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Richard´s paradox: Impredicativity and infinity coinciding

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Impredicativity and infinity coinciding

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”All generalizations are wrong”

Tage Danielsson

Many thanks to Lena, who found many typos and errors when reading through and checking my English, and who also has been most patient with me during this period.

Richard´s paradox:
Impredicativity and infinity coinciding

Contents

Abstract
The principal purpose of this paper is to, on the basis of Richard´s paradox, study how mathematicians over time have discussed some paradoxical reasonings which are more or less connected to that paradox. Hence, we will look upon these questions from a partly historical perspective, but also treat the question why some of these paradoxical reasonings are still around and discussed.

However, when studying this issue, an additional issue emerges, and with that issue also an additional purpose. This additional purpose is to study, with mathematical/logical tools, the issue of the split or discrepancy between classical and constructive/intuitionistic mathematics. This split is so closely related to some of the paradoxical reasonings mentioned above, that it can be explained in terms of the principal questions raised, and, furthermore, the location and the nature of this split can be illuminated by the principal results. Therefore, we will make a synoptic account, both historically and in terms of content, of the split mentioned. Doing this, we will initially focus on two mathematical topics that have been, and still are, regarded as problematic. It is inside the framework of these two topics that answers have been sought for concerning problems revealed by some paradoxical reasoning. The two topics are:

1. Impredicativity (circularity)
2. Quantification over infinite sets

We will see that Richard´s paradox, maybe more than any other, reveals problems belonging to these two topics, not in the sense of problems as being part of the paradox, but rather as being part of the mathematical reasoning from the time of its presentation and up until today. Indeed, Richard´s paradox is the event where these two topics coincide in a way that is very useful for an analysis of them and of the contradictory reasonings connected to them (i.e. is there a common denominator?), but also, as we will show, calls for caution to carefully distinguish between these two topics when they coincide. Additionally as a third topic, the analysis will also have impact on what ontological status we assert to different kinds of mathematical objects.

INTRODUCTION

Impredicativity and infinity
Generally speaking, paradoxes express some kind of contradiction. Sometimes this contradiction is contained within the paradox, e.g. Zeno´s paradoxes, but more recent paradoxes often reveal a contradiction that has its origin not in any flawed reasoning in the paradox itself, but in an external reasoning, e.g. Russell´s paradox.

Apart from Richard´s paradox, we will briefly describe four other paradoxes: Zeno´s paradox of motion, Torricelli´s paradox, Hilbert´s paradox and Russell´s paradox. The purpose of describing them here is that they all contribute, in different ways, to create a background consisting of observations regarding impredicativity and infinity from which we will examine Richard´s paradox and the conclusions that may be drawn from it.

However, initially we will make the two notions above more precise. The word *impredicativity* is derived from ”predicativity”, which is a piece of terminology first appearing in works of Bertrand Russell[1]. The meaning of this word has changed slightly over time, but it is still closely related to circularity. Thus, as an example, we use the adjective *impredicative* on definitions that have elements of their definiendum in their definiens, i.e. in order to define a word, they use that very word in a circular way in the definition. We will already here establish that this can in some contexts cause the possibility of contradictions. How and why this is the case will be treated below (see below e.g. under *Russell´s paradox*).

Less easy to make precise is the concept of infinity. However, it seems easy to define. We can for instance write the progression (a := 1, 2, 3, ...) and we feel comfortable with this expression of the infinity, and that a is a linguistic entity that has an instantiation of the infinity as its reference. It is more difficult when it comes to the properties of the infinite, as well as the meaning and the consequences of the three dots in the end of the expression. This has led to a dichotomy in terms of some of these properties. The main dividing line in this discussion is whether the infinity is to be regarded as some kind of complete set or not, i.e. is the infinity in terms of numbers possible to treat as a completed entity with a given cardinality or not? The two sides of this dichotomy is called *the actual infinite* and *the potential infinite* respectively, and they go back to Aristotle.

Actual infinity and potential infinity

The actual infinite is to be regarded as a completed and actually existing entity, and it presupposes the acceptance of (the corresponding interpretation of) the *axiom of infinity* in ZF (later ZFC)[2]. This axiom is put forward by Georg Cantor (before ZF) and plays an important role in Cantor´s theories. In fact, the whole theory of the existence of different infinite cardinalities is leaning solely on this axiom, which is precisely an axiom and not a conclusion or a derived truth. This axiom states the existence of an infinite set.

The potential infinite on the other hand is to be regarded as an entity that is growing without end, like an eternal process. The concept of potential infinity implies that there is no infinity in the actual sense. This is the way we normally use infinity in calculus; we study what happens when a value *goes towards* infinity. We do not say that there is an actual possibility for the value to finally *reach* infinity.

Aristotle refuted the actual infinite as being paradoxical, which is shown in his critizism of Zeno (see below under *Zeno´s paradox*). Later on, this question was highlighted by Galileo Galilei[3]. In *Two New Sciences* (1638) he is comparing different paradoxical looking infinite sets in such a way that a one-to-one correspondence is settled between e.g. the natural numbers and the squares. However, his conclusions are quite different

1 B Russell (1906), *On some difficulties in the theory of transfinite numbers and order types*, Proc. London Mathematical Society 4, 29-53. Reprinted in 1973, 135-164.

2 Zermelo-Fraenkel´s axiom system of set theory, see below under Hilbert´s paradox.

3 G Galilei: *Dialogues Concerning Two New Sciences*, The Macmillan Company 1914, pp 32-33.

from Cantor´s, who also used a one-to-one correspondence between infinite sets. Here they are expressed by the final statement by Salviati (the book is written in the form of a dialogue):

> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes ”equal,” ”greater,” and ”less,” are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

About Cantor´s use of one-to-one correspondence see below under *Denumerability - one-to-one correspondence.*

A consequence of accepting the actual infinite is (according to what Cantor has shown, see below e.g. under *Hilbert´s paradox*) that the infinite is consisting of different sizes. Cantor´s famous diagonal argument results in the conclusion that the cardinality of the set of all natural numbers is less than the cardinality of the set of all real numbers. In fact, Cantor also, with less success, tried to have proven that the infinite has infinitely many different cardinalities. These claims met objections from many mathematicians, e.g. Leopold Kronecker, Henri Poincaré and others, and later from all the constructivists and intuitionists (Brouwer and others). Poincaré (French mathematician 1854-1912) was very clear[4]:

> *There is no actual infinity;* the Cantorians have forgotten that, and have been trapped by contradictions. [Translation: P Helders]

Today Cantor´s version of the infinite (together with some of its implications) is taught in algebra courses in most of the universities, e.g. Stockholm University, at the same time as the concept of infinity that is used in calculus courses is based on the concept of limit, and thus closer related to the potential infinite[5]. We will in this thesis study arguments that illuminates the fact that the metamathematical assumptions (the actual infinite, expressed by the axiom of infinity and the acceptance of quantification over infinite sets) of Cantor´s argument give rise to famous results and a huge ontology that in fact have no stronger justification than the justification of the axiom of infinity itself and its implications (e.g. that quantification over infinite sets can make sense). So, how strong is the justification of the actual infinite? This is a crucial question in this thesis.

BACKGROUND

Zeno´s paradox of motion: Achilles and the Tortoise

This paradox was refuted by Aristotle, and his refutation has relevance for the descrip-

4 H Poincaré: *Les mathématiques et la logique III*, Rev. métaphys. morale 14 (1906) page 316.
5 About actual and potential infinity in algebra and calculus courses see also pp 37 and 41

tion of the dichotomy of the actual and the potential infinite. As is well known, Zeno from Elea (ca 490 - 430 BC) produced a number of paradoxes concerning e.g. plurality and motion. One of the most famous of these is *Achilles and the Tortoise*. As not being the main topic here, it is presented in a simplified way. Imagine Achilles trying to catch up the Tortoise in a run, where the Tortoise starts with a lead, but Achilles is much faster. When Achilles has reached the starting point of the Tortoise, it still has a lead, because it has advanced to a second point during the time it took for Achilles to reach the starting point of the Tortoise. Then, when Achilles has reached the second point of the Tortoise, the Tortoise has advanced to a third point etc. This goes on forever, and Achilles will never reach the Tortoise. Aristotle reduced this problem to the problem whether it is possible to pass an infinite number of points located along a line during a finite time, while moving along the line. Zeno´s point was that it was not, and he claimed that the fact that it seemed so empirically was an illusion. Aristotle argued[6]:

> For motion…, although what is continuous contains an infinite number of halves, they are not actual but potential halves. …Therefore to the question whether it is possible to pass through an infinite number of units either of time or of distance we must reply that in a sense it is and in a sense it is not. If the units are actual, it is not possible: if they are potential, it is possible.

The paradox is closely related to another paradox from Zeno: *The Dichotomy*. It argues that you never reach a spatial goal, because the distant left can always be divided into to halves, of which you first have to run the first half, and when that is done this pattern repeats forever. At the time of Aristotle it would still take long time before calculus became developed, but Aristotle did point out that as the distances for Achilles to run to point 1, point 2 etc become smaller and smaller, so did the time it took for him to do so. This implies an insight that the series of distances when talking about e.g. halves of the remaining distance (or corresponding time spans) would converge as long as Achilles runs faster than the tortoise, i.e. $0 < n < 1$, where n is the ratio of the two speeds:

$$S = \sum_{k=0}^{\infty} n^k = \frac{1}{1 - n} \quad \text{when } k \to \infty, \quad 0 < n < 1 \text{ and } S \text{ is the sum of time spans.}$$

However, as the running of time is not converging generally, but (in all Newton-based physics) is regarded as just going on linearly, Zeno was wrong. We will not discuss Zeno further, but we use this paradox to show what Aristotle meant with actual and potential infinity, and some properties these concepts have according to him.

Conclusions about Zeno´s paradox: Aristotle refuted the reasoning of Zeno as being paradoxical due to treating the actual infinite as being possible. This was not the entirety of the explanations of the mistake by Zeno, but part of it, and it reflects the view Aristotle held on the infinite.

6 Aristotle: *Physics.* § 263a25-27 and 263b2-5, page 153. Jonathan Barnes, editor, *The Complete Works of Aristotle*, the revised Oxford Translation, vol 1, 1991.

Torricelli´s paradox

Another famous mathematical problem with a paradoxical touch is *Torricelli´s trumpet*, also known as *The Horn of Gabriel* (Evangelista Torricelli 1608-1647, Italian physicist and mathematician). Consider the volume given by rotation of the curve

$$f(x) = \frac{1}{x}$$

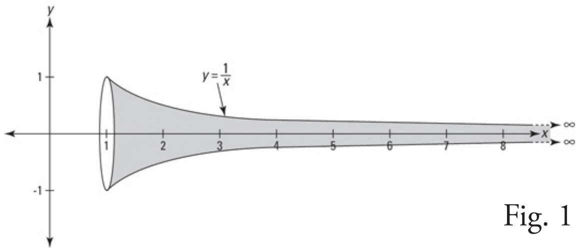

Fig. 1

around the x-axis. If we rotate the part of the curve located from $x = 1$ and towards the infinity ($x \to \infty$) we get a trumpet-like volume. This volume is finite and can be computed as below (1), but at the same time the surface of the trumpet has an infinite size, which is shown in (2):

$$V = \pi \int_1^a \frac{1}{x^2}\, dx = \pi \left(1 - \frac{1}{a}\right) \qquad (a \to \infty) \Rightarrow (V \to \pi) \qquad (1)$$

$$A = 2\pi \int_1^a \frac{\sqrt{1 + \frac{1}{x^4}}}{x}\, dx > 2\pi \int_1^a dx = 2\pi \ln a \qquad (a \to \infty) \Rightarrow (A \to \infty) \qquad (2)$$

This may seem paradoxical. We can reduce the problem to $\mathbb{R}^2$ by considering the length of the curve from $x = 1$ and towards the infinity ($x \to \infty$) compared to the area which will appear between the curve and the x-axis:

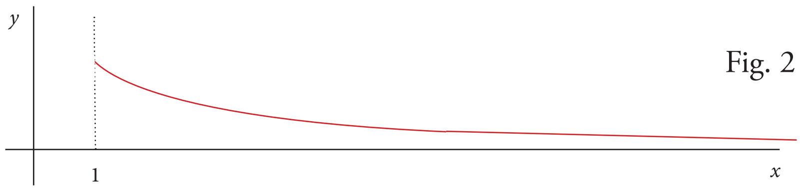

Fig. 2

Somehow the curve must reach $f(x) = 0$ in order for us to get a closed area. For how long must the curve be drawn along the x-axis? Infinitely long. It is only the infinity, as we know it and work with it in calculus, that will give us a closed area. I follows that the curve will have an infinite length *in order to* give us a finite area under the curve.

An intuitive analogy that tells us something about the possible expression of the infinite is the following: Consider the natural number 1. Divide it by three, and express it in decimal form:

0.333...

As we know, the three dots express a never-ending row of the digit 3. Multiply with 3. We get 0.999... This does not look the same as the number 1, but given that we conceive the three dots as an expression of an never-ending process, we get exactly the same as the number 1 that we started with (which we, quite obviously, must get). If we (ever) stop the process, we will not get back to number 1. This can be seen as an intuitive justification of the notion of limit transition.

From an intuitive point of view both limits and infinity are abstract concepts, but the example with division and multiplication by 3 shows us that both limits and the infinite can be conceived in an intuitive meaning. To attribute validity to the seemingly paradoxical reasoning of a finite volume being enclosed by an infinite area, is something that follows from how we regard limits and infinity in calculus.

Conclusions about Torricelli´s paradox: This paradox reveals something crucial about the infinite. No matter how far we go towards infinity, we will never achieve a completed and final length of the trumpet. Whenever we stop the process, we will still be able to find a point closer to the x-axis, i.e. the volume is not closed. Hence, only a truly never-ending process will give us the exact values that can be proven in calculus. This is a view on the infinity that is contrasting to the idea of the axiom of infinity[7]: this is an instantiation of the potential infinite, as opposed to the actual one.

This again reflects the nature of limit transition: the $\mathbb{R}^3$ version of the paradox is solved by the fact that an unlimited x-value is ensuring a diameter of the trumpet that goes towards zero. This again is ensuring that also the surface of the trumpet is limited, as the circumference equals $2\pi r$, and thus the circumference also goes towards zero, from which follows that the area is limited (if a cylinder has no diameter, it has neither volume nor surface area). Hence, the surface of the trumpet is limited as long as the x-value is not restricted. However, in the $\mathbb{R}^2$ version it is more abstract to grasp that the curve will have to have unlimited length in order to ensure a limited area under the curve. This because the length of the curve is not dependent on the distance to the x-axis, as is the case for the area of the trumpet. But again, the concept of limit transition will not give us a curve that never reaches the x-axis; we only have $y \rightarrow 0$, which is the nature of limit transition: only $x \rightarrow \infty$ ensures that $y \rightarrow 0$. This is again an instantiation of the potential infinite, which also reveals something about the nature of that potential infinite.

7 See below under *Hilbert´s paradox.*

Hilbert´s paradox

Let us consider some basic properties of the infinity that was established by Georg Cantor during the second half of the 19th century: One of the axioms in ZF is the *axiom of infinity*, which is a continuation of Cantor´s ideas. This axiom establishes the existence of an infinite set I, which is an inductive set. In formal notation:

$$\exists I\ (\emptyset \in I \wedge \forall x \in I\ ((x \cup \{x\}) \in I))$$

It follows that if the infinite set is I, then $I + 1 = I$. We will come back to the *axiom of infinity* later. Here will we just look at Hilbert´s paradox, from David Hilbert (1862-1943). This paradox has several versions or levels, but here we will just have a short look at the simplest one, a look that does not exclude anything that will be important for our future reasoning:

Imagine a hotel, which has infinitely many rooms. One evening a traveller arrives, and he asks for a room. The porter says that the hotel is full, unfortunately. The traveller gets disappointed by these news, but the porter tells him not to worry. He will get a room. How? asks the traveller. You said the hotell is full. No problem, says the porter. You can take room number 1. OK, but what about the person staying there now? He will move to room number 2, says the porter. OK, but what about that person? He will move to room number three. And so on. Finally everyone has a room....

We look at this anecdote because it says something about the properties of infinite sets that follows from the axiom of infinity. It also says that something is problematic, given that we accept these properties, with the use of expressions like *the hotel ”has” infinitely many rooms* and ”*all” of these rooms are occupied*, as well as ”*everyone” has a room*.

It shall be said that Hilbert, although not at all a constructivist, emphasized the importance of finitistic statements in order to give meaning to more ideal statements[8]. But Hilbert admired Cantor and regarded him as one of the greatest mathematicians ever. Apart from these notes about Hilbert, we will not discuss him or his possible ambitions with the example (and its expansions) above, but just use this famous story about infinity. It takes us directly to the commonly accepted definition of an infinite set[9] (*Elements of Set Theory, Herbert B. Enderton 1977*, page 157):

Def: A set is infinite iff it is equinumerous to a proper subset of itself.

