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Spectral graph theory and graph connectivity

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1 Introduction

This paper aims to discuss the concept of graph connectivity and its relation to the spectrum of a graph. We begin by introducing some basic concepts of graph theory ranging from simple terminology to graph presentations using matrices. Unless otherwise stated, this study will consider simple graphs, meaning they are unweighted and undirected.

Throughout this paper we focus on the properties of the discrete Laplacian matrix of a graph. It can be used to prove many useful properties, including calculating the number of spanning trees of a graph, spectral clustering and much more. We discuss several different special types of graphs and their spectrum; however our main focus in will be the algebraic connectivity of a graph, or rather the second smallest eigenvalue of its Laplacian matrix. This particular eigenvalue gives us information about how well-connected a graph is, which motivates our interest in bounding its value for graphs where it might be difficult to calculate explicitly. For this reason we discuss at length the applications of the Courant-Fischer theorem in bounding the eigenvalues of a matrix and how we can apply those bounds to a path graph. Naturally we could expand this technique to the spectrum of other graphs as well but the path graph gives us a sufficiently good idea of how it is done.

Bounding the eigenvalues of a graph's Laplacian has many applications outside of pure academic activity. In fact, the Laplacian matrix of a graph is instrumental for spectral clustering, commonly used in data science problems today. One such application is the clustering of images, which use the eigenvalues of a graph's Laplacian to find similarities between data points.

2 Basics of graph theory

In this section we will introduce some basic concepts of graph theory which will help us understand some of the more complicated problems in the field. As such, we will first define what a graph is and how it is structured to then define some graph presentations that can help us analyze the graph algebraically.

2.1 What is a graph?

Geometrically a graph is a set of points and lines that connect these points together. Or rather, as mathematicians tend to call them, sets of vertices and edges. In a graph it is common to denote the set of vertices as $V = \{v_1, ..., v_n\}$ and the set of edges as $E = \{e_1, ..., e_m\}$ where n and m are the number of vertices and edges respectively. A common way to illustrate an edge in a graph is as a relation of vertices. For example, for G in Figure 1, its vertices are called $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{e_1, e_2, e_3, e_4\}$. However as we will see soon it is often helpful to think of its edges as pairs of vertices $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$. With this in mind we are ready to define a graph.

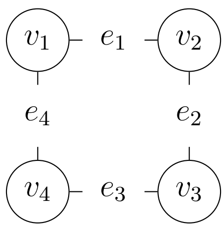


Figure 1: Graph G

Definition 2.1. *A graph G is a set of vertices V together with a set of edges E. Notation*

$$G = (V, E).$$

Let us introduce some terminology to help us structure our work. As stated earlier we can interpret the edges of a graph as pairs of vertices. These pairs of vertices connected by an edge are called adjacent, otherwise two unconnected vertices are called disjoint. For example, look at Figure 1 and notice that (v_1, v_2) are adjacent, while (v_1, v_3) are disjoint. Now we are ready to formally define adjacency and degree of a vertex.

Definition 2.2. *In a graph $G = (V, E)$, two vertices $v_i, v_j \in V$ are adjacent if $\{v_i, v_j\} \in E$.*

Definition 2.3. *The degree $d(v_i)$ of a vertex v_i is the number of vertices in G adjacent to v_i.*

We say that a graph is **complete** if each pair of vertices is connected by an edge, or in other words, if every vertex in a graph G is adjacent to every other vertex. Naturally for a complete graph with n vertices, each vertex has degree $n-1$, since it is adjacent to every other vertex except for itself.

Furthermore we will define four presentations of graphs in the form of four different matrices. One such presentation is called the adjacency matrix of a graph G which we will denote as A_G. As the name suggests it is derived from looking at the adjacency relation of vertices. We will also look at the Laplacian, degree and incidence matrices, all of which we will use later. In order to understand the adjacency matrix we will introduce the so-called adjacency list.

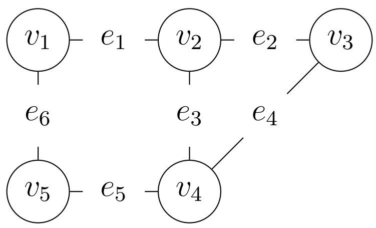


Figure 2: Graph G

Let us for this reason consider graph G in Figure 2. For every vertex v_i in G, with $i = \{1, 2, 3, 4, 5\}$, form a list of all adjacent vertices and call it the adjacency list. In G we see that

- v_1: v_2, v_5,
- v_2: v_1, v_3, v_4,
- v_3: v_2, v_4,
- v_4: v_2, v_3, v_5,
- v_5: v_1, v_4.

Now we can produce the adjacency list, denoted by $L(G)$ as

$$L(G) = \{\{v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3, v_5\}\{v_1, v_4\}\}.$$

Using this list we can easily define the adjacency matrix of a graph and calculate it for our particular graph G in Figure 2.

2.2 Graph presentations

Definition 2.4. *The adjacency matrix A_G of a graph $G = (V, E)$, with the vertex set $V = \{v_1, ...v_n\}$, is a square $n \times n$ matrix with entries $a_{i,j}$ given by*

$$a_{i,j} = \begin{cases} 1 & \text{if} \quad \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

For graph G in Figure 2, the adjacency matrix A_G is given by

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

Notice that if we compare the information in the adjacency matrix A_G with the adjacency list $L(G)$ it is clear that every row represents a vertex in the graph with ones corresponding to adjacent vertices. With the adjacency matrix properly defined we will show how the vertex degrees can be presented as a degree matrix of a graph G. The degree matrix of a graph G is a diagonal matrix which contains information of the degree of each vertex in G. Namely,

Definition 2.5. *The degree matrix D_G of a graph $G = (V, E)$, with entries $d_{i,j}$ given by*

$$d_{i,j} = \begin{cases} d(v_i) & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

For example, for graph G in Figure 2, the degree matrix D_G equals

$$D_G = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} .$$

In fact, we will see later that the degree matrix of a graph has a close relation with the spectrum of a graph.

