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Finite Blaschke products and their properties

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Abstract

A finite Blaschke product $B(z)$ is a special kind of product of finitely many automorphisms of the unit disc, with zeros in a finite set of points on the unit disc. This thesis covers some basic properties regarding finite Blaschke products. Solutions to the equation $B(z) = \omega$ for ω inside, on and outside the unit circle are examined, as well as zeros of $B(z)$ and the derivative $B'(z)$, and their location. In the last section, geometrical properties of the solutions to $B(z) = \omega$ for ω on the unit circle are explored; when the Blaschke product is of degree three, this involves ellipses.

Contents

1 Introduction

Blaschke products are an important class of complex valued functions that are bounded and analytic on the unit disc. They are constructed to have zeros in a finite or infinite set of points $\{a_k\}$ $(k = 1, 2, ...)$ on the unit disc. The importance of these functions in the general theory of bounded functions was first recognized by the Austrian mathematician Wilhelm Blaschke (1885-1962), and hence, they are called Blaschke products [8]. Blaschke products are featured in many different areas of mathematics. They are important in factorization theorems and approximation theorems, among others. Finite Blaschke products have some interesting geometrical properties, which will be the focus of this thesis. The purpose of this thesis is to establish properties and prove different theorems involving finite Blaschke products.

2 Definitions

This section is devoted to definitions related to Blaschke products. First, some basic notation will be introduced; the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, the unit circle $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ and the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Throughout the whole thesis, $\frac{1}{0}$ will be treated as ∞, where ∞ is the point at infinity in the extended complex plane. Similarly, $\frac{1}{\infty}$ will be treated as 0.

2.1 Definitions of Blaschke products

This part is dedicated to defining Blaschke products and related topics that will be used throughout this thesis. In words, a Blaschke product is a special kind of bounded function that maps $\mathbb{D}$ onto $\mathbb{D}$ and is analytic on $\mathbb{D}$. The product is constructed to have zeros in a finite or infinite set of points $a_j \in \mathbb{D}$. If the product is finite, the Blaschke product is also a rational function. Continuing to the more formal definitions, we begin with the definition of a *Blaschke factor*.

Definition 2.1. *For $a_k \in \mathbb{D}$ a Blaschke factor is defined as*

$$b_{a_k}(z) = \begin{cases} \frac{|a_k|}{a_k}\frac{a_k - z}{1-\overline{a_k}z} & \text{if } a_k \neq 0, \\ z & \text{if } a_k = 0. \end{cases}$$

Next, we need to define what a *Blaschke product* is. There are many different but analogous definitions of Blaschke products, which will be briefly discussed in Section 2.2. The definition used throughout this thesis is the following:

Definition 2.2. *Let $a_1, ..., a_n$ be a finite set of points inside $\mathbb{D}$. Then a finite Blaschke product of degree n is defined as*

$$B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k}z}. \tag{1}$$

If $a_j = 0$, define $\frac{|a_j|}{a_j} = -1$ so that each factor in the product above corresponds to the Blaschke factor in Definition (2.1).

The factors $\frac{|a_j|}{a_j}$ $(1 \leq j \leq n)$ normalize the Blaschke product so that it is non-negative at the origin, since $B(0) = \prod_{k=1}^{n} \frac{|a_k|}{a_k}\frac{a_k}{1} = \prod_{k=1}^{n} |a_k| \geq 0$.

An infinite Blaschke product is defined similarly:

Definition 2.3. *Let $a_1, ..., a_n, ...$ be an infinite set of points inside $\mathbb{D}$ that satisfies the Blaschke condition,*

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty.$$

Then an infinite Blaschke product of degree n is defined as

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}.$$

If $a_j = 0$, let $\frac{|a_j|}{a_j} = -1$.

The Blaschke condition makes sure that the infinite Blaschke product converges absolutely for $|z| < 1$, and uniformly on compact subsets of $\mathbb{D}$ (see for instance [7]). For finite Blaschke products, the Blaschke condition will be satisfied for any set of points $a_1, ..., a_n \in \mathbb{D}$. For infinite Blaschke products, consider an arbitrary factor $\frac{|a_k|}{a_k}\frac{a_k - z}{1 - \overline{a_k} z}$. Then we have

$$\begin{aligned} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} &= |a_k| \frac{1 - \frac{z}{a_k}}{1 - \overline{a_k} z} \\ &= |a_k| + \frac{|a_k| \left(1 - \frac{z}{a_k}\right)}{1 - \overline{a_k} z} - \frac{|a_k| (1 - \overline{a_k} z)}{1 - \overline{a_k} z} \\ &= |a_k| + \frac{|a_k| z \left(\overline{a_k} - \frac{1}{a_k}\right)}{1 - \overline{a_k} z} \\ &= |a_k| + \frac{|a_k| z}{a_k} \frac{\overline{a_k} a_k - \frac{a_k}{a_k}}{1 - \overline{a_k} z} \\ &= |a_k| + \frac{|a_k| z}{a_k} \frac{|a_k|^2 - 1}{1 - \overline{a_k} z} \\ &= 1 + (|a_k| - 1) \left(1 + z \frac{|a_k| (|a_k| + 1)}{a_k (1 - \overline{a_k} z)}\right). \end{aligned} \tag{2}$$

An infinite product $\prod_{k=1}^{\infty} (1 + u_k)$ of complex numbers converges absolutely if and only if $\sum_{k=1}^{\infty} |u_k| < \infty$ (see for example Theorem 1 in [13]). Hence, (2) shows us that an infinite Blaschke product will converge absolutely for $z = 0$ if and only if

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty.$$

For $z \in \mathbb{D} \setminus \{0\}$, we have

$$\begin{aligned}
\left| 1 - \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} \right| &= \left| \frac{1 - \overline{a_k} z - |a_k| + \frac{|a_k| z}{a_k}}{1 - \overline{a_k} z} \right| \\
&= \left| \frac{(1 - |a_k|) + z \left(\frac{|a_k|}{a_k} - \overline{a_k} \right)}{1 - \overline{a_k} z} \right| \\
&= \left| \frac{(1 - |a_k|) + z \left(\frac{|a_k|}{a_k} - \frac{|a_k|^2}{a_k} \right)}{1 - \overline{a_k} z} \right| \\
&= (1 - |a_k|) \left| \frac{1 + \frac{|a_k| z}{a_k}}{1 - \overline{a_k} z} \right| \\
&\leq (1 - |a_k|) \frac{1 + \left| \frac{|a_k| z}{a_k} \right|}{1 - |\overline{a_k} z|} \\
&\leq (1 - |a_k|) \frac{1 + |z|}{1 - |z|}.
\end{aligned}$$

Hence, the infinite Blaschke product converges absolutely in $\mathbb{D}$ and uniformly on compact subsets of $\mathbb{D}$ if and only if $\sum_{k=1}^{\infty} (1 - |a_k|) < \infty$. The Blaschke condition is important because if we multiply infinitely many automorphisms of the unit disc that do not approach 1 fast enough is that the product can be zero.

2.2 Other definitions of Blaschke products

As mentioned in the previous section, there are many different but analogous definitions of Blaschke products. They are equivalent to each other up to rotation. A few definitions from different sources will follow below. Only finite Blaschke products will be covered here, since those are the main subject of this thesis.

Definition 2.4 (Daepp, Gorkin, Mortini [2])**.** *Let β be a complex number with $|\beta| = 1$, and let $a_1, ..., a_n$ be complex numbers with modulus less than one. Then*

$$B(z) = \beta \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z} \tag{3}$$

is a Blaschke product of degree n.

Definition 2.5 (Garcia, Mashreghi, Ross [5]). *Let* $\alpha \in \mathbb{R}$, $K \in \mathbb{N} \cup \{0\}$ *and* $a_1, ..., a_n$ *be a set of complex numbers such that* $0 < |a_j| < 1$ *(*$1 \le j \le n$*). Then*

$$B(z) = e^{i\alpha} z^K \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} \tag{4}$$

is a Blaschke product of degree $K + n$.

Definition 2.6 (Fischer [3]). *Let* $a_1, ..., a_n \in \mathbb{D}$. *Then*

$$B(z) = \prod_{k=1}^{n} \left(\frac{-\overline{a_k}}{|a_k|} \right) \left(\frac{z - a_k}{1 - \overline{a_k} z} \right) \tag{5}$$

is a Blaschke product of degree n.

Regardless of which definition is used, the modulus will be the same. Let the points $a_1, ..., a_n$ be fixed, and let the first $0 \le m \le n$ points be zero, and the rest nonzero. Then the modulus of (1), (3) and (5) is

$$|z|^m \prod_{k=m+1}^{n} \frac{|a_k - z|}{|1 - \overline{a_k} z|},$$

and equivalently, the modulus of (4) is

$$|z|^m \prod_{k=1}^{n-m} \frac{|a_k - z|}{|1 - \overline{a_k} z|}.$$

The following figure shows the image of the point $z = 0.4 + 0.7i$ under the Blaschke products described in definitions 2.2, 2.4, 2.5 and 2.6, with $a_1, ..., a_4$ fixed.

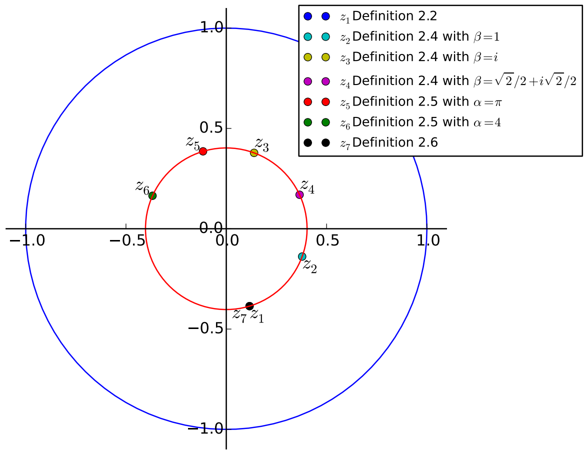

Figure 1: The image of the point $z = 0.4 + 0.7i$ under the Blaschke products from the different definitions 2.2, 2.4, 2.5 and 2.6, using $a_1 = 0.2 + 0.1i$, $a_2 = 0.5i$, $a_3 = 0.7$ and $a_4 = 0.3 - 0.6i$ in all of the definitions.

3 Properties

This section is dedicated to stating and proving some basic properties about finite Blaschke products, mostly about their behaviour inside, on and outside the unit circle.

3.1 Modulus of $B(z)$ on the unit circle

The following proposition concerns the modulus of finite Blaschke products on the unit circle $\mathbb{T}$.

Proposition 3.1. *Let $B(z)$ be a finite Blaschke product of degree n. Then $|B(z)| = 1$ for all $z \in \mathbb{T}$.*

Proof. Notice that if $|z| = 1$ then $z = \frac{1}{\bar{z}}$, since if $z = e^{i\theta}$, then

$$\frac{1}{\bar{z}} = \frac{1}{e^{-i\theta}} = e^{i\theta} = z.$$

Let $z_0 \in \mathbb{T}$. Then

$$\begin{aligned}
|B(z_0)| &= \left| \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z_0}{1 - \overline{a_k} z_0} \right| \\
&= \prod_{k=1}^{n} \left| \frac{|a_k|}{a_k} \right| \left| \frac{a_k - z_0}{1 - \overline{a_k} z_0} \right| \\
&= \prod_{k=1}^{n} \left| \frac{z_0 - a_k}{1 - \overline{a_k} \frac{1}{\overline{z_0}}} \right| \\
&= \prod_{k=1}^{n} \frac{|z_0 - a_k|}{\left| \overline{z_0} \left(1 - \overline{a_k} \frac{1}{\overline{z_0}} \right) \right|} \qquad (6) \\
&= \prod_{k=1}^{n} \frac{|z_0 - a_k|}{|\overline{z_0} - \overline{a_k}|} \\
&= 1.
\end{aligned}$$

In step (6), the property $|z_0| = |\overline{z_0}| = 1$ is used. □

3.2 Modulus of $B(z)$ inside the unit circle

The following proposition concerns the modulus of finite Blaschke products on the open unit disc $\mathbb{D}$.

Proposition 3.2. *Let $B(z)$ be a finite Blaschke product of degree n. Then $|B(z)| < 1$ for all $z \in \mathbb{D}$.*

Proof. The proposition follows from the *maximum modulus principle*. The theorem states that if a function $f(z)$ analytic in a region E and continuous on $\overline{E}$, then $|f(z)|$ attains its maximum value on the boundary of E and not at any interior point (see for instance [12]). Since $B(z)$ is analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, the maximum absolute value is attained on the boundary $\mathbb{T}$, and not in any interior point. Since we know that $|B(z)| = 1$ for all $z \in \mathbb{T}$ and $B(z)$ is not constant, we can conclude that $|B(z)| < 1$ for $z \in \mathbb{D}$. □

3.3 Relation between $B(z)$ and $B(1/\overline{z})$

The following proposition concerns the relation between $B(z)$ and $\overline{B(1/\overline{z})}$ on the extended complex plane.

Proposition 3.3. *Let $B(z)$ be a finite Blaschke product of degree n. Then $B(z) = \dfrac{1}{\overline{B(1/\overline{z})}}$ for all $z \in \widehat{\mathbb{C}}$.*

Proof. Let $B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}$ be a finite Blaschke product. For $z \in \mathbb{C}$ with

$0 < |z| < \infty$, we have

$$\begin{aligned}
\frac{1}{\overline{B(1/\overline{z})}} &= \frac{1}{\overline{\prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k-(1/\overline{z})}{1-\overline{a_k}(1/\overline{z})}}} \\
&= \prod_{k=1}^{n} \frac{\overline{a_k}}{|a_k|} \overline{\frac{1-\overline{a_k}(1/\overline{z})}{a_k-(1/\overline{z})}} \\
&= \prod_{k=1}^{n} \frac{\overline{a_k}}{|a_k|} \frac{1-a_k(1/z)}{\overline{a_k}-(1/z)} \\
&= \prod_{k=1}^{n} \frac{\overline{a_k}}{|a_k|} \frac{\frac{1}{z}(z-a_k)}{\frac{1}{z}(\overline{a_k}z-1)} \\
&= \prod_{k=1}^{n} \frac{\overline{a_k}}{|a_k|} \frac{a_k-z}{1-\overline{a_k}z} \\
&= \prod_{k=1}^{n} \frac{\overline{a_k}a_k}{|a_k|\,a_k} \frac{a_k-z}{1-\overline{a_k}z} \\
&= \prod_{k=1}^{n} \frac{|a_k|^2}{|a_k|\,a_k} \frac{a_k-z}{1-\overline{a_k}z} \\
&= \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k-z}{1-\overline{a_k}z} \\
&= B(z).
\end{aligned}$$

For $z = \infty$ we have

$$\begin{aligned}
B(z) &= \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k-z}{1-\overline{a_k}z} \\
&= \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{z(\frac{a_k}{z}-1)}{z(\frac{1}{z}-\overline{a_k})},
\end{aligned}$$

hence,

$$\begin{aligned}
B(\infty) &= \prod_{k=1}^{n} \frac{|a_k|}{|a_k|^2} \\
&= \prod_{k=1}^{n} \frac{1}{|a_k|}.
\end{aligned}$$

For the right-hand side of the identity, we have

$$\frac{1}{\overline{B(1/\overline{z})}} = \frac{1}{\overline{\prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - (1/\overline{z})}{1 - \overline{a_k}(1/\overline{z})}}}.$$

Thus, by defining $\overline{\infty} = \infty$, we get

$$\begin{aligned}
\frac{1}{\overline{B(1/\overline{\infty})}} &= \frac{1}{\overline{\prod_{k=1}^{n} \frac{|a_k|}{a_k} a_k}} \\
&= \frac{1}{\overline{\prod_{k=1}^{n} |a_k|}} \\
&= \prod_{k=1}^{n} \frac{1}{|a_k|}.
\end{aligned}$$

Thus, $B(\infty) = \dfrac{1}{\overline{B(1/\overline{\infty})}}$. For $z = 0$ we have

$$B(0) = \prod_{k=1}^{n} |a_k|,$$

For the right-hand side of the identity we have

$$\begin{aligned}
\frac{1}{\overline{B(1/\overline{z})}} &= \frac{1}{\overline{\displaystyle\prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - (1/\overline{z})}{1 - \overline{a_k}(1/\overline{z})}}} \\
&= \frac{1}{\displaystyle\prod_{k=1}^{n} \frac{|a_k|}{\overline{a_k}} \frac{\overline{a_k} - 1/z}{1 - a_k/z}} \\
&= \frac{1}{\displaystyle\prod_{k=1}^{n} \frac{|a_k|}{\overline{a_k}} \frac{\frac{1}{z}(\overline{a_k} z - 1)}{\frac{1}{z}(z - a_k)}},
\end{aligned}$$

and thus,

$$\begin{aligned}
\frac{1}{\overline{B(1/\overline{0})}} &= \frac{1}{\displaystyle\prod_{k=1}^{n} \frac{|a_k|}{|a_k|^2}} \\
&= \prod_{k=1}^{n} |a_k|.
\end{aligned}$$

Hence, $B(0) = \dfrac{1}{\overline{B(1/\overline{0})}}$. To clarify, for $z = 0$ and $z = \infty$, the identity in the proposition can be read as $B(0) = \dfrac{1}{B(\infty)}$ and $B(\infty) = \dfrac{1}{B(0)}$ respectively. □

3.4 Modulus of $B(z)$ outside the unit circle

The following proposition concerns the modulus of finite Blaschke products outside the unit circle, i.e. for $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Proposition 3.4. *Let $B(z)$ be a finite Blaschke product of degree n. Then $|B(z)| > 0$ for all $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

Proof. Let $B(z)$ be a finite Blaschke product. From the previous section, we know that $B(z) = \frac{1}{\overline{B(1/\overline{z})}}$ for all $z \in \widehat{\mathbb{C}}$. Suppose $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then $\frac{1}{z} \in \mathbb{D}$, and thus, we know that $|B(1/\overline{z})| < 1$ from Section 3.2. Hence, $|B(z)| > 1$ for $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This holds for $z = \infty$ as well, since we know that $B(\infty) = \prod_{k=1}^{n} \frac{1}{|a_k|} > 1$ from the proof of Proposition 3.3. □

4 Derivative of $B(z)$

This section is dedicated to the derivative of an arbitrary finite Blaschke product. The zeros of $B'(z)$ and their location will be studied in Section 6.2. Since certain results regarding the derivative are very closely related to the solutions to the equation $B(z) = \omega$ for $\omega \in \mathbb{T}$ and the location of the zeros of both $B(z)$ and $B'(z)$, they will be discussed in Section 5 and 6.