This definition implies that an infinite set has a cardinality, i.e. some number of elements. As we shall see, from Cantor and onwards, infinite sets are treaten as having a ”number” of elements. What can be meant by that?

Cantor introduced the famous sign for this ”number”: $\aleph_0$[10]. This is the cardinality of the countable infinity, for example the set of natural numbers. This is the ”biggest” number for any denumerable set[11], and that is why $I + 1 = I$ and subsequently $1 + \aleph_0 = \aleph_0$.

8 A.S. Toelstra and D van Dalen: *Constructivism in Mathematics,* Elsevier 1988, page 25.

9 H B Enderton, *Elements of Set Theory*, Elsevier 1977, page 157.

10 See below under *Diagonalization again - impredicativity and infinity coinciding.*

11 See below under *Denumerability - one-to-one correspondence.*

It follows from the definition above that, for instance, the set A of even integers ≥ 0, as being possible to put in a one-to-one correspondence to the set of $\mathbb{N}$ (by induction), has the same cardinality as the set of $\mathbb{N}$. It follows that both the sets A and $\mathbb{N}$ are infinite, as are infinitely many sets. However, there are according to Cantor infinite sets with a cardinality greater that $\aleph_0$. An example is the set of real numbers $\mathbb{R}$, which has the cardinality $\aleph_1$. $\aleph_0$ and $\aleph_1$ are examples of what Cantor named *transfinite numbers* (see below under *Transfinite numbers*). What is relevant for Hilbert´s paradox is that, as we already have discovered, our normal laws for counting does not hold any longer. In fact, the use of the property "equinumerous" in the definition is a strong implication to treat the cardinality of infinite sets as *numbers*. But:

$$1 + \aleph_0 = \aleph_0 \;\Rightarrow\; 2 + \aleph_0 = \aleph_0 \quad \text{but we know that } 1 \neq 2 \text{ (cancellation)}$$

and: $\aleph_0 \cdot 3 = \aleph_0 \;\Rightarrow\; \aleph_0 \cdot (-5) = -\aleph_0$ but whatabout $\aleph_0 - \aleph_0$ and then
$\aleph_0 - (+\aleph_0) = (-\aleph_0) + \aleph_0$? (commutativity)

The same goes for associative and distributive laws. We may ask: what is then left of the concept *number*, and which are the basic reasons to think of these entities as numbers? How can we know, given that no ordinary counting laws apply, that we can quantify over the set of infinitely many rooms by saying that "all" rooms are occupied? This highlights again the distinction between *actual infinity* and *potential infinity*, where actual infinity is when infinity is thought of as having properties in common with infinite sets, whereas the potential infinity is not at all similar to any kind of number, but rather an unending process, which you cannot quantify over; it does not have any meaning to say "all" of the numbers or objects in an infinite set, as it has no settled cardinality. In fact, already the notion "set" is problematic, as it implies a closed entity, and thus contributes to confusion. We will also see that this kind of quantifying will lead to impredicativity.

It shall also be said that the grounds for the actual infinity seems limited. Starting with Cantor, a huge theory about transfinite numbers[12] was developed. It is still taught in a lot of universities all over the world. But the entire theory is uniquely dependent on the *axiom of infinity* (in combination with Cantor´s diagonal proof, see below under *Diagonalization again - impredicativity and infinity coinciding*) and on the implications of that axiom: that transfinite number exists and have certain properties whose justifications are limited in the sense that they finally depend on one axiom, an axiom that is not obviously true, and intuitively even controversial, which is a property that should be avoided for an axiom[13]. Furthermore, it seems as if part of the justification for the actual infinite (or at least examples of it) consists of the transfinite numbers, at the same time as transfinite numbers would be a result of the existance of the actual infinte. This is also related to ontological questions: What exists? Do transfinite numbers exist?[14] Which are the criterions for mathematical objects to exist, and which are the consequences of ascribing existence to objects merely by stipulation an axiom?

12 See below under *Transfinite numbers.*
13 See below under *Axioms.*
14 See below under *Ontology of mathematical objects - Constructivism and intuitionism.*

We will return to this topic when looking at quantifying over infinite domains (see below under *Denumerability - One-to-one correspondence* and *Richard and Cantor*).
Conclusions about Hilbert´s paradox: the reasoning in Hilbert´s paradox is useful to show the problems with quantifying over infinite sets. Because, when the porter says that the hotell is full, we can ask the question: How can this statement be justified? How does the porter actually *know* that the hotell is full? There exist no criteria for him to know this, logically speaking. It is impossible, not (only) for any time-space-related reason, but (first and foremost) for a logical one. We observe that the logical and the empirical reasons for the non-existence of these criteria are coincining, which is an important source of confusion (see also below under *Ontology of mathematical objects - Constructivism and intuitionism*).

For an all-quantification statement to be meaningful there must exist a method to establish that the actual instantiation of the quantification is valid for the whole set, i.e. that there are no leftovers. The porter, when saying that the hotel is full, must have checked that there are no existing vacant rooms, or no leftover guests:

$$\forall x(Px) \Leftrightarrow \neg\exists x(\neg Px)$$

Is such an observation logically possible if there is no settled number of rooms/guests?

Russell´s paradox

Perhaps the most famous paradox of all is Russell´s paradox (Bertrand Russell, British mathematician 1872-1970). It was formulated in a letter to the Gottlob Frege (German mathematician 1848-1925) in 1903. Frege established the notation used in symbolic logic and he had, inspired by e.g. Cantor´s set theory, set out to create solid and consistent logical foundations for mathematics. Russell showed in his letter that parts of the theory Frege had developed led to contradictions. In the paper *Mathematical Knowledge as Based on the Theory of Types*[15] (1908), Russell presents the paradox, together with a number of similar ones, as follows:

> Let *w* be the class of all those classes that are not members of themselves. Then, whatever class *x* may be, '*x* is a *w*' is equivalent to '*x* is not an *x*'. Hence, giving to *x* the value *w*, '*w* is a *w*' is equivalent to '*w* is not a *w*'.

This paradox created some excitement, because it showed that the attempts to secure mathematics from contradicitions actually created one. But starting already the same year (1908) Zermelo (Ernst Zermelo, German mathematician 1871-1953) formulated set theoretical axioms to avoid this and other related problems by simply banning sets as being capable of including themselves by stating the *axiom of regularity* (which, however, was not added to ZF until 1925 by von Neumann (1903-1957) and 1930 by Zermelo):

$$\forall x(\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \neg\exists z(z \in y \wedge z \in x)))$$

15 B Russell: *Mathematical Knowledge as Based on the Theory of Types,* 1908, American Journal of Mathematics, Vol. 30, No. 3 (Jul., 1908), page 222.

This means: Every non-empty set x contains as an element a set/element y such that y shares no elements with x. Another way to say that every non-empty set x contains something (namely y) that has nothing in common with x is:

$$\forall x(x \neq \emptyset \rightarrow \exists y \in x(y \cap x = \emptyset))$$

Russell himself formulated the *Theory of types* and its fundament, the *axiom of reducibility,* to achieve the same thing. We do not go into details concerning these solutions here, we just note that in both cases no real explanation was provided. Russell pointed out that a set and its extension belong to different ”types” (categories or levels), and it seems as if Russell thought of this fact as a law of nature. Russell, like most mathematicians, logicians and philosophers at that time, held a fairly platonistic view on these subjects, i.e. they were generous concerning assigning existence to abstract objects. According to Russell, the types in the *Theory of types* existed independently of human knowledge.

In fact, even today there have not been too much said about the actual reasons for the paradoxes. The absolutely most common formulation in all literature until today that mention Russell´s paradox is that things like sets that contain themselves should be avoided, but rarely of what reason (except that they lead to contradictions of course). Zermelo´s axiom system, later (in the 1920s) adjusted together with Fraenkel (Abraham Fraenkel, German/Israeli mathematician 1891-1965), got the name ZF, or ZFC (without respectively with the *axiom of choice*). As a treatment for Russell´s paradox (and other similar paradoxes) ZF became more important that Russell´s type theory because the type theory and its foundation, the *axiom of reducibility,* led to other problems, but both meant, and still mean, the curing of the symptoms of a problem, not a diagnosis. Perhaps platonistic views on these problems meant that there was no need for further explanations. And perhaps this was also a point where views on certain mathematical concepts started to split apart; not everyone was satisfied with the explanation that set containing themselves should be avoided because of natural laws. Actually the notion of *predicativity* was implemented by Russell, but for him it was still a field that was existing independently of human activity. However, one person who was at least starting to search for answer within the human knowledge was Henri Poincaré. In *Dernières Pensées* (1913) page 45 (*La Logique de l´Infini*) he gives, in an essay concerning the infinity, an example which says something about how he dealt with logic of sets: he points out that in natural language we may create a type theory just by the needs of getting understood. He is talking about two soldiers, that belong to the same regiment, and therefore we can conclude that they also belonged to the same division, and also to the same brigade. This is nothing but securing that, in order to be able to make the right conclusions, we need to assign to each variable what its domain is, and to carefully make difference between variation of the value of a variable and variation between variables[16].

But the question remains; what is it that assign domains to variables, if it is not natural laws? The answer is that it is what we do when we define things, and most of all, it is what impact the act of defining has on the relation between the words or symbols that we define and their reference, just by what it means to define: **When defining a**

16 See below under *Variables and domains.*

variable, what values that will be possible to assign to it, is restrained by the mere act of defining. Indeed, the axiom of regularity is designed not to *eliminate*, but to *hide* the problems that emerge when not paying attention to this fact.

Generally speaking, defining is to ascribe an asymmetric relation between *definiens* (that, what defines something) and *definiendum* (that, what is defined). This is what we mean by defining. When we define a set in terms of its elements we just give an example of definiens and definiendum. The elements are the definiens, and the set is the definiendum. In order for this to at all be a definition, we need to have an asymmetric relation between these entities. This is not a natural law from outside of the human consciousness that is calling, it is what we do ourselves with the language we are using in order to get understood. When we are defining a set in terms of its extension (its elements), there is already an implicit claim that there is an asymmetric relation between the set and the elements (if the definition is not to be impredicative in the circular sense, which is not the case normally in natural languages or mathematics). If we then propose that this set itself can be one of the elements, then we are saying the opposite (nothing can be in an asymmetric relation to itself). The contradiction is created in that very moment. The relation between definiens and definiendum is very important also when we are looking at Richard´s paradox and diagonalization, as we will see. We will also be able to see the relation between set-theoretic paradoxes and problems occuring when diagonalizating.

Before we go on to Richard´s paradox, we will have a look at an interesting description of the underlying problems exposed by Russell´s paradox given by Jan Ekman in the paper *Self-contradictory Reasoning* (2016)[17]. Ekman begins with giving an intuitive or informal description of his point, starting from another but similar formulation given by Russell:

> Let *t* be the set of all sets not containing themselves. Assume that t contains itself. Hence, by the definition of *t, t* does not contain itself. This contradicts the assumption that *t* contains itself and hence does not contain itself. Since *t* does not cantain itself, it follows from the definition of *t* that *t* contains itself. This is a contradiction.

What Ekman observes is that there are two occurences of the assumption that *t contain itself*, and these two occurences have different meanings, and that these two meanings are not compatible. This is related to what we just saw concerning definitions; when we regard the meaning of the variable ”set” to be a set in terms of consisting of elements, it is one thing, and when we regard the meaning of the variable ”set” to be an element in a set it is another thing. If we by ”set” mean that the variable is an element in itself, then we get a contradiction. If we by ”set” mean another set than the variable, then the reasoning will not lead to any contradiction. And this is logical, because nothing is problematic with sets being elements in sets, as long as this does not obstruct the relations given by the act of defining.

Ekman continues by observing that a conclusion like $A \rightarrow B$ with B being true (and thus also the conclusion being true), tells us more about A than we might think of

17 J Ekman: *Self-contradictory reasoning*, T Piecha and P Schroeder-Heister, editors, *Advances in Proof-Theoretic Semantics* vol 43, Springer 2016, pp 211-.

initially. Given that $A \not\leftrightarrow B$, we know some things about what A cannot be. For example, if $B \leftrightarrow p$, the following is true: $A \leftrightarrow (p \wedge q)$, but $A \leftrightarrow (p \vee q)$ is not (p and q being atomic propositions of A). Informally: From the premise ”it is snowing and the wind is blowing” being true, we can conclude that the conclusion ”it is snowing” is true. Or, contrary, we cannot conclude that it is snowing, if we know that ”it is snowing *or* the wind is blowing” is true. The kind of inference we have says something about the meaning of the proposition (formally: $((A \rightarrow B) \wedge (B \leftrightarrow p)) \rightarrow (A \leftrightarrow (p \wedge q) \wedge \neg(A \leftrightarrow (p \vee q)))$.

Ekman calls this *the meaning forced on the proposition, by steps of the argument*. He says (page 213):

> ...we can explain what a ”self-contradictory argument” is by saying that it is an argument such that the steps of the argument force several meanings on one of the propositions of the argument and that not all of these meanings are compatible. Yet another way to put this is to say that an argument is self-contradictory if and only if the steps of the argument force an *ambiguous meaning* on one of the propositions in the argument. Note that ... the meaning forced on a proposition by an argument is not an interpretation of the proposition, but a constraint on how it may be interpreted.

We can compare this with what we earlier stated about definitions. Analogously, meanings are forced on to propositions (and relations are forced on to and between terms) in an argument by definitions of terms included in the argument. The meanings are constraining how we can interpret these propositions.

Ekman carries on by generalizing and defining what a self-contradictory argument is. He calls the meanings forced on the propositions by the argument *meaning conditions*. He carries on by creating a schematic example, where propositions *X, Y* and *Z* form an argument of the form modus ponens. They have meanings m_x, m_y and m_z. Thus we have modus ponens elimination of implication (the notation ”$m_x : X$” means that the proposition X has the meaning m_x and E stands for ”elimination”):

$$\frac{m_x : X \quad m_y : Y}{m_z : Z}\ (\supset \mathrm{E}) \qquad \text{(Ekman´s def[18]: } (\neg G \equiv G \supset \bot)\text{)} \qquad (D)$$

Now, consider an instantiation of this with propositions *A, B* and *C*. As the argument is on the inference form modus ponens, these propositions will have the meanings *m* and *n* according to the following:

$$\frac{m \Rightarrow n : A \quad m : B}{n : C}\ (\supset \mathrm{E}) \qquad (F)$$

The meaning conditions consist of the relations between the meanings in *F*.

18 J Ekman 2016: Appendix, page 225: It follows that $\supset := (G \supset \bot \equiv \neg G)$.

Def: A deduction D is self-contradictory if there is no assignment of formal meanings m_x, m_y, m_z to the formulas in F such that this assignment satisfies the meaning conditions ($m_x \Leftrightarrow (m \Rightarrow n)$, $m_y \Leftrightarrow m$, $m_z \Leftrightarrow n$).

Ekman is not searching the reason *why* meaning and meaning conditions can be inconsistent (as we are), he is just formalizing the definition of a self-contradictory argument. But we can see the mechanisms that will reveal the contradiction if we have ambiguous meanings of e.g. ”set” in different parts of the argument. We can also see how meanings are forced upon propositions by arguments. It is the same with definitions: A set A being an element is not the same as a set B consisting of elements; these two examples of sets must have different extensions according to the axiom of regularity:

$$A \in B \;\Rightarrow\; \neg(B \in A) \;\wedge\; \neg(B \in B) \;\wedge\; \neg(A \in A)$$

We observe a strong interpretation of e.g. $\neg(B \in B)$ and $\neg(B \in A)$: It does not mean that B ***is*** not an element in B (accidentally), it means that B ***can*** not be a member in B (logically). Therefore, given that $C \Leftrightarrow \neg(B \in B)$, we have that $C \vee \neg C$ cannot not be subject for the *Law of the Excluded Middle* (LEM), which shows the non-universality of LEM[19].

Variables and domains

This is a brief section aimed at clarifying mechanisms behind the occurence of Russell´s paradox. In an intuitive sense, Russell´s paradox is the result of ambiguity of the common language, in a contrasting way to the view of Peano regarding Richard´s paradox[20]. To clarify how the term ”set” can be confused because it can occur as a notation for different (and sometimes incompatible) entities, we will use an analogy. Consider an ordinary second degree equation (or rather the notation for infinitely many such equations):

$$x^2 + ax + b = 0$$

The function $f(x) = x^2 + ax + b$ gives us the curve showing all possible values that function can take (in a suitable coordinate system, with given values for a and b). The two solutions of the equation give us the points of the x-axis where the curve cuts the axis (if it does). We say that x is the variable and a and b are parameters (or constants, meaning that they do not vary for a given equation). This wording is adapted to our wish to investigate the behaviour of a certain second degree function. The behaviour is given by x, and the function by a and b, intuitively speaking. We could say that a and b are variables too, but then we must avoid the risk of mixing up x with a and b, because x is a different variable from a and b. The domains differ. The variation of x is a variation along the curve of a given second degree function, whereas the variation of a and b gives the variation between different second degree functions. Hence, x is varying over one domain and a and b are varying over another, although the *values* of all three variables

19 About LEM see also e.g. page 39 (*Intuitionistic logic vs classical logic*) and page 46 (*Epilogue :Tertium Non Datur - proof by contradiction*)

20 See below under *Richard´s paradox.*

may coincide. Variation of x (between x_1, x_2, x_3) gives different point on the curve, variation of a and b gives variation between $(f(x) = ax + b)_1$, $(f(x) = ax + b)_2$, $(f(x) = ax + b)_3$ etc, i.e. between different curves.

