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Brauer’s Theorem and Beyond

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Brauer's Theorem and Beyond

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Abstract

We provide a proof of Brauer's theorem and a discussion of the implications of the theorem to Artin's holomorphy conjecture. We also define M-groups and prove some basic results about them such as Dade's embedding theorem. We define a quasi-monomial character to be a character a multiple of which is monomial and also provide an example of such a character.

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1 Introduction

In 1946, Brauer proved his famous theorem on induced characters. He was inspired by Artin, who proved a weaker statement and conjectured the theorem. Artin himself was motivated by an application to number theory, where Brauer's theorem proves that a particular family of complex functions has a meromorphic continuation to the entire complex plane. These functions are today known as Artin L-functions. This will be discussed in section 3. Before that, we review some of the representation theory of finite groups in section 2. As this is only a review, very few proofs will be provided. In section 4 we provide a proof of Brauer's theorem.

Artin further conjectured that his L-functions have a holomorphic continuation to the entire complex plane. Here, group theoretic tools do not enable us to settle the conjecture. The most general case of Artin's conjecture is still open today, and is a consequence of a larger set of conjectures, known as the Langlands program. There is however a family of groups where group theory is enough to confirm Artin's conjecture, called M-groups, and they will be discussed in section 5. There we will prove that every M-group is solvable. Finally, in section 6, we will define quasi-monomial characters, a situation where Artin's conjecture holds. We will in section 7 give an example of a quasi-monomial character that is not monomial.

2 Preliminaries from Representation theory

Before we start discussing Brauer's theorem and similar results, we first briefly review some facts about the representation theory of finite groups.

Let G be a finite group. A complex representation of G is a homomorphism $\rho : G \to GL_n(\mathbb{C})$ for som positive integer n. The number n is called the degree of the representation. It is equivalent to give an action of G on a vector space V of dimension n. All representations considered in this text will without further notice be assumed to be complex. We say a representation is faithful if it is an injective homomorphism. If ρ, ρ' of degree n are representations, we say they are isomorphic if there is an element $A \in GL_n(\mathbb{C})$ such that $\rho(g) = A\rho'(g)A^{-1}$ for all $g \in G$. We can also define the direct sum and the tensor product of two representations, denoted $\rho \oplus \rho'$ and $\rho \otimes \rho'$ respectively. This is done in the following way

$$\rho \oplus \rho' = \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho(g)' \end{bmatrix}, \quad \rho \otimes \rho'(g) = \rho(g) \otimes \rho'(g).$$

Here $\rho(g) \otimes \rho'(g)$ denotes the Kronecker product of the two matrices $\rho(g)$ and $\rho'(g)$. We say a representation ρ is irreducible if it cannot be written as the direct sum of two characters of smaller degree. The following is known as Maschke's theorem.

Theorem 2.1. *Any representation ρ can be decomposed in the following way*

$$\rho = \rho_1 \oplus \rho_2 \oplus ... \oplus \rho_n$$

where each ρ_i is irreducible. The ρ_i are uniquely determined up to isomorphism.

Proof. See theorem 2 in [1], chapter 2, page 16. □

If ρ is a representation with irreducible decomposition $\rho = \rho_1 \oplus \rho_2 \oplus ... \oplus \rho_n$, we let ϕ_1 be the direct sum of all representations isomorphic to ρ_1. Similarly, if ρ_i is the first representation not isomorphic to ρ_1, we let ϕ_2 be the direct sum of all representations isomorphic to ρ_i. Continuing like this, we get a decomposition

$$\rho = \phi_1 \oplus \phi_2 \oplus ... \oplus \phi_k.$$

This is known as the canonical decomposition of ρ. The components ϕ_i are known as the isotypic components of ρ. More generally, if a representation has only one isomorphism class in its decomposition, we say it is isotypic.

A character χ of a representation ρ is a function from G to $\mathbb{C}$ defined by $\chi(g) = \text{Trace}(\rho(g))$. A representation is uniquely determined by its character up to isomorphism. This means we can use the two notions interchangeably.

We define irreducible characters and characters of degree n in the expected way. A character is linear if it is of degree 1. We let $Irr(G)$ denote the irreducible characters of G. A character is constant on every conjugacy class of G, and a function with this property is known as a class function. The following is a list of standard properties of characters.

Lemma 2.2. *Let χ, χ' be characters of the representations ρ, ρ' respectively.*

(i) The character of $\rho \oplus \rho'$ is $\chi + \chi'$ and the character of $\rho \otimes \rho'$ is $\chi\chi'$.

(ii) The irreducible characters of G form a basis for the class functions of G. In particular, there are as many irreducible characters of G as there are conjugacy classes in G.

(iii) We have

$$\sum_{\chi \in Irr(G)} \chi(1)^2 = |G|.$$

Proof. Part (i) is proposition 2 in [1], section 2.1, page 11. Part (ii) is theorem 6 in [1], section 2.5, page 19. Part (iii) is Corollary 2 in [1], section 2.4, page 18. □

We now define some standard characters. We let 1_G denote the trivial character of G, defined as $1_G(g) = 1$ for all $g \in G$. We let reg_G denote the regular character of G, defined by

$$\mathrm{reg}_G(g) = \begin{cases} |G| \text{ if } g = 1 \\ 0 \text{ otherwise} \end{cases}.$$

The regular character satisfies

$$\mathrm{reg}_G = \sum_{\chi \in Irr(G)} \chi(1)\chi.$$

In particular, the representation corresponding to reg_G will contain every irreducible character.

We define an inner product $[-,-]_G$ on the class functions of G. If ψ, ϕ are class functions, we have

$$[\psi, \phi]_G = \frac{1}{|G|} \sum_{g \in G} \psi(g^{-1})\phi(g).$$

The irreducible characters of G form an orthonormal basis with respect to this inner product.

Say H is a subgroup of G and ϕ is a representation of H with character λ. Let V be the vector space that H acts on. We can from this information define a representation of G called the induced representation of ϕ. To do this, let

$\{g_1, g_2, ..., g_k\}$ be a set of representatives of the left cosets of H. For each $g \in G$, we have $gg_i = g_{\sigma(g,i)}h_{g,i}$ for $h_{g,i} \in H$. If we let V_i for $1 \leq i \leq k$ be isomorphic copies of V, H has an action on each of them. We let G act on the vector space $\bigoplus_{i=1}^k V_i$ by $g.v_i = h_{g,i}v_{\sigma(g,i)}$. The corresponding representation ρ of G is called the representation induced from the ϕ and is denoted $\operatorname{Ind}_H^G \phi$. The character χ of ρ is denoted $\operatorname{Ind}_H^G \lambda$ and we have

$$\chi(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ sgs^{-1} \in H}} \lambda(sgs^{-1}).$$

We can for any class function f on H define a class function $\operatorname{Ind}_H^G f$ by the formula above.

We can similarly define restriction, but this is much easier. If ρ is a representation of G we can restrict this to a representation of H. We write $\operatorname{Res}_H^G \rho$ for this representation. Similarly, if f is a class function we let $\operatorname{Res}_H^G f$ be the restriction of f to H. It is easily verified using the above formula that if f is a class function on H and h is a class function on G,

$$\operatorname{Ind}_H^G(f)h = \operatorname{Ind}_H^G(f \operatorname{Res}_H^G(h))$$

The basic properties of induction and restriction are given below.

Lemma 2.3. *Let G be a group and let H be a subgroup of G. Also, let χ be a character of G and λ be a character of H.*

(i) Frobenius reciprocity,

$$[\lambda, \operatorname{Res}_H^G \chi]_H = [\operatorname{Ind}_H^G \lambda, \chi]_G.$$

(ii) Inductive properties of induction,

$$\operatorname{Ind}_H^G(\lambda)\chi = \operatorname{Ind}_H^G(\lambda \operatorname{Res}_H^G \chi).$$

(iii) Kernel of induced representations,

$$\ker(\operatorname{Ind}_H^G \lambda) = \bigcap_{g \in G} g \ker(\lambda) g^{-1}.$$

Let G be a group and H, K be subgroups of G. The double cosets $H \backslash G / K$ is defined to be the sets of the form HxK for $x \in G$. They satisfy the following properties.

- The double cosets form a partition of G.
- $|HxK| = |H||K|/|H \cap xKx^{-1}|$.

The following result is known as Mackey's formula.

Theorem 2.4. *Let G be a group, H and K be subgroups of G and ρ be a representation of H. For $s \in G$, we define $H_s = sHs^{-1} \cap K$. We also define ρ_s to be a representation of H_s defined by*

$$\rho_s(h) = \rho(s^{-1}hs).$$

Let S be a set of representatives for the double cosets $K\backslash G/H$. We then have

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \rho \cong \bigoplus_{s \in S} \operatorname{Ind}_{H_s}^K \rho_s.$$

Proof. See Proposition 22 in [1], section 7.3, page 58.