We can now define the Laplacian matrix L_G of a simple graph G as $L_G = D_G - A_G$ where D_G is the degree matrix and A_G is the adjacency matrix. Since G is a simple graph, A_G only contains $1s$ or $0s$. We can now give a formal definition of a Laplacian matrix.

Definition 2.6. *The Laplacian matrix L_G of a graph $G = (V, E)$, with entries $l_{i,j}$ given by*

$$l_{i,j} = \begin{cases} d(v_i) & \textit{if} \quad i = j \\ -1 & \textit{if} \quad \{i, j\} \in E \\ 0 & \textit{otherwise.} \end{cases}$$

From now on we will sometimes say Laplacian instead of Laplacian matrix. As with previous matrix presentation we will calculate the Laplacian of G in Figure 2. We get

$$L_G = D_G - A_G = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Below we will mostly focus on the Laplacian matrix presentation. However for a broader understanding of graphs we will also define the concept of incidence and that of the incidence matrix. In fact, we will see that the Laplacian can be found by studying the incidence matrix, which provide a helpful tool to prove a number of properties that the Laplacian matrix possesses.

Definition 2.7. *A vertex $v \in V$ is incident with an edge $\{v_i, v_j\} \in E$ if either $v = v_i$ or $v = v_j$.*

Much like the adjacency relation we can, using the incidence relation, introduce another matrix presentation that is relevant to our study. To define this vertex-edge incidence matrix, we must first consider what it means for a graph to be oriented. For example, look at the graph in Figure 3. It is almost the same graph as the graph in Figure 2 on page 4 with the only exception being that it is oriented, meaning that the edges are equipped with certain directions. In the graph this is illustrated with arrows on the edges. It is possible to define the incidence matrix without this property, however the resulting matrix will not be as useful for our particular study. As such, we can formally define the incidence matrix of an oriented graph.

Definition 2.8. *The oriented incidence matrix E_G of a graph $G = (V, E)$, with n vertices and m edges is an $n \times m$ matrix with entries*

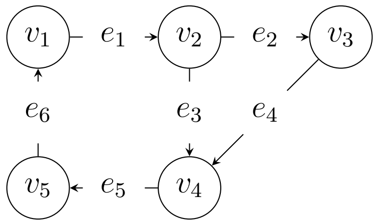


Figure 3: Graph G

E_e, v given by

$$E_e, v = \begin{cases} 1 \quad \textit{if} \quad e = (v, w) \quad \textit{and} \quad v \to w, \\ -1 \quad \textit{if} \quad e = (v, w) \quad \textit{and} \quad v \to w, \\ 0 \quad \textit{otherwise.} \end{cases}$$

Now let us consider the graph G in Figure 3 and the incidence matrix E_G. One has,

$$E_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} .$$

It can easily be shown for an oriented graph that the incidence matrix can be used to calculate the Laplacian matrix. In fact by multiplying the transpose of the incidence matrix by itself we get the Laplacian matrix,

$$E_G^T E_G = L_G.$$

This is a handy way to factorise the Laplacian matrix, which in fact is a tool that can be used to prove a number of useful properties that the Laplacian possesses.

3 Properties of the Laplacian matrix

One of the fundamental properties of a graph is its connectivity. We can use the Laplacian matrix defined on page 4 to study graph connectivity. To do this we first want to provide an alternative definition of the Laplacian matrix that closely relates to its more useful properties. As such, let us consider the Laplacian of a graph on n vertices consisting of just one edge $e = \{v_1, v_2\}$. Using our definition of the Laplacian matrix we get

$$L_e = \begin{pmatrix} 1 & -1 & 0 & & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} .$$

By adding up all such Laplacians we get a new definition of the Laplacian matrix for the whole graph.

Definition 3.1. *For a graph $G = (V, E)$, $L_G = \sum\limits_{e \in E} L_e$.*

Using this definition we will be able to prove a number of properties of the Laplacian matrix by first proving them for one edge and then adding them up. Recall that a matrix A is called symmetric if $A^T = A$, where A^T is the transpose of the matrix. Since it is clear that the Laplacian matrix of a graph is always symmetric we will use this to show additional properties of the Laplacian matrix.

Definition 3.2. *A symmetric matrix M is called positive semi-definite if $\forall x \in \mathbb{R}^n$,*

$$x^T M x \geq 0.$$

In terms of the Laplacian matrix of a graph this implies that all of its eigenvalues are non-negative. As such let us show that the Laplacian matrix of any graph has this property. Consider the Laplacian of an edge

$$L_e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus [zeros].$$

Note that

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} .$$

Remember that a vector v is an eigenvector of a matrix M with the eigenvalue λ if $Mv = \lambda v$ and consequently that $v^T M v = v^T \lambda v$. So for a positive semi-definite matrix we know that $v^T \lambda v$ must also be greater than zero. With this in mind we can show that,

$$x^T L_e x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2,$$

which is greater than or equal to zero. Notice that for the whole Laplacian matrix we get

$$x^T L_G x = x^T (\sum_{e \in E} L_e) x = \sum_{e \in E} x^T L_e x = \sum_{(i,j) \in E} (x_i - x_j)^2.$$

This implies that L_G is in fact positive semi-definite, and that its eigenvalues are real and non-negative.

Theorem 3.1. *For a graph G, every eigenvalue λ of the Laplacian matrix L_G is non-negative.*

Proof. Suppose that λ is an eigenvalue and that $x \in R^n$ is a nonzero eigenvector of λ. Then

$$x^T L_G x = x^T (\lambda x) = \lambda (x^T x).$$

Since $x^T L_G x \geq 0$ from L_G being positive semi-definite and $x^T x > 0$, we have that $\lambda \geq 0$. $\square$

Theorem 3.2. *For any symmetric matrix A, including the Laplacian matrix of a graph, every eigenvalue λ of A is real.*

Proof. Let $Av = \lambda v$ with $v \neq 0$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} \lambda \overline{v}^T x &= \overline{v}^T (\lambda x) = \overline{v}^T A v \\ &= (A^T \overline{v})^T v = (\overline{A} \overline{v})^T v = \overline{\lambda} \overline{v}^T v. \end{aligned}$$

Because $v \neq 0$, then $\overline{v}^T v \neq 0$ and $\lambda = \overline{\lambda}$. $\square$

Now knowing that the eigenvalues of the Laplacian matrix are in fact real and non-negative that L_G has an orthogonal basis consisting of eigenvectors of L_G. Therefore, since G has n vertices, there exists n eigenvalues for L_G. Since they are all non-negative we can conclude that

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n.$$

4 Connectivity

4.1 Graph connectivity

Below we discuss graph connectivity and how the spectrum of the Laplacian matrix can help us to study it. However what does it really mean for a graph to be connected and can we quantify how strongly a graph is connected? We will begin to answer these questions by first defining the concept of a path in a graph.

