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Sobolev norm estimates of the time dependent Schrödinger equation

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1 Introduction

This thesis presents regularity estimates for solutions to the free time dependent fractional Schrödinger equation with initial data using the theory of Fourier transforms. As such our main focus in this thesis are the functions $S^a f$, where

$$(S^a f)[x](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} e^{i|\xi|^a t} \hat{f}(\xi) \mathrm{d}\xi \text{ for } a \geqslant 2. \tag{1}$$

These functions are solutions to the free time dependent fractional Schrödinger equation with initial data f, i.e. they satisfies that

$$i\partial_t u(x,t) = (-\partial_x^2)^{a/2} u(x,t)$$

with $u(x,0) = f(x)$. As is indicated by the factor $\hat{f}$ in the expression for $S^a f$, these solutions are derived by the use of the Fourier transform, which we define as

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx.$$

The problem at hand is to find regularity estimates for $S^a f$ using the initial datum. To tackle this we restrict our focus to problems where we can guarantee the existence of the Fourier transform of our data function. This restriction, while seemingly severe, helps us establish tools and general theory for our estimates which we then can expand to hold for less restricted functions.

We assume throughout this text that our data function f is in the Schwartz space on $\mathbb{R}$, denoted $\mathcal{S}(\mathbb{R})$; a space of smooth functions whose derivatives decrease more rapidly than any polynomial. This space is dense in $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$ and on this space the Fourier transform is a well defined continuous linear operator mapping $\mathcal{S}(\mathbb{R})$ onto itself and for such function we can apply the Plancherel theorem.

In the case of $a = 2$, this is a solution to the partial differential equation called the free time dependent Schrödinger equation, which states that $\partial_x^2 u = i\partial_t u$ [14]. We will derive a solution later in section 2.6 to the similar but time reversed equation $-\partial_x^2 u = i\partial_t u$.

The idea behind the regularity estimates comes from the fact that if you have an operator T and know that $\|Tf\|_B \leqslant C \|f\|_A$ for some number C we can draw the conclusion that if $f \in A$, then $Tf \in B$ and as such we have gained knowledge about Tf. For our estimates a number of different norms will be used, the main ones which are the $L^p(\mathbb{R})$-, $\dot{H}^s(\mathbb{R})$- and $L^\infty(\mathbb{R}, L^p(\mathbb{R}))$-norms, where

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(x)|^p \right)^{1/p} \text{ for } 1 \leqslant p \leqslant \infty,$$

$$\|f\|_{\dot{H}^s(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |\xi|^{2s} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2},$$

and

$$\|f\|_{L^\infty(\mathbb{R},L^p(\mathbb{R}))} = \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |f(x,t)|^p \,\mathrm{d}t \right)^{1/p}.$$

To acquire our result a recurring method throughout this thesis is to observe the effects of dilation of the normed function and different relationships between a function and its Fourier transform, such as the Plancherel theorem and Pitt's inequality.

2 Background

In this text we will use the Fourier transform and its properties on functions in certain spaces to reach conclusions about differential equations. To make sure that all this makes sense we will start with the space of smooth functions such that they and all of their derivatives vanish faster than any polynomial at the infinities. This space is called the Schwartz space and on this the Fourier transform is a well defined bounded linear operator and has some useful properties for us. We will later discuss how to expand to functions beyond this space.

2.1 Schwartz space

The Schwartz space, denoted as $\mathcal{S}(\mathbb{R})$, is the space of all smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that for all natural numbers α and N there is a constant $C_{\alpha,N}$ for which

$$|\partial^\alpha f(x)| \leqslant C_{\alpha,N}(1 + |x|)^{-N}$$

uniformly in $\mathbb{R}$ [1, p.13].

Example 2.1. *The Gaussian bell function $G(x) = e^{-x^2/2}$ belongs to $\mathcal{S}(\mathbb{R})$ which we see by noting that G is smooth and for all natural numbers M we have that*

$$G(x) \leqslant C_M(1 + |x|)^{-M} \tag{2}$$

for some number C_M. And so, given positive integers α, N, there exists numbers C_α from the Leibniz product rule and C_N by (2) above such that

$$\begin{aligned} |\partial^\alpha G(x)| &\leqslant C_\alpha |(1 + |x|)^\alpha G(x)| \\ &\leqslant C_\alpha(1 + |x|)^\alpha C_N(1 + |x|)^{-N-\alpha} = C_{\alpha,N}(1 + |x|)^{-N}, \end{aligned}$$

where $C_{\alpha,N} = C_\alpha \cdot C_N$, showing that $G \in \mathcal{S}(\mathbb{R})$.

We call the functions in $\mathcal{S}(\mathbb{R})$ Schwartz functions and equip the Schwartz space with the semi norms $|f|_{m,\mathcal{S}}$ for $m \in \mathbb{N}$, where

$$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta|\leqslant m} \sup_{x\in\mathbb{R}} \left| x^\alpha \partial^\beta f(x) \right|.$$

These semi norms are defined such as if $|f|_{m,\mathcal{S}} < \infty$ for all $m \in \mathbb{N}_0$ then $f \in \mathcal{S}(\mathbb{R})$ and using them we define convergence in $\mathcal{S}(\mathbb{R})$.

Definition 2.2. *Let* $\{f_k\}_k \subset \mathcal{S}(\mathbb{R})$ *for* $k \in \mathbb{N}$ *be a sequence of functions in* $\mathcal{S}(\mathbb{R})$. *We say that* $\{f_k\}_k$ *converges to* f *in* $\mathcal{S}(\mathbb{R})$ *iff for all* $\varepsilon > 0$ *and all* $m \in \mathbb{N}$ *there is a* $K \in \mathbb{N}$ *such that* $|f_k - f|_{m,\mathcal{S}} < \varepsilon$ *whenever* $k \geqslant K$.

We note that for $p \geqslant 1$ we have that $\mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$. This is a direct result of the definition of the Schwartz space, since for $f \in \mathcal{S}(\mathbb{R})$ there is a number C such that $|f(x)| \leqslant C(1+|x|)^{-2}$ and so we have

$$\|f\|_{L^p(\mathbb{R})}^p = \int_{-\infty}^{\infty} |f(x)|^p \, dx \leqslant C \int_{-\infty}^{\infty} (1+|x|)^{-2p} dx < \infty.$$

A fundamental property of the Schwartz functions is that they vanish rapidly at infinity and an important subspace of $\mathcal{S}(\mathbb{R})$ is $C_0^\infty(\mathbb{R})$ which is the space of smooth functions with compact support, i.e. vanish outside a compact set. Here we define the support of a function f, written as $\operatorname{supp}(f)$, by

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} | f(x) \neq 0\}}.$$

That is, the support of a function is the closure of the set of points on which the function is nonzero.

Example 2.3. *The bump function* φ *defined by*

$$\varphi(x) = \begin{cases} e^{-1/(1-x^2)} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geqslant 1 \end{cases}$$

is a smooth function of compact support. Here $\operatorname{supp}(\varphi) = \overline{(-1,1)} = [-1,1]$.

The space $C_0^\infty(\mathbb{R})$ has the useful property that it is dense in $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$ [15, p.13], and from this and the fact that $\mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$ we can draw the following conclusion.

Lemma 2.4. $\mathcal{S}(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $1 \leqslant p < \infty$.

And so, while we will restrict our results to a space of nicely behaving functions, it is feasible to expand our results to a wider space of functions by density.

2.2 Fourier transform

Our main focus on the Schwartz space is as mentioned the Fourier transform. For functions in the Schwartz space we denote by $\hat{f}$ the Fourier transform of a function f and define the transform as [1, p.9]

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx.$$

We can see that, given $f \in \mathcal{S}(\mathbb{R})$, we have a first indication why this definition makes sense by noting that

$$\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx\right| \leqslant \int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$$

Example 2.5. *The Gaussian bell function $G(x) = e^{-x^2/2}$ is a Schwartz function as seen in example 2.1 and as such we can calculate its Fourier transform. We have that*

$$\begin{aligned}\widehat{G}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x - x^2/2}dx = \int_{-\infty}^{\infty} e^{-(x+i\xi)^2/2-\xi^2/2}dx \\ &= e^{-\xi^2/2}\int_{-\infty}^{\infty} e^{-(x+i\xi)^2/2}dx = e^{-\xi^2/2}\int_{\gamma} e^{-z^2/2}dz,\end{aligned}$$

where γ is the curve in $\mathbb{C}$ parametrized by $z(t) = t + i\xi$ for $t \in \mathbb{R}$. The integrand is analytic and vanishes as $|z| \to \infty$ and so we get that

$$\int_{\gamma} e^{-z^2/2}dz = \int_{-\infty}^{\infty} e^{-z^2/2}dz = \sqrt{2\pi},$$

where we use the known fact that $\int_{-\infty}^{\infty} e^{x^2}dx = \sqrt{\pi}$. This gives us that $\widehat{G}(\xi) = e^{-\xi^2/2} \cdot \sqrt{2\pi} = \sqrt{2\pi}G(\xi)$.

Now, define the operator D as $D = -i\partial$. From the definition above we get the following properties for the Fourier transform.

Theorem 2.6. *Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then*

1. $(\widehat{\alpha\varphi + \beta\psi})(\xi) = \alpha\widehat{\varphi}(\xi) + \beta\widehat{\psi}(\xi)$ *for* $\alpha, \beta \in \mathbb{R}$
2. $(\widehat{D^n\varphi})(\xi) = \xi^n\widehat{\varphi}(\xi)$
3. $D^n\widehat{\varphi}(\xi) = ((\widehat{-x)^n\varphi})(\xi)$
4. $\widehat{\varphi} \in \mathcal{S}(\mathbb{R})$

Proof.

1. From the linearity of integration we have that

$$\begin{aligned}(\widehat{\alpha\varphi + \beta\psi})(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}(\alpha\varphi(x) + \beta\psi(x))dx \\ &= \alpha\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)dx + \beta\int_{-\infty}^{\infty} e^{-i\xi x}\psi(x)dx \\ &= \alpha\widehat{\varphi}(\xi) + \beta\widehat{\psi}(\xi).\end{aligned}$$

2. We begin by noting that since $\varphi \in \mathcal{S}(\mathbb{R})$ we have that $\left[\xi^k e^{-i\xi x}D^m\varphi(x)\right]_{-\infty}^{\infty} = 0$ for all $k, m \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$. Using this, integration by parts now gives us that

$$(\widehat{D^n\varphi})(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}D^n\varphi(x)dx = \xi^n\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x) = \xi^n\widehat{\varphi}(\xi).$$

3. By the Leibniz rule we have that

$$\begin{aligned}D^n\widehat{\varphi}(\xi) &= D^n\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)dx = \int_{-\infty}^{\infty} D^n e^{-i\xi x}\varphi(x)dx \\ &= \int_{-\infty}^{\infty} e^{-i\xi x}(-x)^n\varphi(x)dx = ((\widehat{-x)^n\varphi})(\xi).\end{aligned}$$

4. From Lemma 2.4 we have that if $\varphi \in \mathcal{S}(\mathbb{R})$ then

$$|\widehat{\varphi}(\xi)| = \left|\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)dx\right| \leqslant \int_{-\infty}^{\infty} |\varphi(x)|\,dx = \|\varphi\|_{L^1(\mathbb{R})} < \infty$$

and furthermore we can note that $D^n x^m \varphi \in \mathcal{S}(\mathbb{R})$ for $n, m \in \mathbb{N}_0$. Now, using 2. and 3. above, we have that

$$\begin{aligned}
|\widehat{\varphi}|_{m,\mathcal{S}} &= \sup_{|\alpha|+|\beta|\leqslant m} \sup_{\xi\in\mathbb{R}} \left|\xi^\alpha D^\beta \widehat{\varphi}(\xi)\right| \\
&= \sup_{|\alpha|+|\beta|\leqslant m} \sup_{\xi\in\mathbb{R}} \left|\xi^\alpha (\widehat{(-x)^\beta\varphi})(\xi)\right| \\
&= \sup_{|\alpha|+|\beta|\leqslant m} \sup_{\xi\in\mathbb{R}} \left|(\widehat{D^\alpha(-x)^\beta\varphi})(\xi)\right| \\
&\leqslant \sup_{|\alpha|+|\beta|\leqslant m} \left\|D^\alpha(-x)^\beta\varphi\right\|_{L^1(\mathbb{R})} < \infty.
\end{aligned}$$

□

Using the following boundedness lemma we can make an important observation about the Fourier transform.

Lemma 2.7. *[1, p.14] For $\varphi \in \mathcal{S}(\mathbb{R})$ and $m \in \mathbb{N}$ there is a constant C_m such that*

$$|\widehat{\varphi}|_{m,\mathcal{S}} \leqslant C_m\,|\varphi|_{m+2,\mathcal{S}}$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}
\|\varphi\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\varphi(x)|\,dx = \int_{-\infty}^{\infty} (1+|x|)^{-2}\left|(1+|x|)^2\varphi(x)\right| dx \\
&\leqslant |\varphi|_{2,\mathcal{S}} \int_{-\infty}^{\infty} (1+|x|)^{-2}dx = C\,|\varphi|_{2,\mathcal{S}}\,.
\end{aligned}$$

Using this we have

$$\begin{aligned}
|\widehat{\varphi}|_{0,\mathcal{S}} &= \sup_{\xi\in\mathbb{R}} |\widehat{\varphi}(\xi)| = \sup_{\xi\in\mathbb{R}} \left|\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)dx\right| \\
&\leqslant \sup_{\xi\in\mathbb{R}} \int_{-\infty}^{\infty} |\varphi(x)|\,dx = \|\varphi\|_{L^1(\mathbb{R})} \leqslant C\,|\varphi|_{2,\mathcal{S}}\,.
\end{aligned}$$

Furthermore we have by theorem 2.6 that

$$\xi^\alpha D_\xi^\beta \widehat{\varphi}(\xi) = (\widehat{D_x^\alpha(-x^\beta)\varphi})(\xi)$$

and by the Leibniz rule we have that

$$\left|D_x^\alpha(-x^\beta)\varphi(x)\right| \leqslant \sum_{k\leqslant\alpha} \binom{\alpha}{k} \left|(D^k x^\beta)(D^{\alpha-k}\varphi(x))\right| \leqslant C_{\alpha,\beta}\left|x^\beta D^\alpha\varphi(x)\right|.$$

Using this and taking the supremum over α and β, we have that

$$\sup_{\alpha+\beta\leqslant m}\sup_{\xi\in\mathbb{R}}\left|\xi^{\alpha}D_{\xi}^{\beta}\hat{\varphi}\right| \leqslant \sup_{\alpha+\beta\leqslant m} C_{\alpha,\beta}\left|D_{x}^{\alpha}x^{\beta}\varphi\right|_{2,\mathcal{S}}$$
$$\leqslant \sup_{\alpha+\beta\leqslant m+2}\sup_{\xi\in\mathbb{R}} C_{\alpha,\beta}\left|D_{x}^{\alpha}x^{\beta}\varphi\right| \leqslant C_m\left|\varphi\right|_{m+2,\mathcal{S}}.$$

That is, we have that $|\hat{\varphi}|_{m,\mathcal{S}} \leqslant C_m|\varphi|_{m+2,\mathcal{S}}$, and so we have that $\hat{\varphi}\in\mathcal{S}(\mathbb{R})$ whenever $\varphi\in\mathcal{S}(\mathbb{R})$. □

That is, for a Schwartz function its Fourier transform is bounded by the function itself which, together with the linearity of the transform, helps us establish that the transform is in fact continuous.

