SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Hilbert's Third Problem

av

Klara Wigzell

2020 - No 46

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Hilbert's Third Problem

Klara Wigzell

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Torbjörn Tambour

2020

Abstract

This essay shows and proves a solution to Hilbert's third problem concerning the possible equivalence between volume, equidecomposability and equicomplementability of polyhedra in three-dimensional space.
First, the equivalency between area, equidecomposability and equicomplementability of polygons in the plane is proven through the Wallace-Bolyai-Gerwien Theorem.
Proceeding into three-dimensional space, The Cone Lemma, The Pearl Lemma and Bricard's Condition are presented and proven.
Lastly, three examples of tetrahedra are displayed, which offer a counterexample to the proposition of equivalency of volume and equidecomposability of polygons in three-dimensional space.

Contents

1 Introduction

"...a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guidepost on the mazy paths to hidden truths, and ultimately a reminder of our pleasure in the successful solution." (Newson, 1902)

The quote above is taken from German mathematician David Hilbert's opening speech at the International Congress of Mathematicians year 1900.

To me the quote speaks volumes. The pleasure which is indeed derived from a successful solution is what drove me to the field of mathematics in the first place. A mathematical problem takes the shape of a riddle and the joy you get, if and when, you solve it is a wondrous feeling. When preparing for this essay, I came across the Pearl Lemma. This lemma is a beautiful and seductive example of a riddle which can be solved in such a simple and concrete manner. I knew it had to be a part of the following essay. The essay has drawn heavily from "Proofs from the book" by Aigner & Ziegler (2018) as it presents the proof to Hilbert's third problem in an eloquent way.
I would like to thank my tutor Torbjörn Tambour, my patient mathematician brother Simon and my children who can now call themselves masters of drawing and colouring geometrical shapes.

2 History

Arguably one of the most famous creations in the field of mathematics is Euclid's Elements published around 300b.c. It is the first (known) mathematical work which presents its contents in an axiomatic deductive manner (Sjöberg,1995). This manner of proof construction is now heavily ingrained in the very foundations of modern mathematics .
Famous German mathematician David Hilbert published a complete revision of Euclid's axiom in his "Grundlagen der Geometrie" in 1899. The revision of the axioms stemmed from a need to modernize the original system and is in part still present in our modern-day geometrical axioms.

Students are early on in their learning introduced to the concept of two and even three-dimensional shapes known as polygons and polyhedra.
One of the simplest polygons in Euclidean plane geometry is the triangle, a geometrical figure defined as a two-dimensional shape with three edges. The idea that one can divide a square into smaller pieces and put them back together into a triangle of the same area can be grasped intuitively by most. The Tangram puzzles can be traced as far back as imperial China and the method was likely known by the ancient Greeks (Dupont,2001).

In 1807, 1833 and 1835, three mathematicians showed independently of each other that *equality of area*, *equidecomposability* and *equicomplementability* are equivalent properties of polygons in the Euclidean plane (see section 3.1 for definitions). The aptly named Wallace-Bolyai-Gerwien Theorem proved that equidecomposability and equal area are equivalent and from that follows equicomplementability. The theorem proved that it is possible to use a definition of area without using calculus.

With these questions settled in the plane, the mathematicians turned their focus to three-dimensional space.
The corresponding shape to the triangle in three-dimensional space is the tetrahedron, where a regular tetrahedron describes what we commonly call a pyramid. The volume of a triangular pyramid was calculated by Eudoxos over two thousand years ago by the use of a continuity argument (Karlqvist, 2003). The question of whether a polyhedron's volume has the same relation to equidecomposability as a polygon remained unanswered.

This is then the question that Hilbert posed, namely if the Wallace-Bolyai-Gerwien Theorem can be extended into the third dimension. Some polyhedra are equidecomposable, as shown by Gerling's mirror-image polyhedra from 1844 and Hill's tetrahedra examples from 1896 (Dupont,2001). However, Hilbert believed these polyhedra to be special cases and asked if **any** two polyhedra of the same volume can be decomposed and reassembled into the same cube?
One can suspect that Hilbert did not believe this to be true as he delivered his well-known speech to the Second International Congress of Mathematicians in Paris the year 1900. The speech outlined what came to be known as Hilbert's 23 problems and the third problem concerned the question of volume and equidecomposability.

> "In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, i. e., in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility." (Sah,1979,p.2)

He then proceeded by asking the assembly to specify

> "two tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra." (ibid.)

The third problem did not remain unsolved for long and in that same year, Hilbert's own student Max Dehn presented a solution where he showed that the problem does not have a general solution (Karlqvist,2003).
However, Dehn's solution was deemed complicated and difficult to understand. Thus, it became the subject of much refinement throughout the years and the now classical proof was published by Boltianskii in 1978.

This essay will take a slightly different route when solving Hilbert's third problem relying mainly on the reasoning used in "Proofs From the Book" (Aigner & Ziegler,2018). Before Hilbert even asked the question in 1900, a solution had been published by Bricard four years earlier. However, Bricard's proof was incomplete and incorrect and thus not mentioned by Hilbert.
The essay presents a correct proof for Bricard's Condition through The Cone Lemma - a version of Kagan's integrality argument, and The Pearl Lemma by Benko (2007). Bricard's Condition will then be applied to three tetrahedra in order to present a counterexample, solving Hilbert's third problem.
Note that figures and shapes throughout the essay refers to polygons or polyhedra.

3 The Plane

We will start our journey in the Euclidean plane where polygons, area, decomposability and decomplementability will first be defined. Then the Wallace-Bolyai-Gerwien Theorem showing the equivalence between the three in the plane will be proven.