To put it in another way: variation when a variable is varying between possible values is one thing, variation between variables is another. This is given by context and praxis in our example. But when we are talking about the set of all sets not containing themselves, we are using the one and same variable varying over incompatible domains. The language of set theory was not officially capable of handling this distinction until Zermelo cured the patient with the axiom of regularity in 1930, but without providing a diagnosis.

Axioms

The previous section about variables and domains calls for another clarification, or at least orientation, about the nature and role of axioms. We know axioms from e.g. Euclide. At the time of the ancient greeks, an axiom should be ”evidently true”, thus describing a part of reality that there was no dispute about. This has changed since long ago, and now an axiom is more to be regarded as a rule. We have seen e.g. the *axiom of regularity* and *the axiom of infinity* earlier. We also know for example the axioms of group theory and other parts of abstract algebra. However, although the axioms in ZF can be regarded as rules governing how to compute set theory, we note that, as mentioned above, the *axiom of regularity* was stated to save the set theory from collapsing, which was something mathematicians wanted to avoid, as many of them were believing in set theory as a possible tool to create a foundation of mathematics and logic. Today the hierarchy of what is founding what has been debated over time, and we can read in Jesper Carlström[21]:

> Set theory as the foundation of logic should be avoided since a natural application for logic is precisely set theory.

and L E J Brouwer has said[22]:

> Mathematics is independent from logic; logic is an application of mathematics.

The lack of explanation of the axiom of reducibility (and thus implicitly also of the axiom of regularity and of a diagnosis of Russell´s paradox) is stressed by Per Martin-Löf[23]:

> ...the axiom of reducibility was added on the pragmatic ground that it was needed, although no satisfactory justification (explanation) of it could be provided.

This observation was part of the reason for which Martin-Löf carried out an intuitive type theory in the 1970-1980s. For further discussion of this theory, see also *Epilogue.*

21 J Carlström: *Logic*, Matematiska institutionen, Stockholm universitet 2007/2017, page vii.

22 A S Toelstra/D van Dalen: *Constructivism in Mathematics*, Elsevier 1988, page 21.

23 P Martin-Löf: *Intuitionistic Type Theory,* Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980, page 1.

page 46.

Regarding the axiom of infinity, it actually asserts the existence of a set containing all natural numbers. As we know, according to Cantor the set of all natural numbers has the cardinality $\aleph_0$. The results Cantor put forward as a consequence of his diagonal argument (the cardinality of $\{\mathbb{N}\} = \aleph_0$ and of $\{\mathbb{R}\} = \aleph_1$) should however not be seen as justifications of the axiom of infinity, as this axiom cannot be proved; on the contrary the results mentioned presupposes the axiom to be valid. We note that, although it may no longer be compulsory for an axiom to be *evidently* true, it should have a clear and intuitive explanation. Additionally, it may be undesired for an axiom to directly generate mathematical objects that have no other justification than the axiom itself.

Constructivist mathematicians have rejected the actual infinite. However, today some axiom or rule concerning the infinite is contained in their theories. There are several theories replacing parts of ZF in constructive mathematics (e.g. Martin-Löf), with different degrees and types of acceptance of the concept of the infinite. See also below under *Ontology of mathematical objects - Constructivism and intuitionism* and under *Intuitionistic logic vs classical logic - useability of mathematical objects*. For our present purposes we make the main observation that both axioms mentioned in this section lack general theoretical justification. They are stipulated for purely pragmatic reasons. Although different logical systems freely can accept different axioms, we note a qualitative difference between axioms that have some kind of empirical or other justification, and those who have not.

Impredicative definitions

As we have seen above, what we normally mean by definitions requires them to be predicative. However, it is equally important to note that this observation is limited to the logical system that underlies the concept of linguistic meaning in normal language and also in most mathematics. For example, we may in, say, classical mathematics accept impredicative definitions, but then in set theory, axioms are used to get rid of undesired effects (like contradicitions) of some of them. But it is perfectly possible to create any axiomatic system. Mathematics can be expanded by new axioms and thus produce important results. An example within set theory that is presented in *The Liar*[24] of mathematicians Jon Barwise and John Etchemendy (1986) is called *AFA* (*Anti-Foundation Axiom*) and developed by British mathematician Peter Aczel. This book thoroughly treats the *Liar paradox*, and does so especially by using the *Theory of Hypersets* that is following *AFA* (Aczel). In this theory circularity is allowed and used. Another more well-known mathematical expansion consists of the complex numbers. This concept is created by a definition: $i = \sqrt{-1}$, and this expansion of mathematics has shown extremely useful although completely a product of a stated rule. Note that this is an example of an expansion of mathematics in general, not of an expansion that uses impredicativity.

So, we are not saying here that impredicative definitions are always wrong, per se. The purpose here is to illuminate the mechanisms in set theory that made it non-consistent up to the introduction of the axiom of regularity around 1930, as well as the fact that the reason for the occurance of e.g. Russell´s paradox is not caused by the obstruction of the axiom of regularity, but by the obstruction of the strictly predicative properties that definitions have by default, i.e. unless otherwise stated.

24 J Barwise/J Etchemendy: *The Liar*. Oxford Universiy Press 1987, pp 34-.

RICHARD´S PARADOX

Diagonalization

We are now about to study Richard´s paradox. But before we do that we will consider the concept of *diagonalization*, not to be mixed with the concept with the same name in linear algebra. What we mean here is a method to generate a new number out of an existing list or set of numbers. The method is famous because Georg Cantor used it in 1890 in an even more famous proof (*Cantor´s diagonal proof*). But diagonalization is a central feature also in Richard´s paradox, however in this case it is not used to prove anything, but to show that using diagonalization when quantifying over infinite sets or sets with an non-settled cardinality will lead to contradictions (thus Richard´s diagonalization can be regarded as a critizism of Cantor´s results). Both Cantor and Richard used diagonalization over real numbers with infinitely many decimals.

Consider a list of real numbers, enumerated by natural numbers. Generate a real number that consists of the diagonal (see figure 3). Then change each figure in the diagonal according to some rule, e.g. if a decimal $a_k = 1$ then change it to 2. If $a_k \neq 1$, then change it to 1. We can now ascertain that the real number thus generated from the string of decimals cannot be on the list. If we regard the example in figure 3 the new number would be 0.2111111, which is not and, especially, cannot be on the list.

1	0 , 1 4 0 3 0 3 6
2	0 , 2 3 3 6 0 1 7
3	0 , 3 8 0 0 9 9 3
4	0 , 5 3 3 6 6 4 7
5	0 , 7 6 3 3 2 1 6
6	0 , 8 5 5 4 2 3 4
7	0 , 9 4 5 4 2 0 9

Fig. 3

This way of generating a new number is described in many ways, and with a large number of rules. When Cantor introduced this method he used infinite strings consisting of two letters (*m* and *w*), but the result is the same: a new string is created. What is important is that each decimal digit in the diagonal is changed in some way. Regarding finite sets this is completely uncontroversial. Cantor used the method on infinite sets with the proposed result, among other, that the infinite set of natural numbers is countable (denumerable) while the infinite set of real numbers is uncountable (non-denumerable), and thus having another (greater) cardinality than the set of natural numbers. We will soon return to Cantor.

Presentation of Richard´s paradox

Jules Richard (French mathematician 1862-1956) formulated his paradox in 1905, apparently inspired by Cantor´s proof. There is a large number of related paradoxes, most of them originating from the years after the change of century to 1900. We will not discuss them here, but we can note paradoxes by Berry, Burali-Forti, König and others. Richard´s paradox is far less reproduced than e.g. Russell´s paradox, but made a large impact on mathematicians like Poincaré and Gödel.

Here follows the paradox more or less in Richard´s own words[25]: Assume that we have a set of real numbers defined such that from a list (in alphabetical order) of all permutations of pairs of letters signs and figures, followed by all triples of letters, quadruples of letters etc (the same letter can occur more than once in each permutation, and all permutations have a finite number of letters), we exclude all permutations that do not define a real number. We then get the set E consisting of the numbers u_1 (the first real number defined by a permutation), u_2 (the second real number defined a permutation etcetera). In other words, the set E is the set consisting of all real numbers that can be defined by a finite number of words (or signs). Now consider the following permutation G of letters:

(G) Let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place, $p + 1$ if p is not 8 or 9, and 1 otherwise.

We will now denote the number just defined N. This number N cannot be in E, which is shown by Richard´s diagonalization (using another rule than we did earlier). At the same time N is defined by one of the permutations on the list, namely G. We have a contradiction: N is in E and N is not in E. En passant, we note that Richard´s paradox is reproduced on the English version of Wikipedia in a flawed way. For example, it is said that the list (i.e. E) has infinite length, which is not possible given that the alphabet is finite and that we are not supposed to take into considerations strings where the same letter occures in more than two consecutive positions. We can also see that Richard himself did not assume that the list would be infinite. This becomes clear from the solution of the paradox proposed by himself in 1905:

> Let us show that this contradiction is only apparent. We come back to our permutations. The collection G of letters is one of these permutations; it will appear in my table. But, at the place it occupies, it has no meaning. It mentions the set E, which has not yet been defined. Hence I have to cross it out. The collection G has meaning only if the set E is totally defined, and this is not done except by infinitely many words. Therefore there is no contradiction.

25 J Richard 1905: *Les principes des mathématiques et le problème des ensembles*, in *Revue générale des sciences pures et appliquées* 16, 541. Also in Acta Mathematica 30 (1906), pp 295-6. (English translation in van Heijenoort 1967,143-44.). See also page 49.

Richard also stresses this later[26] (1907):

> It seemed to me easy enough to explain this paradox. Let *G* be the phrase that defines *N*. This phrase is an arrangement of words. Since the elements of *E* come from arrangements of words, in forming the set *E* we will encounter the phrase *G*. Suppose we encounter it at rank *p*. At this moment it does not have meaning, for at this moment the first p - 1 elements of *E* are the only ones defined. Having no meaning, the phrase *G* must be crossed out.

By the time around and after the change of century into the 1900, there was an intense discussion around overall mathematical issues. Slightly earlier Gottlob Frege and, above all, Georg Cantor worked to lay a new foundation for the mathematics through set theory. Bertrand Russell had discovered his paradox in 1903, just before Richard. A number of the related paradoxes appeared around this time. The discussion concerned to a great extent which conclusions were to be drawn from these paradoxes.

Giuseppe Peano (Italian mathematician 1858-1932) had (in 1906) objections[27] on the solution provided by Richard himself:

> But the class *E* is defined in the vocabulary of the common language. Therefore, if we substitute for *E* its definition, the result is that *N* is expressed by means of the vocabulary of the common language alone, and the antinomy remains.

This seems to be meant as an answer on what Richard had written in 1905, where he meant that *E* at the occasion when *N* was defined ”not yet was defined”. It seems unclear what Peano means exactly here. One interpretation is that he meant that if *E* and *N* are defined in the same collection of words, the solution proposed by Richard would not be valid, as the definitions of *E* and *N* would be made simultaneously, and this would disqualify Richard´s explanation that *E* ”not yet” was defined. If we interpret Peano in this way, Richard answered to this in 1907 (page 95):

> But then we can make this remark: the phrase *G* gives rise to a contradiction. Let *p* be its rank in the set *E*; if the phrase *G* defines a number *N*, let *x* be its *n*th digit. The phrase *G* says that the *p*th digit of *N* is equal to $\phi(x)$; so it says that
>
> $$\phi(x) = x$$
>
> But by the definition of $\phi(x)$ we have $\phi(x) \neq x$. Then the phrase *G* says that the *p*th digit of the *p*th number in *E* is different from itself, which is absurd. So we must cross it out.

26 J Richard, 1907: *Sur un paradoxe de la théorie des ensembles et sur l'axiome Zermelo, L'enseignement mathématique* 9, 94-98 (page 95).

27 G Peano 1906: *Super theorema de Cantor-Bernstein et additione, Revista de Mathematica*, VIII, 136-157. (Reprinted in Opere scelte, edizione cremonese, Rome 1957, vol. 1, pp 337-358. This version cited here).

Also en passant, we note that Richard, when writing that G is included in E, is not thorough with G being a *phrase* while E is a *set* consisting of real numbers (G is referring to a number that is included in E, not that it is itself included in E). We note, likewise en passant, that Peano makes the same mistake concerning N when he continues[28]:

> The contradiction lies in the ambiguity of the phrase N. It is necessary to add in an explicit way, 'this phrase included' or 'this phrase excluded'.
> Then we cross out the ambiguous phrase N, and continue on. A little further on we find the phrases:
>
> N' = (phrase N), this phrase excluded
>
> N'' = (phrase N), this phrase included.
>
> N'' does not exist, for the reason given. N' represents a determinate number, belonging to the class E, and clearly different from all other members of E.

Peano stresses that the problem is that crucial parts of Richards paradox are partly formulated in common language[29]:

> Richard's example does not belong to mathematics but to linguistics; an element that is fundamental in the definition of N cannot be defined in an exact way (according to the rules of mathematics). From an element that is not well-defined, we can draw several mutually contradictory conclusions.

Although Peano has a point about the ambiguity of common language in general, Richard´s paradox, and his solution, shows a more important feature of logic. We have seen this feature in Russell´s paradox, but in a slightly different way. As mentioned earlier, Russell´s own explanation of his paradox assumes that it is not possible to break certain natural laws of logic, which made him formulate his theory of types. But, as ascertained earlier, the theory of types does not cure the problem, it only eliminates the symptoms of the paradox. We repeat that one way to actually explain why Russell´s paradox occurs without referring to some platonistic concept of natural laws of logic, is to seek the explanation in how the very act of defining affects, or rather establishes, the relation between the entities involved in the definition. It is essential for defining as an act not to put definiens and definiendum in the same domain. If we do that, then it is not a predicative definition any longer, as we have broken what is by default essential for any definition. Here is a better example of ambiguity of common language than the one Peano referred to; we would be much better off if we did not use expressions like *impredicative definitions*, as this is by default an immediate contradiction in itself. The structure we create when we define things is the structure behind Russell´s theory of types, but it is not independent of human thought. This is not only a ontological fact, but also an expression of the hidden circumstances that cause not only Russell´s paradox, but also Richard´s. Additionally, it has relevance for our understanding of Cantor´s diagonal proof and its assumptions.

28 Peano 1906, page 357.
29 Peano 1906, page 357-358.

In Russell´s paradox the impredicativity occurs in the ambiguity of the using of the word ”set” (occuring both in definiens and in definiendum). In Richard´s paradox it is also definiens and definiendum that are not separated, but in a slightly more delicate way. Additionally, apart from the problem with impredicativity, Richard´s paradox reveals two more problems: a conceptual problem concerning the infinite, and an ontological problem concerning the existence of mathematical objects in general and undefined such objects in particular. All in all, Richard´s paradox highlights three different problems. For the sake of clarity we want to separate these problems carefully, and they are:

(I) The impredicativity occuring when N is both a member of the list and a result generated by the same list.
(II) How is all-quantification possible over infinite sets?
(III) Do defined and undefined mathematical objects have the same ontological status?

These problems are directly related to the three topics described in the abstract of this paper. Here follow some short conclusions of these three problems when looking at them with Richard´s paradox as a background. We will investigate them even further in the following section about Cantor´s diagonal proof.

(I) Maybe it is not clearly expressed by Richard, but his reasoning shows a tendency to assert the origin of the problem to the relation between definiens and definiendum (especially when claiming ”But by the definition of $\phi(x)$ we have $\phi(x) \neq x$)”. He means that N cannot be a part of E, but the reason he is pointing at, is that G have no meaning, as E is ”not yet” defined, and that we consequently must cross out G. This could be interpreted as that Richard means that G lacks meaning in this context. He would have realized that G after all defines a real number, and that G does not lack meaning outside of E. The weakness in this reasoning would then be that the reason for why E cannot contain N is not that G do not have any meaning, but E cannot contain N as E itself constitutes the basis of the definition of N. This is slightly different than what is pointed out as the reason by Richard. It thus seems as, in spite of his paradox, Richard would want to keep the requirement that the quantification over E should involve all numbers that are generated trough all defined permutations, also G and similar constructions. However, we can ascertain that G does not have to be crossed out as long as we make sure that we do not mix N with predicatively defined real numbers. A number like N defined in an impredicative way (which would be the case if $N \in E$), is of a completely different quality than a number not defined in that way, like all elements in E. Note that we are not claiming N not to exist or not being a real number, but just that, as long as we do not have any other, from E independent definition at hands, E will consist of an essentially different kind of real numbers than those defined using E in the definition.

