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On the Hodge, Tate and Mumford-Tate conjectures

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1 Introduction

To a smooth projective algebraic scheme X one can associate on the one hand its Betti and ℓ-adic cohomology, and on the other hand its algebraic cycles. These objects are connected through the cycle class maps, going from the algebraic cycles into the respective cohomologies. The cycle class maps land in a special class of elements in the cohomologies, called the Hodge (respectively Tate) classes. These are elements that are invariant under certain groups, groups which through Tannakian formalism determine the categories generated by the respective cohomologies. The Hodge (respectively Tate) conjecture states that all Hodge classes are algebraic, i.e. comes from the cycle class map, and the Mumford-Tate conjecture asserts that the comparison isomorphism between Betti and ℓ-adic cohomology induces an isomorphism between the group that control the Hodge classes on the one hand, and the group that control the Tate classes on the other.

The aim of this text is to give a rigorous explanation of what the previous paragraph means, and we hope that in doing so provide an introduction to some relevant concepts concerning these conjectures. The text will mainly be focused on introducing the objects in question, taking many of their properties as a black box.

1.1 Outline

In Section 2 we will introduce the notion of a neutral Tannakian category. This is not necessary in order to understand the statement of the three conjectures, but, as we will see, neutral Tannakian categories are present throughout the whole text (in particular, both the Betti cohomology and ℓ-adic cohomology lands in such a category). In this section we introduce the concepts needed to give the notion of a neutral Tannakian category a precise definition, we give a proof of a theorem relating the category of representations of an algebraic group to the group itself, and we state a deep theorem relating any neutral Tannakian category to the category of representations of a certain algebraic group. We also state a useful criterion for determining when an abelian tensor category is neutral Tannakian. After then briefly defining and exemplifying algebraic cycles (Section 3), we will define the two cohomology theories and the more general notion of a Weil cohomology theory (Section 4). In Section 5 we go on to discuss the notion of a Hodge structure. This is fundamental since the Betti cohomology carries such a structure. In this section we also introduce and prove some important results on the Mumford-Tate group. We state without much discussion the notion of a Galois representation and how we obtain such an action on the ℓ-adic cohomology. In Section 6 we thereafter state the three conjectures; the Hodge, Tate and the Mumford-Tate conjecture. After doing so we discuss some examples and sketch some proofs. Although we will not go into much details in the examples, one of the main hopes with presenting them is to indicate how the Mumford-Tate group can be used to study the Hodge conjecture. We also state and prove a short statement on how the three conjectures are related. The final section (Section 7) is then devoted to the concept of motives. This is again mainly a sketch where we mostly define concepts without any proofs. We end that section by stating the so-called motivic Mumford-Tate conjecture(s), and briefly mention how they are related to the three main conjectures of this text. The main hope of the last section on motives is to introduce the concept, and to indicate that it does unify some ideas and objects involved in the Hodge, Tate and the Mumford-Tate conjectures, and that it can serve as a useful tool for the study of them.

1.2 Acknowledgements

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I would also like to clarify that I make no claims of originality in this text. Of course the plan of the exposition and the sporadic intuitive explanations are my own, but none of the definitions, propositions or theorems (or corollaries) are originally due to me. There are several great books and articles from which I have learned whatever is presented in this text, and I have done my best to rightfully explain where one can get a better understanding of each part. The main hope is for this to serve as an overview that can allow for other students without knowledge of the relevant areas to (relatively) quickly get a basic understanding of the three conjectures and the related concepts.

2 Tannakian categories

The purpose of this part is to give a brief introduction to the concept of a neutral Tannakian category. We will present the fundamental notions of a tensor, rigid, and neutral Tannakian category, and state the main theorem on neutral Tannakian categories. Namely, given such a category, the automorphism group of the fibre functor is an affine group scheme, and it determines an equivalence between its category of representations and the given category. This result will help us realise groups arising from Hodge structures and Galois representations as Tannakian automorphism groups. We also sketch a proof of a useful theorem giving a characterisation of Tannakian categories in terms of the trace map. This theorem will then tell us whether the category of pure motives is Tannakian or not, which we will come back to in section (7). All that is written in this section (except Theorem (2.10)) can be found in [3].

2.1 Tensor categories

A **tensor category** is a category $\mathcal{T}$ together with a functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, an identity object $\underline{1} \in \mathcal{T}$ with respect to $\otimes$, and two families of functorial isomorphisms

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z, \quad X, Y, Z \in \mathcal{T} \tag{2.1}$$

respectively

$$\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in \mathcal{T}, \tag{2.2}$$

that form a compatible associativity and commutative constraint. To explain further, the family ϕ above is an **associativity constraint** for $(\mathcal{T}, \otimes)$ if

$$\begin{array}{ccccc} X \otimes \Big(Y \otimes (Z \otimes W)\Big) & \xrightarrow{\mathrm{id}_X \otimes \phi_{Y,Z,W}} & X \otimes \Big((Y \otimes Z) \otimes W\Big) & \xrightarrow{\phi_{X,Y\otimes Z,W}} & \Big(X \otimes (Y \otimes Z)\Big) \otimes W \\ \downarrow{\scriptstyle \phi_{X,Y,Z\otimes W}} & & & & \downarrow{\scriptstyle \phi_{X,Y,Z}\otimes \mathrm{id}_W} \\ (X \otimes Y) \otimes (Z \otimes W) & & \xrightarrow[\phi_{X\otimes Y,Z,W}]{} & & \Big((X \otimes Y) \otimes Z\Big) \otimes W \end{array} \tag{2.3}$$

is commutative for all X, Y, Z, W. The family ψ is a **commutativity constraint** for $(\mathcal{T}, \otimes)$ if

$$\psi_{X,Y} \circ \psi_{Y,X} = \mathrm{id}_{X\otimes Y}, \tag{2.4}$$

for all X, Y. One then says that the two constraints are **compatible** if

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\phi_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\psi_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes \psi_{Y,Z}} & & & & \downarrow{\scriptstyle \phi_{Z,X,Y}} \\ X \otimes (Z \otimes Y) & \xrightarrow[\phi_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & \xrightarrow[\psi_{X,Z}\otimes \mathrm{id}_Y]{} & (Z \otimes X) \otimes Y \end{array} \tag{2.5}$$

is commutative for all X, Y, Z. Also, an **identity** with respect to $\otimes$ is an object U together with an isomorphism $u : U \to U \otimes U$ such that the functor $\mathcal{T} \to \mathcal{T}$ $X \mapsto U \otimes X$ is an equivalence of categories. An identity object is unique up to unique isomorphism, and we will always denote the identity object, as well as the morphism by $\underline{1}$.

The most prototypical example of a tensor category is the category Mod_R of finitely generated modules over a ring R with the usual tensor product. Here the identity is R. A simple non-example is to take Mod_R

but change the associativity constraint to $\phi_{X,Y,Z} : x \otimes (y \otimes z) \mapsto -(x \otimes y) \otimes z$, in which case (2.3) is not commutative.

One can extend the tensor product $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ to $\otimes^I : \mathcal{T}^I \to \mathcal{T}$ for any finite set I in an essentially unique way.

Remark 2.1. By requiring that the tensor operation behaves well with respect to the respective extra structures, one can define an additive and abelian tensor category analogously. In this case $\mathrm{End}(\underline{1})$ is a ring which acts on each object X in $\mathcal{T}$. In fact, in all cases we will consider, $\mathcal{T}$ will be a tensor category over a field k, and we will have $\mathrm{End}(\underline{1}) \cong k$.

Finally, if $\mathcal{T}$ and $\mathcal{T}'$ are tensor categories with defining functorial families of ismorphisms ϕ, ϕ' and ψ, ψ', then we say that a functor $F : \mathcal{T} \to \mathcal{T}'$ is a **tensor functor** if $F(\underline{1}) = \underline{1}'$ and there is a functorial isomorphism $t_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$ such that the following diagrams are commutative:

$$\begin{array}{ccccc} FX \otimes (FY \otimes FZ) & \xrightarrow{\mathrm{id}_{FX} \otimes t_{Y,Z}} & FX \otimes F(Y \otimes Z) & \xrightarrow{t_{X,Y\otimes Z}} & F\Big(X \otimes (Y \otimes Z)\Big) \\ \downarrow{\scriptstyle \phi'_{FX,FY,FZ}} & & & & \downarrow{\scriptstyle F(\phi_{X,Y,Z})} \\ (FX \otimes FY) \otimes FZ & \xrightarrow[t_{X\otimes Y,Z}\otimes \mathrm{id}_{FZ}]{} & F(X \otimes Y) \otimes FZ & \xrightarrow[t_{X\otimes Y,Z}]{} & F\Big((X \otimes Y) \otimes Z\Big) \end{array} \tag{2.6}$$

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{t_{X,Y}} & F(X \otimes Y) \\ \downarrow{\scriptstyle \psi'_{FX,FY}} & & \downarrow{\scriptstyle F(\psi_{X,Y})} \\ FY \otimes FX & \xrightarrow[t_{Y,X}]{} & F(Y \otimes X) \end{array} \tag{2.7}$$

for all $X, Y, Z \in \mathcal{T}$.

2.2 Rigid tensor categories

The next step is to introduce duals. For this, we first introduce invertible objects, and then define internal Hom. We say that an object L in $\mathcal{T}$ is **invertible** if there exists an object, which we denote by L^{-1} when it exists, and an isomorphism $\underline{1} \cong L \otimes L^{-1}$. Such a pair, $(L^{-1}, \delta : L \otimes L^{-1} \xrightarrow{\sim} 1)$, is called an **inverse** to L. The **internal** Hom of $X, Y \in \mathcal{T}$ is, if it exists, the representable object of the functor

$$T \mapsto \mathrm{Hom}(T \otimes X, Y). \tag{2.1}$$

We then denote it by $\underline{\mathrm{Hom}}(X, Y)$. For example, in $\mathcal{T} = \mathsf{Mod}_R$, $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_R(X, Y)$ as R-modules. If $\underline{\mathrm{Hom}}(X, Y)$ exists, then by definition we have a functorial bijection

$$\mathrm{Hom}(T \otimes X, Y) \xrightarrow{\eta_T} \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)), \tag{2.2}$$

and after plugging in $T = \underline{\mathrm{Hom}}(X, Y)$ we denote by $\mathrm{ev}_{X,Y}$ the inverse image of $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ under this bijection. By construction

$$\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y. \tag{2.3}$$

Finally we define the **dual** of $X \in \mathcal{T}$ to be $X^\vee := \underline{\mathrm{Hom}}(X, \underline{1})$, and the map $\mathrm{ev}_X := \mathrm{ev}_{X,\underline{1}} : X^\vee \otimes X \to 1$ is called the **evaluation map**. The name is justified by looking again at Mod_R, where $\mathrm{ev}_X(f \otimes x) = f(x)$ for all modules X and elements $f \in X^\vee$ and $x \in X$.

Suppose now that $\underline{\mathrm{Hom}}(X, Y)$ exists for all X, Y. We want to make the map $X \mapsto X^\vee$ a contravariant functor. For this, note first that the definition of internal Hom tells us that for any T in $\mathcal{T}$ and any morphism

$g : T \otimes X \to Y$, there exists a unique $h : T \to \underline{\mathrm{Hom}}(X, Y)$ such that $g = \mathrm{ev}_{X,Y} \circ (h \otimes \mathrm{id}_X)$. Indeed, for each $a : T \to T'$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(T \otimes X, Y) & \xrightarrow{\eta_T} & \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)) \\ {\scriptstyle \circ(a\otimes \mathrm{id})}\uparrow & & \uparrow{\scriptstyle \circ a} \\ \mathrm{Hom}(T' \otimes X, Y) & \xrightarrow[\eta_{T'}]{} & \mathrm{Hom}(T', \underline{\mathrm{Hom}}(X, Y)) \end{array} . \tag{2.4}$$

With $h := \eta_T(g)$, and $T' = \underline{\mathrm{Hom}}(X, Y)$, $a = h$, the commutativity of (2.4) shows us that

$$\mathrm{ev}_{X,Y} \circ (h \otimes \mathrm{id}_X) = \eta^{-1}_{\underline{\mathrm{Hom}}(X,Y)}(\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}) \circ (\eta_T(g) \otimes \mathrm{id}_X) = \eta_T^{-1}\Big(\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)} \circ \eta_T(g)\Big) = g. \tag{2.5}$$

In particular, to an arbitrary $f : X \to Y$ we let the role of g be played by the morphism $\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) : Y^\vee \otimes X \to 1$, to give us a unique morphism

$${}^t f : Y^\vee = \underline{\mathrm{Hom}}(Y, 1) \to \underline{\mathrm{Hom}}(X, 1) = X^\vee \tag{2.6}$$

such that $\mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f)$. This is called the **transpose** of f. Following up on the prototypical example Mod_R, upon using the usual notation $\langle\, , \rangle_X := \mathrm{ev}_X$, we see that $X^\vee = \mathrm{Hom}_R(X, R)$ and for a morphism $f : X \to Y$, the transpose ${}^t f : Y^\vee \to X^\vee$ is the unique morphism such that $\langle\, {}^t f(y)\, ,\, x\, \rangle_X = \langle\, y\, ,\, f(x)\, \rangle_Y$, for all $x \in X$, $y \in Y^\vee$, as usual.

Now, if f is an isomorphism, we define the **dual** of f to be ${}^t f^{-1} : X^\vee \to Y^\vee$. By construction

$$\mathrm{ev}_X = \mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X) \circ (f^\vee \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) \circ (f^\vee \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (f^\vee \otimes \mathrm{id}_X). \tag{2.7}$$

Note also that, while apologising for the notation, if we in the top row of (2.4) replace X by $X^\vee$, replace T by X, and replace Y by $\underline{1}$, then the morphism $\mathrm{ev}_X \circ \psi_{X,X} : X \otimes X^\vee \to \underline{1}$ is taken by η^{-1} to a morphism

$$X \to X^{\vee\vee}. \tag{2.8}$$

If this is an isomorphism, then X is said to be **reflexive**.

Example 2.2. If we set $T = \otimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i)$, $X = \otimes_{i\in I} X_i$, $Y = \otimes_{i\in I} Y_i$, and g is the map $\otimes_{i\in I} \mathrm{ev}_{X_i}$, we obtain a morphism

$$\otimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}(\otimes_{i\in I} X_i, \otimes_{i\in I} Y_i) \tag{2.9}$$

Two immediate examples following from this are the following:
(a): If we take $Y_i = \underline{1}$ for all i, then we get a morphism

$$\otimes_{i\in I} X_i^\vee \to (\otimes_{i\in I} X_i)^\vee \tag{2.10}$$

(b): If we take $I = \{1, 2\}$, and $X_1 = X$, $Y_1 = \underline{1} = X_2$, and $Y_2 = Y$, then we obtain

$$X^\vee \otimes Y \to \underline{\mathrm{Hom}}(X, Y), \tag{2.11}$$

after using $\underline{\mathrm{Hom}}(\underline{1}, Y) \cong Y$.

We can now introduce the notion of a rigid tensor category.

Definition 2.3. A tensor category is called **rigid** if $\underline{\mathrm{Hom}}(X, Y)$ exists for all pairs (X, Y), all objects are reflexive, and if the morphism (2.9) is an isomorphism for all finite sets I.

Here our reference to Mod_R ends, because there exists finitely generated modules that are not reflexive.

2.3 Neutral Tannakian categories

Now let $\mathcal{T}$ be a rigid, abelian tensor category over k. By that we mean that $\otimes$ is a k-biadditive functor. We say that a tensor functor $\omega : \mathcal{T} \to \mathsf{Vec}_k$ is a **fibre functor** if it is exact, faithful and k-linear. Then

Definition 2.4. A **neutral Tannakian category** over k is a rigid, abelian tensor category that admits a fibre functor.

In such a category, a **Tannakian subcategory** is a strictly full subcategory that is closed under tensor products, duals, and quotients.

We stated in the beginning that the main theorem on neutral Tannakian categories relates such categories to representations of affine group schemes. So let us now turn to such representations, and then state the main theorem as promised.