3.1 Definitions

Polygon

A polygon is defined as a two-dimensional figure limited by any number of straight lines. We describe the polygon by its vertices (corner points) and its edges (the straight lines). Additionally, the space contained within the edges is the *area* of the polygon and described as its face.

Area

Area is the measurement a of the figure F which holds the following qualities:

- $a(F) \geq 0$
- When F and F' share no interior points, their combined area equals the sum of $a(F)$ and $a(F')$. This relation is expressed mathematically as: $a(F \cup F') = a(F) + a(F')$.
- $a(F)$ is invariant when F is rotated or moved.
- The area of a square S with the side measuring 1 is $a(S) = 1$.

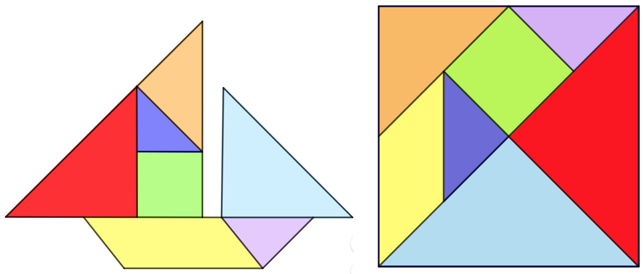
(Karlqvist,2003), (Boltianskii,1978)

Using this definition of area allows us to easily decide the area of any polygon. First, any polygon can be dissected into a finite number of triangles. Secondly, these triangles can be dissected and rearranged into a rectangle which we can easily calculate the area of. Furthermore, it will not matter how the pieces are dissected or rearranged as area is unequivocal (Hartshorne, 1997).

Equidecomposability

Two polygons P and P' in the plane, are defined as equidecomposable when they "can be subdivided into finitely many pieces such that each piece in P is congruent to exactly one piece in P'" (Dupont, 2001, p.1). In other words, a figure F can be dissected and rearranged so that it forms F' consisting of pieces which only differ by some isometry.

Example:
This boat and square are equidecomposable as they can be dissected into congruent non-overlapping pieces.



The boat (B) consists of building blocks red (R), green (G), turquoise (T), dark blue (D), yellow (Y), orange (O) and lilac (L). The square (S) consists of the same congruent pieces.

$$B = R \cup G \cup T \cup D \cup Y \cup O \cup L$$
$$S = R \cup G \cup T \cup D \cup Y \cup O \cup L$$

A generalized definition is thus; Let P and P' be planar polygons and T_i the pieces of the decomposition.

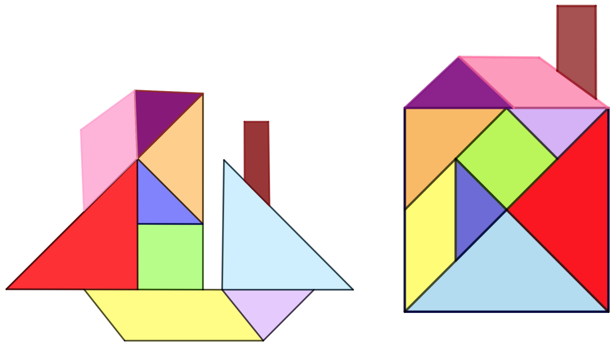
$P = T_1 \cup T_2 \cup T_3 \cup \ldots T_n$
$P' = T'_1 \cup T'_2 \cup T'_3 \cup \ldots T'_n$

The polygons P and P' are then equidecomposable if $T_i = T'_i$ for all i regardless of how the pieces are arranged.

Equicomplementability

Two shapes are defined as equicomplementable if there exists a finite set of figures which when added to each of the original shapes, create two congruent supershapes or equidecomposable supershapes.

Example:
Using the boat and the square from earlier, it is evident that they are also equicomplementable.



We simply add building blocks purple (P), maroon (M) and fuchsia (F) first to the boat (B) creating a new supershape B' and add the same to the square (S), creating S'.

$B' = B \cup P \cup M \cup F$
$S' = S \cup P \cup M \cup F$

B' and S' are also equidecomposable. Using the same method as before, the supershapes can be dissected into congruent non-overlapping pieces.

3.2 The Wallace-Bolyai-Gerwien Theorem

Equipped with our definitions, we can now examine the Wallace-Bolyai-Gerwien Theorem. The theorem states that polygons in the plane are equidecomposable if and only if they have the same area. (Aigner& Ziegler,2010). As a result of this theorem, one can deduce that two polygons are also equicomplementable if and only if they have the same area (Boltianskii,1978).

In order to prove the theorem, we need three lemmas outlined in chapter two of "Hilbert's third problem" (1978) by Boltianskii. The images are adapted from the same.

3.2.1 Lemma 1

If both polygons A and C are equidecomposable with polygon B, then A and C are also equidecomposable.

Proof
Let A, B and C be planar polygons. First decompose B into pieces which can be rearranged to form A, and then decompose B into pieces to form C.

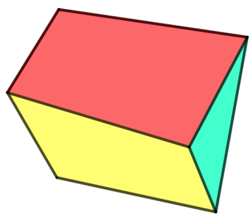


Fig.B

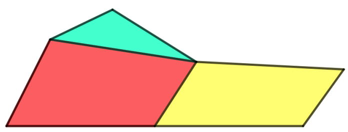


Fig.A

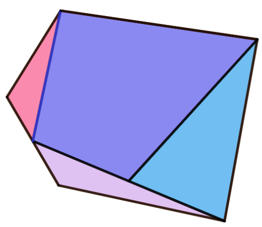


Fig.B

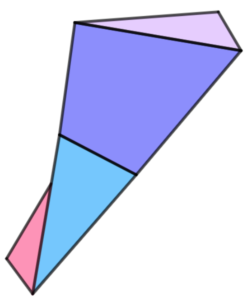


Fig.C

The two sets of lines in each decomposition can then produce a new decomposition consisting of smaller pieces which can be rearranged to give both A and C. The green lines in B are the lines drawn in the first decomposition of B into A. The five polygons in grey are created through these new lines and are then arranged into both A and C, proving the lemma.

