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The Chas-Sullivan product

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Abstract

The Chas-Sullivan product is traditionally defined for a smooth, closed, orientable manifold as a map on the homology of the free loop space of the manifold. In this thesis it is shown that it is possible to generalize the definition to the case where the manifold is neither smooth nor compact. Some calculations for non-closed manifolds, yielding conditions under which the product must be trivial, are included.

1 Introduction

1.1 Background

Let M be an oriented n-dimensional manifold and let LM be its free loop space, i.e. LM is the set

$$\{f : S^1 \longrightarrow M \mid f \text{ is continuous}\}$$

with the compact-open topology (the definition of which is given in Appendix A). The Chas-Sullivan product is a map

$$H_*(LM) \otimes H_*(LM) \longrightarrow H_*(LM)\,.$$

Intuitively, and as presented in the seminal article [CS99] by Chas and Sullivan, the product is defined on the chain level by taking two chains in $C_*(LM)$ which intersect transversely in the images of the basepoint in S^1 and concatenating the pairs of individual loops which have a common basepoint. The case of two 1-chains, which in a nice case will look something like two tubes, is depicted in Figure 1. The images of the basepoint in S^1 will trace out curves on the sides of the tubes, and assuming that these curves intersect transversely in a point (or several points), we get the product of the chains by taking the loops that have their basepoints in the intersection and concatenating them. We thus get that the product of the two 1-chains is a number of disjoint loops, or, equivalently, points in the loop space of M or chains in $C_0(LM)$. This however implicitly assumes something about the ambient space M: the loop product of $\lambda \in H_i(LM)$ and $\mu \in H_j(LM)$ will be an element in $H_{i+j-n}(LM)$ when M is n-dimensional, and will thus be zero whenever $\deg \lambda + \deg \mu$ is less than n. This has to do with the transversality notion. In the case that M is smooth, two smooth maps (think singular chains!) of manifolds $f : N_1 \longrightarrow M$ and $g : N_2 \longrightarrow M$ are said to be transverse whenever for each $x \in \operatorname{im} f \cap \operatorname{im} g$ the differentials

$$Df : TN_1 \longrightarrow TM \text{ and } Dg : TN_2 \longrightarrow TM$$

satisfy that $Df(T_yN_1) + Dg(T_zN_2) = T_xM$ for all $y \in f^{-1}(N_1)$ and $z \in g^{-1}(N_2)$. This can only be the case if $\dim N_1 + \dim N_2 \geq \dim M$. When M is smooth, we may always choose chain representatives for our homology classes in $H_*(LM)$ so that we may speak about transversality in this way. When M is not smooth things are not so easy. There is a notion of transversality ('local flatness') also for general topological manifolds: two maps $f : N_1 \longrightarrow M$ and $g : N_2 \longrightarrow M$ of topological manifolds are said to be transverse if for each $x \in \operatorname{im} f \cap \operatorname{im} g$ there is an open set U containing x which is homeomorphic to $\mathbb{R}^n$ under a homeomorphism taking $\operatorname{im} f \cap U$ to V and $\operatorname{im} g \cap U$ to W, where V and W are linear subspaces in $\mathbb{R}^n$ which satisfy that $V + W = \mathbb{R}^n$. See [Dol72] for details. However, it is in the non-smooth case not necessarily always possible to choose chain representatives that intersect transversely, see Kirby-Siebenmann [KS77]. We may nevertheless define a way to (at least on a homological level) 'intersect' chains on an arbitrary topological manifold in a way which generalizes the smooth concept. Explaining this construction is the goal of the first parts of this text.

The Chas-Sullivan product is traditionally defined only for smooth, closed and orientable manifolds, but, as we just remarked, smoothness can be dispensed with. As it turns out, nor compactness is needed for the definition of the Chas-Sullivan product. It is however unknown to the author whether the product at all can be nontrivial for a non-compact manifold. Whether it might be nontrivial depends mainly on whether the

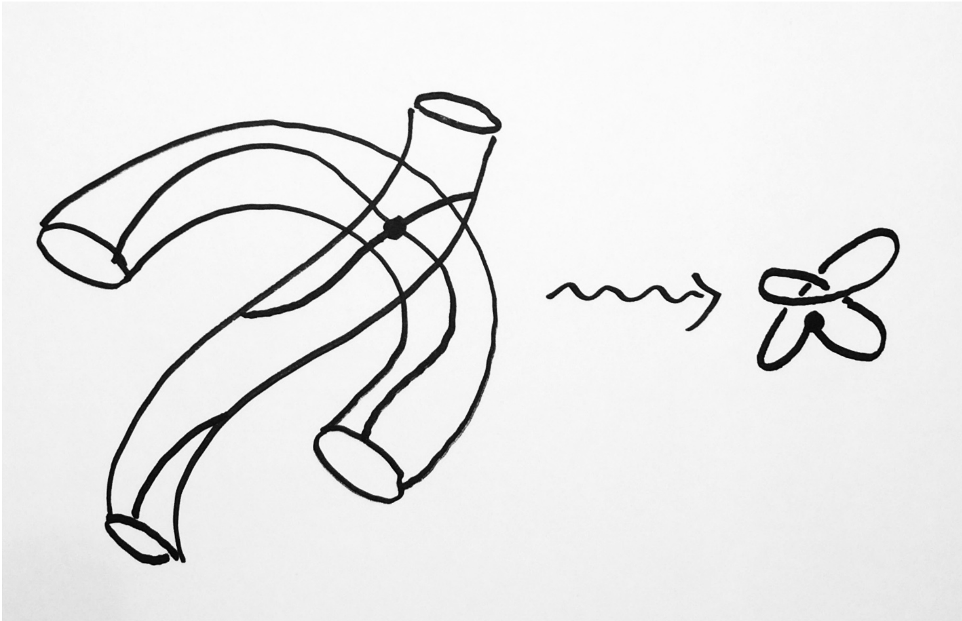

Figure 1: The product of two 1-chains is a 0-chain (in a 2-manifold).

intersection product in turn can be nonzero for a non-compact manifold. A discussion treating this issue as well as partial result in the negative direction is the subject of Section 4.

The structure of this text is as follows. Sections 2 and 3 give some topological background and the definition of the Chas-Sullivan product. These parts are aimed at someone who has knowledge in the field of algebraic topology roughly corresponding to an introductory course at a master's programme level, including material up to the point that cohomology and (relative) cup- and cap-products have been introduced. Section 4 is devoted to some computations and demands more knowledge from the reader, for example familiarity with spectral sequences.

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1.3 Notation and conventions

In this text a manifold is a second-countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$ for some $n \geq 0$. All manifolds are assumed to be connected and with empty boundary. $H_*(-)$ and $H^*(-)$ denotes singular homology and cohomology, respectively, and all coefficients are taken in $\mathbb{Z}$. We denote the group of singular k-chains on a space X by $C_k(X)$. We will write I to denote the unit interval $[0, 1]$. We will represent the circle S^1 as I modulo its endpoints and the basepoint of S^1 is 1 (or rather its equivalence class $\{0, 1\}$).

2 Preliminaries

2.1 Bundles

Definition 2.1. A *fibre bundle* is a quadruple $\xi = (B, E, p, F)$, where $p : E \longrightarrow B$ is a continuous surjective map such that any $b \in B$ has an open neighbourhood U satisfying that $p^{-1}(U)$ is homeomorphic to the product $U \times F$ through a map φ making the following diagram commute:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p|_{p^{-1}(U)}"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}.$$

The spaces B, E and F are called the *base space*, the *total space* and the *fibre* of the bundle, respectively, and the map p is called the *projection map*. The open set U is called a trivializing neighbourhood for b, and the map φ is called a trivializing map for U. We will refer to the different constituents in a fibre bundle as $E(\xi)$, $B(\xi)$, and so on, when this is useful for clarity or brevity. We will often write E_b for the fibre $p^{-1}(b) \cong F$. Sometimes we will use the notation $\xi : E \xrightarrow{p} B$ when the fibre is irrelevant or clear from the context.

Remark 2.2. There exist different conventions for the naming of a fibre bundle. Some texts do not require that a fibre bundle be locally trivial, and instead speak of 'locally trivial fibre bundles'. We will however follow the convention that a fibre bundle is locally trivial in this text.

Let $\xi = (B, E, p, F)$ be a fibre bundle. For two trivializing neighbourhoods U and V for a point $b \in B$, with respective trivializing maps $\varphi : p^{-1}(U) \longrightarrow U \times F$ and $\psi : p^{-1}(V) \longrightarrow V \times F$, we can regard the map $\psi \circ \varphi^{-1}|_{U \cap V} : (U \cap V) \times F \longrightarrow (U \cap V) \times F$. We will mostly be concerned with the case when these maps have some extra structure. Let G be a topological group with a continuous and faithful[1] left action on F. A *G-atlas* for ξ is an open cover $\{U_\alpha\}$ of $B(\xi)$ consisting of trivializing neighbourhoods with respective trivializing maps $\varphi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ such that for any two U_α, U_β in the cover such that $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is given via the G-action on the fibre as

$$\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}(b, f) = (b, \Phi_{\alpha\beta}(b) f)\,,$$

where $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ is a continuous map. Since the action of G on the fibre is assumed to be faithful the map $\Phi_{\alpha\beta}$ is uniquely determined by the map $\varphi_\beta \circ \varphi_\alpha^{-1}$, so $\Phi_{\alpha\beta}$ is not part of the definition of a G-atlas, but only a convenient way of phrasing what we demand of the transition maps. We say that two G-atlases are equivalent if their union is again a G-atlas (and being equivalent is of course an equivalence relation, as one can check if one doubts it).

Definition 2.3. A *G-bundle* is a fibre bundle ξ together with an equivalence class of G-atlases. The group G is called the *structure group* of ξ.

The following example is important enough that we make it a definition of its own.

[1] An action of a group G on a set X is faithful if whenever g and h are distinct elements in G there exists an element $x \in X$ such that $gx \neq hx$.

Definition 2.4. An *n-dimensional real vector bundle* (or *$\mathbb{R}^n$-bundle*) is a fibre bundle with fibre equal to $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$.

Seen in a slightly different way, an $\mathbb{R}^n$-bundle is a fibre bundle whose trivializing functions are linear isomorphisms of n-dimensional real vector spaces when restricted to a fibre. For further details on the subject of fibre bundles, we refer the reader to for example [Hus66] and [Spa66].

We now turn our attention to another sort of bundles, which in a way generalizes the notion of vector bundles.

Definition 2.5. An *n-dimensional microbundle* (or *$\mathbb{R}^n$-microbundle*) is a quadruple $\mathfrak{x} = (B, E, i, p)$, where B and E are topological spaces, called the *base space* and the *total space*, respectively, and $i : B \longrightarrow E$ (the *inclusion*) and $p : E \longrightarrow B$ (the *projection*) are continuous maps satisfying that $p \circ i = \mathrm{id}_B$. The spaces and maps must satisfy the requirements that for each $b \in B$ there exists a subset $U \subseteq B$ containing b and an open subset $V \subseteq E$ such that $i(U) \subseteq V$ and $p(V) \subseteq U$. Moreover, V must be homeomorphic to $U \times \mathbb{R}^n$ under a homeomorphism making the following diagram commute:

$$\begin{tikzcd}
U \arrow[r, "i|_U"] \arrow[dr, "\times 0"'] & V \arrow[r, "p|_V"] \arrow[d] & U \\
& U \times \mathbb{R}^n \arrow[ur, "\mathrm{proj}_1"'] &
\end{tikzcd} .$$

When applicable, we will use the same sort of notation as for fibre bundles (e.g. $E(\mathfrak{x})$) when talking about microbundles.

The following two examples are from Milnor's seminal article [Mil64] on the subject.

Example 2.6. Let M be an n-manifold and consider the maps

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M ,$$

where Δ is the diagonal map and p_1 is projection onto the first factor. Let $x \in M$ be given and choose a neighbourhood U of x homeomorphic to $\mathbb{R}^n$ via $\varphi : U \longrightarrow \mathbb{R}^n$. The set $U \times U \subseteq M \times M$ contains $\Delta(x)$ and is homeomorphic to $U \times \mathbb{R}^n$, and $p_1(U \times U) = U$. Moreover, by defining its middle map to be the homeomorphism

$$(x, y) \longmapsto (x, \varphi(x) - \varphi(y)) ,$$

we get that the diagram

$$\begin{tikzcd}
U \arrow[r, "\Delta|_U"] \arrow[dr, "\times 0"'] & U \times U \arrow[r, "p_1|_{U \times U}"] \arrow[d] & U \\
& U \times \mathbb{R}^n \arrow[ur, "\mathrm{proj}_1"'] &
\end{tikzcd}$$

commutes, showing that $(M, M \times M, \Delta, p_1)$ is a microbundle.

Example 2.7. Let $p : E \longrightarrow B$ be a vector bundle. Then $B \xrightarrow{i} E \xrightarrow{p} B$, where i is the zero section, i.e. the map sending each point $b \in B$ to the point in $p^{-1}(b)$ which corresponds to $0 \in \mathbb{R}^n \cong \{b\} \times \mathbb{R}^n$ under the local trivializations, constitutes a microbundle. This is called the *underlying microbundle* for the vector bundle $E \longrightarrow B$.

The above example shows that any vector bundle gives us a microbundle. The notion of microbundle clearly has a lot in common with that of a fibre bundle with fibre equal to $\mathbb{R}^n$, although with the difference that the trivializations of a microbundle are only local in a neighbourhood around $i(B)$ (hence the name). It however turns out that these concepts share more structure than what is obvious at a first glance. The following is proved in [Kis64].

Theorem 2.8 (Kister-Mazur). *For any $\mathbb{R}^n$-microbundle $\mathfrak{x} = (B, E, i, p)$ there is an open set $U \subseteq E$ containing $i(B)$ such that the restriction $p|_U : U \longrightarrow B$ constitutes a fibre bundle with fibre equal to $\mathbb{R}^n$ and structure group $K_0(n)$, the group of homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ that fix the origin. Moreover, any two fibre bundles thus obtained are isomorphic.*

Remark 2.9. Here an isomorphism of fibre bundles is a homeomorphism between the total spaces of the bundles which preserves fibres and restricts to the identity on the image of the zero section (which coincides the map i in the microbundle), defined as in Example 2.7. The topology on $K_0(n)$ is the compact-open topology. We will refer to a $K_0(\mathbb{R}^n)$-bundle with fibre $\mathbb{R}^n$ as a *topological* $\mathbb{R}^n$*-bundle.*

2.2 Orientations

Let M be a connected d-dimensional manifold. For any $x \in M$, the relative homology group $H_d(M, M \setminus \{x\})$ is isomorphic to $\mathbb{Z}$, as we will show next. For ease of notation we write, following for instance Whitehead [Whi78], $H_i(X \mid x)$ for the group $H_i(X, X \setminus \{x\})$ for a space X and a point $x \in X$. Since M is a manifold there is an open neighbourhood $U \cong \mathbb{R}^d$ of x, and by excision $H_d(M \mid x) \cong H_d(U \mid x) \cong H_d(\mathbb{R}^d \mid 0)$.[2] A part of the long exact sequence in homology for the pair $(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\})$ looks like

$$H_d(\mathbb{R}^d) \longrightarrow H_d(\mathbb{R}^d \mid 0) \longrightarrow H_{d-1}(\mathbb{R}^d \setminus \{0\}) \longrightarrow H_{d-1}(\mathbb{R}^d)\,. \tag{1}$$

In the case that $d \geq 2$ we immediately get that $H_d(\mathbb{R}^d \mid 0) \cong H_{d-1}(\mathbb{R}^d \setminus \{0\}) \cong \mathbb{Z}$, since $\mathbb{R}^d \setminus \{0\} \simeq S^{d-1}$. For the case $d = 1$, the sequence becomes

$$0 \longrightarrow H_1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}\,,$$

where ψ is the map $(a, b) \longmapsto a+b$. Thus $H_1(\mathbb{R}, \mathbb{R}\setminus\{0\})$ is isomorphic to $\ker \psi = \{(a, b) \in \mathbb{Z}^2 \mid b = -a\} \cong \mathbb{Z}$. We remark that (1) also gives that $H_i(\mathbb{R}^d \mid 0)$ is zero for all $i \neq d$, a fact we will use ahead.