Theorem 2.8. *Given $\{\varphi_k\}_k \subset \mathcal{S}(\mathbb{R})$ such that $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R})$ for some $\varphi \in \mathcal{S}(\mathbb{R})$ as $k\to\infty$, then $\widehat{\varphi_k}\to\hat{\varphi}$ in $\mathcal{S}(\mathbb{R})$ as $k\to\infty$, i.e. the Fourier transform is continuous on $\mathcal{S}(\mathbb{R})$.*

Proof. Assume $\varphi_k\to\varphi$ in $\mathcal{S}(\mathbb{R})$ as $k\to\infty$, then $\hat{\varphi}\in\mathcal{S}(\mathbb{R})$ and for all $k\in\mathbb{N}$ we have that $\hat{\varphi}_k\in\mathcal{S}(\mathbb{R})$. Furthermore, we have from lemma 2.7 that there is a constant C_m such that

$$\left|\widehat{(\varphi_k-\varphi)}\right|_{m,\mathcal{S}} \leqslant C_m\left|\varphi_k-\varphi\right|_{m+2,\mathcal{S}}.$$

Now, given $\varepsilon>0$ and $m\in\mathbb{N}$, there is a $K\in\mathbb{N}$ such that

$$\left|\varphi_k-\varphi\right|_{m+2,\mathcal{S}} < \frac{\varepsilon}{C_m}$$

whenever $k\geqslant K$. Thus, using the linearity of the transform, we have that

$$\left|\widehat{\varphi_k}-\hat{\varphi}\right|_{m,\mathcal{S}} = \left|\widehat{(\varphi_k-\varphi)}\right|_{m,\mathcal{S}} \leqslant C_m\left|\varphi_k-\varphi\right|_{m+2,\mathcal{S}} < \varepsilon$$

whenever $k\geqslant K$. □

And so by Theorem 2.6, Lemma 2.7 and Lemma 2.8 we have that on $\mathcal{S}(\mathbb{R})$ the Fourier transform is a continuous linear operator mapping $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. Furthermore, it has the important property of transforming differentiation of a function to multiplication of its transform with a polynomial. This is useful in solving differential equations and we will later use this to define other spaces on which the concept of pseudo differential equations makes sense. Before this, however, the Fourier transform has other important properties we need.

We begin with translations, modulations and dilations, which we for a function φ write as $T_y\varphi$ for a translation of φ by a factor y, where

$$(T_y\varphi)(x) := \varphi(x+y),$$

and $M_y\varphi$ for a modulation of φ by a factor y, where

$$(M_y\varphi)(x) := e^{ixy}\varphi(x),$$

and finally $D_\beta\varphi$ for dilation of φ by a factor β, where

$$(D_\beta\varphi)(x) := \varphi(\beta x).$$

These are connected by the Fourier transform and are useful tools in interpreting the transform and analysing its effect on functions.

Theorem 2.9. *Given $\varphi \in \mathcal{S}(\mathbb{R})$, then*

1. $\widehat{(T_y\varphi)}(\xi) = e^{i\xi y}\widehat{\varphi}(\xi) = (M_y\widehat{\varphi})(\xi)$,
2. $\widehat{(M_y\varphi)}(\xi) = \widehat{\varphi}(\xi - y) = (T_{-y}\widehat{\varphi})(\xi)$,
3. $\widehat{D_\beta\varphi}(\xi) = \frac{1}{\beta}\widehat{\varphi}(\frac{\xi}{\beta}) = \beta^{-1}(D_{\beta^{-1}}\varphi)(\xi)$.

Proof. All three parts relies entirely on a change of variables.

1. Setting $t = x + y$ gives us that

$$\begin{aligned}\widehat{(T_y\varphi)}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}(T_y\varphi)(x)dx = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x+y)dt \\ &= \int_{-\infty}^{\infty} e^{-i\xi(t-y)}\varphi(t)dt = e^{i\xi y}\int_{-\infty}^{\infty} e^{-i\xi t}\varphi(t)dt \\ &= e^{i\xi y}\widehat{\varphi}(\xi) = (M_y\widehat{\varphi})(\xi).\end{aligned}$$

2. Setting $t = x - y$ gives us that

$$\begin{aligned}\widehat{(M_y\varphi)}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}(M_y\varphi)(x)dx = \int_{-\infty}^{\infty} e^{-i\xi x}e^{ixy}\varphi(x)dx \\ &= \int_{-\infty}^{\infty} e^{-ix(\xi-y)}\varphi(x)dx = \widehat{\varphi}(\xi - y) = (T_{-y}\widehat{\varphi})(\xi).\end{aligned}$$

3. Setting $y = \beta x$ gives us that

$$\begin{aligned}\widehat{(D_\beta\varphi)}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}(D_\beta\varphi)dx = \int_{-\infty}^{\infty} e^{-i\frac{\xi}{\beta}\beta x}\varphi(\beta x)dx \\ &= \frac{1}{\beta}\int_{-\infty}^{\infty} e^{-i\frac{\xi}{\beta}y}\varphi(y)dy = \frac{1}{\beta}\widehat{\varphi}\left(\frac{\xi}{\beta}\right).\end{aligned}$$

□

We continue with convolutions. Given φ and ψ in $\mathcal{S}(\mathbb{R})$ the convolution $\varphi * \psi$ is defined as [12, p. 139]

$$(\varphi * \psi)(x) := \int_{-\infty}^{\infty} \varphi(x-t)\psi(t)dt.$$

For the Fourier transform we have that the transform of a convolution of two functions corresponds with a multiplication of their transforms.

Theorem 2.10. *Given φ, ψ in $\mathcal{S}(\mathbb{R})$, then*

$$\widehat{(\varphi * \psi)}(\xi) = \hat{\varphi}(\xi)\hat{\psi}(\xi)$$

Proof. Using Fubini's theorem [7, p. 86] and theorem 2.9 above we have that

$$\begin{aligned}
\widehat{(\varphi * \psi)}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}(\varphi * \psi)(x)dx = \int_{-\infty}^{\infty} e^{-i\xi x} \left(\int_{-\infty}^{\infty} \varphi(x-t)\psi(t)dt \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-i\xi x}(T_{-t}\varphi)(x)dx \right) \psi(t)dt = \int_{-\infty}^{\infty} (M_{-t}\hat{\varphi})(\xi)\psi(t)dt \\
&= \int_{-\infty}^{\infty} e^{-i\xi t}\hat{\varphi}(\xi)\psi(t)dt = \hat{\varphi}(\xi) \int_{-\infty}^{\infty} e^{-i\xi t}\psi(t)dt = \hat{\varphi}(\xi)\hat{\psi}(\xi).
\end{aligned}$$

□

As we will see it's possible when using the Fourier transform to solve differential equations to arrive at a transformed solution that is written as a product of two functions. Using the inverse transform we will soon define we can then use the result in Theorem 2.10 above to arrive at a solution, now written as a convolution.

We can note that while Theorem 2.6 gives us that the Fourier transform maps the Schwartz space to itself the transform does not reverse this action, i.e. in general $\hat{\hat{\varphi}}(x) \neq \varphi(x)$. While this is somewhat inconvenient another useful property of the Fourier transform is the existence of an inversion formula that gives us the inverse transform so that we can regain φ from $\hat{\varphi}$, circumventing this problem. However, to show this we need the next two lemmas, the first called the adjoint or multiplication lemma.

Lemma 2.11. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \varphi(x)\hat{\psi}(x)dx = \int_{-\infty}^{\infty} \hat{\varphi}(x)\psi(x)dx$$

Proof. Using Fubini's theorem we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(x)\hat{\psi}(x)dx &= \int_{-\infty}^{\infty} \varphi(x) \left(\int_{-\infty}^{\infty} e^{-ixy}\psi(y)dy \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x)e^{-ixy}\psi(y)dy \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x)e^{-ixy}dx \right) \psi(y)dy \\
&= \int_{-\infty}^{\infty} \hat{\varphi}(y)\psi(y)dy.
\end{aligned}$$

□

The second lemma, while not directly connected with the Fourier transform, describes a useful property of convolutions.

Theorem 2.12. *Given* $\varphi \in L^1(\mathbb{R})$ *such that* $\int_{-\infty}^{\infty} \varphi(x)dx = 1$, *let* $\varphi_\varepsilon(x) = \varepsilon^{-1} D_{\varepsilon^{-1}} \varphi(x)$. *Then for any* $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$ *we have that*

$$f * \varphi_\varepsilon \to f \text{ in } L^p(\mathbb{R}) \text{ as } \varepsilon \to 0.$$

Proof. Using the change of variables $\tau = \varepsilon^{-1} t$, Fubini's theorem and Minkowski's integral inequality [5, p. 101] we have that

$$\begin{aligned}
\|f * \varphi_\varepsilon - f\|_{L^p(\mathbb{R})} &= \left(\int_{-\infty}^{\infty} |(f * \varphi_\varepsilon)(x) - f(x)|^p \, dx \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-t)\varphi_\varepsilon(t)dt - 1 \cdot f(x) \right|^p dx \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-t)\varphi(\varepsilon^{-1}t)\varepsilon^{-1}dt - 1 \cdot f(x) \right|^p dx \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-\varepsilon\tau)\varphi(\tau)d\tau - \int_{-\infty}^{\infty} \varphi(\tau)d\tau \cdot f(x) \right|^p dx \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} (f(x-\varepsilon\tau) - f(x))\varphi(\tau)d\tau \right|^p dx \right)^{1/p} \\
&\leqslant \int_{-\infty}^{\infty} |\varphi(\tau)| \left(\int_{-\infty}^{\infty} |f(x-\varepsilon\tau) - f(x)|^p \, dx \right)^{1/p} d\tau \\
&= \int_{-\infty}^{\infty} |\varphi(\tau)| \, \|T_{-\varepsilon\tau} f - f\|_{L^p(\mathbb{R})} \, d\tau.
\end{aligned}$$

Now since $\|T_{-\varepsilon\tau} f - f\|_{L^p(\mathbb{R})} \leqslant 2\|f\|_{L^p(\mathbb{R})}$ and $\|T_{-\varepsilon\tau} f - f\|_{L^p(\mathbb{R})} \to 0$ as $\varepsilon \to 0$ we have by Lebesgue dominated convergence theorem that $\|f * \varphi_\varepsilon - f\|_{L^p(\mathbb{R})} \to 0$ as $\varepsilon \to 0$. □

And so we can now state the Fourier inverse formula which gives us a tool to retrieving a transformed function.

Theorem 2.13. *Let* $\varphi \in \mathcal{S}(\mathbb{R})$, *then*

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{\varphi}(\xi) d\xi.$$

Remark: We denote in this formula the action of $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\varphi(\xi)d\xi$ on φ in the right hand side above by the symbol $^\vee$ so that the formula above can be written as

$$\varphi(x) = (\widehat{\varphi})^\vee(x).$$

Proof. Define φ_ε by

$$\varphi_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x - \varepsilon^2 \xi^2/2} \widehat{\varphi}(\xi) d\xi$$

and let

$$g(\xi) = e^{ix\xi}e^{-\varepsilon^2\xi^2/2} = (M_x D_\varepsilon G)(\xi),$$

where G is the Gaussian bell function $G(\xi) = e^{-\xi^2/2}$ as of example 2.5. Then, by Theorem 2.9 on G and since $\widehat{G}(\xi) = \sqrt{2\pi}G(\xi)$ (by example 2.5), we have that

$$\widehat{g}(\zeta) = \varepsilon^{-1}(T_{-x}D_{\varepsilon^{-1}}\widehat{G})(\zeta) = \sqrt{2\pi}\varepsilon^{-1}e^{-(\zeta-x)^2/2\varepsilon^2}.$$

Since

$$\varphi_\varepsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\widehat{\varphi}(\xi)d\xi$$

we now have by Lemma 2.11 that

$$\begin{aligned}\varphi_\varepsilon(x) &= \frac{1}{2\pi\varepsilon}\int_{-\infty}^{\infty}\widehat{g}(\xi)\varphi(\xi)d\xi\\ &= \frac{\sqrt{2\pi}\varepsilon^{-1}}{2\pi}\int_{-\infty}^{\infty} e^{-(\xi-x)^2/2\varepsilon^2}\varphi(\xi)d\xi\\ &= \left(\varphi * \varepsilon^{-1}D_{\varepsilon^{-1}}\frac{G}{\sqrt{2\pi}}\right)(x).\end{aligned}$$

Since $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}G(x)dx = 1$ we now have by Lemma 2.12 that in $L^p(\mathbb{R})$

$$\varphi_\varepsilon(x) \to \varphi \text{ as } \varepsilon \to 0.$$

Finally, Lebesgue dominated convergence gives us that

$$\varphi_\varepsilon \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\widehat{\varphi}(\xi)d\xi \text{ as } \varepsilon \to 0$$

and so we have that

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\widehat{\varphi}(\xi)d\xi.$$

□

With this inversion formula, we now have that the Fourier transform is a continuous bijective mapping on $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$ [12].