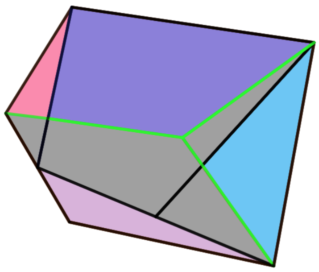


Fig.B

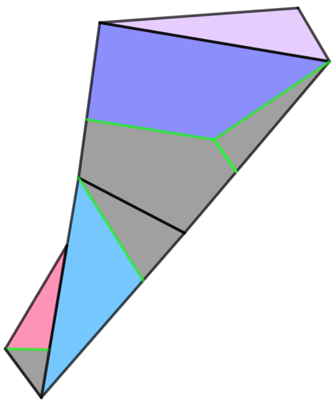


Fig.C

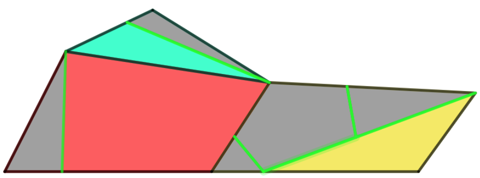
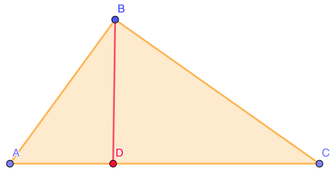


Fig.A

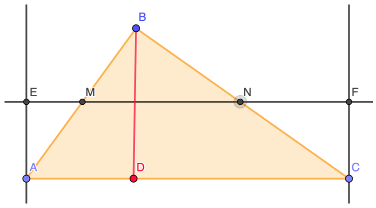
3.2.2 Lemma 2

Every triangle is equidecomposable with some rectangle.

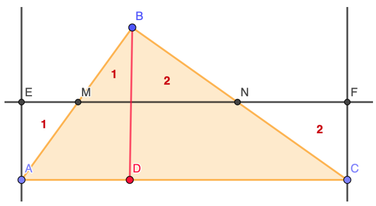
Proof
Let ABC be a triangle and AC its longest side. Then draw the altitude from B to AC, creating the point D. Since AC is the largest side, D will lie in between A and C.



Draw a parallel line MN to AC through the midpoint of BD. Then draw perpendicular lines AE and CF to the line MN.



The triangles marked 1 and 2 in the picture below are congruent and we have thus created the rectangle $AEFC$ which is equidecomposable with $\triangle ABC$.

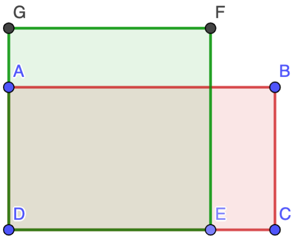


3.2.3 Lemma 3

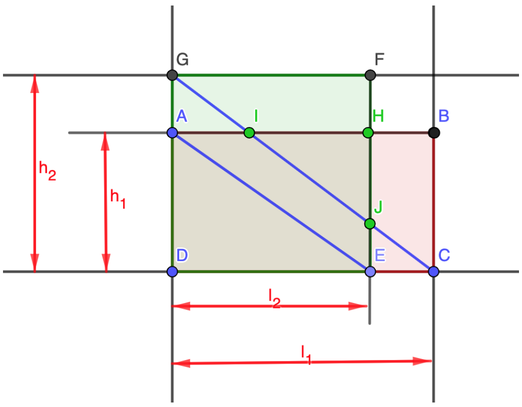
Any two rectangles with the same area are equidecomposable.

Proof
For this proof, we need two examine two different cases.

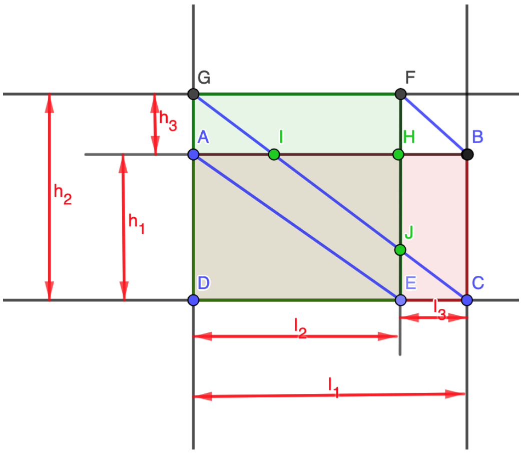
Case 1
Let rectangle $ABCD$ and rectangle $DEFG$ be rectangles of the same area, placed in such a way that they share a right angle at corner D.



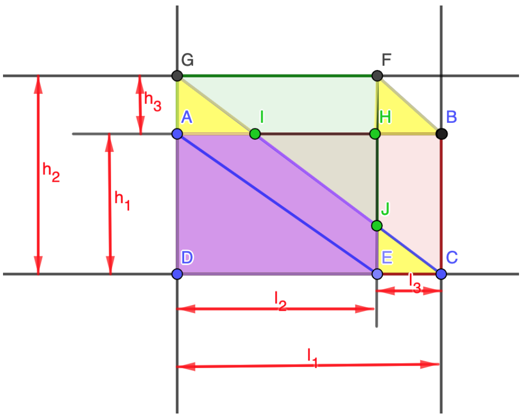
Name the length of the segments $DC = l_1$,$DA = h_1$, $DE = l_2$ and $DG = h_2$. As the rectangles have the same area, we can express the relationship between the lengths as $l_1 h_1 = l_2 h_2$ which is equivalent to $\frac{l_1}{h_2} = \frac{l_2}{h_1}$. Then we draw lines GC and AE, and due to the similarity of $\triangle ADE$ and $\triangle GDC$, these lines are parallel.