2.3.1 Representations of affine group schemes

An **affine group scheme** over k is an affine scheme, G, over k together with regular k-maps $m : G \times G \to G$, $e : 1 \to G$, $\mathrm{inv} : G \to G$ (called multiplication, identity respectively inverse) that turns the underlying set of G into a group. Another, sometimes more useful, way to define an affine group scheme is through the language of functors. Then an affine group scheme over k is a contravariant functor $G : \mathsf{AffSch}_k \to \mathsf{Ab}$ from the category of affine k-schemes to the category of abelian groups such that composing with the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ gives a representable functor. Obtaining this latter description from the former is done by looking at the functor $G : T \mapsto \mathrm{Hom}_{\mathrm{Spec}\, k}(T, G)$. We will interchange between the two notions.

Example 2.5. (a): The **multiplicative group** over k is defined either as the functor that takes a k-algebra R to its multiplicatively group $R^\times$, or as the affine scheme $\mathrm{Spec}\, k[t, t^{-1}]$. It is denoted $\mathbb{G}_{m,k}$, or simply $\mathbb{G}_m$ if k is implicitly understood.

(b): More generally, the **general linear group** GL_n is defined as $\mathrm{Spec}\, k[t_{ij}, \det(t_{ij})^{-1}]_{1 \le i,j \le n}$. As a functor it takes each k-algebra R to the group $GL_n(R) = \mathrm{Aut}_R(k^n \otimes_k R)$.

(b): Generalising further, if V is a vector space of over k, then the **general linear group** of V, denoted $GL(V)$, is defined as the functor taking a k-algebra R to $\mathrm{Aut}_R(V \otimes_k R)$. For a description in terms of an affine scheme see [1].

Just as affine schemes corresponds to rings, affine groups also correspond to a purely algebraic object. Namely, a **bialgebra** over k is a k-algebra A together with maps $\Delta : A \to A \otimes A$, $\epsilon : A \to k$, $S : A \to A$ satisfying the so-called coassociativity axiom, the coidentity axiom, respectively the coinverse axiom:

$$\begin{gathered}
(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta : A \to A \otimes A \rightrightarrows A \otimes A \otimes A \\
(\epsilon \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \epsilon) \circ \Delta = \mathrm{id} : A \to A \otimes A \to k \otimes A \cong A \otimes k \cong A \\
\left(A \xrightarrow{\Delta} A \otimes A \xrightarrow{(S, \mathrm{id}_A)} A\right) = \left(A \xrightarrow{\epsilon} k \to A\right)
\end{gathered} \tag{2.1}$$

The Spec functor then gives an equivalence of categories between the category of affine group schemes over k and the category of k-bialgebras.

With the (bi)algebra structure in mind, we will relate representations of G to so-called comodules. To define these, we say that a **coalgebra** over k is a k-vector space K together with k-linear maps $\Delta : K \to K \otimes K$ and $\epsilon : K \to k$ satisfying the first axioms listed above (coassociativity and coidentity). In particular, each bialgebra is also a coalgebra. Then a **comodule** over K is defined to be a k-vector space V together with k-linear maps $\rho : V \to V \otimes K$ such that $\mathrm{id}_V = (\mathrm{id}_V \otimes \epsilon) \circ \rho$ and $(\mathrm{id}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_K) \circ \rho$. In pictures

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{id}_V} & V \cong V \otimes k \\ \rho\downarrow & \nearrow_{\mathrm{id}_V \otimes \epsilon} & \\ V \otimes K & & \end{array} \quad , \quad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes K \\ \rho\downarrow & & \downarrow\rho\otimes\mathrm{id}_K \\ V \otimes K & \xrightarrow[\mathrm{id}_V \otimes \Delta]{} & V \otimes K \otimes K \end{array} \ . \tag{2.2}$$

For example, each coalgebra A gives a comodule (A, Δ).

Our interest in comodules comes from the fact that for a given affine group scheme $G = \mathrm{Spec}(A)$ over k, and a given k-vector space V, we have a *canonical one-to-one correspondence between A-comodules on V and linear representations of G on V*. Indeed, if $h : G \to GL(V)$ is such a representation, then consider the image of id_A under $h(A) : G(A) \to GL(V)(A) = GL(V \otimes A)$. We then get a k-linear map

$$\rho_h : V \xrightarrow{\sim} V \otimes k \hookrightarrow V \otimes A \xrightarrow{h(A)(\mathrm{id}_A)} V \otimes A \tag{2.3}$$

that actually determines an A-comodule structure on V. Conversely, a comodule structure $\rho : V \to V \otimes A$ gives a representation $h : G \to GL(V)$ by taking a k-algebra R, to the morphism $h(R) : G(R) \to GL(V \otimes R)$ given by taking g in $G(R) \cong \mathrm{Hom}_k(A, R)$ to the automorphism

$$(\mathrm{id}_V \otimes (g, \mathrm{id}_R)) \circ (\rho \otimes \mathrm{id}_R) : V \otimes R \to V \otimes R. \tag{2.4}$$

Under this correspondence, the **regular representation** of G is defined to be that representation corresponding to the A-comodule (A, Δ).

An interesting feature of linear representations of affine group schemes is that they in a nice way come from finite representations. Precisely, note first that if (V, ρ) is a K-comodule, and $v \in V$, then $\rho(v) = \sum_{i=1}^{n} v_i \otimes x_i$, for some $v_i \in V, x_i \in K$, and the comodule generated by $v, v_1, ..., v_n$ is a finite-dimensional sub-comodule of V containing v. Thus, each finite subset of V is contained in a finite-dimensional sub-comodule. This translates to representations; *every linear representation V of G is a directed union of finite-dimensional sub-representations.* Indeed, by the correspondence in the previous paragraph, V is an A-comodule, and the collection of all finite-dimensional sub-comodules is partially ordered by inclusion, directed and has union V, by what was just explained about finite subsets.

So far we have talked about the relation between representations and comodules, one by one. Let us now turn to the whole category of representations of G, and see how we can recover G from it. Denote the just mentioned category by $\mathrm{Rep}_k(G)$. This is a rigid, abelian tensor category, with a fibre functor $\omega : \mathrm{Rep}_k(G) \to \mathsf{Vec}_k$ being the forgetful functor. This gives us a **tensor automorphism group**, $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. For each k-algebra R, it is given by

$$\underline{\mathrm{Aut}}^{\otimes}(\omega)(R) = \left\{ \prod_{X \in \mathrm{Rep}_k(G)} (\lambda_X) : \lambda_X \in \mathrm{Aut}_R(X \otimes R) \ \textit{satisfying (2.6) below} \right\} \tag{2.5}$$

$$\begin{aligned} \lambda_1 &= \mathrm{id}_R \\ \lambda_{X_1} \otimes \lambda_{X_2} &= \lambda_{X_1 \otimes X_2} \\ \lambda_Y \circ (\alpha \otimes 1) &= (\alpha \otimes 1) \circ \lambda_X : X \otimes R \to Y \otimes R \end{aligned} \tag{2.6}$$

for all $X_1, X_2, X, Y \in \mathrm{Rep}_k(G)$ and all G-equivariant maps $\alpha : X \to Y$. We wish to recover G from its category of representations, and we will do so by relating it to this automorphism group.

Proposition 2.6. *Let G be an affine group scheme over k, let $\mathrm{Rep}_k(G)$ be its category of representations over k, with forgetful functor ω. Then $G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega)$.*

Proof. We will prove this over the course of a few pages, where we also introduce new concepts and prove a few results that we will state as lemmas. The main idea is to write G as a limit of "smaller", "finite" subgroups, and prove a corresponding statement on these subgroups, and then pass through the limit to all of G.

The finiteness notion on affine groups that we mentioned is that of an algebraic group. Precisely, an **algebraic group** is an affine group scheme that is finitely generated as an affine scheme. That is, if the group is $G = \mathrm{Spec}(A)$, then A is a finitely generated k-algebra. We then have the following result.

Lemma 2.7. *An affine group scheme G over k is algebraic if and only if it has a finite dimensional faithful representation over k.*

Proof of Lemma (2.7): The forward comes directly since a finite-dimensional, faithful representation $\rho : G \hookrightarrow GL(V)$ embeds G as an algebraic subgroup of $GL(V)$. Now suppose the converse and let V be the regular representation. By our previous discussion, we can write $V = \bigcup_i V_i$ as a directed union of finite-dimensional representations. The fact that G is algebraic implies that it is noetherian as a topological space, whence any decreasing sequence of closed subsets stabilises. In particular, V being faithful implies that $\bigcap_i \ker\Big(G \to GL(V_i)\Big) = \{1\}$, and since each $\ker\Big(G \to GL(V_i)\Big)$ is closed, the fact that G is noetherian implies that $\ker\Big(G \to GL(V_i)\Big) = \{1\}$ for some i. This finishes the proof of Lemma (2.7).

Now, just as we can write any representation as a direct limit of finite-dimensional sub-representations, we want some similar "finiteness-relation" for G. This is obtained in the following.

Lemma 2.8. *Every affine group scheme G over k is the directed inverse limit of algebraic subgroups G_i, $G = \varprojlim G_i$.*

Proof of Lemma (2.8): Since the equivalence functor Spec is contravariant, it turns direct limits of k-bialgebras into inverse limits of k-groups. Thus, we are done if we can show that $A = \varinjlim A_i$, for a k-bialgebra A, and finitely generated sub-bialgebras A_i. But just as each finite subset of an A-comodule is contained in a finite-dimensional sub-comodule, one sees that each finite subset of A is contained in a finitely generated sub-bialgebra (meaning sub-bialgebra that is finitely generated as a k-algebra). Thus, we can write A as a directed union, i.e. $A = \varinjlim A_i$, as wanted. This finishes the proof of Lemma (2.8).

Now we turn to the proof of the proposition.

Proof of Proposition (2.6): Note first that we have a natural map

$$G \to \underline{\mathrm{Aut}}^{\otimes}(\omega) \tag{2.7}$$

given by taking g in $G(R)$ to $(\rho_X(R)(g))_X$, where $\rho_X : G \to GL(X)$ is the morphism corresponding to $X \in \mathrm{Rep}_k(G)$. It is this map we wish to show is an isomorphism. The idea is to use the finiteness relations discussed earlier to show that it restricts to an isomorphism between the algebraic subgroups G_i which G is the limit of, and certain restrictions of ω.

To this end, let $\langle X \rangle$ denote the full subcategory of $\mathrm{Rep}_k(G)$ consisting of those objects isomorphic to a subquotient of some tensor construction of X, (i.e. isomorphic to a subquotient of some object of the form $\oplus_{i=1}^n X^{a_i} \otimes (X^\vee)^{b_i}$ for some $n, a_i, b_i \in \mathbb{N}$). This is indeed a tensor subcategory since the tensor product commutes with colimits, in particular quotients, and since we can take $n = 0$ to get $\underline{1} \in \langle X \rangle$. Let ω_X denote the restriction of ω to $\langle X\rangle$. Each family $(\lambda) \in \underline{\mathrm{Aut}}^{\otimes}(\omega_X)(R)$ is determined by the representation λ_X, by definition of $\langle X\rangle$ and ω_X as the restriction to this subcategory. Thus, $(\lambda) \mapsto \lambda_X$ gives an injection

$$\underline{\mathrm{Aut}}^{\otimes}(\omega_X)(R) \hookrightarrow GL(X \otimes R). \tag{2.8}$$

If we let $G_X \hookrightarrow GL(X)$ be the image of G under the representation X, then the same reasoning as above gives $G_X(R) \hookrightarrow \underline{\mathrm{Aut}}^{\otimes}(\omega_X)(R)$. To obtain the reversed inclusion, Remark 3.2(a) in [2] tells us that it is enough to show that $\underline{\mathrm{Aut}}^{\otimes}(\omega_X)$ is the subgroup of $GL(X)$ that fixes all tensors in V's in $\langle X\rangle$ that are fixed by G_X (equivalently, by definition of G_X, fixed by G). Thus, take an object V in $\langle X\rangle$, $\lambda_V : G \to GL(V)$, and an element v in V fixed by G. Then $\alpha : k \to V$, $a \mapsto av$ is G-equivariant as G acts k-linearly, and hence

$$\lambda_V(v \otimes 1) = \lambda_V \circ (\alpha \otimes 1)(1 \otimes 1) = (\alpha \otimes 1) \circ \lambda_1(1 \otimes 1) = (\alpha \otimes 1)(1 \otimes 1) = v \otimes 1. \tag{2.9}$$

Thus, $G_X = \underline{\mathrm{Aut}}^{\otimes}(\omega_X)$. For each $Y \in \mathrm{Rep}_k(G)$, as $\langle X\rangle \hookrightarrow \langle X \oplus Y\rangle$ (recall that we allow subquotients), we therefore have commutative diagrams

$$\begin{array}{ccc} G_{X\oplus Y} & \xrightarrow{\cong} & \underline{\mathrm{Aut}}^{\otimes}(\omega_{X\oplus Y}) \\ \downarrow & & \downarrow \\ G_X & \xrightarrow{\cong} & \underline{\mathrm{Aut}}^{\otimes}(\omega_X) \end{array} \tag{2.10}$$

where the vertical maps are the "restrictions". From Lemmas (2.7), (2.8) we get $G = \varprojlim G_X$ and then the diagram gives, after taking limits, $G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega)$. ☐

Finally, we state the main theorem on neutral Tannakian categories.

Theorem 2.9. *Let $\mathcal{T}$ be a neutral Tannakian category over k, with $\mathrm{End}(\underline{1}) \cong k$, and fibre functor $\omega : \mathcal{T} \to \mathsf{Vec}_k$. Then $\underline{\mathrm{Aut}}^{\otimes}(\omega) =: G$ is an affine group scheme, and ω gives an isomorphism $\mathcal{T} \cong \mathrm{Rep}_k(G)$.*

Proof. See Theorem 2.11. in [3]. ☐

Although the theorem begins with a neutral Tannakian category and produces an affine group and a category of representations over this group, it can also be used the other way around; we might have a group G and a tensor equivalence $\mathrm{Rep}_k(G) \to \mathcal{T}$, from which we can then obtain information of G as the tensor automorphism group of $\mathcal{T}$.

2.4 A criterion to be neutral Tannakian

In this section we wish to present a criterion for when an abelian, rigid tensor category is neutral Tannakian. This is done through the trace map and the rank, which are defined as follows.

If $\mathcal{T}$ is a rigid tensor category, then for each $X \in \mathcal{T}$, by definition of a rigid tensor category (see (2.11) with $Y = X$) we have an isomorphism $\underline{\mathrm{Hom}}(X, X) \to X^{\vee} \otimes X$. Composing with ev_X gives

$$\underline{\mathrm{Hom}}(X, X) \to \underline{1}. \tag{2.1}$$

If we apply $\mathrm{Hom}(\underline{1}, -)$ to this, we get

$$\mathrm{Tr}_X : \mathrm{Hom}(X, X) \cong \mathrm{Hom}(\underline{1} \otimes X, X) \cong \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(X, X)) \to \mathrm{Hom}(\underline{1}, \underline{1}) = \mathrm{End}(\underline{1}). \tag{2.2}$$

This is how we define the **trace** morphism of X, or simply the trace of X. The **rank** of X is defined to be

$$\mathrm{rk}(X) := \mathrm{Tr}_X(\mathrm{id}_X) \in \mathrm{End}(\underline{1}), \tag{2.3}$$

and it is also sometimes called the **dimension** of X. Finally, we define the **exterior power** of X, denoted $\wedge^n X$, to be the image of the map $a : \sum(-1)^{\mathrm{sgn}(\sigma)}\sigma : X^{\otimes n} \to X^{\otimes n}$, $\sigma \in S_n$.

Using these notions, we now have the following result.