Any map $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ induces a chain map on the long exact sequences of the pairs $(\mathbb{R}^d, \mathbb{R}^d \setminus \{x\})$ and $(\mathbb{R}^d, \mathbb{R}^d \setminus \{f(x)\})$. By choosing (once and for all) a generator γ for $H_{d-1}(\mathbb{R}^d \setminus \{0\})$ we get a choice of generator for $H_{d-1}(\mathbb{R}^d \setminus \{x\})$ for any $x \in \mathbb{R}^d$ by translating the image of a chain representing γ by x. In this way, we may talk about the degree of a map $(\mathbb{R}^d, \mathbb{R}^d \setminus \{x\}) \longrightarrow (\mathbb{R}^d, \mathbb{R}^d \setminus \{f(x)\})$; its degree is the number a in the diagram

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid x) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{d-1}(\mathbb{R}^d \setminus \{x\}) & H_{d-1}(\mathbb{R}^d \setminus \{0\}) \arrow[l, "\text{transl.}"'] \arrow[r] & H_{d-1}(S^{d-1}) \arrow[d, "\cdot a"] \\
H_d(\mathbb{R}^d \mid f(x)) \arrow[r, "\partial"] & H_{d-1}(\mathbb{R}^d \setminus \{f(x)\}) & H_{d-1}(\mathbb{R}^d \setminus \{0\}) \arrow[l, "\text{transl.}"'] \arrow[r] & H_{d-1}(S^{d-1})\,.
\end{tikzcd}$$

[2] We can always choose the homeomorphism $U \cong \mathbb{R}^d$ so that x gets mapped to 0 by composing with a translation.

When chasing through this diagram, we in particular get that any $r : (\mathbb{R}^d, \mathbb{R}^d \setminus \{x\}) \longrightarrow (\mathbb{R}^d, \mathbb{R}^d \setminus \{r(x)\})$ which is the composition of a reflection through a $(d-1)$-dimensional linear subspace and a translation is a degree -1 map. We will also talk about a map $H_d(\mathbb{R}^d \mid x) \longrightarrow H_d(\mathbb{R}^d \mid y)$ as 'multiplication by some number', where that number corresponds to the a in the above diagram.

Definition 2.10. A d-dimensional manifold M is *orientable* if there exists an atlas $\{(U_i, f_i)\}$ for M satisfying that it is possible to compose each f_i with a reflection $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ in such a way that a certain compatibility criterion is satisfied. What we demand is that for all i and j and for each $x \in U_i \cap U_j$ there exists an open neighbourhood $W_{ij} \subseteq U_i \cap U_j$ containing x such that the composition at the top of the diagram

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid f_i(x)) \arrow[rr] \arrow[d, "\text{exc.}"] & & H_d(\mathbb{R}^d \mid f_j(x)) \arrow[d, "\text{exc.}"] \\
H_d(f_i(W_{ij}) \mid f_i(x)) & H_d(W_{ij} \mid x) \arrow[l, "(f_i)_*"'] \arrow[r, "(f_j)_*"] & H_d(f_j(W_{ij}) \mid f_j(x)),
\end{tikzcd}$$

where the two vertical maps are excision isomorphisms, is multiplication with 1. An atlas for M whose coordinate maps satisfy the above criterion is called an oriented atlas for M. An *orientation* for M is a maximal oriented atlas for M.

Remark 2.11. The above definition makes sense because an explicit inverse for the excision map is given by the one induced by the inclusion, so that we have 'the same' generator in $H_d(\mathbb{R}^d \mid f_i(x))$ and $H_d(f_i(W_{ij}) \mid f_i(x))$; we do not introduce any arbitrary reflections or something (which even could have varied with the charts and/or x) that might alter the orientations. Because of the same reason, the above criterion is equivalent to the same requirement for *any* open neighbourhood of x in $U_i \cap U_j$.

A quite reasonable objection to the above definition would be that we only require the compatibility criterion to hold for the coordinate charts belonging to *one* atlas; we certainly want the property of being orientable to be independent of which atlas we choose. We therefore show the following.

Proposition 2.12. *If a manifold M is orientable, any atlas for M can be made into an oriented atlas by composing its coordinate maps with reflections.*

Proof. Let M^d be a manifold, let $\{(U_i, f_i)\}$ be an oriented atlas for M and let $\{(V_i, g_i)\}$ be any atlas for M. Pick a chart (V_k, g_k) in the possibly unoriented atlas for M. Any $x \in V_k$ is in some U_i, and we will get an induced map

$$H_d(\mathbb{R}^d \mid f_i(x)) \longrightarrow H_d(\mathbb{R}^d \mid g_k(x))$$

just like in Definition 2.10 (but using charts from different atlases here). This map is multiplication with ± 1, and we may if necessary compose g_k with a reflection $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ to ascertain that it really is $+1$. Let $\tilde{g}_k$ be the composition of g_k with the right map—a reflection or the identity $\mathbb{R}^d \longrightarrow \mathbb{R}^d$—for making the sign correct. The point x could of course be contained in more than one U_i, so we must make sure that the choice of $\tilde{g}_k$ is independent of which U_i we choose. Let U_j be another chart containing x. We have the following commutative diagram, where $W \subseteq U_i \cap U_j \cap V_k$ is some open set containing x:

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid f_i(x)) \arrow[rrrr] \arrow[dr] \arrow[ddddrr, bend right] & & & & H_d(\mathbb{R}^d \mid f_j(x)) \arrow[dl] \arrow[ddddll, bend left] \\
& H_d(f_i(W) \mid f_i(x)) & & H_d(f_j(W) \mid f_j(x)) & \\
& & H_d(W \mid x) \arrow[ul, "(f_i)_*"'] \arrow[ur, "(f_j)_*"] \arrow[d, "(\tilde{g}_k)_*"] & & \\
& & H_d(\tilde{g}_k(W) \mid \tilde{g}_k(x)) & & \\
& & H_d(\mathbb{R}^d \mid \tilde{g}_k(x)) \arrow[u] & &
\end{tikzcd}.$$

Because $\{(U_i, f_i)\}$ is an oriented atlas, the top arrow is multiplication with 1, and because we have chosen the left curved arrow to be multiplication with 1 as well this means that the right curved arrow has to be multiplication with 1 as well, showing that the choice of U_i did not matter. Adjusting all g_i in this way yields an oriented atlas $\{(V_i, \tilde{g}_i)\}$. To see this, one draws the same diagram as above, but with the fs and the g interchanged; the top arrow will be the map for the orientability criterion for the atlas $\{(V_i, \tilde{g}_i)\}$, and this map will factor through the two bent arrows, which both are multiplication with 1. $\square$

Lemma 2.13. *$M \times M$ is orientable whenever M is.*

Proof. Let $\{(U_i, f_i)\}$ be an oriented atlas for M. We have that $\{(U_i \times U_j, f_i \times f_j)\}$ is an atlas for M since $f_i \times f_j : U_i \times U_j \cong \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$ and any point $(x, y) \in M \times M$ is in some $U_i \times U_j$. Suppose that (x, y) is in $(U_i \times U_j) \cap (U_k \times U_\ell) = (U_i \cap U_k) \times (U_j \cap U_\ell)$, and set $U_{ij} = U_i \cap U_j$ and $x_i = f_i(x)$ (and likewise for y). The cross product map

$$H_d(\mathbb{R}^d \mid x) \otimes H_d(\mathbb{R}^d \mid y) \xrightarrow{\times} H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x, y))$$

is an isomorphism for all x and y in $\mathbb{R}^d$ since $H_i(\mathbb{R}^d \mid x)$ is free for all x and for all i. Letting γ_x be the standard generator for $H_d(\mathbb{R}^d \mid x)$ (in the sense of Definition 2.10), the cross product thus takes the generator $\gamma_x \otimes \gamma_y$ for $H_d(\mathbb{R}^d \mid x) \otimes H_d(\mathbb{R}^d \mid y)$ to *some* generator for $H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x, y))$; we need not concern ourselves with whether this generator is plus or minus the standard generator for $H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x, y)) \cong H_{2d}(\mathbb{R}^{2d} \mid (x, y))$, but only note that the image of the generators under the cross product is consistent in the sense that the diagram

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid x) \otimes H_d(\mathbb{R}^d \mid y) \arrow[r] \arrow[d, "\times"] & H_d(\mathbb{R}^d \mid x') \otimes H_d(\mathbb{R}^d \mid y') \arrow[d, "\times"] \\
H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x, y)) \arrow[r] & H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x', y'))
\end{tikzcd},$$

where the horizontal arrows are induced by the translations taking x to x' and y to y', commutes (this is a special case of the naturality of the cross product). Any translation T of $\mathbb{R}^n$ induces a map $H_n(\mathbb{R}^n \mid x) \longrightarrow H_n(\mathbb{R}^n \mid T(x))$ which is multiplication with 1, so the cross product at least consistently takes $\gamma_x \otimes \gamma_y$ to either plus or minus the standard generator of $H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x, y))$ for all $x, y \in \mathbb{R}^d$. Regard now the diagram in Figure 2. It is everywhere commutative because of naturality of the cross product and the excision isomorphism, and all cross products are isomorphisms since $H_q(f_*(U_{**}) \mid x_*)$ is free for all

q and all choices of indices. The bottom horizontal arrow will take $\gamma_{x_i} \otimes \gamma_{y_j}$ to $\gamma_{x_k} \otimes \gamma_{y_\ell}$ since $\{(U_i, f_i)\}$ is an oriented atlas. Because of the consistency of the cross product discussed above, this means that the top arrow, which precisely is the map determining whether $\{(U_i \times U_j, f_i \times f_j)\}$ is an oriented atlas, will be multiplication with 1. $\square$

Lemma 2.14. *Any open subset of an orientable d-dimensional manifold M is itself an orientable manifold with the subspace topology.*

Proof. Let N be an open subset of M and let $\{(U_i, f_i)\}$ be an oriented atlas for M. Any $x \in N$ will have an open neighbourhood containing it which is homeomorphic via some f_i (assuming that x is in U_i) to an open ball in $\mathbb{R}^d$. Call this open neighbourhood B_i^x, and assume that the ball $f_i(B_i^x)$ has radius ε_i^x. By composing the maps $f_i|_{B_i^x}$ with the map

$$v \longmapsto f_i(x) + \frac{\varepsilon_i^x}{\varepsilon_i^x - \|v - f_i(x)\|}(v - f_i(x))$$

(i.e. by expanding the balls to all of $\mathbb{R}^d$) and calling the resulting maps $\tilde{f}_i^x$, we get an atlas $\{(B_i^x, \tilde{f}_i^x)\}$ for N. We will show that this atlas is oriented. Assume that z is in $B_i^x \cap B_j^y$. By construction, this means that z also is in $U_i \cap U_j$. Let $W \subseteq B_i^x \cap B_j^y \subseteq U_i \cap U_j$ be some open set which contains z. We get the following commutative diagram, whose vertical arrows can be checked to be multiplication with 1.

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid \tilde{f}_i^x(z)) \arrow[rrrr] \arrow[dr] \arrow[dddd, "\cdot 1"] & & & & H_d(\mathbb{R}^d \mid \tilde{f}_j^y(z)) \arrow[dl] \arrow[dddd, "\cdot 1"] \\
& H_d(\tilde{f}_i^x(W) \mid \tilde{f}_i^x(z)) & & H_d(\tilde{f}_j^y(W) \mid \tilde{f}_j^y(z)) & \\
& & H_d(W \mid z) \arrow[ul, "(\tilde{f}_i^x)_*"] \arrow[ur, "(\tilde{f}_j^y)_*"'] \arrow[dl, "(f_i)_*"'] \arrow[dr, "(f_j)_*"] & & \\
& H_d(f_i(W) \mid f_j(z)) & & H_d(f_j(W) \mid f_j(z)) & \\
H_d(\mathbb{R}^d \mid f_i(z)) \arrow[rrrr] \arrow[ur] & & & & H_d(\mathbb{R}^d \mid f_j(z)) \arrow[ul]
\end{tikzcd}$$

Since also the bottom horizontal arrow is multiplication with 1, the top arrow has to be multiplication with 1, so $\{(B_i^x, \tilde{f}_i^x)\}$ is an oriented atlas for N. $\square$

We turn now to a different notion of orientability, that of G-bundles. Whenever the fibre of a bundle is equal to a manifold, we can ask whether the transition maps are orientation-preserving on the fibres in the way just described for manifolds. Since we will be dealing only with bundles which have fibre equal to $\mathbb{R}^n$ and base space equal to a manifold, we give the following slightly restricted definition to make a few proofs later on easier.

Definition 2.15. Let $\xi : E \longrightarrow M$ be a topological $\mathbb{R}^n$-bundle with base space a d-dimensional manifold. We say that ξ is orientable if there exists a cover $\{U_\alpha\}$ of M which consists of trivializing open subsets such that there exists a set of trivializing maps $\varphi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ satisfying that for any α and β, any $x \in U_\alpha \cap U_\beta$ and any $q \in p^{-1}(x)$ the map

$$(\varphi_\beta \circ \varphi_\alpha^{-1})_* : H_d(U_\alpha \cap U_\beta \mid x) \otimes H_n(\mathbb{R}^n \mid v) \longrightarrow H_d(U_\alpha \cap U_\beta \mid x) \otimes H_n(\mathbb{R}^n \mid w)\,,$$

$$
\begin{tikzcd}
H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x_i, y_j)) \arrow[rr] \arrow[d, "\text{exc.}"] & & H_{2d}(\mathbb{R}^d \times \mathbb{R}^d \mid (x_k, y_\ell)) \arrow[d, "\text{exc.}"] \\
H_{2d}(f_i(U_{ik}) \times f_j(U_{j\ell}) \mid (x_i, y_j)) & H_{2d}(U_{ik} \times U_{j\ell} \mid (x, y)) \arrow[l, "(f_i \times f_j)_*"'] \arrow[r, "(f_k \times f_\ell)_*"] & H_{2d}(f_k(U_{ik}) \times f_\ell(U_{j\ell}), (x_k, y_\ell)) \\
H_d(f_i(U_{ik}) \mid x_i) \otimes H_d(f_j(U_{j\ell}) \mid y_j) \arrow[u, "\times"] & H_d(U_{ik} \mid x) \otimes H_d(U_{j\ell} \mid y) \arrow[l, "(f_i)_* \otimes (f_j)_*"'] \arrow[r, "(f_k)_* \otimes (f_\ell)_*"] \arrow[u, "\times"] & H_d(f_k(U_{ik}) \mid x_k) \otimes H_d(f_\ell(U_{j\ell}) \mid y_\ell) \arrow[u, "\times"] \\
H_d(\mathbb{R}^d \mid x_i) \otimes H_d(\mathbb{R}^d \mid y_j) \arrow[rr] \arrow[u, "\text{exc.}"] \arrow[uuu, bend left=60, "\times"] & & H_d(\mathbb{R}^d \mid x_k) \otimes H_d(\mathbb{R}^d \mid y_\ell) \arrow[u, "\text{exc.}"] \arrow[uuu, bend right=60, "\times"']
\end{tikzcd}
$$

Figure 2: Diagram for showing that $M \times M$ is oriented if M is oriented.

where $(x, v) = \varphi_\alpha(q)$ and $(x, w) = \varphi_\beta(q)$ coincides with the map that is induced by the identity on the first factor and the translation $v \longmapsto w$ on the second factor.

Remark 2.16. An important fact here is that the properties of a local trivialization give that the map $\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is the identity when restricted to the first factor; which map that $\varphi_\beta \circ \varphi_\alpha^{-1}$ induces on homology is completely determined by its behaviour on the second factor. Since $\varphi_\beta \circ \varphi_\alpha$ restricted to a fibre is an isomorphism it will induce a map $H_n(\mathbb{R}^n \mid v) \longrightarrow H_n(\mathbb{R}^n \mid w)$ (identifying $\{x\} \times \mathbb{R}^n$ with $\mathbb{R}^n$) which is either plus or minus 1, and one can use this fact to define orientability of a topological $\mathbb{R}^n$-bundle over an arbitrary base space in a way which generalizes the above definition.