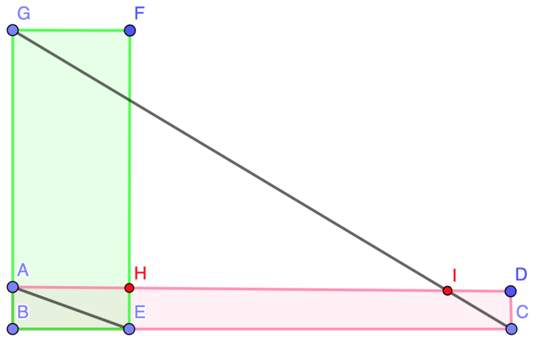
Furthermore, the relationship $\frac{l_1-l_2}{h_2-h_1} = \frac{l_3}{h_3}$ tells us that FB is parallel to AE as $\triangle ADE$ and $\triangle BFH$ are congruent. Consequently, all three lines FB,AE and GC are parallel to one another.



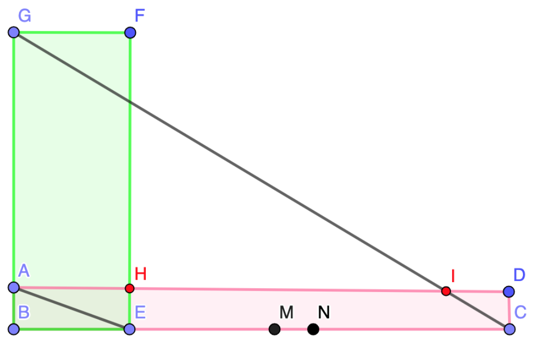
Provided that the segment GC intersects the rectangle $AHED$ we are done. Equidecomposability between our original rectangles is proven as each of them consists of the purple polygon, one of the three yellow congruent triangles and one of the two congruent triangles BCI and FJG as the picture below.



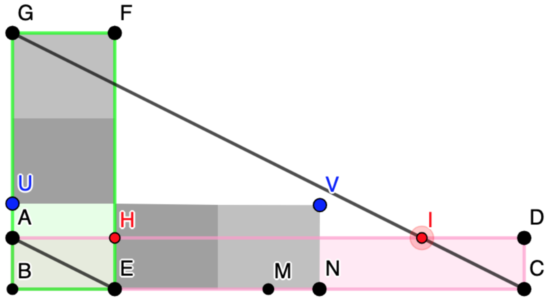
Case 2
The second case is where the rectangles are placed in such a way that the segment GC does **not** intersect the rectangle $ABEH$.



Since $\triangle AHE$ and $\triangle CDI$ are congruent, segments GF, AH and ID are congruent but in this case the length of segment AD is longer than the sum of the two segments AH and ID. Next mark the midpoint of the segment BC and call it M. We then divide BC into k segments congruent to BE until we reach the point N which lies outside of segment BM but inside the segment CM. Here k is the smallest natural number which fulfils this condition (in our case k=3).



Next, subdivide the rectangle $BEGF$ into k pieces by drawing lines parallel to BE and place them along the segment BN.

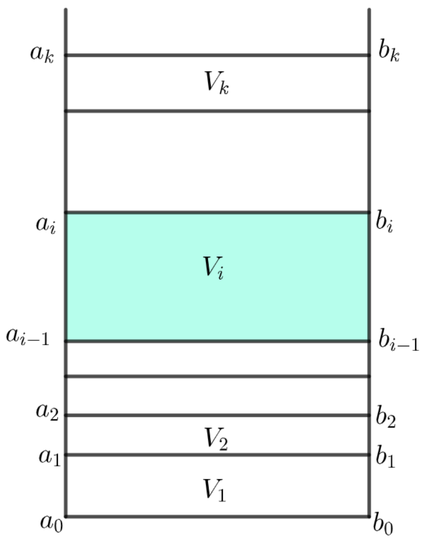


We have now created two rectangles $BEFG$ and $BNVU$ that are equidecomposable as they consist of the same congruent parts. The rectangles $BNVU$ and $ABCD$ are equidecomposable as proven above in case 1.

3.2.4 Proof of The Wallace-Bolyai-Gerwien Theorem

Theorem
Any two polygons of equal area are equidecomposable.

Proof
After proving our three lemmas, the proof of the Wallace-Bolyai-Gerwien Theorem is straightforward.
Let P be any polygon. First decompose P into a finite number of triangles which are in turn equidecomposable to some rectangle R_i according to Lemma 2. By extension we can make the assumption that P is equidecomposable with $R_1 \cup R_2 \cup ... \cup R_k$.
Next, choose any line segment $a_0 b_0$ and draw perpendiculars at its endpoints. Then draw segments parallel to $a_0 b_0$ creating rectangles. We draw these segments in such a way that we create a rectangle $a_{i-1} a_i b_i b_{i-1}$ which has the same area as R_i, where $i = 1, ..., k$. Lemma 3 tells us that R_i is then equidecomposable with this rectangle which we call V_i.



This applies to all rectangles of the decompositions so that $R_1 \cup R_2 \cup ... \cup R_k$ is equidecomposable to $V_1 \cup V_2 \cup ... \cup V_k$. Consequently, Lemma 1 then tells us that P is equidecomposable with rectangle $a_0 a_k b_k b_0$. Thus, we have shown that any polygon is equidecomposable with some rectangle.

Now let P and Q be two polygons of the same area. Since any polygon is equidecomposable to some rectangle, we can find rectangles A and B so that P is equidecomposable with A and Q with B. Since area is invariant, A and B will also have the same area and therefore by Lemma 3, they are also equidecomposable with each other. Since P is equidecomposable with A, A with B and B with Q, then P is equidecomposable with Q by Lemma 1.

