SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Cyclic Sieving Phenomenon on Colorings of Cycle Graphs

av

Basam Al Naschéa

2020 - No K12

Cyclic Sieving Phenomenon on Colorings of Cycle Graphs

Basam Al Naschéa

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Per Alexandersson

2020

Abstract

The authors V. Reiner, D. Stanton, and D.White introduced the Cyclic Sieving Phenomenon in 2004. Since then the phenomenon has been shown to be exhibited by numerous types of combinatorial objects. We show in this thesis that the phenomenon is exhibited by colorings of cycle graphs and one generalization of these.

Acknowledgements

I want to thank my supervisor Per Alexandersson for suggesting this project and helping me through it with invaluable comments and suggestions, and I would also like to thank my friend Benjamin who gave me the starting push I needed to pick this topic. The rest of my thanks go to all my friends, who make school that more fun.

Contents

1 Introduction

In 1996 J. Stembridge introduced the $q = -1$ phenomenon [Ste96] and in 2004 the authors V. Reiner, D. Stanton, and D. White [RSW04] extended this to the so called Cyclic Sieving Phenomenon or CSP for short. The idea is that you have three ingredients, a set of combinatorial objects, a polynomial associated with this set, and a finite cyclic group that is acting on the set. The polynomial evaluated at roots of unity would count the number of fixed points of the set when acted on by the cyclic group. Such polynomials are not hard to produce forcefully but the surprising bit is that they sometimes occur naturally. Instances of this phenomenon on many different types of objects has since been found. For example non-crossing matchings [PPR08], standard Young tableaux [Rho10], and words and multisets [RSW04]. A survey by Bruce E. Sagan is given in [Sag11].

In this work we reveal the occurrence of this phenomenon on colorings of cycle graphs. This work intends on being fairly self contained which is why Chapter 2 is dedicated to recalling necessary preliminaries from combinatorics, group theory, graph theory and q-analogs.

In Chapter 3 we give a starting example of the phenomenon before we define it formally. We then show that CSP is exhibited by colorings of cycle graphs before we move on to generalize this result to the rooted product of a cycle and an arbitrary rooted tree.

2 Preliminaries

2.1 Combinatorics

For any natural numbers n and k it is standard to let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let $\binom{[n]}{k}$ represent the set of subsets of $[n]$ of size k. Then let $\binom{n}{k}$ denote the cardinality of $\binom{[n]}{k}$, which can be shown to equal

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

The following theorem is one of the most basic results in combinatorics.

Theorem 2.1. *(The Binomial Theorem) For any natural numbers n and k we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

In particular we can plug in $x = y = 1$ to retrieve the formula $2^n = \sum_{k=0}^{n} \binom{n}{k}$, which can be made sense of without ever referring to the binomial theorem. Recall simply that the number of subsets of $[n]$ has size 2^n. Now note that $\binom{n}{0}$ counts the number of subsets of $[n]$ of size 0, and $\binom{n}{1}$ counts the subsets of size 1, and so on. Summing over all of these counts all the subsets. Then it comes to no surprise that the two formulas that count the same thing should be equal.

Now let

$$\binom{n}{x_1, \cdots, x_c} := \frac{n!}{x_1! \cdots x_c!}.$$

This next theorem is a generalization of the binomial theorem.

Theorem 2.2. *(The Multinomial Theorem) For $n, c \in \mathbb{N}$ we have*

$$(x_1 + x_2 + \cdots + x_c)^n = \sum_{k_1 + \cdots + k_c = n} \binom{n}{k_1, \cdots, k_c} x_1^{k_1} \cdot \cdots \cdot x_c^{k_c}.$$

In particular by letting all the $x_i = 1$ as before we retrieve the formula

$$c^n = \sum_{k_1 + \cdots + k_c = n} \binom{n}{k_1, \cdots, k_c}.$$

where $k_i \in \mathbb{N}$.

2.2 Graphs

By graph we always mean an undirected simple graph unless otherwise stated. By $V(G)$ and $E(G)$ we mean the set of vertices and edges of G respectively. Two vertices are said to be *adjacent* if they share an edge and the set of adjacent vertices of any vertex is called its neighbours. Any graph which contains no cycles is called a *tree*. For $e \in E(G)$ we let $G - e$ denote the graph G where the edge e has been removed.

Definition 2.3. Let $e = \{v_1, v_2\} \in E(G)$. Let G/e be the resulting graph of removing e, v_1, v_2, adding a new vertex v' and letting all edges of v_1, v_2 be edges of v'. We call G/e an *e-contraction* or *contraction* of G.

Definition 2.4. A *proper coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. Let k be a natural number. A *k-coloring* of G is a proper coloring that uses at most k colors.