Definition 4.1. *A path is a sequence of edges, one following the other where no vertex may appear more than once.*

Using the definition of a path we can easily define what it means for a graph to be connected.

Definition 4.2. *For a non-empty graph G, we say that it is connected if there is a path between any two of its vertices and disconnected otherwise.*

Recall the definition of L_G. It is clear that if all entries of x are the same, then $x^T L_G x$ is zero. Thus consequently, $L_G x = 0$, showing that the constant vectors are eigenvectors with eigenvalue 0. Knowing this, we can start discussing what the eigenvalues of a Laplacian matrix can tell us about the connectivity of a graph.

Proposition 4.1. *Let $G = (V, E)$ be a graph, and let $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be eigenvalues of the Laplacian of G. Then if G is connected if $\lambda_2 > 0$.*

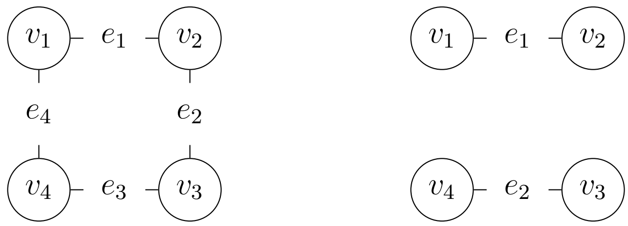
Proof. Assume that G is connected and that x is an eigenvector of L_G with eigenvalue 0. Then we have that

$$xL_G x^T = \sum_{(u,v)\in E} (x_u - x_v)^2 = 0$$

Thus for every pair of vertices connected by an edge, we have $x_u = x_v$. Since for a connected graph every pair of vertices are connected by a path, we conclude that $x_u = x_v$ for all vertices $(u, v) \in V$. Thus x must be a constant vector the multiplicity of eigenvalue 0 is 1. It follows that $\lambda_2 \neq 0$, so $\lambda_2 > 0$ since all eigenvalues are non-negative as shown previously. □

In fact, the multiplicity of the eigenvalue 0 of L_G is exactly the number of connected components in G. We say that a connected component of a graph G is a subgraph of an undirected graph in which any two vertices are connected to each other by a path. For example, look at graph H in Figure 4 and notice that it has two distinct components where vertices are connected by a path, meaning that it has two connected components. Whereas graph G in Figure 4 is connected i.e it has only one component. This leads us to believe that the number of eigenvalues that are 0 in graph G is 1, while in graph H there are 2. This is because if we apply Proposition 4.1 to each connected component in H, we get that they both have an eigenvalue $\lambda_1 = 0$ and a nonzero eigenvalue $\lambda_2 > 0$. The multiplicity of eigenvalue 0 is often called the dimension of the nullspace of a matrix. The second smallest eigenvalue if often referred to as the algebraic connectivity as it has a connection with the overall connectivity of a graph. Let us formally define it as it is relevant to our particular study.

Definition 4.3. *The algebraic connectivity of a graph G, as introduced by Fiedler, is the second smallest eigenvalue of the Laplacian matrix of G.*



(a) Connected graph G (b) Disconnected graph H

Figure 4: Graph components

Corollary 4.1.1. *Let $G = (V, E)$ be a graph. Then the multiplicity of* 0 *as an eigenvalue is the number of connected components of G.*

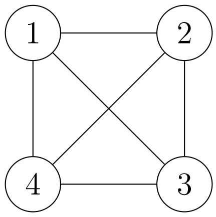
Proof. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), ..., G_k = (V_k, E_k)$ be the number of connected components, or rather connected subgraphs of the graph G. Then by Proposition 4.1 it follows that each connected component has eigenvalue 0 with multiplicity 1. Thus we have that the multiplicity of the eigenvalue 0 of L_G must be the number of such connected components, since they are clearly linearly independent. $\square$

We can now conclude that for a graph G, we have that $\lambda_k = 0$ if and only if G has at least k connected components.

4.2 Examples of Laplacian matrices

Now let us look at some concrete examples of graphs and see how their spectrum looks like when we change the number of edges. The eigenvalues explicitly calculated are found by solving the characteristic equation commonly found in many textbooks of linear algebra. The characteristic equation stated that for a matrix A, its eigenvalues λ can be found from the equation $\det(A - \lambda I) = 0$, where I is the identity matrix.

It is clear from the previous section that this graph is connected. As such the eigenvalue 0 has dimension 1. Since graph G is a complete graph on 4 vertices, the dimension of the eigenvalue 4 is clearly 3, which we prove



(a) Complete graph G

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

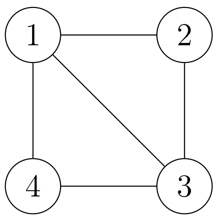
(b) Laplacian matrix of G

Figure 5: Graph G and its corresponding 4×4 Laplacian matrix

later. However this can also be seen algebraically by simply calculating the Laplacians eigenvalues from Figure 5, which are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_2 = \lambda_3 = 4.$$

An interesting observation is that the sum of all vertex degrees is equal to the sum of all eigenvalues. In fact we will see later that this is true for all graphs.