Remember that two polygons which are equidecomposable are also equicomplementable and we have proven the equivalence between area, equidecomposability and equicomplementability in the plane.

4 Three-dimensional Space

After establishing these properties for polygons of the same area in the plane, it is time to investigate if the same holds true in three dimensions for polyhedras of the same volume. First, we will define volume and some properties of polyhedra before moving on to the three proofs derived from "Proofs from the Book" (Aigner & Ziegler, 2018), that will provide us with a way to solve Hilbert's third problem.

4.1 Definitions

Polyhedra

There are several discussions and different definitions of polyhedra. This has led to misunderstandings and confusion in regard to the validity of important mathematical proofs. This essay defines a polyhedron in accordance with Cromwell's definition from "Polyhedra" (1997).

A polyhedron is the union of a finite set of polygons such that:

- Any pair of polygons meet only at their sides or corners.
- Each side of each polygon meets exactly one other polygon along an edge.
- It is possible to travel from the interior of any polygon to the interior of any other.
- Let V be any vertex and let $F_1, F_2, \ldots, F_n$ be the n polygons that meet at V. It is possible to travel over the polygons F_i from one to any other without passing through V.

We describe the polyhedron by its vertices (corner points), edges (line segments connecting certain pairs of vertices) and faces (two-dimensional polygons). Additionally, the polyhedra has a particular interior *volume*.

Volume
Volume is the measurement v of the figure F which holds the following qualities:

- $v(F) \geq 0$
- Provided F and F' are measurable and share no interior points then $F \cup F'$ is measurable and $v(F \cup F') = v(F) + v(F')$.
- v is invariant when F is rotated or moved.
- The volume of the cube C with the side measuring 1 is $v(C) = 1$.

(Karlqvist,2003)

With the definitions in place, let us examine the Cone Lemma, The Pearl Lemma and Bricard's Condition.

4.2 The Cone Lemma

The Cone Lemma is needed in order to prove the Pearl Lemma which in its turn is needed to validate that Bricard's Condition holds true.

The Cone Lemma
If there exists a real positive solution to a homogeneous linear equation system with integer coefficients then there will also exist a positive integer solution.

Proof
Define C as the set of solutions to the homogeneous linear equations $A\mathbf{x} = \mathbf{0}$ where all coordinates of $\mathbf{x}$ are strictly positive and where A is an integer $m \times n$ matrix. C is then called a rational cone; hence the name of the lemma. If C is non-empty, the lemma states that C will also contain integer points.

Next we will examine a subset of C where the solutions all have coordinates larger than or equal to 1 and call it C'. This is easily done, as a positive vector can be multiplied by a suitable positive number to create a vector with coordinates at least 1. Subsequently, it is enough to show that if C' is non-empty it will contain integer points.
Now note that it will suffice to show that a rational solution exists in C' since an integer can be produced by multiplying rationals with their lowest common denominator.

We will do this by showing that there exists a lexicographically smallest real solution $\mathbf{x}$ in C'. A comparison of the vector's elements in order will yield the lexicographically smallest solution. In other words, a vector which has the smallest first element will be the lexicographically smallest solution. If several vectors share this element, a comparison will be made of the following element and so on.

Additionally, this solution will be proven to be rational given that the matrix A is integral. The proof will rely on a method devised by Fourier and Motzkin (Aigner & Ziegler 2010) called "Fourier-Motzkin elimination", together with an induction argument.

The Fourier-Motzkin elimination is a method which enables us to eliminate variables from a system of linear inequalities. Through the elimination of a set of variables a new system of linear inequalities is created, where the solutions in the remaining variables are the same as in the original system with the original variables.

Any linear equation $k_1x_1 + ..., k_rx_r = 0$ can be equivalently enforced by the two inequalities $k_1x_1 + ..., k_rx_r \geq 0$ and $-(k_1x_1 + ..., k_rx_r) \geq 0$. In other words, a system of linear equations $A\mathbf{x} = \mathbf{0}$ is equivalent to the following system of linear inequalities:

$$\begin{cases} A\mathbf{x} \geq \mathbf{0} \\ -A\mathbf{x} \geq \mathbf{0} \end{cases}$$

Thus, we have a new system of linear inequalities which we express as: $A\mathbf{x} \geq \mathbf{0}$.

It will then suffice to prove that any system $A\mathbf{x} \geq \mathbf{b}$ where $x_1, ..., x_n$ are all greater than or equal to 1 and A,b are integral, has a rational lexicographically smallest solution, given that the system has any real solution at all.

For this proof we will rely on induction over n, with n being the number of coordinates contained in $\mathbf{x} = x_1, x_2, ..., x_n$.

First, we prove it for $n = 1$. We have a system of linear inequalities:

$$a_1x_1 \geq b_1, a_2x_1 \geq b_2, ..., a_mx_1 \geq b_m, x_1 \geq 1$$

We know that $a_1, ...a_m$ are integers Solving for x_1 will yield inequalities where x_1 is either $\geq$ or $\leq$ to a rational. Since there exists at least one lower bound ($x_1 \geq 1$) and a solution exists, the smallest solution will be rational. In one dimension, smallest value is equivalent to smallest lexicographical value.

We now assume that it is true for n and we will show that it is true for $n+1$.