First, we need to compute the derivative of an arbitrary finite Blaschke product. Let

$$B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z},$$

and let $B_k(z)$ denote the product of all factors of $B(z)$ except the kth factor, i.e.

$$B_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \overline{a_j} z}. \tag{7}$$

Then, by using the product rule, we get

$$B'(z) = \sum_{k=1}^{n} \left(\frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} \right)' B_k(z).$$

By applying the quotient rule to $\left(\frac{|a_k|}{a_k}\frac{a_k-z}{1-\overline{a_k}z}\right)'$, we get

$$\begin{aligned}\left(\frac{|a_k|}{a_k}\frac{a_k-z}{1-\overline{a_k}z}\right)' &= \frac{|a_k|}{a_k}\left(\frac{(a_k-z)'(1-\overline{a_k}z)-(a_k-z)(1-\overline{a_k}z)'}{(1-\overline{a_k}z)^2}\right)\\ &= \frac{|a_k|}{a_k}\left(\frac{-1(1-\overline{a_k}z)-(a_k-z)(-\overline{a_k})}{(1-\overline{a_k}z)^2}\right)\\ &= \frac{|a_k|}{a_k}\frac{\overline{a_k}z-1+\overline{a_k}a_k-\overline{a_k}z}{(1-\overline{a_k}z)^2}\\ &= -\frac{|a_k|}{a_k}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}.\end{aligned}$$

Hence, the derivative of $B(z)$ is the following

$$B'(z) = -\sum_{k=1}^{n}\left(\frac{|a_k|}{a_k}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}B_k(z)\right). \tag{8}$$

In particular, for any of the points a_j, the product $B_k(a_j)$ will be zero except when $k=j$. This is because for any $k \neq j$, $B_k(z)$ includes the factor $\frac{|a_j|}{a_j}\frac{a_j-z}{1-\overline{a_j}z}$, which will be zero for $z=a_j$. So, for any of the points a_j, we have

$$\begin{aligned}B'(a_j) &= -\frac{|a_j|}{a_j}\frac{1-|a_j|^2}{(1-\overline{a_j}a_j)^2}B_j(a_j)\\ &= -\frac{|a_j|}{a_j}\frac{1-|a_j|^2}{(1-|a_j|^2)^2}B_j(a_j)\\ &= -\frac{|a_j|}{a_j}\frac{1}{1-|a_j|^2}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{a_k-a_j}{1-\overline{a_k}a_j}.\end{aligned}$$

The *logarithmic derivative* of a function f is defined by the formula $\frac{f'}{f}$. Thus, the logarithmic derivative of $B(z)$ is

$$\frac{B'(z)}{B(z)} = \frac{-\sum_{k=1}^{n}\left(\frac{|a_k|}{a_k}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}B_k(z)\right)}{B(z)} \tag{9}$$

$$= -\sum_{k=1}^{n}\left(\frac{\frac{|a_k|}{a_k}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}}{\frac{|a_k|}{a_k}\frac{a_k-z}{1-\overline{a_k}z}}\right) \tag{10}$$

$$= -\sum_{k=1}^{n}\frac{\left(1-|a_k|^2\right)(1-\overline{a_k}z)}{(1-\overline{a_k}z)^2(a_k-z)} \tag{11}$$

$$= -\sum_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)(a_k-z)} \tag{12}$$

$$= \sum_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)(z-a_k)}, \tag{13}$$

for $z \in \widehat{\mathbb{C}} \setminus \{a_j,\ 1/\overline{a_j}:\ 1 \leq j \leq n\}$. The logarithmic derivative $\dfrac{B'(z)}{B(z)}$ is defined at infinity since for all terms where $a_j \neq 0$, we have

$$\frac{1-|a_j|^2}{(1-\overline{a_j}z)(a_j-z)} = \frac{1-|a_j|^2}{z^2\left(\frac{1}{z}-\overline{a_j}\right)\left(\frac{a_j}{z}-1\right)},$$

which, with $z=\infty$, gives us

$$\frac{1-|a_j|^2}{\infty^2\left(\frac{1}{\infty}-\overline{a_j}\right)\left(\frac{a_j}{\infty}-1\right)} = 0,$$

and for all terms where $a_j = 0$, we have

$$\frac{1-|a_j|^2}{(1-\overline{a_j}z)(a_j-z)} = \frac{1}{-z},$$

which, with $z=\infty$, gives us

$$\frac{1}{-\infty} = 0.$$

Thus,

$$\frac{B'(\infty)}{B(\infty)} = \sum_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}\infty)(a_k-\infty)} = 0.$$

In the next section, the solutions to the equation $B(z)=\omega$ will be examined.

5 Solutions to $B(z) = \omega$

This section is dedicated to the solutions to the equation

$$B(z) = \omega,$$

for ω inside, on and outside the unit disc. First we examine the number of solutions to said equation, and how the location of the solutions are affected depending on the choice of ω.

Theorem 5.1 (Garcia, Mashreghi, Ross [6]). *Let $B(z)$ be a finite Blaschke product of degree n. Then for each $\omega \in \widehat{\mathbb{C}}$ the equation $B(z) = \omega$ has exactly n solutions, counted according to multiplicity. If $\omega \in \mathbb{D}$, these solutions belong to $\mathbb{D}$. If $\omega \in \widehat{\mathbb{C}} \setminus \mathbb{D}$, these solutions belong to $\widehat{\mathbb{C}} \setminus \mathbb{D}$. If $\omega \in \mathbb{T}$, these solutions belong to $\mathbb{T}$.*

Proof. From Corollary 4.5 in [5] and the discussion above said corollary, we know that we can write

$$B(z) = \frac{P(z)}{z^n \overline{P(1/\overline{z})}} = \frac{\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n}{\overline{\alpha_n} + \overline{\alpha_{n-1}} z + \cdots + \overline{\alpha_0} z^n},$$

with $\alpha_0, ..., \alpha_{n-1} \in \mathbb{C}$ and $\alpha_n = 1$. The zeros of the polynomial $P(z)$ are the zeros of $B(z)$, and thus, lie in $\mathbb{D}$. Let $\gamma_1, ..., \gamma_n$ denote the zeros of $P(z)$. Then we can rewrite $P(z)$ as

$$P(z) = \alpha_n (z - \gamma_1) \cdots (z - \gamma_n).$$

By expanding the above expression and considering the constant term obtained from that expansion, we get that

$$\alpha_0 = (-1)^n \alpha_n \gamma_1 \gamma_2 \cdots \gamma_n. \tag{14}$$

Since $|\alpha_n| = 1$ and $\gamma_1, ..., \gamma_n \in \mathbb{D}$, the right-hand side in (14) belongs to $\mathbb{D}$. We can rewrite the equation $B(z) = \omega$ as

$$P(z) = \omega z^n \overline{P(1/\overline{z})},$$

or written out

$$\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n = \omega \left(\overline{\alpha_n} + \overline{\alpha_{n-1}} z + \cdots + \overline{\alpha_0} z^n\right). \tag{15}$$

Then we see that the coefficient of z^n is nonzero if $\alpha_n \neq \omega \overline{\alpha_0}$. If $\omega \in \mathbb{D}$, we have that

$$|\omega \overline{\alpha_0}| < |\alpha_n| = 1 \tag{16}$$

since both ω and α_0 are in $\mathbb{D}$. Thus, the coefficient of z^n in (15) is nonzero, and since a polynomial of degree n has exactly n roots, the equation in (15) must have exactly n solutions. Since we know that $B(z) = \dfrac{1}{\overline{B(1/\overline{z})}}$ from Proposition 3.3, we know that $B(z) = \omega$ has exactly n solutions if $\omega \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ as well. For $\omega = \infty$, first note that the equation $B(z) = 0$ have exactly n solutions (namely $a_1, ..., a_n$). From Proposition 3.3, we have that $B(0) = \frac{1}{\overline{B(\infty)}}$. Thus, by the same argument as above, the equation has exactly n solutions if $\omega = \infty$. If $\omega \in \mathbb{T}$, the strict inequality in (16) still holds, since $|\overline{\alpha_0}| < 1$. Hence, $B(z) = \omega$ has exactly n solutions for all $\omega \in \widehat{\mathbb{C}}$. The location of the solutions follow directly from the properties described in Section 3.1, 3.2 and 3.4. □

In some cases, repeated solutions may occur. The most obvious case is when $\omega = 0$, since the solutions to the equation $B(z) = 0$ are $z = a_j$ $(1 \leq j \leq n)$, and all a_j do not need to be distinct. In fact, repeated solutions may occur for $\omega \in \widehat{\mathbb{C}} \setminus \mathbb{T}$. However, for $\omega \in \mathbb{T}$, all solutions must be distinct. Prior to proving that statement, we need the following lemma and proposition.

Lemma 5.2. *Let $f(z)$ be a complex polynomial with $f(\xi) = 0$. Then ξ is a repeated solution if and only if $f(\xi) = f'(\xi) = 0$.*

Proof. Let $f(z)$ be a complex polynomial and suppose ξ is a solution to $f(z) = 0$ with multiplicity m. Then we can write

$$f(z) = (z - \xi)^m g(z)$$

for some polynomial $g(z)$ with $g(\xi) \neq 0$. Then we have

$$f'(z) = m(z - \xi)^{m-1} g(z) + (z - \xi)^m g'(z).$$

Hence, we get that $f(\xi) = f'(\xi) = 0$ if $m \geq 2$, and $f'(\xi) \neq 0$ if $m = 1$. □

Proposition 5.3 (Garcia, Mashreghi, Ross [5]). *If B(z) is a finite Blaschke product, then $B'(z) \neq 0$ for all $z \in \mathbb{T}$.*

Proof. Let $B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}$ be a Blaschke product. From (13), recall that

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{(1 - \overline{a_k} z)(z - a_k)}$$

is the logarithmic derivative of $B(z)$. For each $e^{i\theta} \in \mathbb{T}$, we have

$$\frac{B'(e^{i\theta})}{e^{-i\theta} B(e^{i\theta})} = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{e^{-i\theta}(1 - \overline{a_k} e^{i\theta})(e^{i\theta} - a_k)} \tag{17}$$

$$= \sum_{k=1}^{n} \frac{1 - |a_k|^2}{(e^{-i\theta} - \overline{a_k})(e^{i\theta} - a_k)} \tag{18}$$

$$= \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|e^{i\theta} - a_k|^2}. \tag{19}$$

Since $\left|e^{-i\theta}B(e^{i\theta})\right| = 1$, we have that

$$\begin{aligned}\left|B'(e^{i\theta})\right| &= \left|\sum_{k=1}^{n} \frac{1-|a_k|^2}{|e^{i\theta}-a_k|^2}\right| \\ &= \sum_{k=1}^{n} \frac{1-|a_k|^2}{|e^{i\theta}-a_k|^2} \\ &\neq 0,\end{aligned}$$

which completes the proof. □

Corollary 5.4 (Garcia, Mashreghi, Ross [5]). *If B(z) is a finite Blaschke product of degree n, then for each $\omega \in \mathbb{T}$, the equation $B(z) = \omega$ has exactly n distinct solutions on $\mathbb{T}$.*

Proof. Notice that we can write the equation as

$$h(z) = B(z) - \omega.$$

Then, by Lemma 5.2, we get that ξ is a repeated solution if and only if $h(\xi) = h'(\xi) = 0$. But $h'(z_0) = B'(z_0) \neq 0$ for $z_0 \in \mathbb{T}$, so $h(z) = 0$ cannot have repeated solutions on $\mathbb{T}$, which implies that the equation $B(z) = \omega$ cannot have repeated solutions for $\omega \in \mathbb{T}$. □

6 Location of zeros

This section is dedicated to the location of the zeros of both $B(z)$ and $B'(z)$. First, we need a definition which will be used in both cases.

Definition 6.1 (Rockafellar [9]). *Let $S \subset \mathbb{C}$. The convex hull of S is the intersection of all convex sets containing S. Equivalently, it is the smallest convex set containing S.*

From a corollary in [9], we have that the convex hull of a finite set of points $\{z_1, ..., z_n\} \subset \mathbb{C}$ consists of all vectors of the form

$$\lambda_1 z_1 + \cdots + \lambda_n z_n$$

with $\lambda_j \geq 0$ $(1 \leq j \leq n)$ and $\lambda_1 + \cdots + \lambda_n = 1$. In a two-dimensional space, it might be easier to conceptualize the convex hull of a finite set of points $\{z_1, ..., z_n\}$ from a figure rather than straight from the definition. The figure below shows the convex hull of the set $S = \{z_1, z_2, z_3, z_4, z_5\}$, where $z_1 = (0.5, 0.5)$, $z_2 = (-0.3, 0.1)$, $z_3 = (-0.25, -0.5)$, $z_4 = (0.1, -0.7)$ and $z_5 = (0.5, -0.45)$.

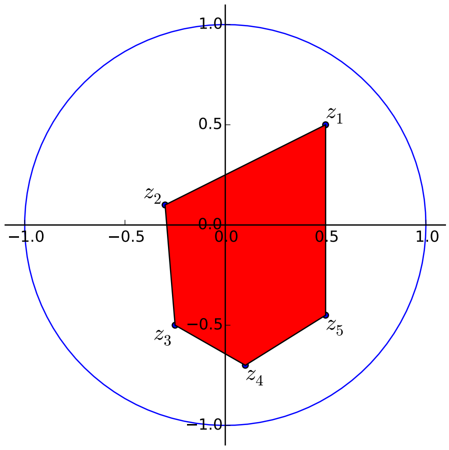

Figure 2: The red region is the convex hull of $\{z_1, z_2, z_3, z_4, z_5\}$.

Now, the location of the zeros of $B(z)$ will be studied in more detail.

6.1 Location of the zeros of $B(z)$

In this section we will study the relation between the location of the zeros of $B(z)$ and the location of the solutions to the equation $B(z) = \omega$ for ω on the unit circle.

We begin by stating a theorem of Gauss and Lucas. The theorem will later be used in the proof of the location of the zeros of finite Blaschke products.