Remember that $H_d(U_\alpha \cap U_\beta \mid x) \otimes H_n(\mathbb{R}^n \mid v) \xrightarrow{\times} H_{d+n}((U_\alpha \cap U_\beta) \times \mathbb{R}^n \mid (x, v))$ is an isomorphism by the Künneth formula; we have no torsion summand since $H_i(\mathbb{R}^n \mid v)$ is free for all i. It is under this isomorphism that the above definition is supposed to be interpreted. Just like for the definition of manifold orientability, we may instead of $U_\alpha \cap U_\beta$ regard any open neighbourhood A of x which is contained in $U_\alpha \cap U_\beta$; we have by excision that $H_d(U_\alpha \cap U_\beta \mid x) \otimes H_n(\mathbb{R}^n \mid v) \cong H_d(A \mid x) \otimes H_n(\mathbb{R}^n \mid v)$.

We thus have two notions of orientability, one for manifolds and one for bundles. In the case that the base space of a topological $\mathbb{R}^n$-bundle is a manifold, it actually holds that both are applicable because of the following.

Proposition 2.17. *If $p : E \longrightarrow M$ is a topological $\mathbb{R}^n$-bundle with base space a d-dimensional manifold, then the total space E is a $(d + n)$-dimensional manifold.*

Proof. Let $\{U_\alpha\}$ be a cover of M consisting of trivializing open sets with respective trivializing maps $\varphi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ and let $\{(V_i, f_i)\}$ be an atlas for M. Let $q \in E$ be given and set $x = p(q)$. Suppose that x is in $U_\alpha \cap V_i$. The set $f_i(U_\alpha \cap V_i)$ is an open subset of $\mathbb{R}^d$ so there is an open ball $B \subseteq f(U_\alpha \cap V_i)$ which contains $f_i(x)$. Set $W = f_i^{-1}(B)$. We then have that q is in $p^{-1}(W)$ and that

$$p^{-1}(W) \overset{\varphi_i}{\cong} W \times \mathbb{R}^n \cong \mathbb{R}^d \times \mathbb{R}^n \cong \mathbb{R}^{d+n},$$

so $p^{-1}(W)$ is a trivializing set for q.

To see that E is Hausdorff, let q and q' be two distinct points in E and set $x = p(q)$ and $x' = p(q')$. Since the base space M is a manifold there are open sets U and U' containing x and x', respectively, which satisfy that $U \cap U' = \varnothing$. We then have that q is in $p^{-1}(U)$ and that q' is in $p^{-1}(U')$, and both $p^{-1}(U)$ and $p^{-1}(U')$ are open sets in E, being inverse images of open sets under the continuous map p. It must hold that $p^{-1}(U) \cap p^{-1}(U') = \varnothing$ since $p(e)$ is in $U \cap U'$ whenever e is in $p^{-1}(U) \cap p^{-1}(U')$. This shows that E is Hausdorff.

To show second countability, let $\{U_i\}$ be an open cover for M consisting of trivializing sets. Since M is second countable (and hence Lindelöf) we may assume that this cover consists of at most countably many sets. We have that $\{p^{-1}(U_i)\}$ is a cover for E and that the sets $U_i \times \mathbb{R}^n$, and hence $p^{-1}(U_i)$, are second countable. Let V be an open set in E. We may write

$$V = V \cap \Big(\bigcup_i p^{-1}(U_i)\Big) = \bigcup_i \Big(V \cap p^{-1}(U_i)\Big).$$

Let $\{A_{i,\alpha}\}$ be a basis for the topology on $p^{-1}(U_i)$. The $A_{i,\alpha}$, being open sets in $p^{-1}(U_i)$ (in the subspace topology), are intersections of $p^{-1}(U_i)$ and respective open sets $A'_{i,\alpha}$ in E, but we must have that $A_{i,\alpha} = A'_{i,\alpha}$ for all i and α, so the $A_{i,\alpha}$ are open also in E. It follows that V is expressible as

$$V = \bigcup_{i,\alpha} A_{i,\alpha}\,,$$

where the $A_{i,\alpha}$ are basis sets for the subspace topologies on the sets $p^{-1}(U_i)$. By the above discussion, all $A_{i,\alpha}$ are open also in E and constitute therefore a basis for the topology on E. This basis is countable, since there are countably many sets $p^{-1}(U_i)$ and countably many basis sets for each i. □

For a topological $\mathbb{R}^n$-bundle ξ with a manifold as its base space we thus can speak both of the orientability of ξ (as a bundle) and of the orientability of $E(\xi)$. As one might suspect, these two notions are connected.

Lemma 2.18. *Let $\xi : E \xrightarrow{p} M$ be a topological $\mathbb{R}^n$-bundle with M an oriented d-dimensional manifold. Then $E(\xi)$ is orientable (as a manifold) if and only if ξ is orientable (as a bundle).*

Proof. Suppose that $\{(U_i, f_i)\}$ is an oriented atlas for M and let $\{V_\alpha\}$ be a cover of M with trivializing subsets with respective trivializing maps $\varphi_\alpha : p^{-1}(V_\alpha) \longrightarrow V_\alpha \times \mathbb{R}^n$. Assume to begin with that E is orientable. Suppose that $V_\alpha \cap V_\beta \neq \varnothing$. Let U_i be a chart for M which has nonempty intersection with $V_\alpha \cap V_\beta$ and let $W \subseteq U_i \cap V_\alpha \cap V_\beta$ be a set homeomorphic to a ball in $\mathbb{R}^d$. Let, like in Proposition 2.17, $F_{i,\alpha}$ and $F_{i,\beta}$ be the respective compositions

$$p^{-1}(W) \xrightarrow{\varphi_\alpha} W \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^d \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^{d+n} \tag{2}$$

and

$$p^{-1}(W) \xrightarrow{\varphi_\beta} W \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^d \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^{d+n} \tag{3}$$

so that $(p^{-1}(W), F_{i,\alpha})$ and $(p^{-1}(W), F_{i,\beta})$ are two charts for E which we without loss of generality may assume are oriented consistently (i.e. $F_{i,\beta} \circ F_{i,\alpha}^{-1}$ induces multiplication with 1 on homology). Let $J_{i,\alpha}$ and $J_{i,\beta}$ be the latter two isomorphisms in (2) and (3), respectively (so that $J_{i,\alpha} \circ \varphi_\alpha = F_{i,\alpha}$ and similarly for β). Let q be a point in $p^{-1}(W)$. Set $(x, v) = \varphi_\alpha(q)$ and $(x, w) = \varphi_\beta(q)$ (assuming implicitly that $p(q) = x$). We have the following diagram, which is commutative because of the naturality of all maps involved.

$$\begin{tikzcd}
H_{d+n}(\mathbb{R}^{d+n} \mid J_{i,\alpha}(x,v)) \arrow[rr] & & H_{d+n}(\mathbb{R}^{d+n} \mid J_{i,\alpha}(x,w)) \\
H_{d+n}(W \times \mathbb{R}^n \mid (x,v)) \arrow[u, "(J_{i,\alpha})_*"] & H_{d+n}(p^{-1}(W) \mid q) \arrow[l, "(\varphi_\alpha)_*"'] \arrow[r, "(\varphi_\beta)_*"] & H_{d+n}(W \times \mathbb{R}^n \mid (x,w)) \arrow[u, "(J_{i,\beta})_*"'] \\
H_d(W \mid x) \otimes H_n(\mathbb{R}^n \mid v) \arrow[u, "\times"] \arrow[rr, "(\varphi_\beta \circ \varphi_\alpha^{-1})_*"] & & H_d(W \mid x) \otimes H_n(\mathbb{R}^n \mid w)\,. \arrow[u, "\times"']
\end{tikzcd}$$

The top horizontal arrow is the map determining whether $(p^{-1}(W), F_{i,\alpha})$ and $(p^{-1}(W), F_{i,\beta})$ are oriented consistently, and we are assuming that this map is multiplication with 1. Because of the consistency of the cross product discussed in the proof of Lemma 2.13, we must have that the bottom horizontal arrow coincides with the map induced by the identity in the first factor and the translation $v \longmapsto w$ in the second factor since $\varphi_\beta \circ \varphi_\alpha^{-1}$ in any case is the identity on the first factor, so ξ is an oriented bundle.

Suppose now that ξ is oriented and let W_i and W_j be $\mathbb{R}^d$-balls contained in $U_i \cap V_\alpha$ and $U_j \cap V_\beta$, respectively, and suppose that $K := p^{-1}(W_i) \cap p^{-1}(W_j)$ is nonempty. We have the following commutative diagram.

$$\begin{tikzcd}
H_{d+n}(\mathbb{R}^d \times \mathbb{R}^n \mid (f_i(x), v)) \arrow[rr] \arrow[d, "\text{exc.}"] & & H_{d+n}(\mathbb{R}^d \times \mathbb{R}^n \mid (f_j(x), w)) \arrow[d, "\text{exc.}"] \\
H_{d+n}(f_i(W_i \cap W_j) \times \mathbb{R}^n \mid (f_i(x), v)) & & H_{d+n}(f_j(W_i \cap W_j) \times \mathbb{R}^n \mid (f_j(x), w)) \\
H_{d+n}(W_i \cap W_j \times \mathbb{R}^n \mid (x, v)) \arrow[u, "(f_i \times \mathrm{id})_*"] & H_{d+n}(K \mid q) \arrow[l, "(\varphi_\alpha)_*"'] \arrow[r, "(\varphi_\beta)_*"] & H_{d+n}(W_i \cap W_j \times \mathbb{R}^n \mid (x, w)) \arrow[u, "(f_j \times \mathrm{id})_*"] \\
H_d(W_i \cap W_j \mid x) \otimes H_n(\mathbb{R}^n \mid v) \arrow[u, "\times"] \arrow[rr, "(\varphi_\alpha^{-1} \circ \varphi_\beta)_*"] & & H_d(W_i \cap W_j \mid x) \otimes H_n(\mathbb{R}^n \mid w) . \arrow[u, "\times"]
\end{tikzcd}$$

The top horizontal arrow is determining whether $(p^{-1}(W_i), F_{i,\alpha})$ and $(p^{-1}(W_j), F_{j,\beta})$ (defined as above and in Lemma 2.17) are oriented consistently.[3] Because ξ is oriented the bottom horizontal arrow is multiplication with 1 on both factors. Since $\{(U_i, f_i)\}$ is an oriented atlas and because of the consistency of the cross product, the generators in the bottom corners are taken to the same generators (in the sense of the discussion in the beginning of this section) in the top corners when mapped along their respective sides. This gives that $(p^{-1}(W_i), F_{i,\alpha})$ and $(p^{-1}(W_j), F_{j,\beta})$ are oriented consistently, and since we may choose an atlas for E consisting solely of charts of this form, this gives that E is oriented. $\square$

Definition 2.19. We will write $K_0^+(\mathbb{R}^n)$ for the group of orientation-preserving homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ that fix the origin. Thus an oriented topological $\mathbb{R}^n$-bundle is the same thing as a $K_0^+(\mathbb{R}^n)$-bundle with fibre $\mathbb{R}^n$.

2.3 Thom classes and the Thom isomorphism

For an oriented topological $\mathbb{R}^n$-bundle ξ, let, for $b \in B(\xi)$, j_b denote the map on cohomology which is induced by the inclusion of $\mathbb{R}^n$ onto the fibre E_b in an orientation-preserving way. Let furthermore E_0 denote the total space of ξ minus the zero section.

Proposition and definition 2.20. For any oriented topological $\mathbb{R}^n$-bundle ξ there exists a unique class $\tau \in H^n(E(\xi), E_0(\xi))$—the *Thom class*—such that, for each $b \in B(\xi)$, $j_b(\tau)$ is the generator in $H^n(\mathbb{R}^n \mid 0)$ that is given by the orientation.[4] Moreover, the map

$$H_{i+n}(E, E_0) \longrightarrow H_i(E) \cong H_i(B)$$

given by $\sigma \longmapsto \tau \frown \sigma$ is an isomorphism—the *Thom isomorphism*.

Existence of τ is shown in for instance [Dol72], but not all its properties are shown there. A reference for them is [Hol66], which even covers the more general case of a not necessarily orientable *micro*bundle. This source is a bit sparse on proofs, though.

[3] The difference between $\mathbb{R}^d \times \mathbb{R}^n$ and $\mathbb{R}^{d+n}$ is immaterial here, although we could of course just as well extend the diagram upwards to an $\mathbb{R}^{d+n}$-level.

[4] By this we mean that a chain representative for $j_b(\tau)$ is the cochain which evaluates to 1 on chain representatives for the standard generator in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$.

2.4 Deformation retracts

The goal of this section is to show that for a fibration (defined below) $E \longrightarrow B$, certain deformation retracts of B lift to deformation retracts in E. We will use this when defining the Chas-Sullivan product; the deformation retract of a topological $\mathbb{R}^n$-bundle over M onto M will lift to the level of loop spaces. Most of the material in this section is taken from [Str66] and [Str68], with a few details and proofs fleshed out to make things more accessible.

Definition 2.21. A *(Hurewicz) fibration* $f : E \longrightarrow B$ is a map satisfying the homotopy lifting property with respect to all spaces X. What this means is that given any commutative square of the form

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, "i"] & E \arrow[d, "f"] \\ X \times I \arrow[r, "h"] & B \end{tikzcd}$$

it is always possible to find a map $\tilde{h} : X \times I \longrightarrow E$ such that $f \circ \tilde{h} = h$.

Definition 2.22. A *(Hurewicz) cofibration* is a map $j : A \longrightarrow X$ satisfying the homotopy extension property for all spaces. This means that given any space Y, a homotopy $f : A \times I \longrightarrow Y$ and a map $F : X \longrightarrow Y$ such that $F|_A = f|_{A\times\{0\}}$, we can always find a map $\tilde{F} : X \times I \longrightarrow Y$ such that $\tilde{F}|_{A\times I} = f$.

Remark 2.23. As explained in Appendix A, we may in the case that a space Z is locally compact and Hausdorff identify a homotopy $h : Z \times I \longrightarrow W$ with a map from Z to W^I, the set of maps from I to W (with the compact-open topology). The statement that $j : A \longrightarrow X$ is a cofibration can then be visualized by the following diagram (where we abuse notation a bit and for instance use the name f also for the map $A \longrightarrow Y^I$):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & Y^I \arrow[d, "\mathrm{proj}_0"] \\ X \arrow[r, "F"] & Y\,. \end{tikzcd}$$

The map proj_0 takes a map $g : I \longrightarrow Y$ to $g(0) \in Y$. The homotopy extension property now is the statement that we can find a map $\tilde{F} : X \longrightarrow Y^I$ such that $\mathrm{proj}_0 \circ \tilde{F} = F$.

An immediate consequence of the definition of a cofibration is that for a cofibration $j : A \longrightarrow X$ which is an inclusion, the inclusion map $(X \times \{0\}) \cup (A \times I) \longrightarrow X \times I$ admits a retraction, i.e. there exists a continuous map $r : X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ such that $r \circ \mathrm{inc.} = \mathrm{id}$. This is because the identity map $(X \times \{0\}) \cup (A \times I) \longrightarrow (X \times \{0\}) \cup (A \times I)$ can be viewed as the gluing of a map $X \cong X \times \{0\} \longrightarrow (X \times \{0\}) \cup (A \times I)$ and a homotopy $A \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ along $A \times \{0\}$, and this homotopy has an extension to all of X. Since this extended homotopy $X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ agrees with the original homotopy (which is the identity) on $A \times I$, we get that the extension is a retraction. If A is closed, the converse also holds,[5] i.e. the inclusion $j : A \hookrightarrow X$ is a

[5] Actually, this holds also if A is not closed (see [Str68]), but that is trickier to prove, and we will not need it here.

cofibration if the inclusion $(X \times \{0\}) \cup (A \times I) \longrightarrow X \times I$ admits a retraction. This is because a homotopy $h : A \times I \longrightarrow Y$ and a map $g : X \cong X \times \{0\} \longrightarrow Y$ which agree on $A \times \{0\}$ may be glued together to a continuous map f since both $X \times \{0\}$ and $A \times I$ are closed in $(X \times \{0\}) \cup (A \times I)$ in this case. Precomposing f with the retraction $X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ then gives an extension of the homotopy h to all of X. We use this characterization of cofibrations to prove the following.