□

Let G be a group and N a normal subgroup. There is a natural action of G on $Irr(N)$. If $g \in G$ and $\lambda \in Irr(N)$, we define

$$g.\lambda(x) = \lambda(g^{-1}xg).$$

We define $I_G(\lambda) = \{g \in G : g.\lambda = \lambda\}$. This is a subgroup of G containing N. Normally, when N is an arbitrary subgroup of G, it is hard to say anything at all about the $\operatorname{Res}_N^G \chi$ for $\chi \in Irr(G)$. However, if N is normal in G there is a lot more to say. Alfred H. Clifford proved in [3] the following two theorems.

Theorem 2.5. *Let G be a group and N a normal subgroup. Let χ be a character of G, and define $\lambda = \operatorname{Res}_N^G \chi$. If ψ is an irreducible constituent of λ, the decomposition of λ into irreducible characters is*

$$\lambda = e \sum_{i=1}^{n} \psi_i$$

where the ψ_i are the distinct elements in the orbit of ψ under the action of G and $e = [\psi, \lambda]$. If the group $I_G(\psi)/N$ is cyclic, $e = 1$. Furthermore, if G/N is cyclic, then each character of N fixed by G extends to a character of G.

Proof. See Theorem 6.2 in [2], page 79.

□

Theorem 2.6. *Let G be a group and N a normal subgroup. If λ is an irreducible character of N and χ is an irreducible constituent of Ind_N^G, there is a character $\phi \in Irr(I_G(\lambda))$ such that $\operatorname{Ind}_{I_G(\lambda)}^G \phi = \chi$.*

Proof. This is Theorem 6.11 in [2], page 82.

□

We will need one more theorem about restrictions to normal subgroups. If $\psi \in Irr(G/N)$, we can extend this character to G by composing the representation corresponding to ψ with the quotient map $\pi : G \to G/N$. This means we can identify $Irr(G/N)$ with a subset of $Irr(G)$. The same can be done for characters that are not irreducible.

Theorem 2.7. *Let G be a group and N a normal subgroup. Let $\chi \in Irr(G)$ satisfy $\operatorname{Res}_N^G \chi = \phi \in Irr(N)$. Then the characters $\psi\chi$ for $\psi \in Irr(G/N)$ are all irreducible. We also have*

$$\operatorname{Ind}_N^G \phi = \operatorname{reg}_{G/N} \chi.$$

Proof. See Theorem 6.17 in [2]. □

Let G be a group acting on a set X. We define a corresponding permutation representation ρ of G, by letting G act on $\mathbb{C}^{|X|}$. If $\{a_x\}_{x\in X} \in \mathbb{C}^{|X|}$, we let G act by the following rule

$$\rho(g)\{a_x\}_{x\in X} = \{a_{g^{-1}.x}\}_{x\in X}.$$

If χ is the character of ρ, we see that in the standard basis $\rho(G)$ consists of permutation matrices, matrices with a single nonzero entry in each row and each column and where that entry is 1. We also see that $\chi(g) = \operatorname{Trace}(\rho(g)) = |Fix(g)|$, where $Fix(g)$ denoted the set of fixpoints of g in X. If the action is transitive, we can say more.

Theorem 2.8. *Say χ is the character of a permutation representation corresponding to a transitive action of G on X. The character χ satisfies $[\chi, 1_G] = 1$, and $\chi - 1_G$ is irreducible if and only if the action of G on X is doubly transitive.*

Proof. We have

$$[\chi, 1_G]_G = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|.$$

By Burnside's lemma, this is 1. We have

$$[\chi, \chi]_G = \frac{1}{|G|}\sum_{g\in G} |Fix(g^{-1})||Fix(g)| = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|^2.$$

Using Burnside's lemma again, this is the number of orbits of the action of G on the set $X \times X$. This action har two orbits if and only if G is doubly transitive. We have

$$[\chi - 1_G, \chi - 1_G] = [\chi, \chi] - 2[\chi, 1_G] + [1_G, 1_G] = [\chi, \chi] - 1$$

The character χ is irreducible if and only if $[\chi, \chi] = 1$, from which the statement follows. □

The character $\mathrm{Ind}_H^G 1_H$ is the character of the permutation representation of G corresponding to the action of G on G/H.

Last but not least, we need some facts about the representations of direct products. If G and G' are groups and χ, χ' are elements in G, G' respectively, we can similarly to the discussion before Theorem 2.7 consider both χ and χ' characters of $G \times G'$ since $G \times G'/G' \cong G$ and $G \times G'/G \cong G'$. The results we need are summarized below.

Lemma 2.9. *Let G, G' be two finite groups and let H, H' be subgroups of G, G' respectively. Let λ and λ' be characters of H and H' respectively. We then have*

$$\mathrm{Ind}_H^G(\lambda)\,\mathrm{Ind}_{H'}^{G'}(\lambda') = \mathrm{Ind}_{H\times H'}^{G\times G'}(\lambda\lambda').$$

Also, if $\psi \in Irr(G \times G')$ we have $\psi = \chi\chi'$ for some $\chi \in Irr(G)$ and $\chi' \in Irr(G')$.

Proof. The second statement is Theorem 4.21 in [2], page 59. For the first statement, we note that

$$\mathrm{Ind}_{H\times H'}^{G\times G'}(\lambda\lambda')(x,x') = \frac{1}{|H \times H'|} \sum_{\substack{(g,g')\in G\times G' \\ (gxg^{-1},g'x'g'^{-1})\in H\times H'}} \lambda(gxg^{-1})\lambda'(g'x'g'^{-1}) =$$

$$= \frac{1}{|H|}\left(\sum_{gxg^{-1}\in H} \lambda(gxg^{-1})\right) \frac{1}{|H'|}\left(\sum_{g'x'g'^{-1}\in H'} \lambda'(g'x'g'^{-1})\right) =$$
$$= \mathrm{Ind}_H^G(\lambda)(x,x')\,\mathrm{Ind}_{H'}^{G'}(\lambda')(x,x').$$

□

3 Artin's Conjecture

In the year 1923, the mathematician Emil Artin published his paper *Uber eine neue Art von L-Reihen* [4]. There he defined what is today called the Artin L-functions. These play an important role in modern mathematics.

Let L/K be a finite Galois extension of algebraic number fields and let ρ be a representation of $\mathrm{Gal}(L/K)$, the galois group of the field extension. We let R and S be the ring of integral elements in the field L and K respectively. If $\mathfrak{p}$ is a prime ideal of R, $\mathfrak{P}$ is a prime ideal of S and $R \cap \mathfrak{P} = \mathfrak{p}$ the field $k = R/\mathfrak{p}$ is naturally identified with a subfield of $l = S/\mathfrak{P}$. We make for brevity the assumption that $\mathfrak{p}$ is unramified, which is true for all but a finite number of $\mathfrak{p}$. In that case there is a natural map

$$\phi : \mathrm{Gal}(l/k) \to \mathrm{Gal}(L/K).$$

The galois group $\mathrm{Gal}(l/k)$ is cyclic and generated by the Frobenius automorphism $\sigma \in \mathrm{Gal}(l/k)$. This automorphism is defined by $\sigma(x) = x^q$ where q is the size of k. We would like to define $\mathbf{Frob}(\mathfrak{p})$ to be $\phi(\sigma)$, but ϕ is not uniquely determined by $\mathfrak{p}$ and depends on $\mathfrak{P}$. We instead defined $\mathbf{Frob}(\mathfrak{p})$ to be the subset of $\mathrm{Gal}(L/K)$ obtained from $\phi(\sigma)$ when $\mathfrak{P}$ varies. This is a conjugacy class, which means the characteristic polynomial of $\rho(\mathbf{Frob}(\mathfrak{p}))$ is well defined. We can then define the Euler factor of $\mathfrak{p}$, denoted $E(\rho, \mathfrak{p}, s)$, to be

$$E(\rho, \mathfrak{p}, s) = \det(I - N(\mathfrak{p})^{-s} \rho(\mathbf{Frob}(\mathfrak{p}))^{-1}.$$

Here $N(\mathfrak{p})$ denotes the absolute norm of $\mathfrak{p}$. The variable s here is allowed to be any complex number, which makes $E(\rho, \mathfrak{p}, s)$ a complex valued function. A slightly more complicated definition is made when $\mathfrak{p}$ is ramified. The Artin L-function of ρ is defined to be

$$L(\rho, s) = \prod_{\mathfrak{p} \subset K} E(\rho, \mathfrak{p}, s).$$

This product converges uniformly in the half plane $\Re(s) > 1$, which means it defines a holomorphic function in that domain.

One of the original motivations Artin had for defining L-functions was for applications to another type of complex function known as Dedekind zeta-functions. They are meromorphic functions that generalize the Riemann zeta-function. For any arithmetic number field K, we can define a corresponding zeta-function ζ_K. A conjecture sometimes called Dedekind's conjecture says that for any field extension L/K, the function ζ_L/ζ_K is holomorphic. This is a quite surprising conjecture to make considering the fact that ζ_K usually has many zeroes. Artin proved in his original paper that if L/K was Galois with $G = \mathrm{Gal}(L/K)$, ζ_L/ζ_K

was a product of Artin L-functions in the following sense

$$\zeta_L/\zeta_K(s) = L(\mathrm{reg}_G - 1_G, s) = \prod_{\substack{\chi\in Irr(G)\\ \chi\neq 1_G}} L(\chi, s)^{\chi(1)}.$$

Artin made a conjecture of his own.