Theorem 6.2 (Gauss, Lucas, as cited in Garcia, Mashreghi, Ross [5])**.** *Let* $z_1, ..., z_n \in \mathbb{C}$ *be distinct and let* $c_1, ..., c_n > 0$*. Then*

$$f(z) = \frac{c_1}{z - z_1} + \cdots + \frac{c_n}{z - z_n} \tag{20}$$

has $n - 1$ *zeros that lie in the convex hull of* $\{z_1, ..., z_n\}$*.*

Proof. Let $z_1, ..., z_n$, $c_1, ..., c_n$ and $f(z)$ be defined as in the theorem. We can see that $f(z)$ has $n-1$ zeros by multiplying (20) with $(z-z_1)\cdots(z-z_n)$. Then we get

$$\begin{aligned} 0 &= \frac{c_1(z - z_1)\cdots(z - z_n)}{z - z_1} + \cdots + \frac{c_n(z - z_1)\cdots(z - z_n)}{z - z_n} \\ &= c_1(z - z_2)\cdots(z - z_n) + \cdots + c_n(z - z_1)\cdots(z - z_{n-1}). \end{aligned}$$

This is clearly a polynomial of degree $n - 1$, and hence, has $n - 1$ zeros. Now we have to prove that the zeros lie in the convex hull of $\{z_1, ..., z_n\}$. Now suppose that ξ is one of the solutions of $f(z) = 0$. Then we have

$$\frac{c_1}{\xi - z_1} + \cdots + \frac{c_n}{\xi - z_n} = 0. \tag{21}$$

Multiplying each term $\frac{c_j}{\xi - z_j}$ in (21) with $\frac{(\overline{\xi} - \overline{z_j})}{(\overline{\xi} - \overline{z_j})}$, we get

$$\frac{c_1 \left(\overline{\xi} - \overline{z_1}\right)}{|\xi - z_1|^2} + \cdots + \frac{c_n \left(\overline{\xi} - \overline{z_n}\right)}{|\xi - z_n|^2} = 0. \tag{22}$$

The expression (22) can also be written as

$$\left(\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2} \right) \overline{\xi} = \overline{z_1} \frac{c_1}{|\xi - z_1|^2} + \cdots + \overline{z_n} \frac{c_n}{|\xi - a_n|^2}.$$

Since $c_1, ..., c_n \in \mathbb{R}$, this is equivalent to

$$\left(\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2} \right) \xi = z_1 \frac{c_1}{|\xi - z_1|^2} + \cdots + z_n \frac{c_n}{|\xi - z_n|^2}.$$

Hence,

$$\begin{aligned} \xi &= \frac{z_1 \frac{c_1}{|\xi - z_1|^2} + \cdots + z_n \frac{c_n}{|\xi - z_n|^2}}{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}} \\ &= \lambda_1 z_1 + \cdots + \lambda_n z_n, \end{aligned}$$

where

$$\lambda_j = \frac{\frac{c_j}{|\xi - z_j|^2}}{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}}$$

for $1 \leq j \leq n$. Since $c_1, ..., c_n > 0$, we have that $0 < \lambda_1, ..., \lambda_n < 1$. We also have that

$$\begin{aligned} \lambda_1 + \cdots + \lambda_n &= \frac{\frac{c_1}{|\xi - z_1|^2}}{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}} + \cdots + \frac{\frac{c_n}{|\xi - z_n|^2}}{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}} \\ &= \frac{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}}{\frac{c_1}{|\xi - z_1|^2} + \cdots + \frac{c_n}{|\xi - z_n|^2}} \\ &= 1. \end{aligned}$$

Hence, ξ lies in the convex hull of $\{z_1, ..., z_n\}$, and the proof is complete. □

Before stating the theorem about the location of the zeros of finite Blaschke products, we need the following lemma:

Lemma 6.3 (Garcia, Mashreghi, Ross [5]). *Let $a_1, ..., a_n \in \mathbb{D}$,*

$$B(z) = z \prod_{k=1}^{n-1} \frac{a_k - z}{1 - \overline{a_k} z}$$

and $\omega \in \mathbb{T}$. Let $\xi_1, ..., \xi_n$ be the n distinct solutions to $B(z) = \omega$. Define

$$\lambda_k = \frac{1}{1 + \displaystyle\sum_{j=1}^{n-1} \frac{1 - |a_j|^2}{|\xi_k - a_j|^2}}$$

for all $1 \leq k \leq n$. Then $\lambda_1, ..., \lambda_n$ satisfy

$$0 < \lambda_1, ..., \lambda_n < 1$$

and

$$\lambda_1 + \cdots + \lambda_n = 1. \tag{23}$$

Moreover,

$$\frac{B(z)/z}{B(z) - \omega} = \frac{(z - a_1) \cdots (z - a_{n-1})}{(z - \xi_1) \cdots (z - \xi_n)} \tag{24}$$

$$= \frac{\lambda_1}{z - \xi_1} + \cdots + \frac{\lambda_n}{z - \xi_n}. \tag{25}$$

Proof. First, observe that

$$\begin{aligned}
\frac{B(z)/z}{B(z) - \omega} &= \frac{\prod_{k=1}^{n-1} \frac{a_k - z}{1 - \overline{a_k} z}}{z \prod_{k=1}^{n-1} \frac{a_k - z}{1 - \overline{a_k} z} - \omega} \\
&= \frac{\prod_{k=1}^{n-1} (a_k - z)}{z \prod_{k=1}^{n-1} (a_k - z) - \omega \prod_{k=1}^{n-1} (1 - \overline{a_k} z)} \\
&= \frac{P(z)}{Q(z)},
\end{aligned}$$

where $P(z)$ is a polynomial of degree $n-1$ with zeros in $a_1, ..., a_{n-1}$ and $Q(z)$ is a polynomial of degree n with zeros in $\xi_1, ..., \xi_n$. Thus, we have

$$\frac{B(z)/z}{B(z) - \omega} = C \frac{(z - a_1) \cdots (z - a_{n-1})}{(z - \xi_1) \cdots (z - \xi_n)}$$

for some constant $C \in \mathbb{C}$ with $C \neq 0$. By multiplying both sides with z and looking at the limit as $z \to \infty$, we get

$$\lim_{z \to \infty} \frac{B(z)}{B(z) - \omega} = \lim_{z \to \infty} C \frac{z^n \left(1 - \frac{a_1}{z}\right) \cdots \left(1 - \frac{a_{n-1}}{z}\right)}{z^n \left(1 - \frac{\xi_1}{z}\right) \cdots \left(1 - \frac{\xi_n}{z}\right)},$$

which gives us that $C = 1$. Let

$$\frac{B(z)/z}{B(z)-\omega} = \frac{\lambda_1}{z-\xi_1} + \cdots + \frac{\lambda_n}{z-\xi_n}.$$

be a partial fraction decomposition. Let j $(1 \leq j \leq n)$ be fixed. By multiplying the preceding equation with $z - \xi_j$ and letting $z \to \xi_j$, we get that

$$\begin{aligned} \lambda_j &= \lim_{z \to \xi_j} \frac{(z-\xi_j)\, B(z)/z}{B(z)-\omega} \\ &= \lim_{z \to \xi_j} \frac{B(z)}{z} \frac{z-\xi_j}{B(z)-\omega}. \end{aligned} \tag{26}$$

Notice that

$$\begin{aligned} \lim_{z \to \xi_j} \frac{z-\xi_j}{B(z)-\omega} &= \lim_{z \to \xi_j} \frac{z-\xi_j}{B(z)-B(\xi_j)} \\ &= \lim_{z \to \xi_j} \frac{1}{\frac{B(z)-B(\xi_j)}{z-\xi_j}} \\ &= \frac{1}{B'(z)}, \end{aligned}$$

by the definition of derivative. Hence, we can rewrite (26) as

$$\begin{aligned} \lambda_j &= \frac{B(\xi_j)}{\xi_j B'(\xi_j)} \\ &= \frac{\overline{\xi_j} B(\xi_j)}{B'(\xi_j)} \\ &= \frac{1}{\frac{B'(\xi_j)}{\overline{\xi_j} B(\xi_j)}} \\ &= \frac{1}{1 + \sum_{k=1}^{n-1} \frac{1-|a_k|^2}{|\xi_j - a_k|^2}}, \end{aligned} \tag{27}$$

where (19) is used in the last step. Thus, we have proven (25). Since the denominator in (27) is greater than 1, we have that

$$0 < \lambda_1, ..., \lambda_n < 1.$$

To prove (23), multiply the expression in (25) with z and let $z \to \infty$. Then we get

$$\begin{aligned}
\lim_{z\to\infty} \frac{\lambda_1 z}{z-\xi_1} + \cdots + \frac{\lambda_n z}{z-\xi_n} &= \lim_{z\to\infty} \frac{z}{z}\left(\frac{\lambda_1}{1-\xi_1/z} + \cdots + \frac{\lambda_n}{1-\xi_n/z}\right) \\
&= \lambda_1 + \cdots + \lambda_n \\
&= \lim_{z\to\infty} \frac{B(z)}{B(z)-\omega} \\
&= \lim_{z\to\infty} \frac{z}{z}\frac{B(z)/z}{B(z)/z+\omega/z} \\
&= 1,
\end{aligned}$$

which completes the proof. □

Now we have enough knowledge to state the theorem about the location of zeros of finite Blaschke products in relation to the solutions to the equation $B(z) = \omega$ for ω on the unit circle.

Theorem 6.4 (Garcia, Mashreghi, Ross [5]). *Let $a_1, ..., a_{n-1} \in \mathbb{D}$,*

$$B(z) = z \prod_{k=1}^{n-1} \frac{a_k - z}{1 - \overline{a_k} z},$$

and $\omega \in \mathbb{T}$. Let $\xi_1, ..., \xi_n$ be the n distinct solutions to $B(z) = \omega$. Then $a_1, ..., a_{n-1}$ belong to the convex hull of $\xi_1, ..., \xi_n$.

Proof. From the previous lemma, we have that

$$\frac{B(z)/z}{B(z)-\omega} = \frac{\lambda_1}{z-\xi_1} + \cdots + \frac{\lambda_n}{z-\xi_n}.$$

On the left-hand side, we can see that the zeros are exactly the points $a_1, ..., a_{n-1}$. From Theorem 6.2, we know that the right-hand side has exactly $n-1$ zeros, and that the zeros lie in the convex hull of $\{\xi_1, ..., \xi_n\}$. Hence, $a_1, ..., a_{n-1}$ lie in the convex hull of $\{\xi_1, ..., \xi_n\}$. □

6.2 Location of the zeros of $B'(z)$

In this section, the location of the zeros of $B'(z)$ will be examined. First, we will look into the number of solutions and roughly where they are located. As we saw in Proposition 5.3, the derivative is nonzero on the unit circle. Hence, the zeros must lie either in $\mathbb{D}$ or in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Before stating the theorem which concerns the number of solutions, how many of them are located in $\mathbb{D}$ and how many are located in $\mathbb{C} \setminus \overline{\mathbb{D}}$, we need a lemma.

Lemma 6.5 (Garcia, Mashreghi, Ross [5]). *Let $B(z)$ be a finite Blaschke product. Then, for every $z \in \widehat{\mathbb{C}} \setminus \{0\}$ with $B(z) \neq 0$, it holds that $B'(z) = 0$ if and only if $B'(1/\overline{z}) = 0$.*

Proof. From Proposition 3.3, we know that

$$B(z)\overline{B(1/\overline{z})} = 1$$

for all $z \in \widehat{\mathbb{C}}$. If we take the derivative of both sides with respect to z, we get

$$B'(z)\overline{B(1/\overline{z})} + B(z)\left(\overline{B(1/\overline{z})}\right)' = 0. \tag{28}$$

For the derivative of $\overline{B(1/\overline{z})}$, first notice that

$$\overline{B'(1/\overline{z})} = -\sum_{k=1}^{n}\left(\frac{|a_k|}{\overline{a_k}}\frac{1-|a_k|^2}{1-\frac{a_k}{z}}\overline{B_k(1/\overline{z})}\right),$$

where $B_k(z)$ is defined as in (7). Now we continue calculating $\left(\overline{B(1/\overline{z})}\right)'$. We have

$$\begin{aligned}
\left(\overline{B(1/\overline{z})}\right)' &= \left(\prod_{k=1}^{n}\frac{|a_k|}{\overline{a_k}}\frac{\overline{a_k}-\frac{1}{z}}{1-\frac{a_k}{z}}\right)' \\
&= -\sum_{k=1}^{n}\left(\frac{|a_k|}{\overline{a_k}}\frac{\overline{a_k}-\frac{1}{z}}{1-\frac{a_k}{z}}\right)'\overline{B_k(1/\overline{z})} \\
&= -\sum_{k=1}^{n}\left(\frac{|a_k|}{\overline{a_k}}\left(\frac{\frac{1}{z^2}\left(1-\frac{a_k}{z}\right)-\frac{a_k}{z^2}\left(\overline{a_k}-\frac{1}{z}\right)}{\left(1-\frac{a_k}{z}\right)^2}\right)\right)\overline{B_k(1/\overline{z})} \\
&= -\sum_{k=1}^{n}\left(\frac{|a_k|}{\overline{a_k}}\left(\frac{1-\frac{a_k}{z}-|a_k|^2+\frac{a_k}{z}}{z^2\left(1-\frac{a_k}{z}\right)^2}\right)\right)\overline{B_k(1/\overline{z})} \\
&= -\frac{1}{z^2}\sum_{k=1}^{n}\left(\frac{|a_k|}{\overline{a_k}}\frac{1-|a_k|^2}{\left(1-\frac{a_k}{z}\right)^2}\overline{B_k(1/\overline{z})}\right) \\
&= -\frac{1}{z^2}\overline{B'(1/\overline{z})}.
\end{aligned}$$

Hence, (28) can be written as

$$B'(z)\overline{B(1/\overline{z})} - \frac{1}{z^2}B(z)\overline{B'(1/\overline{z})} = 0.$$

Since we only consider points where $B(z) \neq 0$, we can see that $B'(z) = 0$ if and only if $B'(1/\overline{z}) = 0$. □

Theorem 6.6 (Garcia, Mashreghi, Ross [5])**.** *Let $B(z)$ be a Blaschke product of degree n, and rewrite it as*

$$B(z) = z^{d_0}\prod_{k=1}^{m}\left(\frac{a_k - z}{1-\overline{a_k}z}\right)^{d_k}, \tag{29}$$

such that $a_1, ..., a_m \in \mathbb{D} \setminus \{0\}$ are distinct numbers and $d_0, ..., d_m$ are positive integers with

$$d_0 + \cdots + d_m = n.$$

Then $B'(z)$ has exactly $n-1$ zeros in $\mathbb{D}$. If $d_0 \neq 0$, the number of zeros of $B'(z)$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is m. If $d_0 = 0$, the number of zeros of $B'(z)$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is less or equal to $m-1$.

Proof. First, suppose that all zeros of $B(z)$ are distinct and that neither $B(z)$ or $B'(z)$ are zero at the origin, that is, $d_0 = 0$ and $m = n$. From (13), we know that $B'(z) = 0$ if and only if

$$\sum_{k=1}^{n} \frac{1 - |a_k|^2}{(1 - \overline{a_k} z)(z - a_k)} = 0.$$

Multiplying by $\prod_{k=1}^{n} (1 - \overline{a_k} z)(z - a_k)$, we get

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \left(\frac{1 - |a_k|^2}{(1 - \overline{a_k} z)(z - a_k)} \prod_{j=0}^{n} (1 - \overline{a_j} z)(z - a_j) \right) \\ &= \sum_{k=1}^{n} \left(\left(1 - |a_k|^2\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} ((1 - \overline{a_j} z)(z - a_j)) \right), \end{aligned}$$

which is a polynomial of degree $2(n-1)$ that does not have any zeros in $\{0, a_1, ..., a_n, 1/\overline{a_1}, ..., 1/\overline{a_n}\}$. By Lemma 6.5, we know that exactly $n-1$ of the zeros are in $\mathbb{D}$ and exactly $n-1$ of the zeros are in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Theorem 5.6 in [5] states that for any finite Blaschke product $B(z)$ of degree n, there is a family $\{B_\epsilon : 0 < \epsilon < \epsilon_0\}$ of Blaschke products with the following properties:

1. each B_ϵ is of degree n,
2. each B_ϵ has distinct zeros,
3. for all ϵ, $B_\epsilon(0) \neq 0$ and $B'_\epsilon(0) \neq 0$, and
4. as $\epsilon \to 0$, B_ϵ converges uniformly to $B(z)$ on compact subsets of $\mathbb{C}$ that do not contain a pole of $B(z)$.

In the general case, this allows us to approximate $B(z)$ by a family B_ϵ of Blaschke products of degree n with distinct zeros and such that both $B_\epsilon(0) \neq 0$ and $B'_\epsilon \neq 0$. Since $\overline{\mathbb{D}}$ is a compact subset of $\mathbb{C}$ that does not contain any pole of $B(z)$, B_ϵ converges uniformly to $B(z)$ on $\overline{\mathbb{D}}$ as $\epsilon \to 0$. This, together with the first part of the proof, gives us that $B(z)$ have $n-1$ zeros in $\mathbb{D}$. In $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, this might not be the case, since $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ contains all poles of $B(z)$. Thus, B'_ϵ may have zeros at the poles of $B(z)$, which means that $B(z)$ may have fewer than $n-1$ zeros in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Therefore, we need to look closer to the number of zeros of $B'(z)$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. First, suppose that $d_0 \neq 0$. Then, by calculating the derivative of (29), we get

$$B'(z) = z^{d_0 - 1} \frac{\prod_{k=1}^{m} (a_k - z)^{d_k - 1}}{\prod_{k=1}^{m} (1 - \overline{a_k} z)^{d_k + 1}} P(z),$$

where

$$P(z) = d_0 \prod_{k=1}^{m} \left((a_k - z) (1 - \overline{a_k} z) \right) + z \sum_{k=1}^{n} \left(d_k \left(|a_k|^2 - 1 \right) \prod_{\substack{j=1 \\ j \neq k}}^{m} \left((a_j - z) (1 - \overline{a_j} z) \right) \right),$$

which is a polynomial of degree $2m$. The polynomial $P(z)$ does not have any zeros in $\{0, a_1, ..., a_n\}$ since

$$P(0) = d_0 \prod_{k=1}^{m} a_k \neq 0$$

and

$$P(a_p) = a_p d_p \left(|a_p|^2 - a \right) \prod_{\substack{j=1 \\ j \neq p}}^{m} \left((a_j - a_p) (1 - \overline{a_j} a_p) \right) \neq 0$$

for any $a_p \in \{a_1, ..., a_n\}$. Since $j_0 + \cdots + j_m = n$, the number of zeros of $B'(z)$ in $\mathbb{C}$ is $n - (m+1) + 2m = n + m - 1$. From Lemma 6.5, we know that the $2m$ zeros of $P(z)$ have the form $\gamma_1, ..., \gamma_m, 1/\overline{\gamma_1}, ..., 1/\overline{\gamma_m}$, where $\gamma_1, ..., \gamma_m \in \mathbb{D} \setminus \{0, a_1, ..., a_n\}$. Thus, the number of zeros in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is exactly m if $d_0 \neq 0$. Now suppose that $d_0 = 0$. Then, by calculating the derivative of (29), we get

$$B'(z) = \frac{\prod_{k=1}^{m} (a_k - z)^{d_k - 1}}{\prod_{k=1}^{m} (1 - \overline{a_k} z)^{d_k + 1}} Q(z)$$

where

$$Q(z) = \sum_{k=1}^{m} \left(d_k \left(|a_k|^2 - 1 \right) \prod_{\substack{j=1 \\ j \neq k}}^{m} \left((a_j - z) (1 - \overline{a_j} z) \right) \right),$$

which is a polynomial of degree at most $2(m-1)$. The polynomial $Q(z)$ does not have any zeros in $\{a_1, ..., a_n\}$ since

$$Q(a_p) = d_p \left(|a_p|^2 - 1 \right) \prod_{\substack{j=1 \\ j \neq p}}^{m} \left((a_j - a_p) (1 - \overline{a_j} a_p) \right)$$

for any $a_p \in \{a_1, ..., a_n\}$. Notice that $Q(z)$ can have zeros at the origin here. Thus, the number of zeros of $B'(z)$ in $\mathbb{C}$ is at most $n - m + 2(m-1) = n+m-2$. From Lemma 6.5, we know that the zeros of $Q(z)$ that are not at the origin have the form $\gamma_1, ..., \gamma_s, 1/\overline{\gamma_1}, ..., 1/\overline{\gamma_s}$ for some s, where $\gamma_1, ..., \gamma_s \in \mathbb{D} \setminus \{0, a_1, ..., a_n\}$. If $Q(z)$ have t zeros at the origin, we have that

$$2s + t = \deg Q \leq 2(m-1),$$

which implies that $s \leq m - 1$. Hence, $B'(z)$ has at most $m - 1$ zeros in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ if $j_0 = 0$, which completes the proof. □