Lemma 2.24. *If $A \subseteq X$ is a closed subset such that its inclusion is a cofibration, there exists a continuous function $\psi : X \longrightarrow I$ such that $A = \psi^{-1}(0)$.*

Proof. Let $r : X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ be a retraction of the inclusion map $(X \times \{0\}) \cup (A \times I) \hookrightarrow X \times I$. Define ψ to be the map

$$\psi(x) = \sup_{t \in I} \left| t - \mathrm{proj}_I(r(x,t)) \right| ,$$

where proj_I is the projection of $X \times I$ onto I. The function ψ is continuous since it is the supremum of a continuous function on a closed interval. To see that $\psi^{-1}(0) = A$, note that r is the identity on $A \times I \subseteq X \times I$. Therefore

$$\psi(x) = \sup_{t \in I} |t - t| = 0$$

for $x \in A$. If on the other hand x is in $X \setminus A$, we have that $\psi(x) > 0$ because of the following. The retraction r is the identity on $X \times \{0\}$. Since A is closed, there is an open neighbourhood $U \subseteq X$ around x which is contained in $X \setminus A$. Because of this there must exist an $\varepsilon > 0$ such that $r(x, [0, \varepsilon])$ is contained in U, and then we have that

$$\psi(x) = \sup_{t \in I} \left| t - \mathrm{proj}_I(r(x,t)) \right| \geq |\varepsilon - 0| = \varepsilon > 0 .$$

□

A fact we will need ahead is the following stronger claim about the inclusion $A \hookrightarrow X$.

Lemma 2.25. *If the inclusion of a subspace $A \subseteq X$ is a cofibration, $(X \times \{0\}) \cup (A \times I)$ is a strong deformation retract of $X \times I$.*

Proof. We know from before that the inclusion $i : (X \times \{0\}) \cup (A \times I) \longrightarrow X \times I$ admits a retraction r. Define a homotopy $D : (X \times I) \times I \longrightarrow X \times I$ by

$$D((x,t),s) = (\mathrm{proj}_X(r(x,(1-s)t)), (1-s)\mathrm{proj}_I(r(x,t)) + st) .$$

If (x,t) is in $A \times I$ we have since r is the identity on $A \times I$ that

$$D((x,t),s) = (\mathrm{proj}_X(x,(1-s)t), (1-s)\mathrm{proj}_I(r(x,t)) + st) = (x,t)$$

for $(x,t) \in A \times I$, so D is a homotopy relative to $A \times I$. For elements of the form $(x,0)$ we get that

$$D((x,0),s) = (\mathrm{proj}_X(r(x,0)), (1-s)\mathrm{proj}_I(r(x,0))) = (x,0) ,$$

so D is stationary also on $X \times \{0\}$. We have further that

$$D((x,t),0) = (\mathrm{proj}_X(r(x,t)), \mathrm{proj}_I(r(x,t))) = r(x,t) = (i \circ r)(x,t)$$

for all (x,t) in $X \times I$ and that

$$D((x,t),1) = (\mathrm{proj}_X(r(x,0)),t) = (x,t) = \mathrm{id}_{X\times I}(x,t)$$

so D is a homotopy from $i \circ r$ to the identity on $X \times I$. Since the range of r is $(X \times \{0\}) \cup (A \times I)$, this shows that $(X \times \{0\}) \cup (A \times I)$ is a strong deformation retract of $X \times I$. $\square$

Theorem 2.26. *Suppose that $f : E \longrightarrow B$ is a fibration and that A is a strong deformation retract of X such that there exists a map $\psi : X \longrightarrow I$ with $\psi^{-1}(0) = A$. Then for any commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{g_1} & E \\ \big\downarrow{\scriptstyle \text{inc.}} & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{g_2} & B\,, \end{array}$$

where $i : A \hookrightarrow X$ is the inclusion, there is a map $g : X \longrightarrow E$ such that $g \circ \mathrm{inc.} = g_1$ and $f \circ g = g_2$.

Proof. Let $r : X \longrightarrow A$ be a retraction and let $R : X \times I \longrightarrow X$ be a homotopy from r to the identity on X relative to A. Define $\tilde{R} : X \times I \longrightarrow X$ by

$$\tilde{R}(x,t) = \begin{cases} R(x, t/\psi(x))\,, & t < \psi(x) \\ R(x,1)\,, & t \geq \psi(x) \end{cases}$$

and regard the diagram

$$\begin{array}{ccccc} X \times \{0\} & \xrightarrow{r} & A & \xrightarrow{g_1} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ X \times I & \xrightarrow{\tilde{R}} & X & \xrightarrow{g_2} & B\,, \end{array}$$

(where $r(x,0)$ means $r(x)$, by slight abuse of notation). Since f is a fibration, there is a map $G : X \times I \longrightarrow E$ such that $f \circ G = g_2 \circ \tilde{R}$ which satisfies that $G(x,0) = (g_1 \circ r)(x,0)$. Define g by

$$g(x) = G(x, \psi(x))\,.$$

We have that

$$(f \circ g)(x) = (f \circ G)(x, \psi(x)) = (g_2 \circ \tilde{R})(x, \psi(x)) =$$
$$= g_2(R(x,1)) = g_2(\mathrm{id}_X(x)) = g_2(x)$$

and since $\psi(x) = 0$ precisely when x is in A and r is the identity on A that

$$(g \circ i)(x) = G(i(x), \psi(i(x))) = G(x,0) =$$

$$= (g_1 \circ r)(x,0) = g_1(x)\,,$$

so g satisfies the assumptions in the theorem. $\square$

Theorem 2.27. *If $f : E \longrightarrow B$ is a fibration and the inclusion $i : A \hookrightarrow X$ of a closed subspace A into X is a cofibration, then for any diagram of the form*

$$\begin{tikzcd}
(X \times \{0\}) \cup (A \times I) \arrow[r, "g"] \arrow[d, "\text{inc.}"] & E \arrow[d, "f"] \\
X \times I \arrow[r, "G"] & B
\end{tikzcd}$$

there is a map $\tilde{G} : X \times I \longrightarrow E$ such that $f \circ \tilde{G} = G$ and which agrees with g on $(X \times \{0\}) \cup (A \times I)$.

Proof. We know from Lemma 2.25 that $(X \times \{0\}) \cup (A \times I)$ is a strong deformation retract of $X \times I$ and Lemma 2.24 gives that there exists a function $\psi : X \longrightarrow I$ with $\psi^{-1}(0) = A$. Define $\Psi : X \times I \longrightarrow I$ by $\Psi(x,t) = t\psi(x)$. We then have that $\Psi^{-1}(0) = (X \times \{0\}) \cup (A \times I)$. Now apply Theorem 2.26. $\square$

Lemma 2.28. *An inclusion $i : A \hookrightarrow X$ is a cofibration if and only if there exist a function $\psi : X \longrightarrow I$ such that $A \subseteq \psi^{-1}(0)$ and a homotopy $h : X \times I \longrightarrow X$ relative to A such that $h|_{X \times \{0\}}$ is the identity on X and $h(x,t)$ is in A if $t > \psi(x)$.*

Proof. Assume first that i is a cofibration. Then we know that there exists a retraction $r : X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$ and the map ψ defined as in Lemma 2.24 satisfies $A \subseteq \psi^{-1}(0)$, although we do not necessarily have equality if A is not closed. Define the homotopy h as

$$h(x,t) = \mathrm{proj}_X(r(x,t))\,.$$

Then we have that h fixes all points in A since r is the identity on $(X \times \{0\}) \cup (A \times I)$. Moreover, we have that if $s > \psi(x)$, then $h(x,s)$ is in A, because otherwise we must have that $r(x,s)$ is in $X \times \{0\}$ (since the range of r is $(X \times \{0\}) \cup (A \times I)$) and then we would have that

$$\big|s - \mathrm{proj}_I(r(x,s))\big| = |s - 0| = s\,,$$

which is precisely to say that $s \leq \psi(x)$, which is a contradiction.

Assume now that ψ and h are as in the formulation of the lemma. To show that $i : A \hookrightarrow X$ is a cofibration it is enough[6] to show the existence of a retraction $r : X \times I \longrightarrow (X \times \{0\}) \cup (A \times I)$. Define r by

$$r(x,t) = \begin{cases} (h(x,t), 0)\,, & t \leq \psi(x) \\ (h(x,t), t - \psi(x))\,, & t > \psi(x)\,. \end{cases}$$

We note that if $t \leq \psi(x)$, $r(x,t)$ is in $X \times \{0\}$ and that $r(x,t)$ is in $A \times I$ if $t > \psi(x)$ by definition of h, so r has the right range. We also need that r is the identity on $(X \times \{0\}) \cup (A \times I)$. This holds on $A \times I$ since $\psi(x) = 0$ for all $x \in A$, so that $r(x,t) = (\mathrm{id}_X(x), t)$ in this case (since $h|_{X \times \{0\}} = \mathrm{id}_X$), and we furthermore have that r is the identity on $X \times \{0\}$ since we then automatically have that $t = 0 \leq \psi(x)$, so that $r(x,t) = (h(x,0),0) = (\mathrm{id}_X(x),0) = (x,0)$. $\square$

[6]We have only shown this for A closed, but we may just as well assume that A really is closed, since this will be the case in all applications later on in this text.

Example 2.29. The zero section of any topological $\mathbb{R}^n$-bundle is a cofibration. To see this we will make use of a result, found for instance in [Dol63] as Corollary 3.2, that implies that any topological $\mathbb{R}^n$-bundle $p : E \longrightarrow B$ strongly deformation retracts onto its zero section via the fibres, in the sense that the deformation retraction $r : E \times I \longrightarrow E$ satisfies that $p \circ r(e,t) = p(e)$ for all $t \in I$ and each $e \in E$. If we then define $\psi : E \longrightarrow \mathbb{R}$ to be the map

$$\psi(e) = \min\{t \in I \mid r(e,t) \in B \subseteq E\}$$

we have that the maps r and ψ satisfy the requirements of the above lemma, so the zero section $B \hookrightarrow E$ is a cofibration.

Theorem 2.30. *If $f : E \longrightarrow B$ is a fibration and $i : A \hookrightarrow B$ an inclusion which is a closed cofibration, then the inclusion $j : f^{-1}(A) \hookrightarrow E$ is a closed cofibration.*

Proof. We note that j is closed since $f^{-1}(A)$ is the inverse image of a closed set under a continuous map. We know from Lemma 2.28 that there exist a function $\psi : B \longrightarrow I$ such that $A \subseteq \psi^{-1}(0)$ and a homotopy $h : B \times I \longrightarrow B$ relative to A such that $h(x,t)$ is in A whenever $t > \psi(x)$. Regard the diagram

$$\begin{tikzcd}
E \times \{0\} \arrow[rr, "\mathrm{id}_E"] \arrow[d, "\text{inc.}"] & & E \arrow[d, "f"] \\
E \times I \arrow[r, "f \times \mathrm{id}_I"] & B \times I \arrow[r, "h"] & B\,.
\end{tikzcd}$$

Since f is a fibration, there exists a map $\tilde{h} : E \times I \longrightarrow E$ such that $f \circ \tilde{h} = h \circ (f \times \mathrm{id}_I)$ and which is the identity on E when restricted to $E \times \{0\} \cong E$. Note that have that $f^{-1}(A) \subseteq (\psi \circ f)^{-1}(0)$ and let $H : E \times I \longrightarrow E$ be the map

$$H(e,t) = \tilde{h}(e, \min\{t, (\psi \circ f)(e)\})\,.$$

We have that $H(e,0) = \tilde{h}(e,0) = \mathrm{id}_E(e) = e$. Furthermore, whenever $t \geq (\psi \circ f)(e)$ we have that $H(e,t) = \tilde{h}(e, (\psi \circ f)(e))$. By definition of $\tilde{h}$ this gives that $f(H(e,t)) = h(f(e),t)$ and by definition of h we have that $h(f(e),t)$ is in A (since we are assuming that $t > \psi(f(e))$). This is precisely to say that $H(e,t)$ is in $f^{-1}(A)$ whenever $t > (\psi \circ f)(e)$, so H and $\psi \circ f$ satisfy the requirements in Lemma 2.28. Therefore $f^{-1}(A) \hookrightarrow E$ is a cofibration. $\square$

We now come to the main theorem of this section.

Theorem 2.31. *If $i : A \hookrightarrow X$ is an inclusion of a closed subspace which is a strong deformation retract of X and $f : E \longrightarrow X$ is a fibration, then $f^{-1}(A)$ is a strong deformation retract of E.*

Proof. Let $r : X \times I \longrightarrow X$ be a homotopy relative to A from id_X to a retraction $X \longrightarrow A$. We like in Example 2.29 get from Lemma 2.28 that i is a cofibration since A is a strong deformation retract of X. Theorem 2.30 gives that the inclusion $f^{-1}(A) \hookrightarrow E$ is a cofibration. Applying Theorem 2.26 to the diagram

$$\begin{tikzcd}
(E \times \{0\}) \cup (f^{-1}(A) \times I) \arrow[rr, "\mathrm{proj}_E"] \arrow[d, "\text{inc.}"] & & E \arrow[d, "f"] \\
E \times I \arrow[r, "f \times \mathrm{id}_I"] & X \times I \arrow[r, "r"] & X\,,
\end{tikzcd}$$

gives the existence of a map $F : E \times I \longrightarrow E$ which satisfies that $f \circ F = r \circ (f \times \mathrm{id}_I)$ (which tells us that $F|_{E\times\{0\}} = f$ and $F|_{E\times\{1\}} \subseteq A$) and that F restricted to $(E \times \{0\}) \cup (f^{-1}(A) \times I)$ is the projection onto the first coordinate (so $F(e,t) = e$ for all $t \in I$ when $e \in f^{-1}(A)$). $\square$

We turn now our attention to free loop spaces (defined for certain manifolds at the very beginning of this text, but the definition is the same for any topological space). There is for any space X an evaluation map $\mathrm{ev} : LX \longrightarrow X$, given by $\mathrm{ev}(\lambda) = \lambda(1)$. We will need the following result, which for instance is found in [Whi78] as Theorem 7.8.

Proposition 2.32. *If $i : A \hookrightarrow X$ is an inclusion of a closed subspace which is a cofibration, then for any space Y the induced map*

$$i^* : \mathrm{Map}(X,Y) \longrightarrow \mathrm{Map}(A,Y)$$

given by $i^*(f) = f \circ i$ is a fibration. Here the function spaces $\mathrm{Map}(-,-)$ have the compact-open topology.

Applying this result to the inclusion $i : \{1\} \hookrightarrow S^1$ (which can be checked to be a cofibration using for example Lemma 2.28) we get for any space X that the induced map

$$i^* : \mathrm{Map}(S^1, X) \longrightarrow \mathrm{Map}(\{1\}, X)$$

is a fibration. The space $\mathrm{Map}(S^1, X)$ is by definition LX, the free loop space of X, and $\mathrm{Map}(\{1\}, M)$ will be homeomorphic to M itself (see Appendix A), so the map i^* is precisely the evaluation map $\mathrm{ev} : \lambda \longmapsto \lambda(1)$. Thus, $\mathrm{ev} : LX \longrightarrow X$ is a fibration.

A product of two fibrations is a fibration. This follows from the fact that a product of homotopies is a homotopy, so homotopy extensions on the individual factors together yield an extension for the product. We hence have that also $\mathrm{ev} \times \mathrm{ev} : LX \times LX \longrightarrow X \times X$ is a fibration for any topological space X. It also holds that the restriction of a fibration $f : E \longrightarrow B$ to any subspace $A \subseteq B$ is a fibration: the extension of any homotopy $Y \times I \longrightarrow A$ will lift to a homotopy in E since we just as well may view the homotopy as a map having image in B, but we will get that the lift has image in $f^{-1}(A)$, so $f|_{f^{-1}(A)}$ is a fibration.