(a) Graph G

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

(b) Laplacian matrix of G

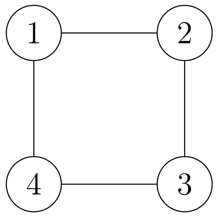
Figure 6: Graph G and its corresponding 4×4 Laplacian matrix

Now let us see what happens with the spectrum of a graph if we remove one of its edges. Before making any calculations we can assume that the graph's algebraic connectivity λ_2 will be less than 4, since we have made it "less" connected. We also know, from Proposition 4.1, that the nullspace has dimension 1 since it is still a connected graph.

If we calculate the eigenvalues of Figure 6 we get

$$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = \lambda_4 = 4.$$

Notice that by removing one edge we reduce the algebraic connectivity by exactly two. We can also observe that the sum of all vertex degrees is again equal to the sum of all eigenvalues. Let us continue by removing another edge.



(a) Graph G

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

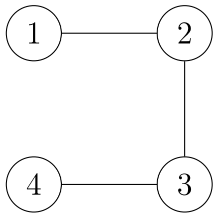
(b) Laplacian matrix of G

Figure 7: Graph G and its corresponding 4×4 Laplacian matrix

The eigenvalues of the Laplacian matrix of graph G in Figure 7 are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 2, \quad \lambda_4 = 4.$$

As with the other graphs we can observe that the eigenvalues reduce in value when we make the graph "less" connected. Let us continue by removing another edge.



(a) Graph G

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(b) Laplacian matrix of G

Figure 8: Graph G and its corresponding 4×4 Laplacian matrix

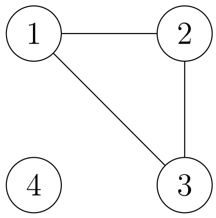
The eigenvalues of the Laplacian matrix of the graph G in Figure 8 are

$$\lambda_1 = 0, \quad \lambda_2 = 2 - \sqrt{2}, \quad \lambda_3 = 2, \quad \lambda_4 = 2 + \sqrt{2}.$$

The graph in Figure 8 has the least number of edges while still being connected and it is clear that if we remove another edge the graph would become disconnected. For that reason, let us look at a disconnected graph to see how the spectrum changes.

In Figure 9 we get that G is no longer connected, implying that the nullspace will now have dimension 2. We can view each connected component in G as subgraphs G_1 with 1 vertex and G_2 with 3 vertices. By Corollary 4.1.1, we have that the nullspace of G must have dimension 2, which confirms our visual assessment. We can also see that since G_2 is a complete graph, its eigenvalues not equal to 0 will be 3. Thus we get that the eigenvalues of G are

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = 3.$$



(a) Disconnected graph G

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Laplacian matrix of G

Figure 9: Graph G and its corresponding 4×4 Laplacian matrix

This can easily be confirmed by solving the characteristic equation.

Now that we have seen some examples of how connectivity is related to the spectrum of a graph's Laplacian matrix it is clear that the second smallest eigenvalue λ_2 is important to the problem. However, as of now, we have no way of evaluating eigenvalues of graphs with complicated Laplacians. In the next section, we will discuss how we can bound the eigenvalues of a matrix. In particular, we are interested in approximating the algebraic connectivity of a graph to see what it can tell us about the graph.

5 Bounding eigenvalues

5.1 Courant-Fischer theorem and Rayleigh quotients

Studying the Laplacian matrix can tell us many things about the connectivity of a graph. As seen previously, we saw that the dimension of the nullspace tell us how many connected components its graph has. However, when discussing connectivity it is also interesting how strongly a graph is connected. The algebraic connectivity is directly related to such questions. Therefore we will now discuss how to bound λ_2 as well as λ_n to find a relation between a graph's spectrum and its connectivity.

We will begin by looking at the general bounds of the sum of a Laplacian's eigenvalues to draw further conclusions from that.

Lemma 5.1. *For a graph G with n vertices of degree d_i, with $i = 1, ..., n$ and a Laplacian L_G with eigenvalues λ_i we have that,*

$$\sum_i \lambda_i = \sum_i d_i \leq n(n-1).$$

Proof. The first two expressions are the trace of L_G so they must be equal. The maximum value of the sum of all vertex-degrees must occur when every vertex is adjacent to every other vertex, that is when each vertex has degree $(n-1)$. The sum of n vertices with $n-1$ degrees is $n(n-1)$. $\square$

We can use the previous lemma to create a statement about the bounds of λ_2 and λ_n.

Lemma 5.2. *With λ_i and d_i as above, we get*

$$\lambda_2 \leq \frac{\sum_i d_i}{n-1},$$

$$\lambda_n \geq \frac{\sum_i d_i}{n-1}.$$

Proof. By the previous lemma and the fact that for a graph $\lambda_1 = 0$, we get that

$$\sum_{i=2}^{n} \lambda_i = \sum_i d_i.$$

Since $\lambda_2 \leq .... \leq \lambda_n$, the bounds follow immediately. $\square$

Now we would also be interested in the upper bounds of the eigenvalues. Let us take a closer look at the Courant-Fischer formula that gives such bounds for a symmetric matrix.

Theorem 5.3. *For any symmetric $n \times n$ matrix A with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and corresponding eigenvectors $v_1, v_2, ..., v_n$, one has*

$$\lambda_1 = \min_{||x||=1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_2 = \min_{\substack{||x||=1 \\ x \perp v_1}} x^T A x = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T A x}{x^T x},$$

$$\vdots$$

$$\lambda_n = \lambda_{max} = \max_{||x||=1} x^T A x = \max_{x \neq o} \frac{x^T A x}{x^T x}.$$

In general, for $1 \leq k \leq n$, let S_k denote the span of $v_1, ..., v_k$, and let $S_k^{\perp}$ denote the orthogonal complement of S_k. Then,

$$\lambda_k = \min_{\substack{||x||=1 \\ x \in S_{k-1}^{\perp}}} x^T A x = \min_{\substack{x \neq 0 \\ x \in S_{k-1}^{\perp}}} \frac{x^T A x}{x^T x}.$$

Proof. Consider the spectral decomposition $A = Q^T \mathbb{A} Q$ of A, where $\mathbb{A}$ is the diagonal matrix of eigenvalues. We observe that

$$\begin{aligned} x^T A x &= x^T Q^T \mathbb{A} Q x \\ &= (Qx)^T \mathbb{A} (Qx). \end{aligned}$$