Theorem 2.10. *Let $\mathcal{T}$ be an abelian, rigid tensor category over a field k of characteristic zero, such that* $\mathrm{End}(\underline{1}) \cong k$. *Then the following are equivalent*

1. *$\mathcal{T}$ is neutral Tannakian;*
2. *For all $X \in \mathcal{T}$,* $\mathrm{rk}(X) \in \mathbb{Z}_{\geq 0}$*;*
3. *For all $X \in \mathcal{T}$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\wedge^n X = 0$.*

Proof. See [25] 7. □

3 Algebraic cycles

This section aims at giving a very brief introduction to the concept of algebraic cycles; we will just state the definition, give two short examples and explain a common way to construct an algebraic cycle from a given scheme (for an in-depth treatment of what we present here, we refer to [4]). This latter aim is important, because both the Hodge and Tate conjecture are existent statements about algebraic cycles.

Let $\mathcal{P}(k)$ be the category of smooth projective algebraic schemes over a field k, and let X be an object in $\mathcal{P}(k)$. Let a **subvariety** denote a reduced and irreducible subscheme $Z \subset X$. The set of **algebraic cycles on** X **of codimension** r is the free abelian group generated by subvarieties of codimension r, denoted $\mathcal{Z}^r(X)$. The set of **algebraic cycles** on X is then defined as

$$\mathcal{Z}^*(X) := \bigoplus_{r \geq 0} \mathcal{Z}^r(X). \tag{3.1}$$

Denote by $[Z]$ the element corresponding to $Z \subset X$ in $\mathcal{Z}^*(X)$. If we at some point extend scalars to Q, then define $\mathcal{Z}^*(X)_{\mathsf{Q}} := \mathcal{Z}^*(X) \otimes \mathsf{Q}$. In order to get a well-defined intersection product and certain functoriality properties, one usually considers $\mathcal{Z}^*(X)_{\mathsf{Q}}$ modulo some "nice" equivalence relation. The exact notion of "nice" is adequate, which we will define in Section (7) on motives. Now, we just introduce one adequate equivalence relation.

We say that $\alpha \in \mathcal{Z}^r(X)$ is **rationally equivalent** to 0, denoted $\alpha \sim_{\text{rat}} 0$, if there exists subvarieties $Z_1, ..., Z_k \subset X \times \mathbb{P}^1$ of codimension r such that the projections $Z_i \to \mathbb{P}^1$ are dominant, and $\alpha = \sum [Z_i(0)] - [Z_i(\infty)]$. Here the notation $Z(0)$ means the following. If $f : Z \to \mathbb{P}^1$ denotes the restriction of the projection $X \times \mathbb{P}^1$, and if $P \in \mathbb{P}^1$ is a closed point, then $Z(P) := f^{-1}(P) = Z \times \operatorname{Spec} \kappa(P) \subset X \times \{P\}$. An **algebraic class of codimension** r is an element of $\mathcal{Z}^r_{\text{rat}}(X) := \mathcal{Z}^r(X)/\sim_{\text{rat}}$. The **algebraic classes** is

$$\mathcal{Z}^*_{\text{rat}}(X) := \bigoplus_{r \geq 0} \mathcal{Z}^r_{\text{rat}}(X). \tag{3.2}$$

With addition coming from the underlying free group, and multiplication as intersection product (see [4] for the definition), this is a ring, which is often called the **Chow ring** of X. In fact, the "obvious" grading coming from the definition makes $\mathcal{Z}^*_{\text{rat}}(X)$ a graded $\mathbb{Z}_{\geq 0}$-algebra. When we extend scalars to $\mathcal{Z}^*_{\text{rat}}(X)_{\mathsf{Q}} := \mathcal{Z}^*(X) \otimes \mathbb{Q}/\sim_{\text{rat}}$ we get a graded Q-algebra.

Example 3.1. Let $X = \mathbb{P}^2_k$. Every subvariety of codimension 1 is determined by an irreducible, homogeneous polynomial $f \in k[x_0, x_1, x_2]$. If two such polynomials, f and g, have the same degree, d, then we can consider $h(x, y) \in k[x_0, x_1, x_2, y_0, y_1]$ defined by

$$h(x, y) := y_0 f + y_1 g. \tag{3.3}$$

This is bihomogeneous of bidegree $(d, 1)$. With the notation $0 = (1 : 0)$ and $\infty = (0 : 1)$, we obtain $h(x, 0) = f$ and $h(x, \infty) = g$. Let $V(-)$ denote the zero loci operation. Then $Z := V(h) \subset X \times \mathbb{P}^1$ is such that $Z(0) = V(f)$, and $Z(\infty) = V(g)$, and hence $[V(f)] - [V(g)] \sim_{\text{rat}} 0$.

This example indicates that one is perhaps not completely wrong to think of rational equivalence as a sort of homotopy relation. Another useful example is the following.

Example 3.2. A cycle of codimension 1 of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by a bihomogeneous polynomial $f(x_0, x_1, y_0, y_1)$ of bidegree $(\deg_x f, \deg_y f)$ and two such are rationally equivalent if and only if they have the same bidegree. From this we get that $\mathcal{Z}^1_{\text{rat}}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{P}^1 \times \{0\} = V(y_1)$ and $\{0\} \times \mathbb{P}^1 = V(x_1)$, where x_1 has bidegree $(1,0)$, and y_1 has bidegree $(0,1)$, we see that a basis for $\mathcal{Z}^1_{\text{rat}}(\mathbb{P}^1 \times \mathbb{P}^1)$ is $\{[\{0\} \times \mathbb{P}^1], [\mathbb{P}^1 \times \{0\}]\} \cong \{(1,0),(0,1)\}$. We have $\Delta_{\mathbb{P}^1} \sim_{\text{rat}} \{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}$. This decomposition of the diagonal will be important in the construction of pure motives in Section (7).

The construction of the Chow ring, and more generally that of algebraic cycles modulo any adequate equivalence relation, is actually functorial. We omit the details, but roughly one gets a contravariant functor $(-)^*$ by taking a morphism $f : X \to Y$ to the map f^* that takes a subvariety of Y, pulls it back to the product $Y \times X$, intersects it with the transpose of the graph of f and then pushes it forward to X. For the covariant functor one does a similar procedure but replacing the transpose of the graph of f with just the graph. As a small remark, however, it is only the pullback (the contravariant functor) that respects the intersection product.

Finally, let us introduce a common way to construct algebraic cycles.

Example 3.3. Suppose $\mathcal{L}$ is a locally free sheaf on X of rank r, with a global, non-trivial, section $s \in \Gamma(X, \mathcal{L})$. The zero subscheme of s, $L(s)$, is the scheme with underlying set consisting of those points $x \in X$ such that $s(x) = 0$ in $\mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Since $\mathcal{L}$ is locally free of rank r, locally around each x in X there is an affine neighbourhood, U, of x, with $\mathcal{L}|_U \cong \mathcal{O}_X|_U^r$, so $s|_U$ corresponds to r functions $f_1, ..., f_r \in \mathcal{O}_X|_U$, and the zero scheme of s can be seen as the set of zeros of the f_i's. In good situations, the codimension of $L(s)$ is exactly r, i.e. each f_i cuts down the dimension by one. In particular, if X is smooth, projective and algebraic over k, and $\mathcal{L}$ is an ample line bundle, then $L(s)$ is a reduced, irreducible subscheme. In this case, the element corresponding to $L(s)$ in $\mathcal{Z}^*_{\text{rat}}(X)$ is called the **ample divisor** (corresponding to $\mathcal{L}$).

4 Cohomology theories

The other "objects" constructed from a scheme $X \in \mathcal{P}(k)$ that we consider are certain cohomology theories. We assume that the reader have seen some (co)homology, in particular singular (co)homology of a topological space. But if not, to put it shortly: (co)homology is a way of "linearising" a space to create algebraic invariants which helps us understand the space in question. Singular (co)homology essentially looks at subpieces of the space in question carved out by continuous functions from a point, a line, a "filled in" triangle, a tetrahedon etc. The idea is, roughly, that this construction at the very least should take into account holes of various dimensions in the space in question, and one important consequence is that homotopically equivalent spaces have isomorphic singular (co)homology.

Just as for the algebraic cycles, we will mainly stick to the definitions, and the cohomology theories we will define are Betti cohomology, ℓ-adic cohomology, and the more general notion of a Weil cohomology. The later is not a cohomology theory per se, but rather a definition of a certain class of such theories, a class both the Betti and the ℓ-adic cohomology theory belongs to. We include its definition partly because it tells us some features of the Betti and the ℓ-adic cohomology, and partly because we want it for the discussion on motives.

4.1 Betti cohomology

A **complex analytic space** is a locally ringed space $(X^{an}, \mathcal{O}_{X^{\mathrm{an}}})$ such that X^{an} locally embeds into $\mathbb{C}^n$, for some n. As such it is endowed with an Euclidean topology and one can therefore consider the singular cohomology on it. The importance of this in the setting of algebraic schemes comes from the following.

Theorem 4.1. *If X is a locally finite type scheme over $\mathbb{C}$, then the functor*

$$Y \mapsto \mathrm{Hom}_{\mathit{Locally\ ringed\ spaces}}(Y, X) \tag{4.1}$$

from the category of analytic spaces into Set *is representable by an analytic space, X^{an}. Furthermore, the underlying set of X^{an} is $X(\mathbb{C})$, the complex points of X.*

In particular, if X is a smooth, projective scheme over a field k, embeddable into $\mathbb{C}$, then $X \times_k \mathbb{C}$ is a locally finite type scheme over $\mathbb{C}$, so to it there corresponds an analytic space, X^{an}. The singular cohomology on this space, which we denote by $H_B(X, \mathbb{Z})$, is called the **Betti cohomology** of X.

Furthermore, with X^{an} comes also a natural morphism $\iota^{\mathrm{an}} : X^{\mathrm{an}} \to X$ of locally ringed spaces. We then have the following important theorem.

Theorem 4.2. *([part of] Serre's GAGA)*
For any coherent sheaf $\mathcal{F}$ on X, and for any n, the natural morphism $\iota^{\mathrm{an}} : X^{\mathrm{an}} \to X$ induces an isomorphism

$$H^n(X, \mathcal{F}) \xrightarrow{\sim} H^n(X^{\mathrm{an}}, (\iota^{\mathrm{an}})^* \mathcal{F}) \tag{4.2}$$

Here $H^n(X, \mathcal{F})$ refers to sheaf cohomology (for the definition of this cohomology theory replace "étale" by "Zariski" in the discussion in the next section, or see [5] Chapter III for a more in-depth treatment). In particular, we have an isomorphism between $H^q(X \times_k \mathbb{C}, \Omega^p_{X \times_k \mathbb{C}/\mathbb{C}})$ and $H^q(X^{\mathrm{an}}, \Omega^p)$, where Ω^1 is the sheaf of relative differentials, and $\Omega^p = \bigwedge^p \Omega^1$.

4.2 ℓ-adic cohomology

The other cohomology theory that we will need is the ℓ-adic cohomology. The underlying idea is that we want a cohomology theory for schemes that shares similar properties to the singular cohomology for topological spaces, but such that we can use it also for schemes over fields of positive characteristic. A main problem is that the Zariski topology is too coarse, and we therefore need to enlarge (and slightly change the definition of) our underlying topology. This leads us to the notion of a site.

Definition 4.3. A **site** is a (small) category $\mathcal{C}$ together with the following information. For each $U \in \mathcal{C}$, there exists a family, $\{\varphi_i : U_i \to U\}_{i \in I}$, of morphisms in $\mathcal{C}$ (called a **covering** of U). The set of all coverings of U is required to satisfy the following conditions:

1. Each isomorphism $\varphi : V \to U$ gives a covering $\{\varphi : V \to U\}$.
2. If $\{\varphi_i : U_i \to U\}_{i\in I}$ is a cover of U and $\{\varphi_{ij} : U_{ij} \to U_i\}_{j\in J}$ is a cover of U_i for all i then $\{\varphi_i \circ \varphi_{ij} : U_{ij} \to U\}_{i,j}$ is also a cover of U.
3. If $\{\varphi_i : U_i \to U\}_{i\in I}$ is a cover of U, and $U' \to U$ is a morphism in $\mathcal{C}$, then the projection morphisms of the respective base changes $\varphi'_i : U'_i \coloneqq U' \times_U U_i \to U'$ form a cover of U', $\{\varphi'_i : U'_i \to U'\}_{i\in I}$.

Remark 4.4. Note that we implicitly require that the base change $U' \times_U U_i$ exists in $\mathcal{C}$. This can intuitively be thought of as a generalisation of the requirement that topologies are stable under intersections.

The **Zariski site** of a scheme X is the "usual" Zariski topology on X. In the language of sites, it is the category X_{Zar} of open immersions $U \hookrightarrow X$ together with all the usual coverings; $\{\varphi_i : U_i \to U\}_{i\in I}$ is a cover of U if $\bigcup_i \varphi_i(U_i) = U$. The **étale site** of X is the category X_{et} of finite, étale morphisms $U \to X$ together with all the coverings $\{\varphi_i : U_i \to U\}_{i\in I}$ such that $\bigcup_i \varphi_i(U_i) = U$. One then says that a **sheaf on** X_{et} is a contravariant functor $\mathcal{F} : X_{\mathrm{et}} \to \mathsf{Ab}$ such that, for each $U \to X$ and each covering $\{U_i \to U\}_{i\in I}$, the sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \tag{4.1}$$

is exact (sheaves on an arbitrary site is defined similarly). We denote the category of sheaves on X_{et} by $\mathsf{S}(X_{\mathrm{et}})$.

Recall that in an abelian category, $\mathcal{A}$, an object I is called **injective** if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact, and the category is said to have **enough injectives** if for each object A in $\mathcal{A}$ there is a monomorphism $A \to I$, for some injective object I. In particular, *the category* $\mathsf{S}(X_{\mathrm{et}})$ *has enough injectives* (for a short proof, see [6] III.1). Furthermore, if $\mathcal{A}$ has enough injectives, $\mathcal{B}$ is another abelian category, and $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then there exists a family of functors $R^iF : \mathcal{A} \to \mathcal{B}$ with the property that

1. $R^0F = F$,
2. $R^iF(I) = 0$ for I injective and $i > 0$,
3. there are boundary maps; each short exact sequence $0 \to A' \to A \to A'' \to 0$ induces a long exact sequence $\cdots R^iF(A) \to R^iF(A'') \xrightarrow{\delta_i} R^{i+1}F(A') \to R^{i+1}F(A) \to \cdots$, and this is functorial.

These are called the **right derived functors** of F.

Using this, we define the **étale cohomology** of X to be the right-derived functors of the global sections functor $\Gamma(X, -) : \mathsf{S}(X_{\mathrm{et}}) \to \mathsf{Ab}$. Denoted

$$H^i_{\mathrm{et}}(X, -) \coloneqq R^i\Gamma(X, -). \tag{4.2}$$

We can in fact give a more explicit description of this. If $\mathcal{F}$ is in $\mathsf{S}(X_{\text{et}})$, then we apply $\Gamma(X,-)$ to an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$. Removing the term $\Gamma(X,\mathcal{F})$, we have a sequence

$$0 \to \Gamma(X,\mathcal{I}^0) \xrightarrow{d^0} \Gamma(X,\mathcal{I}^1) \xrightarrow{d^1} \cdots \tag{4.3}$$

and the étale cohomology of X is the cohomology of this sequence

$$H^i_{\text{et}}(X,\mathcal{F}) := R^i\Gamma(X,\mathcal{F}) = \ker d^{i+1} / \operatorname{im} d^i. \tag{4.4}$$

We now define the **ℓ-adic cohomology** of X (with coefficients in $\mathbb{Q}_\ell$) to be

$$H^i_\ell(X,\mathbb{Q}_\ell) := \Big(\varprojlim_n H^i_{\text{et}}(X,\mathbb{Z}/\ell^n\mathbb{Z})\Big) \otimes \mathbb{Q}_\ell. \tag{4.5}$$

Here too we have a notion of twisting. If we fix some $n \neq \operatorname{char}(k)$ then the operation $\mu : (U \to X) \mapsto \{x \in \Gamma(U,\mathcal{O}_U) : x^n = 1\}$ on X_{et} gives a sheaf. If $\mathcal{F}$ is another sheaf on X_{et} then $\mathcal{F}(r) := \mathcal{F} \otimes \mu^{\otimes r}$, and we define

$$H^i_\ell(X,\mathbb{Q}_\ell(r)) := \Big(\varprojlim_n H^i_{\text{et}}(X,\mathbb{Z}/\ell^n\mathbb{Z}(r))\Big) \otimes \mathbb{Q}_\ell. \tag{4.6}$$

For more on the étale cohomology, we refer to [6].