The next theorem is one of the major theorems for the Fourier transform and one of our main tools when establishing our future results.

Theorem 2.14 (Plancherel theorem)**.** *Given* $\varphi \in \mathcal{S}(\mathbb{R})$*, then*

$$\sqrt{2\pi}\,\|\varphi\|_{L^2(\mathbb{R})} = \|\widehat{\varphi}\|_{L^2(\mathbb{R})}.$$

Proof. We prove this by first proving a more general case, i.e. that

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = 2\pi\int_{-\infty}^{\infty}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi.$$

Using Fubini's theorem and the Fourier inversion formula we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi &= \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} e^{-i\xi x}\overline{\widehat{g}(\xi)}d\xi\right)dx \\ &= \int_{-\infty}^{\infty} f(x)\overline{\left(\int_{-\infty}^{\infty} e^{i\xi x}\widehat{g}(\xi)d\xi\right)}dx = 2\pi\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.\end{aligned}$$

and so for $g(x) = f(x)$ we get the Plancherel theorem. □

This gives us a useful tool in analysing the solutions to differential equations as it allows us to switch between a function and its transform somewhat when analysing the regularity of the function. Furthermore we can make a direct observations from the Plancherel theorem and the fact that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Corollary 2.15. *There is a unique continuous extension of the Fourier transform from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$, and as such the Plancherel theorem is extended to functions in $L^2(\mathbb{R})$.*

Proof. Given $f \in L^2(\mathbb{R})$, by Lemma 2.4 there is a sequence $\{\varphi_k\}_k$ in $\mathcal{S}(\mathbb{R})$ converging to f in $L^2(\mathbb{R})$. We have by Lemma 2.6 that $\{\widehat{\varphi}_k\}_k$ is also a sequence in $\mathcal{S}(\mathbb{R})$ and by linearity of the transform and the Plancherel theorem we have that

$$\|\widehat{\varphi}_m - \widehat{\varphi}_n\|_{L^2(\mathbb{R})} = \left\|\widehat{\varphi_m - \varphi_n}\right\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\,\|\varphi_m - \varphi_n\|_{L^2(\mathbb{R})},$$

and so we have that $\|\widehat{\varphi}_m - \widehat{\varphi}_n\|_{L^2(\mathbb{R})} \to 0$ as $m, n \to \infty$. By the completeness of $L^2(\mathbb{R})$ we now have that there is a function g_φ in $L^2(\mathbb{R})$ such that $\widehat{\varphi}_k \to g_\varphi$ in $L^2(\mathbb{R})$.
Now, given another sequence $\{\psi_k\}_k$ in $\mathcal{S}(\mathbb{R})$ such that $\psi_k \to f$ in $L^2(\mathbb{R})$ we have by the same argument as above that there is a function g_ψ in $L^2(\mathbb{R})$ such that $\widehat{\psi}_k \to g_\psi$ in $L^2(\mathbb{R})$. Furthermore, by the triangle inequality and Plancherel theorem, we have that

$$\begin{aligned}\left\|\widehat{\varphi}_n - \widehat{\psi}_n\right\|_{L^2(\mathbb{R})} &= \sqrt{2\pi}\,\|\varphi_n - \psi_n\|_{L^2(\mathbb{R})} \\ &\leqslant \sqrt{2\pi}(\|\varphi_n - f\|_{L^2(\mathbb{R})} + \|f - \psi_n\|_{L^2(\mathbb{R})}) \to 0 \text{ as } n \to \infty\end{aligned}$$

and so

$$\|g_\varphi - g_\psi\|_{L^2(\mathbb{R})} \leqslant \|g_\varphi - \widehat{\varphi}_m\|_{L^2(\mathbb{R})} + \left\|\widehat{\psi}_n - g_\psi\right\|_{L^2(\mathbb{R})} + \left\|\widehat{\varphi}_m - \widehat{\psi}_m\right\|_{L^2(\mathbb{R})}.$$

Letting $m, n \to \infty$ now gives us that

$$\|g_\varphi - g_\psi\|_{L^2(\mathbb{R})} = 0$$

i.e. $g_\varphi = g_\psi$ in $L^2(\mathbb{R})$. We now have that given f in $L^2(\mathbb{R})$ there exists a unique function $\mathcal{F}f$ in $L^2(\mathbb{R})$ such that for every sequence $\{\varphi_k\}_k$ in $\mathcal{S}(\mathbb{R})$ converging to f in $L^2(\mathbb{R})$ the sequence $\{\widehat{\varphi}_k\}_k$ in $\mathcal{S}(\mathbb{R})$ converges to $\mathcal{F}f$ in $L^2(\mathbb{R})$.

Finally, we have that

$$\begin{aligned}\|\mathcal{F}f\|_{L^2(\mathbb{R})} &\leqslant \|\widehat{\varphi}_k - \mathcal{F}f\|_{L^2(\mathbb{R})} + \|\widehat{\varphi}_k\|_{L^2(\mathbb{R})} \\ &= \|\widehat{\varphi}_k - \mathcal{F}f\|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{2\pi}} \|\varphi_k\|_{L^2(\mathbb{R})}\end{aligned}$$

and so, by letting $k \to \infty$ above we have that

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})} \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_{L^2(\mathbb{R})}.$$

Furthermore

$$\begin{aligned}\|f\|_{L^2(\mathbb{R})} &\leqslant \|\varphi_k - f\|_{L^2(\mathbb{R})} + \|\varphi_k\|_{L^2(\mathbb{R})} \\ &= \|\varphi_k - f\|_{L^2(\mathbb{R})} + \sqrt{2\pi} \|\widehat{\varphi}_k\|_{L^2(\mathbb{R})}.\end{aligned}$$

and so, by once again letting $k \to \infty$ we have that

$$\|f\|_{L^2(\mathbb{R})} \leqslant \sqrt{2\pi} \|\mathcal{F}f\|_{L^2(\mathbb{R})}.$$

That is, for f in $L^2(\mathbb{R})$ we have that

$$\|f\|_{L^2(\mathbb{R})} = \sqrt{2\pi} \|\mathcal{F}f\|_{L^2(\mathbb{R})}.$$

□

This gives us that there is a extension $\mathcal{F}$ for the Fourier transform from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$ such that for $\varphi \in \mathcal{S}(\mathbb{R})$ we have that $\mathcal{F}\varphi = \widehat{\varphi}$. As such we will from now on also use $\mathcal{F}$ to indicate the Fourier transform on functions in $L^2(\mathbb{R})$.

While we now have a space of nicely behaving functions for which the Fourier transform makes sense and has a number of useful properties, not all functions are nicely behaving, e.g. we can't calculate a Fourier transform of or even differentiate all functions, and so to make the transform a bit more useful the restriction to $\mathcal{S}(\mathbb{R})$ need to be relaxed, why the next step is needed.

2.3 Tempered distributions

To generalize the idea of the Fourier transform, we introduce the space of tempered distributions. This space, denoted $\mathcal{S}'(\mathbb{R})$, is the dual space of $\mathcal{S}(\mathbb{R})$, i.e. the set of all linear and continuous mappings $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ in the sense of that f is a tempered distribution if its linear and there exists a constant C and an interger k such that $|f(\varphi)| \leqslant |\varphi|_{k,\mathcal{S}}$ for all $\varphi \in \mathcal{S}(\mathbb{R})$ [2, p. 18]. This allows us to define the Fourier transform on objects in a wider sense than before while retaining and expanding on the core properties of the transform.

Example 2.16. *The Dirac delta function, δ_0, is classically defined to be such that $\delta_0(x) = 0$ if $x \neq 0$, and $\int_{-\infty}^{\infty} \delta_0(x)dx = 1$, i.e. it spikes at zero enough to* *give its integral a value of 1. This is not actually a function, but it is a tempered distribution and as such defined as*

$$\delta_0(\varphi) := \varphi(0) \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

To see that δ_0 is indeed a tempered distribution we have that for all $\varphi \in \mathcal{S}(\mathbb{R})$ and natural numbers k we have that

$$|\delta_0(\varphi)| = |\varphi(0)| \leqslant |\varphi|_{k,\mathcal{S}}.$$

There is a more general case δ_x, the translation of δ_0 by a factor x, which gives us

$$\delta_x(\varphi) := \varphi(x) \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

A core property of this space is that a function f that behaves nicely with all functions in $\mathcal{S}(\mathbb{R})$ (called test functions) can be identified with a tempered distribution T_f by the duality product [1]

$$T_f(\varphi) = \langle T_f, \varphi\rangle := \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

This is linear by the linearity of integration and continuous by the Lebesgue convergence theorem. The important niceness of f here is that this integral above need to make sense. Indeed if $f \in L^1(\mathbb{R})$ is at most of polynomial growth (i.e. $(1+|x|)^{-N}f(x) \in L^1(\mathbb{R})$ for some $N \in \mathbb{N}$), then T_f is a tempered distribution[1].

Remark: For functions f we will write $\langle f, \varphi\rangle$ instead of $\langle T_f, \varphi\rangle$ when identifying f as a tempered distribution and there is no risk of ambiguity.

To see that this duality product does indeed give us a tempered distribution we can note that given $f \in L^1(\mathbb{R})$ of at most polynomial growth of degree k (as described above), then we have that there is a number D such that $\left\|(1+|\cdot|)^k f\right\|_{L^1(\mathbb{R})} \leqslant D$. Furthermore, given $\varphi \in \mathcal{S}(\mathbb{R})$ we have that for $k \in \mathbb{N}$ there exists a number C_k such that $\sup_{x\in\mathbb{R}} \left|(1+|x|)^k\varphi(x)\right| \leqslant C_k |\varphi|_{k,\mathcal{S}}$. We now have that

$$\begin{aligned}
|\langle f, \varphi\rangle| &= \left|\int_{-\infty}^{\infty} f(t)\varphi(t)dt\right| = \left|\int_{-\infty}^{\infty} (1+|t|)^{-k} f(t)(1+|t|)^k\varphi(t)dt\right| \\
&\leqslant \sup_{x\in\mathbb{R}} \left|(1+|x|)^k\varphi(x)\right| \left|\int_{-\infty}^{\infty} (1+|t|)^{-k} f(t)dt\right| \\
&\leqslant \sup_{x\in\mathbb{R}} \left|(1+|x|)^k\varphi(x)\right| \left\|(1+|\cdot|)^k f\right\|_{L^1(\mathbb{R})} \leqslant D \cdot C_k |\varphi|_{k,\mathcal{S}}
\end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$.

2.3.1 Properties of tempered distributions

The tempered distribution behaves linearly [6, p. 6], i.e. for $\lambda, \mu \in \mathbb{C}$, $f, g \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\langle \lambda f + \mu g, \varphi\rangle = \alpha\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle$$

and we define what we mean by differentiation of a tempered distribution by:

Definition 2.17. *If $f \in \mathcal{S}'(\mathbb{R})$, then the derivative of order m of f, $\partial^m f$, is*

$$\langle \partial^k f, \varphi \rangle = (-1)^k \langle f, \partial^k \varphi \rangle$$

Example 2.18. *This definition coincides with the classical differentiation of differentiable functions. Let f be a differentiable function such that its derivative is polynomially bounded. Then, using integration by parts, we get that*

$$\begin{aligned} \langle \partial T_f, \varphi \rangle = -\langle f, \partial \varphi \rangle &= -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx \\ &= \overbrace{\left[f'(x)\varphi(x) \right]_{-\infty}^{\infty}}^{=0} + \int_{-\infty}^{\infty} f'(x)\varphi(x)dx = \langle T_{f'}, \varphi \rangle \end{aligned}$$

and so we se that the differentiation of the tempered distribution defined by f correspond to the tempered distribution defined by the derivative of f, as we would expect if we want to avoid ambiguity.

This definition of differentiation has the benefit of moving the differentiation to the smooth test functions, thus not only giving us a way to differentiate tempered distributions but also expand up on which functions that are differentiable (as identified as a tempered distribution). As such, since they relax the criteria of differentiation, these derivatives are called *distributional derivatives.*

Example 2.19. *The function*

$$f(x) = \begin{cases} 1, & \text{if } n \leqslant x \leqslant n+1 \\ 0, & \text{else} \end{cases}$$

has the distributional derivative $\delta_n - \delta_{n+1}$ since we have that

$$\begin{aligned} \langle \partial f, \varphi \rangle = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx &= -\int_{n}^{n+1} \varphi'(x)dx = \varphi(n) - \varphi(n+1) \\ = \langle \delta_n, \varphi \rangle - \langle \delta_{n+1}, \varphi \rangle &= \langle \delta_n - \delta_{n+1}, \varphi \rangle \end{aligned}$$

for all test functions φ.

2.3.2 Fourier transform of tempered distributions

One of the main reasons for us to introduce tempered distributions is to expand the use of the Fourier transform. To do this we define the Fourier transform of $f \in \mathcal{S}'(\mathbb{R})$ as the distribution identified through passing the transform over to the test functions. That is, given $f \in \mathcal{S}'(\mathbb{R})$, we define the Fourier transform of f, by [1]

$$\langle \mathcal{F}[f], \varphi \rangle := \langle f, \mathcal{F}[\varphi] \rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R})$$

and, similarly, for the inverse

$$\langle \mathcal{F}^{-1}[f], \varphi \rangle := \langle f, \mathcal{F}^{-1}[\varphi] \rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Since our test functions are Schwartz functions this definition lets us keep properties of the Fourier transform while allowing us to calculate a Fourier transform of any tempered distribution [1, p .24], which contains functions that are not necessarily compatible with the first idea of the Fourier transform, such as the constant functions.

Example 2.20. *The Fourier transform of the Dirac delta function δ_0:*
We have that

$$\langle \mathcal{F}[\delta_0], \varphi \rangle = \langle \delta_0, \mathcal{F}[\varphi] \rangle = \mathcal{F}[\varphi](0) = \int_{-\infty}^{\infty} 1 \cdot \varphi(x) dx = \langle 1, \varphi \rangle$$

and so $\mathcal{F}[\delta_0]$ is identified with the constant function 1.