That we have an assymmetric relation between N and any number in E is expressed when Richard says that E ”not yet” is a defined set. It is tempting to use some time concept here, proposing that *first* E must be defined, *then* we can define N. However, we will avoid that here, and instead we denote the numbers in E being *more basic* than the number N.

The description of *N* as being *different* than the elements in *E* can be modified or avoided by adding them, anyway, to *E*, in spite of them being defined in a different way. Thus we would get rid of the qualitative distinction just mentioned, but instead we will have a situation where *E* is constantly growing. This makes it impossible to quantify over *E*, as this set never can be definitively defined. Henri Poincaré writes in1913[30]:

> In summary the classification of the numbers can be fixed only after the selection of the sentences is completed, and this selection can be completed only after the classification is determined, so that neither the classification nor the selection can ever be terminated.

Poincaré was among the first to more specifically point at impredicativity as a result of our own acts of defining. He was very critical both to Cantor and to Russell, and his results and the polemic he exercised had an important impact on a growing reaction against the in some mathematical contexts very appreciating view on Cantor´s work. This would later delevop into constructivism and intuitionism within the mathematics. In fact, this development had already begun by this time.

Poincaré agreed with Richard´s solution of the paradox, but delevoped it. Some years earlier, 1906, he expressed his basic view[31], which was that we cannot define *E* in terms of *E* itself. The numbers in *E* are more basic than those that can be defined by using *E*, and although Poincaré around 1906 claimed that *G* cannot define any number at all because the definition of a member in *E* cannot be dependent of *E* (circularity), he denotes in spite of this such definitions ”non-predicative”. Again in 1908 he made it completely clear, also capturing the possibilities of quantification over *E*[32]:

> *E* is the set of all numbers one can define with finitely many words, without introducing the set *E* itself. Without this (restriction) the definition of *E* will contain a vicious circle; one cannot define *E* through the set *E* itself.

Thus it seems as Poincaré successively (1906 and 1913) developed his standpoint away from the idea that *G* does not define anything at all (lacks meaning) towards making a distinction between *predicative* and *non-predicative* classifications (Poincaré´s own therminology[33]).

The concept of impredicativity is today described more detailed, also with substantial historical references, by Solomon Feferman (2005). He points out similarities between Poincaré and Russell in terms of impredicativity: both pointed at circularity (vicious circle). But for Poincaré the concept of definition was part of the analysis, which led him away from the more platonistic Russell. Additionally, Poincaré´s reasoning developed in the direction of refuting the existence of an actual infinity, see e.g. page 5 above. Fefer-

30 H Poincaré: *Dernières Pensées*, 1913, Dover Publications Inc 1963, page 46.

31 H Poincaré: *Les mathématiques et la logique*, 1906, Revue de métaphysique et de morale 14, page 307.

32 A S Toelstra and D van Dalen: *Constructivims in Mathematics*, page 19 (from H Poincaré: *Les Derniers Efforts des Logisticiens*, 1908).

33 H Poincaré, 1913, page 47.

man writes[34]:

> ... Poincaré came up with two distinct diagnoses of the source of the paradoxes via what he regarded as "typical" examples. The first was that there is in each case a vicious circle in the purported definition. For example, in the case of Richard (1905), since each definition of a real number via its decimal expansion can be written out using a finite number of symbols, the set *D* of definable real numbers is countable. Then by Cantor's diagonal construction one can define a real number *r* which is distinct from each member of *D*; but since *r* is defined, it is a member of *D*, which is a contradiction. According to Poincaré, in this case the vicious circle lies in trying to produce the object *r* in *D* by reference to the supposed totality of objects in *D*; indirectly, then, *r* is defined in terms of itself, as one of the objects in *D*. Poincaré´s second diagnosis is distinct in its emphasis, namely that the source of each paradox lies in the assumption of the "actual" or "completed" infinite.

(II) How can all-quantification over infinite sets be possible? This is a justified question, as this type of quantification is used over and over again from the presentation of the *axiom of infinity* and onwards. What does "all" real numbers mean? What does it mean that two infinite sets are "equinumerous"? What does it mean that the Hilbert hotel is "full"? To understand this we have to study the concept of mapping, or establishing a one-to-one relation between elements in different sets. This is completely uncontroversial in the finite case, but what about the infinite case? This question does not look less interesting when we realize that this operation together with the acceptance of the *axiom of infinity* is the total foundation of the whole theory of transfinite numbers. We will study this closely in the next sections (*Diagonalization again - impredicativity and infinty coinciding* and *Denumerability - one-to-one correspondence*). Here we will highlight the way this issue emerges in Richard´s paradox.

Actually, there is no quantification over infinite sets in Richard´s paradox, as the set *E* is in fact finite. But the quantification over *E* in Richard´s paradox will for us serve as an example, a background to the examination of the infinite case in next section. Remember, the list of permutations is considered to contain *all* linguistic expressions that define a real number. Still, the paradox shows that the list did not contain *G*, because we need to have *E* in order to give a menaing to *G*. This shows clearly that the word "all" does not have an unambiguous interpretation. What we mean with "all" is constantly changing. This is not due to weakness in common language (as e.g. Peano proposed); it would be exactly the same in a formal language (i.e. the all-quantification $\forall$ and the word "all" would both lack meaning). This is something we need to bear in mind as we approach the next section of this paper. The fact that all-quantification in some context is doomed to be ambiguous is extremely important for our interpretation of these contexts, at least as long as avoiding contradictions is something we aim for.

In fact, the ambiguity of quantification over infinite sets (or any other context containing non-defined variables) is a result we will ascertain when we investigate the constructive approach to mathematics. L E J Brouwer wrote on quantification[35], already

34 S Feferman: *Predicativity*, Oxford Handbook of Philosophy of Mathematics and Logic, Oxford University Press, 2005, pp 590-624,

35 L E J Brouwer: *Over de Grondslagen der Wiskunde*, 1907/Toelstra and van Dalen:

in 1907:

> The mistake which so many people, thinking that they could reason logically about other objects than mathematical structures built by themselves, and overlooked, that wheresoever logic uses the word *all* or *every*, this word, in order to make sense, tacitly involves the restriction: insofar as belonging to a mathematical structure which is supposed to be constructed *beforehand*. [Italics by P Helders]

This fact has consequences for the third problem arising from Richard´s paradox. See also below under e.g. *Ontology of mathematical objects - Constructivism and intuitionism*.

(III) A platonistic view on mathematics would, at least in a historical context, claim that all mathematical objects that we use are discovered, not invented. The objects of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$ or roots, equations or polynoms are all eternally and actually existant independent of mankind. A constructivistic view on the other hand would be to regard all mathematical objects as only potentially existant. We need to define them in order to have them, and the ontological status of such an object is closely related to whether it is defined or not. From a constructivistic point of view, E contains all *so far* defined real numbers, and this may change, but not until a new definition is made. This way the word ”all” would have a less ambiguous meaning. Richard´s use of the words ”not yet” can be seen as an expression of what later came to be regarded as a constructivistic standpoint. And we can see that the distinction between the actual infinity and the potential infinity is closely related to the question of the ontological status of mathematical objects.

Diagonalization again - impredicativity and infinity coinciding

Georg Cantor had already in 1874 produced a proof of his theorem that the totality of real numbers were of a greater cardinality than that of the natural numbers, i.e. the real numbers were non-denumerable. However, this proof was leaning on other assumptions regarding irrational numbers, and to an important extent counteracted by Leopold Kronecker, who also was in power to decide whether Cantor´s results were to be published or not in Crelle´s *Journal*. Kronecker did apparently cause a delay of publication, but in 1878 these results of Cantor were published under the title *Beitrag*. Earlier, in 1874, he presented the results to, among others, Kronecker. At that time the title was *On a Property of the Collection of All Real Algebraic Numbers*. In his detailed biography of Cantor, Joseph Warren Dauben asks why Cantor used this title[36] (as the algebraic numbers were not the most relevant part of the results, and thus somewhat misleading as a part of the title). Dauben proposes the answer that Cantor did this to conceal the real purpose: to show that the real numbers in general were non-denumerable (i.e. uncountable), in fear of being refused by Kronecker.

In 1891 Cantor presented a new proof of the same theorem, this time avoiding some problems in the first version, problems that Kronecker had exposed and that we will not

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Constructivism in Mathematics, 1988, Elsevier, page 22.

36 J Warren Dauben: *GEORG CANTOR - His Mathematics and Philosophy of the Infinite*, Harvard University Press 1979, reprinted by Princeton University Press 1990, page 67

investigated here[37]. This proof is today known as *Cantor´s diagonal proof.*

In a paper published 1998 in *The Bulletin of Symbolic Logic*, the English mathematician Wilfrid Hodges (1941-) defends Cantor´s proof against the numeruous apparently ”hopeless” objections to the proof that he had recieved as an editor throughout the years[38]. As we are going to object to the assumptions of the proof of Cantor (and thus to the conclusions of it), for the sake of objectivity we here choose to use Hodges formulation of Cantor´s proof, which is very clear:

> (1) We claim first that for every map f from the set {1,2,... } of positive integers to the open unit interval (0,1) of the real numbers, there is some real number which is in (0,1) but not in the image of f.
> (2) Assume that f is a map from the set of positive integers to (0,1).
> (3) Write 0, $a_{n1}\ a_{n2}\ a_{n3}$ ... for the decimal expansion of $f(n)$, where each a_{ni} is a numeral between 0 and 9. (Where it applies, we choose the expansion which is eventually 0, not that which is eventually 9.)
> (4) For each positive integer n, let b_n be 5 if $a_n \neq 5$, and 4 otherwise.
> (5) Let b be the real number whose decimal expansion is 0, $b_1\ b_2 b_3$ ...
> (6) Then b is in (0,1).
> (7) If n is any positive integer, then $b_n \neq a_{nn}$, and so $b \neq f(n)$. Thus b is not in the image of f.
> (8) This proves the claim in (1).
> (9) We deduce that there is no surjective map from the set of positive integers to the set (0, 1).
> (10) Since one can write down a bijection between (0, 1) and the set of real numbers (and a bijection between the positive integers and the natural numbers, if we want the latter to include 0), it follows that there is no surjective map from the set of natural numbers to the set of real numbers.
> (11) So there is no bijection between these two sets; in other words, they have different cardinalities.

The proof above, here in the formulation of W Hodges, is the very place where impredicativity and quantifying over infinity coincides. Concerning the impredicativity, we have seen in the section about Richard´s paradox how it is impossible to generate a member of a set by the set itself if we want to avoid contradictions. Consider (1) above: as the diagonalization method is able to, for any n, produce a new real number out of any set of real numbers over which it is possible to quantify, we will at any time we make the mapping have the situation that we will find a new real number which is in (0,1). The reason for this must not be that the numbers of reals are greater than the number of naturals, but that it is an essential part of the act of definition, that definiens and definiendum cannot be the same (see above under *Russell´s paradox* and under *Variables and domains*). If they are, the definition will be impredicative[39]. So one conclusion here is about the *reason why* a new real number can be generated. Additionally, as we noted in the section about Richard´s paradox, the new real number is not yet defined when the mapping takes place, but rather defined by the mapping and the following changes of

37 See Dauben 1979: pages 66-70

38 W Hodges: *An Editor Recalls Some Hopeless Papers,* The Bulletin of Symbolic Logic, Vol. 4, No. 1 (Mar., 1998), pp. 1-16.

39 See above under *Impredicative definitions*.

figures in the diagonal. The proof claims that no matter how long we continue, we will always find a new real number. Yes, but also: no matter how long we continue, we will not reach infinity. In order to exist, the mapping must be executed at some time, and then we can just add the new real number to the range as we add a new natural number to the domain of this mapping. That process is quite similar to the process described under *Torricelli´s paradox* above (pp 7-8): No matter how far we go along the *x*-axis, we will still have some space left between the curve and the axis. For every map from the domain of *x*-values to the range of *y*-values, there is some *y*-value under the curve that is not in the image of the domain (comparing *Torricelli´s paradox* with (1) page 26).

As we see, the discussion about this proof involves assumptions concerning the nature of infinity. Quite obviously, a necessary assumption for the proof is the actual infinite, i.e. the axiom of infinity, and its implication that we can quantify over infinite sets. Consider (11) above. The conclusion that the cardinalities are different presupposes LEM (*the Law of the Excluded Middle*): as both sets in the bijection are supposed to have settled cardinalities, these cardinalities are either the same or not. But whatabout if these sets have no settled cardinalities, as a description based on the view of the potential infinite would propose? Not only would LEM not be applicable (becuase if we have no cardinalities then the question if they are the same or not has no meaning)[40], but also: if there are no infinite cardinalities, then there will be no bijection. But from the observation that there is no bijection we can conclude either that the sets have no cardinalities (because then there can be no bijection) or that they have cardinalities and these cardfinalities must be different. This choice is dependent of whether it is the potential infinite or the actual infinite that is assumed.

So what about that question? Is there an actual infinity? There is a famous saying in philosophy that we cannot prove the existence of God, but we cannot disprove it either. So, although we may explain the occurence of additional new real numbers when using diagonalization over a set of real numbers by the act of definition, it is not easy to establish that this explanation is better than the explanation that the totality of real numbers is greater than the one of natural numbers. It seems as that we would have to lean on the assumption of the potential infinity similarly as the opposite assumption is needed for the Cantor proof to be able to produce its conclusion.

Yet, let us consider what other results we can obtain by the method of diagonalization together with the assumption of an actual infinite. Cantor himself noted that also the totality of infinite binary strings has a greater cardinality than the natural numbers (given the actual infinite). The cardinality of the set of natural numbers equals $\aleph_0$ and the cardinality of the set of reals and the number of infinite binary strings equals $\aleph_1$. What about the cardinality of the set of natural numbers with infinitely many digits? We can immediately ascertain that this set has the cardinality $\aleph_1$, a conclusion achieved by assuming the actual infinite and the using of diagonalization, exactly as in the Cantor proof. But are these objects really natural numbers? Given the assumption of an actual infinite, yes: Let *S* be the subset of $\mathbb{N}$ consisting of denary strings of the digits 0-9 not beginning with a zero and having infinitely many digits. There is no property of infinitely long real numbers required for the diagonalization result that will not be valid for natural numbers

40 See also below: *Epilogue: Tertium Non Datur - Proof by contradiction*

of infinitely many digits. Additionally, it must be, and is, possible to diagonalize over this subset, as the the following holds for these numbers:

Let n be a number in $\mathbb{N}$, and let $f(n)$ be its number of digits. Then

$$f(n) = \text{floor}(\log_{10}(n)+1)) \;\Rightarrow\; f(n) \to \infty \text{ when } n \to \infty \quad (n = \{1, 2, 3, \ldots\}) \qquad (3)$$

Thus, although $n \to \infty$ faster than $f(n) \to \infty$, with the assumptions held by Cantor in his proof, we have that $n = \aleph_0 \Rightarrow f(n) = \aleph_0$, and therefore $n = \aleph_0 \Rightarrow n = f(n)$.

The result of this is that we, by diagonalization over S, can prove the following:

$$\text{card}(S) > \text{card}(\mathbb{N})$$

Since $S \subset \mathbb{N}$ by definition, this result is absurd, which should lead to rejection of the axiomatic, metamathematical premises, and therefore also of the conclusions, of Cantor´s diagonalization proof as stated on page 26.

Is the concept of the set of natural numbers with infinitely many digits absurd too? If so, the objection would not hold of course. But if we regard (3) as true, this will lead us to the concept mentioned. If we want to aviod that we would need axiomatic or metamathematically stated limitations.

The general conclusion which we get from this reasoning is that Cantor´s diagonal proof, given the metamathematical premises used by Cantor, proves that not only the set of real numbers has a greater cardinality that the set of all numbers in $\mathbb{N}$, but that all sets of any kind of infinite strings (binary, denary, multinary) have a greater cardinality that the set of natural numbers in $\mathbb{N}$, with some of those strings representing natural numbers as shown in (3) above, thus being proper subsets of $\mathbb{N}$.