Definition 2.5. For any natural number k let $\chi_G : \mathbb{N} \to \mathbb{N}$ where $\chi_G(k)$ is the number of k-colorings of G. We call $\chi_G(k)$ the *chromatic polynomial* of G.

We will simply write $\chi(k)$ when the graph is understood from context. We should hastily mention that the chromatic polynomial of a graph really is a polynomial. In order to work out the chromatic polynomial we will need a fact that relates the chromatic polynomial of G with that of graphs that are smaller than G. It is called the deletion contraction formula and will be an indispensable tool when we wish to compute the chromatic polynomial.

Theorem 2.6. *(The Deletion-Contraction Formula) Let $e \in E(G)$. The chromatic polynomial of G satisfies $\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$.*

A rooted graph is one in which one vertex has been distinguished as the root and the notion of rooted products will be useful when we define the Mistletoe-Graphs later on. If G is a graph and H is a rooted graph with root h_1 then we glue one copy of H to each vertex of G by equating one h_1 with each $g \in V(G)$. The resulting graph is called the rooted product of G and H. More formally we say the following.

Definition 2.7. (Rooted product of graphs) Let G be a graph and let H be a rooted graph with $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{h_1, \ldots, h_m\}$, and assume that h_1 is the root of H. We define the *rooted product* of G and H to be

$$G \odot H := (V, E)$$

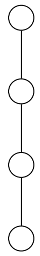
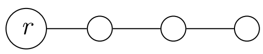
where

$$V = \{(g_i, h_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and

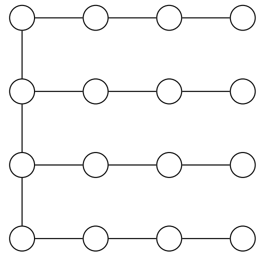
$$E = \{\{(g_i, h_1), (g_k, h_1)\} : \{g_i, g_k\} \in E(G)\} \cup \bigcup_{i=1}^{n} \{\{(g_i, h_j), (g_i, h_k)\} : \{h_j, h_k\} \in E(H)\}.$$

Example 2.8. *Let*

$G =$  *and* $H =$ 

where the vertex labeled r is the root of H. Then

$G \odot H =$



2.3 Groups

In this section we recall some things from group theory that will be needed later on.

The order of a finite group G is its cardinality and is typically written as $|G|$. The order of an element $g \in G$ is the least $n \in \mathbb{N}$ such that $g^n = 1$, we write this as $|g|$. The following is a basic result in group theory.

Theorem 2.9. *(Lagrange's Theorem) The order of any element $g \in G$ divides $|G|$.*

Recall now that a group is called cyclic if it is generated by one element. We write $\mathfrak{S}_n$ to mean the symmetric group on n elements. We typically use cycle notation when refering to the elements of $\mathfrak{S}_n$. For example, the element $\pi = (1, 2, 3)(4, 5, 6) \in \mathfrak{S}_7$ refers to the permutation sending 1 to 2, 2 to 3, 3 to 1, 4 to 5, 5 to 6 to 6 to 4 and 7 to 7.

Definition 2.10. An n:th root of unity is a solution to the equation $x^n - 1 = 0$.

One can show that n:th roots of unity have the form $e^{\frac{2\pi ik}{n}}$ where $k \in [n]$.

Theorem 2.11. *The n:th roots of unity form a cyclic group under multiplication.*

This last theorem now allows us to start talking about the orders of the roots of unity.

Definition 2.12. A *primitive n:th root of unity* is a root of unity with order n.

One more concept we would like to define which will be central later on is group actions on finite sets.

Definition 2.13. (Group Action) If G is a group with identity e, and X is a set then a *group action* is a function $\varphi : G \times X \to X$ such that for all $g, h \in G$ and all $x \in X$ the following two axioms hold.

- $\varphi(e, x) = x$
- $\varphi(gh, x) = \varphi(g, \varphi(h, x))$

When we say cyclic group action we mean a group action where the acting group is cyclic.

2.4 q-analogs

A q-analog of an expression is another expression in terms of q such that the original expression is obtained when $q \to 1$, an example would be to let the q-analog of 1 to be q. Although this satisfies the definition of being a q-analog, it might not be particularly useful or interesting. Typically one is interested in q-analogs that arise naturally in some sense but what exactly constitutes "natural" is not very well defined, however it has been established that the following definition certainly fits the criteria.

Definition 2.14. The q-analog of any $n \in \mathbb{N}$ is defined to be

$$[n]_q := \frac{1-q^n}{1-q}.$$

The identity $\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$ shows that $\lim_{q\to 1}[n]_q = n$. From this definition we can go on to define the q-factorial.

Definition 2.15. The q-factorial is defined as

$$[n]_q! := [1]_q[2]_q \cdots [n]_q.$$

From this we can further go on to define the q-multinomial.