Remark 4.5. One can also define singular cohomology as a right-derived functor. In particular, it too takes short exact sequences to long exact sequences in a functorial manner.

4.3 Weil cohomology

(In this section we follow the definition in [15]).

The two cohomology theories introduced above are examples of so-called Weil cohomologies. In general, a **Weil cohomology theory** with coefficients in Q is a contravariant functor

$$H : \mathcal{P}(k) \to \mathsf{GrVec}_{\mathsf{Q}}^{\mathbb{Z}_{\geq 0}} \tag{4.1}$$

from the category $\mathcal{P}(k)$ of smooth projective schemes over k, to the category of $\mathbb{Z}_{\geq 0}$-graded vector spaces over Q, respecting the monoidal structures of the two categories, and satisfying the following conditions.

1. We have $\dim_{\mathsf{Q}} H^2(\mathbb{P}^1) = 1$.

   Remark 4.6. We define the **Tate twist** (with respect to H) on $\mathsf{GrVec}_{\mathsf{Q}}$ to be $V(r) := V \otimes H^2(\mathbb{P}^1)^{\otimes}(-r)$, for $r \in \mathbb{Z}$, and $-$ refer to the tensor of the dual.

2. For each $X \in \mathcal{P}(k)$ of pure dimension d, there exists a Q-linear map $\operatorname{Tr}_X : H^{2d}(X)(d) \to \mathsf{Q}$, such that the composition

   $$H^i(X) \times H^{2d-i}(X)(d) \to H^{2d}(X)(d) \xrightarrow{\operatorname{Tr}_X} \mathsf{Q} \tag{4.2}$$

   is a perfect pairing. We require Tr_X to be an isomorphism if X is geometrically connected, and in all cases we require that it satisfies $\operatorname{Tr}_{X\times Y} = \operatorname{Tr}_X \operatorname{Tr}_Y$.

3. For each $X \in \mathcal{P}(X)$ and all $r \geq 0$, we have maps $\operatorname{cl}^r_X : \mathcal{Z}^r_{\text{rat}}(X) \to H^{2r}(X)(r)$, that are

   (a) contravariant in X,

   (b) satisfies $\operatorname{cl}^{r+s}_{X\times Y}(\alpha \times \beta) = \operatorname{cl}^r_X(\alpha) \otimes \operatorname{cl}^s_Y(\beta)$, and

(c) when X is pure of dimension d, $\mathrm{Tr}_X \circ \mathrm{cl}_X^d$ gives the degree map (for closed points $P_i \in X$, $\deg \sum_i n_i[P_i] \coloneqq \sum_i [\kappa(P_i) : k]$, where $\kappa(P_i)$ is the field corresponding to P_i).

Remark 4.7. Since H respects the monoidal structures there is indeed an isomorphism $H(X \times Y) \cong H(X) \otimes H(Y)$.

The second condition is often called **Poincaré duality**, the isomorphism $H(X \times Y) \cong H(X) \otimes H(Y)$ is referred to as the **Künneth isomorphism** and the map cl_X is called the **cycle class map**. These have the important consequence that elements in $\mathcal{Z}_{\mathrm{rat}}(X \times X)$ give functions $H(X) \to H(X)$. Indeed, if $e \in \mathcal{Z}_{\mathrm{rat}}^{d_X}(X \times X)$, then the cycle class map gives an element $e' \in H^{2d_X}(X \times X)$. The Kunneth isomorphism gives an element $e'' \in \oplus_{i+j=2d_X} H^{2d_X - i}(X) \otimes H^i(X)$. By Poincaré duality, $H^{2d_X - i}(X) = H^i(X)^\vee$, whence we get an element $e''' \in \oplus_i H^i(X)^\vee \otimes H^i(X) \cong \oplus_i \mathrm{Hom}(H^i(X), H^i(X)) \cong \mathrm{Hom}(H(X), H(X))$. In short:

$$\begin{aligned} \mathcal{Z}_{\mathrm{rat}}^{\dim X}(X \times X) &\to H^{2d_X}(X \times X) && \text{(cycle class map)} \\ &\cong \bigoplus_i H^{2d_X - i}(X) \otimes H^i(X) && \text{(Kunneth isomorphism)} \\ &\cong \bigoplus_i H^i(X)^\vee \otimes H^i(X) && \text{(Poincaré)} \\ &\cong \mathrm{Hom}(H(X), H(X)). \end{aligned} \tag{4.3}$$

4.4 Cycle class map

We will take all of the properties of a Weil cohomology for granted for the Betti and the ℓ-adic cohomology, but let us at least sketch how the cycle class map is constructed for divisors, that is, for elements in $\mathcal{Z}_{\mathrm{rat}}^1(X)$. Let $X \in \mathcal{P}(k)$ as before and let $\underline{\mathbb{Z}}$ denote the constant sheaf on X with values in $\mathbb{Z}$. On the analytic space corresponding to X we have an exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0 \tag{4.1}$$

so if we apply $H_B(X, -)$ on this we get

$$\cdots \to H_B^1(X, \mathcal{O}_X^\times) \xrightarrow{\delta} H_B^2(X, \underline{\mathbb{Z}}) \to \cdots \tag{4.2}$$

Now, using Cech cohomology one can show that $H^1(X, \mathcal{O}_X^\times)$ is isomorphic to the Picard group of X, which, since X is smooth, is isomorphic to $\mathcal{Z}_{\mathrm{rat}}^1(X)$. We then extend scalars $H_B^2(X, \mathbb{Z}) \to H_B^2(X, \mathbb{Q})$. Combining δ with this extension of scalars gives the cycle class map in degree 1.

Remark 4.8. The Picard group of a scheme X, denoted $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. For a brief introduction to Cech cohomology see [16] 18, and to get a map from the Cech cohomology group to $\mathrm{Pic}(X)$ use that locally free sheaves are determined by their transition functions, and use the relations defining the Cech cohomology to define transition functions.

For the ℓ-adic cohomology, we have the so-called **Kummer sequence**

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 1. \tag{4.3}$$

Here n is a positive integer that is invertible in $\mathcal{O}_X$, $\mathbb{G}_m \in \mathsf{S}(X_{\mathrm{et}})$ is the sheaf $U \mapsto \Gamma(U, \mathcal{O}_X|_U)^\times$ and μ_n is thus the sheaf $\ker\left(\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m\right)$. While the left-exactness of this sequence follows from definition, the exactness to the right is not as obvious (see [6] II.2). If $Z \hookrightarrow X$ is a smooth[1] subvariety, then one can define a notion of étale cohomology with support on Z, denoted $H_Z(X, -)$ (see [6] III.1), and for each sheaf $\mathcal{F} \in \mathsf{S}(X_{\mathrm{et}})$, there is a long exact sequence

$$\cdots H^r_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(X - Z, \mathcal{F}|_{X-Z}) \xrightarrow{\delta^r_{Z,\mathcal{F}}} H^{r+1}_Z(X, \mathcal{G}) \to \cdots \tag{4.4}$$

Now, $H^0(X - Z, \mathbb{G}_m) \cong \Gamma(X - Z, \mathcal{O}_{X-Z})^\times$ by definition of cohomology as a right-derived functor, and again we have $H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$. Similarly $H^1(X - Z, \mathbb{G}_m) \cong \mathrm{Pic}(X - Z)$. We then have the following commutative diagram (see [6] VI.6)

$$\begin{array}{ccccccc} H^0(X - Z, \mathbb{G}_m) & \longrightarrow & H^1_Z(X, \mathbb{G}_m) & \longrightarrow & H^1(X, \mathbb{G}_m) & \longrightarrow & H^1(X - Z, \mathbb{G}_m) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \Gamma(X - Z, \mathcal{O}_{X-Z})^\times & \xrightarrow{\mathrm{ord}_Z} & \mathbb{Z} & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X - Z) \end{array} \tag{4.5}$$

where $H^1_Z(X, \mathbb{G}_m) \cong \mathbb{Z}$ follows from the 5-Lemma. Now, apply $H^1_Z(X, -)$ to the Kummer sequence and compose with $H^2_Z(X, \mu_n) \to H^2(X, \mu_n)$ (putting in $\mathcal{F} = \mu_n$ in (4.4)) to get a map

$$\mathbb{Z} \cong H^1_Z(X, \mathbb{G}_m) \to H^2_Z(X, \mu_n) \to H^2(X, \mu_n). \tag{4.6}$$

The image of the cycle class map of Z is defined to be the image of 1 under this map (to get the cycle class map to ℓ-adic cohomology rather than étale we do this for all n and take the corresponding object in the limit).

Given the cycle class map in degree 1 one can define it for all $r \geq 0$ by using the notion of Chern classes. For a thorough treatment of this we refer to [17] and for a sketch on how one goes from what we have to the general cycle class map see [3] 1. (or, for the étale cohomology, [6] VI.9).

[1]We add this requirement here already for simplicity.

5 Hodge structures and Galois representations

A particularly interesting feature of the two cohomology theories (Betti and ℓ-adic) is that they come with some extra structure. The Betti cohomology carries a Hodge structure and the ℓ-adic cohomology comes with a Galois representation. In this section we will give a general introduction to the notion of a Hodge structure, and in particular introduce a group called the Mumford-Tate group and show some important results regarding it. This will relate back to Section (2). In the end of this section we will also explain how we get a Galois structure on the ℓ-adic cohomology, and define groups coming form this Galois structure that play analogous roles to the Mumford-Tate group.

For a more in-depth treatment on Hodge structures we refer to [18] and [19], from where we got most of the material.

5.1 Hodge structures

A real **Hodge structure** is a real vector space V together with a decomposition of its complexification

$$V_{\mathbb{C}} = V \otimes \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}, \tag{5.1}$$

such that $\overline{V^{p,q}} = V^{q,p}$. A rational Hodge structure is defined similarly. If $V^{p,q} = 0$ for all p, q such that $p + q \neq n$, then the Hodge structure is said to be (pure) of weight n. It is worth pointing out the "obvious" fact that a Hodge structure V can be written as $V = \bigoplus_n V^{(n)}$, where $V^{(n)}$ is pure of weight n. Let $\mathbb{R}\mathsf{HS}$ and $\mathbb{Q}\mathsf{HS}$ denote the category of real respectively rational Hodge structures. The morphisms in these categories are linear maps between the underlying vector spaces such that their complexification respects the grading. For example, for rational Hodge structures V and W, $\mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}(V, W)$ consists of those linear maps $f : V \to W$ such that $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$ for all p, q. Hence, if $V \in \mathbb{Q}\mathsf{HS}$ is pure of weight n, and $W \in \mathbb{Q}\mathsf{HS}$ is pure of weight m, then $\mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}(V, W) = 0$ whenever $n \neq m$. This definition of morphisms makes it clear that cokernels exists; with f as before, $\operatorname{coker} f$ is the cokernel of f as a linear map together with the grading $(W/\operatorname{im} f) \otimes \mathbb{C} = \bigoplus_{p,q} W^{p,q}/f_{\mathbb{C}}(V^{p,q})$. All other axioms for an abelian category follow similarly from the category of finite dimensional vector spaces, thus $\mathbb{R}\mathsf{HS}$ and $\mathbb{Q}\mathsf{HS}$ are abelian categories. In fact, they are rigid tensor categories; if V and W are Hodge structures of weight n respectively m, then define $V \otimes W$ to be $V \otimes W$ as vector spaces, with the decomposition

$$(V \otimes W)_{\mathbb{C}} = \bigoplus_{a+b=m+n} (V \otimes W)^{a,b}, \qquad (V \otimes W)^{a,b} := \bigoplus_{\substack{p+p'=a\\ q+q'=b}} V^{p,q} \otimes W^{p',q'}. \tag{5.2}$$

An important example is that of twists; $\mathbb{Q}(n)$ is the Hodge structure with underlying vector space $(2\pi i)^n\mathbb{Q}$ and complexification $\mathbb{C} = \mathbb{Q}(n)^{-n,-n}$ of weight $-2n$, and then the **Tate twist** of a Hodge structure V is $V(n) := V \otimes \mathbb{Q}(n)$. We see also that the identity with respect to this tensor product is $\underline{1} := \mathbb{Q}(0)$. For the dual Hodge stucture, we simply let $V^\vee$ be the dual as a vector space together with the decomposition

$$(V^\vee)_{\mathbb{C}} = \bigoplus_{p+q=-n} (V^\vee)^{p,q}, \qquad (V^\vee)^{p,q} = (V^{-p,-q})^\vee. \tag{5.3}$$

We see that taking duals negates the weight of the Hodge structure, and taking tensor products adds up the two weights. One can also check that $\underline{\mathrm{Hom}}(V, W) = V^\vee \otimes W$.

The forgetful functor to $\mathsf{Vec}_{\mathbb{Q}}$ (respectively $\mathsf{Vec}_{\mathbb{R}}$) is a fibre functor, thus the category of Hodge structures is in fact neutral Tannakian. However, the category is not semisimple, and we therefore want to introduce a way of taking "orthogonal complements".

To this end, let V be a rational Hodge structure of weight n. Consider the endomorphism $C_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ given by $\sum_{p+q=n} x_{p,q} \mapsto \sum_{p+q=n} i^{p-q} x_{p,q}$, where $x_{p,q} \in V^{p,q}$. Since $\overline{V^{p,q}} = V^{q,p}$, $C_{\mathbb{C}}$ restricts to $C := C_{\mathbb{R}} \in GL(V_{\mathbb{R}})$. This is called the **Weil operator**. We use it to define a **polarisation** of a rational Hodge structure of weight n to be a morphism of Hodge structures

$$\varphi : V \otimes V \to \mathbb{Q}(-n) \tag{5.4}$$

such that the bilinear map

$$\varphi_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R} \tag{5.5}$$

given by $\varphi_{\mathbb{R}}(x, y) := (2\pi i)^n \varphi(Cx \otimes y)$ is symmetric and positive-definite. As mentioned before, the main point of such a polarization is that the complement $V^{\perp} = \{v \in V : \varphi_{\mathbb{R}}(v, w) = 0 \text{ for all } w\}$ is again a Hodge structure. Hence, the category of polarizable Hodge structures, denoted $\mathbb{Q}\mathsf{HS}^{\mathrm{pol}}$, is a semisimple, neutral Tannakian category.

We end this section with the most relevant example for this text.

Example 5.1. (Remark: In this example we omit for simplicity all twists.)
Suppose X is a smooth, projective, algebraic scheme over $k \hookrightarrow \mathbb{C}$ of dimension d_X, and let $0 \leq n \leq d_X$. Then the Betti cohomology $H^n_B(X, \mathbb{Q})$ is a polarizable $\mathbb{Q}$-Hodge structure of weight n. The Hodge decomposition is given by

$$H^n_B(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega^p). \tag{5.6}$$

The polarisation is defined as follows. First, if L is an ample divisor on X, let η denote its image in $H^2_B(X, \mathbb{Q})$ under the cycle class map. By taking the cup-product with η this gives a morphism $H(X) \to H(X)$, called **the Lefschetz morphism** corresponding to η, also denoted L, which can be shown gives an isomorphism $L^i : H^{d_X - i}(X) \to H^{d_X + i}(X)$. This decomposes $H^n_B(X, \mathbb{Q})$ into two parts, the **primitive cohomology** and the non-primitive. Intuitively, the non-primitive consists of all those $x \in H^n_B(X, \mathbb{Q})$ that can be obtained from L, and the primitive part is those that cannot be obtained from L. More precisely, the non-primitive cohomology (in degree $n-k$) is

$$H^{n-k}_{B,\text{non-prim}} := \operatorname{im}\left(L : H^{n-k-2}_B(X) \to H^{n-k}_B\right) \tag{5.7}$$

and the primitive cohomology (in degree $n-k$) is

$$P^{n-k} = H^{n-k}_{B,\text{prim}} := \ker\left(L^{k+1} : H^{n-k}(X) \to H^{n+k+2}(X)\right). \tag{5.8}$$

We then have $H^{n-k}_B(X) \cong H^{n-k}_{B,\text{prim}}(X) \oplus H^{n-k}_{B,\text{non-prim}}(X)$ which gives the **Lefschetz decomposition**

$$H^n_B(X) = \bigoplus_i L^i H^{n-2i}_{B,\text{prim}}(X). \tag{5.9}$$

We define the polarisation $\varphi_{\text{prim}} : H^n_{B,\text{prim}}(X) \times H^n_{B,\text{prim}}(X) \to \mathbb{Q}(-n)$ as $(x, y) \mapsto (-1)^n L^{d_X - n} x \cup y$, and then extend it through (5.9) to $H^n_B(X)$.