Denumerability - one-to-one correspondence

Other consequences of the metamathematical assumption of the axiom of infinity concern the establishment of a bijection between two infinite sets. Cantor elaborates this concept already in the 1874 proof published in 1878. We read in Dauben[41]:

> Cantor also introduced the following partition of linear sets, which today would be described as a disjoint union:
>
> $$a \equiv \{a', a'', \ldots, a^{(v)}, \ldots\}.$$
>
> The symbol $\equiv$ was used to indicate that the intersection of any two of the elements $a^{(v)}$ and $a^{(v')}$ was always empty. Thus a could be considered the disjoint union of linear sets $a^{(v)}$, while from any set a a series of mutually disjoint sets $a^{(v)}$ could be formed. Finally, Cantor introduced his new concept of equivalence and wrote, whenever two domains a and b were of equal power, that $a \sim b$. Of particular note is the theorem that Cantor then gave concerning the relation of two infinite sequences of disjoint elements:

41 J W Dauben 1979, page 62.

Theorem E: If $a', a'', \ldots, a^{(v)}, \ldots$ is the finite or infinite sequence of variables or constants which have no pair-wise connection (the elements of the sequence are all different or disjoint), and if $b', b'', \ldots, b^{(v)}, \ldots$ is another sequence of the same character, then every variable $a^{(v)}$ of the first sequence corresponds to a definite variable $b^{(v)}$ of the second, and if these corresponding variables are always equivalent to each other, i.e., if $a^{(v)} \sim b^{(v)}$, then it is also always true that $a \sim b$ if

$$a \equiv \{a', a'', \ldots, a^{(v)}, \ldots\}$$
$$b \equiv \{b', b'', \ldots, b^{(v)}, \ldots\}$$

This is, as far as Dauben presents it, never proven (only established), and dependent on the metamathematical assumption of the axiom of infinity, and on the assumed property of an infinite set that it has an actual cardinality. We can agree that $a^{(v)} \sim b^{(v)}$ but the establishment of Theorem E never describes how equinumerousity between a and b can be granted without presupposing that a and b have the same cardinality (which obviously would be circular reasoning). On the other hand it is used to prove the Theorem F that followed Theroem E. Theorem F claims that it is possible to map open intervals onto closed ones, but this proof is leaning on Theorem E.

This is a description of a one-to-one correspondence, or bijection. The validity of Theorem E is dependent of the meaning of the three dots after $a^{(v)}$ and $b^{(v)}$ in the two sets a and b respectively. What meaning that should be ascribed to these dots are dependent of whether we consider the infitiny as being actual or potential. It should be mentioned that with saying that the *corresponding variables are always equivalent to each other*, we interpret Cantor as meaning equality in terms of ordinal number. But still, when claiming a bijection we normally, i.e. for finite sets, demand equinumerousity; that there are no elements left in any of the sets. How can this be established for infinite sets? Only by metamathematical presumptions or axioms that state the actual infinite. Is the concept of equinumerousity really meaningful when talking about infinite sets? We have to make the metamathematical choice between two things: either the presumption of the actual infinite, which has the consquence e.g. that we get mathematical objects (like $\aleph_0$ and an infinity of other Alephs) for which we need an axiom and new counting rules, or, on the other hand; the potential infinite, where concepts like *equinumerous, greater than, all of* or a *proper subset of a set being equinumerous to the set itself* concerning infinity or infinite sets become without meaning, and where we have no theory of transfinite numbers left. This choice shows the split between classicism and constructivism in mathematics[42]. There are, as shown above, arguments for each way of regarding these issues, but we here claim that the argument following (3) above (about diagonalization over natural numbers with infinitely many digits) shows a problem concerning the assumption of the axiom of infinity that is not explained, and that there is no corresponding or similar counterargument concerning the constructivist view.

Richard and Cantor

Richard managed to show already in 1905, as we have seen, that diagonalization can

42 See below under *Ontology of mathematical objects - constructivism.*

cause impredicativity if we claim that an all-quantification over a set also can include elements defined by that set, at a certain time or position in the process of diagonalization. In fact, the two examples (Cantor and Richard) are very similar, and that one is called a paradox and the other a proof say more about how they entered the scene respectively, than about the nature of these examples. The only difference is that in Richard´s case the domain is not infinite (but it has a crucial feature in common with infinite sets; it lacks absolute cardinality, at least according to the constructivist view). We have noted that quantifying over infinite sets can cause impredicativity, because the notion of ”all” is not settled for infinite sets unless you accept the axiom of infinity. However, it is still not possible to quantify over finite sets if they can be expanded by definition of extensions of the set based on elements in the set itself. This means that we can see the nature of impredicativity when quantifying over sets which we enlarge by diagonalization already without involving infinity. Thus, Richard´s paradox reveals another problem of the metamathematical assumptions of Cantor´s diagonal proof: it is not only a question of how to treat infinity, it is also a question of impredicativity in terms of definitions, which is what the two examples have in common with each other and with e.g. Russell´s paradox (see also above page 19). The two aspects mentioned (the question of quantification over sets with no settled cardinality and the question of impredicativity) are coinciding in these examples, which surely is a reason for confusion. An analysis of these examples of diagonalization must indentify the two aspects, and separate them.

Transfinite numbers

In this section we will make some notes regarding the concept of transfinite numbers. We have mentioned them earlier in this paper (see e.g. *Hilbert´s paradox*). The term itself was used in Cantor´s *Beiträge zur Begründung der transfiniten Mengenlehre*[43] (1895-1897), the paper in which he developed the earlier results (the date for coining this term is on *https://en.wikipedia.org/wiki/Transfinite_number* incorrectly stated to be 1915; that was the year of the English translation, which is clear already from the original title of Cantor´s paper). As we have seen, according to Cantor, the first transfinite number is $\aleph_0$ which equals the cardinality of the set of all natural numbers and also of all infinitely denumerable sets. We have also seen that an infititely denumerable set is an infinite set whose elements according to Cantor can be put in to a one-to-one correspondence (bijective relation) with the set of all natural numbers. In section 6 of *Beiträge* he seeks to make the concept of $\aleph_0$ more precise. He had in the previous sections shown, together with a number of other non-controversial theorems concerning finite sets, that for a finite set with cardinality v, we have that $v \neq v + 1$. In earlier sections he had also gone over all the counting laws for finite mathematics. But, in section 6, leaning on the bijection idea he had shown for finite sets, he begins[44]:

> Aggregates with finite cardinal numbers are called ”finite aggregates”, all others we will call ”transfinite aggregates” and their cardinal numbers ”transfinite cardinal numbers”.

43 English translation by P E B Jourdain The Open Court Publishing Company 1915.

44 G Cantor: *Beiträge zur Begründung der transfiniten Mengelehre* (1895-1897), English translation by P E R Jourdain The Open Court Publishing Company 1915, page 104.

The first example of a transfinite cardinal aggregate is given by the totality of finite cardinal numbers v; we call its cardinal number (§1) "Aleph-zero" and denote it by $\aleph_0$; thus we define

$$(1) \qquad \aleph_0 = \overline{\overline{\{v\}}}.$$

That $\aleph_0$ is a *transfinite* number, that is to say, not equal to any finite number μ, follows from the simple fact that, if to the aggregate $\{v\}$ is added a new element e_0, the union aggregate $(\{v\}, e_0)$ is equivalent to the original aggregate $\{v\}$. For we can think of this reciprocally univocal correspondence between them: to the element e_0 of the first correspond the element 1 of the second, and to the element v of the first corresponds the element $v + 1$ of the second. By §3 we thus have

$$(2) \qquad \aleph_0 + 1 = \aleph_0$$

The rest of section 6 consists of justifications of this, but with no exception they are all leaning only on definitions and bijection as a method to count infinite sets. As we noted before, the transfinite numbers also require new counting laws[45], e.g.:

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + v = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0, \quad \aleph_0 \cdot 2 = \aleph_0, \quad \aleph_0 \cdot v = \aleph_0, \quad \aleph_0 \cdot \aleph_0 = \aleph_0$$

Again, before continuing, let us ask the question: What is left of the concept of number that was stated by earlier (and later) attempts to provide foundations for arithmetic and mathematics, or of the intuitive concept of number?

For finite sets we have that the cardinality of the set of subsets of a set with cardinality s equals 2^s. Cantor uses this to establish the next transfinite number $\aleph_1$. To do this he introduces a second number class, which is defined:

> Definition: The second numbers class $Z(\aleph_0)$ is the totality $\{\alpha\}$ of all order types α of well-ordered sets of cardinality $\aleph_0$.

To understand this we need to see Cantor´s view on the relation between ordinal numbers and cardinal numbers of elements of a set. In the finite case he had already established that the properties of the ordinal numbers must coincide with the properties of the cardinal numbers. However, it was quite different in the infinite case[46]:

> It is entirely different with transfinite ordinal numbers; to one and the same transfinite cardinal number a there is an infinite number of ordinal numbers, which comprise a homogeneous, coherent system, which we call the "number class $Z(\aleph_0)$." It is a part of the type class $\{\alpha\}$.

We will not do any more details, but Cantor claimed showing that the totality of numbers of the second number class was not denumerable and that $\{\alpha\} \neq \aleph_0$. With earlier results and conditions from finite applications, the cardinal number of $Z(\aleph_0)$ equaled $\aleph_1$,

45 G Cantor (1915) page 106.
46 G Cantor (1915) page 159.

the second smallest transfinite number, which corresponded to the cardinality of the set of all real numbers, and also equaled $2^{\aleph_0}$, thus:

$$2^{\aleph_0} = \aleph_1$$

This last assumption leaned not only on similar relations between a set and the cardinality of the set of its subsets in the finite case, but also on another assumption: that there is no transfinite number between $\aleph_0$ and $\aleph_1$; therefore, as it had been proven (but only by the diagonal argument) that the cardinality of the set of all real numbers is bigger that the one of the natural numbers, the cardinality of the set of all real numbers must equal another transfinite cardinal number than $\aleph_0$, i.e. $\aleph_1$. This assumption is the famous continuum hypothesis, a hypothesis that Cantor never managed to prove. Later Kurt Gödel (1940) and Paul Cohen (1963) showed that the continuum hypothesis could not be disproved (Gödel) nor proved (Cohen) within ZFC. The normal conclusion when a proposition is neither provable nor disprovable is that the proposition is meaningless, which means that there is no ontology or anything else that can justify that such a proposition is true or false. We will end this section by quoting J W Dauben (1979)[47]:

> Cantor´s presentation of the principles of transfinite set theory in the *Beitrag* was elegant but ultimately disappointing. One might have thought that at long last, having given the extensive and rigorous foundations of the transfinite ordinal numbers of the second number class, Cantor would then have gone on to discuss the higher cardinal numbers in some detail. In particular one might have expected him to fulfill his promise made in Part I to establish not only the entire succession of transfinite cardinal numbers $\aleph_0$, $\aleph_1$, ... $\aleph_v$, ... , but to prove as well the existence of $\aleph_w$, and to show that in fact there was no end to the ever-increasing sequence of transfinite alephs. But instead ... the proofs concerning the power of $Z(\aleph_0)$ were presented, but were then left to impress the reader in silence ... The entire manner of Cantor´s handling of the transfinite cardinals in the *Beiträge* was fundamentally unsatisfying....
>
> By the time he wrote the Beiträge, the solution of the continuum hypothesis seemed as elusive as ever, despite the tantalizing hope that coverings, which led to the formulation $2^{\aleph_0} = \aleph_1$, might provide the key for which Cantor had searched so long. But by 1897 the discovery of the paradoxes of set theory, his inability to establish directly the comparability of all cardinal numbers, and the lack of any proof that every set could be well-ordered seemed to leave him with no alternative than the one he chose: rather than to produce complete, absolutely certain solutions to the outstanding problems his set theory had raised, he was forced to accept something less.

The problems of this theory pointed out by Dauben are added to the foundational considerations connected to the fact that the theory is dependent on the metamatemathical assumptions of Cantor´s diagonal proof as well as the observation that the theory and the huge ontology it creates has no other justification.

47 J W Dauben 1979, page 217.

Cantor and God

The purpose of this section is to highlight some religious concerns that Cantor held. The reason for dealing with this at this point in the present thesis, is that might shed some light on why Cantor made the choices he did, for instance regarding what is quoted from Dauben just above. Obviously, religion must not be an obstacle for science generally speaking. Nevertheless, if religion affects scientific objectivity and the status of evidence-based results, this should be taken into consideration, not only to keep science clean, but also to provide psychological explanations for why unscientific results may have been put forth.

J W Dauben´s biography over Cantor constitutes a rich source for our understanding of Cantor and his results. Several sections of this book covers Cantor´s thoughts about the role God had in mathematics and in Cantor´s work. The research of Dauben is based on publications, diaries, private letters and testimonies.

We read on page 231[48]:

> One must distinguish numbers as they are in and of themselves, in and of the Absolute Intelligence, and those same numbers as they appear in our limited discursive mental capacity and are defined (in different ways) by us for systematic or pedagogic purposes ... The (cardinal numbers) are all independent from one another (taken absolutely), all are equally good and equally necessary metaphysically.[47]

> Just as God had earlier confirmed the necessary reality of the transfinite numbers, Cantor made a similar appeal to ensure the correctness of his own belief in the independence of the cardinal numbers. Once more Cantor had invoked God to guarantee the absolute truth of the principles upon which set theory was based.

And on page 232[49]:

> Moreover, Cantor also believed in the absolute truth of his set theory because it had been revealed to him, as he once told Mittag-Leffler, from God directly.[50]

The final chapter of Dauben´s book is named *Epilogue: The Significance of Cantor´s Personality*. Together with chapter 6 in the same book, we have access to a vaste amount of information concerning Cantor´s metaphysical ideas. The reason to mention this here, is that these ideas are not only relevant to Cantor´s mathematical ideas, but also that this of course invokes judgements concerning Cantor´s work from a standpoint of theory of science and of scientificity. We will only make one more short observation about this: Cantor held a view that the natural sciences produced too approximate results. He was convinced that the number of elementary particles must be regarded as absolutely infinite in order for natural sciences to be able to explain natural phenomena. Addition-

48 J W Dauben 1979. The footnote 47 refers to quoted letter from Cantor to Veronese, October 6 1890, Cantor´s letter-book for 1890 through 1895.

49 J W Dauben. The footnote 50 refers to letter from Cantor to Mittag-Leffler, December 23 1883, archives of Institut Mittag-Leffler, Djursholm, Sweden.

ally, he thought that these particles were absolutely extensionless, spatially speaking. If this was assumed, according to Cantor the step to connect natural science to set theory was immediate. Cantor also was inspired by Leibniz and the *Monadology* with its rather extreme and mysticism-influenced ontological assumptions[50].

How would set theory clarify any results of natural science? Gösta Mittag-Leffler (Swedish mathematician 1846-1927), who was a friend and benefactor of Cantor, asked Cantor to give some indications about the set theory´s utility in other branches of science. Cantor prepared some paragraphs to be added at the end of the paper *Über verschiedene Theoreme aus der Theorie der Punctmengen in einem n-fach ausgedehnten stetigen Raume Gn: Zweite Mittheilung* (January 1885) that was to be published in *Acta Mathematica* vol. 7 (1885). As Cantor was very concerned to relate God, his creation and science in a way that was non-heretical, he stressed that bringing his set theory into the real world did not, in spite of the actually infinite nature of the universe and his own assumption that corporal and aetherical monads were related to each other as powers equivalent to the transfinite cardinals $\aleph_0$ and $\aleph_1$, mean that God could not have created the world in another way. As Cantor put in his manuscript for *Acta Mathematica*:

> It ought not to be said that *other* (kinds of) matter could not have been created (or even that they have not been created) by the Creator, but only that those two *substrates* seem to be sufficient to *explain* all the *perceived appearances thus far observed* (in Nature.)

The Editor of *Acta Mathematica*, Gustaf Eneström (1852-1923), wrote to Cantor and persuaded him to actually omit most of this last sentence from the manuscript before printing it[51] (and to change the last part of it), and this sentence is therefore not present in the published text. Eneström was obviously uncomfortable with Cantor apparently claiming that his ”discovery” of $\aleph_0$ and $\aleph_1$ had revealed a feature more eternal than the actual creation of God (which could have been done in another way) and thus that the relation between $\aleph_0$ and $\aleph_1$ could be interpreted as the common denominator for all possible worlds, and maybe also with that this relation was the ”explanation of all the perceived appearances thus far observed”. Eneström preferred to have no reference to the Creator.

Again, these notes on Cantor´s personality and beliefs are added here to provide an objective view, or at least a glance of it, in order to be able to judge whether the theory of transfinite numbers are scientifically sound or not (see also the section *Scientificity* below).

Below (fig. 4) is a scanned image of the omitted sentence in the original manuscript[52]. Translation from German (see the quotation above) as in J W Dauben: *GEORG*

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50 See J W Dauben 1979, section *Cantor´s world Hypotheses*, pp 291-295.

51 J W Dauben 1979, note 83 on page 295: From the manuscript of *Zweite Mittheilung* intended for Acta Mathematica vol.7, 1885. Institut Mittag-Leffler, Djursholm, Sweden, and note 84 on page 295: From Eneström to Cantor, April 1884; the original is in the Kungl. Vetenskapakademiens Bibliothek, Stockholm, Sweden. See also appendices II and III page 53-54.

