = y$ and $\delta h = (f, 0, ..., 0)$. If we verify that h is a DGL-morphism, then $f \sim 0$ by h. Thus we want to show $\partial h = h\partial$. Now $\mathcal{L}_2$ is inductive, i.e $\partial_{a_0} a_{012} \in \widehat{\mathbb{L}}(\dot{\Delta}^2)$. Further h is zero on all generators except a_{012} and a_{12}. Clearly

$$\partial_{h(a_0)} h(a_{012}) = [h(a_0), h(a_{012})] + \partial y = 0 + x = x.$$

Further
$$h(\partial_{a_0} a_{012}) = h(a_{12} - a_{02} + a_{01}) + h(\partial^{\geq 2}_{a_0} a_{012}) = x + h(\partial^{\geq 2} a_{012}).$$
Now let z be any term of $\partial^{\geq 2} a_{012}$. Either z contains a factor which is not a_{12}, or all factors are a_{12}. In the first case $h(z) = 0$ since $z \in \widehat{\mathbb{L}}(\dot{\Delta}^2)$ and h is zero on the factor which is not a_{12}. In the other case z contains the factor $[a_{12}, a_{12}]$, but this is equal to zero by the properties of the Lie bracket since $|a_{12}| = 0$. Hence $h(\partial^{\geq 2}_{g(a_0)}) = 0$. Similarly we can see directly from the definitions of $\partial a_{i_0 i_1}$ and ∂a_{i_0} that h commutes with the differential, and so it is a DGL morphism. Surjectivity follows from the fact that $f : \mathcal{L}_1 \to L$ defined as $f(a_{01}) = x$ and zero on all other generators is a DGL morphism. This follows easily from the explicit formula of ∂a_{01}.

Now suppose that $n \geq 2$.

Injectivity: Let $[f] \in \pi_n\langle L\rangle$ and $f : \mathcal{L}_n \to L$ some representative of $[f]$. Suppose that $\varphi([f]) = [f(a_{0\dots n})] = [0]$, that is $f(a_{0\dots n}) \in \operatorname{im}\partial_n$. To conclude injectivity we want to show that $f \sim 0$. Set $f(a_{0\dots n}) = x$. We have that $\partial_n y = x$ for some $y \in L_n$. Define a map $h : \mathcal{L}_{n+1} \to L$ by $h(a_{0\dots n+1}) = y$, $h(a_{1\dots n+1}) = x$ and zero elsewhere. We note that

$$\delta h = (f, 0, 0, ..., 0)$$

and so $f \sim 0$ follows if we show that h is a DGL morphism. Hence we need to verify that $\partial h = h\partial$. By definition we have that h vanishes on n-simplicies of length greater than one. Since $\mathcal{L}_{n+1}$ is inductive we have

$$h(\partial a_{0\dots n+1}) = h(a_{1\dots n+1}) + h(\partial_{>1} a_{0\dots n+1}) = x + 0 = \partial y = \partial h(a_{0\dots n+1}).$$

Similar arguments apply when showing $h(\partial a_{1\ldots n+1}) = \partial h(a_{1\ldots n+1})$, and so it follows that h is a DGL-morphism.

Surjectivity: Let $x \in \ker \partial_n$ and define $f : \mathcal{L}_n \to L$ by $f(a_{0\ldots n}) = x$ and zero elsewhere. This f is a DGL-morphism due to the inductivity of $\mathcal{L}_n$. Further f satisfies $f \in \pi_n\langle L \rangle$ and $\varphi(f) = [x]$, which finishes the proof. □

The adjoint functor $\mathcal{L}$ and examples

Using the cosimplicial DGL $\mathcal{L}_\bullet$, they define in [1] a functor

$$\mathcal{L} : \mathbf{sSet} \to \mathbf{DGL}$$

which is adjoint to the realization functor $\langle \cdot \rangle$. This functor is homotopy equivalent to the Quillen functor λ described in the introduction on finite simplicial complexes. We will not study the construction of $\mathcal{L}$ here, but use some of its properties to calculate some homotopy groups. Recall that a finite simplicial complex K is a topological space consisting of a finite collection of topological simplices such that

- Each face of $\sigma \in K$ is also in K.
- The intersection of two simplices $\sigma, \tau \in K$ is a face to both σ and τ.