Conjecture 3.1. *(Artin's conjecture) Let L/K be a finite Galois extension of algebraic number fields, and let $G = \mathrm{Gal}(L/K)$. For every character χ of G that does not contain 1_G as one of its irreducible constituents, $L(\chi, s)$ has a holomorphic extension to the entire complex plane.*

Dedekind's conjecture follows from Artin's conjecture when L/K is Galois. When we in the sequel mention that a complex-valued function has a holomorphic extension, we mean a holomorphic extension to the entire complex plane.

Much can be said about Artin's conjecture using only group theoretic tools and basic properties of L-functions. Below is a list of such basic properties.

Theorem 3.1. *Let $K \subset F \subset L$ be arithmetic number fields such that L/K is Galois. Let ρ, ρ' be representations of $\mathrm{Gal}(L/K)$ and let λ be a representation of $\mathrm{Gal}(F/K)$.*

(i) $L(\rho + \rho', s) = L(\rho, s)L(\rho', s)$.

(ii) *If* $\lambda' = \mathrm{Ind}_{\mathrm{Gal}(F/K)}^{\mathrm{Gal}(L/K)}(\lambda)$, $L(\lambda, s) = L(\lambda', s)$.

(iii) *If ρ is a nontrivial linear representation, $L(\rho, s)$ has a holomorphic extension.*

Proof. Part (i) and (ii) is **L2** and **L4** in [5], page 233. Part (iii) is Theorem 2 in [5], page 234. □

We can see that Artin's conjecture holds for L-functions corresponding to linear characters and using the properties above, we can "spread" this to other L-functions. To make this more precise we make the following definition.

Definition 3.1. *Let G be a finite group. We say a character χ of G is **monomial** if it is induced by a linear character of some subgroup of G.*

Using theorem 3.1, if a character χ is a positive sum of monomial characters not induced by the identity representation, the corresponding L-function has a holomorphic extension. We could hope every character is of this form. Unfortunately, this is not the case, and a counterexample will be given in section 4. Artin conjectured something weaker. He believed every character in a group G is a linear combination of monomial characters with integer coefficients, although not necessarily positive coefficients. This implies that every Artin L-function has a meromorphic extension. This was proven by Brauer, and a proof will be given in the next section. The most general case of Artin's conjecture is still open today, it is for example not known when $\mathrm{Gal}(L/K) \cong A_5$. A discussion of the known cases of Artin's conjecture can be found in [6].

It is important to note that $L(1_{\mathrm{Gal}(F/K)}, s) = \zeta_K$, which is known to have a meromorphic extension that is holomorphic except for $s = 1$, where it has a simple pole.

4 Brauer's theorem

In the following section, we let G be a finite group and $|G| = n$. We say a class function f on G is a virtual character if it is a linear combination of characters with integer coefficients. We let $R(G)$ denote the abelian group of the virtual characters of G, also known as the group ring of G. We want to prove the following theorem.

Theorem 4.1. *Every character of a group G is a linear combination with integer coefficients of monomial characters.*

To begin with we make the following definition.

Definition 4.1. *A group G is called* ***supersolvable*** *if there is a composition series of G,*

$$1 = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_n = G,$$

where each G_i is normal in G and where G_{i+1}/G_i is cyclic of prime order.

The following theorem gives us a fairly large family of groups that are supersolvable.

Theorem 4.2. *A nilpotent group is supersolvable*

Proof. This is theorem 14 in [1], chapter 8, page 64.

□

The family of supersolvable groups will be a source of monomial characters in our proof of Brauer's theorem. We start with some results about such groups.

Lemma 4.3. *Let G be a group and H a central subgroup of G. If G/H is cyclic, then G is abelian.*

Proof. Let gH be a generator of G/H. If $a, b \in G$, we must have $aH = g^n H$ and $bH = g^m H$ for some n, m. But then $a = g^n h$ and $b = g^m h'$ for $h, h' \in H$. This means

$$ab = g^n h g^m h' = g^{m+n} h h' = g^m h' g^n h = ba.$$

□

We will use the following fact from group theory.

Lemma 4.4. *Let G be a nonabelian supersolvable group. Then G has a normal abelian subgroup N, strictly containing $Z(G)$.*

Proof. Let $H = G/Z(G)$, and let $\pi : G \to H$ be the quotient homomorphism. The group H is supersolvable which means it has a composition series

$$1 = H_0 \subset H_1 \subset H_2 \subset ... \subset H_n = H,$$

where each H_i is normal in H and where H_{i+1}/H_i is cyclic of prime order. The group $N = \pi^{-1}(H_1)$ is normal in G and strictly contains $Z(G)$. Since H_1 is cyclic and $H_1 = N/Z(G)$, Lemma 4.3 shows that N is abelian. □

The main theorem about supersolvable groups we will be using is the following.

Theorem 4.5. *Let ρ be an irreducible representation of a supersolvable group G. Then ρ is monomial.*

Proof. We can assume G is nonabelian. We can also assume ρ is faithful by replacing G by $G/\ker(\rho)$, and ρ by the corresponding representation ρ' of $G/\ker(\rho)$. We can make this assumption since ρ is monomial if and only if ρ' is. By lemma 3.3, G has a normal abelian subgroup N strictly containing $Z(G)$. Let V be the space G acts on by ρ. Consider the representation $\rho_N = \mathrm{Res}_N^G \rho$ of N and let $V = \oplus_{i=1}^n V_i$ be the canonical decomposition of ρ_N into its isotypic components. Using that N is normal in G we get for every $g \in G$ and $a \in N$

$$\rho(a)\rho(g)V_i = \rho(g)\rho(g^{-1}ag)V_i = \rho(g)V_i.$$

In particular, the space $\rho(g)V_i$ is N-stable for every i. If $W_i', W_i \subset V_i$ are isomorphic and irreducible as representations of N, $\rho(g)W_i$ and $\rho(g)W_i'$ must be isomorphic and irreducible as well. This means $\rho(g)V_i$ is isotypic, which means G permutes the V_i. This action must be transitive, since the direct sum of the elements in an orbit otherwise would form a G-stable subspace of V, contrary to the fact that ρ is irreducible.

Assume first $V_1 = V$. Then ϕ is an isotypic representation of N, and since N is abelian we can choose a basis such that $\phi(a)$ is a scalar matrix for every $a \in N$. This means $\phi(N)$ is in the center of $\rho(G)$. But since ρ is faithful and N strictly contains $Z(G)$ this is a contradiction.

This means $V_1 \neq V$, so the group $H = \{g \in G : \rho(g)V_1 = V_1\}$ is a proper subgroup of G. This implies that the action of G on the V_i is isomorphic to G/H, which means ρ is induced by the representation $\rho_H = \mathrm{Res}_H^G \rho$. By induction on the order of G and using that a subgroup of a supersolvable group is supersolvable, we get that ρ_H is monomial which means $\rho_H = \mathrm{Ind}_C^H(\lambda)$ for λ a linear character of a subgroup H of G. But this means

$$\rho = \mathrm{Ind}_H^G(\mathrm{Ind}_C^H(\lambda)) = \mathrm{Ind}_C^G(\lambda).$$

□

To show that every character is an integral linear combination of monomial characters, we will find an explicit set S of subgroups of G that induces these monomial characters. We want to choose this set such that it consists only of supersolvable groups, as then every irreducible character of a group belonging to S is monomial. This means by the inductive properties of induction that it is enough to show that every character of G is an integral linear combination of characters induced by characters of S.

Definition 4.2. *Let p be a prime number. A subgroup of G is called p-elementary if it is the of the form $C \times P$, where C is a cyclic group of order prime to p and P is a p-group.*

An elementary p-group is a direct product of a cyclic group and a p-groups. Since p-groups and cyclic groups are nilpotent and nilpotency is preserved under the operation direct product we see that p-elementary groups are nilpotent. In particular means it is supersolvable. We let X_p be the set of p-elementary subgroups of G.

Let n be the order of G, and ζ_n a primitive n:th root of unity. In the course of the following proof, it will be useful to work with the group $R(G) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n]$. The ring $\mathbb{Z}[\zeta_n]$ has a $\mathbb{Z}$-basis given by $\{1, \zeta_n, \zeta_n^2, \zeta_n^3, ..., \zeta_n^{n-1}\}$. This means the $\mathbb{Z}$-module $\mathbb{Z}[\zeta_n]$ is free. A free $\mathbb{Z}$-module is faithfully flat, since a map tensored with $\mathbb{Z}^n$ is just the direct sum of the original map n times. This means the map

$$\mathrm{Ind} : \bigoplus_{\substack{p \text{ prime} \\ H \in X_p}} R(H) \to R(G)$$

is surjective if and only if the map

$$\mathbb{Z}[\zeta_n] \otimes \mathrm{Ind} : \bigoplus_{\substack{p \text{ prime} \\ H \in X_p}} \mathbb{Z}[\zeta_n] \otimes R(H) \to \mathbb{Z}[\zeta_n] \otimes R(G)$$

is surjective. To show Theorem 4.1, it is thus enough to show that every character of G is a $\mathbb{Z}[\zeta_n]$-linear combination of character induced by p-elementary subgroups of G. We show something stronger.