The above results give in particular that any neighbourhood U of the diagonal $\Delta \subseteq M \times M$ which strongly deformation retracts onto Δ lifts to a strong deformation retraction

$$(\mathrm{ev} \times \mathrm{ev})^{-1}(U) \longrightarrow (\mathrm{ev} \times \mathrm{ev})^{-1}(\Delta)\,.$$

We will use this result when defining the Chas-Sullivan loop product later on. Intuitively, this deformation retraction is a tool for moving the chains in $C_*(LM \times LM)$ so that their basepoints intersect.

3 Products

3.1 The intersection product

Let M be an oriented n-dimensional manifold. The Kister-Mazur theorem together with Example 2.6 tell us that there around the diagonal embedding $M \xrightarrow{\Delta} M \times M$ exists a neighbourhood U which is isomorphic to a topological $\mathbb{R}^n$-bundle over $\Delta(M) \cong M$. Since U is an open subset of the oriented manifold $M \times M$, U will by Lemma 2.14 itself be an oriented manifold with the subspace topology. Since also the base space of the bundle $U \longrightarrow \Delta(M)$ is oriented, we get from Lemma 2.18 that this bundle in fact must be a $K_0^+(\mathbb{R}^n)$-bundle, and we thus have a Thom class in $H^n(U, U \backslash \Delta(M))$. This accommodates for the following.

Definition 3.1. With notation as above, the *intersection product* on $H_*(M)$ is $(-1)^{n(n-i)}$ times the composition

$$H_k(M) \otimes H_\ell(M) \xrightarrow{\times} H_{k+\ell}(M \times M) \longrightarrow H_{k+\ell}(M \times M, \Delta^c) \longrightarrow$$

$$\longrightarrow H_{k+\ell}(U, \Delta^c) \xrightarrow{\tau \frown} H_{k+\ell-n}(U) \xrightarrow{\cong} H_{k+\ell-n}(M)\,,$$

where the second map is the one induced by projection on the chain level (as for instance in the long exact sequence of the pair $(M \times M, \Delta^c)$), the third map is the excision isomorphism, and the last map is induced by a retraction from U onto $\Delta(M) \cong M$. We denote the intersection product of $a \otimes b \in H_*(M) \otimes H_*(M)$ by $a \bullet b$.

Remark 3.2. In the case that M is smooth and also closed, so that we have the Poincaré duality isomorphism $P : H^{n-k}(M) \longrightarrow H_k(M)$, given by cap product with the fundamental class $[M] \in H_n(M)$, i.e.

$$P(\varphi) = \varphi \frown [M]\,,$$

the intersection product is dual to the cup product in cohomology in the sense that

$$(\varphi \smile \psi) \frown [M] = (\varphi \frown [M]) \bullet (\psi \frown [M])\,.$$

See [Bre93] for proofs and details. It is this fact—or rather our wanting to have a neat formula—that explains the factor $(-1)^{n(n-i)}$ in the definition of the intersection product.

3.2 Loop concatenation

For any pointed topological space (X, x), we can form its loop space $\Omega X := \{\omega : (S^1, 1) \longrightarrow (X, x)\}$ (which has the compact-open topology). Any pair of loops $(\omega_1, \omega_2) \in \Omega X \times \Omega X$ determines another loop $\omega_1 * \omega_2$ by

$$(\omega_1 * \omega_2)(t) = \begin{cases} \omega_1(2t), & 0 \le t < 1/2 \\ \omega_2(2t-1), & 1/2 \le t \le 1\,, \end{cases}$$

just like in the definition of the product on the fundamental group of X. We thus have a product on ΩX, the *loop product*, which however in general is neither commutative nor associative.

For the free loop space LX of X, we can regard the subset $\mathcal{L}_X \subseteq LX \times LX$ consisting of pairs of loops (λ, μ) such that $\lambda(1) = \mu(1)$. Let $\gamma : \mathcal{L}_X \longrightarrow LX$ be the map $(\lambda, \mu) \longmapsto \lambda * \mu$, where $\lambda * \mu$, just as for the loop product on ΩX, is given by

$$(\lambda * \mu)(t) = \begin{cases} f(2t), & 0 \leq t < 1/2 \\ g(2t-1), & 1/2 \leq t \leq 1\,. \end{cases}$$

Definition 3.3. We say that the *concatenation homomorphism* is the map

$$\gamma_* : H_*(\mathcal{L}_X) \longrightarrow H_*(LX)\,.$$

3.3 The Chas-Sullivan product

We are now ready to define the main concept in this text, the Chas-Sullivan product on the homology of the free loop space of an oriented manifold. The idea behind the construction is to lift the sequence of maps that constitute the intersection product on the homology of the manifold to the level of loop spaces, and to combine the result with the concatenation homomorphism. Let M be an oriented n-dimensional manifold. We know that there is an open neighbourhood U of the diagonal in $M \times M$ which is isomorphic to an oriented topological $\mathbb{R}^n$-bundle over the diagonal. Set $\tilde{U} = (\mathrm{ev} \times \mathrm{ev})^{-1}(U)$ and $\tilde{\Delta} = (\mathrm{ev} \times \mathrm{ev})^{-1}(\Delta)$, so $\tilde{U}$ is an open neighbourhood around the embedding $\tilde{\Delta} \hookrightarrow LM \times LM$. We get an induced map

$$(\mathrm{ev} \times \mathrm{ev})^* : H^*(M \times M, \Delta^c) \longrightarrow H^*(LM \times LM, \tilde{\Delta}^c)\,,$$

and by excision we get a map

$$(\mathrm{ev} \times \mathrm{ev})^* : H^*(U, \Delta^c) \longrightarrow H^*(\tilde{U}, \tilde{\Delta}^c)\,,$$

so we get an induced class $(\mathrm{ev} \times \mathrm{ev})^*(\tau) \in H^*(\tilde{U}, \tilde{\Delta}^c)$ from the Thom class $\tau \in H^*(U, \Delta^c)$.

The above gives the first maps in the sequence defining the Chas-Sullivan product:

$$H_i(LM) \otimes H_j(LM) \xrightarrow{\times} H_{i+j}(LM \times LM) \longrightarrow H_{i+j}(LM \times LM, \tilde{\Delta}^c) \longrightarrow$$

$$\longrightarrow H_{i+j}(\tilde{U}, \tilde{\Delta}^c) \xrightarrow{\mathrm{ev}^*(\tau)\frown} H_{i+j-n}(\tilde{U})\,,$$

where—just as for the intersection product on $H_*(M)$—the second map is projection and the third is excision.

The results in Section 2.4 give that that $\tilde{\Delta}$ is a strong deformation retract of $\tilde{U}$, so the inclusion $\tilde{\Delta} \hookrightarrow \tilde{U}$ induces an isomorphism on homology. Let

$$\rho : H_{i+j-n}(\tilde{U}) \longrightarrow H_{i+j-n}(\tilde{\Delta})$$

be the inverse of this isomorphism, and add ρ to the above sequence. We have that $\tilde{\Delta}$ is precisely the subset of $LM \times LM$ that consists of pairs of loops (λ, μ) such that $\lambda(1) = \mu(1)$, so we have the concatenation homomorphism $\gamma_* : H_*(\tilde{\Delta}) \longrightarrow H_*(LM)$, which we use to finish the sequence of maps defining the Chas-Sullivan product:

Definition 3.4. The Chas-Sullivan product is (with notation as above) $(-1)^{n(n-i)}$ times the composition

$$H_i(LM) \otimes H_j(LM) \xrightarrow{\times} H_{i+j}(LM \times LM) \longrightarrow$$

$$\longrightarrow H_{i+j}(LM \times LM, \tilde{\Delta}^c) \longrightarrow H_{i+j}(\tilde{U}, \tilde{\Delta}^c) \xrightarrow{\mathrm{ev}^*(\tau)\frown}$$

$$\xrightarrow{\mathrm{ev}^*(\tau)\frown} H_{i+j-n}(\tilde{U}) \xrightarrow{\rho} H_{i+j-n}(\tilde{\Delta}^c) \xrightarrow{\gamma_*} H_{i+j-n}(LM)\,.$$

We will denote the Chas-Sullivan product of $\lambda \otimes \mu \in H_i(LM) \otimes H_j(LM)$ by $\lambda \circ \mu$.

Remark 3.5. As it is formulated here, the Chas-Sullivan product does not respect the grading. It is therefore common to define the *loop homology* (or *string homology*) $\mathbb{H}_*(LM) := H_{*+n}(LM)$, where n is the dimension of M. In this way we get that the Chas-Sullivan product is a map

$$\mathbb{H}_i(LM) \otimes \mathbb{H}_j(LM) = H_{i+n}(LM) \otimes H_{j+n}(LM) \longrightarrow H_{i+j+n}(LM) = \mathbb{H}_{i+j}(LM)\,.$$

The factor $(-1)^{n(n-i)}$ is present because we have the same factor in the definition of the intersection product. However, there exist different sign conventions for the Chas-Sullivan product in the literature.

In the case that the underlying manifold M is smooth and closed the Chas-Sullivan product is well studied. In this case we have that $\mathbb{H}_*(LM)$ with the Chas-Sullivan product is a graded commutative algebra. Moreover, there is a rotation action of the circle on the loop space given as a map by

$$\begin{aligned} c : S^1 \times LM &\longrightarrow LM \\ (t, \lambda) &\longmapsto \lambda_t\,, \end{aligned}$$

where $\lambda_t(s) := \lambda(s+t)$. This gives an induced map $c_* : H_*(S^1 \times LM) \longrightarrow H_*(LM)$ and by precomposing with the cross product we get a map

$$H_*(S^1) \otimes \mathbb{H}_*(LM) = H_*(S^1) \otimes H_{*+n}(LM) \longrightarrow H_{*+n}(LM) = \mathbb{H}_*(LM)\,.$$

Letting σ be the generator in $H_1(S^1) \cong \mathbb{Z}$, we get a map

$$\Delta : \mathbb{H}_*(LM) \longrightarrow \mathbb{H}_{*+1}(LM)$$

given by[7]

$$\Delta(a) = c_*(\sigma \times a)\,.$$

From naturality of the cross product we get that

$$\Delta^2(a) = \Delta(c_*(\sigma \times a)) = c_*(\sigma \times c_*(\sigma \times a)) = c_*((\mathrm{id}_{S^1} \times c)_*(\sigma \times (\sigma \times a)))$$

and associativity of the cross product gives that $\sigma \times (\sigma \times a) = (\sigma \times \sigma) \times a$. This implies that $\Delta^2(a) = 0$ for all $a \in H_*(LM)$ since $\sigma \times \sigma$ is an element in $H_2(S^1) = 0$.

It is shown in [CS99] that one can use the operator Δ to give $\mathbb{H}_*(LM)$ the structure of a Gerstenhaber-algebra. See also [CV05] for details on this.

[7] The symbol 'Δ' does not here have anything to do with the diagonal it usually denotes in this text. It just happens to be the conventional symbol used for denoting this operator.

4 Example calculations

One method for calculating the Chas-Sullivan product, proved in [CJY02], is the following:

Theorem 4.1 (Cohen-Jones-Yan, 2002)**.** *For M a simply-connected orientable manifold, there is a spectral sequence with $E^2_{i,j} = H_{i+n}(M; H_j(\Omega M))$ which converges to $\mathbb{H}_*(LM)$. Furthermore, the Chas-Sullivan product on $\mathbb{H}_*(LM)$ is induced by the product on the E^2-page which is given by the intersection product on $H_*(M; H_*(\Omega M))$ with the pairing of the coefficients that is given by the Pontryagin product on $H_*(\Omega M)$.*

Remark 4.2. As it is stated in the original article, the theorem asserts that $E^2_{-i,j} = H^i(M; H_j(\Omega M))$. The authors obtain this result from what we have stated here by applying Poincaré duality in the very last step of the derivation, and in the case that M is closed, the statements are equivalent. Since we here are also concerned with non-closed manifolds, we choose to present this more general version of the theorem. We note here also that the theorem as stated in the original source [CJY02] assumes that M is closed and smooth. However, an inspection of the proof they give there gives that these assumptions are not needed for the existence of the spectral sequence.

On the chain level, the product on the E^2-page in the spectral sequence for $\mathbb{H}_*(LM)$ is thus given as $(a \otimes \lambda)(b \otimes \mu) = (a \bullet b) \otimes (\lambda \cdot \mu)$ for $a, b \in C_*(M)$ and $\lambda, \mu \in C_*(\Omega M)$, where '$\cdot$' denotes the Pontryagin product induced by the loop product.

To compute the Chas-Sullivan product using Theorem 4.1, we must know both the intersection product for the manifold and the Pontryagin ring for its loop space. We will study a certain fibration to gain knowledge about the intersection product. Note that the map $H_{i+j}(M \times M) \longrightarrow H_{i+j}(M \times M, \Delta^c)$ which is used in the definition of the intersection product fits into the long exact sequence for the pair $(M \times M, \Delta^c)$:

$$\cdots \longrightarrow H_{i+j}(\Delta^c) \longrightarrow H_{i+j}(M \times M) \longrightarrow H_{i+j}(M \times M, \Delta^c) \longrightarrow H_{i+j-1}(\Delta^c) \longrightarrow \cdots .$$

Moreover, the diagonal complement $\Delta^c = \{(x, y) \in M \times M \mid x \neq y\}$ is precisely the (ordered) configuration space[8] $F_2(M)$. We will try to calculate the homology of $F_2(M)$, thus enabling us to draw conclusions about the long exact sequence of the pair $(M \times M, \Delta^c)$ and in particular about the map $f : H_{i+j}(M \times M) \longrightarrow H_{i+j}(M \times M, \Delta^c)$.

For a manifold M, let M_* denote M with a point removed, with the subspace topology. Note that it does not matter which point we remove from M in the sense that the spaces obtained always will be homeomorphic to each other.

Lemma 4.3. *Let M be an orientable n-dimensional manifold such that $H_{n-1}(M)$ is finitely generated.[9] Then the homology groups of M_* are as follows.*

$$\begin{aligned} H_i(M_*) &= H_i(M), && \text{for } 0 \leq i \leq n-2 \\ H_{n-1}(M_*) &= \begin{cases} H_{n-1}(M)\,, & \text{if } M \text{ is compact} \\ H_{n-1}(M) \oplus \mathbb{Z}\,, & \text{if } M \text{ is non-compact} \end{cases} \end{aligned}$$

(and $H_n(M_*) = 0$ since M_* is non-compact).

[8] This fact will not be used much, but makes notation and relations to previous results clearer.

[9] This is always satisfied for closed M if $n \neq 4$, and also for $n = 4$ at least if M is smoothable, see [KS77].

Proof. Take a set $B \subseteq M$ which is homeomorphic to $\mathbb{R}^n$ and which contains the point that is removed in M_*. Since $B \cap M_* = B \setminus \{\text{point}\} \cong \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, the Mayer-Vietoris sequence for $M = M_* \cup B$ will look as follows.

$$H_{i+1}(M) \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(B) \oplus H_i(M_*) \longrightarrow H_i(M) \longrightarrow H_{i-1}(S^{n-1})\,.$$

Assume first that $n \geq 2$. Interesting things might happen only when $i = n-1$ and when $i = n$: for $1 \leq i \leq n-2$ every third group in the sequence is zero and $H_i(B) = 0$, and M_* is connected when $n \geq 2$, so $H_0(M_*) = \mathbb{Z} = H_0(M)$. Since M_* is non-compact, we know that $H_n(M_*) = 0$, so we get the exact sequence

$$0 \longrightarrow H_n(M) \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(M_*) \longrightarrow H_{n-1}(M) \longrightarrow 0\,.$$

Corollary 7.13 in [Bre93] gives us that the torsion subgroup of $H_{n-1}(N)$ is zero for any orientable n-dimensional manifold N, so $H_{n-1}(M)$ is in fact free, since it is assumed to be finitely generated. In the case that M is compact, so that $H_n(M) = \mathbb{Z}$, the sequence thus looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow{f} A \longrightarrow \mathbb{Z}^k \longrightarrow 0$$

for some torsion-free $\mathbb{Z}$-module A and some non-negative integer k. Since A is torsion-free the integer a has to be ± 1, and from this it follows that $A = \mathbb{Z}^k$, i.e. that $H_{n-1}(M_*) = H_{n-1}(M)$.