For the rest of this section, the focus will be on the solutions that are located on the unit disc. There is an old theorem by Gauss and Lucas that gives a geometrical connection between the zeros of a complex polynomial $P(z)$ and its derivative $P'(z)$:

Theorem 6.7 (Gauss, Lucas, as cited in Garcia, Mashreghi, Ross [5]). *Let $P(z)$ be a complex polynomial. Then the zeros of $P'(z)$ lie in the convex hull of the zeros of $P(z)$.*

There is a similar theorem regarding finite Blaschke products:

Theorem 6.8 (Cassier, Chalendar [1]). *Let $B(z)$ be a finite Blaschke product. Then the zeros of $B'(z)$ that lie inside the unit disc are included in the convex hull of $\{0\} \cup \{a_1, ..., a_n\}$.*

Proof. Let $B(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}$ be a finite Blaschke product. Recall the logarithmic derivative

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{(1 - \overline{a_k} z)(z - a_k)}$$

from (13). With partial fraction decomposition we get

$$\begin{aligned} \frac{B'(z)}{B(z)} &= \sum_{k=1}^{n} \left(\frac{A}{1 - \overline{a_k} z} + \frac{B}{z - a_k} \right) \\ &= \sum_{k=1}^{n} \left(\frac{A(z - a_k) + B(1 - \overline{a_k} z)}{(1 - \overline{a_k} z)(z - a_k)} \right). \end{aligned}$$

For an arbitrary term in the sum, we have that

$$1 - |a_j|^2 = A(z - a_j) + B(1 - \overline{a_j} z),$$

which gives us that $A = \overline{a_j}$ and $B = 1$ for each j $(1 \leq j \leq n)$. Thus, we have

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \left(\frac{\overline{a_k}}{1 - \overline{a_k} z} - \frac{1}{a_k - z} \right).$$

Now let $\xi \in \{z: \ B'(z)=0\} \cap \overline{\mathbb{D}} \setminus \{a_1, ..., a_n\}$. Then we know that ξ satisfies

$$\sum_{k=1}^{n} \left(\frac{\overline{a_k}}{1-\overline{a_k}\xi} - \frac{1}{a_k - \xi} \right) = 0,$$

i.e.

$$\begin{aligned}
0 &= \sum_{k=1}^{n} \left(\frac{\overline{a_k}\left(1-a_k\overline{\xi}\right)}{\left|1-\overline{a_k}\xi\right|^2} - \frac{\overline{a_k}-\overline{\xi}}{\left|a_k-\xi\right|^2} \right) \\
&= \sum_{k=1}^{n} \left(\frac{a_k\left(1-\overline{a_k}\xi\right)}{\left|1-a_k\overline{\xi}\right|^2} - \frac{a_k-\xi}{\left|a_k-\xi\right|^2} \right) \\
&= \sum_{k=1}^{n} \left(\frac{a_k}{\left|1-a_k\overline{\xi}\right|^2} - \frac{\left|a_k\right|^2\xi}{\left|1-a_k\overline{\xi}\right|^2} - \frac{a_k}{\left|a_k-\xi\right|^2} + \frac{\xi}{\left|a_k-\xi\right|^2} \right).
\end{aligned}$$

Thus, we have

$$\xi \sum_{k=1}^{n} \left(\frac{1}{\left|a_k-\xi\right|^2} - \frac{\left|a_k\right|^2}{\left|1-a_k\overline{\xi}\right|^2} \right) = \sum_{k=1}^{n} a_k \left(\frac{1}{\left|a_k-\xi\right|^2} - \frac{1}{\left|1-a_k\overline{\xi}\right|^2} \right). \tag{30}$$

The left-hand side in (30) can be rewritten as

$$\begin{aligned}
\xi \sum_{k=1}^{n} \left(\frac{1}{\left|a_k-\xi\right|} - \frac{\left|a_k\right|^2}{\left|1-a_k\overline{\xi}\right|^2} \right) &= \xi \sum_{k=1}^{n} \frac{\left|1-a_k\overline{\xi}\right|^2 - \left|a_k\right|^2\left|a_k-\xi\right|^2}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{\left(1-a_k\overline{\xi}\right)\left(1-\overline{a_k}\xi\right) - \left|a_k\right|^2\left(a_k-\xi\right)\left(\overline{a_k}-\overline{\xi}\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{1-\left|a_k\right|^4 + \overline{a_k}\xi\left(\left|a_k\right|^2-1\right) + a_k\overline{\xi}\left(\left|a_k\right|^2-1\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{\left(\left|a_k\right|^2-1\right)\left(\overline{a_k}\xi + a_k\overline{\xi} - 1 - \left|a_k\right|^2\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{\left(1-\left|a_k\right|^2\right)\left(1+\left|a_k\right|^2 - \overline{a_k}\xi - a_k\overline{\xi}\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{\left(1-\left|a_k\right|^2\right)\left(\left|1-\overline{a_k}\xi\right|^2 + \left|a_k\right|^2 - \left|a_k\right|^2\left|\xi\right|^2\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2} \\
&= \xi \sum_{k=1}^{n} \frac{\left(1-\left|a_k\right|^2\right)\left(\left|1-\overline{a_k}\xi\right|^2 + \left|a_k\right|^2\left(1-\left|\xi\right|^2\right)\right)}{\left|a_k-\xi\right|^2\left|1-a_k\overline{\xi}\right|^2}.
\end{aligned}$$

The right-hand side in (30) can be rewritten as

$$\begin{aligned}\sum_{k=1}^{n} a_k\left(\frac{1}{|a_k-\xi|^2}-\frac{1}{\left|1-a_k\overline{\xi}\right|^2}\right) &= \sum_{k=1}^{n} a_k\frac{\left|1-a_k\overline{\xi}\right|^2-|a_k-\xi|^2}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}\\ &= \sum_{k=1}^{n} a_k\frac{(1-\overline{a_k}\xi)\left(1-a_k\overline{\xi}\right)-(a_k-\xi)\left(\overline{a_k}-\overline{\xi}\right)}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}\\ &= \sum_{k=1}^{n} a_k\frac{1+|a_k|^2|\xi|^2-|a_k|^2-|\xi|^2}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}\\ &= \sum_{k=1}^{n} a_k\frac{\left(1-|a_k|^2\right)\left(1-|\xi|^2\right)}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}.\end{aligned}$$

Hence, we have

$$\xi\sum_{k=1}^{n}\frac{\left(1-|a_k|^2\right)\left(|1-\overline{a_k}\xi|^2+|a_k|^2\left(1-|\xi|^2\right)\right)}{|a_k-\xi|^2\left|1-a_k\overline{\xi}\right|^2}=\sum_{k=1}^{n} a_k\frac{\left(1-|a_k|^2\right)\left(1-|\xi|^2\right)}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}.$$

Since $\xi\in\{z:\ B'(z)=0\}\cap\overline{\mathbb{D}}\setminus\{a_1,...,a_n\}$, we have that $|\xi|\leq 1$. Suppose that $|\xi|=1$. Then the right-hand side is 0, and we would get

$$\sum_{k=1}^{n}\frac{\left(1-|a_k|^2\right)|1-\overline{a_k}\xi|^2}{|a_k-\xi|^2\left|1-a_k\overline{\xi}\right|^2}=0.$$

But each term on the left-hand side is positive, so we must have that $|\xi|<1$ if $\xi\in\{z:\ B'(z)=0\}\cap\overline{\mathbb{D}}\setminus\{a_1,...,a_n\}$. Now let

$$\rho_k=\frac{\frac{\left(1-|a_k|^2\right)\left(1-|\xi|^2\right)}{\left|1-a_k\overline{\xi}\right|^2|a_k-\xi|^2}}{\sum_{j=1}^{n}\frac{\left(1-|a_j|^2\right)\left(|1-\overline{a_j}\xi|^2+|a_j|^2\left(1-|\xi|^2\right)\right)}{|a_j-\xi|^2\left|1-a_j\overline{\xi}\right|^2}}.$$

Then we have that $\xi=\sum_{k=1}^{n}a_k\rho_k$. Notice that

$$\begin{aligned}|1-\overline{a_j}\xi|^2+|a_j|^2\left(1-|\xi|^2\right) &= (1-\overline{a_j}\xi)\left(1-a_j\overline{\xi}\right)+|a_j|^2-|a_j|^2|\xi|^2\\ &= 1-a_j\overline{\xi}-\overline{a_j}\xi+|a_j|^2\\ &= \left(|a_j|^2-a_j\overline{\xi}-\overline{a_j}\xi+|\xi|^2\right)+1-|\xi|^2\\ &= |a_j-\xi|^2+1-|\xi|^2.\end{aligned}$$

Since we have

$$|1-\overline{a_j}\xi|^2+|a_j|^2\left(1-|\xi|^2\right)=|a_j-\xi|^2+1-|\xi|^2,$$

we must have

$$1 - |\xi|^2 \leq |1 - \overline{a_j}\xi|^2 + |a_j|^2 \left(1 - |\xi|^2\right), \tag{31}$$

since

$$|1 - \overline{a_j}\xi|^2 + |a_j|^2 \left(1 - |\xi|^2\right) - |a_j - \xi|^2 \not> |1 - \overline{a_j}\xi|^2 + |a_j|^2 \left(1 - |\xi|^2\right).$$

Thus, given the inequality in (31), we get

$$\sum_{k=1}^{n} \rho_k = \frac{\sum_{k=1}^{n} \left(\frac{\left(1 - |a_k|^2\right)\left(1 - |\xi|^2\right)}{|a_k - \xi|^2 \left|1 - a_k\overline{\xi}\right|^2} \right)}{\sum_{j=1}^{n} \left(\frac{\left(1 - |a_j|^2\right)\left(|1 - \overline{a_j}\xi|^2 + |a_j|^2 \left(1 - |\xi|^2\right)\right)}{|a_j - \xi|^2 \left|1 - a_j\overline{\xi}\right|^2} \right)}$$

$$= \frac{\sum_{k=1}^{n} \left(\left(1 - |a_k|^2\right)\left(1 - |\xi|^2\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(|a_j - \xi|^2 \left|1 - a_j\overline{\xi}\right|^2\right) \right)}{\sum_{k=1}^{n} \left(\left(1 - |a_k|^2\right)\left(|1 - \overline{a_k}\xi|^2 + |a_k|^2 \left(1 - |\xi|^2\right)\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(|a_j - \xi|^2 \left|1 - a_j\overline{\xi}\right|^2\right) \right)}$$

$$\leq 1.$$

Hence, we can conclude that the zeros of $B'(z)$ inside $\mathbb{D}$ lie inside the convex hull of $\{0\} \cup \{a_1, ..., a_n\}$. □

For finite Blaschke products, there is a refinement to Theorem 6.8 which involves hyperbolic geometry. This thesis will not go into depth about hyperbolic geometry, but here is a brief introduction. Hyperbolic geometry is a non-Euclidean geometry, that is, the contrary to the parallel postulate in Euclid's Elements is assumed. The parallel postulate states that given a line l and a point p that is not on the line, there is one and only one line that is parallel to l that goes through the point p. In hyperbolic geometry, that statement is replaced with "given a line l and a point p not on the line, there is more than one line that is parallel to l and goes through the point p". A hyperbolic line between z_1 and z_2 is given by

$$t \to \frac{z_1 - \frac{z_1 - z_2}{1 - \overline{z_1}z_2}t}{1 - \overline{z_1}\frac{z_1 - z_2}{1 - \overline{z_1}z_2}t},$$

and can be written as the parametrization

$$\gamma \frac{\alpha - z}{1 - \overline{\alpha}z} = t,$$

where $t \in [-1, 1]$, $\gamma \in \mathbb{T}$ and $\alpha \in \mathbb{D}$ (see for instance [4]). The figure below shows an example of a hyperbolic line segment between two points:

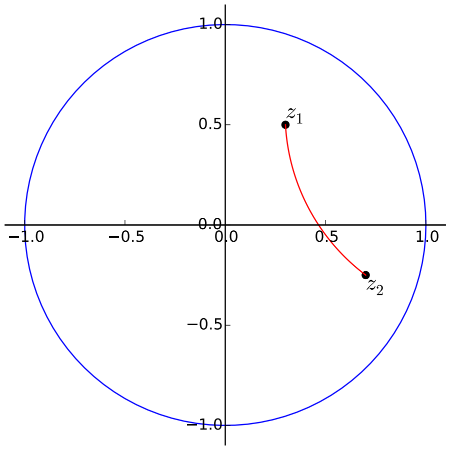

Figure 3: A hyperbolic line segment between the points z_1 and z_2.

Now we can define what a hyperbolically convex set is.

Definition 6.9. *A set $A \subset \mathbb{D}$ is hyperbolically convex if*

$$z_1, z_2 \in A \text{ and } t \in [0, 1] \implies \frac{z_1 - \frac{z_1 - z_2}{1 - \overline{z_1} z_2} t}{1 - \overline{z_1} \frac{z_1 - z_2}{1 - \overline{z_1} z_2} t} \in A.$$

We also have the following definition of a hyperbolic convex hull:

Definition 6.10 (Garcia, Mashreghi, Ross [5])**.** *The hyperbolic convex hull of a finite set of points $\{z_1, ..., z_n\} \subset \mathbb{D}$ is the smallest hyperbolic convex set that contains $\{z_1, ..., z_n\}$.*

The definitions can be difficult to visualize at first glance. Let S be the same set as in Figure 4 in Section 6, i.e. $S = \{z_1, z_2, z_3, z_4, z_5\}$ where $z_1 = (0.5, 0.5)$, $z_2 = (-0.3, 0.1)$, $z_3 = (-0.25, -0.5)$, $z_4 = (0.1, -0.7)$ and $z_5 = (0.5, -0.45)$. If we plot the segment

$$\frac{z_i - \frac{z_i - z_j}{1 - \overline{z_i} z_j} t}{1 - \overline{z_i} \frac{z_i - z_j}{1 - \overline{z_i} z_j} t}$$

with $0 \le t \le 1$ and $(i, j) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 1)$, we get the following figure:

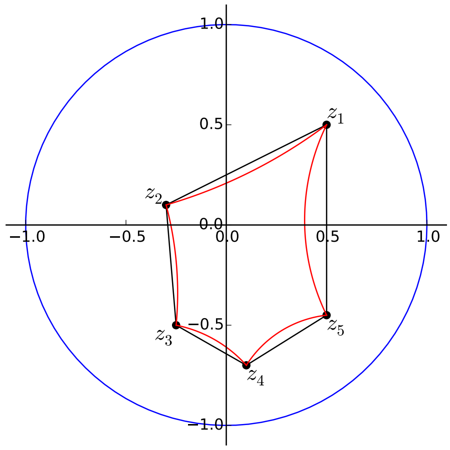

Figure 4: The black is the outline of the convex hull of S. The red is the outline of the hyperbolic convex hull of S.