52 Thanks to Hans Ringström (Deputy Director), Institut Mittag-Leffler, Djursholm, Sweden.

CANTOR - His Mathematics and Philosophy of the Infinite. For a scanned copy of the whole page 15 in the manuscript from 1885 see page 53. For a copy of the published version see page 54.

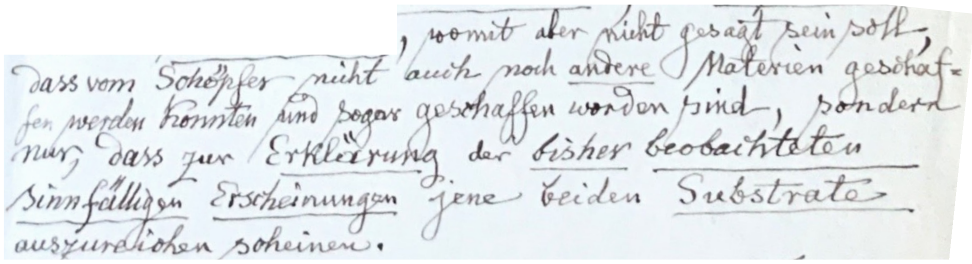
, womit aber nicht gesagt sein soll,
dass vom Schöpfer nicht auch noch andere Materien geschaf-
fen werden konnten und sogar geschaffen worden sind, sondern
nur, dass zur Erklärung der bisher beobachteten
sinnfälligen Erscheinungen jene beiden Substrate
auszureichen scheinen.

Fig 4

Some other voices

It must be said that the mainstream perception of this problem is (still) that Cantor´s assumptions were right. As one of many examples, in *Numbers and Infinity* (1986) by mathematicians Ernst Sondheimer and Alan Rogerson[53] we read:

> Cantor...showed (starting around 1873) that the attitude of Cauchy and Weierstrass in rejecting 'completed' infinite sets from mathematics was over-cautious, and that it was indeed possible to give clear and rigorous rules for calculating with such sets, provided one was willing to admit that these rules may differ from those of finite sets...

Sondheimer and Roberson continue by regarding how Cantor puts objects in different infinite sets in a one-to-one correspondence between the objects of the different sets. They note that Cantor´s conclusion when ordinalwise pairing e.g. the set of natural numbers with the set of positive even integers was that the two sets were equinumerous; they have *the same* number of elements (italics by Sondheimer/Rogerson). When discussing the one-to-one corresponance method to compare finite sets they write:

> But this method works also for infinite sets! ... This number, which is clearly not finite, is our first transfinite number: it is denoted by $\aleph_0$ ... It is not to be called 'infinity': that is too vague a concept.
>
> The conclusion that our two sets have "the same" number of elements is a straightforward deduction from our pairing principle, and yet it seems absurd. After all, the even integers do not include the elements 1, 3, 5, ... (technically, the even integers form a *proper subset* of the set of all the integers), so there are surely *fewer* of them than the numbers in the set of *all* the integers! Well - it all depends on what you mean by 'fewer'! The confusion and apparent paradoxes in this subject arise from the transfer of everyday language, acquired from experience with finite collections, to infinite sets where we must train ourselves to work strictly with the mathematical rules of the game even though they lead to surprising results.

53 E Sondheimer and A Rogerson: *Numbers and Infinity, a Historical Account of Mathematical Concepts*, Dover Publications 1982/2006, page 149.

The intuitionistic view on these comments would be to dispute the justification of the use of *the same* (number of elements). It is not justified to use expressions like ”fewer”, ”more numerous”, ”the same number” etcetera in these contexts. Sondheimer/Rogerson themselves seem to be trapped in applying finite concepts to infinite sets, which was their explanation why e.g. Weierstrass and Cauchy were wrong (which is proven). Again, one-to-one correspondence and, especially bijection, require equinumerousity, which for infinte sets is granted only by the *axiom of infinity* (interpreted as that there is an infinite set with a specific cardinal number). Infinite sets do not, according to the intuitionistic or constructive view, have a given number of elements, and that is the very property that makes them infinite. This is a purely logical standpoint that has nothing to do with everyday language. As far as everyday language concerns, what can be said is that the everyday expression ”set” is misleading when used on infinite aggregates, as it implies finity but certainly has different properties that a finite set. In fact, Brouwer noticed in 1908 that the *Law of the Excluded Middle* was abstracted from finite situations, then extended without justifiction to statements about infinite collections[54]. About LEM and infinite proper subsets of infinite sets, see below under *Intuitionistic logic vs classical logic - useability of mathematical propositions*.

Sondheimer and Rogerson also states (about Cantor´s diagonal proof and about the role of transfinite numbers in mathematics today[55]):

> The proof is very simple, and very ingenious.
>
> ... the subject of transfinite numbers has become a major branch of mathematics.

As stated earlier, the conclusions of the proof are not stronger than the axiom of infinity, and whether the theory of transfinite numbers is a major branch in mathematics can be discussed. However, these kind of statements are not rare.

Sondheimer/Rogerson also propose another, more geometrical method to demonstrate that the infinite set of rationals is smaller than the infinite set of reals. Think of the real numbers in [0, 1] as points on a line. Assume they are denumerable. Enclose one of the points in an arbitrary interval, say 1/10. Then enclose a second point in an interval of $1/10^2$, and so on. The total length all the intervals summed will have the limit 1/9, which we can see by the following computation (very similar to how we compute what happens in Zeno´s paradox of motion; see earlier description of that paradox on page 4):

$$\frac{10^{-1}}{1 - 10^{-1}} = \frac{1}{9}$$

This indicates that the set of intervals, which is infinitely denumerable, cannot cover the whole length of [0, 1]. Sondheimer/Rogerson conclude:

54 L E J Brouwer: *The Unreliability of the Logical Principles*, 1908, English translation in Heyting (ed.) 1975: 107–111. Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/logic-intuitionistic/, 2020.

55 Sondheimer/Rogerson 1982 pp 153-154.

It shows strikingly that the set of rationals numbers is a smaller infinite set than the set of reals.

Sondheimer/Rogerson avoid the word ”proof” in this section, and stay to the less obliging ”demonstration”. And, in fact, objections seem reasonable. It suffices to change the set of points from being the set of all real numbers in [0, 1] to the set of all rational numbers of [0, 1] (whatever we mean by ”all” here, we mean the same as in the example of Sondheimer/Rogerson). Then, by the same logic as in Sondheimer/Rogerson´s example, we will have shown that the set of rational numbers in the interval [0, 1] is strictly smaller that itself (as the totality of intervals cannot cover the whole set of rational numbers in [0, 1]), which is absurd. So whatever conclusion one can draw from their example, it is not that the set of real numbers is bigger that the set of rational numbers. The phenomenon that the totality of intervals in this example cannot cover anything beyond the point 1/9 (if put together next to each other) is related to basic properties of converging series, and to what is meant by covergence and divergence in the light of potential infinity, as taught in calculus courses.

One other voice is a person that did not accept Cantor´s diagonal argument: Ludwig Wittgenstein (1889-1951). We will later make some, for our discussion relevant, observations concerning Wittgenstein´s view on the concept of meaning (see below under *Intuitionistic logic vs classical logic - useability of mathematical propositions*). Here is just a quotation showing Wittgenstein´s standpoint[56]:

> If it were said: Consideration of the diagonal procedure shews you that the concept ‘real number’ has much less analogy with the concept ‘cardinal number’ than we, being misled by certain analogies, are inclined to believe, that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the ‘set’ of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus.

On the other side of the spectrum we have e.g. Wilfrid Hodges (1998) defending Cantor´s proof[57]:

> Cantor’s argument is short and lucid. It has been around now for over a hundred years. Probably every professional mathematician alive today has studied it and found no fallacy in it. So there is every temptation to imagine that anybody who writes a paper attacking it must be of dangerously unsound mind.
> In English-speaking philosophy (and much European philosophy too) you are taught not to take anything on trust, particularly if it seems obvious and undeniable.You are also taught to criticise anything said by earlier philosophers. Mathematics is not like that; one has to accept some facts as given and not up for argument.

56 L Wittgenstein: *Remarks on the Foundations of Mathematics* II §22 1956/1978 Revised Edition, Oxford: Basil Blackwell, G.H. von Wright, R. Rhees and G.E.M. Anscombe (eds.); translated by G.E.M Anscombe.

57 W Hodges: *An Editor Recalls Some Hopeless Papers,* The Bulletin of Symbolic Logic, Vol. 4, No. 1 (Mar., 1998), page 3.

CONCLUSIONS

Summary of conclusions

We have studied Richard´s paradox and a number of related topics. The observations we have made concern diagonalization as method to prove propositions involving quantification over domains from which we produce objects that are defined by the domains but not located in them. We have also made observations that concern justifications of proofs involving the infinite, and more precisely the distinction between actual and potential infinity. These observations lead to two main conclusions:

1. When diagonalizing, it is essential that the object defined (and thus produced) by diagonilizing cannot be a member of the domain that defined them. This is a direct consequence of the act of defining (see above under *Russell´s paradox*). It follows that the conclusion of Cantor´s diagonal argument, i.e. that the set of real numbers have greater cardinality than the set of natural numbers, must not be the reason why an object that is defined and produced by diagonalization cannot be found in the domain that defined them.

2. The property that is required for sets in order to be able to establish a bijective one-to-one correnspondance between the elements of the sets respectively (i.e. actual cardinality) is not proven to exist for infinite sets. It is only stated by the axiom of infinity, which is an axiom without justification, and which postulates that there is an actual infinite number that corresponds to the cardinality of an infinite set. This axiom leans on the assumption that the properties for finite sets must also be properties for infinite sets. Therefore, as the method of bijection is fundamental for Cantor´s diagonal argument, rejecting bijection as a method to compare infinite sets is to reject the conclusion of Cantor´s diagonal argument.

An immediate consequence of 2 is that all-quantification over infinite sets is not justified (only postulated), as this too would lean on the assumption that infinite sets have an actual number of elements. Again, as the possibility of such a quantification is an assumption for Cantor´s diagonal argument, conclusions of the argument are directly dependent of the assumption that infinite sets have actual cardinalities.

If we regard these conclusions as serious objections to the assumptions of Cantor´s diagonal proof, i.e. the axiom of infinity (interpreted as the postulate of existence of infinite sets with a specific cardinality), then some important consequences follow. One of those is that, as the theory of transfinite numbers has no stronger (or other) justification than Cantor´s proof and the axiom of infinity; objections to the conclusions of the proof are also objections to the whole theory of transfinite numbers. This theory would then be regarded as a platonistic and superfluous ontology, without any empirical content. This would also be strengthened by some views on the concept of meaning that is put forth in intuitionistic logic: in order for linguistic statements (here including mathematical statements) to have a meaning, they must have a use[58]. They must be used consistently in some kind of linguistic community, and that is in short the very essence of meaning. We note that, concerning the theory of transfinite numbers, the lack of useability coincides with the other objections earlier mentioned. A second consequnce is that the concept of denumerability for infinite sets will lack justification and meaning.

Other consequences of the conclusions are presented below under *Consequences - Extension of the conclusions.*

Scienticifity

In this summary it should be noted, that in addition to the conclusions themselves, they should be viewed upon towards the background that is presented in the section *Cantor and God*. Dauben points out[59]:

> ... the analysis of personality, in particular when creative individuals are concerned, can reveal a great deal about the nature of intellectual discovery.

Generally, we demand scientificity when scientific results are published and maybe also used in university courses. Although, as have already been mentioned, religion is not per se an obstacle for science, religious statements should not be mixed up with scientific conclusions. When judging Cantor´s results we must bear in mind that he did apparently not distinguish completely between scientific results and religious (at times nearly mysticistic or science-fiction-like) results or views. The doubts concerning the context of Cantor´s diagonal proof that are put forth here do not become easier to neglect when considering the fact that Cantor apparently had non-scientific (non-objective) reasons or wills to achieve certain results. At least, we must demand the same degree of evidence-based scientificity concerning axiomatic assumptions Cantor´s theories as we do concerning other theories.

Ontology of mathematical objects - Constructivism and intuitionism

Ontology is the study of being, i.e. the study of the concept of existence. What does it mean to claim that something exists? Are there different degrees of existence? For example, do the objects in $\mathbb{C}$ exist in the same sense as the objects in $\mathbb{N}$? Especially: do undefined mathematical objects exist in the same sense as defined mathematical objects? The natural numbers are defined recursively by e.g. Peano in the 19th century, and Dedekind and others refined these definitions together with definitions of mathematical operations. But although the level of abstraction can be different for different mathematical objects, nobody has objections to ascribing existence to well defined mathematical objects, at least not in mathematics. When we come to undefined objects it is different.

58 I.e. M Dummett: *The Philosophical Basic of Intuitionistic Logic*, 1973, *Philosophy of Mathematics* (Ed. P Benacerraf), Cambridge University Press 1983), pages 216-217.

59 J W Dauben 1979, page 271.

Although e.g. real numbers can be generally described, it is not possible to state a rule that captures them recursively in the same way as for instance the naturals. This difference was noticed by Leopold Kronecker (1823-1891), who in a lecture already in 1886 said:

> God made the integers, the rest is the work of man.

This was contrasting to Cantor, who around the same time said that God had created the transfinite numbers for Cantor himself to discover[60]. This reflects the early split between what would be the classical, logicist or formalist line (e.g. Cantor, Dedekind, Frege and later Hilbert) and the pre-intuitionists that later developed intuitionism (e.g. Kronecker, Poincaré and later Brouwer and, partly, Herman Weyl). The quotation on page 23 is from Brouwer´s PhD thesis in 1907, and shows clearly that a general description was not regarded as sufficient for the intuitionists. Objects without any method or rule to define them were regarded as non-existing and the linguistic entities that represented them did not have any meaning.

We can highlight the constructivistic/intuitionistic view with an often quoted example[61]:

> **Proposition:** There exist two irrational numbers a and b such that a^b is a rational number.
> **Proof:** $(\sqrt{2})^{\sqrt{2}}$ is either rational, and then we take $a = b = \sqrt{2}$, or $(\sqrt{2})^{\sqrt{2}}$ is a irrational, and then we can take $a = (\sqrt{2})^{\sqrt{2}}$, $b = \sqrt{2}$ ($\Rightarrow a^b = 2$ in both cases)□.

If we read ”there exists” as ”we can construct”, then there is no method given to construct a as a real number. It is a counterexample, but without any possibility to, in this context, compute a. There is no positive proof that can assure that a is an irrational. So, we do not know, after having made the proof, whether $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational.

A constructive way to prove the proposition would be to use to Gelfonds theorem[62]: if a and b are algebraic numbers with $a \neq 0, 1$, and with b irrational then a^b is transcendental. From this follows that $(\sqrt{2})^{\sqrt{2}}$ is irrational, and we can use the second half of the proof. It can also be proven in a constructive way without using Gelfonds theorem by giving a counterexample that does not, as the example above, lean on the law of the excluded middle (LEM): Let $a = \sqrt{2}$ and $b = \log_2 25$. Then $a^b = 5$. We have that $\log_2 25$ is irrational, because if it was rational then $25^m = 2^n$ where $\log_2 25 = m/n$. But 25^m is odd and 2^n is even, so m/n is not a rational □. Concerning LEM see also below under *Intuitionistic logic* vs *classical logic - useability of mathematical propositions*.

With this example as background we can in a general way summarize what we mean by constructivism. It is a more narrow approach to mathematics than the classical view. Mathematics is thought of as an activity of human mind. The ontological status of mathematical objects is dependent on human mind, not on natural laws, which would be the more platonistic view emphasized by parts of classical mathematics. In this paper constructivism is a similar concept to intuitionism, but generally with weaker limitations

60 See above page 31.

61 A S Toelstra and D van Dalen: *Constructivism in Mathematics*, page 7.

62 See e.g. https://en.wikipedia.org/wiki/Gelfond-Schneider_theorem (september 2020)

of the basics of the classical mathematics than in the intuitionistic approach. There are different levels of constructivism, that correspond to different degrees of restriction of the basic classical mathematics. One very basic and important property of constructivism and intuitionism is that LEM is not universally allowed[63].

With classical mathematics we mean the broader approach. Mathematical objects are, or at least were, thought of as being discovered by mathematicians rather than invented by them, and LEM is an important basic axiom. It is interesting to note that in the end of the 19th century the meaning of the expression *classical mathematics* was the ideas that Cantor opposed to[64]. His ideas were regarded as the modern and new mathematics replacing the old, at that time classical, school from e.g. Cauchy, Weierstrass and the other creators of what came to be the calculus. However, today, by classical mathematics we mean the view accepting LEM (universally) and Cantor´s ideas, as well as the calculus that has its origin in the work of Cauchy, Weierstrass and others. Concerning the infinity, these two parts of nowadays classical mathematics support somewhat different views.