Example 2.21. *The fourier transform of the constant function $f(x) = 1$:*
We have that

$$\langle \mathcal{F}[1], \varphi \rangle = \langle 1, \mathcal{F}[\varphi] \rangle = \int_{-\infty}^{\infty} \widehat{\varphi}(\xi) d\xi = \varphi(0) = \langle \delta_0, \varphi \rangle$$

and so $\mathcal{F}[1]$ is the δ_0 distribution.

The Fourier transform keeps the property of being a linear and continuous operator in $\mathcal{S}'(\mathbb{R})$ in the sense of the following lemma given by Bahoiuri et al [2, prop. 1.23].

Lemma 2.22. *Given a sequence $\{f_k\}_k$ in $\mathcal{S}'(\mathbb{R})$ that converges to f in $\mathcal{S}'(\mathbb{R})$, then $\{\mathcal{F}[f_k]\}_k$ converges to $\mathcal{F}[f]$.*

As stated before, an important tool of the Fourier transform is that of convolutions and for tempered distributions we define them by once again passing along the operation to the test functions. As such we can define a convolution between a distribution and a Schwartz function as follows.
Let $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, then $f * g \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\langle f * g, \varphi \rangle \quad := \quad \langle f, \tilde{g} * \varphi \rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R})$$

where $\tilde{g}(x) := g(-x)$ for all $x \in \mathbb{R}$.

Example 2.23. *Convolution with δ_0. Let $g \in \mathcal{S}(\mathbb{R})$ then for all $\varphi \in \mathcal{S}(\mathbb{R})$*

$$(\delta_0 * g)(\varphi) = \delta_0(\tilde{g} * \varphi) = (\tilde{g} * \varphi)(0) = \int_{-\infty}^{\infty} g(x - 0)\varphi(x) dx = g(\varphi)$$

and so as distributions we have that $\delta_0 * g = g$.

As we will see later, if we use the Fourier transform to acquire solutions to a differential equation it, it is possible to get a solution in the form of a convolution with certain functions (called good kernels [12, p. 139]). To help us establish these solution we observe a property of some sequences of distributions that has a delta-like behaviour which, together with the property of δ_0 we saw in the example above, will be of use for us.

Lemma 2.24. *[6, p.12] Let $\beta(x)$ be a nonnegative function with $\int_{-\infty}^{\infty} \beta(x)dx = 1$ and let*

$$\beta_\varepsilon(x) = \frac{1}{\varepsilon}\beta(\frac{x}{\varepsilon}).$$

Then $\beta_\varepsilon(\varphi) \to \delta_0(\varphi)$ as $\varepsilon \to 0$ for all $\varphi \in \mathcal{S}(\mathbb{R})$.

Proof. We have that

$$\begin{aligned}\beta_\varepsilon(\varphi) - \delta_0(\varphi) &= \int_{-\infty}^{\infty} \beta_\varepsilon(x)\varphi(x)dx - \varphi(0) \\ &= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \beta(\frac{x}{\varepsilon})\varphi(x)dx - \varphi(0) \\ &= \int_{-\infty}^{\infty} \beta(x)\varphi(\varepsilon x)dx - \int_{-\infty}^{\infty} \beta(x)dx\varphi(0) \\ &= \int_{-\infty}^{\infty} \beta(x)(\varphi(\varepsilon x) - \varphi(0))dx\end{aligned}$$

Now since $|\varphi(\varepsilon x) - \varphi(0)|$ is bounded and $\varphi(\varepsilon x) \to \varphi(0)$ as $\varepsilon \to 0$ for all $x \in \mathbb{R}$ we have from Lebesgue convergence theorem [7, p. 54] that

$$\left|\int_{-\infty}^{\infty} \beta(x)(\varphi(\varepsilon x) - \varphi(0))dx\right| \leqslant \int_{-\infty}^{\infty} \beta(x)\left|\varphi(\varepsilon x) - \varphi(0)\right| dx \to 0 \text{ as } \varepsilon \to 0.$$

□

2.4 Pseudo differential operators and Sobolev spaces

A function f on $\mathbb{R}$ is said to be locally $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, written as $L^p_{loc}(\mathbb{R})$, if

$$\left(\int_K |f(x)|^p\, dx\right)^{1/p} < \infty$$

for all compact $K \subset \mathbb{R}$ [1, p. 204]. Clearly every function in $L^p(\mathbb{R})$ is in $L^p_{loc}(\mathbb{R})$ while the reverse is not true.

Example 2.25. *On $\mathbb{R}$, every nonzero constant function is in $L^p_{loc}(\mathbb{R})$ but not in $L^p(\mathbb{R})$.*

Example 2.26. *On $\mathbb{R}$, every continuous function is in $L^p_{loc}(\mathbb{R})$ by the existence of upper and lower bound on compact intervals for such functions.*

Example 2.27. *The function $f(x) = x^{-1}$ is in $L^p_{loc}(\mathbb{R}\backslash\{0\})$.*

2.4.1 Pseudo differential operators

We defined the differential operator D acting on a function f as $Df := -i\partial f$ and as such we had the property $\widehat{(D\varphi)} = \xi\hat{\varphi}$ for $\varphi \in \mathcal{S}(\mathbb{R})$. Using this we can draw the conclusion that $D\varphi = (\xi\hat{\varphi})^\vee$ by the Fourier inversion theorem (Theorem 2.13). Extending on this, for the linear differential operator $P(x, D) = \sum_k a_k D^k$ we define the symbol $P(x, \xi)$ by replacing D^k with ξ i.e.

$$P(x,\xi) = \sum_k a_k \xi^k$$

and as such we have that

$$(P(x,D)\varphi)(x) = (P(x,\xi)\hat{\varphi})^\vee(x)$$

That is, we can represent a differential operator with a polynomial by the use of the inverse Fourier transform. To further extend upon this we can define the class of operators called pseudo differential operators $P(x, D)$ by using a wider set of signs $P(x, \xi)$ than polynomials, where as before

$$P(x,D)\varphi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} P(x,\xi)\hat{\varphi}(\xi)d\xi.$$

Using this idea of pseudo differential operators we can create an understanding of our results in this thesis by applying this to the following spaces.

2.4.2 Sobolev spaces

A function $f : \mathbb{R} \to \mathbb{R}$ is called homogenous of degree $s \in \mathbb{R}$ if $f(tx) = t^s f(x)$ for all $t > 0$ and $x \neq 0$ [1, p. 34]. By this definition, the function $(1 + |\xi|)^s$ is not homogenous (also called inhomogenous) while the function $|\xi|^s$ is homogeneous of degree s. From this, for $s \in \mathbb{R}$, the space of tempered distributions f such that $\hat{f} \in L^2_{loc}(\mathbb{R})$ and

$$\|f\|^2_{H^s(\mathbb{R})} := \int_{-\infty}^{\infty} (1 + |\xi|^2)^s \left|\hat{f}(\xi)\right|^2 d\xi < \infty$$

is called the **inhomogenous Sobolev space** $H^s(\mathbb{R})$ and the space of tempered distributions f such that $\hat{f} \in L^1_{loc}(\mathbb{R})$ and

$$\|f\|^2_{\dot{H}^s(\mathbb{R})} := \int_{-\infty}^{\infty} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi < \infty$$

is called the **homogenous Sobolev space** $\dot{H}^s(\mathbb{R})$[2, p. 38,25]. These two spaces, while quite similar in their definitions, are not the same space but are linked in that for positive s we have that $H^s(\mathbb{R}) \subset \dot{H}^s(\mathbb{R})$ and for negative s the reverse is true i.e. $\dot{H}^s(\mathbb{R}) \subset H^s(\mathbb{R})$. We will in this thesis concentrate our findings to Schwartz functions, which are in all $\dot{H}^s(\mathbb{R})$, and use the homogenous Sobolev norm to acquire our results. As such we will try to understand how to interpret the norm.

We can begin by noting that by Plancherel theorem we have that

$$\|f\|^2_{\dot{H}^s(\mathbb{R})} = \int_{-\infty}^{\infty} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi = \left\|\xi^s \hat{f}\right\|_{L^2(\mathbb{R})} \leqslant \sqrt{2\pi}\, \|D^s f\|_{L^2(\mathbb{R})}$$

where D^s is the pseudo differential operator represented by the symbol ξ^s, i.e. $D^s f = \mathcal{F}^{-1}[\xi^s \widehat{f}]$, and so we have that

$$f \in \dot{H}^s(\mathbb{R}) \iff D^s f \in L^2(\mathbb{R}).$$

As such, we can interpret this in the following way. If $s = k$ is a positive integer and $f \in \dot{H}^k(\mathbb{R})$ then the derivative of order k of f is in $L^2(\mathbb{R})$. So for positive integers k we have that $\dot{H}^k(\mathbb{R})$ is the space of distributions whose kth derivative is in $L^2(\mathbb{R})$.
Now, what happens if s is negative? We start with the case $s = -1$ and by noting that

$$f \in \dot{H}^{-1}(\mathbb{R}) \iff \widehat{F} := |\xi|^{-1} \widehat{f} \in L^2(\mathbb{R}).$$

To get an understanding of what this means we need to understand F, the inverse of $\widehat{F}$ defined above. Firstly we note that

$$DF(x) = (\xi \widehat{F})^{\vee}(x) = (\xi |\xi|^{-1} \widehat{f})^{\vee}(x) = (\operatorname{sgn}(\xi)\widehat{f})^{\vee}(x).$$

Now define $\sigma(\xi) := \operatorname{sgn}(\xi)$ and let $G := \sigma(D)F$. Then $G \in L^2(\mathbb{R})$ and

$$DG(x) = (\xi\sigma(\xi)\widehat{F})^{\vee}(x) = (\sigma(\xi)^2 \widehat{f})^{\vee}(x) = f(x).$$

And so $\dot{H}^{-1}(\mathbb{R})$ consists of those distributions that are a distributional derivative of a function in L^2.
This reasoning can be expanded on for when s is a negative integers.

If we want to understand what happens when s in a non-integer number then the idea is to expand on these results to pseudo differential operators. This is, however, outside the scope of this thesis.

2.5 Solving the Heat equation

We have now established a theory we can apply to find solutions to some partial differential equations. At times this requires assuming that the function and its initial data are in $\mathcal{S}'(\mathbb{R})$. To help with some calculations of our solutions we have the following useful lemma.

Lemma 2.28. *[2, p.23] Let z be a nonzero complex number with nonnegative real part and let $f_z(x) = e^{-zx^2}$ for $x \in \mathbb{R}$. Then $f_z \in \mathcal{S}(\mathbb{R})$ if $Re(z) > 0$ and*

$$\mathcal{F}[f_z](\xi) = \frac{\sqrt{\pi}}{\sqrt{z}} e^{-\frac{\xi^2}{4z}}$$

where $\sqrt{z} = |z|^{1/2} e^{i(\arg z)/2}$ and $\arg z \in [-\pi/2, \pi/2]$.
For z above with $Re(z) = 0$, the result holds in a distributional way.

Proof. Firstly, assume that $Re(z) > 0$ and let $z = a + bi$ with $a > 0$. Then $f_z(x) = e^{-ax^2} e^{-ibx^2}$ and so $|f_z| \leqslant |G_a(x)|$, where $G_a(x) = e^{-ax^2}$, a Gaussian bell function. Since $G_a \in \mathcal{S}(\mathbb{R})$ by Example 2.1 and $|\partial^\alpha f| \leqslant C_\alpha (1 + |x|)^\alpha |f_z|$

we have that $|f_z|_{m,\mathcal{S}} \leqslant |G_a|_{m,\mathcal{S}} < \infty$, and so $f_z \in \mathcal{S}(\mathbb{R})$.

If $\text{Im}(z) = 0$ (i.e. z is a positive real number), then we have that

$$\begin{aligned}\mathcal{F}[f_z](\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} f_z(x)dx = \int_{-\infty}^{\infty} e^{-i\xi x - zx^2} dx \\ &= e^{-\frac{\xi^2}{4z}} \int_{-\infty}^{\infty} e^{-z(x+i\frac{x}{2z})^2} dx = e^{-\frac{\xi^2}{4z}} \int_{\gamma_1} e^{-zw^2} dw,\end{aligned}$$

where γ_1 is the line in $\mathbb{C}$ parametrized as $w(s) = s + i\frac{\xi^2}{2z}$, $-\infty < s < \infty$. The function e^{-zw^2} is analytical on $\mathbb{C}$ and vanishes as $|w| \to \infty$. Cauchy's integral theorem [10, p. 187] now gives us that we can change the integration over γ_1 to integrating over γ_2, the curve parametrized by $w(s) = s$, $-\infty < s < \infty$ (i.e. the real line in $\mathbb{C}$), giving us that

$$\int_{\gamma_1} e^{-zw^2} dw = \int_{\gamma_2} e^{-zw^2} dw = \int_{-\infty}^{\infty} e^{-zs^2} ds = \frac{\sqrt{\pi}}{\sqrt{z}},$$

and thus we have that

$$\mathcal{F}[f_z](\xi) = \frac{\sqrt{\pi}}{\sqrt{z}} e^{-\frac{\xi^2}{4z}}.$$

Both $\mathcal{F}[f_z]$ and $\frac{\sqrt{\pi}}{\sqrt{z}} e^{-\frac{\xi^2}{4z}}$ are analytical (w.r.t. z) on $D = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ and since they are equal on the part of D that are strictly real then they are equal on all of D [10, p. 297].

Now let z s.t. $\text{Re}(z) = 0$, i.e. $z = it$ for $t \in \mathbb{R}$, and let $\{z_n\}_n \subset D$ be a sequence s.t. $z_n \to it$ as $n \to \infty$, then by Lebesgue's dominated convergence theorem [7, p. 54] we have for all $\varphi \in \mathcal{S}(\mathbb{R})$ that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-z_n x^2} \varphi(x)dx = \int_{-\infty}^{\infty} e^{-itx^2} \varphi(x)dx$$

and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{z_n}} e^{-\frac{\xi^2}{4z_n}} \varphi(x)dx = \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{it}} e^{-\frac{\xi^2}{4it}} \varphi(x)dx.$$

Now, since $\mathcal{F}\left[e^{-z_n x^2}\right](\xi) = \frac{\sqrt{\pi}}{\sqrt{z_n}} e^{-\frac{\xi^2}{4z_n}}$ for each z_n we have by Lemma 2.22 that as tempered distributions

$$\mathcal{F}[f_z](\xi) = \frac{\sqrt{\pi}}{\sqrt{z}} e^{-\frac{\xi^2}{4z}}.$$

□

The heat equation is a partial differential equation that describes how heat disperses over time and is what inspired Joseph Fourier to establish the theory that grew into Fourier analysis. In one space dimension with initial heat given by $f \in \mathcal{S}'(\mathbb{R})$ the equation is as follows [12, p. 146].