Theorem 6.11 (Garcia, Mashreghi, Ross [5])**.** *Let $B(z)$ be a finite Blaschke product. Then the zeros of $B'(z)$ inside $\mathbb{D}$ belong to the hyperbolic convex hull of the zeros of $B(z)$.*

Proof. Let $B(z)$ be a finite Blaschke product with zeros in $a_1, ..., a_n \in \mathbb{D}$. First, we need to introduce some notation that will be used throughout this proof. First, we have the lower half of the unit disc,

$$\mathbb{D}_- = \mathbb{D} \cap \{z : \ \mathrm{Im}(z) < 0\},$$

and the upper half of the unit disc,

$$\mathbb{D}_+ = \mathbb{D} \cap \{z : \ \mathrm{Im}(z) > 0\}.$$

Suppose that all zeros of $B(z)$ are in $\mathbb{D}_+ \cup (-1, 1)$. With (13), we get that

$$\mathrm{Im}\left(\frac{B'(z)}{B(z)}\right) = \sum_{k=1}^{n} \mathrm{Im}\left(\frac{1 - |a_k|^2}{(1 - \overline{a_k}z)(z - a_k)}\right). \tag{32}$$

Let $\alpha \in \mathbb{D}_+$ be fixed, and consider the function

$$\varphi(z) = \frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)(z - \alpha)}.$$

Now we will study the image of $\mathbb{D}_-$ under φ. Since φ is analytic on $\overline{\mathbb{D}_-}$, we only need to study the simple closed curve that constitutes the boundary of $\mathbb{D}_-$, that is $\mathbb{T}_- \cup [-1,1]$, where

$$\begin{aligned}\mathbb{T}_- &= \{e^{i\theta}: \ -\pi \leq \theta \leq 0\} \\ &= \mathbb{T} \cap \{z: \ \operatorname{Im}(z) \leq 0\}.\end{aligned}$$

For $\mathbb{T}_-$, we have

$$\begin{aligned}\varphi(e^{i\theta}) &= \frac{1-|\alpha|^2}{(1-\overline{\alpha}e^{i\theta})(e^{i\theta}-\alpha)} \\ &= \frac{1-|\alpha|^2}{e^{i\theta}(e^{-i\theta}-\overline{\alpha})(e^{i\theta}-\alpha)} \\ &= \frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}e^{-i\theta}.\end{aligned}$$

Hence, the image of $\mathbb{T}_-$ under φ is in $\mathbb{C}_+ \cup \mathbb{R} = \mathbb{C} \setminus \mathbb{C}_-$. For $t \in [-1,1]$ we have

$$\begin{aligned}\varphi(t) &= \frac{1-|\alpha|^2}{(1-\overline{\alpha}t)(t-\alpha)} \\ &= \frac{\left(1-|\alpha|^2\right)(1-\alpha t)(t-\overline{\alpha})}{(1-\overline{\alpha}t)(1-\alpha t)(t-\alpha)(t-\overline{\alpha})} \\ &= \frac{1-|\alpha|^2}{|(1-\overline{\alpha}t)(t-\alpha)|^2}(1-\alpha t)(t-\overline{\alpha}).\end{aligned}$$

For the calculation of the imaginary part of φ in $[-1,1]$, recall that

$$\operatorname{Im}(z) = \frac{z-\overline{z}}{2i}$$

Thus, we get

$$\begin{aligned}\operatorname{Im}(\varphi(t)) &= \frac{\varphi(t)-\overline{\varphi(t)}}{2i} \\ &= \frac{\frac{1-|\alpha|^2}{|(1-\overline{\alpha}t)(t-\alpha)|^2}\left((1-\alpha t)(t-\overline{\alpha})-(1-\overline{\alpha}t)(t-\alpha)\right)}{2i} \\ &= \frac{\frac{1-|\alpha|^2}{|(1-\overline{\alpha}t)(t-\alpha)|^2}\left(\alpha-\overline{\alpha}-t^2(\alpha-\overline{\alpha})\right)}{2i} \\ &= \frac{1-|\alpha|^2}{|(1-\overline{\alpha}t)(t-\alpha)|^2}\left(1-t^2\right)\frac{\alpha-\overline{\alpha}}{2i} \\ &= \frac{1-|\alpha|^2}{|(1-\overline{\alpha}t)(t-\alpha)|^2}\left(1-t^2\right)\operatorname{Im}(\alpha).\end{aligned}$$

Hence, the image of $[-1,1]$ under φ is also in $\mathbb{C}_+ \cup \mathbb{R}$. Thus, φ maps $\mathbb{D}_-$ to a simple closed curve in $\mathbb{C}_+$. Since φ is analytic on $\overline{\mathbb{D}_-}$, we can conclude that φ maps $\mathbb{D}_-$ into $\mathbb{C}_+$. Since we have assumed that all zeros of $B(z)$ are in $\mathbb{D}_+$, (32) and the conclusions we have done about the function φ gives us

$$z \in \mathbb{D}_- \implies \operatorname{Im}\left(\frac{B'(z)}{B(z)}\right) > 0.$$

Thus, $B'(z)$ do not have any zeros in $\mathbb{D}_-$. By continuity, since all of the zeros of $B(z)$ are assumed to be in $\mathbb{D}_+ \cup (-1,1)$, all zeros of $B'(z)$ that are inside the unit disc must also be in $\mathbb{D}_+ \cup (-1,1)$. Now, consider the automorphism

$$\tau_\alpha = \frac{\alpha - z}{1 - \overline{\alpha} z} \in \operatorname{Aut}(\mathbb{D}).$$

According to Lemma 3.11 in [5], for a finite Blaschke product $B(z)$ of degree n, both $\tau_\alpha \circ B$ and $B \circ \tau_\alpha$ are finite Blaschke products of degree n. Let $f = B \circ \tau_\alpha$. Then f is a finite Blaschke product. We have that

$$\begin{aligned}
\tau_\alpha^2(z) &= \frac{\alpha - \left(\frac{\alpha - z}{1-\overline{\alpha}z}\right)}{1 - \overline{\alpha}\left(\frac{\alpha - z}{1-\overline{\alpha}z}\right)} \\
&= \frac{\dfrac{\alpha\,(1-\overline{\alpha}z) - (\alpha - z)}{1-\overline{\alpha}z}}{\dfrac{(1-\overline{\alpha}z) - \overline{\alpha}\,(\alpha - z)}{1-\overline{\alpha}z}} \\
&= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \overline{\alpha}z - |\alpha|^2 + \overline{\alpha}z} \\
&= \frac{z\left(1 - |\alpha|^2\right)}{1 - |\alpha|^2} \\
&= \mathrm{id},
\end{aligned}$$

where id is the identity mapping $\mathrm{id}(z) = z$. Thus, the function f has zeros in $\tau_\alpha(a_1), ..., \tau_\alpha(a_n) \in \mathbb{D}$. Let $d_1, ..., d_{n-1}$ denote the zeros of $B'(z)$ inside $\mathbb{D}$. Then the zeros of f' are in $\tau_\alpha(d_1), ..., \tau_\alpha(d_{n-1})$ (by the same argument as above). Since we have assumed that $a_1, ..., a_n$ lie in $\mathbb{D}_+ \cup (-1,1)$, we can choose $\alpha \in \mathbb{D}$ such that

$$\operatorname{Im}(\tau_\alpha(a_1)), ..., \operatorname{Im}(\tau_\alpha(a_n)) \leq 0,$$

then the conclusions drawn from the function φ previously in the proof gives us that

$$\operatorname{Im}(\tau_\alpha(d_1)), ..., \operatorname{Im}(\tau_\alpha(d_{n-1})) \leq 0.$$

From this, we can conclude that if the zeros of $B(z)$ are on one side of the hyperbolic line

$$\frac{\alpha - z}{1 - \overline{\alpha}z} = t, t \in [-1, 1],$$

then the zeros of $B'(z)$ that are inside $\mathbb{D}$ are on the same side of said line. The same holds if we replace τ_α by a rotation, and since the intersection of all such hyperbolic lines forms the hyperbolic convex hull, we can conclude that all zeros of $B'(z)$ that lie in $\mathbb{D}$ lie in the hyperbolic convex hull of $a_1, ..., a_n$. □

7 Blaschke products and ellipses

This section is dedicated to the connection between finite Blaschke products and ellipses, which is obtained by exploring the geometrical properties of the solutions to the equation $B(z) = \omega$ for different $\omega \in \mathbb{T}$. First, we need to define what a *Möbius transformation* is, which will be used in both Section 7.1 and 7.2.

Definition 7.1 (Saff, Snider [10]). *A Möbius transformation is a function of the form*

$$f(z) = \frac{az + b}{cz + d}$$

with $ad \neq bc$.

A Möbius transformation that maps $\mathbb{D}$ to $\mathbb{D}$ has the form

$$\beta \frac{z - \alpha}{1 - \overline{\alpha}z}, \tag{33}$$

where $|\beta| = 1$ and $\alpha \in \mathbb{D}$. From [11], we know that there exists a Möbius transformation $z = M(\omega)$, which is of the form described in (33), such that $B(z) = B(M(\omega)) = C(z)$, where $C(z)$ is another Blaschke product. Then the points $\omega_1, ..., \omega_n$ for which $a_k = M(\omega_k)$ are the zeros of $C(z)$. Therefore, by composing a Blaschke product with an appropriate Möbius transformation, we can assume that the Blaschke product has a zero at the origin. Now we have enough information to study the results obtained by Blaschke products of various degrees.

7.1 Blaschke products of degree two

In this section, we only consider Blaschke products of degree two, i.e.

$$B_2(z) = \frac{|a_1|}{a_1} \frac{a_1 - z}{1 - \overline{a_1}z} \frac{|a_2|}{a_2} \frac{a_2 - z}{1 - \overline{a_2}z}.$$

By composing $B_2(z)$ with a Möbius transformation, we can assume our Blaschke product of degree two has the form

$$B(z) = z \frac{a - z}{1 - \overline{a}z},$$

where $a \neq 0$. The main goal in this section is to examine what will happen if we draw a line connecting the two solutions of the equation

$$B(z) = \omega$$

for different values of $\omega \in \mathbb{T}$. First, we start off by examining an example. Let $B(z) = z\frac{(0.3-0.7i)-z}{1-(0.3+0.7i)z}$. If we solve the equation $B(z) = 1$ and draw a line between the two solutions ξ_1 and ξ_2, we achieve the following figure:

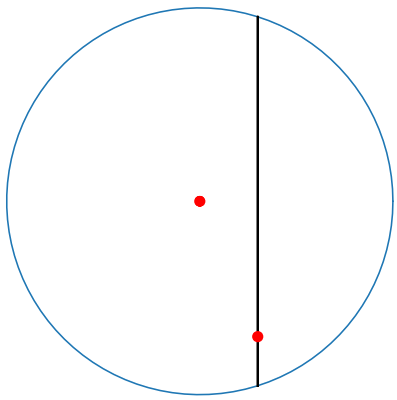

Figure 5: Lines connecting the two solutions to the equation $B(z) = z\frac{(0.3-0.7i)-z}{1-(0.3+0.7i)z} = 1$. The red dots are the zeros of $B(z)$.

Now let

$$\Gamma = \left\{\pm 1, \pm i, \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{4} \pm \frac{\sqrt{14}}{4}i, -\frac{\sqrt{2}}{4} \pm \frac{\sqrt{14}}{4}i, \frac{\sqrt{14}}{4} \pm \frac{\sqrt{2}}{4}i, -\frac{\sqrt{14}}{4} \pm \frac{\sqrt{2}}{4}i\right\}.$$

Plotting a figure in the same way as in Figure 5, but for all $\omega \in \Gamma$, we get the following figure:

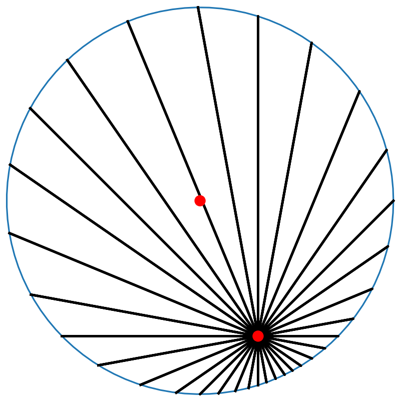

Figure 6: Lines connecting the two solutions to the equation $B(z) = z\frac{(0.3-0.7i)-z}{1-(0.3+0.7i)z} = \omega$, for $\omega \in \Gamma$. The red dots are the zeros of the Blaschke product $B(z)$.

By removing the zero at the origin, we can see that all lines pass through the non-origin zero of $B(z)$, which we can see in the following two figures:

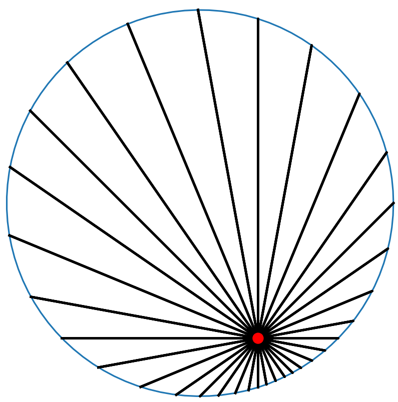

Figure 7: Lines connecting the two solutions to the equation $B(z) = z\frac{(0.3-0.7i)-z}{1-(0.3+0.7i)z} = \omega$, for $\omega \in \Gamma$. The red dot is the non-origin zero of $B(z)$.

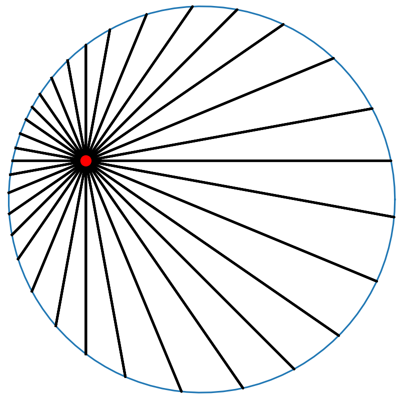

Figure 8: Lines connecting the two solutions to the equation $B(z) = \omega$, where $\omega \in \Gamma$ and $B(z) = z\frac{(-0.6+0.2i)-z}{1-(-0.6-0.2i)z}$. The red dot is the non-origin zero of $B(z)$.

From the figures above, it seems like the line connecting the two solutions of $B(z) = z\frac{a-z}{1-\overline{a}z} = \omega$ with $\omega \in \mathbb{T}$ and $a \neq 0$ passes through the point a. This is actually the case for all such Blaschke products, as shown in the theorem below.

Theorem 7.2 (Daepp, Gorkin, Mortini [2])**.** *Let $B(z) = z\frac{a-z}{1-\overline{a}z}$ with $a \neq 0$. For $\omega \in \mathbb{T}$, let ξ_1 and ξ_2 be the two distinct solutions to the equation $B(z) = \omega$. Then the line joining ξ_1 and ξ_2 passes through a. Conversely, for any line L that passes through a, the two points ξ_1 and ξ_2 where the line L intersects $\mathbb{T}$ satisfy $B(\xi_1) = B(\xi_2)$.*

Proof. Let $\omega = e^{i\theta}$ and consider the equation

$$z\frac{a-z}{1-\overline{a}z} = e^{i\theta}.$$

From Section 3.1 we know that $|z| = 1$, so we have that $z = 1/\overline{z}$. Hence,

$$\begin{aligned} e^{i\theta} &= z\frac{a-z}{1-\overline{a}z} \\ &= \frac{1}{\overline{z}}\frac{a-z}{1-\overline{a}z} \\ &= \frac{a-z}{\overline{z}-\overline{a}\,|z|^2} \\ &= \frac{a-z}{\overline{z}-\overline{a}}. \end{aligned} \tag{34}$$

Now let $\xi_j = a + r_j e^{i\theta_j}$ ($j \in \{1,2\}$), where r_j, θ_j are positive real numbers. Substituting ξ_j ($j \in \{1,2\}$) into the equation $\frac{a-z}{\bar{z}-\bar{a}} = e^{i\theta}$, we get the equation

$$\begin{aligned} e^{i\theta} &= \frac{a - \left(a + r_j e^{i\theta_j}\right)}{\left(\bar{a} + r_j e^{-i\theta_j}\right) - \bar{a}} \\ &= \frac{-r_j e^{i\theta_j}}{r_j e^{-i\theta_j}} \\ &= -e^{2i\theta_j}. \end{aligned}$$

This yields the two solutions $\xi_1 = a + ie^{i\frac{\theta}{2}}$ and $\xi_2 = a - ie^{i\frac{\theta}{2}}$. Recall that the three points a, ξ_1 and ξ_2 will be on the same line if and only if

$$a - \xi_2 = c(\xi_2 - \xi_1) \tag{35}$$

for some $c \in \mathbb{R}$ (see for instance [10]). Substituting with $\xi_1 = a + ie^{i\frac{\theta}{2}}$ and $\xi_2 = a - ie^{i\frac{\theta}{2}}$ in (35), we get

$$\begin{aligned} a - \left(a - r_2 e^{i\frac{\theta}{2}}\right) &= c\left(\left(a - r_2 e^{i\frac{\theta}{2}}\right) - \left(a + r_1 e^{i\frac{\theta}{2}}\right)\right), \\ r_2 e^{i\frac{\theta}{2}} &= -c\left(r_2 + r_1\right) e^{i\frac{\theta}{2}}. \end{aligned}$$

So, with $c = -\frac{r_2}{r_1+r_2}$, we have $a - \xi_2 = c\left(\xi_2 - \xi_1\right)$. Since $r_1, r_2 \in \mathbb{R}$, we have that $c \in \mathbb{R}$. Hence, a, ξ_1 and ξ_2 lie on the same line. Thus, the first part of the theorem is proven.

For the second part of the proof, suppose that ξ_1 and ξ_2 are points on the unit circle such that

$$a - \xi_2 = c\left(\xi_2 - \xi_1\right) \tag{36}$$

for some $c \in \mathbb{R}$. Recall that we know that $B(z) = \frac{a-z}{\bar{z}-\bar{a}}$ on the unit circle from (34). From (36) we can derive that $\overline{\xi_2} - \bar{a} = -c\left(\overline{\xi_2} - \overline{\xi_1}\right) = c\left(\overline{\xi_1} - \overline{\xi_2}\right)$. Hence,

$$\begin{aligned} B(\xi_2) &= \frac{a - \xi_2}{\overline{\xi_2} - \bar{a}} \\ &= \frac{c\left(\xi_2 - \xi_1\right)}{c\left(\overline{\xi_1} - \overline{\xi_2}\right)} \\ &= \frac{\xi_2 - \xi_1}{\overline{\xi_1} - \overline{\xi_2}}. \end{aligned} \tag{37}$$

From (36), we can derive that

$$a - \xi_1 = (c+1)\left(\xi_2 - \xi_1\right)$$

and

$$\begin{aligned} \overline{\xi_1} - \bar{a} &= -\left(c+1\right)\left(\overline{\xi_2} - \overline{\xi_1}\right) \\ &= \left(c+1\right)\left(\overline{\xi_1} - \overline{\xi_2}\right). \end{aligned}$$

Thus,

$$\begin{aligned} B(\xi_1) &= \frac{a-\xi_1}{\overline{\xi_1}-\overline{a}} \\ &= \frac{(c+1)(\xi_2-\xi_1)}{(c+1)\left(\overline{\xi_1}-\overline{\xi_2}\right)} \\ &= \frac{\xi_2-\xi_1}{\overline{\xi_1}-\overline{\xi_2}}. \end{aligned} \tag{38}$$

From (37) and (38), we can see that $B(\xi_1) = B(\xi_2)$. □

In the next section, we will look into the somewhat more interesting geometrical result which appears with Blaschke products of degree 3.