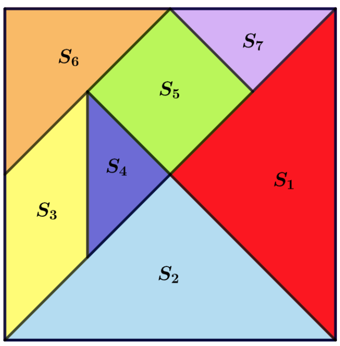
First let's look at all the inequalities that involve x_{n+1} where $\mathbf{x} = (x_1, x_2, ..., x_{n+1})$. We then solve the inequalities for x_{n+1}, which gives us a set of lower and/or upper bounds expressed in $x_1, ..., x_n$ and $a_1, ...a_{n+1}$ for x_{n+1}.
Next, we create a new system of inequalities $A'\mathbf{x}' \geq \mathbf{b}', \mathbf{x}' \geq \mathbf{1}$, where $\mathbf{x}'$ has n variables. From the original system we include the inequalities where x_{n+1} was not present and also all inequalities obtained when taking an upper bound on x_{n+1} and requiring it to be greater than or equal to a lower bound on x_{n+1}. Note that the first inequalities are integral, and the second type of inequalities are easily transformed to integral. Since the original system has a solution, there also exists a solution for our new system.
Our induction assumption then tells us that a smallest lexicographical solution exists for this system and it is rational.
Insert this solution into the system of inequalities solved for x_{n+1} (inequalities without x_{n+1} are trivially true). Choose the lowest value for x_{n+1} which fulfils these inequalities and we have a lexicographically smallest solution for the original system.
Consider our system of inequalities, $A\mathbf{x} \geq \mathbf{0}$ and $-A\mathbf{x} \geq \mathbf{0}$. Denote the smallest lexicographical solution in the first inequality as x_1 and in the second x_2. We are left with $A(x_1 - x_2) \geq 0$. However, since $x_1 - x_2$ is lexicographically smaller than x_1 we must have $x_1 - x_2 = 0$.
Thus, we have proven that the system $A\mathbf{x} \geq b$ has a unique positive smallest lexicographical solution.

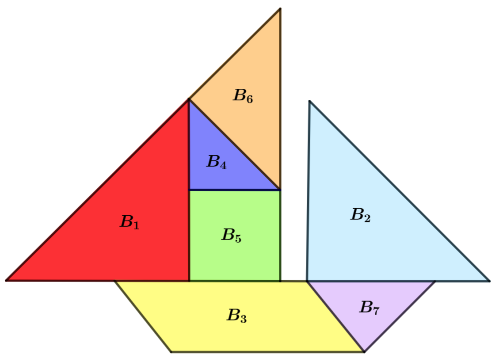
4.3 The Pearl Lemma

This relatively new lemma, courtesy of David Benko in 2007, provides an interesting and simplified method of how to prove Bricard's Condition. Then Bricard's Condition can be implemented as a direct route to solving Hilbert's third problem.

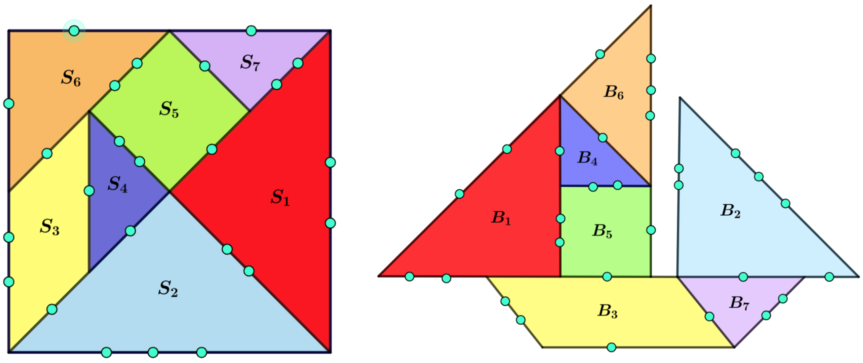
Before we start with the lemma, the definition of a *segment* is needed. Each edge of a piece in the decomposition of a polygon or polyhedron can be divided into smaller sections by either vertices or edges of other pieces. These sections are *segments*. If you place the segments end to end, the result is the length of the edge. The endpoints of segments in a decomposition of a polygon are given by vertices whereas the endpoints of segments in a decomposed polyhedron can also be given by the crossing of two edges. Interior points of a given segment in both two and three dimensions all belong to the same edge or edges (Aigner & Ziegler,2010).
For our proof we will use the two-dimensional square and boat from earlier.

The Pearl Lemma
Given S and B equidecomposable, with $S = S_1 \cup ... \cup S_n$ and $B = B_1 \cup ... \cup B_n$ where S_k is congruent to B_k.





Then one can place a number of pearls on all the segments in S and B given the above decompositions in such a way that each edge of a piece S_k receives the same number of pearls as the corresponding edge of B_k.



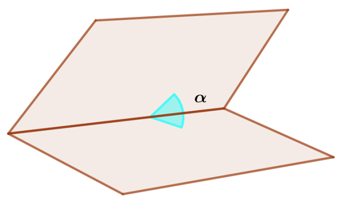
An interesting thing to observe is that the total number of pearls in S is not equal to the total number of pearls in B.

Now observe an edge of a piece, for example S_1. The number of pearls on one edge of S_1 is equal to the sum of the number of pearls on the adjacent edges of S_5 and S_7. The number of pearls on the edge of the corresponding piece B_1 in B is the sum of pearls on the adjacent edges of B_5 and B_4.
In the third dimension, the edges can consist of multiple segments which may or may not be consistent, but the number of pearls placed on the edge of a piece must still be the same for the two decompositions. If these sums are equal for the entire decomposition, then the lemma is true.
In other words, we are faced with a number of linear equation systems where the variables to solve for are the pearls on each segment. Since the pearls need to be whole we need positive integer solutions.
The Cone Lemma gives us the solution. It states that if positive real solutions to the linear equation systems exist, then integer solutions exist. In this case, there exists a positive real solution; namely the lengths of the segments. Consequently, the Pearl Lemma is proven.