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) \\ u(x,0) = f(x) \end{cases}$$

To solve this assume that for all t we have $u(x,t) \in \mathcal{S}'(\mathbb{R})$. We can now use the Fourier transform on this equation (with respect to x) and get

$$\mathcal{F}[\partial_t u](\xi,t) = \mathcal{F}[\partial_x^2 u](\xi,t)$$

and so, using the theory of differentiation and the Fourier transform from Theorem 2.6 on the right hand side and Leibniz integral rule on the left hand side we have that

$$\partial_t \widehat{u}(\xi,t) = -\xi^2 \widehat{u}(\xi,t)$$

and thus that

$$\widehat{u}(\xi,t) = A(\xi)e^{-\xi^2 t}.$$

This gives us that $\widehat{u}(\xi,0) = A(\xi)$ and from the initial condition we have that $\widehat{u}(\xi,0) = \widehat{f}(\xi)$, so our solution becomes

$$\widehat{u}(\xi,t) = e^{-\xi^2 t}\widehat{f}(\xi).$$

Now for $t > 0$, by Lemma 2.28 above with $z = t$ we have

$$\begin{aligned} \mathcal{H}_t(x) := \mathcal{F}^{-1}\left[e^{-\xi^2 t}\right](x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-\xi^2 t}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}e^{-\eta^2 t}d\eta = \frac{1}{2\pi}\mathcal{F}\left[e^{-\xi^2 t}\right](x) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \end{aligned}$$

and so, by Theorem 2.10, we get our solution as

$$u(x,t) = (\mathcal{H}_t * f)(x)$$

and we are left with confirming that $u(x,t) \to f(x)$ as $t \to 0$.
Now, we note that if $\beta(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$ then $\int_{-\infty}^{\infty} \beta(x)dx = 1$ and since $\mathcal{H}_t(x) = \frac{1}{\sqrt{4t}}\beta(\frac{x}{\sqrt{4t}})$ we have that $\mathcal{H}_t(x) \to \delta_0$ as $t \to +0$ by lemma 2.24. And so we have, for fixed x, that in $\mathcal{S}'(\mathbb{R})$

$$u = \mathcal{H}_t * f \to \delta_0 * f = f \text{ as } t \to +0.$$

2.6 Solving the time dependent Schrödinger equation

Our main focus in this thesis is the equation proposed by Erwin Schrödinger in the early 20th century to describe a wave equation for matter. The need for this equation came from the discoveries in particle physics where, under certain conditions a particle has a wavelike behaviour[4]. The equation presented by Schrödinger was

$$-i\hbar\partial_t u(x,t) = (\frac{\hbar^2}{2m}\Delta + V(x,t))u(x,t)$$

where $x \in \mathbb{R}^n$, $t > 0$ [3]. Solutions to this equation describes probabilities of particles being located in certain regions. We will now consider the case of one space dimension so that the Laplacian Δ of the space variables is ∂_x^2 and we will omit the positive physical terms $\hbar$ and $2m$ (for clarity) and assume that the potentiality term V is zero. As such the free time dependent Schrödinger equation in one space dimension [13] with initial datum $f \in \mathcal{S}'(\mathbb{R})$ states that

$$\begin{cases} i\partial_t u(x,t) = -\partial_x^2 u(x,t) \\ u(x,0) = f(x). \end{cases} \tag{3}$$

To solve this we begin a similar approach as for the heat equation above and using the Fourier transform (with respect to x) gives us that

$$\partial_t \widehat{u}(\xi,t) = -i\xi^2 \widehat{u}(\xi,t)$$

and so our solution becomes

$$\widehat{u}(\xi,t) = \widehat{g}_t(\xi)\widehat{f}(\xi)$$

where $\widehat{g}_t(\xi) = e^{-it\xi^2}$.

First we note that we have for finite $t > 0$ and $a \in \mathbb{N}$ that $|\partial_\xi^a \widehat{g}_t| \leqslant C_a(1+|\xi|)^a$ and given $h \in \mathcal{S}(\mathbb{R})$ then for any $\alpha, N \in \mathbb{N}$ there is a $C_{\alpha,N}$ such that

$$|\partial_\xi^\alpha h| \leqslant C_{\alpha,N}(1+|\xi|)^{-N}$$

and so there is a $C_{\alpha,N}$ such that

$$\sum_{a \leqslant \alpha} |\partial_\xi^a h| \leqslant C_{\alpha,N}(1+|\xi|)^{-N-\alpha}$$

thus, for any $\alpha, N \in \mathbb{N}$, Leibniz rule gives us

$$\begin{aligned} |\partial_\xi^\alpha(\widehat{g}_t h)| &= \sum_{\substack{a+b=\alpha \\ a,b \geqslant 0}} C_{a,b}|(\partial_\xi^a \widehat{g}_t)(\partial_\xi^b h)| \\ &\leqslant (1+|\xi|)^\alpha \sum_{0 \leqslant b \leqslant \alpha} C_b|\partial_\xi^b h| \leqslant C_{\alpha,N}(1+|\xi|)^\alpha(1+|\xi|)^{-\alpha-N} \\ &\leqslant C_{\alpha,N}(1+|\xi|)^{-N} \end{aligned}$$

and so for all $h \in \mathcal{S}(\mathbb{R})$ we have $\widehat{g}_t h \in \mathcal{S}(\mathbb{R})$.

We have that

$$|\widehat{g}_t - 1| = \left|e^{-it\xi^2} - 1\right| = \left|\int_0^1 e^{-its\xi^2} ds(it\xi^2)\right| \leqslant t\,|\xi|^2$$

and, for $t < 1$, we have for $\alpha \geqslant 1$ that $|\partial_\xi^\alpha \widehat{g}_t| \leqslant C_\alpha t(1+|\xi|)^\alpha$. Given $m \in \mathbb{N}$

then, using the same method as above for the sum, we have that

$$
\begin{aligned}
|\widehat{g}_t h - h|_{m,\mathcal{S}} &= \sup_{\xi\in\mathbb{R}} \sup_{\alpha+\beta\leqslant m} ||\xi|^\beta \partial_\xi^\alpha (\widehat{g}_t(\xi)h(\xi) - h(\xi))| \\
&= \sup_{\xi\in\mathbb{R}} \sup_{\alpha+\beta\leqslant m} |\xi|^\beta |(\widehat{g}_t - 1)\partial_\xi^\alpha h + \sum_{\substack{a+b=\alpha \\ a>0,\, b\geqslant 0}} C_{a,b}(\partial_\xi^a \widehat{g}_t)(\partial_\xi^b h)| \\
&\leqslant \sup_{\xi\in\mathbb{R}} \sup_{\alpha+\beta\leqslant m} |\xi|^\beta t|\xi|^2 |\partial_\xi^\alpha h| + t(1+|\xi|)^\alpha \sum_{0\leqslant b\leqslant\alpha} C_b |\partial_\xi^b h| \\
&\leqslant \sup_{\xi\in\mathbb{R}} \sup_{\alpha+\beta\leqslant m} tC_{\alpha,\beta}|\xi|^{\beta+2}(1+|\xi|)^{-\beta-3} + tC_\alpha(1+|\xi|)^\alpha(1+|\xi|)^{-\alpha-1} \\
&\leqslant \sup_{\xi\in\mathbb{R}} \sup_{\alpha+\beta\leqslant m} tC_{\alpha,\beta}(1+|\xi|)^{-1} + tC_\alpha(1+|\xi|)^{-1} \leqslant tD
\end{aligned}
$$

and so for all $h \in \mathcal{S}(\mathbb{R})$ we have $|\widehat{g}_t h - h|_{m,\mathcal{S}} \to 0$ as $t \to 0$.

Now, given $f \in \mathcal{S}(\mathbb{R})$ then $\widehat{f} \in \mathcal{S}(\mathbb{R})$ and since $\widehat{u}(\xi,t) = \widehat{g}_t(\xi)\widehat{f}(\xi)$ and $\widehat{u}(\xi,0) = \widehat{f}(\xi)$ we have from our previous result, with $h = \widehat{f}$, that $\widehat{u}(\xi,t) \to \widehat{f}(\xi)$ in $\mathcal{S}(\mathbb{R})$ as $t \to 0$. Since $\mathcal{F}^{-1}$ is continuous, we have $u(x,t) = \mathcal{F}^{-1}[\widehat{g}_t\widehat{f}](x) \to \mathcal{F}^{-1}[\widehat{f}](x) = f(x)$ as $t \to 0$. That is, $u \to f$ in $\mathcal{S}(\mathbb{R})$ as $t \to 0$.

Let $\tilde{f}(x) = f(-x)$, then $\tilde{g}_t = g_t$ and, from the results above, for all $\varphi \in \mathcal{S}(\mathbb{R})$ we have $g_t * \varphi \to \varphi$ in $\mathcal{S}(\mathbb{R})$ as $t \to 0$ and so, using convolutions in $\mathcal{S}'(\mathbb{R})$, we have

$$(g_t * f)(\varphi) = f(\tilde{g}_t * \varphi) = f(g_t * \varphi).$$

So by the continuity of f and the results above we have that

$$f(\tilde{g}_t * \varphi) \to f(\varphi) \text{ as } t \to 0$$

i.e.

$$g_t * f \to f \text{ as } t \to 0 \text{ in } \mathcal{S}'(\mathbb{R}).$$

Now, if $f \in L^2(\mathbb{R})$, then, using Plancherel

$$
\begin{aligned}
\|u\|_{L^2(\mathbb{R})} &= (2\pi)^{-1/2} \|\widehat{u}\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \left\|\widehat{g}_t\widehat{f}\right\|_{L^2(\mathbb{R})} \\
&= (2\pi)^{-1/2} \left\|\widehat{f}\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}
\end{aligned}
$$

and so for $t > 0$ we have $u \in L^2(\mathbb{R})$.

Finally, using Plancherel again, we have

$$
\begin{aligned}
\|u - f\|_{L^2(\mathbb{R})} &= (2\pi)^{-1/2} \left\|\widehat{u} - \widehat{f}\right\|_{L^2(\mathbb{R})} \\
&= (2\pi)^{-1/2} \left\|(\widehat{g}_t - 1)\widehat{f}\right\|_{L^2(\mathbb{R})}
\end{aligned}
$$

and so, since $\widehat{g}_t - 1 \to 0$ as $t \to 0$, Lebesgue dominated convergence theorem gives us that $\|u - f\|_{L^2(\mathbb{R})} \to 0$ as $t \to 0$, i.e. $u \to f$ in $L^2(\mathbb{R})$ as $t \to 0$.

We can now summarize our solution as

$$u(x,t) = (g_t * f)(x) = \int_{-\infty}^{\infty} g_t(x-y)f(y)dy \tag{4}$$

where, using the symmetry of $\widehat{g}_t$, we have from Lemma 2.28 above that

$$\begin{aligned} g_t(x) &= \mathcal{F}_\xi^{-1}[e^{-it\xi^2}](x) = \frac{1}{2\pi}\mathcal{F}_\xi[e^{-it\xi^2}](-x) = \frac{1}{2\pi}\frac{\sqrt{\pi}e^{-\frac{x^2}{4it}}}{\sqrt{it}} \\ &= \frac{e^{-\frac{x^2}{4it}}}{\sqrt{i4\pi t}} \end{aligned}$$

or, from $\widehat{u}(\xi,t) = \widehat{g}_t(\xi)\widehat{f}(\xi)$ we have

$$u(x,t) = \mathcal{F}^{-1}[\widehat{g}_t(\xi)\widehat{f}(\xi)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-it\xi^2}\widehat{f}(\xi)d\xi$$

We can note that if we reverse time in our solution above, we have a solution to the similar equation $i\partial_t u = \partial_x^2 u$, and we can see that indeed $u(x,-t) = \left(S^2 f\right)[x](t)$ as stated in the introduction.

2.6.1 Regularity estimates for the free time dependent Schrödinger

We have now established a solution to the Schrödinger equation and the next step is to analyse the behaviour of it, i.e. what we can expect from such a solution based on the initial conditions of the equation. We will do this by use of the following theorem (Theorem 7.1.12 in [8]).

Theorem 2.29 (Riesz-Thorin interpolation theorem)**.** *Let A be a linear operator A mapping $L^{p_0}(\mathbb{R})$ to $L^{q_0}(\mathbb{R})$ and $L^{p_1}(\mathbb{R})$ to $L^{q_1}(\mathbb{R})$. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|Ax\|_{L^{q_0}(\mathbb{R})} \leqslant C_0 \|x\|_{L^{p_0}(\mathbb{R})}$$

and

$$\|Ax\|_{L^{q_1}(\mathbb{R})} \leqslant C_1 \|x\|_{L^{p_1}(\mathbb{R})}.$$

Then for all $0 < \theta < 1$ we have that

$$\|Ax\|_{L^{q}(\mathbb{R})} \leqslant C_0^{1-\theta}C_1^{\theta} \|x\|_{L^{p}(\mathbb{R})},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Using this theorem we can now acquire our first estimate. For solutions u to the Schrödinger equation (3), as derived above, with initial datum f we have the following result.