When M is non-compact, and thus $H_n(M) = 0$, we get the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}^k \longrightarrow 0\,,$$

which splits since $\mathbb{Z}^k$ is free, so $A = \mathbb{Z}^{k+1}$, which is to say that $H_{n-1}(M_*) = H_{n-1}(M) \oplus \mathbb{Z}$.

The case $n = 1$ remains. However, the only manifolds (that are connected and have empty boundary) in dimension 1 are $\mathbb{R}$ and S^1, and the statement in this case thus amounts to saying that $\mathbb{R} \setminus \{*\}$ has two components, while S^1 remains connected—but becomes non-compact—after removing a point. $\square$

Remark 4.4. It should be noted that the above lemma holds also if we instead of assuming that $H_{n-1}(M)$ is finitely generated assume that $H_{n-1}(M)$ is projective. The proof goes through in the same way in this case.

Lemma 4.5. *For an n-manifold M, $p : F_2(M) \longrightarrow M$ given by $p(x, y) = x$ is a fibre bundle, with fibre equal to $M \setminus \{x\}$.*

Proof. Let $x \in M$ be given and take a neighbourhood U of x which is homeomorphic to $\mathbb{R}^n$ via φ and suppose for simplicity that $\varphi(x) = 0$. Let B_1 and B_2 be the inverse images under φ of the open balls centered at the origin in $\mathbb{R}^n$ with radii 1 and 2, respectively. Let $h : (\mathbb{R}^n \setminus \{0\}) \times I \longrightarrow \mathbb{R}^n \setminus \{0\}$ be the homotopy

$$h(v,t) = \begin{cases} \left(\|v\|(1-t) + \frac{2+\|v\|}{2}t\right)\frac{1}{\|v\|}v\,, & \|v\| < 2 \\ v\,, & \|v\| \geq 2\,. \end{cases}$$

What we do here is that we take the annulus between radii 0 and 2 and squish it into the annulus between 1 and 2, leaving everything else unchanged.

Suppose now that we remove x from M. We can then precompose φ^{-1} with h, thus getting a homotopy $F : (\mathbb{R}^n \setminus \{0\}) \times I \longrightarrow U$ which deforms U into $U \setminus \overline{B}_1$ and is the identity on the complement of B_2 for all $t \in I$. Define $\Phi : M \setminus \{x\} \longrightarrow M \setminus \{x\}$ as

$$\Phi(y) = \begin{cases} y\,, & y \in U^c \\ F(\varphi(y), 1)\,, & y \in U \setminus \{x\}\,. \end{cases}$$

One can verify that Φ is a homeomorphism onto its image.

We will now define a trivializing map $p^{-1}(B_1) \longrightarrow B_1 \times (M \setminus \{x\})$. The set $p^{-1}(B_1)$ is by definition $\{(y, y') \in M \times M \mid y \in B_1,\ y' \neq y\}$. Let $g : p^{-1}(B_1) \longrightarrow B_1 \times (M \setminus \{x\})$ be the map $(y, y') \longmapsto (y, \Phi(y'))$. Since Φ misses precisely all points in B_1 this is a homeomorphism onto its image, and since Φ was a homeomorphism from $M \setminus \{x\}$ to its image, it follows that $(\mathrm{id}_{B_1} \times \Phi^{-1}) \circ g : p^{-1}(B_1) \longrightarrow B_1 \times (M \setminus \{x\})$ is a homeomorphism, and it is compatible with p and projection onto the first coordinate, so B_1 is a trivializing set for x. $\square$

It should be noted that the fibre bundle $M_* \longrightarrow F_2(M) \longrightarrow M$ is a special case of the so-called Fadell-Neuwirth fibre bundle (defined in [FN62])

$$\begin{aligned} F_k(M) &\longrightarrow F_\ell(M) \\ (x_1, \ldots, x_k) &\longmapsto (x_1, \ldots, x_\ell)\,, \end{aligned}$$

where $F_i(M)$ is the ordered configuration space of i points in M and $k \geq \ell$. The Leray-Serre spectral sequence for the bundle $M_* \longrightarrow F_2(M) \longrightarrow M$, which also is known as the Fadell-Neuwirth spectral sequence,[10] has $E^2_{i,j} = H_i(M; H_j(M_*))$, and converges to $H_*(F_2(M))$ (the coefficients $H_j(M_*)$ are to be interpreted as local coefficients in the case that M is not simply-connected). Let us look a little closer at the E^2-page of this spectral sequence. The universal coefficient theorem gives that the total complex $\mathrm{Tot}(E^2)$ of the E^2-page of this spectral sequence is given for simply-connected M in degree k as

$$\mathrm{Tot}_k(E^2) := \bigoplus_{i+j=k} H_i(M; H_j(M_*)) \cong$$

$$\cong \bigoplus_{i+j=k} \big(H_i(M) \otimes H_j(M_*) \oplus \mathrm{Tor}(H_{i-1}(M), H_j(M_*))\big)\,. \tag{4}$$

On the other hand, the Künneth formula gives that we for the product manifold $M \times M$ have that

$$H_k(M \times M) \cong \bigoplus_{i+j=k} \big(H_i(M) \otimes H_j(M) \oplus \mathrm{Tor}(H_{i-1}(M), H_j(M))\big)\,. \tag{5}$$

These two expressions look awfully similar, and we will try to figure out exactly how similar they are. Assume that $H_{n-1}(M)$ is projective,[11] so that we may apply Lemma 4.3, and that M is simply-connected, so that the isomorphism in (4) is valid. The Tor functor commutes with direct sums in the sense that

$$\mathrm{Tor}(-, A \oplus B) = \mathrm{Tor}(-, A) \oplus \mathrm{Tor}(-, B)$$

[10] Or rather a special case thereof, but we will refer to it as *the* Fadell-Neuwirth spectral sequence for brevity.

[11] Or finitely generated, in which case it is free and hence projective, as seen in the proof of lemma 4.3.

for all $\mathbb{Z}$-modules A and B. This gives that

$$\mathrm{Tor}(-, H_j(M_*)) = \mathrm{Tor}(-, H_j(M))\,,$$

since the only possible difference between $H_j(M_*)$ and $H_j(M)$ is a $\mathbb{Z}$-summand, and $\mathrm{Tor}(A, \mathbb{Z}) = 0$ for all $\mathbb{Z}$-modules A. We therefore want to compare the summands

$$\bigoplus_{i+j=k} H_i(M) \otimes H_j(M) \qquad \text{and} \qquad \bigoplus_{i+j=k} H_i(M) \otimes H_j(M_*)$$

in the right-hand sides of (4) and (5), respectively. In the case that M is closed, the difference will arise when $j = n$ and we have that

$$H_{k-n}(M) \otimes H_n(M) = H_{k-n}(M) \otimes \mathbb{Z} \cong H_{k-n}(M)\,,$$

while $H_{k-n}(M) \otimes H_n(M_*) = 0$. If M is open, on the other hand, things will change where $j = n-1$, and we have that

$$\begin{aligned} H_{k-(n-1)}(M) \otimes H_{n-1}(M_*) &= H_{k-(n-1)}(M) \otimes \Big(H_{n-1}(M) \oplus \mathbb{Z}\Big) = \\ &= \Big(H_{k-(n-1)}(M) \otimes H_{n-1}(M)\Big) \oplus \Big(H_{k-(n-1)}(M) \otimes \mathbb{Z}\Big) = \\ &= \Big(H_{k-(n-1)}(M) \otimes H_{n-1}(M)\Big) \oplus \mathbb{Z}\,. \end{aligned}$$

Thus, when M is closed, $\mathrm{Tot}_k(E^2)$ is an '$H_{k-n}(M)$-summand' smaller than $H_k(M \times M)$, and when M is open we instead have that $\mathrm{Tot}_k(E^2)$ is $H_k(M \times M) \oplus H_{k-(n-1)}(M)$. In short:

$$\mathrm{Tot}_k(E^2) \oplus H_{k-n}(M) = H_k(M \times M) \tag{6}$$

when M is closed, and

$$\mathrm{Tot}_k(E^2) = H_k(M \times M) \oplus H_{k-(n-1)}(M) \tag{7}$$

when M is open, provided that $H_{n-1}(M)$ is projective.

We now return to the long exact sequence for the pair $(M \times M, \Delta^c)$. Using the Thom isomorphism $H_k(M \times M, \Delta^c) \cong H_{k-n}(M)$, we can rewrite the sequence as

$$\cdots \longrightarrow H_k(M \times M) \longrightarrow H_{k-n}(M) \longrightarrow H_{k-1}(\Delta^c) \longrightarrow H_{k-1}(M \times M) \longrightarrow \cdots\,.$$

Assume now that the spectral sequence for the Fadell-Neuwirth fibration $F_2(M) = \Delta^c \longrightarrow M$ collapses at the E^2-page, and that we do not have any extension issues. If this is the case, then the above sequence will look like

$$\cdots \longrightarrow H_k(M \times M) \longrightarrow H_{k-n}(M) \longrightarrow \mathrm{Tot}_{k-1}(E^2) \longrightarrow H_{k-1}(M \times M) \longrightarrow \cdots\,.$$

Let us therefore study this sequence a bit. In the closed case, the sequence has the pattern

$$\cdots \longrightarrow \mathrm{Tot}_k(E^2) \longrightarrow \mathrm{Tot}_k(E^2) \oplus H_{k-n}(M) \longrightarrow H_{k-n}(M) \xrightarrow{\partial} \cdots\,, \tag{8}$$

repeating every third group (the boundary map is marked to keep track of where we really are in the long exact sequence for $(M \times M, \Delta^c)$), and when M is open we get the pattern

$$\cdots \longrightarrow H_{k-n+1}(M) \xrightarrow{\partial} H_{k-n+1}(M) \oplus H_k(M \times M) \longrightarrow H_k(M \times M) \longrightarrow \cdots\,. \tag{9}$$

The above patterns are of the form $A \xrightarrow{f} A \oplus B \xrightarrow{g} B$ and thus very reminiscent of (split) short exact sequences, but we do not know anything about the injectivity and surjectivity of f and g. In the general case, f need not be injective. Take for instance $A = B = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and let f be the map

$$(a_1, a_2, a_3, \ldots) \longmapsto \big((a_1 + a_2, a_3, \ldots), (0, \ldots)\big)$$

and g projection onto B. Then $\operatorname{im} f = \ker g$, but f is not injective. We however have the following result:

Lemma 4.6. *If $A \xrightarrow{f} A \oplus B \xrightarrow{g} B$ is an exact sequence (without zeros at the ends!) of free and finitely generated modules over a PID, then f is in fact injective.*

Remark 4.7. It however need not hold that g is surjective, as is seen from the sequence

$$\mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{[2\ -2]} \mathbb{Z}\,.$$

Proof. Since the images and kernels of f and g are submodules of free modules over a PID, they are free modules. We thus get short exact sequences

$$0 \longrightarrow \ker f \longrightarrow A \longrightarrow \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \ker g \longrightarrow A \oplus B \longrightarrow \operatorname{im} g \longrightarrow 0$$

which are split, so that

$$A \cong \ker f \oplus \operatorname{im} f \qquad \text{and} \qquad A \oplus B \cong \ker g \oplus \operatorname{im} g\,.$$

Putting this together, we get that

$$(\ker f \oplus \operatorname{im} f) \oplus B \cong \ker g \oplus \operatorname{im} g\,.$$

Since $\operatorname{im} f = \ker g$, we get that

$$\operatorname{rank}(\ker f \oplus B) = \operatorname{rank}(\operatorname{im} g)$$

which since the rank of B is greater than or equal to that of $\operatorname{im} g$ implies that $\operatorname{rank}(\ker f) = 0$. Since $\ker f$ is free, this implies that $\ker f = 0$, so f is injective. □

If we now make quite a few assumptions about the homology of the manifold M, we can draw some conclusions: if $H_i(M)$ is free and finitely generated for all i, also $H_i(M \times M)$ and $\operatorname{Tot}_i(E^2)$ will be free and finitely generated for all i, so the above lemma applies. This means that every third map in the long exact sequences in (8) and (9) will be zero, so the sequences will split into short exact sequences of the forms

$$0 \longrightarrow \operatorname{Tot}_k(E^2) \longrightarrow \operatorname{Tot}_k(E^2) \oplus H_{k-n}(M) \longrightarrow H_{k-n}(M) \xrightarrow{\partial} 0$$

(for M closed) and

$$0 \longrightarrow H_{k-n+1}(M) \xrightarrow{\partial} H_{k-n+1}(M) \oplus H_k(M \times M) \longrightarrow H_k(M \times M) \longrightarrow 0$$

(for M open), respectively. Now *if* the Fadell-Neuwirth spectral sequence collapses at the E^2-page and $H_i(M)$ is free and finitely generated for all i (which also gives that there cannot be any extension issues), these short exact sequences in fact give us the behaviour of the long exact sequence of the pair $(M \times M, \Delta^c)$. In this case we immediately get that the map $H_i(M \times M) \longrightarrow H_i(M \times M, \Delta^c)$ is zero when M is open. Combining this with Theorem 4.1, we get the following result.

Proposition 4.8. *If M is an open manifold such that $H_*(M)$ is free and of finite type and the Fadell-Neuwirth spectral sequence for the fibration $F_2(M) \longrightarrow M$ collapses at the E^2-page, the intersection product on $H_*(M)$ and the Chas-Sullivan product on $H_*(LM)$ are trivial.*

This might be a quite specialized result, but it for example applies to $\mathbb{R}^n$ minus a finite number of points, since the Fadell-Neuwirth spectral sequence then has to collapse by dimensional reasons.

On the other hand, when M is closed, we get that the map $H_i(M \times M) \longrightarrow H_i(M \times M, \Delta^c)$ is surjective whenever the sequence collapses at the E^2-page, and since all other maps constituting the intersection product are injective (or even isomorphisms), we may get nonzero results in this case, which we may compute using the above results.