Definition 2.16. For natural numbers $\alpha_1, \cdots, \alpha_c$ we define the q-multinomial as

$$\begin{bmatrix} n \\ \alpha_1, \alpha_2, \cdots, \alpha_c \end{bmatrix}_q := \frac{[n]_q!}{[\alpha_1]_q![\alpha_2]_q! \cdots [\alpha_c]_q!}.$$

As a special case we retrieve the q-binomial by letting $c = 2$ and putting $\alpha_2 = n - \alpha_1$, though one typically writes this as $\begin{bmatrix} n \\ \alpha \end{bmatrix}_q$. This expression turns out to be a polynomial in q with natural numbers as coefficients, the proof of which can be read in [Sta11].

These next two theorems will be our main tools when we evaluate some of the q-analogs that are of interest to us. The first theorem proves quite useful when we wish to evaluate q-binomials at roots of unity, the second theorem lets us rewrite q-multinomials as products of q-binomials.

Theorem 2.17. *(q-Lucas Theorem) [Dés82] Let ξ_d be a primitive d:th root of unity over the complex numbers. Let $n = n_0 d + n_1$, $k = k_0 d + k_1$ where $0 \le n_0, k_0 \le d$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\xi_d} = \binom{n_0}{k_0} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{\xi_d}$$

Theorem 2.18. *Let $k_1, \cdots, k_c$ be natural numbers. Then the following equality holds.*

$$\begin{bmatrix} n \\ k_1, \cdots, k_c \end{bmatrix}_q = \begin{bmatrix} n \\ k_1 \end{bmatrix}_q \begin{bmatrix} n - k_1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} n - k_1 - \cdots - k_{c-1} \\ k_c \end{bmatrix}_q.$$

The q-Lucas theorem was first proved in [Oli65]. For readers interested in further reading on the topic of q-analogs we recommend [Sta11].

3 Cyclic Sieving Phenomenon

3.1 Introduction by example

The cyclic sieving phenomenon has its roots, pun intended, in Stembridge's $q = -1$ phenomenon [Ste96] and was first introduced in a paper published in 2004 by V. Reiner, D. Stanton, and D. White [RSW04] as a generalization of that. In short words one could say that cyclic sieving occurs when evaluating a polynomial which has natural number coefficients and is associated with a family of combinatorial objects at roots of unity gives the number of fixed points of that family under a cyclic group action. Perhaps it is best to start off with an example.

Example 3.1. *Consider the set* $X = \binom{[6]}{2}$*. Recall from Chapter 2.1 that* $\binom{[6]}{2}$ *has* $\binom{6}{2}$ *elements and recall from Definition* 2.16 *that one naturally occuring q-analog we can associate is* $\begin{bmatrix}6\\2\end{bmatrix}_q$*. In order to evaluate this term we first write it out more explicitly and obtain*

$$f(q) = \begin{bmatrix}6\\2\end{bmatrix}_q = \frac{[6]_q!}{[2]_q![4]_q!} = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= 1 + q + 2q^2 + 2q^3 + 4q^4 + 2q^5 + 2q^6 + q^7 + q^8.$$

Let $\xi = \exp(\frac{2\pi i}{6})$*, then we get*

$$f(\xi) = 0$$
$$f(\xi^2) = 0$$
$$f(\xi^3) = 3$$
$$f(\xi^4) = 0$$
$$f(\xi^5) = 0$$
$$f(\xi^6) = 15$$

Now consider the element $\pi = (1, 2, 3, 4, 5, 6) \in \mathfrak{S}_n$ *which generates the cyclic group* $\langle\pi\rangle = C_6$*. We let* π *act on* X *by cyclic shift, meaning that for any element* $t = \{a, b\} \in X$ *we have* $\pi \cdot t = \{\pi(a), \pi(b)\}$*. Then letting* π *act on* X *fixes no elements. Likewise with* π^2, π^4 *and* π^5*, whereas* π^6 *on the other hand fixes all 15 elements. Lastly the three elements* $\{1, 3\}, \{2, 4\}$ *and* $\{3, 6\}$ *are fixed by* π^3*. Where have we seen this enumeration before? When we let* π^d *act on* X *the number of fixed points are given by* $f(\xi^d)$*.*

This motivates the following formal definiton.