Theorem 2.30. *For $p, q \in \mathbb{R}$ such that $1 \leqslant p \leqslant 2$ and $1/q + 1/p = 1$ there is a number C independent of f such that*

$$\|u(\cdot,t)\|_{L^{q}(\mathbb{R})} \leqslant C\,|t|^{1/2-1/p}\,\|f\|_{L^{p}(\mathbb{R})}.$$

Proof. As we saw in equation (4) above, we can write u as a convolution $u = g_t * f$, where $g_t(x) = \dfrac{e^{-\frac{x^2}{4it}}}{\sqrt{i4\pi t}}$ is the kernel and f is our initial datum. Using this, i.e. that

$$u(x,t) = (g_t * f)(x) = \int_{-\infty}^{\infty} (4\pi it)^{-1/2} e^{-(x-y)^2/(4it)} f(y)dy$$

we can note that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqslant (4\pi)^{-1/2} |t|^{-1/2} \int_{-\infty}^{\infty} |f(y)|\, dy = (4\pi)^{-1/2} |t|^{-1/2} \|f\|_{L^1(\mathbb{R})}. \quad (5)$$

and that the Plancherel theorem and Theorem 2.10 gives us that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \left\|\widehat{g}_t \widehat{f}\right\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \left\|\widehat{f}\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \quad (6)$$

The Riesz–Thorin interpolation theorem, with $q_0 = \infty$, $q_1 = 2$, $p_0 = 1$, and $p_1 = 2$, now gives us that

$$\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leqslant (4\pi)^{1/2-1/p} |t|^{1/2-1/p} \|f\|_{L^p(\mathbb{R})},$$

where $1 < p < 2$ and $1/q + 1/p = 1$. The cases $p = 1$ and $p = 2$ comes from equation (5) and (6). □

2.6.2 The free fractional Schrödinger equation

The fractional Schrödinger equation in one space dimension is given by the pseudo differential equation

$$-i\partial_t u = (-\partial_x^2)^{a/2} u \quad (7)$$

where $(-\partial_x^2)^{a/2}$ is defined as the pseudo differential operation given by the Fourier inverse of the symbol $|\xi|^a$ [3]. This equation with initial datum $u(x,0) = f(x)$ has the solution $S^a f$ as given in the introduction, i.e.

$$(S^a f)[x](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} e^{i|\xi|^a t} \widehat{f}(\xi) \mathrm{d}\xi.$$

We will not derive these solutions as we did for the case $a = 2$ as this is beyond the scope of this thesis. We can however show that when a is a positive even integer we have that $S^a f$ satisfies equation (7). To do this, let $a = 2k$ for some $k \in \mathbb{N}$. Then the operator $(-\partial_x^2)^{a/2}$ corresponds with the differential operator $(-1)^k \partial_x^{2k} = (-1)^k \partial_x^a$ and we have that

$$(-1)^k \partial_x^a (S^a f) = (-1)^k i^a \xi^a (S^a f) = \xi^a (S^a f) = -i\partial_t (S^a f).$$

That is, $S^a f$ is a solution to equation (7).

3 Regularity estimates for the free time dependent fractional Schrödinger equation

The problem at hand is to find, for $a \geqslant 2$, regularity estimates for the functions $S^a f$ based on the data function f. What this means is that we are searching for expressions using norms on the form of $\|g\|_A \leqslant \|h\|_B$ so that $g \in A$ whenever $h \in B$. In our case this means that we want to give us an understanding of how $S^a f$ behaves in terms of f. As we saw in section 2.6.1 we have already established an estimate for $a = 2$ in that for $1 \leqslant p \leqslant 2$ with $1/p + 1/q = 1$ we have that

$$\left\|S^2 f\right\|_{L^q(\mathbb{R})} \leqslant C\,|t|^{1/2-1/p}\,\|f\|_{L^p(\mathbb{R})}\,.$$

To get that estimate we used that our solution could be written as a convolution between a function and the initial data. As we don't have this expression for $S^a f$ we cannot use quite the same approach. However, the fractional Schrödinger equation is a pseudo differential operator and our initial data function is assumed to be a Schwartz function on $\mathbb{R}$ and as such is in $\dot{H}^s(\mathbb{R})$ for $s \in \mathbb{R}$. From this we argue our first choice of estimates using the homogeneous Sobolev norm. Before we begin, however, we make the following useful observation about how $S^a f$ behaves under dilation.

Lemma 3.1. *Let $\alpha, \beta > 0$ then*

$$(S^a f)\,[\alpha x](\beta^a t) = (S^a f_\beta)\,[\frac{\alpha}{\beta}x](t)$$

where $f_\beta(x) = D_\beta f(x)$.

Proof. Let $\alpha, \beta > 0$ and note that $\widehat{f_\beta}(\xi) = \frac{1}{\beta}\widehat{f}(\frac{\xi}{\beta})$ from Theorem 2.6. Using this, we have that

$$\begin{aligned}
(S^a f)\,[\alpha x](\beta^a t) &= \int_{-\infty}^{\infty} e^{i\xi\alpha x} e^{i|\xi|^a\beta^a t}\widehat{f}(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} e^{i\beta\xi\frac{\alpha}{\beta}x} e^{i|\beta\xi|^a t}\widehat{f}(\frac{1}{\beta}\beta\xi)\frac{1}{\beta}\beta\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} e^{i\xi\frac{\alpha}{\beta}x} e^{i|\xi|^a t}\frac{1}{\beta}\widehat{f}(\frac{\xi}{\beta})\mathrm{d}\xi = \int_{-\infty}^{\infty} e^{i\xi\frac{\alpha}{\beta}x} e^{i|\xi|^a t}\widehat{f_\beta}(\xi)\mathrm{d}\xi \\
&= (S^a f_\beta)\,[\frac{\alpha}{\beta}x](t).
\end{aligned}$$

□

3.1 Estimate using the Plancherel theorem

We will now establish an estimate using the homogeneous Sobolev norm. We start with using the effects of dilation on $S^a f$ (given in Lemma 3.1) to derive the necessary conditions for this estimate to hold. We then continue to establish that this is indeed a valid estimate with the use of the Plancherel theorem (Theorem 2.14).

Proposition 3.2. *Assume that there exists a number C independent of f such that*

$$\|S^a f\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} \leqslant C\,\|f\|_{\dot{H}^s(\mathbb{R})}\,. \tag{8}$$

Then $s = \dfrac{1-a}{2}$.

Proof. Assume that there exists a number C independent of f such that

$$\|S^a f\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} \leqslant C\,\|f\|_{\dot{H}^s(\mathbb{R})}\,. \tag{9}$$

Let $f_\beta(x) = f(\beta x)$. Now by our assumption and Lemma 3.1 we have that

$$\begin{aligned}
\|S^a f\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} &= \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|(S^a f)\,[x](t)\right|^2 \mathrm{d}t \\
&= \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|(S^a f)\,[x](\beta^a t)\right|^2 \beta^a \mathrm{d}t \\
&= \beta^a \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|(S^a f_\beta)\,[\frac{x}{\beta}](t)\right|^2 \mathrm{d}t \\
&= \beta^a \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left|(S^a f_\beta)\,[x](t)\right|^2 \mathrm{d}t \\
&\leqslant \beta^a C \int_{-\infty}^{\infty} \left|\widehat{f_\beta}(\xi)\right|^2 |\xi|^{2s}\,\mathrm{d}\xi \\
&= \beta^{a-1+2s} C \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 |\xi|^{2s}\,\mathrm{d}\xi \\
&= \beta^{a-1+2s} C\,\|f\|^2_{\dot{H}^s(\mathbb{R})}\,.
\end{aligned}$$

This needs to be independent of β so the β^{a-1+2s} term need to vanish. This gives us that $s = \frac{1-a}{2}$. □

Now that we have a necessary condition on s for our estimate to hold we can establish that this is indeed a valid estimate. To do so we view $S^a f$ as a Fourier transform with frequency $-t$. This enables us to use the Plancherel theorem to reach the following result.

Theorem 3.3. *There exists a number C independent of f such that*

$$\|S^a f\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} \leqslant C\,\|f\|_{\dot{H}^{(1-a)/2}(\mathbb{R})}\,.$$

Proof. Assume that the function f is such that $\operatorname{supp}(\widehat{f}) \subset [0, \infty)$. We have that

$$
\begin{aligned}
(S^a f)[x](t) &= \int_{-\infty}^{\infty} e^{ix\xi} e^{i|\xi|^a t} \widehat{f}(\xi) \mathrm{d}\xi = \int_0^{\infty} e^{ix\xi} e^{i|\xi|^a t} \widehat{f}(\xi) \mathrm{d}\xi \\
&= \begin{bmatrix} \xi = \eta^{1/a} \\ \mathrm{d}\xi = \dfrac{1}{a}\eta^{1/a-1}\mathrm{d}\eta \end{bmatrix} = \int_0^{\infty} e^{i\eta^{1/a}x} e^{i\eta t} \widehat{f}(\eta^{1/a}) \frac{1}{a} \eta^{1/a-1} \mathrm{d}\eta \\
&= \int_0^{\infty} e^{-i\eta(-t)} \frac{e^{i\eta^{1/a}x} \widehat{f}(\eta^{1/a})}{a\eta^{1-1/a}} \mathrm{d}\eta = \widehat{g_x}(-t)
\end{aligned}
$$

where

$$
g_x(\eta) = \frac{e^{i\eta^{1/a}x} \widehat{f}(\eta^{1/a})}{a\eta^{1-1/a}}.
$$

By the Plancherel theorem we have that there exists a number A such that

$$
\|\widehat{g_x}\|_{L^2(\mathbb{R})}^2 = aA \|g_x\|_{L^2(\mathbb{R})}^2 .
$$

Using this we have that

$$
\begin{aligned}
\|S^a f\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}))}^2 &= \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(S^a f)[x](t)|^2 \,\mathrm{d}t \\
&= \sup_{x \in \mathbb{R}} \int_0^{\infty} |\widehat{g_x}(-t)|^2 \,\mathrm{d}t = \sup_{x \in \mathbb{R}} aA \int_0^{\infty} |g_x(\eta)|^2 \,\mathrm{d}\eta \\
&= \sup_{x \in \mathbb{R}} aA \int_0^{\infty} \left| \frac{e^{i\eta^{1/a}x} \widehat{f}(\eta^{1/a})}{a\eta^{1-1/a}} \right|^2 \mathrm{d}\eta \\
&= A \int_0^{\infty} \left|\eta^{1/a-1}\right| \left|\widehat{f}(\eta^{1/a})\right|^2 \frac{\mathrm{d}\eta}{a\eta^{1-1/a}} \\
\begin{bmatrix} \xi^a = \eta \\ \mathrm{d}\xi = \dfrac{\mathrm{d}\eta}{a\eta^{1-1/a}} \end{bmatrix} &= A \int_0^{\infty} |\xi|^{2(\frac{1-a}{2})} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = A \|f\|_{\dot{H}^{\frac{1-a}{2}}(\mathbb{R})}^2 .
\end{aligned}
$$

Assume now instead that f is a function such that $\operatorname{supp}(\widehat{f}) \subset (-\infty, 0]$. We have that

$$
\begin{aligned}
(S^a f)[x](t) &= \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) \mathrm{d}\xi = \int_{-\infty}^{0} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) \mathrm{d}\xi \\
&= \int_0^{\infty} e^{-ix\xi} e^{it|\xi|^a} \widehat{f}(-\xi) \mathrm{d}\xi = \int_0^{\infty} e^{-ix\xi} e^{it|\xi|^a} \tilde{\widehat{f}}(\xi) \mathrm{d}\xi,
\end{aligned}
$$

where $\tilde{\widehat{f}}(\xi) = \widehat{f}(-\xi)$. We have that $\operatorname{supp}(\tilde{\widehat{f}}) \subset [0, \infty)$ and the result now follows analogously as for $\widehat{f}$ with nonnegative support by noting that

$$
\int_0^{\infty} |\xi|^{2(\frac{1-a}{2})} \left|\tilde{\widehat{f}}(\xi)\right|^2 \mathrm{d}\xi = \int_{-\infty}^{0} |\xi|^{2(\frac{1-a}{2})} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = \|f\|_{\dot{H}^{\frac{1-a}{2}}(\mathbb{R})}^2 .
$$

Thus we have that for f such that $\operatorname{supp}(\widehat{f}) \subset (-\infty, 0]$ there exists a number B independent of f such that

$$\|S^a f\|^2_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} = B \|f\|^2_{\dot{H}^{\frac{1-a}{2}}(\mathbb{R})}.$$

Now let f be a function with non empty support for its Fourier transform and let the two functions $\widehat{f_+}$ and $\widehat{f_-}$ divide $\widehat{f}$ such that $\operatorname{supp}(\widehat{f_+}) \subset [0,\infty)$, $\operatorname{supp}(\widehat{f_-}) \subset (-\infty, 0]$ and $\widehat{f_+} + \widehat{f_-} = \widehat{f}$. We have for fixed x, by the Minkowski inequality[5] and the linearity of integration, that

$$\begin{aligned}
&\left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) \mathrm{d}\xi \right\|^2_{L^2(\mathbb{R})} = \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \left(\widehat{f_+}(\xi) + \widehat{f_-}(\xi) \right) \mathrm{d}\xi \right\|^2_{L^2(\mathbb{R})} \\
&= \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_-}(\xi) \mathrm{d}\xi + \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_+}(\xi) \mathrm{d}\xi \right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \left(\left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_-}(\xi) \mathrm{d}\xi \right\|_{L^2(\mathbb{R})} + \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_+}(\xi) \mathrm{d}\xi \right\|_{L^2(\mathbb{R})} \right)^2.
\end{aligned}$$

This gives us that the is a $C = A + B$ independent of f such that

$$\begin{aligned}
\|S^a f\|^2_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} &= \sup_{x\in\mathbb{R}} \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) \mathrm{d}\xi \right\|^2_{L^2(\mathbb{R})} \\
&= \sup_{x\in\mathbb{R}} \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \left(\widehat{f_-}(\xi) + \widehat{f_+}(\xi) \right) \mathrm{d}\xi \right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \sup_{x\in\mathbb{R}} \left(\left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_-}(\xi) \mathrm{d}\xi \right\|_{L^2(\mathbb{R})} \right. \\
&\qquad \left. + \left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{it|\xi|^a} \widehat{f_+}(\xi) \mathrm{d}\xi \right\|_{L^2(\mathbb{R})} \right)^2 \\
&\leqslant (A+B)^2 \|f\|^2_{\dot{H}^{(1-a)/2}(\mathbb{R})} = C^2 \|f\|^2_{\dot{H}^{(1-a)/2}(\mathbb{R})}.
\end{aligned}$$

□

Observation
If we consider the case $a = 2$, i.e. the regularity of $\left(S^2 f\right)[x](t)$ we get from the result above that there is a number C independent of f such that

$$\left\|S^2 f\right\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} \leqslant C \|f\|_{\dot{H}^{-1/2}(\mathbb{R})}.$$

As briefly discussed in Section 2.4.2 this states that it is enough that $D^{-1/2} f \in L^2(\mathbb{R})$ (where $D^{-1/2}$ is the operator represented by the symbol $|\xi|^{-1/2}$) for $S^2 f$ to be in $L^\infty(\mathbb{R}, L^2(\mathbb{R}))$.