Theorem 4.6. *Let p be a prime and let $n = p^k m$ where p does not divide m. Let V_p be the image of the map*

$$\mathbb{Z}[\zeta_n] \otimes \mathrm{Ind} : \bigoplus_{H \in X_p} \mathbb{Z}[\zeta_n] \otimes R(H) \to \mathbb{Z}[\zeta_n] \otimes R(G).$$

Then $m \in V_p$.

The set of all possible values of m as p varies between all prime divisors of n is a set of relatively prime integers. By Bezout's idenitity, we find that $1 \in V_p$.

This means

$$1 = \sum_{\substack{\chi \in Irr(H) \\ H \in X_p}} a_\chi \operatorname{Ind}_H^G \chi$$

for $a_\chi \in \mathbb{Z}[\zeta_n]$. If now $f \in \mathbb{Z}[\zeta_n] \otimes R(G)$, we have

$$f = 1 \cdot f = \sum_{\substack{\chi \in Irr(H) \\ H \in X_p}} a_\chi \operatorname{Ind}_H^G(\chi) f = \sum_{\substack{\chi \in Irr(H) \\ H \in X_p}} a_\chi \operatorname{Ind}_H^G(\chi \operatorname{Res}_H^G(f)).$$

This means the map $\mathbb{Z}[\zeta_n] \otimes \operatorname{Ind}$ is surjective. In particular, Theorem 4.5 implies Theorem 4.1. To prove Theorem 4.5 we will need a few lemmas.

Theorem 4.7. *Let f be a class function on G with integer values divisible by n. Then f is a $\mathbb{Z}[\zeta_n]$-linear combination of characters induced by cyclic subgroups of G.*

Proof. Let C be a cyclic subgroup of G and let $|C| = c$. We define the function $\theta_C : C \to \mathbb{Z}$ by

$$\theta_C(s) = \begin{cases} c \text{ if } s \text{ generates } C \\ 0 \text{ otherwise} \end{cases}.$$

This is a class function. We have

$$\operatorname{Ind}_C^G(\theta_C)(s) = \frac{1}{c} \sum_{\substack{t \in G \\ tst^{-1} \in C}} \theta_C(tst^{-1}) = \frac{1}{c} \sum_{\substack{t \in G \\ tst^{-1} \text{ gen. } C}} c = \sum_{\substack{t \in G \\ tst^{-1} \text{ gen. } C}} 1.$$

But tst^{-1} generates a cyclic subgroup of G for all $t \in G$. Thus

$$\sum_{C \subset G} \operatorname{Ind}_C^G(\theta_C)(s) = \sum_{t \in G} 1 = n.$$

Let $f = nh$ where h is a class function with integer values. Then

$$f = nh = \sum_C \operatorname{Ind}_C^G(\theta_C) h = \sum_C \operatorname{Ind}_C^G(\theta_C \cdot \operatorname{Res}_C^G(h)).$$

The class function $\theta_C \cdot \operatorname{Res}_C^G(h)$ have values divisible by c. Also, any character ψ of C have values consisting entirely of n:th roots of unity, so

$$[\theta_C \cdot \operatorname{Res}_C^G(h), \psi]_C \in \mathbb{Z}[\zeta_n].$$

Thus, $\theta_C \cdot \operatorname{Res}_C^G(h)$ is an $\mathbb{Z}[\zeta_n]$-linear combination of characters of C. Since Ind is $\mathbb{Z}[\zeta_n]$-linear, the statement follows.

□

Let $s \in G$ has order mp^r where p does not divide m. Because m and p^r are relatively prime, there are integers x and y such that $xm + yp^r = 1$. If we define $s_p = s^{xm}$ and $s'_p = s^{yp^r}$ then s_p has order p^r, s'_p has order m and $s_p s'_p = s'_p s_p = s$. The elements s_p and s'_p are called the p-component and p'-component of s respectively.

Lemma 4.8. *For any $\chi \in \mathbb{Z}[\zeta_n] \otimes R(G)$ with integer values, we have*

$$\chi(s) \equiv \chi(s'_p) \mod p.$$

Proof. Let C be the cyclic subgroup generated by s and let $\chi' = \mathrm{Res}_C^G \chi$. Since C is abelian, we have

$$\chi' = \sum a_i \chi_i, \quad a_i \in \mathbb{Z}[\zeta_n]$$

where every χ_i is a character of degree one. Consider the element s^{p^r}. We have

$$s^{p^r} = (s_p s'_p)^{p^r} = s_p^{p^r} s_p'^{p^r} = (s'_p)^{p^r}$$

We then have

$$\chi(s)^{p^r} = \chi'(s)^{p^r} = \left(\sum a_i \chi_i(s)\right)^{p^r} \equiv \sum a_i^{p^r} \chi_i(s^{p^r}) \equiv$$

$$\equiv \sum a_i^{p^r} \chi_i((s'_p)^{p^r}) \equiv \chi(s'_p)^{p^r} \mod p\mathbb{Z}[\zeta_n].$$

But since χ has integer values, and $p\mathbb{Z}[\zeta_n] \cap \mathbb{Z} = p\mathbb{Z}$, we have

$$\chi(s)^{p^k} \equiv \chi(s'_p)^{p^k} \mod p \Rightarrow \chi(s) \equiv \chi(s'_p) \mod p.$$

□

Lemma 4.9. *If $s \in G$ is an element of G of order relatively prime to p, there is a $\phi_s \in V_p$ with integer values such that*

$$\phi_s(s) \not\equiv 0 \mod p$$

but $\phi_s(t) = 0$ if t is an element of G of order relatively prime to p not conjugate to s.

Proof. For any $s \in G$ of order relatively prime to p, we let C be the cyclic group generated by s, P be a sylow-p subgroup of $Cent_G(s)$ and $H = CP \cong C \times P$. Also say $c = |C|$ and $p^a = |P|$. Then we can define χ_s as a function on C by $\chi_s(t) = \delta_{st} c$. Here δ denotes the Kronecker delta. The following is true

$$\sum_{\chi \in Irr(C)} \chi(s^{-1})\chi(t) = \sum_{\chi \in Irr(C)} \chi(s^{-1}t) = \begin{cases} \sum_{i=1}^{c} \zeta_c^i = 0 \text{ if } s \neq t \\ c \text{ if } s = t \end{cases}.$$

In particular

$$\chi_s = \sum_{\chi \in Irr(C)} \chi(s^{-1})\chi$$

which means $\chi_s \in \mathbb{Z}[\zeta_n] \otimes R(C)$. We now define ψ_s as a function on H to be χ_s composed with the quotient homomorphism $H \to C$. The function $\psi_s \in \mathbb{Z}[\zeta_n] \otimes R(H)$. If $t \in G$ has order prime to p and $u \in G$ then utu^{-1} has order prime to p, so the only way $utu^{-1} \in H$ is if $utu^{-1} \in C$. If t is not conjugate to s, then by the definition of ψ_s, we have

$$\mathrm{Ind}_C^G(\psi_s)(t) = \frac{1}{p^a c} \sum_{\substack{u \in G \\ utu^{-1} \in H}} \psi_s(utu^{-1}) = \frac{1}{p^a c} \sum_{\substack{u \in G \\ utu^{-1} \in C}} \chi_s(utu^{-1}) = 0.$$

Since the order usu^{-1} is equal to the order of s, $usu^{-1} \in H$ if and only if $usu^{-1} \in C$. This means

$$\mathrm{Ind}_C^G(\psi_s)(s) = \frac{1}{p^a c} \sum_{\substack{u \in G \\ usu^{-1} \in H}} \psi_s(usu^{-1}) =$$

$$= \frac{1}{p^a c} \sum_{usu^{-1} = s} \chi_s(s) = \frac{1}{p^a} \sum_{usu^{-1} = s} 1 = \frac{|Cent_G(s)|}{p^a}.$$

This is an integer and we see that it is not congruent to zero modulo p as p^a is the largest power of p dividing $Cent_G(s)$. Because of this, the induced character above will have integer values.

□

We can finally prove our proposition.