5.1.1 Hodge structures as representations

Firstly, recall that the **character group** of an algebraic group G over k is the group of homomorphisms $X^*(G) := \operatorname{Hom}(G, \mathbb{G}_{m,k})$, and the **cocharacter group** is $X_*(G) := \operatorname{Hom}(\mathbb{G}_{m,k_s}, G)$. Further, a **torus** over k is an algebraic group T over k such that $T \times k_s$ is isomorphic to $(\mathbb{G}_{m,k_s})^r$ for some $r \in \mathbb{N}$, called the **rank of the torus**. Thus, if T is a torus of rank r, then $X^*(T) \cong \mathbb{Z}^r$. A useful fact of representations of a torus is the following:

Proposition 5.2. *Suppose T is a torus over k. Then any representation ρ of $T \otimes k_s$ on a k_s vector space V_{k_s} splits the vector space into a direct sum of its character spaces $V_{k_s}(\lambda) = \{v \in V_{k_s} : \rho(t) \cdot v = \lambda(t)v\}$, $\lambda \in X^*(T)$. In short*

$$\Big(\rho : T \otimes k_s \to GL(V_{k_s})\Big) \rightsquigarrow V_{k_s} = \bigoplus_{\lambda \in X^*(T)} V_{k_s}(\lambda). \tag{5.10}$$

This can be reversed: any such decomposition gives a representation: $\rho(t) \cdot \sum v_\lambda := \sum \lambda(t) v_\lambda$, $v_\lambda \in V_{k_s}(\lambda)$.

In our case, we are interested in the Deligne torus, defined as follows. If X is a scheme over S', which is a scheme over S, then the **Weil restriction** of X from S' to S is defined as the scheme that represents the functor

$$\mathrm{Res}_{S'/S}(X) : Y/S \mapsto \mathrm{Hom}_{S'}(Y \times_S S', X). \tag{5.11}$$

The Weil restriction of $\mathbb{G}_{m,\mathbb{C}}$ from $\mathbb{C}$ to $\mathbb{R}$ is called the **Deligne torus**, and we denote it by $\mathbb{S}$. Since $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}[x]/(x^2+1) \cong \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i) \cong \mathbb{C} \times \mathbb{C}$ we see that

$$\mathbb{S}(\mathbb{C}) = \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{C}}(\mathrm{Spec}\,\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{Spec}\,\mathbb{C}[t, t^{-1}]) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[t, t^{-1}], \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong (\mathbb{C} \otimes \mathbb{C})^\times \cong \mathbb{C}^\times \times \mathbb{C}^\times \tag{5.12}$$

whence the name Deligne *torus* is justified. Furthermore, this computation shows that $X^*(\mathbb{S}) \cong \mathbb{Z}^2$ is generated by two elements, z and $\overline{z}$. On points, z is defined as projection onto the first coordinate, and $\overline{z}$ as the projection on the second. The reason for the notation comes from the following composition

$$\mathbb{C}^\times \cong \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \xrightarrow{z, \overline{z}} \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^\times. \tag{5.13}$$

Since this fixes $\mathbb{R}$, both z and $\overline{z}$ lies in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. That is, one is identity and the other acts by conjugation. Let also μ denote the cocharacter given on points by $z \mapsto (z, 1)$, $\mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) \to \mathbb{S}(\mathbb{C})$. Thus, $z \circ \mu = \mathrm{id}_{\mathbb{C}}$ and $\overline{z} \circ \mu = 1$. As a remark, a similar computation as in (5.12) shows that $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times$ for any $\mathbb{R}$-algebra A. In particular, $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$.

Now suppose V is a real vector space. Since z and $\overline{z}$ generates $X^*(\mathbb{S})$, each $\lambda \in X^*(\mathbb{S})$ can be written as $z^{-p}\overline{z}^{-q}$ for some $p, q \in \mathbb{Z}$. Thus Proposition (5.2) tells us that giving a real Hodge structure on V is the same as giving a representation $\rho : \mathbb{S} \to GL(V)$ of the Deligne torus, with the correspondence

$$v \in V^{p,q} \iff \rho(z) \cdot v = z^{-p}\overline{z}^{-q} v \text{ for all } z \in \mathbb{C}^\times \cong \mathbb{S}(\mathbb{R}), \tag{5.14}$$

where the minus sign is just a convention. Under this correspondence, a sub-Hodge structure corresponds to a sub-representation.

Note that if V is of weight n, and $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ denotes the cocharacter that corresponds to the inclusion on real points, $\mathbb{R}^\times \cong \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \to \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$, then $\rho \circ w : \mathbb{G}_{m,\mathbb{R}} \to GL(V)_{\mathbb{R}}$ is given on $\mathbb{R}$-points by $a \mapsto a^{-n}\,\mathrm{id}$. Also, since a rational Hodge structure V is just a $\mathbb{Q}$-vector space with a real Hodge structure on $V_{\mathbb{R}}$, we obtain a representation $\rho : \mathbb{S} \to GL(V_{\mathbb{R}})$ defined over $\mathbb{Q}$.

5.1.2 Hodge classes

We define a **Hodge class** (of a $\mathbb{Q}$-Hodge structure V of weight 0) to be an element $v \in V$ such that v lies in $V^{0,0}$ in the Hodge decomposition.

Example 5.3. Suppose V and W are two Hodge structures of weight n. By the definition of tensor products and duals, we see that

$$(V^\vee \otimes W)_{\mathbb{C}} = \bigoplus_{a+b=0} (V^\vee \otimes W)^{a,b}, \tag{5.15}$$

and the Hodge classes are the elements in $V^\vee \otimes W$ belonging to

$$(V^\vee \otimes W)^{0,0} = \bigoplus_{\substack{p+p'=0 \\ q+q'=0}} (V^\vee)^{p,q} \otimes W^{p',q'} = \bigoplus_{\substack{p=-p' \\ q=-q'}} (V^{-p,-q})^\vee \otimes W^{p,q} = \bigoplus_{p+q=n} (V^{p,q})^\vee \otimes W^{p,q}. \tag{5.16}$$

Hence, for each Hodge class $\alpha \otimes \beta \in V^\vee \otimes W$ purely of type (p,q) in the previous computation, and for each $x \in V_{\mathbb{C}}$, α is non-trivial only on the part of x in $V^{p,q}$. Hence, $\alpha(x)\beta \in W^{p,q}$, and thus the morphism $f_{\alpha\otimes\beta} : V \to W$ given by $f_\alpha(v) = \alpha(v)\beta$ does indeed respect the Hodge decomposition. We can (make this rigorous and) reverse this process, and thus obtain

$$\mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}(V,W) = \Big\{ \text{ Hodge classes in } V^\vee \otimes W \Big\}. \tag{5.17}$$

Sometimes Hodge classes are defined on Hodge structures of weight $2p$ as the elements in the underlying vector space purely of type (p,p) in the Hodge decomposition. Given such a Hodge structure V, we can twist it to $V(2p)$, obtaining a Hodge structure of weight 0, and then the Hodge classes (in our sense) are exactly those $v \in V(2p)$ of form $v = (2\pi i)^{2p} v'$, for $v' \in V$ a Hodge class in the other sense. We will actually use both notions, but a main reason for adopting the definition for Hodge structures of weight 0 is to make Hodge classes invariant under the Mumford-Tate group, which we turn to now.

5.1.3 Mumford-Tate groups

This section introduces one of the main objects used to study the Hodge conjecture. The idea is that since the category of Hodge structures is neutral Tannakian, so is each tensor subcategory generated by an object. Hence, all information about the Hodge structure is contained in the automorphism group of the (restriction of the) fibre functor, and in the case of $\mathbb{Q}$-Hodge structures, this group is easily described using the representation of the Deligne torus. We proceed to the definition.

If V is a $\mathbb{Q}$-Hodge structure, and $\rho : \mathbb{S} \to GL(V)_{\mathbb{R}}$ is the corresponding representation of the Deligne torus, then the **Mumford-Tate group** (of V) is defined to be the smallest $\mathbb{Q}$-algebraic subgroup $G \subset GL(V)$ such that ρ factors through $G_{\mathbb{R}}$. We denote it by $MT(V)$. This group is connected (else we replace it by the connected component of identity) and if V is polarisable, then it is also reductive. From the Tannakian viewpoint, the key property of this group is that it determines the invariants of tensor constructions of V. More precisely

Proposition 5.4. *Let V be a $\mathbb{Q}$-Hodge structure with Mumford-Tate group, $MT(V)$. Let $T \coloneqq V^{\otimes a_1} \otimes (V^\vee)^{\otimes b_1} \oplus \cdots \oplus V^{\otimes a_s} \otimes (V^\vee)^{\otimes b_s}$ be a tensor construction of V. Then a subspace $W \subset T$ is a sub-Hodge structure if and only if W is invariant under the action of $MT(V)$.*

Proof. Let $\rho : \mathbb{S} \to GL(V)$ be the representation corresponding to the Hodge structure V, let ρ^T be the representation corresponding to T and let $\mathrm{Stab}_{GL(V)}(W) \subset GL(V)$ be the subgroup that stabilises W

$$\mathrm{Stab}_{GL(V)}(W) \coloneqq \{g \in GL(V) : g \cdot W \subset W\}. \tag{5.18}$$

If W is a sub-Hodge structure, then it is a sub-representation of ρ^T, so in particular it is stable under the action of ρ. Indeed, ρ acts on W through ρ^T as ρ^T is obtained from ρ (through the tensor construction). Hence, $\rho_{\mathbb{R}}$ factors through $\mathrm{Stab}_{GL(V)}(W)_{\mathbb{R}}$ so the definition of a Mumford-Tate group implies that $MT(V) \subset \mathrm{Stab}_{GL(V)}(W)$.

Conversely, suppose $MT(V) \subset \mathrm{Stab}_{GL(V)}(W)$. Then $\rho_{\mathbb{R}}$ factors through $\mathrm{Stab}_{GL(V)}(W)_{\mathbb{R}}$ so W is a sub-representation, and thus sub-Hodge structure. ∎

Corollary 5.5. *Let V and T be as above. An element $t \in T$ is a Hodge class if and only if it is stable under the action of $MT(V)$.*

Proof. Consider $T \oplus \mathbb{Q}(0)$ and the subspace $L := \mathbb{Q} \cdot (t, 1)$. Since the Hodge decomposition of a direct sum of Hodge structures is just the direct sum of the respective decompositions, we see that $L_{\mathbb{C}} = \left(\bigoplus_{\substack{p+q=n \\ (p,q)\neq(0,0)}} T^{p,q} \right) \oplus (T^{0,0} \oplus \mathbb{C})$. Thus, if L is a sub-Hodge structure, the line $L_{\mathbb{C}}$ must lie in one of these components, and since $(t_{\mathbb{C}}, 1)$ is in L we see that it must be in $T^{0,0} \oplus \mathbb{C}$. That is, t must be a Hodge class. Conversely, if t is a Hodge class, then $L_{\mathbb{C}} \subset T^{0,0} \oplus \mathbb{C}$ and thus L is a sub-Hodge structure. In short, L is a sub-Hodge structure if and only if t is a Hodge class. Now, suppose $MT(V)$ fixes t. Then L is invariant under the action of $MT(V)$ (as it acts by identity on $\underline{1} = \mathbb{Q}(0)$), so L is a sub-Hodge structure by Proposition (5.4) and thus t is a Hodge class. Conversely, suppose t is a Hodge class. Then L is a sub-Hodge structure, and hence again we refer to Proposition (5.4) to see that L is stable under the action of $MT(V)$. But $MT(V)$ acting as identity on $\mathbb{Q}(0)$ shows that t must be fixed by $MT(V)$ (else, if $MT(V) \cdot t = \alpha t$, then $MT(V) \cdot (t, 1) = (\alpha t, 1) \notin \mathbb{Q} \cdot (t, 1)$, contradicting L being a sub-Hodge structure, and thus a sub-representation of $MT(V)$). □

Remark 5.6. When one defines Hodge classes as elements of type (p, p) (in a Hodge structure of weight $2p$), the previous result does not hold. This is because for $z \in \mathbb{C}^\times$, the Mumford-Tate group acts on the (p, p) part of the Hodge decomposition by z^{-2p}, i.e. non-trivially. The rational subgroup of $GL(V)$ (for such a Hodge structure V) that fixes the elements of type (p, p) is called the **special Mumford-Tate group**, denoted $SMT(V)$. From this definition it is immedate that $SMT(V) = MT(V)$ when $p = 0$, and one can show that $MT(V) = (\mathbb{G}_m \cdot \mathrm{id}_V)SMT(V)$ otherwise. For example, if $MT(V) = GL(V)$, then $SMT(V) = SL(V)$.

Remark 5.7. Note that, by using $\underline{\mathrm{Hom}}(V, W) = V^\vee \otimes W$ and (5.17), the previous Corollary implies that $\mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}(V, W) = \underline{\mathrm{Hom}}(V, W)^{MT(V \oplus W)}$. In particular, one can show that $MT(V \oplus V) \cong MT(V)$ (acting diagonally on $V \oplus V$), so we get

$$\mathrm{End}_{\mathbb{Q}\mathsf{HS}}(V) = \underline{\mathrm{End}}(V)^{MT(V)}. \tag{5.19}$$

Using this identity together with Proposition (5.4) and a result on representations of reductive algebraic groups, we can obtain the important corollary below.

Corollary 5.8. *Let V be a $\mathbb{Q}$-Hodge structure, and let $\langle V \rangle \subset \mathbb{Q}\mathsf{HS}$ be the full tensor subcategory generated by V. Then $\langle V \rangle \cong \mathrm{Rep}_{\mathbb{Q}}(MT(V))$.*

Proof. See [19], Corollary (4.5). □

5.2 Galois representations

Now let k be a finitely generated field of arbitrary characteristic. Let ℓ be a prime different from $\mathrm{char}(k)$. A **Galois representation** (on a $\mathbb{Q}_\ell$-vector space V) is simply a homomorphism $\Gamma_k := \mathrm{Gal}(k_s/k) \to GL(V)$ that is continuous with respect to the Krull topology on Γ_k and the ℓ-adic topology on $GL(V)$. The category of such representations is denoted $\mathrm{Rep}(\Gamma_k)_{\mathbb{Q}_\ell}$ and is a neutral Tannakian category.

Let $\rho : \Gamma_k \to GL(V)$ be such a representation. A somewhat analogous group to the Mumford-Tate group is the Zariski closure of the image of ρ, which we denote by

$$G_{\ell,\rho}(V) := \overline{\mathrm{im}\,\rho}. \tag{5.1}$$

Just as the Mumford-Tate group of a Hodge structure V determines sub-Hodge structures of all tensor constructions of V, the group $G_{\ell,\rho}$ determines all sub-representations of all tensor constructions ρ (reference [7]). In the Tannakian formalism, this gives $\langle V \rangle \cong \mathrm{Rep}_{\mathbb{Q}_\ell}(G_{\ell,\rho})$.