4.4 Bricard's Condition

The final proof needed is Bricard's Condition and it will allow us to give a solution to Hilbert's third problem. Four years before Hilbert posed the question to the congress of mathematicians, Bricard published a solution. However, the proof given was incorrect. Now armed with the Pearl Lemma, we can show that Bricard's Condition is in fact correct.

For this proof we need to define a *dihedral angle*. In a polyhedron we call each polygon a face of the polyhedron and the dihedral angle α is defined as the angle between two adjacent faces. To locate a dihedral angle, choose a point on the shared edge as your starting point. Then draw a line perpendicular to the edge in each of the faces and the dihedral angle is the angle between the two (Cromwell,1997).



Theorem - Bricard's Condition
Let A and B be two equidecomposable polyhedra with the dihedral angles $\alpha_1, ..., \alpha_q$ respectively $\beta_1, ..., \beta_r$. Then there exists an integer k and positive integers n_i and m_i such that

$$n_1\alpha_1 + ... + n_q\alpha_q = m_1\beta_1, ..., m_r\beta_r + k\pi$$

This theorem also holds more generally for equicomplementable polyhedra.

Proof
According to the definition of equidecomposability, the decompositions of A and B yield congruent pieces. Now we can use the Pearl Lemma and place a number of pearls on each segment accordingly. Then define $\sum_1$ as the sum of all dihedral angles measured at every pearl in the decomposition of A and $\sum_2$ as the sum of all dihedral angles measured at every pearl in the decomposition of B. Depending on where the pearl is placed, the following

rules apply.

(i) If a number of pearls are placed on the same edge of a piece, the dihedral angle will be added several times to the sum.

(ii) If a pearl is contained in more than one piece all dihedral angles for each of the pieces will be added to the sum. Depending on the segment's location, the addition yields four different angles.
If the segment:

- lies on the edge of the polyhedron P we get a dihedral angle α_j.
- is not on an edge of P but in its boundary, the angle yielded is π.
- lies in the interior of P the angles yielded are either 2π or π.

We can now express all sums of the dihedral angles at the pearls in A as follows:

$$\sum_{1} = n_1\alpha_1 + ... + n_q\alpha_q + k_1\pi$$

where $n_1, ..., n_q$ are positive integers and k_1 is non-negative.
We then apply the same reasoning to polyhedron B and are left with the expression:

$$\sum_{2} = m_1\beta_1 + ... + m_q\beta_q + k_2\pi$$

where $m_1, ..., m_q$ are positive integers and k_2 is non-negative.
The Pearl Lemma tells us that each pair of congruent pieces in the decompositions will have the same number of pearls on their corresponding edges and as the pieces are congruent, we will measure the same dihedral angles. Thus, the sum of dihedral angles of the edges of the two decompositions of A and B are equal and we are left with a difference of an integer k multiplied by π. By defining $k \in \mathbb{Z}$ as $k_2 - k_1$ we have expressed Bricard's Condition for equidecomposability through the use of the Pearl Lemma.

$$n_1\alpha_1 + ... + n_q\alpha_q + k_1\pi = m_1\beta_1 + ... + m_q\beta_q + k_2\pi$$

The next step is to prove that Bricard's Condition holds true for polyhedra which are equicomplementable. We create two new polyhedra A' and B' from our original polyhedra by adding congruent pieces to both. Then we decompose A' and B' in 2 different ways. First we divide them into their original polyhedra A and B together with the added congruent pieces $A'_i = B'_i$.

$$A' = A \cup A'_1 \cup A'_2 \cup \ldots A'_n$$

and

$$B' = B \cup B'_1 \cup B'_2 \cup \ldots B'_n$$

Then we create another decomposition of A' and B' where A''_i is congruent to B''_i.

$$A' = A''_1 \cup A''_2 \cup \ldots A''_n$$

and

$$B' = B''_1 \cup B''_2 \cup \ldots B''_n$$

We are left with four decompositions of the two equidecomposable A' and B'. Once again, we apply pearls to all segments of each decomposition in accordance with the Pearl Lemma with an added constraint. The number of pearls placed on the edges of both decompositions of A' must be the same and likewise for the two decompositions of B'. As before, the sums of the angles at the pearls are calculated, yielding sums $\sum'_1$ and $\sum'_2$ corresponding to the first two decompositions and sums $\sum''_1$ and $\sum''_2$ for the two second decompositions.

Let's examine the dihedral angle sums of the second decompositions first. As each piece A''_i is congruent to each piece B''_i, we have already proven that the angle sums $\sum''_1$ and $\sum''_2$ are equal.

Next, let's look at the dihedral angle sums $\sum_1'$ and $\sum_1''$. These angles correspond to the same polyhedron A' but in different decompositions. In compliance with the added constraint and the placement of the pearls through the Pearl Lemma, the same number of pearls have been placed on the edges of both decompositions. Thus, following the reasoning from earlier we can express the relation between the sums as $\sum_1' = \sum_1'' + \lambda\pi$. We can apply the same reasoning to the relationship between the sums $\sum_2'$ and $\sum_2''$ which yields the expression $\sum_2' = \sum_2'' + \iota\pi$.
Since the angle sums $\sum_1''$ and $\sum_2''$ are equal we simply substitute these in our two relationships and we can now express the relationship between $\sum_1'$ and $\sum_2'$ as

$$\sum_2' = \sum_1' + k\pi$$

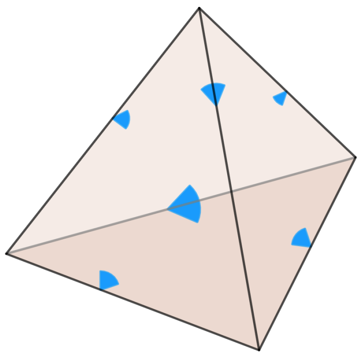
Recall that the dihedral angle sums $\sum_1'$ and $\sum_2'$ contain the sums of the angles at the pearls from our original polyhedra A and B respectively, as well as congruent pieces A_j' and B_j'. Since these are identical, we subtract their angles sums from both sides of the equation. What we are left with is Bricard's condition where A and B are the sole contributors to the angle sums except for a difference of an integer multiplied by π.
Bricard's Condition is thus proven to hold for equidecomposable and equicomplementable polyhedra alike and has provided us with the tools needed in order to solve Hilbert's third problem.