Proof of Theorem 4.6. Let s_i be a set of representatives from the conjugacy classes of elements of order prime to p. If we let $\phi_i = \phi_{s_i}$ we have by lemma 3.8

$$\phi_i(s_i) \not\equiv 0 \mod p, \quad \phi_i(s_j) \equiv 0 \mod p, \quad i \neq j.$$

Let $\phi = \sum_i \phi_i$. It is clear that ϕ has integer values, and

$$\phi(s) \not\equiv 0 \mod p$$

For any $s \in G$ of order relatively prime to p. Let $s \in G$ be arbitrary, and $s = s_p s_p'$ the decomposition of s into it's p-component and p'-component. By Lemma 3.7

$$\phi(s) \equiv \phi(s_p') \mod p \quad \Rightarrow \quad \phi(s) \not\equiv 0 \mod p, \quad s \in G.$$

If we let $N = |(\mathbb{Z}/p^a\mathbb{Z})^*|$, such that $\phi(s)^N \equiv 1 \mod p^a$ for all $s \in G$, then $m(\phi^N - 1)$ has integer values divisible by n. By Theorem 3.6, $m(\phi^N - 1) \in V_p$. But then so is $m = m\phi^N - m(\phi^N - 1)$, and our proposition is proven. □

5 M-groups

It was noted in section 2 that if a character is a linear combination of monomial characters with positive coefficients not induced from a trivial character, then any Artin L-function corresponding to a representation with that character has a holomorphic extension. There are however characters without this property. To give such an example we first prove the following.

Lemma 5.1. *If χ is an irreducible character of a group G that is a linear combination of monomial characters with positive real coefficients, then $m\chi$ is monomial for some positive integer m.*

Proof. Say

$$\chi = \sum_{i=1}^{n} a_i \lambda_i$$

where each λ_i is monomial and each a_i is a positive real number. If $\{\chi = \chi_1, \chi_2, \dots \chi_k\}$ are the irreducible characters of G, we must have

$$\lambda_i = \sum_{j=1}^{k} b_{ij} \chi_j$$

where the b_{ij} are nonnegative integers. Combining these two identities, we get

$$\chi = \sum_{j=1}^{k} \chi_j \sum_{i=1}^{n} a_i b_{ij}$$

Using the linear independence of the irreducible characters of G, we find that $\sum_{i=1}^{n} a_i b_{ij} = 0$ for $j \neq 1$, which means $b_{ij} = 0$ for $j \neq i$. This means $\lambda_1 = a_1 b_{11} \chi_1$. Because every character is a linear combination of irreducible characters with integral coefficients the real number $a_1 b_{11}$ must be an integer. □

We now give an example of a character that does not satisfy the criterion of the lemma.

Consider the group $G = A_5$. Because G is a permutation group, the group admits a natural permutation representation with character ϕ. Because G is doubly transitive as a permutation group, this permutation representation splits into the trivial representation and an irreducible representation ρ of degree 4 with character χ by Lemma 2.8. I claim the character χ is not the direct sum of monomial characters. If it would be, we would by the lemma above have $m\chi = \lambda$ for some monomial character λ. The character λ would be induced by a linear character of a subgroup H of order $15/m$. The group G has no subgroup of order 15, since such a subgroup would give rise to a transitive action of G on the 4 cosets. This would means there exists a homomorphism $\pi : G \to S_4$ with nontrivial image, which is impossible since G is simple. This means there are three remaining cases

(i) The group H has order 3 and $m = 5$.

(ii) The group H has order 5 and $m = 3$.

(iii) The group H has order 1 and $m = 15$.

Case 3 cannot happen since the only character induced by the trivial group is the regular representation. If H has order 5, it must be generated by a 5-cycle. But then if $h \in H$ is not the identity, $\phi(h) = 0$, since if $g \in G$, $\phi(g)$ is the number of fixpoints of the permutation g. If λ is the character of H that induces 3χ, then by Frobenius reciprocity

$$[\phi, 3\chi]_G = [\mathrm{Res}^G_H \phi, \lambda]_H = \frac{\lambda(1)\phi(1)}{|H|} = 1.$$

But $[\phi, 3\chi]_G = 3[\phi, \chi]_G$ is 3 times an integer, a contradiction. If H has order 3, it must be generated by a 3-cycle. In this case $\phi(h) = 2$ for $h \in H$ and $h \neq 1$. We again let λ be the linear character of H inducing 5χ and let s be the 3-cycle that generates H. By Frobenius reciprocity we have

$$[\phi, 5\chi]_G = [\mathrm{Res}^G_H \phi, \lambda]_H = \frac{1}{|H|}(5\lambda(1) + 2\lambda(2) + 2\lambda(3)) = 1.$$

This is also a contradiction since $[\phi, 5\chi]_G = 5[\phi, \chi]_G$ is an integer divisible by 5.

This means Artin's conjecture is in this case not entirely solved by group theoretic methods. To help us determine exactly when Artin's theorem can be settled using only representation theory, we make the following definition.

Definition 5.1. *A finite group G is called an **M-group** if every irreducible character of G is monomial.*

In such a group every character is a sum of monomial characters, which means M-groups satisfy Artin's conjecture. We first show some basic properties about M-groups.

Theorem 5.2. *The following holds*

(i) If G is an M-group and N is a normal subgroup, G/N is and M-group.

(ii) The groups H and K are M-groups if and only if $H \times K$ is an M-group.

Proof. If ρ is a representation of G/N, we get a representation ρ_G of G by composing ρ with the canonical map $\phi : G \to G/N$. The representation ρ_G is irreducible if and only if ρ is, since reducibility only depends on the image of ρ. Because G is an M-group, ρ is monomial. Since monomiality only depends on the image of ρ, ρ_G has to be monomial as well. This proves (i).

The if follows from part (ii) of lemma 2.9. Because both H and K are quotients of $H \times K$, the only if part follows from part (i). $\square$

We could hope that the family of M-groups is large, so that we know Artin's conjecture for most groups. This is far from the case. The following is a result originally proven by Taketa in [8], showing that the family of M-groups is very limited.

Theorem 5.3. *An M-group is solvable.*

Proof. Let G be an M-group, and let $d_1 < d_2 < ... < d_n$ be the distinct degrees of the irreducible characters of G. Let χ be an irreducible character of degree d_i. We prove by induction on i that $G^{(i)} \subseteq \ker(\chi)$.

If $i = 1$, the character χ is linear, which means $G^{(1)} \subseteq \ker(\chi)$. If instead $i \geq 2$, since G is an M group there is a subgroup H of G and a linear character λ of H such that $\mathrm{Ind}_H^G(\lambda) = \chi$. The permutation representation ρ corresponding to G acting on G/H cannot be irreducible, since it contains the trivial representation. Let ϕ be the character of a nontrivial irreducible constituent of ρ. It follows that

$$\phi(1) < \dim(\rho) = [G : H] = \chi(1).$$

By induction, $G^{(i-1)} \subseteq \ker(\phi)$. But then,

$$G^{(i-1)} \subseteq \ker(\phi) \subseteq \ker(\rho) = H$$

since $G^{(i-1)}$ is a normal subgroup of G. By Lemma 2.3.(iii), we have

$$G^{(i)} \subseteq [G^{(i-1)}, G^{(i-1)}] \subseteq [H, H] = H^{(1)} \subseteq \ker(\lambda) \subseteq \ker(\mathrm{Ind}_H^G \lambda) = \ker(\chi).$$

What remains to be showed is that

$$\bigcap_{\chi \in Irr(G)} \ker(\chi) = \{1\}.$$

Because every representation of G is a direct sum of irreducible representations, it is enough to show that there is a faithful representation of G. The regular representation is an example of such a representation. $\square$

We move on to prove another important result, known as Dade's embedding theorem.

Theorem 5.4. *Every solvable group G is isomorphic to a subgroup of an M-group M.*

We mostly follow [9] in our proof. To prove the theorem we need to construct a fairly large amount of M-groups. For this, we use the wreath product.

Let G and H be finite groups and X a set that G acts on from the right. If we associate every coordinate of the group $H^{|X|}$ with an element of X, G acts on $H^{|X|}$ from the right by permuting the coordinates. If we let $\phi : G \to Aut(H^{|X|})$ be the corresponding homomorphism, we define the wreath product of G and H, denoted $H \operatorname{wr}_X G$, to be $H^{|X|} \rtimes_\phi G$. If $X = G$ and the action of G is by right multiplication on itself, $H \operatorname{wr}_X G$ is called the regular wreath product of G and H and will be denoted $H \operatorname{wr} G$.

A common situation in group theory is to have an exact sequence

$$1 \to N \to G \to H \to 1.$$

This is another way of saying $H \cong G/N$. Given N and G, the group structure of H is uniquely determined. The same is true about N if we are given G and H. But if we know N and H, we are still far from determining G. We say G is an extension of H by N. A solvable group is exactly a group that is given by inductively extending the trivial group by groups of the form C_p for p a prime. To find all such extensions is in it's most general form a very difficult problem, known as the extension problem. While the extension problem is far from being solved, there are partial results, such as the following theorem. It is known as the universal extension theorem.