Since by definition Q is orthogonal it suffices to consider the case when $A = \mathbb{A}$ is a diagonal matrix with eigenvalues in the diagonal. Then we can write

$$x^T A x = \sum_{i=1}^{n} \lambda_i x_i^2.$$

Notice that when A is diagonal, the eigenvectors of A are $v_k = e_k$, i.e $(e_k)_i = 1$ if $i = k$ and 0 otherwise. Then the condition $x \in S_{k-1}^{\perp}$ implies $x \perp e_i$ for $i = 1, ..., k-1$, so $x_i = 0$. Therefore, for $x \in S_{k-1}^{\perp}$ with $|| \, x \, ||= 1$, we have

$$\begin{aligned} x^T A x &= \sum_{i=1}^{n} \lambda_i x_i^2 \\ &= \sum_{i=k}^{n} \lambda_i x_i^2 \geq \lambda_k \sum_{i=k}^{n} x_i^2 \\ &= \lambda_k \, || \, x \, ||^2 = \lambda_k. \end{aligned}$$

On the other hand, plugging in $x = e_k$ yields $x^T A x = (e_k)^T A e_k = \lambda_k$. This shows that

$$\lambda_k = \min_{\substack{||x||=1 \\ x \in S_{k-1}^{\perp}}} x^T A x.$$

The same argument holds to show that λ_{max} holds, but instead we plug in $x = e_n$, giving us

$$\lambda_{max} = \max_{||x||=1} x^T A x.$$

□

Since this is true for any symmetric $n \times n$ matrix we can in particular apply this fact to the Laplacian matrix of a graph. Since we defined the Laplacian on page 9 in its quadratic form we can use the Courant-Fischer theorem to get a very helpful expression of its eigenvalues. Such expression of eigenvalues is called the Rayleigh quotient.

Corollary 5.3.1. *Let $G = (V, E)$ be a graph and let L_G be the Laplacian of G. We already know that $\lambda_1 = 0$ and that v_1 is the vector with entries equal to 1. Then by the Courant-Fischer formula,*

$$\lambda_2 = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T L_G x}{x^T x} = \min_{\substack{x \neq 0 \\ x \perp 1}} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2},$$

$$\lambda_{max} = \max_{x \neq 0} \frac{x^T L_G x}{x^T x} = \max_{x \neq 0} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i \in V} x_i^2}.$$

The Rayleigh quotient is a helpful tool to aproximate the eigenvalues of matrices that are usually difficult to diagonalize. However in many problems we don't need the exact value of λ to evaluate its connectivity. Instead we just construct a vector with a small Rayleigh quotient to find an aproximation of the algebraic connectivity of a graph. Similarly one could construct a vector with a large Rayleigh quotient to find a lower bound of the largest eigenvalue of a graph. As such, we can easily find a lower bound of λ_n of the Laplacian of a graph.

Lemma 5.4. *Let $G = (V, E)$ be a graph with $V = \{1, 2, ..., n\}$ vertices and $u \in V$. If u has degree d, then*

$$\lambda_n(G) \geq d.$$

Proof. By the Courant-Fischer theorem we have that

$$\lambda_n(G) = \max_{x \neq 0} \frac{x^T L_G x}{x^T x}.$$

Now let $x = e_u$ where $e_1, e_2, ..., e_n$ is the standard basis. Applying the Rayleigh quotient we get that,

$$\frac{e_u^T L_G e_u}{e_u^T e_u} = \frac{\sum\limits_{(u,v)\in E} (x_u - x_v)^2}{\sum x_u^2} = \frac{d}{1} = d.$$

So $\lambda_n(G) \geq \frac{x^T L_G x}{x^T x} = d$. □

This bound however is not very precise and does not tell us much about the connectivity of a graph. Let us therefore improve it slightly.

Lemma 5.5. *Let $G = (V, E)$ be a graph with $V = \{1, 2, ..., n\}$ vertices and $u \in V$. If u has degree d, then*

$$\lambda_n(G) \geq d + 1.$$

Proof. The argument is similar to that in the previous proposition, with the difference that instead we consider the vector x given by,

$$x_i = \begin{cases} d & \text{if} \quad i = u \\ -1 & \text{if} \quad \{i, u\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that,

$$\frac{x^T L_G x}{x^T x} = \frac{\sum\limits_{(u,v)\in E} (x_u - x_v)^2}{\sum x_u^2}$$

$$= \frac{d(d-(-1))^2}{d(-1)^2 + d^2} = \frac{(d+1)^2}{d+1}$$

$$= d+1.$$

So $\lambda_n(G) \geq \frac{x^T L_G x}{x^T x} = d+1$. □

Now with λ_n adequately bound we turn our attention to λ_2. This however proves to be a difficult eigenvalue to approximate generally, so we will begin by looking at some special types of graphs and their spectrum.

5.2 Spectra of some types of graphs

Definition 5.1. *The path graph, P_n on n vertices is a graph $G = (V, E)$ where $V = \{1, 2, ..., n\}$ and $E = \{\{i, i+1\} \mid 1 \leq i < n\}$.*

Definition 5.2. *The cycle graph, C_n on n vertices is the graph $G = (V, E)$ where $v = \{1, 2, ..., n\}$ and $E = \{\{i, i+1\} \mid 1 \leq i \leq n\} \cup \{1, n\}$.*

Proposition 5.6. *The Laplacian of the cycle graph C_n on n vertices has eigenvalues $2 - 2\cos\left(\frac{2\pi k}{n}\right)$ and eigenvectors of the form*

$$x_i(k) = \cos\left(\frac{2\pi k i}{n}\right),$$

$$y_i(k) = \sin\left(\frac{2\pi k i}{n}\right),$$

where $x_i(k)$ denotes the $i-th$ component of the eigenvector for the $k-th$ eigenvalue, $k \leq \frac{n}{2}$.