3.1.1 Replacing $|\xi|^a$ with a differentiable and injective function φ

In the previous section, the only properties of $|\xi|^a$ that were used were that it is differentiable, homogeneous and injective (after splitting $\mathbb{R}$ into a positive and a negative side). To continue the process of generalization we now search for estimates when replacing $|\xi|^a$ with a differentiable and injective function φ such that $\varphi'(\xi) \neq 0$ for every $\xi \in \mathbb{R}$, defining the function

$$(S^\varphi f)[x](t) = \int_{-\infty}^{\infty} e^{ix\xi} e^{it\varphi(\xi)} \widehat{f}(\xi) \mathrm{d}\xi.$$

Repeating the process of viewing our function as a Fourier transform established in the previous sections and by the use of the Plancherel theorem we obtain the following result.

Theorem 3.4. *There exists a number C independent of f such that*

$$\|S^\varphi f\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} \leqslant C \left(\int_{-\infty}^{\infty} |\varphi'(\xi)|^{-1} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \right)^{1/2}.$$

Note: The function $\varphi(\xi) = |\xi|^a$ is differentiable (since $a \geqslant 2$) and injective for $\xi \geqslant 0$ or $\xi \leqslant 0$, and so we have that $|\varphi'(\xi)|^{-1} = a^{-1} |\xi|^{1-a}$ as expected from our previous results.

Proof. Let φ be a differentiable and injective function such that $\varphi'(\xi) \neq 0$ for every $\xi \in \mathbb{R}$. We have

$$\begin{aligned}
(S^\varphi f)[x](t) &= \int_{-\infty}^{\infty} e^{ix\xi} e^{i\varphi(\xi)t} \widehat{f}(\xi) \mathrm{d}\xi = \begin{bmatrix} \varphi(\xi) = \zeta \\ \mathrm{d}\xi = \frac{\mathrm{d}\zeta}{\varphi'(\varphi^{-1}(\zeta))} \end{bmatrix} \\
&= \int_{-\infty}^{\infty} e^{i\zeta t} \frac{e^{i\varphi^{-1}(\zeta)x} \widehat{f}(\varphi^{-1}(\zeta))}{\varphi'(\varphi^{-1}(\zeta))} \mathrm{d}\zeta \\
&= \widehat{g_x}(-t)
\end{aligned}$$

where

$$g_x(\zeta) = \frac{e^{i\varphi^{-1}(\zeta)x} \widehat{f}(\varphi^{-1}(\zeta))}{\varphi'(\varphi^{-1}(\zeta))}.$$

This gives us, by the Plancherel theorem (theorem 2.14), that

$$\begin{aligned}
\|S^\varphi f\|^2_{L^\infty(\mathbb{R},L^2(\mathbb{R}))} &= \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |(S^\varphi f)[x](t)|^2 \mathrm{d}t \\
&= \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |\widehat{g_x}(-t)|^2 \mathrm{d}t \leqslant \sup_{x\in\mathbb{R}} C \int_{-\infty}^{\infty} |g_x(\zeta)|^2 \mathrm{d}\zeta \\
&= \sup_{x\in\mathbb{R}} C \int_{-\infty}^{\infty} \left| \frac{e^{i\varphi^{-1}(\zeta)x} \widehat{f}(\varphi^{-1}(\zeta))}{\varphi'(\varphi^{-1}(\zeta))} \mathrm{d}\zeta \right|^2 \mathrm{d}\zeta \\
&= C \int_{-\infty}^{\infty} \left| \frac{\widehat{f}(\varphi^{-1}(\zeta))}{\varphi'(\varphi^{-1}(\zeta))} \mathrm{d}\zeta \right|^2 \mathrm{d}\zeta \\
&= C \int_{-\infty}^{\infty} |(\varphi'(\varphi^{-1}(\zeta)))^{-1}| \left|\widehat{f}(\varphi^{-1}(\zeta))\right|^2 \frac{\mathrm{d}\zeta}{\varphi'(\varphi^{-1}(\zeta))} \\
\begin{bmatrix} \varphi(\xi) = \zeta \\ \mathrm{d}\xi = C\frac{\mathrm{d}\zeta}{\varphi'(\varphi^{-1}(\zeta))} \end{bmatrix} &= C \int_{-\infty}^{\infty} |\varphi'(\xi)|^{-1} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi.
\end{aligned}$$

□

We can note that $\int_{-\infty}^{\infty} |\varphi'(\xi)|^{-1} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = \left\|(\varphi'(\xi))^{-1/2}\widehat{f}\right\|_{L^2(\mathbb{R})}$. And so by Plancherel theorem we have that our function $S^{\varphi}f$ is in $L^{\infty}(\mathbb{R}, L^2(\mathbb{R}))$ if we have that $\Phi f \in L^2(\mathbb{R})$, where Φ is the pseudo differential operator represented by the symbol $(\varphi')^{-1/2}$.

3.2 Estimating using Pitt's inequality

We will now use a more general result than the Plancherel theorem known as Pitt's inequality to acquire a similar estimate to that of the previous section. This time as a weighted $L^q(\mathbb{R})$-norm of our function.
Pitt's inequality states that there exists a weighted norm estimate between the function and its Fourier transform in the following manner.

Theorem 3.5 (Pitt's inequality [9, p. 729])**.** *Let $p \leqslant q$, $0 \leqslant \alpha < 1 - \frac{1}{p}, 0 \leqslant \gamma < \frac{1}{q}$ and $\alpha - \left(1 - \frac{1}{p}\right) = \gamma - \frac{1}{q}$. Then there exists a number C independent of f such that*

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^q |\xi|^{-\gamma q} \,\mathrm{d}\xi\right)^{1/q} \leqslant C \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p} \,\mathrm{d}x\right)^{1/p}.$$

The conditions on α, γ, p and q given in Pitt's inequality are sufficient conditions for the inequality to hold. We can, for functions in $\mathcal{S}(\mathbb{R})$, expand this result by stating a necessary condition for these parameters. They can be found by one again observing the effect of dilution on these functions and their Fourier transform, as stated in Theorem 2.9.

Lemma 3.6. *If there exists a number C independent of f such that*

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^q |\xi|^{-\gamma q} \,\mathrm{d}\xi\right)^{1/q} \leqslant C \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p} \,\mathrm{d}x\right)^{1/p}$$

then $\alpha - (1 - \frac{1}{p}) = \gamma - \frac{1}{q}$.

Proof. Assume that

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^q |\xi|^{-\gamma q} \,\mathrm{d}\xi\right)^{1/q} \leqslant C \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p} \,\mathrm{d}x\right)^{1/p}.$$

Now, let $f_\beta(x) = f(\beta x)$. Our assumption and Theorem 2.9 gives us that

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^q |\xi|^{-\gamma q}\,\mathrm{d}\xi\right)^{1/q} &= \left(\int_{-\infty}^{\infty} \left|\widehat{f}(\frac{\beta\xi}{\beta})\right|^q \left|\frac{\beta\xi}{\beta}\right|^{-\gamma q} \frac{\beta\mathrm{d}\xi}{\beta}\right)^{1/q} \\
&= \beta^{\gamma+1-\frac{1}{q}} \left(\int_{-\infty}^{\infty} \left|\frac{1}{\beta}\widehat{f}(\frac{\xi}{\beta})\right|^q |\xi|^{-\gamma q}\,\mathrm{d}\xi\right)^{1/q} \\
&= \beta^{\gamma+1-\frac{1}{q}} \left(\int_{-\infty}^{\infty} \left|\widehat{f_\beta}(\xi)\right|^q |\xi|^{-\gamma q}\,\mathrm{d}\xi\right)^{1/q} \\
&\leqslant \beta^{\gamma+1-\frac{1}{q}} C \left(\int_{-\infty}^{\infty} |f_\beta(x)|^p |x|^{\alpha p}\,\mathrm{d}x\right)^{1/p} \\
&= \beta^{\gamma+1-\frac{1}{q}} C \left(\int_{-\infty}^{\infty} |f(\beta x)|^p |x|^{\alpha p}\,\mathrm{d}x\right)^{1/p} \\
&= \beta^{\gamma+1-\frac{1}{q}} C \left(\int_{-\infty}^{\infty} |f(x)|^p \left|\frac{x}{\beta}\right|^{\alpha p} \frac{\mathrm{d}x}{\beta}\right)^{1/p} \\
&= \beta^{\gamma+1-\frac{1}{q}-\alpha-\frac{1}{p}} C \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha p}\,\mathrm{d}x\right)^{1/p}.
\end{aligned}$$

For this to hold the β-term need to vanish, and so we get that

$$\alpha - \left(1 - \frac{1}{p}\right) = \gamma - \frac{1}{q}.$$

□

And so, with these conditions inherited from the use of Pitt's inequality, we can establish a weighted norm estimate for $S^a f$ by following the procedure of the previous section. As such we begin with the necessary conditions.

Proposition 3.7. *Let $q \geqslant 2$, and $0 \leqslant \gamma < \frac{1}{q}$. Assume there exists a number C independent of f such that*

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |(S^a f)\,[x](t)|^q |t|^{-\gamma q}\,\mathrm{d}t\right)^{1/q} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R})}.$$

Then $s = \dfrac{1}{2} + a\gamma - \dfrac{a}{q}$.

Proof. Assume there exists a number C independent of f such that

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |(S^a f)\,[x](t)|^q |t|^{-\gamma q}\,\mathrm{d}t\right)^{1/q} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R})}.$$

Using this and Lemma 3.1, we have for $f_\beta(x) = f(\beta x)$ that

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} |(S^a f)\,[x](t)|^q\,|t|^{-\gamma q}\,\mathrm{d}t\right)^{1/q} &= \left(\int_{-\infty}^{\infty} |(S^a f)\,[x](\beta^a t)|^q\,|\beta^a t|^{-\gamma q}\,\beta^a \mathrm{d}t\right)^{1/q} \\
&= \beta^{\frac{a}{q}-a\gamma}\left(\int_{-\infty}^{\infty} \left|(S^a f_\beta)\,[\frac{x}{\beta}](t)\right|^q |t|^{-\gamma q}\,\mathrm{d}t\right)^{1/q} \\
&= \beta^{\frac{a}{q}-a\gamma}\left(\int_{-\infty}^{\infty} |(S^a f_\beta)\,[x](t)|^q\,|t|^{-\gamma q}\,\mathrm{d}t\right)^{1/q} \\
&\leqslant \beta^{\frac{a}{q}-a\gamma} C\left(\int_{-\infty}^{\infty} \left|\widehat{f_\beta}(\xi)\right|^2 |\xi|^{2s}\,\mathrm{d}\xi\right)^{1/2} \\
&= \beta^{\frac{a}{q}-a\gamma} C\left(\int_{-\infty}^{\infty} \left|\frac{1}{\beta}\hat{f}(\frac{\xi}{\beta})\right|^2 |\xi|^{2s}\,\mathrm{d}\xi\right)^{1/2} \\
&= \beta^{\frac{a}{q}-a\gamma-\frac{1}{2}} C\left(\int_{-\infty}^{\infty} \left|\hat{f}(\frac{\xi}{\beta})\right|^2 \left|\beta\frac{\xi}{\beta}\right|^{2s} \frac{\mathrm{d}\xi}{\beta}\right)^{1/2} \\
&= \beta^{\frac{a}{q}-a\gamma-\frac{1}{2}+s} C\left(\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 |\xi|^{2s}\,\mathrm{d}\xi\right)^{1/2} \\
&= \beta^{\frac{a}{q}-a\gamma-\frac{1}{2}+s} C\,\|f\|_{\dot{H}^s(\mathbb{R})}\,.
\end{aligned}$$

This inequality need to be independent of β, so we need the β-term to vanish i.e. that $s = \frac{1}{2} + a\gamma - \frac{a}{q}$. ☐

What is left now is to establish the estimate.

Theorem 3.8. *Let $q \geqslant 2$, and $0 \leqslant \gamma < \frac{1}{q}$, then there exists a number C independent of f such that*

$$\sup_{x\in\mathbb{R}}\left(\int_{-\infty}^{\infty} |t|^{-\gamma q}\,|(S^a f)\,[x](t)|^q\,\mathrm{d}t\right)^{1/q} \leqslant C\,\|f\|_{\dot{H}^{\frac{1}{2}+a\gamma-\frac{a}{q}}(\mathbb{R})}\,.$$

Proof. Assume $q \geqslant 2$, $0 \leqslant \gamma < \frac{1}{q}$ and f such that $\operatorname{supp}(\hat{f}) \subset [0,\infty)$. As in the proof of Theorem 3.3 we begin by observing that

$$(S^a f)\,[x](t) = \hat{g_x}(-t)$$

where

$$g_x(\eta) = \begin{cases} \frac{e^{i\eta^{1/a}x}\hat{f}(\eta^{1/a})}{a\eta^{1-1/a}} & \text{if } \eta \geqslant 0, \\ 0 & \text{if } \eta < 0. \end{cases}$$