Theorem 5.5. *Let H and N be finite groups, and say G is an extension of H by N. Then G is isomorphic to a subgroup of the group $N \operatorname{wr} H$.*

Proof. In the following proof we identify N with a subgroup of G and H with G/N. Let $|H| = k$ and let $g_1, g_2, g_3, ..., g_k$ be a set of representatives for the left cosets of N in G. For every $g \in G$ there is a permutation $\sigma(g)$ and an element $(n_{g,1}, n_{g,2}, ..., n_{g,k}) \in N^k$ such that $g_i g = n_{g,i} g_{\sigma(g)(i)}$. This is because N is normal in G, such that $g_{\sigma(g)(i)}N = Ng_{\sigma(g)(i)}$. We define a function $\phi : G \to N \operatorname{wr} H$ given by

$$\phi(g) = ((n_{g,1}, n_{g,2}, ..., n_{g,k}), gN).$$

We first show that this is a homomorphism. Let $g, h \in G$. First note that by the definition of the regular wreath product, we have for $gN \in H$ that

$$(n_1, n_2, ..., n_k).gN = (n_{\sigma(g)^{-1}(1)}, n_{\sigma(g)^{-1}(2)}, ..., n_{\sigma(g)^{-1}(k)}).$$

This means

$$\phi(g)\phi(h) = ((n_{g,1}, n_{g,2}, ..., n_{g,k}), gN)((n_{h,1}, n_{h,2}, ..., n_{h,k}), hN) =$$

$$= ((n_{g,1}, n_{g,2}, ..., n_{g,k})((n_{h,1}, n_{h,2}, ..., n_{h,k}).(gN)^{-1}), gNhN) =$$

$$= ((n_{g,1}n_{h,\sigma(g)(1)}, n_{g,2}n_{h,\sigma(g)(2)}, ..., n_{g,\sigma(g)(k)}), ghN).$$

What remains to be showed is that $n_{g,i}n_{h,\sigma(g)(i)} = n_{gh,i}$. We have by definition $g_i gh = n_{gh,i}g_{\sigma(gh)(i)}$. But we also have

$$g_i gh = n_{g,i}g_{\sigma(g)(i)}h = n_{g,i}n_{h,\sigma(g)(i)}g_{\sigma(gh)(i)}.$$

This proves that ϕ is a homomorphism.

We now prove ϕ is injective. Assume $\phi(g) = \phi(h)$. This means $gN = hN$, so g and h lie in the same coset. By the definition of σ, we must have $g_{\sigma(g)(1)} = g_{\sigma(h)(1)}$. The identity $\phi(g) = \phi(h)$ also implies $n_{g,1} = n_{h,1}$. But then

$$g = g_1^{-1}n_{g,1}g_{\sigma(g)(1)} = g_1^{-1}n_{h,1}g_{\sigma(h)(1)} = h.$$

□

Theorem 5.6. *Let G be an M-group and let C_p be a cyclic group of prime order. The group $H = G \operatorname{wr} C_p$ is an M-group.*

Proof. Let χ be a character of H. The group G^p is a normal subgroup of H of index p. This means H/G^p is cyclic, so by theorem 2.5 $\phi = \operatorname{Res}^H_{G^p} \chi$ either splits into p linear characters or is irreducible. If $\phi = \sum_{i=1}^p \psi_i$, for ψ_i irreducible, then by theorem 2.6 $I_H(\phi) = G^p$ and any ψ_i will induce χ. But G is an M-group, which means G^p is an M-group as well. This means ψ_i is monomial, and by the inductive properties of monomiality, so is χ.

Say instead ϕ is irreducible. By lemma 2.9, we have

$$\phi = \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_p,$$

where each ϕ_i is a character of G. We also must have $I_H(\phi) = H$, which means H fixes ϕ. But C_p acts on G^p by transitively permuting the coordinates. This means we must have $\phi_1 = \phi_2 = ... = \phi_n$. Since G is an M-group, ϕ_1 is induced by a linear character λ_1 of a subgroup K of G. This means $\lambda = \lambda_1 \otimes \lambda_1 \otimes ... \otimes \lambda_1$ is a character of the group K^p that induces ϕ. Because K^p is normalized by C_p, the product $H' = K^pC_p$ is a group. The index $[K^p : H'] = p$, so by theorem λ extends to a linear character μ of H'.

First, by theorem 2.7,

$$\operatorname{Ind}^H_{K^p} \lambda = \operatorname{Ind}^H_{G^p}(\operatorname{Ind}^{G^p}_{K^p} \lambda) = \operatorname{Ind}^H_{G^p}(\phi) = \operatorname{reg}_{H/G^p} \chi.$$

On the other hand, we also have

$$\operatorname{Ind}^H_{K^p} \lambda = \operatorname{Ind}^H_{H'}(\operatorname{Ind}^{H'}_{K^p} \lambda) = \operatorname{Ind}^H_{H'}(\operatorname{reg}_{H'/K^p} \mu) = \operatorname{reg}_{H/G^p} \operatorname{Ind}^H_{H'} \mu.$$

This means $\operatorname{reg}_{H/G^p} \chi$ and $\operatorname{reg}_{H/G^p} \operatorname{Ind}_{H'}^{H} \mu$ have the same irreducible constituents. In particular, χ is an irreducible constituent of $\operatorname{reg}_{H/G^p} \operatorname{Ind}_{H'}^{H} \mu$. But by theorem 2.7, these are all of the form $\psi \operatorname{Ind}_{H'}^{H} \mu$ for $\psi \in Irr(H/G^p)$. Because $|H/G^p| = p$, ψ is linear. But then

$$\chi = \psi \operatorname{Ind}_{H'}^{H} \mu = \operatorname{Ind}_{H'}^{H}(\mu \operatorname{Res}_{H'}^{H}),$$

which means χ is monomial, $\square$

We are now ready to prove the embedding theorem.

Proof of 5.4. Because G is solvable, G has a composition series of the following form

$$1 = G_1 \trianglelefteq G_2 \trianglelefteq .... \trianglelefteq G_k = G,$$

such that $G_{i+1}/G_i \cong C_{p_i}$ where p_i is a prime for each i. We can inductively define the groups M_i by the formula

$$M_1 = 1, \quad M_{i+1} = M_i \operatorname{wr} C_{p_i}.$$

By theorem 5.5, G_i is isomorphic to a subset of M_i, and by theorem 5.6, all M_i are M-groups. This means we can set $M = M_k$. $\square$

6 Quasi-monomial Characters and The Unitary Groups

As we have seen, if a character of a group is monomial any Artin-L functions corresponding to that representation will have a holomorphic extension. We can come to the same conclusion from a weaker hypothesis.

Definition 6.1. *Let χ be a character of a finite group G. We say χ is quasi-monomial if there is a positive integer m such that $m\chi$ is monomial.*

Say we have an Artin-L function $L(\chi, s)$ where χ is quasi-monomial. By Brauer's theorem on monomial characters, $L(\chi, s)$ has a meromorphic extension. If $m\chi$ is monomial and not induced by a trivial representation

$$L(m\chi, s) = L(\chi, s)^m$$

has a holomorphic extension. But then $L(\chi, s)$ must have a holomorphic extension as well. It is by this argument enough for a character to be quasi-monomial for conclusions about Artin's conjecture to be drawn. We can of course go ahead and define the notion of a QM-group where every irreducible character is quasi-monomial. There is however no known example of a QM-group that is not an M-group. Because $\ker(m\chi) = \ker(\chi)$ for any character χ, the proof of theorem 4.3 applies without change to QM-groups, which means every QM-group is solvable. This means we can not prove Artin's conjecture for nonsolvable groups only by introducing QM-groups.

The remainder of this text will focus on providing an example of a quasi-monomial character that is not monomial and conjecture an infinite family of such examples. This, contrary to most of the material in this text, is the authors own discovery (these examples may of course already be very well known). Before we do this, we first have to introduce the special unitary groups.

Let p be a prime number and $q = p^k$ be a prime power. We let $\mathbb{F}_q$ denote the finite field with q elements and $\overline{\mathbb{F}_q}$ be an algebraic closure. There is an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ given by $\sigma(x) = x^q$. If $A \in GL_n(\overline{\mathbb{F}_q})$, we define $\sigma(A)$ to be σ applied to each entry of A. We now define the frobenius endomorphism $F : GL_n(\overline{\mathbb{F}_q}) \to GL_n(\overline{\mathbb{F}_q})$ by

$$F(A) = \sigma((A^t)^{-1}).$$

The fixed points of this endomorphism form a group called the unitary group. This group is denoted $U(n, q)$. Because $F^2(A) = \sigma^2(A)$, the matrices in $U(n, q)$ have entries in $\mathbb{F}_{q^2}$. In particular, it is a finite group. We define the special unitary group, denoted $SU(n, q)$, to be the elements of $U(n, q)$ with determinant one. A very important fact about $SU(n, q)$ (that we will not use) is the following.