7.2 Blaschke products of degree three

In this section, we only consider Blaschke products of degree three, i.e.

$$B_3(z) = \frac{|a_1|}{a_1}\frac{a_1-z}{1-\overline{a_1}z}\frac{|a_2|}{a_2}\frac{a_2-z}{1-\overline{a_2}z}\frac{|a_3|}{a_3}\frac{a_3-z}{1-\overline{a_3}z}.$$

By composing $B_3(z)$ with a Möbius transformation, we can assume our Blaschke product of degree three has the form

$$B(z) = z\frac{a-z}{1-\overline{a}z}\frac{b-z}{1-\overline{b}z}.$$

The main goal of this section is to examine what will happen if we draw lines connecting the three solutions of the equation

$$B(z) = \omega$$

for many different values of $\omega \in \mathbb{T}$. First, we begin with an example. Let $B(z) = z\frac{0.5-z}{1-0.5z}\frac{(-0.4-0.5i)-z}{1-(-0.4+0.5i)z}$. If we solve the equation $B(z) = 1$ and draw lines that connects the solutions, we get the following figure:

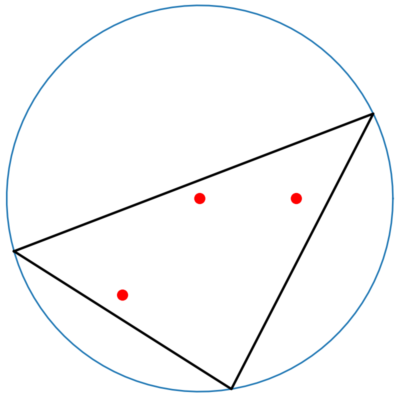

Figure 9: Lines connecting the three solutions to the equation $B(z) = z\frac{0.5-z}{1-0.5z}\frac{(-0.4-0.5i)-z}{1-(-0.4+0.5i)z} = 1$. The red dots are the zeros of $B(z)$.

Now let

$$\Gamma = \left\{\pm 1, \pm i, \tfrac{\sqrt{2}}{2} \pm \tfrac{\sqrt{2}}{2}i, -\tfrac{\sqrt{2}}{2} \pm \tfrac{\sqrt{2}}{2}i, \tfrac{\sqrt{2}}{4} \pm \tfrac{\sqrt{14}}{4}i, -\tfrac{\sqrt{2}}{4} \pm \tfrac{\sqrt{14}}{4}i, \tfrac{\sqrt{14}}{4} \pm \tfrac{\sqrt{2}}{4}i, -\tfrac{\sqrt{14}}{4} \pm \tfrac{\sqrt{2}}{4}i\right\}.$$

Plotting a figure in the same way as in Figure 9, but for all $\omega \in \Gamma$, we get the following figure:

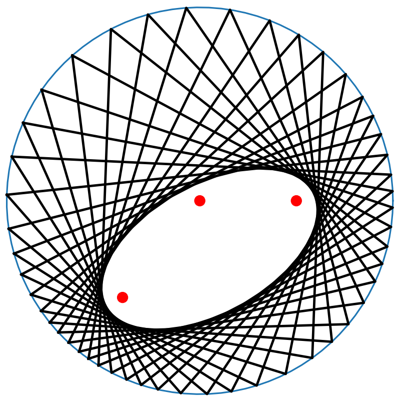

Figure 10: Lines connecting the three solutions to the equation $B(z) = z\frac{0.5-z}{1-0.5z}\frac{(-0.4-0.5i)-z}{1-(-0.4+0.5i)z} = \omega$, where $\omega \in \Gamma$. The red dots are the zeros of $B(z)$.

The lines connecting the solutions to the equation $B(z) = \omega$ seem to be tangents of an ellipse, and if we remove the zero at the origin, it seems like the non-origin zeros of $B(z)$ are the foci of the ellipse:

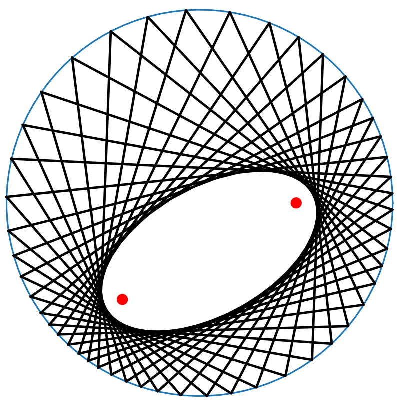

Figure 11: Same as Figure 10, but with the zero at the origin removed.

This seems to be a consistent finding. Two more examples follow in the figures below.

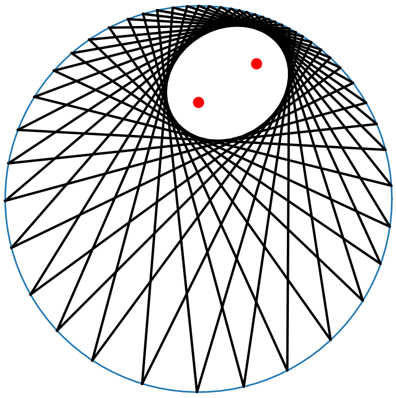

Figure 12: Lines drawn between the solutions to the equation $B(z) = \omega$ for $\omega \in \Gamma$ with $B(z) = z\frac{(0.3+0.7i)-z}{1-(0.3-0.7i)z}\frac{(0.5i)-z}{1-(-0.5i)z}$.

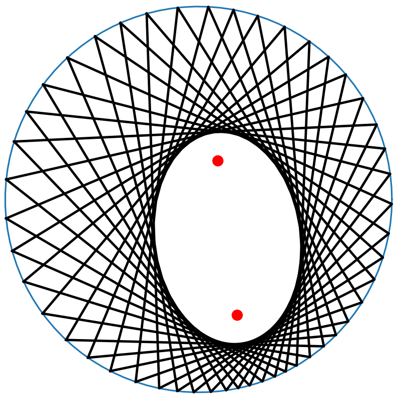

Figure 13: Lines drawn between the solutions to the equation $B(z) = \omega$ for $\omega \in \Gamma$ with $B(z) = z\frac{(0.2-0.6i)-z}{1-(0.2+0.6i)z}\frac{(0.1+0.2i)-z}{1-(0.1-0.2i)z}$.

Before stating the relevant theorems, recall the definition of an ellipse.

Definition 7.3. *Let a and b be complex numbers and c be a positive, real number. Then*

$$|z-a|+|z-b|=c$$

is the equation of an ellipse in the complex plane, and the points a and b are its foci.

Before we continue, we need to mention the notation of angles that will be used for the upcoming theorem. An angle $\angle(p_1, p_2, p_3)$ will be defined using three points, with the vertex in the middle. One thing that is worth mentioning is that $\angle(a, b, c) = -\angle(c, b, a)$, since one of the angles is viewed clockwise and the other one is counter clockwise. If we consider a point p on the ellipse E and draw a line from each focus to the point p. Then the angles that arise when said lines intersect the tangent of E at the point p are congruent. That is, for any point p on an ellipse E and two arbitrary points ξ_1 and ξ_2 on the tangent of E at the point p, the angles $\angle(a, p, \xi_1)$ and $\angle(b, p, \xi_2)$ are congruent. Since one of the angles has to be clockwise and the other counter-clockwise (with respect to this notation of angles), we have that $\angle(a, p, \xi_1) = -\angle(b, p, \xi_2)$ for all such points.

Now we can formally state and prove the main result of this section.

Theorem 7.4 (Daepp, Gorkin, Mortini [2])**.** *Let $B(z) = z\left(\frac{a_1-z}{1-\overline{a_1}z}\right)\left(\frac{a_2-z}{1-\overline{a_2}z}\right)$ with $a_1 \neq a_2$ and $a_2 \neq 0$. For any $\omega \in \mathbb{T}$, let ξ_1, ξ_2 and ξ_3 be the three distinct solutions to the equation $B(z) = \omega$. Let*

$$F(z) = \frac{B(z)/z}{B(z)-\omega} = \frac{\lambda_1}{z-\xi_1} + \frac{\lambda_2}{z-\xi_2} + \frac{\lambda_3}{z-\xi_3}.$$

Then the line L joining ξ_1 and ξ_2 is the tangent to the ellipse E given by

$$|z-a_1|+|z-a_2| = |1-\overline{a_1}a_2|$$

at the point $p = \frac{\lambda_1\xi_2+\lambda_2\xi_1}{\lambda_1+\lambda_2}$. Conversely, for each point p on the ellipse E, the tangent of E at p intersects $\mathbb{T}$ at two distinct points ξ_1 and ξ_2 with $B(\xi_1) = B(\xi_2)$.

Proof. First, notice that $F(a_j) = 0$ for $j = 1, 2$, since $B(a_j) = 0$. Thus, we have

$$\begin{aligned} 0 &= F(a_j) \\ &= \frac{\lambda_1}{a_j - \xi_1} + \frac{\lambda_2}{a_j - \xi_2} + \frac{\lambda_3}{a_j - \xi_3} \\ &= \frac{\lambda_3}{a_j - \xi_3} + \frac{\lambda_1 (a_j - \xi_2) + \lambda (a_j - \xi_1)}{(a_j - \xi_1)(a_j - \xi_2)} \\ &= \frac{\lambda_3}{a_j - \xi_3} + \frac{a_j (\lambda_1 + \lambda_2) - \frac{(\lambda_1 \xi_2 + \lambda_2 \xi_1)}{(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)}{(a_j - \xi_1)(a_j - \xi_2)} \\ &= \frac{\lambda_3}{a_j - \xi_3} + \frac{(a_j - p)(\lambda_1 + \lambda_2)}{(a_j - \xi_1)(a_j - \xi_2)}. \end{aligned}$$

Therefore, we have

$$\frac{-\lambda_3}{a_j - \xi_3} = (\lambda_1 + \lambda_2) \frac{(a_j - p)}{(a_j - \xi_1)(a_j - \xi_2)}$$

From Lemma 6.3, we know that $0 < \lambda_1, \lambda_2, \lambda_3 < 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$, thus

$$\lambda_3 \left| \frac{1}{a_j - \xi_3} \right| = (1 - \lambda_3) \left| \frac{(a_j - p)}{(a_j - \xi_1)(a_j - \xi_2)} \right|,$$

which we can rewrite as

$$|p - a_j| = \frac{\lambda_3}{1 - \lambda_3} \left| \frac{(a_j - \xi_1)(a_j - \xi_2)}{(a_j - \xi_3)} \right|. \tag{39}$$

If we use (39) in the expression $\frac{1}{|1 - \overline{a_1} a_2|} |p - a_1| + \frac{1}{|1 - \overline{a_1} a_2|} |p - a_2|$ we get

$$\frac{|p - a_1|}{|1 - \overline{a_1} a_2|} + \frac{|p - a_2|}{|1 - \overline{a_1} a_2|} = \frac{\lambda_3}{1 - \lambda_3} \left| \frac{(a_1 - \xi_1)(a_1 - \xi_2)}{(1 - \overline{a_1} a_2)(a_1 - \xi_3)} \right| + \frac{\lambda_3}{1 - \lambda_3} \left| \frac{(a_2 - \xi_1)(a_2 - \xi_2)}{(1 - \overline{a_1} a_2)(a_2 - \xi_3)} \right| \tag{40}$$

$$= \frac{\lambda_3}{1 - \lambda_3} \left(\left| \frac{(a_1 - \xi_1)(a_1 - \xi_2)}{(1 - \overline{a_1} a_2)(a_1 - \xi_3)} \right| + \left| \frac{(a_2 - \xi_1)(a_2 - \xi_2)}{(1 - \overline{a_1} a_2)(a_2 - \xi_3)} \right| \right). \tag{41}$$

Since $B(z) = z \frac{a_1 - z}{1 - \overline{a_1} z} \frac{a_2 - z}{1 - \overline{a_2} z}$ and ξ_1,ξ_2 and ξ_3 are the three distinct solutions to $B(z) = \omega$ for some $\omega \in \mathbb{T}$, we have

$$B(z) - \omega = \frac{(z - \xi_1)(z - \xi_2)(z - \xi_3)}{(1 - \overline{a_1} z)(1 - \overline{a_2} z)}.$$

Since $B(a_j) = 0$ for $i = 1, 2$, we get

$$\begin{aligned} |B(a_j) - \omega| &= 1 \\ &= \left| \frac{(a_j - \xi_1)(a_j - \xi_2)(a_j - \xi_3)}{(1 - \overline{a_1} a_j)(1 - \overline{a_2} a_j)} \right|. \end{aligned}$$

Using the above expression in respective term in (41), together with the fact that $|1-\overline{a_1}a_2| = |1-a_1\overline{a_2}|$, we get

$$\frac{1}{|1-\overline{a_1}a_2|}|p-a_1| + \frac{1}{|1-\overline{a_1}a_2|}|p-a_2| = \frac{\lambda_3}{1-\lambda_3}\left(\left|\frac{\left(1-|a_1|^2\right)}{(a_1-\xi_3)^2}\right| + \left|\frac{\left(1-|a_2|^2\right)}{(a_2-\xi_3)^2}\right|\right) \tag{42}$$

$$= \frac{\lambda_3}{1-\lambda_3}\left(\frac{1-|a_1|^2}{|a_1-\xi_3|^2} + \frac{1-|a_2|^2}{|a_2-\xi_3|^2}\right) \tag{43}$$

From Lemma 6.3, we have that

$$\lambda_j = \frac{1}{1+\sum_{k=1}^{2}\frac{1-|a_k|^2}{|\xi_j-a_k|^2}}$$

for $j = 1, 2$, which implies

$$\begin{aligned}\frac{1}{\lambda_j} &= 1+\sum_{k=1}^{2}\frac{1-|a_k|^2}{|\xi_j-a_k|^2}\\ &= 1+\sum_{k=1}^{2}\frac{1-|a_k|^2}{|a_k-\xi_j|^2}.\end{aligned}$$

Hence, we can write (43) as

$$\begin{aligned}\frac{1}{|1-\overline{a_1}a_2|}|p-a_1| + \frac{1}{|1-\overline{a_1}a_2|}|p-a_2| &= \frac{\lambda_3}{1-\lambda_3}\left(\frac{1-|a_1|^2}{|a_1-\xi_3|^2} + \frac{1-|a_2|^2}{|a_2-\xi_3|^2}\right)\\ &= \frac{\lambda_3}{1-\lambda_3}\left(1+\sum_{k=1}^{2}\frac{1-|a_k|^2}{|a_k-\xi_3|^2}-1\right)\\ &= \frac{\lambda_3}{1-\lambda_3}\left(\frac{1}{\lambda_3}-1\right)\\ &= \frac{\lambda_3}{1-\lambda_3}\frac{1-\lambda_3}{\lambda_3}\\ &= 1.\end{aligned}$$

Thus, we can conclude that

$$|p-a_1| + |p-a_2| = |1-\overline{a_1}a_2|,$$

which means that p lies on the ellipse E. Since

$$\begin{aligned}\frac{p-\xi_1}{\xi_1-\xi_2} &= \frac{\frac{\lambda_1\xi_2+\lambda_2\xi_1}{\lambda_1+\lambda_2}-\xi_1}{\xi_1-\xi_2}\\ &= \frac{\lambda_1\xi_2+\lambda_2\xi_1-\lambda_1\xi_1-\lambda_2\xi_1}{(\xi_1-\xi_2)(\lambda_1+\lambda_2)}\\ &= \frac{\lambda_1(\xi_2-\xi_1)}{(\xi_1-\xi_2)(\lambda_1+\lambda_2)}\\ &= \frac{-\lambda_1}{\lambda_1+\lambda_2}\in\mathbb{R},\end{aligned}$$

we know that p lies on the line L that goes through ξ_1 and ξ_2. So now we have to prove that the line L is the tangent to the ellipse E at the point p. To do that, we have to prove that

$$\angle(a_1,p,\xi_1) = -\angle(a_2,p,\xi_2) \tag{44}$$

where the minus sign is because one of the angles is viewed clockwise and the other counter-clockwise, as mentioned before the theorem. The angles can be expressed as the argument of complex numbers, and that is how we will prove the equality in (44). Recall that for two arbitrary complex numbers z_1 and z_2,

$$\arg(z_1z_2) = \arg(z_1)+\arg(z_2),$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2).$$

Thus, we can express the angles in (44) can be expressed as

$$\angle(a_1,p,\xi_1) = \arg\left(\frac{a_1-p}{\xi_1-p}\right),$$

and

$$-\angle(a_2,p,\xi_2) = -\arg\left(\frac{a_2-p}{\xi_2-p}\right).$$

Before we prove (44), consider the function $\frac{\lambda_1}{z-\xi_1}+\frac{\lambda_2}{z-\xi_2}$. Then p is a zero of that function, since

$$\begin{aligned}\frac{\lambda_1}{p-\xi_1}+\frac{\lambda_2}{p-\xi_2} &= \frac{\lambda_1}{\frac{\lambda_1\xi_2+\lambda_2\xi_1}{\lambda_1+\lambda_2}-\xi_1}+\frac{\lambda_2}{\frac{\lambda_1\xi_2+\lambda_2\xi_1}{\lambda_1+\lambda_2}-\xi_2}\\ &= \frac{\lambda_1(\lambda_1+\lambda_2)}{\lambda(\xi_2-\xi_1)}+\frac{\lambda_2(\lambda_1+\lambda_2)}{\lambda_2(\xi_1-\xi_2)}\\ &= \frac{\lambda_1+\lambda_2}{\xi_2-\xi_1}-\frac{\lambda_1+\lambda_2}{\xi_2-\xi_1}\\ &= 0.\end{aligned}$$

Also, recall from (25) that

$$\frac{B(z)/z}{B(z)-\omega} = \frac{(z-a_1)(z-a_2)}{(z-\xi_1)(z-\xi_2)(z-\xi_3)}.$$

Hence, we have that

$$\begin{aligned}\frac{\lambda_3}{p-\xi_3} &= \frac{\lambda_1}{p-\xi_1} + \frac{\lambda_2}{p-\xi_2} + \frac{\lambda_3}{p-\xi_3} \\ &= F(p) \\ &= \frac{(p-a_1)(p-a_2)}{(p-\xi_1)(p-\xi_2)(p-\xi_3)}.\end{aligned}$$

Finally, we can prove (44):

$$\begin{aligned}\angle(a_1,p,\xi_1) + \angle(a_2,p,\xi_2) &= \arg\left(\frac{a_1-p}{\xi_1-p}\right) + \arg\left(\frac{a_2-p}{\xi_2-p}\right) \\ &= \arg\left(\frac{(a_1-p)(a_2-p)}{(\xi_1-p)(\xi_2-p)}\right) \\ &= \arg\left(\frac{(p-a_1)(p-a_2)}{(p-\xi_1)(p-\xi_2)}\right) \\ &= \arg\left((p-\xi_3)\,F(p)\right) \\ &= \arg\left((p-\xi_3)\,\frac{\lambda_3}{(p-\xi_3)}\right) \\ &= \arg(\lambda_3) \\ &= 0,\end{aligned}$$

since λ_3 is a positive real number.