Now let $\alpha = \frac{1}{2} + \gamma - \frac{1}{q}$. From the restriction on q and γ we get that $0 \leqslant \alpha < \frac{1}{2}$ and so using Pitt's inequality (Theorem 3.5) with $p = 2$, we have that there is a number C independent of f such that

$$\begin{aligned}
\left\| \left(\int_{-\infty}^{\infty} |(S^a f)[\cdot](t)|^q |t|^{-\gamma q} \,\mathrm{d}t \right)^{1/q} \right\|_{L^\infty(\mathbb{R})} &= \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |(S^a f)[x](t)|^q |t|^{-\gamma q} \,\mathrm{d}t \right)^{1/q} \\
&= \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |\widehat{g}_x(-t)|^q |t|^{-\gamma q} \,\mathrm{d}t \right)^{1/q} \\
&\leqslant \sup_{x\in\mathbb{R}} C \left(\int_{-\infty}^{\infty} |g_x(\eta)|^2 |\eta|^{2\alpha} \,\mathrm{d}\eta \right)^{1/2} \\
&= \sup_{x\in\mathbb{R}} C \left(\int_{-\infty}^{\infty} \left| \frac{e^{i\eta^{1/a}x} \widehat{f}(\eta^{1/a})}{a\eta^{1-1/a}} \right|^2 |\eta|^{2\alpha} \,\mathrm{d}\eta \right)^{1/2} \\
&= C \left(\int_{-\infty}^{\infty} \left| \widehat{f}(\eta^{1/a}) \right|^2 |\eta|^{2\alpha+\frac{2}{a}-2} \,\mathrm{d}\eta \right)^{1/2} \\
\begin{bmatrix} \eta & = \xi^a \\ \mathrm{d}\eta & = a\xi^{a-1}\mathrm{d}\xi \end{bmatrix} &= C \left(\int_{-\infty}^{\infty} \left| \widehat{f}(\xi) \right|^2 |\xi|^{2\alpha a+2-2a} \xi^{a-1} \mathrm{d}\xi \right)^{1/2} \\
&= C \left(\int_{-\infty}^{\infty} \left| \widehat{f}(\xi) \right|^2 |\xi|^{2(\alpha a-\frac{a}{2}+\frac{1}{2})} \,\mathrm{d}\xi \right)^{1/2} \\
&= C \|f\|_{\dot{H}^{\alpha a-\frac{a}{2}+\frac{1}{2}}(\mathbb{R})}.
\end{aligned}$$

Using that $\alpha = \frac{1}{2} + \gamma - \frac{1}{q}$ we have that

$$\alpha a - \frac{a}{2} + \frac{1}{2} = (\frac{1}{2} + \gamma - \frac{1}{q})a - \frac{a}{2} + \frac{1}{2} = \frac{1}{2} + a\gamma - \frac{a}{q},$$

and so for f such that $\mathrm{supp}(\widehat{f}) \subset [0, \infty)$ there exists a number C independent of f such that

$$\left\| \left(\int_{-\infty}^{\infty} |(S^a f)[\cdot](t)|^q |t|^{-\gamma q} \,\mathrm{d}t \right)^{1/q} \right\|_{L^\infty(\mathbb{R})} \leqslant C \|f\|_{\dot{H}^{\frac{1}{2}+a\gamma-\frac{a}{q}}(\mathbb{R})}.$$

Now assume instead that $\mathrm{supp}(\widehat{f}) \subset (-\infty, 0]$. We have that

$$(S^a f)[x](t) = \int_0^\infty e^{-ix\xi} e^{it|\xi|^a} \tilde{\widehat{f}}(\xi) \mathrm{d}\xi,$$

where $\tilde{\widehat{f}}(\xi) = \widehat{f}(-\xi)$. The result now follows analogously with the one above and we have that

$$\begin{aligned}
\left\| \left(\int_{-\infty}^{\infty} |(S^a f)[\cdot](t)|^q |t|^{-\gamma q} \,\mathrm{d}t \right)^{1/q} \right\|_{L^\infty(\mathbb{R})} &\leqslant C \left(\int_{-\infty}^{\infty} \left| \tilde{\widehat{f}}(\xi) \right|^2 |\xi|^{2(\alpha a-\frac{a}{2}+\frac{1}{2})} \,\mathrm{d}\xi \right)^{1/2} \\
&= C \|f\|_{\dot{H}^{\alpha a-\frac{a}{2}+\frac{1}{2}}(\mathbb{R})} = C \|f\|_{\dot{H}^{\frac{1}{2}+a\gamma-\frac{a}{q}}(\mathbb{R})}.
\end{aligned}$$

Finally, given a function f such that $\mathrm{supp}(\widehat{f})$ is not a subset of neither $[0, \infty)$ nor $(-\infty, 0]$. Then, using the same method of the previous section, we divide $\widehat{f}$

into a positive, $\widehat{f_+}$, and a negative, $\widehat{f_-}$, part such that $\hat{f} = \widehat{f_+} + \widehat{f_-}$. And so, using the Minkowski inequality, we now get

$$\begin{aligned}
&\left\| \left(\int_{-\infty}^{\infty} \left| (S^a f)\,[\cdot](t) \right|^q |t|^{-\gamma q}\,\mathrm{d}t \right)^{1/q} \right\|_{L^\infty(\mathbb{R})} \\
&\quad \leqslant \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{i\xi x} e^{i|\xi|^a t} \widehat{f_+}(\xi) d\xi \right|^q |t|^{-\gamma q}\,\mathrm{d}t \right)^{1/q} \\
&\qquad + \sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{i\xi x} e^{i|\xi|^a t} \widehat{f_-}(\xi) d\xi \right|^q |t|^{-\gamma q}\,\mathrm{d}t \right)^{1/q} \\
&\quad = C \|f\|_{\dot{H}^{\alpha a - \frac{a}{2} + \frac{1}{2}}(\mathbb{R})} = C \|f\|_{\dot{H}^{\frac{1}{2} + a\gamma - \frac{a}{q}}(\mathbb{R})}
\end{aligned}$$

□

Note This result aligns with our previous findings for $q = 2$ and $\gamma = 0$. Furthermore this can once again give us an L^2-estimate by noting that for $s = 0$ the $\dot{H}^s$-norm is the L^2 i.e. for $\frac{1}{2} + a\gamma - \frac{a}{q} = 0$, or rather $\gamma = \frac{1}{q} - \frac{1}{2a}$ we have the following estimate.

Corollary 3.9. *Let $2 \leqslant q \leqslant 2a$. Then there exists a number C independent of f such that*

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| (S^a f)\,[x](t) \right|^q |t|^{q/2a-1} \mathrm{d}t \right)^{1/q} \leqslant C \|f\|_{L^2(\mathbb{R})}.$$

Note: In the case $a = 2$ we have, by this corollary, for $2 \leqslant q \leqslant 4$ that

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| \left(S^2 f\right) [x](t) \right|^q |t|^{q/4-1} \mathrm{d}t \right)^{1/q} \leqslant C \|f\|_{L^2(\mathbb{R})}.$$

If we want to make a direct comparison with the results in the previous section, i.e. a comparison with the $\dot{H}^s$-norm in Theorem 3.8 for $s = (1-a)/2$. To achieve this we use that $\gamma = \frac{1}{q} - \frac{1}{2}$, giving us the following estimate.

Corollary 3.10. *Let $2 \leqslant q \leqslant 2a$. There exists a number C independent of f such that*

$$\sup_{x\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |t|^{q/2-1} \left| (S^a f)\,[x](t) \right|^q \mathrm{d}t \right)^{1/q} \leqslant C \|f\|_{\dot{H}^{(1-a)/2}(\mathbb{R})}.$$

This is to be compared with the estimate in Theorem 3.3, which states that

$$\|S^a f\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}))} \leqslant C \|f\|_{\dot{H}^{(1-a)/2}(\mathbb{R})}$$

and we can see that we have expanded the results from the previous section.

3.2.1 Stronger estimate

As we could see in Corollary 3.9 above, we have established limits on q for these estimates to hold, namely that $2 \leqslant q \leqslant 2a$. Comparing these results to Strichartz estimates for one space dimension, which for $q \leqslant 2 + 2a$ states that [11, p. 369]

$$\left(\iint_{K \subset \mathbb{R}^2} \left| (S^a f)\,[x](t) \right|^q \mathrm{d}x\mathrm{d}t \right)^{1/q} \leqslant A \left\| f \right\|_{L^2(\mathbb{R})},$$

we can see that, while interesting, our results are locally surpassed by these estimates, e.g. we can for $a = 2$ at most get a estimate for $q = 4$, while similar for Strichartz is $q = 6$.
To tackle that our estimate is somewhat limited in comparison to other results we restrict ourselves to an even stricter class of test functions, namely those that have a Fourier transform with compact support on the real axis, i.e. f such that $\hat{f} \in C_0^\infty(\mathbb{R})$, with the added restriction that the support cannot contain zero.

We begin by noting the following result for these functions.

Lemma 3.11. *Let f such that $\operatorname{supp}(\hat{f}) \subset [b, c] \not\ni 0$, then there exists a number C independent of f such that*

$$\left\| f \right\|_{\dot{H}^s(\mathbb{R})} \leqslant C \left\| f \right\|_{L^2(\mathbb{R})}.$$

Note: While C is independent of f it is not independent of s, as we see in the proof below.

Proof. Assume that f such that $\operatorname{supp}(\hat{f}) \subset [b, c] \not\ni 0$. Let $C = \sqrt{2\pi} \max(|b|^s, |c|^s)$. Now, using the Plancherel theorem (Theorem 2.14), we have

$$\begin{aligned}
\left\| f \right\|_{\dot{H}^s(\mathbb{R})} &= \left(\int_{-\infty}^{\infty} |\xi|^{2s} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} = \left(\int_b^c |\xi|^{2s} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&\leqslant \frac{C}{\sqrt{2\pi}} \left(\int_b^c \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} = \frac{C}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&= \frac{C}{\sqrt{2\pi}} \left\| \hat{f} \right\|_{L^2(\mathbb{R})} = C \left\| f \right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

□

Now, this lemma gives an indication of the results to come. What we can see is that an estimate based on the homogeneous Sobolev norm of f can, for these functions, be replaced with a estimate based on a $L^2(\mathbb{R})$-norm of f without adding restrictions on s, as we previously did to get a $L^2(\mathbb{R})$-norm. In theorem 3.8 our weighted estimate is restricted by the term s in the $\dot{H}^s(\mathbb{R})$-norm in that we established that $s = \frac{1}{2} + a\gamma - \frac{a}{q}$ for this estimate to hold. This restriction can now be removed leaving q bound only by the conditions necessary for the use of Pitt's inequality, giving us the following result.

Theorem 3.12. *Let $K = [x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2$, $q \geqslant 2$ and f be such that $\operatorname{supp}(\widehat{f}) \subset [b, c]$ for numbers b, c such that $0 \notin [b, c]$. Then there exists a number C independent of f such that*

$$\left(\iint_K |(S^a f)[x](t)|^q \, \mathrm{d}x\mathrm{d}t\right)^{1/q} \leqslant C \|f\|_{L^2(\mathbb{R})}$$

Proof. Assume that $K = [x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2$, $q \geqslant 2$ and f such that $\operatorname{supp}(\widehat{f}) \subset [b, c] \not\ni 0$. We have

$$\left(\iint_K |(S^a f)[x](t)|^q \, \mathrm{d}x\mathrm{d}t\right)^{1/q} \leqslant \left(\sup_{x \in [x_1, x_2]} A \int_{t_1}^{t_2} |(S^a f)[x](t)|^q \, \mathrm{d}t\right)^{1/q}$$

where $A = |x_2 - x_1|$.
Now let $0 \leqslant \gamma < 1/q$ and $B = \left(\max\{|t|^{\gamma q}, t \in [t_1, t_2]\}\right)^{1/q}$, then

$$\begin{aligned}\left(\sup_{x \in [x_1, x_2]} \int_{t_1}^{t_2} |(S^a f)[x](t)|^q \, \mathrm{d}t\right)^{1/q} &\leqslant B \left(\sup_{x \in [x_1, x_2]} \int_{t_1}^{t_2} |(S^a f)[x](t)|^q \, |t|^{-\gamma q} \, \mathrm{d}t\right)^{1/q} \\ &\leqslant B \left(\sup_{x \in [x_1, x_2]} \int_{-\infty}^{\infty} |(S^a f)[x](t)|^q \, |t|^{-\gamma q} \, \mathrm{d}t\right)^{1/q} \\ &\leqslant B \left(\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(S^a f)[x](t)|^q \, |t|^{-\gamma q} \, \mathrm{d}t\right)^{1/q}\end{aligned}$$

And so by Theorem 3.8 and by Lemma 3.11 we have that there is a C independent of f such that

$$\left(\iint_K |(S^a f)[x](t)|^q \, \mathrm{d}x\mathrm{d}t\right)^{1/q} \leqslant C \|f\|_{L^2(\mathbb{R})}.$$

□

And so $S^a f$ is locally L^q for $q \geqslant 2$ as long as the initial datum f is in $L^2(\mathbb{R})$ and has a Fourier transform of compact support not containing zero.

4 Final remarks

We have established regularity estimates for the functions $S^a f$ and $S^\varphi f$ under the assumption that the initial data function f is a Schwartz function. As we have stated in Corollary 2.15 we can extend the Fourier transform and Plancherel theorem to $L^2(\mathbb{R})$, and as such it seems possible to, by density, expand our result to the case of functions in $L^2(\mathbb{R})$.

References

[1] Helmut Abels. Pseudodifferential and singular integral operators. De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012. An introduction with applications.

[2] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. Fourier analysis and nonlinear partial differential equations, volume 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011.

[3] Saleh Baqer and Lyubomir Boyadjiev. Fractional Schrödinger equation with zero and linear potentials. Fract. Calc. Appl. Anal., 19(4):973–988, 2016.

[4] H. Benson. University Physics. Wiley, 1996.

[5] Donald L. Cohn. Measure theory. Birkhäuser Boston, Inc., Boston, MA, 1993. Reprint of the 1980 original.

[6] Gregory Eskin. Lectures on linear partial differential equations, volume 123 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[7] Avner Friedman. Foundations of modern analysis. Dover Publications, Inc., New York, 1982. Reprint of the 1970 original.

[8] Lars Hörmander. The analysis of linear partial differential operators. I, volume 256 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[9] Benjamin Muckenhoupt. Weighted norm inequalities for the Fourier transform. Trans. Amer. Math. Soc., 276(2):729–742, 1983.

[10] E. B. Saff and A. D. Snider. Fundamentals of Complex Analysis, third edition. Pearson Education Inc., 2003.

[11] Elias M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[12] Elias M. Stein and Rami Shakarchi. Fourier analysis, volume 1 of Princeton Lectures in Analysis. Princeton University Press, Princeton, NJ, 2003. An introduction.

[13] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. Proc. Amer. Math. Soc., 102(4):874–878, 1988.

[14] B. G. Walther. Homogeneous estimates for oscillatory integrals. Acta Math. Univ. Comenian. (N.S.), 69(2):151–171, 2000.

[15] M. W. Wong. An introduction to pseudo-differential operators, volume 6 of Series on Analysis, Applications and Computation. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, third edition, 2014.