Definition 3.2. (Cyclic Sieving Phenomenon) [RSW04] Let X be a set of combinatorial objects. Let $C_n = \langle g\rangle$ be a finite cyclic group of order n, and

$f(q) \in \mathbb{N}[q]$. Then we say that the triple $(X, C_n, f(q))$ *exhibits the cyclic sieving phenomenon* if for all $d \in \mathbb{N}$ we have

$$|\{x \in X : g^d \circ x = x\}| = f\left(\exp\left(2\pi i \frac{d}{n}\right)\right).$$

So $f(q)$ encodes the number of fixed points of C_n acting on X. More specifically we let $\xi = \exp \frac{2\pi i}{n}$ and we look at $f(\xi^d)$. Then we take g which generates C_n and look at the number of fixed points of g^d acting on every $x \in X$. If the number of fixed points is given by $f(\xi^d)$ for every $d \in \mathbb{N}$ then we say that this triple exhibits the cyclic sieving phenomenon. We will sometimes say CSP for short. Typically the polynomial $f(q)$ is a generating function and in our case we strictly look at q-analogs of counting functions of our sets.

Lastly before we move on we would like to mention a special type of CSP on a family of combinatorial objects. The idea is that sometimes fixed points of this family will be in bijection with smaller members of that family, then we say that the CSP is Lyndon-like. Formally we say the following.

Definition 3.3. (Lyndon-like CSP) [ALP19] Let $\{(X_n, C_n, f_n(q))\}_{n=1}^{\infty}$ be a family of instances of CSP. The family is *Lyndon-like* if for all $n \geq 1$,

$$f_{n/m}(1) = f_n\left(e^{\frac{2\pi i}{m}}\right), \text{ whenever } m|n.$$

The name Lyndon-like stems from the Lyndon words which can be thought of as building blocks for all other words. They are lexicographically smaller than all their rotations and can be rotated and concatenated to produce any other word.

3.2 Cycle Graphs

Now that we have an idea of what the cyclic sieving phenomenon is we start to wonder where this might appear. Typically there are two ways of attacking these kinds of problems. One is that you are looking at a family of combinatorial objects which has a natural q-analog associated to it and you try to find a cyclic group action acting on it. The other is that you instead have a natural cyclic group action and try to find a nice q-analog. The latter is the approach we are taking with the family of colourings of cycle graphs. The motivation behind this is that cycle graphs have the natural cyclic group action of rotation. This leads us to look at colourings on cycle graphs and rotations of these. What are the fixed points of these colorings under the cyclic action of rotation?

Consider a small example when $n = 6$, we number the vertices in clockwise order starting with 1 and suppose we intend on rotating by 3 steps clockwise, meaning that 1 takes the place of 4 and vice versa. Then for any particular coloring the vertex pair $(1, 4)$ would need to be the same color, in order for that particular coloring to be fixed under rotation by three steps. Same with the pairs $(2,5)$ and $(3,6)$. So in order for a coloring of C_6 to be a fixed point under three-step rotation we would need the graph to consist of two identically colored pieces of size three in the manner just described. Furthermore since vertex 3 is connected to vertex 4 we also require that 1 and 3 have different colors, this amounts to the same as if 1 and 3 had an edge between them. Altogether this leads us to the idea that the amount of fixed points of C_n under rotation by d steps are the same as the amount of colorings of a graph of size d whenever d divides n. If d does not divide n then there can be no fixed points, because then C_n could not possibly be colored into n/d identically colored copies. This is the motivation behind Theorem 3.5, but we first need to state a crucial piece of information about the cycle graphs.

Theorem 3.4. *The chromatic polynomial of the cycle graph C_n on n vertices is $(k-1)^n + (-1)^n(k-1)$.*

Proof. Let $n = 3$. Then the first vertex can be colored using k colors. The second one can use $(k-1)$ and the third one uses $(k-2)$ colors. Altogether $k(k-1)(k-2)$ which is equal to $(k-1)^3 + (-1)^3(k-1)$. Now assume this holds by induction on the number of vertices. We claim that the chromatic polynomial of C_n with any one edge removed is $k(k-1)^{n-1}$. To see this notice that such a graph is just a path. For any path of length n we can color the starting vertex using k colors, and the remaining $n-1$ vertices will only have $k-1$ colors available which in total gives $k(k-1)^{n-1}$. By Theorem 2.6 and by induction we have

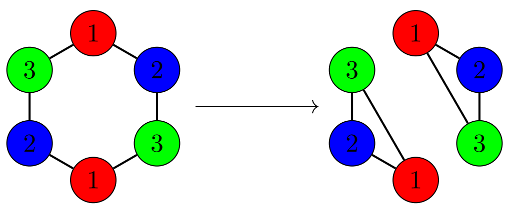
$$\begin{aligned}\chi(k) &= k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) \\ &= (k-1)^{n-1}(k-1) + (-1)^n(k-1) \\ &= (k-1)^n + (-1)^n(k-1).\end{aligned}$$

□

Theorem 3.5. *The number of fixed points of the colorings with k colors on C_n under rotation by n/d steps is given by*

$$(k-1)^{n/d} + (k-1)(-1)^{n/d}. \tag{1}$$

Note that this expression is the same as the chromatic polynomial of the cycle graph of size n/d.