Proof. To prove this notice that the Laplacian of a cycle graph with n vertices will be of the form,

$$L_{C_n} = \begin{pmatrix} 2 & -1 & 0 & & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & & \vdots \\ & \vdots & & \ddots & -1 \\ -1 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Now let λ be a eigenvalue of x_k. Then x should satisfy the following relation,

$$x_k\lambda = 2x_k(u) - x_k(u+1) - x_k(u-1).$$

Now we can verify this with a simple computation using the proposed formula for the eigenvalues and eigenvectors of the graph.

$$\begin{aligned}
x_k\lambda_u &= 2x_k(u) - x_k(u+1) - x_k(u-1) \\
&= 2\cos\left(\frac{2\pi ku}{n}\right) - \cos\left(\frac{2\pi k(u-1)}{n}\right) - \cos\left(\frac{2\pi k(u+1)}{n}\right) \\
&= 2\cos\left(\frac{2\pi ku}{n}\right) - \cos\left(\frac{2\pi ku}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi ku}{n}\right)\sin\left(\frac{2\pi k}{n}\right) \\
&\quad - \cos\left(\frac{2\pi ku}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi ku}{n}\right)\sin\left(\frac{2\pi k}{n}\right) \\
&= 2\cos\left(\frac{2\pi ku}{n}\right) - 2\cos\left(\frac{2\pi ku}{n}\right)\cos\left(\frac{2\pi k}{n}\right) \\
&= \cos\left(\frac{2\pi ku}{n}\right)(2 - 2\cos\left(\frac{2\pi k}{n}\right)) \\
&= x_k(u)(2 - 2\cos\left(\frac{2\pi k}{n}\right)).
\end{aligned}$$

With $2 - 2\cos\left(\frac{2\pi k}{n}\right)$ being the eigenvalues of the cycle graph we have shown the relation. The computation for y_k follows similarly. □

Proposition 5.7. *The Laplacian of the path graph P_n has the same eigenvalues as C_{2n}, excluding 2. That is P_n has eigenvalues $2-2\cos\left(\frac{\pi k}{n}\right)$ and the associated eigenvectors,*

$$x_k(u) = \cos\left(\frac{\pi ku}{n} - \frac{\pi k}{n}\right),$$

for $0 \leq k < n$.

Proof. To prove this, we treat P_n as a quotient of C_{2n} by identifying vertex i of P_n with both vertices i and $2n+1-i$ of C_{2n}. Then we find an eigenvector v of C_{2_n} such that $v_i = v_{2n+1-i}$ for all vertices i of C_{2n}. Then

$$x = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

is an eigenvector of P_n. Now notice that the Laplacian L_{P_n} has the form

$$\begin{pmatrix} 1 & -1 & 0 & & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & & \vdots \\ & \vdots & & \ddots & -1 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

So if λ is an eigenvalue and x is an eigenvector of P_n, then it must satisfy

$$x_1(u) - x_1(u+1) = \lambda x_1,$$

$$x_n - x_{n-1} = \lambda x_n \quad \text{and}$$
$$2x_k(u) - x_k(u+1) - x_k(u-1) = \lambda x_k, \quad \forall \quad 1 < k < n.$$

From the argument in Proposition 5.6 our x satisfies the last condition, so we must check the first two conditions.

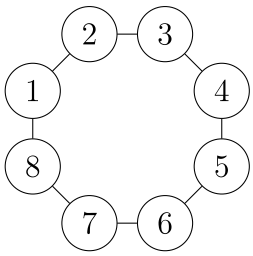


Figure 10: Graph C_8

Consider the graph C_8 in Figure 10 and notice that applying the condition that we can identify each vertex i with both vertex i in the path graph and as $2n+1-i$ in the cycle graph. These will be pairs of vertices as can be seen in the figure. Now it becomes clear that we can do the following calculations:

$$\lambda x_1 = 2x_1 - x_2 - x_{2n} = 2x_1 - x_2 - x_1 = x_1 - x_2,$$

$$\lambda x_n = 2x_n - x_{n+1} - x_{n-1} = 2x_n - x_n - x_{n-1} = x_n - x_{n-1}.$$

So our x satisfies the above conditions. Lastly we must check if there exists an eigenvector v of C_{2n} that satisfies $v_i = v_{2n+1-i}$, so that we can derive our x from it. Therefore let

$$v_i(k) = \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right),$$

then

$$
\begin{aligned}
v_{2n+1-i}(k) &= \cos\left(\frac{\pi k(2n+1-i)}{n} - \frac{\pi k}{2n}\right) \\
&= \cos\left(\frac{\pi k(4n+2-2i-1)}{2n}\right) \\
&= \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right) = v_i(k),
\end{aligned}
$$

which satisfies our definition of v. Since

$$
v_i(k) = \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right)
$$

$$
= \cos\left(\frac{\pi k}{2n}\right)\cos\left(\frac{2\pi k i}{2n}\right) + \sin\left(\frac{\pi k}{2n}\right)\sin\left(\frac{2\pi k i}{2n}\right),
$$

we have that $v \in Span(\{x(k), y(k)\}$ where $x(k)$ and $y(k)$ are the eigenvectors of C_{2n} following from Proposition 5.6. The associated eigenvalues are thus $\lambda_k = 2 - 2\cos(\frac{\pi k}{n})$ where $1 \leq k \leq n$. $\square$

Definition 5.3. *A complete graph, K_n on n vertices is a graph $G = (V, E)$ where $V = \{1, 2, ..., n\}$ and $E = \{\{i, j\} \mid i \neq j, i, j \in V\}$.*

Proposition 5.8. *The multiplicity of the eigenvalue 0 of the complete graph K_n is 1 and eigenvalue n with multiplicity $n-1$.*

Proof. We have mentioned how a complete graph looks visually where these results are quite intuitive. However the formal proof follows directly from Lemma 5.1 which states that the sum of all degrees are equal to the sum of all eigenvalues. We know that the eigenvalue $\lambda_1 = 0$, and that the sum of all degrees of a complete graph is $n(n-1)$. Therefore, from page 3 in [1], the multiplicity of the eigenvalue n must be $n-1$. $\square$

Now let us define the bipartite graph, which interpolates between other types of graphs defined earlier.