As mentioned, from the constructivist standpoint, quantification over sets containing non-defined entities is not possible. It follows that the constructivist standpoint emphasizes a different ontological status for defined objects compared to undefined. When we say ”all” reals, this does not mean anything else than all reals that are already defined, or can be defined by a stated rule without using as definiens a definiendum already defined by the same rule. The constructivist view supports the idea that a number is a mental construction and not an object with an existence independent from human mind, so, when we execute Cantor´s diagonal proof, we can think of it as a list of all constructed reals enumerated by their ordinal numbers. When we execute the diagonalization, the list will be extended with the new number. In fact, this shows that the lack of possibility to make a list of all real numbers is not (only) a problem of time or space, but (first and foremost) a problem of logical possibility. We cannot make the list, because for each real number that we want to put on the list, we need to *know* what to write (even, in some sense, for objects with infinite number of digits). For an undefined object this is not an incidental (room/space-type) impossibility, but a logical one. Hence, there are three different (but related) reasons for why we cannot quantify over all reals in Richard´s paradox or in Cantor´s diagonal proof:

1. Additional numbers can be created out of the numbers in the set over which we quantify, but they cannot be member of that set (impredicativity).
2. We cannot quantify over sets with no established cardinality.
3. We cannot quantify over objects that do not have an established existence (ontological status of mathematical objects).

If we accept (even only one of) these assumptions, then from the constructivist point of view there is no possibility to accept the conclusions of Cantor´s diagonal proof.

63 See below under *Intuitionistic logic vs classical logic etc*.
64 Encyclopedia Britannica: *https://www.britannica.com/biography/Georg-Ferdinand-Ludwig-Philipp-Cantor.*

CONSEQUENCES - EXTENSIONS OF THE CONCLUSIONS

Intuitionistic logic vs classical logic - useability of mathematical propositions

The present paper has from the top until here circled around one feature, that has been highlighted with several examples and viewed upon from different points of view. This feature is the split, or discrepancy, between classical logic and constructive/intuitionistic logic. All the problems we have met are closely related to this split. We have on more than one occasion mentioned constructive mathematics. When we have mentioned constructivism we have made examples of applications of intuitionistic logic. Constructivism and intuitionism exist in different levels, but what they have in common is the idea that classical mathematics (and logic) need to be restricted. If not, we will have contradictions and paradoxes, along with an unjustified and superfluous ontology, as well as lack of useability for some mathematical contexts. Basically the cause of this is that classical mathematics has its origin in practise in the finite realm. As is already mentioned above, this finding was the entry for e.g. Brouwer, who started to investigate the consequences of classical mathematics making to strong statements (i.e. promised to much), which was around 1907-1908. Here we will only make a short and basic sketch of what intuitionistic logic is.

A general difference between classical logic and intuitionistic logic is that in intuitionism, logic is not providing a foundation of mathematics, but is a part of mathematics. In the history of logic there is a branch called logicism, which hold the opposite view, i.e. that the foundations of mathematics is to be found within logic. This branch is a part of (the foundation of) classical mathematics. In intuitionism there is no foundation of mathematics; it is regarded as a mental activity, where meaning is dependent of use and useability.

The most well-known and basic difference between classical and intuitionistical logic is that the latter do not accept the ancient law *Tertium non Datur*, in english *The law of the excluded middle* (LEM). Already Aristotle commented on that law[65]. It is not easy to follow his argumentation, but he concluded that this law must be valid, even for some propositions that we could not have knowledge about (they were either true or false independently of our knowledge). Anyway, formally LEM is denoted as follows:

$$A \vee \neg A \qquad (1)$$

This means that either a proposition is true, or its negation is. Classical logic holds this as true, as well as

$$A \rightarrow \neg\neg A \quad \text{and} \quad \neg\neg A \rightarrow A; \quad \text{thus also } A \leftrightarrow \neg\neg A \qquad (2)$$

65 Aristotle: *De Interpretatione*, chapter 9.

This is called double negation elimination (DNE). As intuitionistic logic refutes (1), it must refute (2), because the double negation of LEM is provable in intuitionistic logic:

$$\neg\neg(A \vee \neg A) \qquad (3)$$

and if (2) where true then (1) would follow from (3).

The rejection of LEM has huge consequences for mathematics, of which we have already commented some (see above: *Ontology of mathematical objects - Constructivism and intiutionism*).

It must be said that intuitionistic logic normally does not *always* reject LEM and DNE. These two rules can most often be used on a case to case basis. It is the universal acceptance that is rejected. This is important to bear in mind, because it is not in the everyday realm that intuitionistic logic has stronger demands concerning rules than classical logic. It is in the realms that are very abstract. We will focus on propositions concerning the infinite, and more precisely the axiom of infinity.

It is especially among intuitionists and constructivists that objections have been raised against this axiom. In 1946 Herman Weyl wrote[66]:

> ... classical logic was abstracted from the mathematics of finite sets and their subsets .... Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of [Cantor's] set theory ...

We have also already noted that this axiom lacks all sorts of self-evidence, although this aspect is today the same for many axioms in mathematics. Additionally, there is no use for this axiom. It does not lead to any useful mathematics, only to the concept of denumerability for infinite sets and then on to the theory of transfinite numbers, and this theory is not useful (and therefore not used) in mathematics. We have also seen that Brouwer already in 1908 held the same view as Weyl later did (see above page 32). These circumstances put together would point in the direction that the infinite is a subject where implications from the finite set theory should be practiced with caution, or avoided. Subsequently, intuitionistic logic does not accept this axiom. And an intuitionistic definition of an infinite set (if at all accepted as a mathematical object) would *not* be

Def: A set is infinite iff it is equinumerous to a proper subset of itself[67].

Because from this definition we can derive that for an infinite set we have $\omega + 1 = \omega$, but the intuitionist would at most accept $\neg\neg(\omega + 1 = \omega)$, which would then *not* be the same thing. The reason for this is that concepts like cardinality and equinumerousity concerning infinite sets have no meaning in intuitionistic mathematics. An infinite set can be

66 H Weyl 1946: *Mathematics and logic,* editor by P Pesic *Levels of Infinity*, Dover Publications 1930/2012, page 141.

67 See above page 8.

regarded as equinumerous to everything and to nothing.

The difference between (1) and (3) (and thus also between $(\omega + 1 = \omega)$ and $\neg\neg(\omega + 1 = \omega)$ above) is also highlighted by Toelstra[68]:

> In fact, we cannot hope to refute any individual instance of PEM [LEM], that is to say we cannot find a mathematical statement A such that $\neg(A \vee \neg A)$. This is impossible, since $\neg\neg(A \vee \neg A)$ holds universally: it is an intuitionistic logical law. This may be seen on the basis of the BHK-interpretation as follows. Suppose
>
> $$c \text{ establishes } \neg(A \vee \neg A). \qquad (1)$$
>
> Hence,
>
> $$\text{if } d \text{ establishes } A \vee \neg A,\ c(d) \text{ proves } \perp .$$
>
> Clearly, there are operations $e \mapsto e_0$, $f \mapsto f_1$ such that
>
> $$e \text{ establishes } A \;\Rightarrow\; e_0 \text{ establishes } A \vee \neg A,$$
>
> $$f \text{ establishes } \neg A \;\Rightarrow\; f_1 \text{ establishes } A \vee \neg A$$
>
> Therefore if (1) holds, $g : e \mapsto c(e_0)$ (i.e. $g := \lambda e.\ c(e_0)$) is a construction for $A \to \perp \equiv \neg A$ and $h : f \mapsto c(f_1)$ (or $h := \lambda f.c(f_1)$) for $\neg A \to \perp$. Therefore $h(g)$ is a construction for $\perp$, so the mapping $c \mapsto h(g)$ establishes $\neg\neg(A \vee \neg A)$.

When describing the most basic properties of intuitionistic logic it is essential to observe the concepts of *use* and *meaning*, and their relation. Ludwig Wittgenstein (1889-1951) elaborated this relation in a broader sense, i.e. not only concerning mathematics but for language in general (and thus also for mathematics). Hardly any other person have had the same impact on our understanding of the mutual dependance in which these two concepts stand. It is particularly interesting to observe that Wittgenstein himself held a fairly platonistic view on these matters (at least in some fields) in his early period[69]. By that time he thought that linguistic meaning had its explanation exterior to the human mind, and this was an opinion (or rather *the* opinion) he later changed[70]. Although not being a mathematician, his work has great concern for mathematics as being important for metamathematics. In his later works he described the meaning of linguistic expression as coinciding with (or, rather, defined by) its use and useability (not in the strict pragmatic way, but as a feature of interaction in a linguistic community). This view was to a great extent coinciding with the view i.e. Brouwer held concerning mathematics half a century before Wittgenstein´s later work was (posthumously) published in 1953 (in fact, some researchers claim that Wittgenstein was greatly influenced by Brouwer[71]).

68 A S Toelstra and D van Dalen 1988 page 11.

69 See L Wittgenstein: *Tractatus Logico Philosophicus* 1921, Chiron Academic Press 2016

70 See L Wittgenstein: *Philosophical Investigations*, Macmillan Publishing Company 1953

71 M Marion: *Wittgenstein and Brouwer*, Synthese , Nov., 2003, Vol. 137, No. 1/2, History

So, the relation between meaning and use is essential for the views that opposed the ideas that had been forwarded by Cantor in late 19th century, and here we can see an important part of what constituated the split between (what was to be called) classical mathematics and constructivist mathematics occuring in the beginning of the 20th century.

One of the mathematicians who supported Cantor´s ideas was David Hilbert. Apart from exposing problems with the infinite (see above: *Hilbert´s paradox*), he is particularly famous for the following words[72]:

> From the paradise, that Cantor created for us, no-one shall be able to expel us.

However, he saw problems with parts of Cantor´s results concerning the infinite, and he was leading a development of the so called formalism (or finitism), which briefly seeked to establish mathematical propositions in finite terms, but still keeping the set theory as the base for the foundations of mathematics. Formalism also held the classically oriented view that mathematics was a purely abstract science dealing with universal (and eternal) truths that had no dependence of human mind concerning their ontological status. The debate during the first half of the 20th century was, at times, intense. Another mathematician who in this context is interesting is Herman Weyl (1885-1955), because he was a student of Hilbert, whom he admired, but he was also much affected by the writings of Brouwer. A simplified description of Weyl´s development was that he first held on to Hilbert´s ideas, later opposed Hilbert and supporting Brouwer´s ideas, and finally he, which is well expressed in his late writings, came to hold a view somewhere in between. This later standpoint was, however, closer to constructivist ideas that to classical. In Weyl´s *Levels of Infinity* the editor Peter Pesic in the preface highlights this by formulating the question[73] (see also above under *Ontology of mathematical objects - Constructivism*).

> *What sort of existence* does $(\sqrt{2})^{\sqrt{2}}$ have?

He continues:

> In the early 20s, Weyl went so far as to set his own fundamental approach aside and acclaim Brouwer as ”die Revolution.”[12] Hilbert, in contrast, advocated a purely abstract formalism in which mathematics became a meaningless game played with symbols, utterly detached from intuition and hence untained by human fallacies and illusions. Hilbert thought thereby to assure at least the non-contradictoriness of mathematics, leaving for the future to prove its consistency through iron-clad logical means that would owe nothing to mere intuition.[13] For him, Weyl´s metamorphosis into an acolyte of intuitionism verged on betrayal of what Hilbert thought was the essential mathematical project, which

of Logic (Nov., 2003), Springer, pp. 103-127

72 D Hilbert: *Über das Unendlische*, 1926, Matematische Annalen 95(1), page 170.

73 P Pesic: *Preface* to H Weyl: *Levels of Infinity* (Ed. P Pesic), Dover Publications 2012, page 3-4. The notes 12 and 13 refer to a number of writings of Weyl, Scholz and Feferman, and to selections of Hilbert writings by Benacerraf, Putnam, Ewald and Mancosu (page 13).

> included the ”paradise” (as Hilbert called it) of Cantor´s transfinite numbers. But by the mid-1920s, Weyl´s enthusiasm for intuitionism had given way to a more measured view of ”the revolution,” which he (along with Hilbert) judged would leave in ruins too much beautiful and important mathematics that could not be proved using intuitionistically pure arguments.

Weyl himself, in the essay *Mathematics and Logic* (1946, section *Levels or no levels*[74]*? The constructive and the axiomatic standpoints,* page 137) critizises the results of Russell concerning the solution of his paradox, and at the same time of Cantor´s set theory. Weyl refutes the Russellian hierarchy of types as a mathematical system no longer founded on logic:

> ...but on a sort of logician´s paradise, a universe endowed with an ”ultimate furniture” of rather complex structure and governed by quite a number of sweeping axioms of closure. The motives are clear, but belief in this transcendental world taxes the strength of our faith hardly less than the doctrines of the early Fathers of the Church or of the scholastic philosophers of the Middle Ages.

This critizism concerns the problems we are dealing with in the present paper, and is clearly of a constructivistic character. When it comes to infinity, Weyl offers substantial argumentation in the essay *Levels of Infinity* (1930, the first essay in the mentioned book edited by P Pesic). Just a few fragments of Weyl´s conclusions[75]:

> Indeed, set theory proceeds even more radically: it uses the expressions ”*there exists*” and ”*for all*” unrestrainedly....
>
> The method of set theory has taken over not only analysis but also arithmetic... and promised to reduce them to general logical concepts like ”*there exists*,” ”*for all*, ”*one-to-one correspondence*.”

And from the conclusions in the end of that essay[76]:

> 3) The infinite is accessible to mind and intuition in the form of a field of possibilities opening to infinity, as with the always further continuable sequences of numbers; but
> 4) The completed, actual infinite as a closed realm of absolute existence cannot be given to the mind.

Maybe the soundest way of regarding mathematics is located somewhere in between classical and intuitionistic mathematics. But where then more precisely? That is yet a question that will not be treated in the present thesis. However, let us just take a short glimpse in that direction: consider the first proof from page 40 under *Ontology of*

74 The title of this chapter has nothing to do with the title of P Pesic´s essay collection of Weyl´s writings (and the title of the first of those essays). Here is meant the levels in Russell´s type theory, not the different levels of the infinite proposed by Cantor.
75 H Weyl: *Levels of Infinity*, edited by P Pesic, Dover Publications 1930/2012, page 25.
76 H Weyl 1930/2012, pp 29-30.

mathematical objects - Constructivism and intuitionism. This proof is grounded on the assumption that $(\sqrt{2})^{\sqrt{2}}$ is either a rational number or it is not. Thus, the proof leans on the proposition that LEM is true. But compare this proof and its founding proposition (LEM) with any proposition containing for example that the set of all even natural numbers is denumerable or that this set has a specific number of elements, or that we can speak of ”all elements” in this set. These two propositions would be substantially different. It is not the same thing to claim LEM in the case of this proof that it is to claim it for the second kind of proposition. Remember that one way to look upon constructive mathematics and the rejection of LEM is that this rejection only concerns the universal application of LEM[77]. This is so because, as an example, the first proposition (the proof on page 40) has meaning, whereas the second has not. Maybe this distinction is a suitable starting point from which to find out where to draw the line between classical logic and intuitionistic logic, because the location of this line is closely related to the location of the line between meaningful and meaningless propositions. In fact we can suspect that this is the same thing.

So, again, useablility (in the non-pragmatic interpretation of the word) might be the tool with which we can locate the best way to go between LEM and its rejection, as well as between classical and constructive mathematics. After all, to claim that there is only the choice of totally accepting LEM or totally rejecting LEM, is per se a non-constructivistic claim (that is already presupposing LEM), and thus the solution with a weaker rejection of LEM than e.g. Brouwer´s strict intuitionism could be a possible way for constructive mathematics. This view would bring us very far from Cantor, nearly as far from Russell, a bit less far from Hilbert, even less far from Brouwer (on the other side) and pretty close to Weyl.

Epilogue: *Tertium Non Datur* - proof by contradiction

We have already several times touched upon LEM - *Tertium Non Datur - The Law of the Excluded Middle*. This last section has the purpose to highlight the impact that different applications of LEM have on our attempts to establish the location of the split between classical and constructive/intuitionistic mathematics. Additionally, it has the purpose to highlight the impact that the concept of meaning has on this application. These observations will end this paper, and hopefully establish a point from where it is possible to point at future directions in this matter.

When summing up this thesis, three concepts (and their relative relations) emerge as being important for our understanding of the split mentioned. These three concept are:

1. Impredicativity
2. Infinity
3. Meaning

What is their relation? In fact, *meaning* (or rather *lack of meaning*) is the common denominator. Both impredicativity and infinity are concepts represented by linguistic constructions where meaning is not always evident. Not only in the way that meaning

77 See above page 40.

can be ambiguous. But also because meaning sometimes does not exist. And at the same time, most often meaning really is there, clear and evident. It is when we examine the other cases that we discover why we have this split, and also where it may be located. So which are these other cases?