Proof. The proof of the general case is the same as the above example except for a difference in notation. We omit writing it out. □

Thus we also have a nice formula for counting the number of fixed points. The question is, can we find a q-analog associated to the chromatic polynomial which also happens to count these fixed points when evaluated at the appropriate roots of unity?

This is a rhetorical question, the answer is yes in this case. To produce a q-analog we look at the terms of the chromatic polynomial individually. Recall from Theorem 2.2 that we can write k^n as a sum of multinomial terms and recall from Definition 2.16 that we defined the q-multinomial. This suggests the following definition.

Definition 3.6. Let $[k : n]$ denote the expression k^n where k and n are natural numbers and let $c_1, \cdots, c_k \in \mathbb{N}$. Then we define the q-analog of $[k : n]$ to be

$$[k : n]_q \coloneqq \sum_{c_1+\cdots+c_k=n} \begin{bmatrix} n \\ c_1, \cdots, c_k \end{bmatrix}_q.$$

When we let $q \to 1$ this becomes a sum of regular multinomials which from Theorem 2.2 can be seen to equal k^n. We mentioned in Chapter 2.4 that the q-multinomials turn out to be polynomials with natural coefficients and this property is retained when we sum over several such polynomials. Now we are ready to state the q-analog we will be working with.

Definition 3.7. Let $n \in \mathbb{N}$, $k \in \mathbb{N}$, let $\chi_{C_n}(x)$ be the chromatic polynomial of the cycle graph of size n, and let

$$\sigma(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}.$$

Then we define the q-analog of $\chi(x)$ to be

$$Cyc_n(k : q) \coloneqq [k-1 : n]_q + q^{\sigma(n)}(k-1)(-1)^n \tag{2}$$

We now have all three ingredients that we require for a CSP with X being the set of colorings of cycle graphs of size n and C_n being the cyclic group acting by rotation on X. We begin by showing a very useful lemma.

Lemma 3.8. Let ξ be an n:th root of unity of order d where d divides n. Then

$$[k-1 : n]_\xi = (k-1)^{n/d}.$$

Proof. Recall from Theorem 2.18 that we can write

$$[k-1 : n]_q = \sum_{c_1+\cdots+c_{k-1}=n} \begin{bmatrix} n \\ c_1 \end{bmatrix}_q \begin{bmatrix} n-c_1 \\ c_2 \end{bmatrix}_q \cdots \begin{bmatrix} n-\cdots-c_{k-2} \\ c_{k-1} \end{bmatrix}_q. \tag{3}$$

Let ξ be an n:th root of unity of order d. By Theorem 2.9 we know that d divides n so we can write $n = dn_0$ and $c_i = db_i + r_i$ for some $n_0, b_i, r_i \in \mathbb{N}$. Then by the q-Lucas Theorem 2.17 we can write the first factor of the right hand side of (3) evaluated at ξ as

$$\begin{bmatrix} n \\ c_1 \end{bmatrix}_\xi = \binom{n_0}{b_1} \begin{bmatrix} 0 \\ r_1 \end{bmatrix}_\xi .$$

We see then that for $r_1 \neq 0$ the whole term becomes 0, so we may assume that $r_1 = 0$. Then we may write $n_0 = n/d$ and $b_1 = c_1/d$. The next factor we may express in the same fashion as

$$\begin{bmatrix} n - c_1 \\ c_2 \end{bmatrix}_\xi = \binom{n_0 - b_1}{b_2} \begin{bmatrix} 0 \\ r_2 \end{bmatrix}_\xi .$$

Again we see that we must have $r_2 = 0$ if we want there to be a factor left, and we can continue onward in the same fashion with all the remaining factors. This allows us to write

$$\sum_{c_1+\cdots+c_{k-1}=n} \begin{bmatrix} n \\ c_1 \end{bmatrix}_\xi \begin{bmatrix} n - c_1 \\ c_2 \end{bmatrix}_\xi \cdots \begin{bmatrix} n - \cdots - c_{k-2} \\ c_{k-1} \end{bmatrix}_\xi =$$

$$\sum_{c_1+\cdots+c_{k-1}=n} \binom{n_0}{b_1} \binom{n_0 - b_1}{b_2} \cdots \binom{n_0 - \cdots - c_{k-2}}{b_{k-1}} =$$

$$\sum_{c_1+\cdots+c_{k-1}=n} \binom{n_0}{b_1, b_2, \cdots, b_{k-1}} =$$

$$\sum_{c_1+\cdots+c_{k-1}=n} \binom{n/d}{c_1/d, c_2/d, \cdots, c_{k-1}/d}. \tag{4}$$

Now we can use the multinomial Theorem 2.2 to obtain

$$\sum_{c_1+\cdots+c_{k-1}=n} \binom{n/d}{c_1/d, c_2/d, \cdots, c_{k-1}/d} = (k-1)^{n/d}.$$