These propositions are due to [1].

Proposition 5.9. *If K is a finite simplicial complex, then $K \subset \Delta^n$ for some n, and $\mathcal{L}(K)$ is trivially isomorphic to the complete sub DGL $(\widehat{\mathbb{L}}(V), \partial) \subset \mathcal{L}_n$.*

Proposition 5.10. *If K is a simply connected finite simplicial complex, then for every vertex $a \in K$ we have that*

$$H_n(\mathcal{L}(K), \partial_a) \cong \pi_{n+1}(K) \otimes \mathbb{Q}, \quad \text{for } n \geq 1.$$

Example 5.11. The n-dimensional disc $D^n \subset \mathbb{R}^n$ is homeomorphic to the topological n-simplex Δ^n. We have that $\mathcal{L}(|\Delta^n|) = (\widehat{\mathbb{L}}(\Delta^n), \partial)$ by proposition 5.9. Now $H_k(\widehat{\mathbb{L}}(\Delta^n), \partial_{a_0}) = 0$ due to corollary 4.12. Thus we gather

$$\pi_k(D^n) \otimes \mathbb{Q} \cong \pi_k(|\Delta^n|) \otimes \mathbb{Q} \cong H_{k-1}(\Delta^n) = 0$$

by proposition 5.10.

Example 5.12. Calculation of $\pi_k(S^n) \otimes \mathbb{Q}$ for $k \geq 2$. We have that $|\dot{\Delta}^n|$ is a finite simplicial subcomplex of $|\Delta^n|$, so $\mathcal{L}(|\dot{\Delta}^n|) \cong (\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial)$ by proposition 5.9. Further we have that $H(\dot{\Delta}^{n+1}, \partial_{a_0}) = \widehat{\mathbb{L}}(\partial_{a_0} a_{0\ldots n+1})$ from corollary 2.5 ii) [1]. For simplicity we may write $\widehat{\mathbb{L}}(\partial_{a_0} a_{0\ldots n+1}) = \widehat{\mathbb{L}}(u)$ for some generator u of degree $n-1$ and $\partial u = 0$. The components of the graded Lie algebra are only $V_{n-1} = \mathbb{Q}u$ and $V_{2n-2} = \mathbb{Q}[u, u]$ when n

is even, since otherwise $[u, u] = 0$ due to the identities of the graded Lie bracket. Thus it is an easy calculation gives that

$$H_k(\widehat{\mathbb{L}}(\dot{\Delta}^{n+1}), \partial_{a_0}) = \begin{cases} \mathbb{Q} & \text{for } k = n-1 \text{ and } k = 2n-2 \text{ when } n \text{ eben} \\ 0 & \text{else.} \end{cases}$$

Now it is well known that the n-dimensional sphere S^n is homeomorphic to the boundary of the $n+1$ topological simplex $|\Delta^{n+1}|$. By proposition 5.10 the k:th rational homology of $|\dot{\Delta}^{n+1}|$ for $k \geq 2$ is given by

$$\pi_k(|\dot{\Delta}^{n+1}|) \otimes \mathbb{Q} \cong H_{k-1}(\widehat{\mathbb{L}}(\dot{\Delta}^{n+1}), \partial_{a_0}) = \begin{cases} \mathbb{Q} & \text{for } k = n \text{ and } k = 2n-1 \text{ when } n \text{ even} \\ 0 & \text{else.} \end{cases}$$

Thus we gather $\pi_k(S^n) \otimes \mathbb{Q} \cong H_{k-1}(\widehat{\mathbb{L}}(\dot{\Delta}^{n+1}))$. In particular we gather that the non-torsion part of $\pi_k(S^n)$ is $\mathbb{Z}$ for $k = n$, $\mathbb{Z}$ when $k = 2n-1$ and n even, and zero otherwise. This is the expected outcome and is confirmed in the literature.

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