Theorem 6.1. *For $n \geq 4$, the group $SU(n,q)/Z(SU(n,q))$ is simple.*

This is a special case of a much more general family of simple groups, given by taking the points of simple algebraic groups over a finite field. In fact, almost all simple groups are of this form. It is conjectured (by me) that $SU(n,2)$ gives a family of examples of quasi-monomial characters that are not monomial. More precisely

Conjecture 6.1. *The group $SU(n,2)$ has for $n \geq 3$ an irreducible character χ, such that χ is not monomial but 2χ is. For $n \geq 4$, χ is of degree*

$$3\prod_{i=2}^{n-1}(2^i - (-1)^i)$$

if n is even and

$$\frac{1}{2^{n-1}-1}\prod_{i=2}^{n}(2^i - (-1)^i)$$

if n is odd.

We will in the next section prove the conjecture in the special case $n = 3$.

7 The special Case of SU(3,2)

This section will be dedicated to proving the following

Theorem 7.1. *The group $SU(3,2)$ has an irreducible character χ of degree 6, such that χ is not monomial but 2χ is.*

We use the following fact

Theorem 7.2. *The order of $SU(n,q)$ is*

$$q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - (-1)^i).$$

Proof. See theorem 14.3.2 in [10] $\square$

This means $|SU(3,2)| = 216 = 2^3 \cdot 3^3$, which by Burnside's $p^a q^b$-theorem means $SU(3,2)$ is solvable. Next we study the Sylow structure of our group. Let D be the subgroup of diagonal matrices of $SU(3,2)$. These consists of three elements of $\mathbb{F}_4^*$ on the diagonal, and the first two uniquely determines the third since the determinant has to be 1. Conversely, any choice of the two first elements gives us an element of $SU(3,2)$. This means

$$|D| = |\mathbb{F}_4^*|^2 = 9.$$

Consider also the group C generated by

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The group C has order 3 and normalizes D, since

$$a \operatorname{diag}(t_1, t_2, t_3) a^{-1} = \operatorname{diag}(t_3, t_1, t_2).$$

Since $D \cap C = 1$, we have $DC \cong D \rtimes C$, and in particular $R = DC$ is a sylow-3 subgroup of SU(3,2). By the following lemma, it is in fact the only sylow-3 subgroup.

Lemma 7.3. *The group $SU(3,2)$ has one normal sylow-3 subgroup.*

Proof. By the Sylow theorems, there can be either four or one Sylow-3 subgroups of $SU(3,2)$. If there are 4 sylow-3 subgroups, we have $N_{SU(3,2)}(P) = |SU(3,2)|/4 = 2 \cdot 3^3$. In particular, the elements of order a power of 2 that normalizes P must have order 2 by Lagrange's theorem. Let α be a third root of unity in $\mathbb{F}_4$, and consider the element

$$b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}.$$

The matrix b has order 4, and we have

$$bab^{-1} = \operatorname{diag}(1, \alpha^2, \alpha),$$

$$b \operatorname{diag}(\alpha, \alpha, \alpha)b^{-1} = \operatorname{diag}(\alpha, \alpha, \alpha)$$
$$b \operatorname{diag}(1, \alpha, \alpha^2)b^{-1} = a.$$

These 3 elements generate R. Since b takes these elements by conjugation into R, the element b must normalize R. Because b has order 4, this means there cannot be four sylow-3 subgroups, which means there is one. In particular, SU(3,2) has a normal sylow-3 subgroup. □

We now consider the sylow-2 subgroups of $SU(3, 2)$. Let α be a third root of unity in $\mathbb{F}_4$. Consider the four matrices

$$\overline{e} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, i = \begin{bmatrix} \alpha & 1 & \alpha^2 \\ 1 & 1 & 1 \\ \alpha^2 & 1 & \alpha \end{bmatrix}, j = \begin{bmatrix} \alpha & \alpha^2 & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha^2 & \alpha \end{bmatrix}, k = \begin{bmatrix} \alpha & \alpha & \alpha^2 \\ \alpha^2 & 1 & \alpha^2 \\ \alpha^2 & \alpha & \alpha \end{bmatrix}.$$

Let $Q = \langle \overline{e}, i, j, k \rangle$. A direct calulation shows that $\overline{e}^2 = 1$ and $i^2 = j^2 = k^2 = ijk = \overline{e}$. This is exactly a presentation of Q_8, the quaternion group. Because $\overline{e}, i, j, k$, are all distinct we must have $Q \cong Q_8$. This means every sylow-2 subgroup of SU(3,2) has a unique element of order 2.

Consider the subgroup P of permutation matrices in $SU(3, 2)$ and note that $P \cong S_3$. Also consider the group Z generated by $\operatorname{diag}(\alpha, \alpha, \alpha)$ of order 3. The group Z is central and since $P \cap Z = 1$, we have $PZ \cong P \times Z$. We call this product H. This will be the subgroup that induces the character we are interested in. Before we construct our representation, we prove som important facts about our group H.

Lemma 7.4. *The following holds:*

(i) An element $g \in SU(3, 2)$ that normalizes H also normalizes P.

(ii) Let $r \in P$ be of order 3. If $g \in SU(3, 2)$ normalizes $\langle p \rangle$, then $g \in H$.

(iii) $N_{SU(3,2)}(H) = H$.

Proof.

(i) Say $g \in SU(3,2)$ normalizes H. Because Z is central, we have $gZg^{-1} = Z$. We have $gPZg^{-1} = gPg^{-1}Z$, so we must have $gPg^{-1} \subset PZ$. Let $s \in P$ has order 2. Because the only elements of H of order 2 lies in P, we must have $gsg^{-1} \in P$. Because the elements of order 2 generate P, g must normalize P.

(ii) Say $g \in SU(3,2)$ normalizes $\langle r \rangle$. There are two elements of order three in P and since these are squares of one another we may w.l.o.g. assume

$$r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix r has three eigenspaces generated by $v_1 = (1, \alpha, \alpha^2), v_2 = (1, \alpha^2, \alpha)$ and $v_3 = (1,1,1)$ with eigenvalues $1, \alpha$, and α^2 respectively. Because g normalizes $\langle p \rangle$, we either have $gpg^{-1} = p$ or $gpg^{-1} = p^2$. If v is an eigenvector of p with eigenvalue λ we have either

$$pgv = gpv = \lambda gv,$$

or

$$pgv = gp^2v = \lambda^2 gv.$$

This means g fixes the space generated by v_1 and permutes or fixes the eigenspaces generated by v_1 and v_2. This means g in the basis (v_1, v_2, v_3) is either diagonal or of the form

$$g = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix}.$$

Because $v_1, v_2, v_3 \in (\mathbb{F}_4)^3$ the entries of g in the basis (v_1, v_2, v_3) lies in $\mathbb{F}_4$, and because g must have determinant 1 we have a total of 18 choices for g. But all elements of H normalize $\langle r \rangle$, and $|H| = 18$. This means $g \in H$.

(iii) Let $g \in N_{SU(3,2)}(H)$. By part (i), g normalizes P. But then g also normalizes the group of elements of order three in P, which by part (ii) means $g \in H$.

$\square$

If we compose the quotient map $H \to Z$ with any nontrivial representation of Z we get a representation corresponding to a linear character λ of H. Now consider $\chi = \mathrm{Ind}_H^G(\lambda)$. I claim this is twice an irreducible character. To prove this we use theorem 2.4.

We apply the the theorem to our situation by letting $G = SU(3, 2)$ and $H = K$. We let S be a set of representatives of the double cosets $H\backslash SU(3, 2)/H$, and define $H_s = s^{-1}Hs \cap H$. Using frobenius reciprocity, we have

$$[\rho, \rho]_G = [\lambda, \operatorname{Res}_H^G \operatorname{Ind}_H^G \lambda]_H = \sum_{s \in S} [\lambda, \operatorname{Ind}_{H_s}^K \lambda_s]_H$$

Using frobenius reciprocity again, we get

$$\sum_{s \in S} [\lambda, \operatorname{Ind}_{H_s}^H \lambda_s]_{H_s} = \sum_{s \in S} [\operatorname{Res}_{H_s} \lambda, \lambda_s]_{H_s}.$$

Because both λ and λ_s is linear, $[\operatorname{Res}_{H_s} \lambda, \lambda_s]_{H_s}$ is either 0 or 1 depending on whether λ_s and $\operatorname{Res}_{H_s} \lambda$ are equal.