Definition 5.4. *A bipartite graph $G = (V, E)$ is a graph on n vertices where the vertices are partitioned into independent sets V_1 and V_2 such that $V_1 \cup V_2 = V$.*

We can see in Figure 11 an example of how a bipartite graph looks like. It is partitioned into two sets such that no vertices in each set are adjacent to each other. Now let us look at the complete bipartite graph, which has a lot of useful applications.

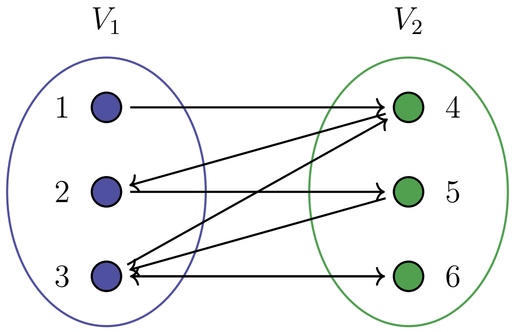


Figure 11: Bipartite graph G

Definition 5.5. *A complete bipartite graph K_{n_1,n_2} is a bipartite graph in which each vertex in V_1 is adjacent to each vertex in V_2. The number of vertices in the graph K_{n_1,n_2} is $n = n_1 + n_2$ where the number of edges are $n_1 * n_2$.*

The general form of the Laplacian matrix of a complete bipartite graph is

$$L_{K_{n_1,n_2}} = \begin{pmatrix} n_1 I_{n_2\times n_2} & -E_{n_2\times n_1} \\ -E_{n_1\times n_2} & n_2 I_{n_1\times n_1} \end{pmatrix} \quad ,$$

where I is the identity matrix and E is the matrix with only ones.

Proposition 5.9. *The Laplacian of the complete bipartite graph K_{n_1,n_2} has eigenvalues of $0, n-n_1, n-n_2$ and n with multiplicity $1, n_1-1, n_2-1$ and 1 respectively.*

The proof can be found in [5], Theorem 2 and involves using the complement of K_{n_1,n_2}. In particular, let us look at the graph $K_{4,2}$ as an example. Its Laplacian equals

$$L_{4,2} = \begin{pmatrix} 4I_{2,2} & -E_{2,4} \\ -E_{4,2} & 2I_{4,4} \end{pmatrix} \quad .$$

From Proposition 5.9 we get that the eigenvalues of the Laplacian of $K_{4,2}$ will be,

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = 2, \quad \lambda_5 = 4, \quad \lambda_6 = 6,$$

which can easily be confirmed by calculating the eigenvalues explicitly. This result coincides well with what we have discovered about the spectrum of graphs up until now. Looking at the algebraic connectivity λ_2 we can see that it is exactly 2, or rather exactly the number of vertices in the set of vertices with the least amount of vertices. In fact, for a graph $K_{4,3}$ the algebraic connectivity would be 3, which can be confirmed with Proposition 5.9 and is shown in [3], "Old and new results on algebraic connectivity".

This result on bipartite graphs can be extended to multipartite graphs, showing that similarly from Proposition 5.9 we can find the eigenvalues of a graph $K_{n_1,n_2,...,n_k}$ with k independent sets of vertices $V_1, V_2, ..., V_k$. Such eigenvalues are $0, n-n_k, n-n_{k-1}, ..., n-n_1$ and n with multiplicity $1, n_k-1, n_{k-1}-1, ..., n_1-1$ and k respectively.

Now let us explicitly bound the algebraic connectivity of the path graph on n vertices to see how it could be done using the tools we have described so far.

5.3 Bounding λ_2 of a path graph

Consider the Rayleigh quotient again in order to find an upper bound of λ_2. We have

$$\lambda_2 = \min_{\substack{x\neq 0\\ x\perp v_1}} \frac{x^T L_G x}{x^T x} = \min_{\substack{x\neq 0\\ x\perp 1}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i\in V} x_i^2}.$$

When choosing a vector to get a Rayleigh quotient the path graph is nice to evaluate since each subsequent vector follows the graph's edges quite predictably, which makes it considerably easier to use the Rayleigh quotient. As such we will find an upper bound of the algebraic connectivity of a path graph.

Proposition 5.10. *Let P_n be a path graph, then $\lambda_2 \leq \frac{12}{n(n+1)}$.*

Proof. Consider the vector u such that $u_i = (n+1) - 2i$ for $1 \leq i \leq n$ and the vector v_1 with entries equal to 1. Then we get that

$$u \cdot v_1 = \sum_i (n+1) - 2i = 0,$$

which we have seen previously means that $u \perp 1$. Then, by the Rayleigh quotient we have that

$$\lambda_2(P_n) = \frac{\sum\limits_{1\leq i<n} (u_i - u_{i+1})^2}{\sum\limits_i (u_i)^2}$$

$$= \frac{\sum\limits_{1\le i<n} ((n+1-2i)-(n+1-2(i+1)))^2}{\sum\limits_i (n+1-2i)^2} =$$

$$= \frac{2^2(n-1)}{\sum\limits_i (n+1-2i)^2}.$$

The denominator $\sum\limits_i (n+1-2i)^2$ is clearly of order n^3. By calculating the sum we get,

$$\sum_i (n+1-2i)^2 = \frac{(n+1)n(n-1)}{3}.$$

Thus,

$$\lambda_2(P_n) = \frac{2^2(n-1)}{n(n^2-1)/3} = \frac{12}{n(n+1)}.$$

Therefore we get a rough upper bound of the algebraic connectivity of a path graph with n vertices by

$$\lambda_2 \le \frac{12}{n(n+1)}.$$

□

Now we try to get a lower bound for λ_2 of P_n. For this we will need another technique. We begin by introducing a special partial order on symmetric $n\times n$-matrices. For two symmetric $n\times n$ matrices A, B we say that $A \succeq B$ if the matrix $A-B$ is positive semi-definite. So if $A \succeq B$, then $x^T Ax \ge x^T Bx$ for all x. This notion can be applied to Laplacian matrices of graphs as well. We say that for a graph G, $G \succeq H$ if $L_G \succeq L_H$ is true. This notion will be most useful when discussing some multiple of an edge graph, much like what we saw when discussing properties of the Laplacian matrix in section 2.