The case of interest for us is the following. Let X be a smooth, projective algebraic scheme over k_s, and let $X^{(K)}$ be a model of X, i.e. a scheme $X^{(K)}$ over a finitely generated field $k \subset K \subset k_s$ such that $X^{(K)} \times k_s \cong X$. By applying the functors Spec, base change, and $H_\ell^{2r}(-, \mathbb{Q}_\ell(r))$ on each element $g \in \Gamma_k$, we obtain a representation

$$\rho_{\ell,r} : \Gamma_k \to GL(H_\ell^{2r}(X^{(K)} \times k_s, \mathbb{Q}_\ell(r))) \cong GL(H_\ell^{2r}(X, \mathbb{Q}_\ell(r)). \tag{5.2}$$

We denote $G_{\ell,\rho_{\ell,r}}(H_\ell^{2r}(X_{k_s}, \mathbb{Q}_\ell(r)))$ simply by G_ℓ, and its connected component of identity by G_ℓ^0. À priori these groups depend on the choice of model $X^{(K)}$, but [7] Remark 2.2.2(i) explains why G_ℓ^0 is independent of these choices. Corollary (5.5) then motivates a **Tate class** to be defined as an element $x \in H_\ell$ that is stable under the action of G_ℓ^0.

6 The Hodge, Tate, and Mumford-Tate conjectures

We are now in a position to state the conjectures.

Conjecture 6.1. (Hodge conjecture)
Let $k \hookrightarrow \mathbb{C}$ a field, and let $X \in \mathcal{P}(k)$. The cycle class map

$$\mathrm{cl}_B : \mathcal{Z}^r_{\sim_{rat}}(X) \otimes \mathbb{Q} \longrightarrow \left\{ \text{ Hodge classes in } H^{2r}_B(X, \mathbb{Q}(r)) \right\} \tag{6.1}$$

is surjective for all r.

Conjecture 6.2. (Tate conjecture)
Let k be a finitely generated field, and let $X \in \mathcal{P}(k_s)$. The cycle class map

$$\mathrm{cl}_\ell : \mathcal{Z}^r_{\sim_{rat}}(X) \otimes \mathbb{Q}_\ell \longrightarrow \left\{ \text{ Tate classes in } H^{2r}_\ell(X, \mathbb{Q}_\ell(r)) \right\} \tag{6.2}$$

is surjective and the group G^0_ℓ is reductive, for all r.

Remark 6.3. If we ignore the twisting, e.g. if we look at $H^{2r}_B(X, \mathbb{Q})$ instead of $H^{2r}_B(X, \mathbb{Q}(r))$, then the statements of the respective conjectures are the same, we just need to recall the other definition of Hodge classes (mentioned in the end of Section (5.1.1)). In the examples we sketch below, we will ignore all twists.

Finally, if again k is finitely generated and embeddable into $\mathbb{C}$, we then have a comparison isomorphism $H_B \otimes \mathbb{Q}_\ell \to H_\ell$ which gives an isomorphism $\gamma_r : GL(H^{2r}_B(X_{\mathbb{C}}, \mathbb{Q}(r)) \otimes \mathbb{Q}_\ell \to GL(H^{2r}_\ell(X_{\mathbb{C}}, \mathbb{Q}_\ell(r))$.

Conjecture 6.4. (Mumford-Tate conjecture)
The map γ_r restricts to an isomorphism

$$\gamma_r|_{MT} : MT(H^{2r}_B(X_{\mathbb{C}}, \mathbb{Q}(r)) \otimes \mathbb{Q}_\ell \to G^0_\ell(H^{2r}_\ell(X_{\mathbb{C}}, \mathbb{Q}_\ell(r)) \tag{6.3}$$

for all $r \geq 0$.

Remark 6.5. There are some serious technical and important details that we have omitted here. To put it shortly, when X is defined over k, we do not à priori have a Galois action on $X_{\mathbb{C}}$, but rather on $X_{\overline{k}}$. It is then a deep result on étale cohomology (see [6] VI.2) that for a fixed embedding $\overline{k} \hookrightarrow \mathbb{C}$ we have an isomorphism between then étale cohomology on $X_{\mathbb{C}}$ and the étale cohomology on $X_{\overline{k}}$ which induces the Galois action on $H_\ell(X_{\mathbb{C}}, \mathbb{Q}_\ell)$.

Let us sketch or state a few cases that are known.

6.1 Examples

Firstly, if $r = 0$ or $r = \dim X$ then the two conjectures are true immediately as $H_B \cong \mathbb{Q}(0)$ and $H_\ell \cong \mathbb{Q}_\ell(0)$ are generated by the class of X and the class of a closed point, respectively.

6.1.1 Lefschetz theorem on divisor classes

When $r = 1$, the Hodge conjecture is known to be true. This result is called the **Lefschetz theorem on divisor classes**. Let us write a sketch of the proof.

From Section (4.4) we see that the cycle class map is obtained from the exact sequence

$$\cdots H^1(X, \mathcal{O}_X^\times) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \xrightarrow{\alpha} H^2(X, \mathcal{O}_X) \to \cdots . \tag{6.1}$$

obtained from the exponential sequence. Now, the Hodge decomposition of $H^2(X,\mathbb{Q})$ is (upon extending scalars to $\mathbb{C}$) $H^0(X,\Omega^2) \oplus H^1(X,\Omega^1) \oplus H^2(X,\Omega^0)$. We have $\underline{\mathbb{C}} \hookrightarrow \mathcal{O}_X \cong \Omega^0$, and the map α is exactly the projection map $H^2_B(X,\mathbb{C}) \twoheadrightarrow H^2(X,\Omega^0)$. Thus, the Hodge classes, $H^1(X,\Omega^1)$, lie in the kernel of α. Since the sequence (6.1) is exact, the proof is completed.

Using that the Hard Lefschetz theorem holds for Betti cohomology, the Lefschetz theorem on divisor classes implies that the Hodge conjecture is true also for $r = \dim X - 1$, and thus we know it to be true for all smooth, projective, algebraic schemes X over $k \hookrightarrow \mathbb{C}$ of dimension smaller than 4.

6.1.2 Powers of elliptic curves

The point of this section is mainly to indicate how the Mumford-Tate group can be used to prove the Hodge conjecture in some cases. This approach is particularly useful in the case when the scheme is an abelian variety, that is, a projective algebraic variety that is also an algebraic group.

Remark 6.6. (a) : The category of complex abelian varieties is equivalent to the category of polarisable integral Hodge structures of type $\{(-1,0),(0,-1)\}$, i.e. integral Hodge structures H such that $H_{\mathbb{C}} = H^{-1,0} \oplus H^{0,-1}$ (an integral Hodge structure is defined analogously to real and rational Hodge structures).

(b): The category of complex abelian varieties up to isogeny is equivalent to the category of polarisable rational Hodge structures of type $\{(-1,0),(0,-1)\}$.

The equivalence comes from taking an abelian variety A to $H_1(A,\mathbb{Q}) = H^1(A,\mathbb{Q})^\vee$.

Given an abelian variety A, one says that the Mumford-Tate group of A is the Mumford-Tate group of $H_1(A,\mathbb{Q})$, denoted $MT(A)$. By the remark, this is the Mumford-Tate group of a polarisable Hodge structure, and as such it is reductive. We know also that it is connected, and that it contains the homotheties (indeed, we noticed in Section (5.1.1) that if $\rho : \mathbb{S} \to GL(H_1(A,\mathbb{Q}))$ is the representation corresponding to $MT(A)$, then each $z \in \mathbb{C}^\times$ maps through the composition $\mathbb{G}_m(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) \to GL(H_1(A,\mathbb{Q}))$ to $z^{-n} \cdot \mathrm{id}$, and by definition of the Mumford-Tate group this lies in $MT(A)$). Using this information one can sometimes get an explicit description of what $MT(A)$ is.

Let us sketch an example (which can be found in [7] 2.1.13 and further explained in [9] section 3). Let $A = E^k$ be a fixed product of an elliptic curve (i.e. a smooth, projective algebraic curve of genus one). There are finitely many possible types for the endomorphism algebra $\mathrm{End}(E) \otimes \mathbb{Q}$, one of which is $\mathbb{Q}$. Suppose we are in this situation. Since $H_1(E,\mathbb{Q})$ has dimension 2, $GL(H_1(E,\mathbb{Q})) \cong GL_2(\mathbb{Q})$. It is known that the only connected, reductive subgroup $G \subset GL_2(\mathbb{Q})$ that contains the homotheties and satisfies $\mathrm{End}(\mathbb{Q}^2)^G \cong \{\mathbb{Q} \cdot \mathrm{id}\}$ is $GL_2(\mathbb{Q})$. Thus, if we can show that $\mathrm{End}(H_1(E,\mathbb{Q}))^{MT(E)} \cong \mathbb{Q} \cdot \mathrm{id}$, then we know that $MT(E) = GL(H_1(E,\mathbb{Q}))$. To this end, note that the equivalence of categories gives an isomorphism

$$\mathbb{Q} \cdot \mathrm{id} \cong \mathbb{Q} \cong \mathrm{End}(E) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}\Big(H_1(E,\mathbb{Q}), H_1(E,\mathbb{Q})\Big). \tag{6.2}$$

Furthermore, from the identity in (5.19) we see that

$$\mathrm{End}(H_1(E,\mathbb{Q}))^{MT(E)} \cong \mathrm{Hom}_{\mathbb{Q}\mathsf{HS}}\Big(H_1(E,\mathbb{Q}), H_1(E,\mathbb{Q})\Big). \tag{6.3}$$

Combining the two equations we conclude from the previous discussion that $MT(E) = GL(H_1(E,\mathbb{Q}))$.

Now let us throw in two more black boxes; (i) the whole cohomology ring, $H(A,\mathbb{Q})$, of A is generated by $H^1(E,\mathbb{Q})$ through the exterior power, i.e. $H^*(A,\mathbb{Q}) = \bigwedge^*(H^1(E,\mathbb{Q})^{\oplus k})$, (ii) for any rational vector space V, the ring of $SL_2(\mathbb{Q})$-invariants in $\bigwedge^\bullet V^{\otimes k}$ is generated by the invariant elements of degree 2 (a proof of (i) can be found in [21], and an explanation for (ii) can be found in [8]).

Why do we want (i) and (ii)? Since $H^*(A,\mathbb{Q}) = \bigwedge^*(H^1(E,\mathbb{Q})^{\oplus k})$, the elements of degree two are exactly $H^2(A,\mathbb{Q})$, and we know that the elements in $H^2(A,\mathbb{Q})$ that are stable under $SMT(E)$ are exactly the Hodge classes in degree two. From the Lefschetz theorem on divisor classes, we also know that they are algebraic. Thus, since $MT(E) = GL_2(\mathbb{Q})$ implies that $SMT(E) = SL_2(\mathbb{Q})$, (ii) implies that the Hodge classes in degree 2 generate all elements invariant under the action of $SMT(E)$, namely, all Hodge classes. Hence, the Hodge conjecture is true for A.

In fact, the Hodge conjecture is true for arbitrary (finite) products of elliptic curves, and in [9] there is a more detailed discussion on both this example and others, using a similar approach.

6.1.3 Discussion on the Tate conjecture

Here we just state a few known cases of the Tate conjecture, which we abbreviate by (TC). For a more in-depth and complete discussion of the results stated here, please see [10].

The (TC) is true for divisor classes (i.e. $r = 1$) of abelian varieties over finite fields (proven by Tate in [11]) and for divisor classes of abelian varieties over number fields (proven by Faltings in [12]). The proof of the Lefschetz theorem on divisor classes indicates that the dimensions of the $H^{p,q}$-terms for $p \neq q$ can give a lot of information. There is a notion called Kuga-Saka correspondence which tells us that if a Hodge structure H with a decomposition $H_{\mathbb{C}} = H^{p,q} \oplus H^{p+q,p+q} \oplus H^{q,p}$, satisfies $\dim H^{p,q} = \dim H^{q,p} = 1$, then it is the direct sum of the Hodge structure corresponding to an abelian variety. The knowledge of abelian varieties can therefore help in other cases as well. For instance, (TC) is true for K3 surfaces over number fields (proved by Tankeev in [23] using this technique). Moonen also shows that for varieties over $\mathbb{C}$ which satisfy $\dim H^{2,0} = 1$, upon making some restrictions on their moduli, the Tate conjecture for divisor classes is true (see [22]). The (TC) is also true for divisors on irreducible hyperkähler varieties (definition omitted) over number fields (see [24]). There has also been significant progress in proofs of certain K3 surfaces over finite fields, but we leave that discussion to [10].

6.1.4 Relations between the conjectures

A particularly interesting feature of the Mumford-Tate conjecture is that it connects the other two conjectures.

Proposition 6.7. *If the Mumford-Tate conjecture is true for all $r \geq 0$, then the Hodge conjecture is equivalent to the Tate conjecture.*

Proof. Suppose the Mumford-Tate conjecture is true. Let $r \geq 0$ be fixed, let $H_B \coloneqq H_B^{2r}(X,\mathbb{Q}(r))$, $H_\ell \coloneqq H_\ell^{2r}(X,\mathbb{Q}_\ell(r))$, and let $\alpha : H_B \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H_\ell$ be the isomorphism giving $\gamma_r : MT(H_B) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} G_\ell^0$. Then γ_r acts by $f \mapsto \alpha \circ f \circ \alpha^{-1}$, and hence each Hodge class x maps under α to a Tate class. Indeed, each g in G_ℓ^0 is of the form $\gamma_r(f)$ for some $f \in MT(H_B) \otimes \mathbb{Q}_\ell$, and thus

$$g \cdot \alpha(x) = \alpha \circ f \circ \alpha^{-1}\alpha(x) = \alpha(x) \tag{6.4}$$

as f fixes the Hodge class x. Since α commutes with the cycle class maps, this shows the proposition. □

In particular, by the Lefschetz theorem on divisor classes, we see that the Mumford-Tate conjecture would imply the Tate conjecture for $r = 1$.

7 Motives

Up to this point we have, with respect to some X in $\mathcal{P}(k)$, had a parallel discussion on Hodge structures and Betti cohomology on the one hand, and Galois actions and ℓ-adic cohomology on the other. These two sides we have seen share certain features. On both sides it is conjectured (Hodge respectively Tate) that certain classes in the cohomology are algebraic, and on both of these sides we have seen that these classes are controlled by certain groups. Furthermore, on both sides we are working with neutral Tannakian categories and the groups that control the classes just mentioned are in fact the tensor automorphism groups of the fibre functors.

The idea of motives (partly) comes from this connection and further deepens it. It was Grothendieck that envisioned, upon noting the many similarities between different Weil cohomologies, that there should exist some "universal cohomology theory" through which they all factor. This envisioned category can also be explained as sort of a "universal Tannakian category".

In the context of the three conjectures presented in this text, a category of motives would serve two particularly important functions. Firstly, the hope is that in this "universal" category called motives, we have a notion of "motivated classes", analogous to the Hodge respectively Tate classes, capturing the information of both. One would then hope that it would be possible to break down the Hodge and Tate conjectures into two steps; show first that Hodge (respectively Tate) classes are motivated classes, then show that motivated classes are algebraic. The second important function such a category would serve in the context of these conjectures is to give a unified "universal", "motivic" Galois theory. This Galois theory would then serve as a tool to understand the different groups (the Mumford-Tate group and what we denoted G_ℓ and G_ℓ^0) coming from the different cohomology theories.

This last section hopes to introduce the concept of motives in three steps. First, we will define the notion of pure motives (as we mentioned was originally constructed by Grothendieck). Connecting to Theorem (2.10) we will see that this category is not (unconditionally) a neutral Tannakian cateogory. We will thus turn to another approach of constructing such a category, which was done by Yves André (see [13]). We sketch the construction of the category referred to as "André's category of motives". Then we use this to define the notion of a motivic Galois group, and state a motivic version(s) of the Mumford-Tate conjecture. This section will mostly be a collection of definitions and propositions without proofs, but we hope that the reader will find it useful as a final connection of the conjectures presented, as well as a brief look into one of the tools people use to try and solve these conjectures.

For further reference, we particularly promote [15] for a thorough treatment of the concept. The short text [20] is a good first read, [7] explains the motivic Mumford-Tate conjectures great and [13] treats in detail the construction of André's category of motives.

For all that follows, let $\mathcal{P}(k)$ denote the category of smooth projective schemes over a field k.