□

Theorem 3.9. *(Main result) The triple $(X, C_n, Cyc_n(k : q))$ exhibits the cyclic sieving phenomenon.*

Proof. Let d be any natural number which divides n. Our goal is to show that $Cyc_n(k : q)$ evaluated at primitive d:th roots of unity equals

$$(k-1)^{n/d} + (-1)^{n/d}(k-1). \tag{5}$$

We first want to evaluate $Cyc_n(k : q)$ at roots of unity. To start consider $[k-1 : n]_q$ which is the first term of (2). By Lemma 3.8 this evaluates to $(k-1)^{n/d}$ which equals the first term of (5).

Now we turn our attention to the term

$$q^{\sigma(n)}(k-1)(-1)^n$$

from (2). Assume that n is even. By putting $q \to \exp(\frac{2\pi i}{d})$, a primitive d:th root of unity, we obtain

$$e^{\pi i \cdot n/d} \cdot (k-1)(-1)^n = (k-1)(-1)^{n+n/d}$$

We claim that the expression $n + n/d$ will always have the same parity as n/d. To see this write $(-1)^n = 1$ which in turn means that $(-1)^n = (-1)^{n+n/d}$. Now assume instead that n were odd. Then we would get the expression $(k-1)(-1)^n$. Likewise we claim that n and n/d have the same parity. Write $n = kd$ where both k and d are odd. Then $n/d = kd/d = k$ is also odd. Thus we can write

$$(k-1)(-1)^{n+n/d} = (k-1)(-1)^{n/d}.$$

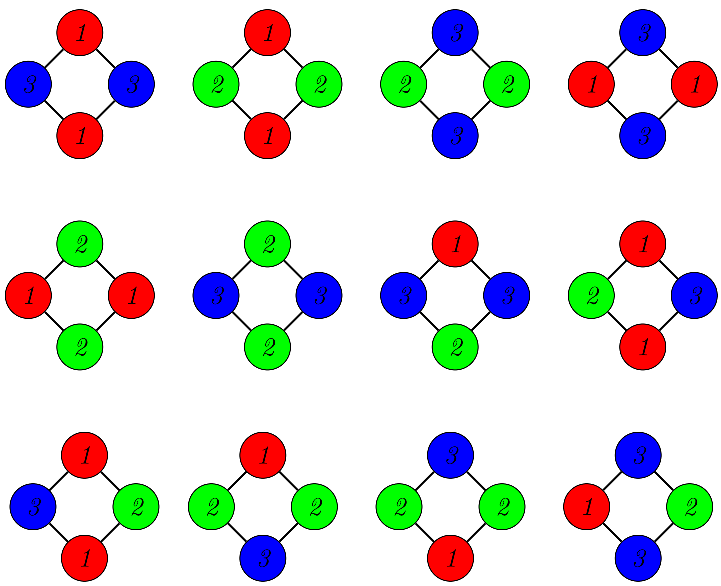
which equals the second term of (5). Altogether this gives us

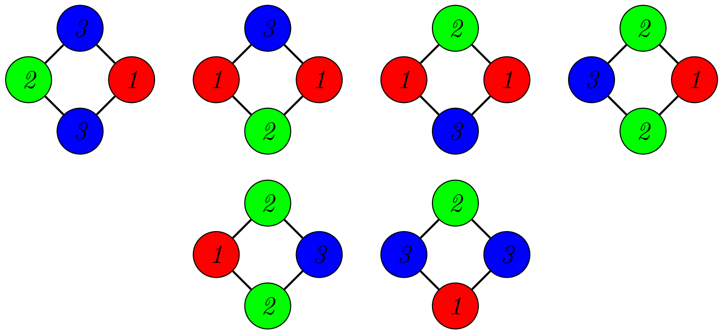
$$Cyc_n(k:\xi) = (k-1)^{n/d} + (k-1)(-1)^{n/d}.$$

□

Note that the CSP in Theorem 3.9 is Lyndon-like. Let us consider a digestible example.

Example 3.10. *Let $n = 4$ and $k = 3$. Then the set of colorings of the graph C_4 are the following*





We see that the first 6 of these are the ones that have any non-trivial rotational symmetry with rotation by 2 steps, and the rest do not. Altogether we have the following.