Lemma 5.11. *If G and H are two graphs with n vertices such that*

$$c \cdot L_G \succeq L_H, \quad c > 0,$$

then

$$c \cdot \lambda_2(G) \ge \lambda_2(H).$$

Proof. Applying Courant-Fischer formula we see that

$$c \cdot \lambda_2(L_G) = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{cx^T(L_G)x}{x^T x}$$

$$= \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T(cL_G)x}{x^T x} \geq \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T(L_G)x}{x^T x}.$$

By definition of the above partial order we saw that

$$\min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T(L_G)x}{x^T x} \geq \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T(L_H)x}{x^T x} = \lambda_2(H),$$

which proves the lemma. □

With this tool we will be able to find a lower bound of $\lambda_2(P_n)$ by comparing it to $\lambda_2(K_n)$. However first we need to look at some inequalities that will help us to understand how the path graph can be compared to the complete graph. Consider for that reason the path graph P_n from vertex 1 to vertex n and let $G_{1,n}$ be the graph with just one edge $(1, n)$. Let all of these edges be unweighted.

Lemma 5.12. *In the above notation,*

$$(n-1)P_n \succeq G_{1,n}.$$

Proof. We need to show that for every $x \in \mathbb{R}^n$,

$$(n-1)\sum_{i=1}^{n-1}(x_{i+1} - x_i)^2 \geq (x_n - x_1)^2.$$

For $1 \leq i \leq n-1$, set

$$\Delta(i) = x_{i+1} - x_i.$$

Notice that $(x_n - x_1)$, the inequality on the right-hand side, can be rewritten as;

$$\sum_{i=1}^{n-1}(x_{i+1} - x_i) = x_2 - x_1 + x_3 - x_2 + ... + x_{n-1} - x_{n-2} + x_n - x_{n-1} = x_n - x_1.$$

Then the inequality becomes

$$(n-1)\sum_{i=1}^{n-1}\Delta(i)^2 \geq (\sum_{i=1}^{n-1}\Delta(i))^2.$$

This however is precisely the Cauchy-Schwartz inequality that follows from the fact that the inner product of two vectors is at most the product of their norms. With the vector v with entries equal to one and using Δ we see that,

$$(n-1)\sum_{i=1}^{n-1}\Delta(i)^2 =\| v \|^2\| \Delta \|= (\| v \|\| \Delta \|)^2$$

$$\geq (v^T\Delta)^2 = (\sum_{i=1}^{n-1}\Delta(i))^2.$$

□

With Lemma 5.12 we can show that some multiple of the path graph P_n is at least the complete graph K_n. To this end, see

$$L_{K_n} = \sum_{j<i} L_{G_{i,j}}.$$

Proposition 5.13. *For a path graph P_n,*

$$\lambda_2(P_n) \geq \frac{6}{n^2-1}.$$

Proof. We will prove this by comparing the path graph P_n to the complete graph K_n. Suppose $K_n = (V, E)$ where $V = (1, ..., n)$. Then for every edge $(i, j) \in E$ in K_n, we apply Lemma 5.12 to show that

$$(j-i)P_n \succeq (j-i)\sum_{k=1}^{j-1} G_{k,k+1} \succeq G_{i,j}.$$

This says that $G_{i,j}$ is at most $(j-i)$ times the part of the path connecting i to j and that this part of the path is less than the whole. Then summing over all pairs of i, j with $i < j$, we get

$$\sum_{i<j}(j-i)P_n \succeq \sum_{i<j} G_{i,j} = K_n.$$

Notice that

$$\sum_{1\leq i<j\leq n}(j-i) = \sum_{k=1}^{n-1} k(n-k)$$

$$= n\sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2$$

$$= n\frac{(n-1)n}{2} - \frac{n(n-1)(2n-1)}{6}$$
$$= \frac{n^3}{6} - \frac{n}{6} = \frac{n(n^2-1)}{6}.$$

Therefore, we get that

$$\frac{n(n^2-1)}{6} P_n \succeq K_n.$$

From Lemma 5.11, we have

$$\frac{n(n^2-1)}{6} \lambda_2(P_n) \geq \lambda_2(K_n).$$

Proposition 5.8 implies that $\lambda_2(K_n) = n$, therefore

$$\frac{n(n^2-1)}{6} \lambda_2(P_n) \geq n$$
$$\iff \lambda_2(P_n) \geq \frac{6}{n^2-1}.$$

□

We can readily see that this lower bound has the same order as our previous rough upper bound of λ_2. Thus we now have a pretty good bounding of $\lambda_2(P_n)$, namely

$$\frac{6}{n^2-1} \leq \lambda_2(P_n) \leq \frac{12}{n(n-1)}.$$

6 The algebraic connectivity

Now that we have seen some examples of the spectra of graphs and evaluated their connectivity it is clear that the algebraic connectivity is an important characteristic of a graph. As opposed to the vertex and edge connectivity it is more concerned with the global structure whereas the vertex and edge connectivity's are more concerned with the smallest vertex or edge cut. As seen from some of the examples in chapter 4 and from some of the other calculations we have done, it is clear that λ_2 ranges from n for a complete graph and decreases as the graph becomes less and less connected. Our results with bounding the path graph, which is a very "weak" family of graphs, tells us that when the number of vertices become very large, the algebraic connectivity approaches 0. This in contrast with the complete graph that has an ever increasing algebraic connectivity as the order increases. This becomes very natural when you consider some of the propositions outlined in Section 4 that discuss the concept of connected components and the impact that they have on the graphs overall connectivity.

As the path graph is only one family of graphs this study could naturally be expanded to other types of graphs using similar techniques. One could also expand it by further discussing the problem of maximizing the algebraic connectivity of certain families of graphs with constraints on the number of edges and vertices.

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