7.1 Adequate equivalence relations

Firstly, let us introduce the notion of an adequate equivalence relation. Recall that $\mathcal{Z}^*(X)$ is the group of algebraic cycles. An equivalence relation $\sim$ on $\mathcal{Z}^*(X)$ is said to be an **adequate equivalence relation** if it satisfies three properties. Firstly, it should respect the linear structure of $\mathcal{Z}^*(X)$. If we are looking at $\mathcal{Z}^*(X) \otimes \mathsf{Q}$ for some scalars Q, then this also means that $\sim$ should respect the Q-linearity. Secondly, we want to be able to intersect cycles. Precisely, for each α, β in $\mathcal{Z}^*(X)$ there must be some α' in $\mathcal{Z}^*(X)$ with $\alpha \sim \alpha'$ such that α' and β intersects properly (i.e. for each subvariety Z occuring as a term in α'

and each subvariety W occuring as a term in β, we have $\dim Z \cap W = \dim Z + \dim W - \dim X$). Finally, the third requirement is a kind of functoriality condition. It says that for any Y in $\mathcal{P}(k)$, and any γ in $\mathcal{Z}^*(X \times Y)$ and α in $\mathcal{Z}^*(X)$ such that γ and $(p_X^{XY})^*\alpha$ intersects properly, we have that $\alpha = 0$ implies that $\gamma_*(\alpha) := (p_Y^{XY})_*(\gamma \cdot (p_X^{XY})^*\alpha) = 0$. Here $p_X^{XY} : X \times Y \to X$ is the projection map, and similarly for all such notations (to come).

We will not discuss in depth any class of algebraic cycles, but we mention four common examples of adequate equivalence relations.

Example 7.1. (a): Rational equivalence was defined above. If one replaces $\mathbb{P}^1$ by any smooth projective curve, and proceeds verbatim, one gets **algebraic equivalence**.

(b): One says that α is **homologically equivalent** to 0, denoted $\alpha \sim_{\text{hom}} 0$, if $\text{cl}(\alpha) = 0$.

(c). One says that $\alpha \in \mathcal{Z}^r_{\text{rat}}(X)$ is **numerically equivalent** to 0, if, for each $\beta \in \mathcal{Z}^{\dim X - r}_{\text{rat}}(X)$, one has $\deg \alpha \cdot \beta = 0$. To explain this, note first that by definition, we can choose α, β such that α and β intersects properly, e.g. if $\alpha = [Z]$, $\beta = [V]$, then $\alpha \cdot \beta = [Z \cap V] = [P]$ is a closed point. In general, this gives us $\alpha \cdot \beta = \sum n_i[P_i]$, for finitely many closed points $P_i \in X$. The **degree** of such a sum is defined to be

$$\deg\left(\sum n_i[P_i]\right) := \sum n_i[\kappa(P_i) : k], \tag{7.1}$$

where $\kappa(P_i)$ is the field corresponding to the closed point P_i.

One says that an equivalence relation $\sim_1$ is **finer** than another such relation $\sim_2$ if $\alpha \sim_1 \beta$ implies $\alpha \sim_2 \beta$, for all cycles α, β. In this case one also says that $\sim_2$ is **coarser** than $\sim_1$. If we denote by $\sim_1 \succeq \sim_2$ the statement that $\sim_1$ is finer than $\sim_2$, then we have

$$\sim_{\text{rat}} \;\succeq\; \sim_{\text{alg}} \;\succeq\; \sim_{\text{hom}} \;\succeq\; \sim_{\text{num}} . \tag{7.2}$$

Now fix an $X \in \mathcal{P}(k)$, fix a Weil cohomology theory H with coefficients in Q, fix an adequate equivalence relation $\sim$ and denote $\mathcal{Z}^*(X)_{\mathsf{Q}}/\sim$ by $\mathcal{Z}^*_\sim(X)_{\mathsf{Q}}$. We proceed to define the categor(ies) of motives.

7.2 Pure motives

The idea is to tweak the category $\mathcal{P}(k)$ by enlarging the set of morphisms through the notion of algebraic cycles and through categorical procedures force it to be (pseudo-)abelian with duals. This is expressed in the following three steps.

STEP 1. Recall from (4.3) that an element in $\mathcal{Z}^{\dim X}_\sim(X \times Y)_{\mathsf{Q}}$ induces an element in $\text{Hom}(H(X), H(Y))$. With this in mind, the **category of correspondences** modulo $\sim$ with coefficients in Q is defined to be the category, denoted $\mathcal{C}_\sim(k)_{\mathsf{Q}}$, which has the same objects as $\mathcal{P}(k)$, and morphisms

$$\text{Hom}_{\mathcal{C}_\sim(k)_{\mathsf{Q}}}(X, Y) := \mathcal{Z}^{\dim X}_\sim(X \times Y)_{\mathsf{Q}}. \tag{7.1}$$

The composition law is given by

$$\begin{aligned} &\text{Hom}_{\mathcal{C}_\sim(k)_{\mathsf{Q}}}(X, Y) \times \text{Hom}_{\mathcal{C}_\sim(k)_{\mathsf{Q}}}(Y, Z) \to \text{Hom}_{\mathcal{C}_\sim(k)_{\mathsf{Q}}}(X, Z) \\ &\qquad\qquad (\alpha, \beta) \mapsto \beta \circ \alpha := (p_{XZ}^{XYZ})_*\Big((p_{XY}^{XYZ})^*\alpha \cdot (p_{YZ}^{XYZ})^*\beta\Big) \end{aligned} \tag{7.2}$$

which is well-defined for all adequate equivalence relations coarser than (or equal to) rational equivalence (see [4] 16). Through the group structure of the algebraic cycles, this is an additive category, with direct sum $X \oplus Y := X \coprod Y$, and by defining $X \otimes Y := X \times_k Y$ it is also a tensor category. We also have a contravariant embedding $\mathcal{P}(k) \to \mathcal{C}_\sim(k)_{\mathsf{Q}}$ given by sending X to X and $f : Y \to X$ to the transpose of its graph, ${}^t\Gamma_f$ (the tranpose is just the composition $Y \xrightarrow{\Gamma_f} Y \times X \to X \times Y$).

However, this category is not abelian, and to (try and) change that, we pass to the pseudo-abelian envelope. In short, we formally add the images of idempotents.

STEP 2. Let $\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$ denote the category with objects (X, e) where X is in $\mathcal{P}(k)$ and e is an idempotent in $\text{Hom}_{\mathcal{C}_\sim(k)_{\mathsf{Q}}}(X, X)$, i.e. $e^2 = e$. If $(X, e), (X', e')$ are two objects in $\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$, then the set of morphisms between (X, e) and (X', e') is

$$\text{Hom}_{\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}}\Big((X, e), (X', e')\Big) := e' \circ \mathcal{Z}_\sim^{\dim X}(X \times X') \circ e, \tag{7.3}$$

where composition is the one previously defined in $\mathcal{C}_\sim(k)_{\mathsf{Q}}$. We embed $\mathcal{C}_\sim(k)_{\mathsf{Q}}$ into $\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$ by taking X to (X, Δ_X) and $f : X \to Y$ to $\Delta_Y \circ f \circ \Delta_X$, where Δ_X denotes the diagonal morphism $\Delta_X : X \to X \times X$. Composing with $\mathcal{P}(k) \to \mathcal{C}_{\text{rat}}(k)_{\mathsf{Q}}$ this gives a contravariant embedding $h : \mathcal{P}(k) \to \mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$. The category $\mathcal{M}_{\text{rat}}^{\text{eff}}(k)_{\mathsf{Q}}$ is called the category of **pure effective motives** modulo $\sim$ with coefficients in k.

Remark 7.2. Oen can use the above construction to show that to any additive category $\mathcal{A}$ (the role of which was played by $\mathcal{C}_\sim(k)_{\mathsf{Q}}$ above) there exists a pesudo-abelian category $\mathcal{A}^\#$ and a covariant additive functor $\mathcal{A} \to \mathcal{A}^\#$ that is universal for all such functors from $\mathcal{A}$ to pseudo-abelian categories.

But we want our final category of motives to be Tannakian, so in particular we must have a concept of duals.

STEP 3. The embedding $h : \mathcal{P}(k) \to \mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$ takes $\mathbb{P}^1$ to $(\mathbb{P}^1, \Delta_{\mathbb{P}^1})$, and as can be seen from Example (3.2) we have a decomposition

$$h(\mathbb{P}^1) = h^0(\mathbb{P}^1) + h^2(\mathbb{P}^1). \tag{7.4}$$

The object $h^2(\mathbb{P}^1)$ is the so-called **Lefschetz motive**, also denoted $\mathbb{L}$. The procedure of formally inverting $\mathbb{L}$ gives us the category of **pure motives** modulo $\sim$ with coefficients in Q, which we denote by $\mathcal{M}_\sim(k)_{\mathsf{Q}}$. The objects are now triples, (X, e, r), where X and e are as in $\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}}$, and r is an integer (this is thought of as the degree of twisting). The morphisms between two objects (X, e, r) and (X', e', r') are defined to be

$$\text{Hom}_{\mathcal{M}_\sim(k)_{\mathsf{Q}}}\Big((X, e, r), (X', e', r')\Big) := e' \circ \mathcal{Z}_\sim^{r'-r}(X \times X') \circ e. \tag{7.5}$$

We also have an embedding $\mathcal{M}_\sim^{\text{eff}}(k)_{\mathsf{Q}} \to \mathcal{M}_\sim(k)_{\mathsf{Q}}$, which gives an embedding (still denoted h) $h : \mathcal{P}(k) \to \mathcal{M}_\sim(k)_{\mathsf{Q}}$. One usually denotes the object (X, e, r) by $eh(X)(r)$, and we will use both notations. The tensor and direct sum of pure motives is the same as for pure effective motives (and we add up the degrees of twisting), and the dual of (X, e, r) is $(X, {}^te, \dim X - r)$.

7.3 Bridge

What can be said about the category of pure motives? Firstly, by sending (X, e, r) to $eH(X)(r)$ where e denotes the induced morphism on cohomology, we do get that Weil cohomologies factor through $\mathcal{M}_{\text{rat}}(k)$. By definition, this also holds for all adequate equivalence relations finer than (or equal to) homological equivalence. However, it is not certain that Weil cohomologies factor through $\mathcal{M}_{\text{num}}(k)$ since we do not know if numerical equivalence implies homological equivalence. This (partly) motivates the so-called

Conjecture 7.3. (Standard Conjecture D)
Numerical equivalence implies homological equivalence.

This was phrased by Grothendieck and is still unknown in general. It is however known that the Hodge conjecture would imply the Standard Conjecture D.

What more? For all adequate equivalence relations we do get that $\mathcal{M}_{\sim}(k)$ is an additive, pseudo-abelian rigid tensor category. We could then ask if $\mathcal{M}_{\sim}(k)$ semisimple abelian, and in [14] Theorem 1 it is shown that this is true if and only if the equivalence relation is numerical equivalence.

The category of pure numerical motives $\mathcal{M}_{\text{num}}(k)$ is not neutral Tannakian either. This is because the rank of an object $\mathsf{M} = (X, \Delta_X, 0)$ is

$$\operatorname{rk}(\mathsf{M}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H^i(X), \tag{7.1}$$

which can be negative (e.g. take X to be an elliptic curve of genus $g \geq 2$).

In $\mathsf{Vec}_{\mathsf{Q}}$ the cohomology $H(X)$ has a decomposition, a grading,

$$H(X) = H^0(X) \oplus \cdots \oplus H^{2\dim X}(X), \tag{7.2}$$

and to this corresponds the so-called **Künneth projectors**

$$\pi^i : H(X) \twoheadrightarrow H^i(X). \tag{7.3}$$

(Even though we do not know if H factors through $\mathcal{M}_{\text{num}}(k)$) we would like to think that this grading comes from within $\mathcal{M}_{\text{num}}(k)$. Thus, we want that

$$(X, \Delta_X, 0) = \bigoplus_i (X, \pi^i, 0) \tag{7.4}$$

for some idempotent correspondences (also denoted) π^i that are pair-wise orthogonal, i.e. $\pi^i \circ \pi^j = 0$ for $i \neq j$. For example, we saw in Example (3.2) that we indeed have this decomposition for $X = \mathbb{P}^1$. If this was true in general, then (following the procedure sketched in the proof of Theorem 7.6) one could "easily" alter $\mathcal{M}_{\text{num}}(k)$ to become a neutral Tannakian category. However, such a decomposition is not known to exist in general, which motivates the Standard Conjecture C.

Conjecture 7.4. (Standard Conjecture C)
The Künneth projectors are algebraic.

There is also another important Standard Conjecture, which says

Conjecture 7.5. (Standard Conjecture B)
The Lefschetz involution (see (7.5) below) is algebraic.

The Lefschetz involution is defined as follows. Let X in $\mathcal{P}(k)$ be irreducible of dimension d, let η denote the image of an ample divisor in $H^2(X)$, let L denote the Lefschetz morphism corresponding to the ample divisor (recall Example (5.1)) and assume the strong Lefschetz theorem holds for H. That is, $L^i : H^{\dim X - i}(X) \to H^{\dim X + i}(X)$ is an isomorphism. Recall the Lefschetz decomposition $H^n(X) = \bigoplus_i L^i H^{n-2i}_{\text{prim}}(X)$. The **Lefschetz involution** corresponding to L, denoted $*_L$, is defined as the morphism that on $x = \sum_i L^i x_{n-2i}$ in $H^n(Z)$, acts as

$$*_L\, x \coloneqq L^{\dim X - n} x = \sum_i L^{\dim X + i - n} x_{n-2i}. \tag{7.5}$$

It is known that Conjecture B implies the other two in characteristic zero (of k), and when k is arbitrary it does so when combined with a similar conjecture for the Hodge involution (definition omitted). Since the

Hodge conjecture is known to imply Conjecture B in characteristic zero, the Hodge conjecture implies all the Standard conjectures. For the étale cohomology the Tate conjecture implies Conjecture B for all k, and thus it implies all Standard conjectures for k of characteristic zero. For more on the relations between the Hodge and Tate conjectures and the Standard conjectures, see [15].

Next we turn to a sketch of André's construction of pure motives. To put it simply, the idea is to force the Lefschetz involution to be algebraic.

7.4 Andrés category of motives

Now let X in $\mathcal{P}(k)$ be irreducible. Since a reoccurring problem has been the lack of (ways to construct) algebraic cycles, the idea is to formally add the Lefschetz involution of (the image under the cycle class map of) algebraic cycles. Precisely, for each other Y in $\mathcal{P}(k)$ irreducible, if L_X and L_Y are ample divisors, let L be the ample divisor on $X \times Y$ corresponding to $(p_X^{XY})^* L_X + (p_Y^{XY})^* L_Y$, and let $* = *_L$. Then we define a **motivated cycle** to be an element of the form

$$(p_X^{XY})_* \Big(\operatorname{cl}(\alpha) \cup * \operatorname{cl}(\beta) \Big) \tag{7.1}$$

where α, β are algebraic cycles on $X \times Y$. Let $A^*_{\text{mot}}(X)$ denote the set of all motivated cycles (if we look at algebraic cycles with coefficients in E, then we denote by $A^*_{\text{mot}}(X)_E$ the corresponding motivated cycles). By definition of the pushforward this is a subalgebra of $H^{2*}(X)$ relative to the cup-product, and it is stable under both push-forward and pullbacks, i.e. $(p_X^{XY})^*(A^*_{\text{mot}}(X)) \subset A^*_{\text{mot}}(X \times Y)$ and $(p_X^{XY})_*(A^*_{\text{mot}}(X \times Y)) \subset A^*_{\text{mot}}(X)$ (see [13] Proposition 2.1). Thus, the composition

$$\begin{aligned} A^*_{\text{mot}}(X \times Y) \times A^*_{\text{mot}}(Y \times Z) &\to A^*_{\text{mot}}(X \times Z) \\ (f, g) &\mapsto (p_{XZ}^{XYZ})_* \Big((p_{XY}^{XYZ})^* f \cup (p_{YZ}^{XYZ})^* g \Big) \end{aligned} \tag{7.2}$$

is well-defined.