4.1 Intersection product on $H_*(S^n)$

We will calculate the intersection product for $H_*(S^n)$ $(n \geq 1)$ using the results from earlier in this section. To begin with we remind ourselves that we want to compute $(-1)^{n(n-i)}$ times the composition

$$H_i(S^n) \otimes H_j(S^n) \xrightarrow{\times} H_{i+j}(S^n \times S^n) \longrightarrow H_{i+j}(S^n \times S^n, \Delta^c) \longrightarrow$$
$$\longrightarrow H_{i+j}(U, \Delta^c) \xrightarrow{\tau\frown} H_{i+j-n}(U) \cong H_{i+j-n}(S^n)\,.$$

Since $H_i(S^n) = \mathbb{Z}$ if i is 0 or n and zero otherwise, the Künneth formula gives that the cross product $H_*(S^n) \otimes H_*(S^n) \longrightarrow H_*(S^n \times S^n)$ is an isomorphism, so that

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z}\,, & i = 0,\ 2n \\ \mathbb{Z}^2\,, & i = n \\ 0\,, & \text{otherwise} \end{cases}$$

and we by Lemma 4.3 (or by noting that $S^n \setminus \{*\} \cong B^n$) have that

$$H_i(S^n \setminus \{*\}) = \begin{cases} \mathbb{Z}\,, & i = 0 \\ 0\,, & \text{otherwise}\,. \end{cases}$$

This gives that the E^2-page of the Fadell-Neuwirth spectral sequence for the fibration $F_2(S^n) \longrightarrow S^n$ has

$$E^2_{i,j} = H_i(S^n; H_j(S^n \setminus \{*\})) = \begin{cases} H_i(S^n)\,, & j = 0 \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and the sequence collapses at the E^2-page by dimensional reasons. By the above discussion, the map $H_i(S^n \times S^n) \longrightarrow H_i(S^n \times S^n, \Delta^c)$ is surjective. Also by dimensional reasons, the product can only be nonzero if $i + j = n$ or $i + j = 2n$. In the case that $i+j = n$, we have that the generators for $H_n(S^n \times S^n) \cong \mathbb{Z}^2$ are given by the images under the cross product of $\sigma \otimes u$ and $u \otimes \sigma$, where u and σ are the generators for $H_0(S^n)$ and $H_n(S^n)$, respectively. We know by the Künneth formula that $u \times \sigma$ and $\sigma \times u$ are linearly independent in $H_n(S^n \times S^n)$, and let us choose the isomorphism $H_n(S^n \times S^n) \cong \mathbb{Z}^2$ so that $\sigma \times u$ corresponds to $(1, 0)$ and $u \times \sigma$ to $(0, 1)$. We have from the long exact sequence

$$\cdots \longrightarrow H_n(\Delta^c) \longrightarrow H_n(S^n \times S^n) \longrightarrow H_n(S^n \times S^n, \Delta^c) \longrightarrow \cdots$$

that the kernel of the map $H_n(S^n \times S^n) \longrightarrow H_n(S^n \times S^n, \Delta^c)$ is the image of the map $H_n(\Delta^c) \longrightarrow H_n(S^n \times S^n)$, which is induced by the inclusion. For the specific case of spheres, we actually have that the complement of the diagonal in $S^n \times S^n$ is homotopy equivalent to S^n itself. In fact, the set

$$\{(x, -x) \in S^n \times S^n \mid x \in S^n\} \cong S^n$$

is a strong deformation retract of Δ^c. We will give an explicit formula for this deformation retraction. Let (x, y) be a point in Δ^c. We may write

$$y = \lambda x + \sqrt{1 - \lambda^2}\, x_y^{\perp} \,, \tag{10}$$

for some $\lambda \in [-1, 1)$ and some $x_y^{\perp} \in S^n$ which is orthogonal to x, and this expression for y is uniquely determined for a given x. Define a homotopy $h : \Delta^c \times I \longrightarrow \Delta^c$ by

$$h((x, y), t) = \left(x\,,\ (\lambda - (\lambda + 1)t)x + \sqrt{1 - (\lambda - (\lambda + 1)t)^2}\, x_y^{\perp}\right) .$$

The map $y \longmapsto x_y^{\perp}$ can be checked to depend continuously on both x and y, so h is continuous. For a point of the form $(x, -x) \in \Delta^c$ we have that the λ in (10) equals -1. Hence

$$h((x, -x), t) = \left(x\,,\ (-1)x + \sqrt{1 - (-1)^2}\, x_{-x}^{\perp}\right) = (x, -x)\,,$$

so h is stationary on $\{(x, -x)\}$. We get by insertion of $t = 0$ that $h|_{\Delta^c \times \{0\}} = \mathrm{id}_{\Delta^c}$ and we have that $h((x, y), 1) = (x, -x)$ for all $(x, y) \in \Delta^c$. This shows that $(S^n \times S^n) \setminus \Delta$ strongly deformation retracts onto the set of points of the form $(x - x)$.

Let $A : S^n \longrightarrow \Delta^c$ be the map $x \longmapsto (x, -x)$, so that A induces an isomorphism $H_n(S^n) \longrightarrow H_n(\Delta^c)$. Denoting the inclusion $\Delta^c \hookrightarrow S^n \times S^n$ by i, we get the commutative diagram

$$\begin{tikzcd} H_n(\Delta^c) \arrow[r, "i_*"] & H_n(S^n \times S^n) \\ H_n(S^n) \arrow[u, "A_*"] \arrow[r, "(i \circ A)_*"] & H_n(S^n \times S^n)\,. \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

For a sphere of degree n, the antipodal map has degree $(-1)^{n+1}$, so we get that the image of a generator in $H_n(\Delta^c)$ under the map i_* is $\pm(1, 1)$ if n is odd and $\pm(1, -1)$ if n is even. This gives that the map $H_n(S^n \times S^n) \longrightarrow H_n(S^n \times S^n, \Delta^c)$ is given by a matrix of the form

$$\pm\begin{bmatrix} 1 & -1 \end{bmatrix}$$

when n is odd, and by a matrix of the form

$$\pm\begin{bmatrix} 1 & 1 \end{bmatrix}$$

when n is even. The intersection product involves the factor $(-1)^{n(n-i)}$, where i is the degree of the left element in the product. We thus introduce a minus sign in the product of two classes in the homology for the n-sphere precisely when n is odd and the left element is u and the right one is σ. This gives that we may regard the map $H_n(S^n \times S^n) \longrightarrow H_n(S^n \times S^n, \Delta^c)$ as given by

$$\pm\begin{bmatrix} 1 & 1 \end{bmatrix}$$

for all n and need not care about the factor $(-1)^{n(n-i)}$ any longer. We however still need to determine whether the sign in front of the matrix is positive or negative. We will return to this question shortly.

Before solving the sign question, we turn to the case $i+j=2n$, which in practice means that $i=j=n$, because everything else will be zero. Since both $H_{2n}(S^n \times S^n)$ and $H_{2n}(S^n \times S^n, \Delta^c)$ are isomorphic to $\mathbb{Z}$ and the map $H_{2n}(S^n \times S^n) \longrightarrow H_{2n}(S^n \times S^n, \Delta^c)$ is surjective, this map must in fact be an isomorphism, so *all* maps in the intersection product are isomorphisms in this case. We thus get that $\sigma \bullet \sigma = \pm\sigma$.

Now to the sign question. We can choose the neighbourhood U of the diagonal in $S^n \times S^n$ as

$$S^n \times S^n \setminus \{(x,-x) \mid x \in S^n\},$$

and the projection map p onto the diagonal is $(x,y) \longmapsto (x,x)$. We then get that

$$p^{-1}(x,x) = \{(x,y) \mid y \in B^n\} \cong \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n,$$

and p can be checked to be locally trivial over $\Delta(S^n) \cong S^n$ minus any point.

An explicit representative for σ is given by the chain $s = s_0 - s_1$ consisting of two simplices s_0 and s_1 which are mapped homeomorphically to one hemisphere each in such a way that $s_0|_{\partial\Delta_n} = s_1|_{\partial\Delta_n}$ (see [Hat00]). To make things easier later on, we choose the simplices s_0 and s_1 to be oriented consistently with the chosen (standard) orientation of the sphere in the sense that the projection of s_0 onto the hyperplane P through the intersection of s_0 and s_1 is a chain representative for the standard orientation on $H_{n-1}(P \mid 0)$. Let c be a chain representative of the generator $u \in H_0(S^n)$. We may without loss of generality assume that the image of c is contained in the image of the interior of s_0. A representative for the class $u \times \sigma \in H_n(S^n \times S^n)$, and hence also for its coset in $H_n(S^n \times S^n, \Delta^c)$, is given by the chain $c \times s \in C_n(S^n \times S^n)$ given by

$$(c \times s)(t_0, \ldots, t_n) = (c(1), s(t_0, \ldots, t_n))$$

(see [Dol72] for details). When mapping $u \times \sigma$ further to $H_n(U, \Delta^c)$ by the excision isomorphism we thus thankfully do not need to perform any barycentric subdivision, but only discard the simplex s_1, since it (and only it) intersects the point opposite to the image of c. A representative for the image of $u \times \sigma$ in $H_n(U, \Delta^c)$ is thus the chain $c \times s_0$, which is defined as

$$(c \times s_0)(t_0, \ldots, t_n) = (c(1), s_0(t_0, \ldots, t_n)).$$

It is now that the Thom class $\tau \in H^n(U, \Delta^c)$ comes in to play. By definition, the cochain representative for τ evaluates to 1 on chain representatives for the chosen generators for $H_n(p^{-1}(x,x) \mid (x,x)) \cong H_n(\mathbb{R}^n \mid 0)$. Let

$$j_x : p^{-1}(x,x) \longrightarrow U$$

be the inclusion of the fibre $p^{-1}(x,x)$. We have the following diagram, which is commutative because of the definition of a pullback.

$$\begin{tikzcd}
C_n(p^{-1}(x,x), p^{-1}(x,x) \setminus \{(x,x)\}) \arrow[r, "(j_x)_\#"] \arrow[d, "j_x^\#(\tau)"] & C_n(U, \Delta^c) \arrow[d, "\tau"] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z}\,.
\end{tikzcd}$$

(We abuse notation a bit here and write τ also for the cochain representative of the Thom class.) A chain representative for the generator in $H^n(p^{-1}(x,x) \mid (x,x))$ is given by any n-simplex which is oriented in the right way and which intersects (x,x) in its interior, so for example s_0 will do the job. We have that

$$((j_x)_\#(s_0))(t_0,\ldots,t_n) = ((x,x), s_0(t_0,\ldots,t_n))\,,$$

and thus that $(j_x)_\#(s_0)$ represents the same homology class in $H_n(U,\Delta^c)$ as $c \times s$ does. This gives that the cap product $\tau \frown (u \times \sigma)$ on the (co)chain level is given as the 0-chain which is given by

$$(\tau \frown (c \times s))(1) = (\tau(c \times s))(c(1), s(0,\ldots,0,1)) = +1 \cdot (c(1), \text{some point})\,,$$

and as a consequence represents $+1$ times the generator in $H_0(U)$. The last map $H_0(U) \longrightarrow H_0(S^n)$ is the inverse of the isomorphism which is induced by the inclusion onto the diagonal, so we have 'the same' generators in both groups and hence get that $u \bullet \sigma$ is equal to u.

It now follows immediately that also $\sigma \bullet u$ is equal to $+u$, since $u \times \sigma$ and $\sigma \times u$ were seen to have the same image in $H_n(S^n \times S^n, \Delta^c)$ (with our treatment of the minus signs, that is).

We leave out the discussion of the sign of $\sigma \bullet \sigma$, but merely state that $\sigma \bullet \sigma = +\sigma$. One could give a similar argument as the above for this, but for example the expression for the chain representative of $\sigma \times \sigma$ would be way more complicated. We spare ourselves from this.

Summarizing our results, we have for our generators u and σ in $H_*(S^n)$ that $\sigma^2 = \sigma$, $u^2 = 0$ (by dimensional reasons) and that $\sigma \bullet u = u \bullet \sigma = u$. This can be phrased as follows.

Proposition 4.9. *The homomorphism induced by the mapping*

$$u \longmapsto x\,, \qquad \sigma \longmapsto 1$$

is a ring isomorphism from $H_*(S^n)$ with the intersection product to $\mathbb{Z}[x]/\langle x^2\rangle$, which, however, is not a homomorphism of graded rings.

Remark 4.10. The above result is an example of the more general phenomenon mentioned in Remark 3.2 that for a smooth and closed manifold $[M]$ the intersection product will be Poincaré dual to the cup product in cohomology. We indeed have that $(H_*(M), \bullet)$ and $(H^*(M), \smile)$ are isomorphic as rings—via Poincaré duality!

4.2 Chas-Sullivan product on $H_*(LS^n)$

We will now give an account of how to calculate the Chas-Sullivan product on $H_*(LS^n)$ using Theorem 4.1 (for $n \geq 2$, since we must have a simply-connected space in order to be able to use Theorem 4.1). To do this calculation, we need the following, which is proved in [BS53].

Proposition 4.11. *With the Pontryagin product induced by loop concatenation, the homology $H_*(\Omega S^n)$ is for $n \geq 2$ isomorphic as a graded algebra to the tensor algebra $T(x)$, where x has degree $n-1$.*

Remark 4.12. The generator in $H_{n-1}(\Omega S^n)$ that corresponds to x under this isomorphism is given (up to a choice of sign) by the image of the generator in $\pi_n(S^n) \cong \mathbb{Z}$ under the composition

$$\pi_n(S^n) \xrightarrow{\cong} \pi_{n-1}(\Omega S^n) \xrightarrow{\cong} H_{n-1}(\Omega S^n)\,.$$

The two isomorphisms here are, in order of appearance, the canonical isomorphism $\pi_n(S^n) \cong \pi_{n-1}(\Omega S^n)$ and the Hurewicz homomorphism, which is an isomorphism since ΩS^n is $(n-2)$-connected.

Since the spectral sequence for the fibration $LS^n \longrightarrow S^n$ has $E^2_{i,j} = H_{i+n}(S^n; H_j(\Omega S^n))$, the sequence can thus only have nontrivial differentials on the E^n-page. An exposition of how to compute these differentials is found in [McC90]; the results are that d_n is equal to the zero map unless n is even, in which case it is given as

$$d_n(1 \otimes x^{\otimes k}) = 0$$

(which follows directly by dimensional reasons) and

$$d_n(\sigma \otimes x^{\otimes k}) = \begin{cases} -2x^{\otimes k+1}\,, & k \text{ odd} \\ 0\,, & k \text{ even}\,, \end{cases}$$

where $x \in H_{n-1}(\Omega S^n)$ is the generator for $H_*(\Omega S^n)$ as an algebra and σ is the generator in $H_n(S^n)$ and we make the identifications $H_i(S^n; H_j(\Omega S^n)) \cong H_i(S^n) \otimes H_j(\Omega S^n) \cong H_j(\Omega S^n)$ via the universal coefficient theorem. By $x^{\otimes k}$ we mean the k-fold tensor product $x \otimes \ldots \otimes x$, which is the generator in $H_{k(n-1)}(\Omega S^n)$. Let u and ω be the generators in $H_0(S^n)$ and $H_0(\Omega S^n)$, respectively.

We begin with the odd case, where the sequence collapses at the E^2-page. It can be shown (as is done in [CJY02]) that there for dimensional reasons cannot be any extension issues. We thus get that

$$\mathbb{H}_k(LS^n) = \mathrm{Tot}_k(E^2) = \bigoplus_{i+j=k} H_{i+n}(S^n) \otimes H_j(\Omega S^n)$$

as a graded module, and the product is given on the generators by

$$(a \otimes x^{\otimes k}) \circ (b \otimes x^{\otimes \ell}) = (a \bullet b) \otimes x^{\otimes (k+\ell)}\,.$$

This gives an algebra isomorphism

$$\mathbb{H}_*(LS^n) \cong \Lambda(a) \otimes \mathbb{Z}[w]\,,$$

where $w := \sigma \otimes x$ and $a := u \otimes \omega$ and $\Lambda(a)$ is the exterior algebra on a. The degree of w (not in $\mathbb{H}_*(LS^n)$ but in $H_*(LS^n)$, where we have shifted all degrees by $-n$) is $\deg \sigma + \deg x - n = n + (n-1) - n = n-1$. Likewise, the degree of a is $0 + 0 - n = -n$.

In the case that n is even, we get that the sequence collapses at the E^3-page and have that

$$E^\infty_{i,j} = \begin{cases} \mathbb{Z}[u \otimes \omega]\,, & i = n\,,\ j = 0 \\ \mathbb{Z}[\sigma \otimes x^{\otimes 2k}]\,, & i = 0\,, j = 2k(n-1) \\ \mathbb{Z}[u \otimes x^{\otimes (2k+1)}]\,, & i = n\,, j = (2k+1)(n-1) \\ \mathbb{Z}[u \otimes x^{\otimes 2k}]/\langle 2(u \otimes x^{\otimes 2k})\rangle\,, & i = n\,,\ j = 2k(n-1)\,,\ k > 0 \\ 0\,, & \text{else} \end{cases}$$

as $\mathbb{Z}$-modules.