Since the only two lines that go through the point p and make congruent angles to the lines from the foci and through p are the tangent and the normal, we know that the line L must be either the tangent or the normal. However, the normal would not satisfy the equation (44). Hence, L is the tangent of the ellipse E at the point p.

Now we can continue to the proof of the second part of the theorem. Let p be a point on E, and let L be the tangent of E at p. Then L intersects $\mathbb{T}$ at exactly two distinct points ξ_1 and ξ_2. Recall that there are exactly two points $\zeta_1, \zeta_2 \in \mathbb{T}$ with $\xi_1 \neq \zeta_1 \neq \zeta_2$ and $B(\xi_1) = B(\zeta_1) = B(\zeta_2)$. From the first part of the proof, we know that the two lines that go through ξ_1 and ζ_j (for $j = 1, 2$ respectively) are tangents to the ellipse E. We also know that there are exactly two tangents to E that goes through ξ_1. Hence, we know that L must be one of the two tangents, and thus we have that $\xi_2 = \zeta_1$ or $\xi_2 = \zeta_2$, which completes the proof. $\square$

7.3 Blaschke products of higher degree

The previous sections may give rise to the question of what happens for Blaschke products of higher degree than three. Before the theorem concerning Blaschke products of higher degree, a simple lemma will be stated.

Lemma 7.5. *Let $\omega \in \mathbb{T}$. Then the mapping*

$$\upsilon = \frac{\omega z}{z-1} \tag{45}$$

has an inverse

$$z = \frac{\upsilon}{\upsilon - \omega}. \tag{46}$$

We also have that $|\upsilon| = 1$ if and only if $\mathrm{Re}(z) = \frac{1}{2}$.

Proof. We begin with the inverse. By multiplying with the denominator in (45), we get

$$\upsilon z - \upsilon = \omega z.$$

by moving all terms with z to the left-hand side and factoring out z, we get

$$z(\upsilon - \omega) = \upsilon,$$

which gives us (46). Now we will continue to prove that the mapping in (45) satisfies $|\upsilon| = 1$ if and only if $\mathrm{Re}(z) = \frac{1}{2}$. We have

$$\begin{aligned} 1 = |\upsilon| &= \left| \frac{\omega z}{z-1} \right| \\ &= |\omega| \left| \frac{z}{z-1} \right| \\ &= \left| \frac{z}{z-1} \right|. \end{aligned}$$

Hence, we have

$$|z-1| = |z|.$$

By squaring both sides and simplifying the expression, we get

$$z + \overline{z} = 1.$$

If we let $z = x + yi$, we get that

$$1 = x + yi + x - yi,$$

which implies that $\mathrm{Re}(z) = \frac{1}{2}$. ☐

Theorem 7.6 (Daepp, Gorkin, Mortini [2]). *Let $B(z) = z\prod_{k=1}^{n-1} \frac{a_k - z}{1-\overline{a_k}z}$ be a Blaschke product with n distinct zeros. For $\omega \in \mathbb{T}$, let ξ_r be any of the n points that satisfies $B(\xi_r) = \omega$. Then there exists λ_r with $0 < \lambda_r < 1$ and another Blaschke product*

$$C(z) = z\prod_{k=1}^{n-2} \frac{c_j - z}{1 - \overline{c_j}z}$$

such that

$$\frac{B(z)/z}{B(z)-\omega} = \frac{\lambda_r}{z-\xi} + (1-\lambda_r)\,\frac{C(z)/z}{C(z)-\omega} \tag{47}$$

and

$$\sum_{k=1}^{n-1} \frac{1}{\prod_{j\neq k}|1-\overline{a_k}a_j|}\,|(a_k - c_1)\cdots(a_k - c_{n-2})| = 1. \tag{48}$$

Proof. We begin with the statement in (47). Let $\xi_1, ..., \xi_n$ be the n solutions to $B(z) = \omega$. Then we have the partial fraction expansion

$$\frac{B(z)/z}{B(z)-\omega} = \sum_{k=1}^{n} \frac{\lambda_k}{z-\xi_k}.$$

From Lemma 6.3, we know that $0 < \lambda_1, ..., \lambda_n < 1$ and $\sum_{k=1}^{n} \lambda_k = 1$. Now, for some $\xi_r \in \{\xi_1, ..., \xi_n\}$, let

$$\begin{aligned} R(z) &= \frac{1}{1-\lambda_r}\left(\frac{B(z)}{B(z)-\omega} - \frac{\lambda_r z}{z-\xi_r}\right) \\ &= \frac{1}{1-\lambda_r}\left(\sum_{k=1}^{n}\left(\frac{\lambda_k z}{z-\xi_k}\right) - \frac{\lambda_r z}{z-\xi_r}\right) \\ &= \frac{1}{1-\lambda_r}\sum_{\substack{k=1\\k\neq r}}^{n}\left(\frac{\lambda_k z}{z-\xi_r}\right). \end{aligned}$$

The function $R(z)$ is analytic everywhere except at the points ξ_j for $j \neq r$, where it has simple poles. Now recall that $|B(z)| = 1$ if $z \in \mathbb{T}$, and for any $z \in \mathbb{C}$, we have that $\text{Re}(z) = \frac{z+\overline{z}}{2}$. Thus, for $z \in \mathbb{T}$, we have

$$\begin{aligned}
\text{Re}\left(\frac{B(z)}{B(z)-\omega}\right) &= \frac{\frac{B(z)}{B(z)-\omega} + \frac{\overline{B(z)}}{\overline{B(z)}-\overline{\omega}}}{2} \\
&= \frac{\frac{B(z)\left(\overline{B(z)}-\overline{\omega}\right)+\overline{B(z)}(B(z)-\omega)}{(B(z)-\omega)\left(\overline{B(z)}-\overline{\omega}\right)}}{2} \\
&= \frac{\frac{2-B(z)\overline{\omega}-\overline{B(z)}\omega}{2-B(z)\overline{\omega}-\overline{B(z)}\omega}}{2} \\
&= \frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
\text{Re}\left(\frac{z}{z-\xi_r}\right) &= \frac{\frac{z}{z-\xi_r} + \frac{\overline{z}}{\overline{z}-\overline{\xi_r}}}{2} \\
&= \frac{\frac{z\left(\overline{z}-\overline{\xi_r}\right)+\overline{z}(z-\xi_r)}{(z-\xi_r)\left(\overline{z}-\overline{\xi_r}\right)}}{2} \\
&= \frac{\frac{2-z\overline{\xi_r}-\overline{z}\xi_r}{2-z\overline{\xi_r}-\overline{z}\xi_r}}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, for $z \in \mathbb{T}$, we have

$$\begin{aligned}
\text{Re}\left(R(z)\right) &= \text{Re}\left(\frac{1}{1-\lambda_r}\left(\frac{B(z)}{B(z)-\omega} - \frac{\lambda_r z}{z-\xi_r}\right)\right) \\
&= \frac{1}{1-\lambda_r}\left(\text{Re}\left(\frac{B(z)}{B(z)-\omega}\right) - \lambda_r \,\text{Re}\left(\frac{z}{z-\xi_r}\right)\right) \\
&= \frac{1}{1-\lambda_r}\left(\frac{1}{2} - \frac{\lambda_r}{2}\right) \\
&= \frac{1}{1-\lambda_r}\frac{1}{2}\left(1-\lambda_r\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Now let

$$C(z) = \frac{\omega R(z)}{R(z)-1}. \tag{49}$$

From Lemma 7.5, we know that the function $C(z)$ has modulus one on the unit circle. Now suppose that $z \in \mathbb{D}$. Then, for every j, we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{z}{z-\xi_j}\right) &= \frac{\frac{z}{z-\xi_j} + \frac{\overline{z}}{\overline{z}-\overline{\xi_j}}}{2} \\
&= \frac{1}{2}\left(\frac{z\left(\overline{z}-\overline{\xi_j}\right)+\overline{z}\left(z-\xi_j\right)}{\left(z-\xi_j\right)\left(\overline{z}-\overline{\xi_j}\right)}\right) \\
&= \frac{1}{2}\left(\frac{|z|^2 - z\overline{\xi_j} - \overline{z}\xi_j + |z|^2}{|z|^2 - z\overline{\xi_j} - \overline{z}\xi_j + 1}\right) \\
&\leq \frac{1}{2}.
\end{aligned}$$

Thus, we have that $\operatorname{Re}(R(z)) \leq \frac{1}{2}$ for $z \in \mathbb{D}$, since

$$\begin{aligned}
\operatorname{Re}(R(z)) &= \operatorname{Re}\left(\frac{1}{1-\lambda_r}\sum_{\substack{k=1\\k\neq r}}^{n}\frac{\lambda_k z}{z-\xi_k}\right) \\
&= \frac{1}{1-\lambda_r}\sum_{\substack{k=1\\k\neq r}}^{n}\left(\lambda_k \operatorname{Re}\left(\frac{z}{z-\xi_k}\right)\right) \\
&\leq \frac{1}{2}\frac{1}{1-\lambda_k}\sum_{\substack{k=1\\k\neq r}}^{n}\lambda_k \\
&= \frac{1}{2}\frac{1}{1-\lambda_r}\left(1-\lambda_r\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Since $R(z)$ is analytic everywhere except at its poles, which are at the $n-1$ points ξ_j for $j \neq r$, this implies that $C(z)$ is analytic on $\mathbb{D}$. Since $\operatorname{Re}(R(z)) = \frac{1}{2}$ on $\mathbb{T}$, $|C(z)| = 1$ for $z \in \mathbb{T}$. Corollary 4.2 in [5] states that if a function is analytic on $\mathbb{D}$, extends continuously on $\overline{\mathbb{D}}$ and is has modulus one on $\mathbb{T}$, then the function is a finite Blaschke product. Thus, we have that $C(z)$ is a finite Blaschke product. Since $R(z)$ has $n-1$ poles, $C(z)$ must have degree $n-1$. If we rewrite (49) as the inverse described in Lemma 7.5, we get

$$R(z) = \frac{C(z)}{C(z)-\omega}.$$

Hence, we have that

$$\frac{C(z)}{C(z)-\omega} = \frac{1}{1-\lambda_r}\left(\frac{B(z)}{B(z)-\omega} - \frac{\lambda_r z}{z-\xi_r}\right),$$

and thus

$$\frac{B(z)/z}{B(z)-\omega} = \frac{\lambda_r}{z-\xi_r} + (1-\lambda_r)\,\frac{C(z)/z}{C(z)-\omega},$$

which completes the proof for the statement in (47). Now we will continue to the statement in (48). First note that the zeros of $C(z)$ are the zeros of $R(z)$. Since we have

$$\frac{1}{1-\lambda_r}\left(\sum_{\substack{k=1\\k\neq r}}^{n} \frac{\lambda_k z}{z-\xi_k}\right) = \frac{z}{1-\lambda}\left(\sum{}_{\substack{k=1\\k\neq r}}^{\;n} \frac{\lambda_k}{z-\xi_k}\right),$$

we know that the zeros of $C(z)$ that are not at the origin are exactly the zeros of $\sum_{\substack{k=1\\k\neq r}}^{n} \frac{\lambda_k}{z-\xi_k}$. Hence,

$$\begin{aligned}
\frac{C(z)/z}{C(z)-\omega} &= \frac{1}{1-\lambda_r}\left(\frac{B(z)/z}{B(z)-\omega} - \frac{\lambda_r}{z-\xi_r}\right)\\
&= \frac{1}{1-\lambda_r}\left(\sum_{k=1}^{n} \frac{\lambda_k}{z-\xi_k} - \frac{\lambda_r}{z-\xi_r}\right)\\
&= \frac{1}{1-\lambda_r}\left(\sum_{\substack{k=1\\k\neq r}}^{n} \frac{\lambda_k}{z-\xi_k}\right)\\
&= \frac{\prod_{k=1}^{n-2}(z-c_k)}{\prod_{\substack{k=1\\k\neq r}}^{n}(z-\xi_k)}.
\end{aligned}$$

The rest of the proof will mostly follow the proof of Theorem 7.4, but letting $\frac{C(a_j)/a_j}{C(a_j)-\omega}$ having the role that $\frac{a_j-p}{(a_j-\xi_1)(a_j-\xi_2)}$ had in that proof. For any a_j, we have

$$\begin{aligned}
0 &= \frac{B(a_j)/a_j}{B(a_j)-\omega}\\
&= \frac{\lambda_r}{a_j-\xi_r} + (1-\lambda_r)\,\frac{C(a_j)/a_j}{C(a_j)-\omega}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\lambda_r\left|\frac{1}{a_j-\xi_r}\right| &= (1-\lambda_r)\left|\frac{C(a_j)/a_j}{C(a_j)-\omega}\right|\\
&= (1-\lambda_r)\left|\frac{\prod_{k=1}^{n-2}(a_j-c_k)}{\prod_{\substack{k=1\\k\neq r}}^{n}(a_j-\xi_k)}\right|.
\end{aligned}$$

Thus,

$$\left|\prod_{k=1}^{n-2}(a_j - c_k)\right| = \frac{\lambda_r}{1-\lambda_r}\left|\frac{\prod_{\substack{k=1\\k\neq r}}^{n}(a_j-\xi_k)}{(a_j-\xi_r)}\right|. \tag{50}$$

Recall that

$$|B(a_j)-\omega| = |\omega| = 1$$

and that we can write

$$B(z)-\omega = \frac{\prod_{k=1}^{n}(z-\xi_k)}{\prod_{k=1}^{n-1}(1-\overline{a_k}z)}.$$