Rotations	*Fixed points*
1	*0*
2	*6*
3	*0*
4	*18*

Table 1

Now we look at $Cyc_4(k : q)$ *with* $k = 3$. *We get that*

$$\begin{aligned} Cyc_4(3 : q) &= [2 : 4]_q + q^{\sigma(4)}(2)(-1)^4 \\ &= \sum_{c_1+c_2=4} \begin{bmatrix} 4 \\ c_1, c_2 \end{bmatrix}_q + 2q^2 \\ &= \sum_{c=0}^{4} \begin{bmatrix} 4 \\ c \end{bmatrix}_q + 2q^2 \end{aligned} \tag{6}$$

where the first equality is by definition and the second one is due to what we mentioned in Definition 2.16. *Now we are left with a sum that we want to unwind, for* $c = 0$ *and* $c = 4$ *the terms become 1. We see also that*

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q = \frac{[4]_q!}{[1]_q![3]_q!} = [4]_q = 1 + q + q^2 + q^3.$$

Lastly we have

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q!}{[2]_q![2]_q!} = \frac{[4]_q[3]_q}{[2]_q} &= \frac{(1 + q + q^2 + q^3)(1 + q + q^2)}{1 + q} \\ &= 1 + q + 2q^2 + q^3 + q^4. \end{aligned}$$

Putting all this together we get that (6) *is equal to*

$$2q^2 + (2) + (2 + 2q + 2q^2 + 2q^3) + (1 + q + 2q^2 + q^3 + q^4)$$

$$= 5 + 3q + 6q^2 + 3q^3 + q^4.$$

The only thing left to do is to evaluate at roots of unity. Let $\xi = \exp(2\pi i/4)$*. We get*

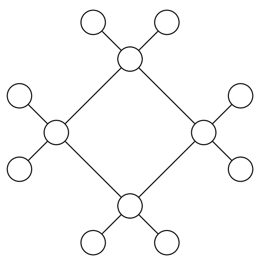
$$Cyc_4(3 : \xi^1) = 0$$
$$Cyc_4(3 : \xi^2) = 6$$
$$Cyc_4(3 : \xi^3) = 0$$
$$Cyc_4(3 : \xi^4) = 18$$

which matches what we had in Table 1 and is exactly what Theorem 3.9 *tells us it should be.*

3.3 Mistletoe-Graphs

We have now studied the cyclic sieving phenomenon on cycle graphs and have discovered and proved an instance of this relating to colorings and rotations. We might then ask ourselves, can we generalize this result somehow? This leads us to look at the Mistletoe-Graph. Recall how we defined rooted products of graphs in Definition 2.7.

Definition 3.11. (Mistletoe-Graph) Let T be any rooted tree of size j and let C_n be the cycle graph of size n. The rooted product $C_n \odot T$ we call *Mistletoe-graph* or $M(T, n)$.



An example of a mistletoe-graph

Note that the graph $M(T, n)$ has size $(j-1)n$ and that the cycle graph is an instance of a Mistletoe-Graph with $(j, n) = (1, n)$. Now we would like to inspect the number of colorings of these graphs. We will see in Theorem 3.13 that the problem requires the following lemma.

Lemma 3.12. Any tree T of size j has chromatic polynomial

$$\chi_T(k) = k(k-1)^{j-1}.$$

Proof. Consider any tree T of size j and let any $v_1 \in V(T)$ be the starting point for our coloring. This vertex can be colored in k ways. The neighbours $v_2, \ldots, v_i$ of v_1 can each be colored using $k-1$ colors. Their neighbours can in turn also be colored using $k-1$ colors each. Since there are no cycles in trees this process goes on in the same manner until we have colored the entire tree. Altogether we get $k(k-1)^{j-1}$ □

Theorem 3.13. *Let T be a rooted tree with j vertices. Then the Mistletoe-graph $M(T,n)$ has chromatic polynomial*

$$(k-1)^{(j-1)n}[(k-1)^n + (k-1)(-1)^n]$$

Proof. Firstly we color the cycle part which by Theorem 3.4 can be done in $(k-1)^n + (k-1)(-1)^n$ ways. For each vertex of the cycle we can apply the same argument as in the proof of Lemma 3.12. We color the neighbours in $k-1$ ways each and color their neighbours in $k-1$ ways each as well, and so on, until all the tree parts of $M(T,n)$ have been colored. Since there are n tree parts we get in total

$$(k-1)^{(j-1)n}[(k-1)^n + (k-1)(-1)^n].$$

□

Then how many of these are fixed under rotation by n/d steps?

Theorem 3.14. *Let T be a rooted tree with j vertices. Then the number of fixed points of the k-colorings of $M(T,n)$ under rotation by n/d steps is given by*

$$(k-1)^{(j-1)n/d}[(k-1)^{n/d} + (k-1)(-1)^{n/d}]. \tag{7}$$

Proof. We can apply essentially the same argument as in 3.5. □

Next we want to consider a natural q-analog associated to this. One that comes to mind is

$$Mis_{j,n}(k;q) = [k-1 : (j-1)n]_q \cdot ([k-1:n]_q + q^{\sigma(n)}(-1)^n(k-1)). \tag{8}$$

With this q-analog and C_n acting as before by rotation we are ready to state our next theorem.