Now let $X = \coprod X_i$ and $Y = \coprod Y_j$ in $\mathcal{P}(k)$ be arbitrary, where each X_i and each Y_j is irreducible, and let $r = (r_{ij})$, $r_{ij} \in \mathbb{Z}$. Then the **motivated correspondences** of degree r between X and Y is

$$\operatorname{Corr}^r_{\text{mot}}(X, Y) \coloneqq \bigoplus_{i,j} A^{\dim X_i + r_{ij}}_{\text{mot}}(X_i \times Y_j), \tag{7.3}$$

and the algebra of motivated correspondences is

$$\operatorname{Corr}^*_{\text{mot}}(X, Y) \coloneqq \bigoplus_{r} \operatorname{Corr}^r_{\text{mot}}(X, Y). \tag{7.4}$$

When we look at algebraic cycles with coefficients in E the correspondences are denoted $\operatorname{Corr}^*_{\text{mot}}(X, Y)_E$. As noted, we have a composition law on this (by composing on each irreducible component at a time).

Finally, the **André category of (pure) motives** over k with coefficients in E is the category $\mathsf{Mot}(k)_E$ with objects being triples $\mathsf{M} = (X, e, r)$, where $X \in \mathcal{P}(k)$, $e \in \operatorname{Corr}^0_{\text{mot}}(X \times X)_E$ an idempotent, and r an integer, and with morphisms

$$\operatorname{Hom}_{\mathsf{Mot}(k)_{\mathbb{Q}}} \Big(e\mathsf{h}(X)(r), e'\mathsf{h}(X')(r') \Big) = e' \circ \operatorname{Corr}^{r'-r}_{\text{mot}}(X, X')_E \circ e. \tag{7.5}$$

The embedding $\mathsf{h} : \mathcal{P}(k) \to \mathsf{Mot}(k)$ is given by $X \mapsto (X, \Delta_X, 0)$ as before, and we also denote (X, e, r) by $e\mathsf{h}(X)(r)$.

From now on, fix k to be of characteristic zero, fix H to be one of the classical Weil cohomologies (i.e. Betti, de Rham or étale), and let the coefficients of the algebraic cycles above be $\mathbb{Q}$. For simplicity let $\mathsf{Mot}(k) = \mathsf{Mot}(k)_{\mathbb{Q}}$. We then have the following.

Theorem 7.6. *André's category of pure motives with coefficients in $\mathbb{Q}$ is an abelian, semisimple neutral Tannakian category.*

Proof. We only state the relevant objects.

For $\mathsf{M} = (X, e, r)$ and $\mathsf{N} = (Y, f, s)$, the tensor product and internal Hom are

$$\begin{aligned} \mathsf{M} \otimes \mathsf{N} &\coloneqq (X, e, r) \otimes (Y, f, s) \coloneqq (X \times Y, e \times f, r + s) \\ \underline{\mathrm{Hom}}(\mathsf{M}, \mathsf{N}) &\coloneqq (X \times Y, {}^t e \times f, \dim X - r + s). \end{aligned} \tag{7.6}$$

Since $\underline{1} \coloneqq (\operatorname{Spec} k, \Delta_{\operatorname{Spec} k}, 0)$ the dual of M is thus seen to be

$$\mathsf{M}^\vee = \underline{\mathrm{Hom}}(e\mathsf{h}(X)(r), \underline{1}) = (X, {}^t e, \dim X - r) = {}^t e\mathsf{h}(X)(\dim X - r). \tag{7.7}$$

There is also a notion of twisting, defined as

$$\mathsf{M}(n) \coloneqq \mathsf{M} \otimes \mathbb{Q}(n), \tag{7.8}$$

where $\mathbb{Q}(n) \coloneqq (\operatorname{Spec} k, \Delta_{\operatorname{Spec} k}, n)$. There is an "obvious" commutativity constraint, namely $(X \times Y, e \times f, r + s) \cong (Y \times X, f \times e, r + s)$, coming from the isomorphism $X \times Y \cong Y \times X$. However, using this would give the same problem regarding the rank as with the pure motives defined earlier. To solve this, André (see [13] Proposition 2.2) proved that the motivic correspondences contains the Künneth components of the diagonal, so for each $\mathsf{M} = eH(X)(r)$ one can define a $\mathbb{Z}$-grading by

$$\mathsf{M} = \bigoplus_n \mathsf{M}^n, \quad \mathsf{M}^n = (X, e, r)^n \coloneqq (X, e\pi^{n+2r}, n), \tag{7.9}$$

where the π^i's are the Künneth components of the diagonal. Then the "obvious" commutativity constraint $\psi_{\mathsf{M},\mathsf{N}} : \mathsf{M} \otimes \mathsf{N} \to \mathsf{N} \otimes \mathsf{N}$ can be written as

$$\bigoplus_{i,j} \psi_{\mathsf{M}_i,\mathsf{N}_j} : \bigoplus_{i,j} \mathsf{M}^i \otimes \mathsf{N}^j \to \bigoplus_{i,j} \mathsf{N}^j \otimes \mathsf{M}^i, \tag{7.10}$$

and to force this to give a non-negative rank one changes $\psi_{\mathsf{M}_i,\mathsf{N}_j}$ to $\psi'_{\mathsf{M}_i,\mathsf{N}_j} \coloneqq (-1)^{ij} \psi_{\mathsf{M}_i,\mathsf{N}_j}$ and

$$\psi'_{\mathsf{M},\mathsf{N}} \coloneqq \bigoplus_{i,j} \psi'_{\mathsf{M}_i,\mathsf{N}_j}. \tag{7.11}$$

This produces

$$\mathrm{rk}(\mathsf{M}) = \sum_{i \geq 0} \dim eH^i(X) \geq 0. \tag{7.12}$$

For a more detailed explanation and computations, please see [14] Lemma 1 and Corollary 2. In [13] Proposition 3.3 (combined with [14] Lemma 2) it is also shown that the category is semisimple abelian, so by Theorem (2.10) this shows that $\mathsf{Mot}(k)$ is neutral Tannakian. ☐

Finally, the **motivic cohomology** of X in $\mathcal{P}(k)$ is defined as the image of X under the functor $\mathsf{h} : X \mapsto (X, \Delta_X, 0)$ and $f : X \to Y \mapsto \Delta_X \circ {}^t\Gamma_f \circ \Delta_Y$. Further, we define $H_{\mathrm{mot}} : \mathsf{Mot}(k)_{\mathsf{Q}} \to \mathsf{Vec}_{\mathsf{Q}}$ by $H_{\mathrm{mot}}(e\mathsf{h}(X)(r)) \coloneqq eH(X)(r)$, where e is the endomorphism $H(X) \to H(X)$ obtained from the Künnet formula and Poincaré duality. This lets us finally state that through this procedure, each classical Weil cohomology H with coefficients in Q factors through $\mathsf{Mot}(k)$,

$$\begin{array}{ccc} \mathcal{P}(k) & \xrightarrow{\quad H \quad} & \mathsf{Vec}_{\mathsf{Q}} \\ & {\scriptstyle \mathsf{h}} \searrow \quad \nearrow {\scriptstyle H_{\mathrm{mot}}} & \\ & \mathsf{Mot}(k)_{\mathsf{Q}} & \end{array} \quad . \tag{7.13}$$

The functor H_{mot} is often referred to as the **realisation functor** of the cohomology theory H (the term "realisation functor" is not specific to André's category of motives). For example, when H is the Betti cohomology, then $H_{\mathrm{mot}}(e\mathsf{h}(X)(r)) = eH_B(X, \mathbb{Q}(r))$.

7.5 Motivic Galois groups and motivic Mumford-Tate conjecture(s)

Let H be a Weil cohomology theory. Because we have a fibre functor $H_{\mathrm{mot}} : \mathsf{Mot}(k)_{\mathsf{Q}} \to \mathsf{Vec}_{\mathsf{Q}}$ we can consider its automorphism group, which is called the **motivic Galois group**, denoted $\mathcal{G}_{\mathrm{mot},H,k} =: \mathcal{G}_H$. For each M in $\mathsf{Mot}(k)_{\mathsf{Q}}$ we can also restrict H_{mot} to the tensor category $\langle \mathsf{M} \rangle$ generated by M, and this gives us the **motivic Galois group of** M, denoted $\mathcal{G}_H(\mathsf{M})$. By the main theorem on Tannakian categories, Theorem (2.9), we see that

$$\mathsf{Mot}(k)_{\mathsf{Q}} \cong \mathrm{Rep}_{\mathsf{Q}}(\mathcal{G}_H) \tag{7.1}$$

and similarly

$$\langle \mathsf{M} \rangle \cong \mathrm{Rep}_{\mathsf{Q}}(\mathcal{G}_H(\mathsf{M})). \tag{7.2}$$

Remark 7.7. Under this correspondence, $\mathcal{G}_H(\mathsf{M})$ can also be viewed as the image of the representation $\mathcal{G}_H \to GL(H_{\mathrm{mot}}(\mathsf{M}))$. In particular, a motive M is on the one hand a "purely" geometric object, being a triple (X, e, r) consisting of a scheme and a (generalised version of an) endomorphism of the scheme, and on the other hand it is a "purely" algebraic object, i.e. from the above equivalence the motive M is the representation $\mathcal{G}_H \to GL(H_{\mathrm{mot}}(\mathsf{M}))$ corresponding to it.

Remark 7.8. Another important remark is that, if H and H' are two classical Weil cohomology theories, with coefficients in Q respectively Q' and with a comparison isomorphism $H \otimes \mathsf{Q}'' \cong H' \otimes \mathsf{Q}''$ for a common extension $\mathsf{Q} \subset \mathsf{Q}'' \supset \mathsf{Q}'$, then, for each $\mathsf{M} \in \mathsf{Mot}(k)$, the motivic Galois groups $\mathcal{G}_H(\mathsf{M})$ and $\mathcal{G}_{H'}(\mathsf{M})$ are isomorphic upon tensoring with Q''. The isomorphism is the restriction of the map $GL(H_{\mathrm{mot}}(\mathsf{M})) \otimes \mathsf{Q}'' \xrightarrow{\sim} GL(H'_{\mathrm{mot}}(\mathsf{M})) \otimes \mathsf{Q}''$ coming from the comparison theorem.

For a motive $\mathsf{M} = (X, e, r)$, a **motivated cycle** or a **motivated class** is defined to be an element x in $H_{\mathrm{mot}}(\mathsf{M}) = eH(X)(r)$ that is of the form $x = e(\psi)$ for some $\psi \in A^r_{\mathrm{mot}}(X)_{\mathsf{Q}} \subset H^{2r}(X)(r)$, and these are exactly the elements fixed by the action of $\mathcal{G}_H(\mathsf{M})$. By the definition of motivated cycles through cycle class maps, and since pushforwards are morphisms of Hodge structures, it is immediate that when H_{mot} is the Betti realisation functor, motivated classes are Hodge classes (and similarly for Tate classes). The main issue is to show the converse. We state an instance of this after defining the motivic Mumford-Tate conjecture(s) now.

7.5.1 Motivic Mumford-Tate conjecture(s)

Now we fix our attention to the Betti and ℓ-adic realisation functors, denoted H_B and H_ℓ respectively. Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$, fix an algebraic closure $k \hookrightarrow \overline{k}$, and fix an embedding $\overline{\sigma} : \overline{k} \hookrightarrow \mathbb{C}$ such that $\overline{\sigma}|_k = \sigma$. Upon base changing along these maps, we thus obtain functors

$$\mathsf{Mot}(k) \to \mathsf{Mot}(\overline{k}) \to \mathsf{Mot}(\mathbb{C}), \tag{7.3}$$

which, for each M in $\mathsf{Mot}(k)$, give group homomorphisms in the other direction

$$\mathcal{G}_{H_B}(\mathsf{M}_{\mathbb{C}}) \xrightarrow{\sim} \mathcal{G}_{H_B}(\mathsf{M}_{\overline{k}}) \hookrightarrow \mathcal{G}_{H_B}(\mathsf{M}). \tag{7.4}$$

See [3] Proposition 6.22(b) and Proposition 6.23(b) for a proof of the isomorphism and inclusion. Because motivated classes under the Betti realisation functor are Hodge classes this gives us an inclusion

$$MT(H_B(\mathsf{M})) \subset \mathcal{G}_{H_B}(\mathsf{M}_{\mathbb{C}}) \subset GL(H_B(\mathsf{M})). \tag{7.5}$$

Similarly, for a fixed prime ℓ, we have an inclusion

$$\mathcal{G}_{H_\ell}(\mathsf{M}_{\overline{k}}) \hookrightarrow \mathcal{G}_{H_\ell}(\mathsf{M}) \tag{7.6}$$

that is an isomorphism on identity components. Using again that motivated classes under the ℓ-adic realisation are Tate classes, we get an inclusion

$$G_\ell^0(H_\ell(\mathsf{M})) \subset \mathcal{G}_{H_\ell}(\mathsf{M}_{\overline{k}}) \subset GL(H_\ell(\mathsf{M})). \tag{7.7}$$

There are three so-called motivic Mumford-Tate conjectures, two of them are isomorphism statements about the groups above, and the third is an isomorphism statement of the groups occuring in the Mumford-Tate conjecture but in the motivic setting.

Conjecture 7.9. (Hodge classes are motivated)
With notations as above, we have an equality in (7.5)

$$MT(H_B(\mathsf{M})) = \mathcal{G}_{H_B}(\mathsf{M}_{\mathbb{C}}) \tag{7.8}$$

Conjecture 7.10. (Tate classes are motivated)
With notations as above, we have an equality in (7.7)

$$G_\ell^0(\mathsf{M}) = \mathcal{G}_{H_\ell}(\mathsf{M}_{\overline{k}}). \tag{7.9}$$

Conjecture 7.11. (Motivic Mumford-Tate conjecture)
With notations as above, the isomorphism $GL(H_B(\mathsf{M})) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} GL(H_\ell(\mathsf{M}))$ coming from the comparison isomorphism $H_B(X, \mathbb{Q}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H_\ell(X_{\mathbb{C}})$ restricts to an isomorphism

$$MT(H_B(\mathsf{M})) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} G_\ell^0(\mathsf{M}). \tag{7.10}$$

7.5.2 Discussion on the motivic conjectures

The Hodge (respectively Tate) conjecture imply that Hodge (respectively Tate) classes are motivated, and it is clear that if two of Conjecture 7.9, 7.10 or 7.11 are true, then so is the third (this follows essentially from Remark (7.8)). Since $\mathcal{P}(k)$ embeds into $\mathsf{Mot}(k)$, we note also that the motivic Mumford-Tate conjecture implies the Mumford-Tate conjecture. A much more deep result is proved by Deligne (see [2]), which states the following.

Theorem 7.12. *If M in $\mathsf{Mot}(\mathbb{C})$ is an abelian motive, then its Hodge classes (under the Betti realisation functor) are motivated, i.e. Conjecture (7.9) is true for M, i.e.*

$$MT(H_B(\mathsf{M})) = \mathcal{G}_{H_B}(\mathsf{M}). \tag{7.11}$$

Here an **abelian motive** is a motive in the full Tannakian subcategory of $\mathsf{Mot}(\mathbb{C})$ generated by objects of the form $(X, \Delta_X, 0)$ where X is an abelian variety or equal to $\operatorname{Spec} L$ for a finite separable field extension $L/\mathbb{C}$. For the record, if M is in the full Tannakian subcategory generated by $(\operatorname{Spec} L, \Delta_{\operatorname{Spec} L}, 0)$ as above, then it is called an **Artin motive**. Similar definitions of these notions apply for all k for which $\mathsf{Mot}(k)$ is defined.

An immediate consequence of this theorem (upon plugging in $\mathsf{M} = (X, \Delta_X, 0)$) is that, if we take the isomorphism $H_B \otimes \mathbb{Q}_\ell \to H_\ell$ as an identification, then the combination of (7.11) and (7.10) gives an inclusion

$$G_\ell^0 \subset MT(H_B). \tag{7.12}$$

Thus, for abelian varieties, the Mumford-Tate conjecture is equivalent to the weaker statement that $MT(H_B) \subset G_\ell^0$.

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