We now compute $[\chi, \chi]$. To do this, we discuss the structure of $H\backslash SU(3, 2)/H$. If $s \in SU(3, 2)$ is such that $|HsH| = 18$, we must have $sHs^{-1} \cap H = H$, which means s normalizes H. But H is its own normalizer, which means $s \in H$ and $HsH = H$. This means there is a unique double coset in $H\backslash SU(3, 2)/H$ of order 18. Because Z is central, we must have $Z \in s^{-1} \cap H$ for any $s \in SU(3, 2)$. This means 3 must divide $|sHs^{-1} \cap H|$, which means $|HsH|$ must divide $|H|^2/3 = 108$. This means there are only three possibilities for $|HsH|$ when $s \notin H$:

- The size of HsH is 108. This does not actually occur.
- The size of HsH is 54. In this case $|sHs^{-1} \cap H| = 6$, which means $sHs^{-1} \cap H = \langle t \rangle Z$ for some element of order 2 in P. Since all elements in H lie in P, we must have $t = st's^{-1}$ for some $t' \in P$ of order 2. This means s centralizes t or takes t to another element of order 2 in P. In either case, it normalizes $\ker(\operatorname{Res}_{H_s} \lambda)$ and centralizes Z. This means $\lambda_s = \operatorname{Res}_{H_s} \lambda$ in this case, which means this double cosets contributes 1 to $[\chi, \chi]$.
- The size of HsH is 36. In this case $|sHs^{-1} \cap H| = 9$, which means $sHs^{-1} \cap H = \langle p \rangle Z$ for some element of order 3 in P. If s normalized $\langle p \rangle$ we would by lemma (ii) have $s \in H$. But then $HsH = H$ which has order 18, a contradiction. This means s does not normalize $\langle p \rangle$, which means $sps^{-1} = zp'$ for $z \in Z$ and $p' \in \langle p \rangle$ and where $z \neq 1$. But then $\lambda(p) = 1 \neq \lambda(z) = \lambda_s(p)$. This means $\lambda_s \neq \lambda$ in this case which means this double coset contribute 0 to $[\chi, \chi]$.

Given that these are the only possibilities, there are three ways to partition SU(3,2) into double cosets. There can be

1. one coset of size 18, three cosets of size 54 and one coset of size 36,
2. one coset of size 18, one coset of size 54 and four cosets of size 36,
3. one coset of size 18, one coset of size 36, one coset of size 54 and one coset of size 108.

We now prove that case 2 and case 3 cannot happen by showing the following

Lemma 7.5. *There is more than one element of $H\backslash SU(3,2)/H$ of size 54.*

Proof. By the remarks above, we see that an element $g \in SU(3,2)$ is in a double coset of size 54 if and only if it takes an element of order 2 in P to an element of order 2 in P by conjugation and is not in H. Also, an element not in H cannot centralize two elements in P, as it would then centralize P since the group is generated by any two elements of order 2, and then also centralize H.

Let s be an element of order 2 in P. The element s must be contained in a Sylow-2 subgroup of $SU(3,2)$. These are isomorphic to the quaternion group, Q_8, which contain a unique element of order 2 that is central. This means $|Cent_{SU(3,2)}(s)| \geq 8$. But any element of Z centralizes s, and $|Z| = 3$. This means $|Cent_{SU(3,2)}(s)| \geq 8 \cdot 3 = 24$. Because $Cent_{SU(3,2)}(s) \cap H = \langle s \rangle Z$ which has order 6, and there is 3 possible candidates for s, there is at least

$$(24 - 6) \cdot 3 = 54$$

elements in $SU(3,2)$ that lie in a double coset of size 54. But if we for example let

$$s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad s'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

then $s's(s')^{-1} = s''$. This implies the set $s'Cent_{SU(3,2)}(s)$ takes s to s'' by conjugation. Because this is a set of size 24, there must be at least one element not in H in there, which gives an additional element that lie in a double coset of size 54. □

The only remaining case is case 1. This means $[\chi, \chi] = 4$. The character χ is now either twice an irreducible character or the sum of four distinct irreducible characters. Let R be a sylow-3 subgroup of $SU(3,2)$ and let Q be a sylow-2 subgroup such that $Q \cap H \neq \{1\}$. Because R is normal in $SU(3,2)$, we must have $SU(3,2) \cong R \rtimes Q$. If we let $q \in Q$ be an element of order 2 in Q, the group $\langle q \rangle$ is central in Q. This means $N = R\langle q \rangle$ is a normal subgroup of $SU(3,2)$ of index 4 such that $H \subset N$. We define

$$\phi = \operatorname{Ind}_H^N \lambda.$$

The group N has order 54, and this means among the double cosets HnH for $n \in N$, one has to have size 18 and one must have size 36. The one of size 18 contribute 1 to $[\phi, \phi]$, while the one of size 36 contributes zero. This means $[\phi, \phi] = 1$, so ϕ is irreducible. By Frobenious reciprocity, we have

$$4 = [\chi, \chi] = [\operatorname{Res}_N^G \phi, \phi].$$

This means $\operatorname{Res}_N^G \phi = 4\phi$. Say χ is a sum of four distinct irreducible characters, and call one of them θ. Then θ and all other irreducible components of χ must restrict to ϕ on N. By theorem 2.7 we have $\chi = \operatorname{Ind}_N^G \phi = \rho_{G/N}\theta$. This means $\chi(g)$ is a multiple of four whenever it is an integer, since $\rho_{G/N}$ obtains either value 4 or value 0. But consider for example an element s in H of order 2. By our calculations, there are $18 + 54 = 72$ elements $g \in G$ such that $gsg^{-1} \in H$ and for any such g, $\lambda(gsg^{-1}) = \lambda(s) = 1$. This means

$$\chi(s) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gsg^{-1} \in H}} \lambda(gsg^{-1}) = \frac{72 \cdot \lambda(s)}{|H|} = 2.$$

This is not a multiple of 4, a contradiction.

What remains to be proven is that $\chi/2$ is not monomial. We first show a lemma.

Lemma 7.6. *All elements in $SU(3,2)$ of order 3 that does not lie in Z have centralizers of order 9 and are conjugate. There are no elements in $SU(3,2)$ of order 9.*

Proof. Let s be an element of P of order 3. By lemma 7.4.(ii), the centralizer of s lies in H. This means it must be $\langle s \rangle Z$ which has order 9. The size of the conjugacy class of s must be $|SU(3,2)|/|\langle s \rangle Z| = 24$. All elements of order 3 lies in the unique sylow-3 subgroup R of $SU(3,2)$ by lemma 6.3, and that group has order 27. This means all elements in $R - Z$ must be conjugate. This also means all elements of R are of order 3. □

Since $\chi/2$ is of degree 6 this means it is enough to show the following.

Lemma 7.7. *The group $SU(3,2)$ has no subgroup of index 6.*

Proof. Let K be such a group. It has order $2^2 \cdot 3^2$. Because $SU(3,2)$ has a normal Sylow-3 subgroup, K must have a normal Sylow-3 subgroup as well. Call this group R'. If $r \in R' - Z$, we conclude by Lemma 7.6 that r has a centralizer of order 9. But R' is of order 9 and is thus abelian, which means $R' = Cent_{SU(3,2)}(r)$. In particular, $Z \in R'$ and $R' = \langle r \rangle Z$. Let Q be a Sylow-2 subgroup of K. Because Q is contained in a Sylow-2 subgroup of $SU(3,2)$, it can contain at most one element of order 2, which means Q must contain elements of order 4. Let s be an element of Q of order 4. The elements s must normalize R', which means $srs^{-1} = zr^i$ for $z \in Z$ and $i \in \{1,2\}$. If $i = 2$, we see that $s^2rs^{-2} = z^3r^4 = r$, but this is a contradiction since $s^2 \notin R' = Cent_{SU(3,2)}(r)$. This means $i = 1$, which implies $r = s^4rs^{-4} = z^4r = zr$. This means $z = 1$, which means s normalizes $\langle r \rangle$. By Lemma 7.6, r is conjugate to an element r' of P. By Lemma 7.4.(ii), the normalizer of $\langle r' \rangle$ is contained in H, but since H

has order 18 it contains no element of order 4. This is a contradiction.

□

I have verified by computer that this is the smallest example of a group G with a quasi-monomial character that is not monomial. A complete proof of conjecture 6.1 seems very difficult for general n, although I have discovered a proof that $SU(n, 2)$ has an irreducible character of the given degree using Deligne-Lusztig theory.

References

[1] J. Serre."*Linear Representations of Finite Groups*". Springer, NY, 1977.

[2] Isaacs, I. Martin. "*Character Theory of Finite Groups.*" (1976).

[3] Clifford, A. H. (1937), "Representations Induced in an Invariant Subgroup", Annals of Mathematics, Second Series, Annals of Mathematics, 38 (3):

[4] E. Artin: Uber eine neue Art von L-reihen, *Hamb. Abh.*, (1923), 89-108.

[5] Lang, Serge. "*Algebraic Number Theory.*" (1971).

[6] Prasad D., Yogananda C.S. (2000) A Report on Artin's Holomorphy Conjecture. In: Bambah R.P., Dumir V.C., Hans-Gill R.J. (eds) Number Theory. Hindustan Book Agency, Gurgaon.

[7] D. S. Dummit and R. M. Foote. *Abstract Algebra, Third Edition*. John Wiley & Sons, Inc., NJ, 2004.

[8] K. Taketa. Über die Gruppen, deren Darstellungen sich sämtlich auf monomiale Gestalt. Proc. Imp. Acad., Vol. 6, No. 2(1930), 31–33.

[9] McHugh, John, "*Monomial Characters of Finite Groups*" (2016). Graduate College Dissertations and Theses. 572.

[10] Roger W. Carter, "*Finite groups of Lie type. Conjugacy classes and complex characters*". Bull. Amer. Math. Soc. (N.S.) 17 (1987)