Putting (50) into the expression (48), we get

$$\begin{aligned}
\sum_{k=1}^{n-1}\left(\frac{1}{\prod\limits_{\substack{j=1\\j\neq k}}^{n-1}|1-\overline{a_k}a_j|}\left|\prod_{j=1}^{n-2}(a_k-c_j)\right|\right) &= \sum_{k=1}^{n-1}\left(\frac{1}{\prod_{\substack{j=1\\j\neq k}}^{n-1}|1-\overline{a_k}a_j|}\frac{\lambda_r}{1-\lambda_r}\left|\frac{\prod_{\substack{j=1\\j\neq r}}^{n}(a_k-\xi_j)}{(a_k-\xi_r)}\right|\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\sum_{k=1}^{n-1}\left(\frac{1}{|a_k-\xi_r|}\frac{\prod_{\substack{j=1\\j\neq r}}^{n}|a_k-\xi_j|}{\prod_{\substack{j=1\\j\neq k}}^{n-1}|1-\overline{a_k}a_j|}\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\sum_{k=1}^{n-1}\left(\frac{1-|a_k|^2}{|a_k-\xi_r|^2}\frac{\prod_{j=1}^{n}|a_k-\xi_j|}{\prod_{j=1}^{n-1}|1-\overline{a_k}a_j|}\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\sum_{k=1}^{n-1}\left(\frac{1-|a_k|^2}{|a_k-\xi_r|^2}|B(a_k)-\omega|\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\sum_{k=1}^{n-1}\left(\frac{1-|a_k|^2}{|a_k-\xi_r|^2}\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\left(\frac{1}{\lambda_r}-1\right)\\
&= \frac{\lambda_r}{1-\lambda_r}\frac{1-\lambda_r}{\lambda_r}\\
&= 1,
\end{aligned}$$

which completes the proof. □

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A Appendix 1

The following code was used to make Figure 1.

```
import matplotlib.pyplot as plt
import numpy as np
from math import sqrt, pi
from cmath import exp
import os

def plot_unit_circle():
    '''
    Plots the unit circle and sets the axes in
    the right position.
    '''
    fig = plt.figure()
    ax =  fig.add_subplot(1,1,1)

    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')

    ax.spines['right'].set_color('none')
    ax.spines['top'].set_color('none')

    ax.xaxis.set_ticks_position('bottom')
    ax.yaxis.set_ticks_position('left')

    plt.locator_params(axis = 'x',nbins=7)
    plt.locator_params(axis = 'y',nbins=7)

    t1 = np.linspace(0,np.pi*2,100)
    plt.plot(np.cos(t1), np.sin(t1), linewidth=1)
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)
    plt.gca().set_aspect('equal', adjustable='box')

def plot_circle(r):
    '''
    Plots a circle with radius r.
    '''
    t = np.linspace(0,np.pi*2,100)
    plt.plot(r*np.cos(t), r*np.sin(t),
             linewidth=1, color='r')

def b_thesis(z, a_list):
    '''
    The definition used in the thesis.
    '''
    prod = 1
    for a_k in a_list:
        if a_k == 0:
            prod *= -1 * (a_k-z)/(1-a_k.conjugate()*z)
        else:
            prod *= abs(a_k)/a_k * (a_k-z)/(1-a_k.conjugate()*z)
    return prod
```

```
def b_daepp(z, beta, a_list):
    '''
    The definition of finite Blaschke products used in [2].
    '''
    prod = beta
    for a_k in a_list:
        prod *= (z-a_k)/(1-a_k.conjugate()*z)
    return prod

def b_garcia(z, alpha, a_list):
    '''
    The definition of finite Blaschke products used in [5].
    '''
    prod = exp(1j*alpha)
    for a_k in a_list:
        if a_k == 0:
            prod *= z
        else:
            prod *= abs(a_k)/a_k * (a_k-z)/(1-a_k.conjugate()*z)
    return prod

def b_fisher(z, a_list):
    '''
    The definition of finite Blaschke products used in [3].
    '''
    prod = 1
    try:
        for a_k in a_list:
            prod *= (-a_k.conjugate()/abs(a_k))
            prod *= (z-a_k)/(1-a_k.conjugate()*z)
    except ZeroDivisionError:
        print('This definition is not defined at the origin.')
    return prod

def plot_blaschke(points, labels, colors):
    '''
    Plots B(z) for the different definitions of Blaschke
    products and saves the plot in a pdf file.
    '''
    def name_file(filename):
        '''
        Returns a filename that doesent already exist.
        '''
        i = 0
        while True:
            if os.path.isfile(filename+str(i)+'.pdf'):
                i += 1
            else:
                break
        return filename+str(i)+'.pdf'
    plt.clf()

    #Plot the unit circle and set axes to the desired position.
    plot_unit_circle()
    x_values = [i.real for i in points]
    y_values = [i.imag for i in points]
    visited = []
```

```
    for i in range(len(points)):
        plt.plot(x_values[i], y_values[i],colors[i]+'o',
                 label=r'$z_{}$'.format(i+1)+labels[i])
        if x_values[i] >= 0 and (round(x_values[i],2), round(y_values[i],2)) not in visited:
            horizontal = 'left'
        else:
            horizontal = 'right'
        if y_values[i] >= 0 and (round(x_values[i],2), round(y_values[i],2)) not in visited:
            vertical  = 'bottom'
        else:
            vertical = 'top'
        plt.annotate(r'$z_{}$'.format(i+1), (x_values[i], y_values[i]),
                     horizontalalignment=horizontal, verticalalignment=vertical,
                     fontsize='large')
        visited.append((round(x_values[i],2), round(y_values[i],2)))
    plot_circle(abs(points[0]))
    plt.legend(bbox_to_anchor=[0.65,0.65], loc='lower left', prop={'size':10})
    try:
        filename = name_file('blaschke_definitions')
        plt.savefig(filename)
        print('The figure is saves as', filename)
    except:
        print('Something went wrong.')

if __name__=='__main__':
    z = 0.4+0.7j
    colors = ['b', 'c', 'y', 'm', 'r', 'g', 'k']
    a_list = [0.2+0.1j, 0.5j, 0.7, 0.3-0.6j]
    points = [b_thesis(z,a_list),
              b_daepp(z,1,a_list),
              b_daepp(z,1j,a_list),
              b_daepp(z,sqrt(2)/2+sqrt(2)/2*1j,a_list),
              b_garcia(z, pi, a_list),
              b_garcia(z,4, a_list),
              b_fisher(z,a_list)]
    labels = ['Definition 3.2',
              'Definition 3.4 with 'r'$\beta=1$',
              'Definition 3.4 with 'r'$\beta=i$',
              'Definition 3.4 with 'r'$\beta=\sqrt{2}/2+i\sqrt{2}/2$',
              'Definition 3.5 with 'r'$\alpha=\pi$',
              'Definition 3.5 with 'r'$\alpha=4$',
              'Definition 3.6']
    plot_blaschke(points, labels, colors)
```

A Appendix 2

The following code was used to make Figure 4.

```
import matplotlib.pyplot as plt
import numpy as np
import os

def plot(x_values, y_values, convex_hull=False):
    def name_file(filename):
        '''
        Returns a filename that doesent already exist.
        '''
        i = 0
        while True:
            if os.path.isfile(filename+str(i)+'.pdf'):
                i += 1
            else:
                break
        return filename+str(i)+'.pdf'
    plt.clf()
    fig = plt.figure()
    ax =  fig.add_subplot(1,1,1)

    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')

    ax.spines['right'].set_color('none')
    ax.spines['top'].set_color('none')

    ax.xaxis.set_ticks_position('bottom')
    ax.yaxis.set_ticks_position('left')

    t = np.linspace(0,np.pi*2,100)
    plt.plot(np.cos(t), np.sin(t), linewidth=1)
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)
    plt.gca().set_aspect('equal', adjustable='box')

    plt.scatter(x_values,y_values)

    for i in range(len(x_values)):
        horizontal = 'left' if x_values[i] >= 0 else 'right'
        vertical   = 'bottom' if y_values[i] >= 0 else 'top'
        plt.annotate(r'$z_{}$'.format(i+1),
        (x_values[i],y_values[i]), horizontalalignment=horizontal,
        verticalalignment=vertical, fontsize='x-large')

    if convex_hull:
        ax.fill(x_values, y_values, 'r')
    try:
        filename = name_file('convex_hull')
        plt.savefig(filename)
        print('The figure was saves as',filename)
    except:
        print('Something went wrong.')
```

```python
if __name__=='__main__':
    plot([0.5, -0.3, -0.25, 0.1, 0.5],
         [0.5, 0.1, -0.5, -0.7, -0.45],
         convex_hull=True)
```

A Appendix 3

The following code was used to make Figure 5-13.

```
import matplotlib.pyplot as plt
import numpy as np
from sympy.solvers import solve
from sympy import Symbol
from math import sqrt
import os

def plot_unit_circle():
    '''
    Plots the unit circle and sets the axes in the right position.
    '''
    fig = plt.figure()
    ax =  fig.add_subplot(1,1,1)

    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')

    ax.spines['right'].set_color('none')
    ax.spines['left'].set_color('none')
    ax.spines['top'].set_color('none')
    ax.spines['bottom'].set_color('none')

    ax.set_xticks([], [])
    ax.set_yticks([], [])

    plt.locator_params(axis = 'x',nbins=7)
    plt.locator_params(axis = 'y',nbins=7)

    t1 = np.linspace(0,np.pi*2,100)
    plt.plot(np.cos(t1), np.sin(t1), linewidth=1)
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)
    plt.gca().set_aspect('equal', adjustable='box')

def name_file(filename):
    '''
    Returns a filename that doesent already exist.
    '''
    i = 0
    while True:
        if os.path.isfile(filename+str(i)+'.pdf'):
            i += 1
        else:
            break
    return filename+str(i)+'.pdf'

def solve_equation(equation):
    '''
    Finds the zeros of the given equation.
    '''
    z = Symbol('z')
    return solve(equation, z)
```

```
def unit_circle_solutions(blaschke_product, modulus_one):
    '''
    Solves the equation blaschke_product(z)=lambda for
    lambda in modulus_one.
    '''
    solution_list = []
    for one in modulus_one:
        s = solve_equation(blaschke_product+'-'+one)
        solutions = [i.as_real_imag() for i in s]
        solution_list.append(solutions)
    return solution_list

def plot_ellipse(blaschke_product, modulus_one, remove_origo):
    '''
    Plots a figure with lines between the solutions to
    blaschke_product(z)=w for
    w in modulus_one.
    '''
    plt.clf()
    plot_unit_circle() # Plots the unit circle and removes the axes

    #Find the solutions to the equation B(z)=lambda
    point_list = unit_circle_solutions(blaschke_product, modulus_one)
    #Plot the lines between the solutions
    for lambda_solution in point_list:
        x_list = [i[0] for i in lambda_solution]
        y_list = [i[1] for i in lambda_solution]
        x_list.append(x_list[0])
        y_list.append(y_list[0])
        plt.plot(x_list, y_list, 'k-')

    #Finding the zeros of the Blaschke product
    blaschke_zeros = solve_equation(blaschke_product)
    blaschke_zeros = [i.as_real_imag() for i in blaschke_zeros]
    for zero in blaschke_zeros: #Plot the zeros
        if zero != (0,0) or remove_origo is False:
            plt.plot(zero[0], zero[1], 'ro')
    try:
        filename=name_file('ellipse')
        plt.savefig(filename)
        print('The figure is saved as', filename)
    except:
        print('Something went wrong.')

if __name__=='__main__':
    #Blaschke products
    B1 = 'z*((0.3-0.7*I)-z)/(1-(0.3+0.7*I)*z)'
    B2 = 'z*((-0.6+0.2*I)-z)/(1-(-0.6-0.2*I)*z)'
    B3 = 'z*((0.5-z)/(1-0.5*z))*((z-(-0.4-0.5*I))/(1-(-0.4+0.5*I)*z))'
    B4 = 'z*(((0.3+0.7*I)-z)/(1-(0.3-0.7*I)*z)*((0.5*I)-z))/(1-(-0.5*I)*z)'
    B5 = 'z*((0.2-0.6*I)-z)/(1-(0.2+0.6*I)*z)*((0.1+0.2*I)-z)/(1-(0.1-0.2*I)*z)'

    modulus_one = ['(1)','(1*I)', '(-1)', '(-1*I)']
    modulus_one += ['(sqrt(2)/2+sqrt(2)/2*I)', '(sqrt(2)/2-sqrt(2)/2*I)',
                    '(-sqrt(2)/2+sqrt(2)/2*I)','(-sqrt(2)/2-sqrt(2)/2*I)']
    modulus_one += ['(sqrt(2)/4+sqrt(14)/4*I)','(sqrt(2)/4-sqrt(14)/4*I)',
                    '(-sqrt(2)/4+sqrt(14)/4*I)','(-sqrt(2)/4-sqrt(14)/4*I)']
```

```
modulus_one += ['(sqrt(14)/4+sqrt(2)/4*I)','(sqrt(14)/4-sqrt(2)/4*I)',
                '(-sqrt(14)/4+sqrt(2)/4*I)','(-sqrt(14)/4-sqrt(2)/4*I)']

plot_ellipse(B1,['(1)'],False) #Figure 4
plot_ellipse(B1,modulus_one,False) # Figure 5
plot_ellipse(B1,modulus_one,True) # Figure 6
plot_ellipse(B2,modulus_one,True) # Figure 7

plot_ellipse(B3,['(1)'],False) # Figure 8
plot_ellipse(B3,modulus_one,False) # Figure 9
plot_ellipse(B3,modulus_one,True) # Figure 10
plot_ellipse(B4,modulus_one,True) # Figure 11
plot_ellipse(B5,modulus_one,True) # Figure 12
```

A Appendix 4

Appendix 4

The following code was used to make Figure 6.2.

```
import matplotlib.pyplot as plt
import numpy as np
import os

def plot_unit_circle():
    '''
    Plots the unit circle and sets the axes in
    the right position.
    '''
    fig = plt.figure()
    ax =  fig.add_subplot(1,1,1)

    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')

    ax.spines['right'].set_color('none')
    ax.spines['top'].set_color('none')

    ax.xaxis.set_ticks_position('bottom')
    ax.yaxis.set_ticks_position('left')

    plt.locator_params(axis = 'x',nbins=7)
    plt.locator_params(axis = 'y',nbins=7)

    t1 = np.linspace(0,np.pi*2,100)
    plt.plot(np.cos(t1), np.sin(t1), linewidth=1)
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)
    plt.gca().set_aspect('equal', adjustable='box')

def plot_points(point_list):
    '''
    Plots the points. The argument plot_list is a
    list of tuples with the complex numbers real
    and imaginary part.
    '''
    for point in point_list:
        plt.plot(point[0], point[1], 'ko')

    for i in range(len(point_list)):
        horizontal = 'left' if point_list[i][0] >= 0 else 'right'
        vertical   = 'bottom' if point_list[i][1] >= 0 else 'top'
        plt.annotate(r'$z_{}$'.format(i+1),
                     (point_list[i][0],point_list[i][1]),
                     horizontalalignment=horizontal,
                     verticalalignment=vertical, fontsize='x-large')
```

```
def plot_convex_hull_outline(point_list):
    '''
    Plots the outline of the convex hull of the points
    in point_list. The argument plot_list is a list of
    tuples with the complex numbers real and imaginary part.
    '''
    point_list.append(point_list[0])
    x_list = [i[0] for i in point_list]
    y_list = [i[1] for i in point_list]
    plt.plot(x_list, y_list, 'k-')

def hyperbolic_convex(t, z1, z2):
    h1 = (z1-((z1-z2)/(1-np.conjugate(z1)*z2))*t)
    h2 = (1-np.conjugate(z1)*((z1-z2)/(1-np.conjugate(z1)*z2))*t)
    return h1/h2

def plot_hyperbolic(point_list):
    '''
    Plots the outline of the hyperbolic convex hull
    of the points in point_list.
    '''
    def name_file(filename):
        '''
        Returns a filename that doesent already exist.
        '''
        i = 0
        while True:
            if os.path.isfile(filename+str(i)+'.pdf'):
                i += 1
            else:
                break
        return filename+str(i)+'.pdf'
    plt.clf()
    #Plot the unit circle and set the axes in the desired positions
    plot_unit_circle()
    separated_list = [(i.real, i.imag) for i in point_list]
    plot_points(separated_list) #Plots the points in point_list
    #Plots the outline of the convex hull
    plot_convex_hull_outline(separated_list)

    #Plot the outline of the convex hull
    point_list.append(point_list[0])

    # Plot convex hull outline
    t2 = np.linspace(0,1,100)
    for i in range(len(point_list)-1):
        plt.plot(hyperbolic_convex(t2, point_list[i],
                 point_list[i+1]).real,
                 hyperbolic_convex(t2, point_list[i],
                 point_list[i+1]).imag, 'r-', linewidth=1)

    try:
        filename=name_file('hyperbolic_convex_hull')
        plt.savefig(filename)
        print('The figure was saved as', filename)
    except:
        print('Something went wrong.')
```

```
if __name__=='__main__':
    plot_hyperbolic([0.5+0.5j,-0.3+0.1j,-0.25-0.5j,
                     0.1-0.7j, 0.5-0.45j])
```

A Appendix 5

The following code was used to make Figure 6.2.

```
import matplotlib.pyplot as plt
import numpy as np
import os

def plot_unit_circle():
    '''
    Plots the unit circle and sets the axes in the right position.
    '''
    fig = plt.figure()
    ax =  fig.add_subplot(1,1,1)

    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')

    ax.spines['right'].set_color('none')
    ax.spines['top'].set_color('none')

    ax.xaxis.set_ticks_position('bottom')
    ax.yaxis.set_ticks_position('left')

    plt.locator_params(axis = 'x',nbins=7)
    plt.locator_params(axis = 'y',nbins=7)

    t1 = np.linspace(0,np.pi*2,100)
    plt.plot(np.cos(t1), np.sin(t1), linewidth=1)
    plt.xlim(-1.1, 1.1)
    plt.ylim(-1.1, 1.1)
    plt.gca().set_aspect('equal', adjustable='box')

def plot_points(point_list):
    '''
    Plots the points. The argument plot_list is a list of tuples
    with the complex numbers real and imaginary part.
    '''
    for point in point_list:
        plt.plot(point[0], point[1], 'ko')

    for i in range(len(point_list)):
        horizontal = 'left' if point_list[i][0] >= 0 else 'right'
        vertical   = 'bottom' if point_list[i][1] >= 0 else 'top'
        plt.annotate(r'$z_{}$'.format(i+1),
                     (point_list[i][0], point_list[i][1]),
                     horizontalalignment=horizontal,
                     verticalalignment=vertical,
                     fontsize='x-large')

def hyperbolic_convex(t, z1, z2):
    h1 = (z1-((z1-z2)/(1-np.conjugate(z1)*z2))*t)
    h2 = (1-np.conjugate(z1)*((z1-z2)/(1-np.conjugate(z1)*z2))*t)
    return h1/h2
```

```
def plot_hyperbolic_line(z_1, z_2):
    '''
    Plots a hyperbolic line segment.
    '''
    def name_file(filename):
        '''
        Returns a filename that doesent already exist.
        '''
        i = 0
        while True:
            if os.path.isfile(filename+str(i)+'.pdf'):
                i += 1
            else:
                break
        return filename+str(i)+'.pdf'
    plt.clf()
    plot_unit_circle() #Plots the unit circle and sets the
    axes in the desired positions

    t2 = np.linspace(0,1,100)
    plot_points([(z_1.real,z_1.imag),(z_2.real, z_2.imag)])
    plt.plot(hyperbolic_convex(t2,z_1,z_2).real,
             hyperbolic_convex(t2,z_1,z_2).imag,
             'r-', linewidth=1)

    try:
        filename=name_file('hyperbolic_line_segment')
        plt.savefig(filename)
        print('The figure was saved as', filename)
    except:
        print('Something went wrong.')


if __name__=='__main__':
    plot_hyperbolic_line(0.3+0.5j, 0.7-0.25j)
```