We have that $E^{\infty}_{*,*}$ is generated as a $\mathbb{Z}$-algebra by the elements (or rather their images under d_n)

$$a := u \otimes \omega, \quad b := u \otimes x \quad \text{and} \quad v := \sigma \otimes x^{\otimes 2} .$$

To see this, note that generators for $E^{\infty}_{i,j}$ for $i = 0$ and positive j are expressible either as $b \circ v^k$ or as $a \circ v^k$ for some k; for $i = n$ elements of the form v^k are generators for all nonzero $E^{\infty}_{i,j}$; and a is a generator for $E^{\infty}_{0,0}$. Since $E^{\infty}_{0,j}$ is 2-torsion for $j = 2k(n-1)$, $k > 0$, we get that $2(a \circ v^k) = 0$ for $k > 0$. By dimensional reasons $a^2 = a \circ b = 0$. All this gives that we have an isomorphism of algebras

$$E^{\infty} \cong (\Lambda(b) \otimes \mathbb{Z}[a, v])/\langle a^2, b^2, 2(a \circ v^k)\rangle ,$$

where $\deg v = n + 2(n-1) - n = 2(n-1)$ and $\deg b = 0 + (n-1) - n = -1$ (and the degree of a is still $-n$). When $n > 2$, we like in the even case cannot have any extension issues by dimensional reasons, so we get then that

$$\mathbb{H}_*(LS^n) \cong (\Lambda(b) \otimes \mathbb{Z}[a, v])/\langle a^2, a \circ b, 2(a \circ v^k)\rangle .$$

This holds also in the case $n = 2$, but to see this one has to study the filtration used when defining the spectral sequence in more closely. We refer the reader to [CJY02] for details. We summarize the above in the following proposition, where we for completeness include also the case $n = 1$, a proof of which also can be found in [CJY02].

Proposition 4.13. *For n odd there is for $n \geq 3$ an algebra isomorphism*

$$\mathbb{H}_*(LS^n) \cong \Lambda(a) \otimes \mathbb{Z}[w] ,$$

where the product on $H_*(LS^n)$ is the Chas-Sullivan product and $\deg a = -n$ and $\deg w = n-1$. For n even the Chas-Sullivan product on $H_*(LS^n)$ gives rise to an algebra isomorphism

$$H_*(LS^n) \cong (\Lambda(b) \otimes \mathbb{Z}[a, v])/\langle a^2, b^2, 2(a \circ v^k)\rangle ,$$

where $\deg b = -1$, $\deg a = -n$ and $\deg v = 2n - 2$. We also have that the Chas-Sullivan product gives an algebra isomorphism

$$H_*(LS^1) \cong \Lambda(b) \otimes \mathbb{Z}[t, t^{-1}] ,$$

where $\deg b = -1$ and $\deg t = \deg t^{-1} = 0$.

4.3 Concluding remarks

We have shown that the Chas-Sullivan product exists on the homology of the free loop space of any orientable manifold. However, the only result we have reached concerning its behaviour when the manifold is non-closed merely says that the product must be zero in some cases. It could be of interest to determine whether there exists an open manifold which admits a non-trivial Chas-Sullivan product. A further question one could study (assuming a positive answer to the first) is what properties the string homology of the loop space get in the non-closed case. For instance, the unit in the intersection ring on the homology of a closed manifold is the fundamental class of the manifold, and there cannot exist a unit when the manifold is not compact, so the behaviour of the string homology is not necessarily as nice in this case. Furthermore, one could also try to calculate the product in the case that the underlying manifold is not smoothable.

5 Appendices

5.1 Appendix A: The compact-open topology

The compact-open topology on a function space is used in several places in this text, in particular in the definition of one of the main object of interest, the free loop space of a manifold. We therefore in this appendix give the definition of the compact-open topology and list a few of its properties. It should be noted that there exist a few variants for how to define the compact-open topology; which definition one might prefer to use depends in part on what assumptions one makes about the domain and codomain of the functions. We give here the perhaps most general definition.

Definition 5.1. Let X and Y be topological spaces and let $\mathrm{Map}(X, Y)$ be the set of continuous functions from X to Y. The compact-open topology on $\mathrm{Map}(X, Y)$ is generated by sets of the form

$$\{f : X \longrightarrow Y \mid f(K) \subseteq U, \text{ for some compact } K \subseteq U \text{ and } U \subseteq Y\},$$

in the sense that all finite intersection of sets of this form constitute a basis for the compact-open topology on $\mathrm{Map}(X, Y)$.

We get directly from the definition that $\mathrm{Map}(\{*\}, X) \cong X$ for any topological space X, where $\{*\}$ is the one-point space. To see this, note that $U_f := \{f : \{*\} \longrightarrow X \mid f(*) \subseteq U\}$ is an open set in $\mathrm{Map}(\{*\}, X)$ for any set U in a basis for the topology on X. We may just as well identify U with U_f by identifying a point in $x \in U$ with the map that takes the point $*$ to x, thereby getting an explicit homeomorphism.

For any two topological spaces X and Y we have a map

$$e : \mathrm{Map}(X, Y) \times X \longrightarrow Y$$

given by $e(f, x) = x$. One reason for choosing the compact-open topology as the topology on $\mathrm{Map}(X, Y)$ is that we have the following fact, which is proved in for instance [AGP02].

Proposition 5.2. *If X and Y are topological spaces and X is locally compact and Hausdorff, the compact-open topology is the coarsest topology on* $\mathrm{Map}(X, Y)$ *such that the map e is continuous.*

We also have the following result, also proved in [AGP02]:

Proposition 5.3. *If X, Y and Z are topological spaces and Y is locally compact and Hausdorff, the map*

$$F : \mathrm{Map}(X \times Y, Z) \longrightarrow \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

given by

$$\big(F(f)(x)\big)(y) = f(x, y)$$

is a homeomorphism (when the mapping spaces have the compact-open topology).

Since the circle S^1 is locally compact and Hausdorff, this gives in particular that paths in the free loop space $\mathrm{Map}(S^1, X)$ (i.e. maps $I \longrightarrow LX$) of a space X corresponds homeomorphically to maps $I \times S^1 \longrightarrow X$, which is the same thing as homotopies of maps from S^1 to X.

For further details on the compact-open topology we refer the reader directly to [AGP02].

5.2 Appendix B: Equivalence of notions of orientability

The definitions of orientability and orientations of a manifold that are used in this text are somewhat idiosyncratic. They were chosen over the more traditional ones involving the orientation bundle to make a few proofs easier. Also the definition of bundle orientability is different from the traditional one, but this is more a question about the usual notion only covering the case of vector bundles. It should be shown that the notions used in the text are equivalent to the old ones where applicable. We therefore recall the construction of the orientation bundle of a d-dimensional manifold M. As a set, the orientation bundle $\Theta = \Theta(M)$ is the disjoint union

$$\bigsqcup_{x \in M} H_n(M \mid x)\,.$$

and its projection map $H_n(M \mid x) \longrightarrow M$ is given by taking an element in $H_n(M \mid x)$ to $x \in M$. We show in the beginning of Section 2.2 that $H_n(M \mid x) := H_n(M, M \setminus \{x\})$ is isomorphic to $\mathbb{Z}$. Define a basis for a topology on $\Theta(M)$ as follows. For a closed set $K \subseteq M$ let

$$j_{x,K} : H_n(M, M \setminus K) \longrightarrow H_n(M \mid x)$$

be the map which is induced by the inclusion $(M, M \setminus K) \hookrightarrow (M, M \setminus \{x\})$. For an open set $A \subseteq M$ and an element $\alpha \in H_n(M, M \setminus \overline{A})$, define $A_\alpha \subseteq \Theta(M)$ to be the set

$$\{j_{x,\overline{A}}(\alpha) \mid x \in A\}\,.$$

Take sets of the form A_α (with A open) as the basis for the topology on $\Theta(M)$.

We now can prove the following:

Proposition 5.4. *A manifold is orientable in the sense of Section 2 if and only if it is orientable in the sense that there exists a nonzero section of* $\Theta(M)$.

Proof. Let M be a d-dimensional manifold and assume to begin with that M is orientable in the sense of Section 2. This means that there exists an atlas $\{(U_i, f_i)\}$ for M such that the top horizontal arrow in the diagram

$$\begin{tikzcd}
H_d(\mathbb{R}^d \mid f_i(x)) \arrow[rr] \arrow[d, "\text{exc.}"] & & H_d(\mathbb{R}^d \mid f_j(x)) \arrow[d, "\text{exc.}"] \\
H_d(f_i(W_{ij}) \mid f_i(x)) & H_d(W_{ij} \mid x) \arrow[l, "(f_i)_*"'] \arrow[r, "(f_j)_*"] & H_d(f_j(W_{ij}) \mid f_j(x))
\end{tikzcd}$$

is multiplication with 1 for any i and j, any $x \in U_i \cap U_j$ and any open neighbourhood W_{ij} of x which is contained in $U_i \cap U_j$. Let x in M be given and suppose that x is in U_i. We then get a choice of generator for $H_n(M \mid x)$ by mapping the standard generator for $H_n(\mathbb{R}^d \mid f_i(x))$ to $H_n(U_i \mid x)$ with $(f_i)_*^{-1}$ and then further to $H_n(M \mid x)$ with the inverse of the excision isomorphism. Define a section

$$s : M \longrightarrow \Theta(M)$$

as the map which takes a point x to the generator for $H_n(M \mid x)$ which is obtained in this way. Because of the naturality of the excision isomorphism and the above diagram, the definition is independent of which chart that contains x we are using.

It remains to be shown that s is continuous. Let A_α be a set in the basis for the topology on $\Theta(M)$ and suppose that the image of s has nonempty intersection with

A_α, more specifically that $s(x_0)$ is in A_α. Let B be an $\mathbb{R}^d$-ball containing x_0 which is contained in the intersection of A (which by assumption is an open set) with a chart U_0. The homology groups

$$H_n(M, M \setminus B) \text{ and } H_n(M \mid x_0)$$

are isomorphic through the map $H_n(M, M \setminus B) \longrightarrow H_n(M \mid x_0)$ which is induced by the inclusion and the map $j_{x_0,\overline{A}}$ factors as

$$H_n(M, M \setminus A) \longrightarrow H_n(M, M \setminus B) \longrightarrow H_n(M \mid x_0)$$

since the inclusions factor on the manifold level. For any $y \in B$, the diagram

$$\begin{tikzcd}
& H_n(M, M \setminus B) \arrow[dl] \arrow[dr] & \\
H_n(M \mid y) \arrow[d, "\text{exc.}"] & & H_n(M \mid x_0) \arrow[d, "\text{exc.}"] \\
H_n(B \mid y) \arrow[d, "(f_0)_*"] & & H_n(B \mid x_0) \arrow[d, "(f_0)_*"] \\
H_n(f_0(B) \mid f_0(y)) & & H_n(f_0(B) \mid f_0(x_0)) \\
H_n(\mathbb{R}^d \mid f_0(y)) \arrow[u, "\text{exc.}"] \arrow[rr, "(\text{translation})_*"] & & H_n(\mathbb{R}^d \mid f_0(x_0)) \arrow[u, "\text{exc.}"]
\end{tikzcd}$$

is commutative since we in a sense have the same generator in the topmost three groups: we may choose a chain representative for the generator in $H_n(M, M \setminus B)$ and this chain will represent the images of the generator in $H_n(M \mid y)$ and $H_n(M \mid x_0)$. By definition, this means that $j_{y,\overline{A}}(\alpha)$ is equal to $s(y)$, so s is 'locally constant', in a sense, over $\mathbb{R}^d$-balls.

Let now z be any other point in the same component (which is to say path-component since M is a manifold) of A as x_0. Any path from x_0 to z can be covered by a finite number of $\mathbb{R}^d$-balls, and from this it follows, like it did for y, that $s(z)$ must be equal to $j_{z,\overline{A}}(\alpha)$, and s is actually "locally constant" over any component[12] of M. As a consequence, we get that the inverse image under s of A_α is equal to the union of the components of A that have nonempty intersection with the image of s. This is an open set in M, so s is continuous.

Assume now that we are given a continuous section $s : M \longrightarrow \Theta(M)$ and a possibly unoriented atlas $\{(U_i, f_i)\}$ for M. Given $x \in M$ we get for each U_i containing x a generator of $H_n(\mathbb{R}^d \mid f_i(x))$ by mapping $s(x) \in H_n(M \mid x)$ through the composition

$$H_n(M \mid x) \xrightarrow{\text{exc.}} H_n(U_i \mid x) \xrightarrow{(f_i)_*} H_n(f_i(U_i) \mid f_i(x)) \xrightarrow{\text{exc.}^{-1}} H_n(\mathbb{R}^d \mid f_i(x))\,.$$

If the generator of $H_n(\mathbb{R}^d \mid f_i(x))$ thus obtained is not the standard generator, define $\tilde{f}_i$ to be the composition of f_i with the reflection through a (any!) hyperplane in $\mathbb{R}^d$. Otherwise, set $\tilde{f}_i = f_i$. This gives that the atlas $\{(U_i, \tilde{f}_i)\}$ satisfies the orientability criterion given in definition 2.10. $\square$

[12]We *do* only concern ourselves with connected manifolds in the rest of the text, but this result holds in general.

Concerning the definition of bundle orientability (definition 2.15 in section 2.2), we here merely note that it is a generalization of the (maybe more familiar) definition of orientability of a vector bundle. This definition in turn says that a vector bundle $E \longrightarrow B$ is orientable if there is a cover $\{(U_\alpha, \varphi_\alpha)\}$ of B consisting of trivializing neigbourhoods with trivializing maps such that the composition $\varphi_\beta \circ \varphi_\alpha^{-1}$ (which we remind ourselves is a linear isomorphism restricted to each fibre E_x such that x is in $U_\alpha \cap U_\beta$) is a map with positive determinant restricted to any fibre over $U_\alpha \cap U_\beta$.

5.3 Appendix C: This is really the Chas-Sullivan product as you know it

Since the definition of the Chas-Sullivan product given here differs from previous ones (and in fact is a generalization), we show here that it actually coincides with the old definitions where applicable. In [HW17] an expression for the chain level description of the Chas-Sullivan product is given as

$$a \circ b = (-1)^{(n-i)n} \mathrm{concat}\Big(R_{CS}\left((\mathrm{ev} \times \mathrm{ev})^{\#}(\tau) \frown (a \times b)\right)\Big)$$

for $a \in C_i(LM)$ and $b \in C_j(LM)$ for an n-dimensional, smooth, closed and orientable manifold M. We will walk through this formula step by step and compare it to Definition 3.4.

The first map above is just the cross product, so this step is precisely the same as in this text.

The second map above is cap product with the element $(\mathrm{ev} \times \mathrm{ev})^{*}(\tau)$, where τ is the Thom class of the tangent bundle on M. As the authors explain in an appendix, this cap product should really be read as the composition

$$C_*(M \times M) \longrightarrow C_*(M \times M, \Delta^c) \longrightarrow C_*(U, \Delta^c) \xrightarrow{\tau \frown} C_*(U)\,,$$

where U is a neighbourhood around the diagonal in $M \times M$ which is diffeomorphic to TM under a map taking the diagonal to the zero section. More precisely, the authors give an explicit diffeomorphism from a subset of the tangent bundle TM of M to $U \subseteq M \times M$ as

$$(x, v) \longmapsto (x, \exp_x(v))\,,$$

where $\exp_x$ denotes the exponential map $T_xM \longrightarrow M$. The domain of definition of this map is all $(x, v) \in TM$ such that $\|v\|$ is smaller that the injectivity radius of M (for some chosen metric on M). The projection map of U onto $\Delta(M)$ is thus also in their construction projection onto the first coordinate. Lemma 11.7 in [MS74] says that the Thom class in $H^n(U, \Delta^c)$ (or rather the element in $H^n(M \times M, \Delta^c)$ corresponding to it under the excision isomorphism) is uniquely characterized by that its image under the map

$$H^n(M \times M, \Delta^c) \longrightarrow H^n(M \mid x)$$

that is induced by the inclusion $(M, M \setminus \{x\}) \hookrightarrow (M \times M, \Delta^c)$ given by $y \longmapsto (x, y)$ is the generator in $H^n(M \mid x)$ that is given by the orientation (cf. Appendix B) for all $x \in M$. This is equivalent to the characterization of the Thom class given in Definition 2.20, so the map $(\mathrm{ev} \times \mathrm{ev})^{\#}(\tau) \frown$ induces the same map on homology as the one used in the corresponding step(s) in the definition in this text.

The map R_{CS} is a retract of $(\mathrm{ev} \times \mathrm{ev})^{-1}(U)$ onto $LM \times_M LM$ and will thus, no matter how it is defined, induce the map ρ in 3.4 on homology, and the map called "concat" here is precisely the concatenation homomorphism γ_*.

This shows that the definition given in this text agrees with the old ones where applicable.

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