Theorem 3.15. *Let X be the set of k-colorings of M(T,n) where T is any rooted tree on j vertices. Then the triple $(X, C_n, Mis_{j,n}(k;q))$ exhibits the cyclic sieving phenomenon.*

Proof. Let ξ be a root of unity of order d. Our goal is to evaluate $Mis_{j,n}(k;q)$ at ξ and if it happens to equate to the same as (7) then we are happy. First note that $Mis_{j,n}(k;q)$ consists of two factors. The factor

$$[k-1:n]_q + q^{\sigma(n)}(-1)^n(k-1)$$

is the same one we had in Theorem 3.9 and we already know that it becomes

$$(k-1)^{n/d} + (k-1)(-1)^{n/d}$$

when evaluated at ξ.

Next consider the other factor

$$[k-1:(j-1)n]_\xi.$$

We can apply Lemma 3.8 and get that this is equal to

$$(k-1)^{(j-1)n/d}.$$

Altogether we now have

$$(k-1)^{(j-1)n/d}[(k-1)^{n/d}+(k-1)(-1)^{n/d}].$$

□

Note that the CSP in Theorem 3.15 is Lyndon-like.

4 Future work

During Chapter 3.3 we took an interesting leap from working with cycle-graphs to the more generalized mistletoe-graphs. This was one out of many possible leaps we could have taken. When we look back we realize that there was nothing about trees which we used other than their chromatic polynomial. So one question we might ask ourselves is whether this CSP could have been generalized to colorings of rooted products of cycle graphs with something other than rooted trees. One that comes to mind for example is the rooted product of a cycle graph with another cycle graph.

One might also consider graphs different from cycles altogether but which still have some type of structure that preserves the legality of colorings when acted on by rotation. In any such case where there is a clear action of rotation the more difficult task seems to be in finding a desired q-analog associated with the set.

Lastly we would like to mention something about statistics. Given a set X of combinatorial objects, a combinatorial statistic is a function $st : X \to \mathbb{N}$. The generating polynomial f of st is defined as $\sum_{x \in X} q^{st(x)}$. Some of the most common ones are the major index and the inversion number which can both be read about in [Sta11]. So the question we would like to ask is whether there is some statistic on our sets which generate the polynomials we have been working with. This is a typical question when working with CSP polynomials.

References

[ALP19] Per Alexandersson, Svante Linusson, and Samu Potka. The cyclic sieving phenomenon on circular Dyck paths. *The Electronic Journal of Combinatorics*, 26:1–32, October 2019. https://doi.org/10.37236/8720 doi:10.37236/8720.

[Dés82] Jacques Désarménien. Un analogue des congruences de Kummer pour les q-nombres d'Euler. *European Journal of Combinatorics*, 3(1):19–28, March 1982. https://doi.org/10.1016/s0195-6698(82)80005-x doi:10.1016/s0195-6698(82)80005-x.

[Oli65] Gloria Olive. Generalized powers. *The American Mathematical Monthly*, 72(6):619, June 1965. https://doi.org/10.2307/2313851 doi:10.2307/2313851.

[PPR08] T. Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades. Promotion and cyclic sieving via webs. *Journal of Algebraic Combinatorics*, 30(1):19–41, September 2008. https://doi.org/10.1007/s10801-008-0150-3 doi:10.1007/s10801-008-0150-3.

[Rho10] Brendon Rhoades. Cyclic sieving, promotion, and representation theory. *Journal of Combinatorial Theory, Series A*, 117(1):38–76, January 2010. https://doi.org/10.1016/j.jcta.2009.03.017 doi:10.1016/j.jcta.2009.03.017.

[RSW04] Victor Reiner, Dennis Stanton, and Dennis E. White. The cyclic sieving phenomenon. *Journal of Combinatorial Theory, Series A*, 108(1):17–50, October 2004. https://doi.org/10.1016/j.jcta.2004.04.009 doi:10.1016/j.jcta.2004.04.009.

[Sag11] Bruce E. Sagan. The cyclic sieving phenomenon: a survey. In Robin Chapman, editor, *Surveys in Combinatorics 2011*, pages 183–234. Cambridge University Press, 2011. https://doi.org/10.1017/cbo9781139004114.006 doi:10.1017/cbo9781139004114.006.

[Sta11] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, second edition, 2011. https://doi.org/10.1017/CBO9781139058520 doi:10.1017/CBO9781139058520.

[Ste96] John R. Stembridge. Canonical bases and self-evacuating tableaux. *Duke Mathematical Journal*, 82(3):585–606, March 1996. https://doi.org/10.1215/s0012-7094-96-08224-1 doi:10.1215/s0012-7094-96-08224-1.