I. When *logical impossibility* is taken for accidental falsehood.
II. When *definitions* are impredicative.
III. When *quantification* is executed in contexts where quantification makes no sense.
IV. When *existence* is used as a predicate that is not well-defined.

Examples of I - IV:

I. Let B be a set and C a proposition, and let $C \Leftrightarrow \neg(B \in B)$. Here C seems obviously true, but it must always be taken into consideration that there are no conditions under which it can be false, i.e. truth is granted without any meaning conditions. There is a qualitative difference between logical and accidental truth, i.e. LEM is superfluous in the case of logical truth (in fact, truth can be seen a feature that is possible only for empirical statements). Using LEM here implies (falsely) that the truth of C is an accidental one, as a meaningful use of LEM must be a use on propositions that *can* be both true or false. So, ultimately this is the question if logical truth (with non-empirical premises in the logically true conclusion) has a qualititive difference from empiricial truth, as when we have e.g. the following proposition: *That A = A is either true or false*. This is an instantiation of LEM, but clearly different from how we use it in mathematics and elsewhere. Another example is (11) in W Hodges formulation of Cantor´s diagonal argument (page 26 above): if we assume that infinite sets do not have any cardinality, then the conclusion will not follow, because LEM cannot be applied to the apparent lack of equinumerousity. That we cannot observe equinumerousity does neither mean that it exists, nor that it does not exist, because there is no cardinality.

To dig further into this question would require elaboration of the relation between meaning and truth, a relation which is in fact crucial for inuitionistic logic. An example of someone who has presented results of such digging is Michael Dummett[78].

II. When N is said to be an element in E (using Richard´s terminology) although defined by E.

III. When E is said to consist of *all* real numbers that can be defined by permutations of letters, or when the set of real numbers over which we are diagonalizing in Cantor´s examples are said to contain *all* real numbers, or whenever talking about equinumerousity regarding infinite sets. See also the quotation below from Per Martin-Löf concerning quantification over infinite domains.

IV. When quantifying over the set of all real numbers claiming that this set also contain numbers impredicatively defined from the set itself. There is a qualitative difference

78 M Dummett: *Truth and Other Enigmas*, Harvard University Press 1978

between objects that are defined (or possible to define with a stated method) and objects not defined.

What role does LEM play in these cases? Since meaning is essential for justified use of LEM, it follows that justified use of LEM can be affected when one or more of the cases I-IV listed are actual. From this observation follows that any proof by contradiction in these cases will also be affected, i.e. if such proofs are used under these conditions we risk to get impredicative or false results (e.g. Cantor´s diagonal proof). However, LEM can always be used on a case to case basis, but that requires cautious considerations when we are dealing with cases belonging to I - IV. **To decide in which cases LEM can be used, is to clarify where constructive/intuituitionistic mathematics draw the line for what is possible in mathematics, and on the other side of that line is still some classical mathematics left, which cannot be used for anything**. Hence, intuitionistic mathematics is the proper subset of classical mathematics that is *always* useful. It is reasonable to seek the answer to why the location of the line between classical and intuitionsitic mathematics is still disputed in the assumption that this subset is nearly identical to the set of classical mathematics, but only nearly. It would be of interest, in a future paper, to draw this line taking into consideration the ontology of an intuitionistic type theory, e.g. Martin-Löf. The purpose would be to clarify our presentations of abstract mathematical concepts in mathematics in a consistent way which is also compatible with computer science. An intuitionistic type theory can be seen as a theory that abolishes axioms whose purpose is to justify superfluous ontology in mathematics, and thus also abolishes that kind of ontology. Therefore, this thesis ends with a short explanation from Per Martin-Löf of the reasons for producing an intuitionistic type theory[79]:

> The principal problem that remained after *Principia Mathematica* was completed was, according to its authors, that of justifying the axiom of reducibility (or, as we would now say, the impredicative comprehension axiom). The ramified theory of types was predicative, but it was not sufficient for deriving even elementary parts of analysis. So the axiom of reducibility was added on the pragmatic ground that it was needed, although no satisfactory justification (explanation) of it could be provided. The whole point of the ramification was then lost, so that it might just as well be abolished. What then remained was the simple theory of types. Its official justification (Wittgenstein, Ramsey) rests on the interpretation of propositions as truth values and propositional functions (of one or several variables) as truth functions. The laws of the classical propositional logic are then clearly valid, and so are the quantifier laws, as long as quantification is restricted to finite domains. However, it does not seem possible to make sense of quantification over infinite domains, like the domain of natural numbers, on this interpretation of the notions of proposition and propositional function. For this reason, among others, what we develop here is an intuitionistic

79 P Martin-Löf: *Intuitionistic Type Theory*, Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980, page 1-2. Notes in the quotation: 1): A. Hoare, *An axiomatic basis of computer programming*, Communications of the ACM, Vol. 12, 1969, pp. 576–580 and 583. 2): W. Dijkstra, *A displine of Programming*, Prentice Hall, Englewood Cliffs, N.J., 1976. 3): Martin-Löf, *Constructive mathematics and computer programming*, Logic, Methodology and Philosophy of Science VI, Edited by L. J. Cohen, J. Los, H. Pfeiffer and K. P. Podewski, North-Holland, Amsterdam, 1982, pp. 153–175.

theory of types, which is also predicative (or ramified). It is free from the deficiency of Russell's ramified theory of types, as regards the possibility of developing elementary parts of mathematics, like the theory of real numbers, because of the presence of the operation which allows us to form the cartesian product of any given family of sets, in particular, the set of all functions from one set to another.
In two areas, at least, our language seems to have advantages over traditional foundational languages. First, Zermelo-Fraenkel set theory cannot adequately deal with the foundational problems of category theory, where the category of all sets, the category of all groups, the category of functors from one such category to another etc. are considered. These problems are coped with by means of the distinction between sets and categories (in the logical or philosophical sense, not in the sense of category theory) which is made in intuitionistic type theory. Second, present logical symbolisms are inadequate as programming languages, which explains why computer scientists have developed their own languages (FORTRAN, ALGOL, LISP, PASCAL, . . .) and systems of proof rules (Hoare[1], Dijkstra[2] , . . .). We have show elsewhere[3] how the additional richness of type theory, as compared with first order predicate logic, makes it usable as a programming language.
[Colorings by P Helders]

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LETTRE

A Monsieur le rédacteur de la Revue Générale des Sciences.

PAR

I. RICHARD

à DIJON.

Dans son numéro du 30 mars 1905, la *Revue* signale certaines contradictions qu'on rencontre dans la théorie générale des ensembles.

Il n'est pas nécessaire d'aller jusqu'à la théorie des nombres ordinaux pour trouver de telles contradictions. En voici une qui s'offre dès l'étude du continu, et à laquelle plusieurs autres se ramèneraient probablement:

Je vais définir un certain ensemble de nombres, que je nommérai l'ensemble E, à l'aide des considérations suivantes:

Ecrivons tous les arrangements deux à deux des vingt-six lettres de l'alphabet français, en rangeant ces arrangements par ordre alphabétique, puis, à la suite tous les arrangements trois à trois, rangés par ordre alphabétique, puis, à la suite, ceux quatre à quatre, etc. Ces arrangements peuvent contenir la même lettre répétée plusieurs fois, ce sont des arrangements avec répétition.

Quel que soit l'entier p, tout arrangement des vingt-six lettres p à p se trouvera dans ce tableau, et comme tout ce qui peut s'écrire avec un nombre fini de mots est un arrangement de lettres, tout ce qui peut s'écrire se trouvera dans le tableau dont nous venons d'indiquer le mode de formation.

La définition d'un nombre se faisant avec des mots, et ceux-ci avec des lettres, certains de ces arrangements seront des définitions de nombres. Biffons de nos arrangements tous ceux qui ne sont pas des définitions de nombres.

Acta mathematica. 30. Imprimé le 23 mai 1906.

Soit u_1 le premier nombre défini par un arrangement, u_2 le second, u_3 le troisième, etc.

On a ainsi, rangés dans un ordre déterminé, *tous les nombres définis à l'aide d'un nombre fini de mots.*

Donc: Tous les nombres qu'on peut définir à l'aide d'un nombre fini de mots forment un ensemble dénombrable.

Voici maintenant où est la contradiction. On peut former un nombre n'appartenant pas à cet ensemble.

«Soit p la $n^{\text{ième}}$ décimale du $n^{\text{ième}}$ nombre de l'ensemble E; formons un nombre ayant zéro pour partie entière, et pour $n^{\text{ième}}$ décimale $p+1$, si p n'est égal ni à huit, ni à neuf, et l'unité dans le cas contraire.» Ce nombre N n'appartient pas à l'ensemble E. S'il était le $n^{\text{ième}}$ nombre de l'ensemble E, son $n^{\text{ième}}$ chiffre serait le $n^{\text{ième}}$ chiffre décimal de ce nombre, ce qui n'est pas.»

Je nomme G le groupe de lettres entre guillemets.

Le nombre N est défini par les mots du groupe G, c'est à dire par un nombre fini de mots; il devrait donc appartenir à l'ensemble E. Or, on a vu qu'il n'y appartient pas.

Telle est la contradiction.

Montrons que cette contradiction n'est qu'apparente. Revenons à nos arrangements. Le groupe de lettres G est un de ces arrangements; il existera dans mon tableau. Mais, à la place qu'il occupe, il n'a pas de sens. Il y est question de l'ensemble E, et celui-ci n'est pas encore défini. Je devrai donc le biffer. Le groupe G n'a de sens que si l'ensemble E est totalement défini, et celui-ci ne peut l'être que par un nombre infini de mots. *Il n'y a donc pas contradiction.*

On peut encore remarquer ceci: L'ensemble de l'ensemble E et du nombre N forme un autre ensemble. Ce second ensemble est dénombrable. Le nombre N peut être intercalé à un certain rang k dans l'ensemble E, en reculant d'un rang tous les autres nombres de rang supérieur à k. Continuons à appeler E l'ensemble ainsi modifié. Alors le groupe de mots G définira un nombre *N' différent de N*, puisque le nombre N occupe maintenant le rang k, et que le $k^{\text{ième}}$ chiffre de N' n'est pas égal au $k^{\text{ième}}$ chiffre du $k^{\text{ième}}$ nombre de l'ensemble E.

Next page: Scanned copy of the manuscript from January 1885, written by Cantor to be published by Acta Mathematica in December 1885, on a direct request from Gösta Mittag-Leffler that Cantor should develop how the theory of transfinite numbers was useful for the natural sciences. Yellow frame by P Helders.

45.

Die im Vorstehenden zu einer Art von Abschluss gelangten Untersuchungen über Punctmengen habe ich von Anfang an nicht bloss aus speculativem Interesse, sondern zugleich im Hinblick auf Anwendungen unternommen, welche ich mir davon in der mathematischen Physik und in anderen Wissenschaften versprach.

Die Hypothesen, welche den meisten theoretischen Untersuchungen über Naturerscheinungen zu Grunde gelegt werden, haben mich niemals sehr befriedigt und ich glaube dies dem Umstande zuschreiben zu müssen, dass die Theoretiker ~~oft~~ meistens entweder über die letzten Elemente der Materie eine völlige Unbestimmtheit herrschen lassen oder dass sie dieselben als sogenannte Atome von zwar sehr kleinem, aber doch nicht gänzlich verschwindendem Rauminhalte annehmen.

Mir unterlag es keinem Zweifel, dass, um zu einer befriedigenderen Naturerklärung zu gelangen, die letzten oder eigentlichen einfachen Elemente der Materie in actual unendlicher Zahl vorauszusetzen und in Bezug auf das Räumliche als völlig ausdehnungslos und streng punctuell zu betrachten sind und ich wurde in dieser Ansicht bestärkt, indem ich bemerkte, dass so hervorragende Physiker, wie Faraday, Ampère, Wilh. Weber und von Mathematikern neben Anderen Cauchy dieselbe Ueberzeugung vertreten haben.

Um aber diese Grundanschauung zur Durchführung bringen zu können, schienen mir allgemeine Untersuchungen über Punctmengen, wie ich sie angestellt habe, vorhergehen zu müssen.

Ich nenne im Anschluss an Leibniz die einfachen Elemente der Natur, aus deren Zusammensetzung im gewissen Sinne die Materie hervorgeht, Monaden oder Einheiten (Man vergleiche namentlich die beiden Leibnizschen Abhandlungen: „La Monadologie", Edit. Erdmann, pag. 705 oder Edit. Dutens, tom. II, pag. 20 und „Principes de la nature et de la grâce, fondés en raison", Edit. Erdmann, pag. 714 und Edit. Dutens, tom. II, pag. 32) und gehe von der Ansicht aus, ~~mit~~ ~~durch~~ mit welcher ich mich in Uebereinstimmung mit der heutigen Physik zu befinden glaube, dass zwei specifisch verschiedene, auf einander wirkende Materien und demgemäss auch zwei verschiedene Classen von Monaden neben einander zu Grunde zu legen sind, die Körpermaterie und die Aethermaterie, die Körpermonaden und die Aethermonaden, womit aber nicht gesagt sein soll, dass vom Schöpfer nicht auch noch andere Materien geschaffen werden konnten und sogar geschaffen worden sind, sondern nur, dass zur Erklärung der bisher beobachteten sinnfälligen Erscheinungen jene beiden Substrate auszureichen scheinen.

Auf diesem Standpuncte ergiebt sich als die erste Frage, hieran scheinen aber weder Leibniz noch die Späteren gedacht zu haben, welche Mächtigkeiten jenen beiden Materien in [...] sofern sie als Mengen von Körper- [...] zukommen

Page 123 from Acta Mathematica vol. 7, dec 1885. This is the printed version of the manuscript of page 53 above. The sentence with omitted content is in yellow frame.

eine völlige Unbestimmtheit herrschen lassen oder dass sie dieselben als sogenannte *Atome* von zwar *sehr kleinem* aber doch *nicht gänzlich verschwindendem Rauminhalte* annehmen. Mir unterlag es keinem Zweifel, dass, um zu einer befriedigenderen *Naturerklärung* zu gelangen, die letzten oder eigentlichen *einfachen* Elemente der Materie in *actual unendlicher Zahl* vorauszusetzen und in Bezug auf das *Räumliche* als *völlig ausdehnungslos* und *streng punctuell* zu betrachten sind; ich wurde in dieser Ansicht bestärkt, indem ich bemerkte, dass in der neueren Zeit so hervorragende *Physiker*, wie FARADAY, AMPÈRE, WILH. WEBER und von *Mathematikern* neben Anderen CAUCHY dieselbe Überzeugung vertreten haben.

Um aber diese *Grundanschauung* zur Durchführung bringen zu können, schienen mir allgemeine Untersuchungen über *Punctmengen*, wie ich sie angestellt habe, vorhergehen zu müssen. Ich nenne im Anschluss an LEIBNIZ die *einfachen* Elemente der Natur, aus deren Zusammensetzung in gewissem Sinne die *Materie* hervorgeht, *Monaden* oder *Einheiten* (man vergleiche namentlich die beiden LEIBNIZ'schen Abhandlungen: »*La Monadologie*», Edit. ERDMANN, p. 705 oder Edit. DUTENS, T. II, p. 20 und »*Principes de la nature et de la grâce, fondés en raison*», Edit. ERDMANN, p. 714 und Edit. DUTENS, T. II, p. 32) und gehe von der Ansicht aus, mit welcher ich mich in Übereinstimmung mit der heutigen Physik zu befinden glaube, dass *zwei specifisch verschiedene, auf einander wirkende Materien* und demgemäss auch *zwei* verschiedene Classen von *Monaden* neben einander zu Grunde zu legen sind, die *Körpermaterie* und die *Aethermaterie*, die *Körpermonaden* und die *Aethermonaden*, indem diese beiden *Substrate* zur *Erklärung* der *bisher beobachteten sinnfälligen Erscheinungen* auszureichen scheinen.

Auf diesem Standpuncte ergiebt sich als die *erste Frage*, woran aber weder LEIBNIZ noch die Späteren gedacht haben, welche *Mächtigkeiten* jenen beiden Materien in Ansehung ihrer Elemente, sofern sie als *Mengen* von *Körper-* resp. *Aethermonaden* zu betrachten sind, zukommen; in dieser Beziehung habe ich mir schon vor Jahren die *Hypothese* gebildet, dass die *Mächtigkeit* der *Körpermaterie* diejenige ist, welche ich in meinen Untersuchungen die *erste* Mächtigkeit nenne, dass dagegen die *Mächtigkeit* der *Aethermaterie* die *zweite* ist.

Für diese Ansicht und Meinung lassen sich *sehr viele* Gründe ins Feld führen, wie ich bei einer späteren Gelegenheit auseinandersetzen will;