4.5 Examples

In order to answer Hilbert's question regarding the possible equivalence of equality of volume, equidecomposability and equicomplementability, all that is needed is to show an example where the assumption is false. Three examples will be presented and examined.

4.5.1 The Regular Tetrahedron - Example 1

The first example is that of a regular tetrahedron T_1 which is a type of convex polyhedron. A regular tetrahedron has four congruent faces and six straight edges of the same length. We are interested in the dihedral angles of T_1 which are all angle α in order to examine whether or not T_1 is equidecomposable or equicomplementable with a cube C.



A Regular Tetrahedron with angles α

Since the tetrahedron is regular, we view the vertices as vectors **a**, **b**, **c** and **d** and assign them with the coordinates (1,1,1),(1,−1,1),(−1,1,−1),(−1,−1,1). Use of the dot product formula yields the answer $\alpha = \arccos(\frac{1}{3})$.

The next step is to calculate the dihedral angles β of a cube C which are trivially $\beta = \frac{\pi}{2}$.

In order for T_1 and C to be either equidecomposable or equicomplementable, Bricard's Condition requires that;

$$n_1\alpha_1 + ... + n_q\alpha_q = m_1\beta_1, ..., m_r\beta_r + k\pi$$

In our case this translates to;

$$n_1 \arccos(\frac{1}{3}) = m_1 \frac{\pi}{2} + k\pi$$

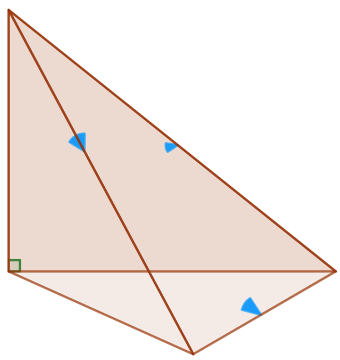
which is equivalent to;

$$k = \frac{1}{\pi} n_1 \arccos(\frac{1}{3}) - m_1 \frac{1}{2}$$

In order to meet the requirements of Bricard's Condition, k needs to be an integer. However, since $\frac{1}{\pi}\arccos(\frac{1}{3})$ is an irrational number (Aigner & Ziegler, 2018, chapter 8, theorem 3) this is not possible. Hence, we have shown that a regular tetrahedron is neither equidecomposable nor equicomplementable with a cube.

4.5.2 The Trirectangular Tetrahedron - Example 2

For the second example we will examine a different type of tetrahedron T_2 and its potential equidecomposability or equicomplementability with a cube. Let T_2 be a tetrahedron constructed by three orthogonal edges of equal length v, sharing one vertex.



A Trirectangular Tetrahedron with three blue angles λ and three angles $\frac{\pi}{2}$

The tetrahedron T_2 consists of six dihedral angles, three of those are simply $\frac{\pi}{2}$ due the orthogonal edges. The remaining three dihedral angles λ still need to be calculated.
This time we use the Pythagorean theorem to first calculate the diagonal

sides to $\sqrt{2v}$. Then we use the method from the example above and the dot product formula yields the answer $\lambda = \arccos(\frac{1}{\sqrt{3}})$.

As before, if T_2 is equidecomposable or equicomplementable with a cube with dihedral angles $\frac{\pi}{2}$, the following equation must be true;

$$n_1 \arccos(\frac{1}{\sqrt{3}}) + n_2 \frac{\pi}{2} = m_1 \frac{\pi}{2} + k\pi$$

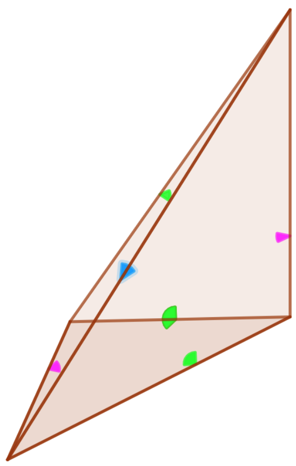
Then we rearrange the equation in order to solve k.

$$k = \frac{1}{\pi} n_1 \arccos(\frac{1}{\sqrt{3}}) + (n_2 - m_1)\frac{1}{2}$$

Once again, we are faced with an impossible solution under the condition that k needs to be an integer as $\frac{1}{\pi} \arccos(\frac{1}{\sqrt{3}})$ is irrational (Aigner & Ziegler, 2018, chapter 8, theorem 3). Thus, we have shown that this type of tetrahedron is neither equidecomposable nor equicomplementable with a cube, just as in the first example.

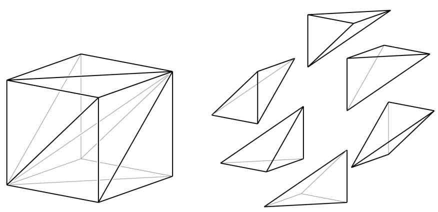
4.5.3 A Tetrahedron with an Orthoscheme - Example 3

With example three, we can finally solve Hilbert's third problem by presenting two tetrahedra of the same volume that are neither equidecomposable nor equicomplementable with each other. First let us create a third tetrahedron T_3 which is a three-edged orthoscheme. In other words, three of its consecutive edges are mutually orthogonal. These edges have the same length v as the trirectangular tetrahedron in example two.



A Birectangular Tetrahedron or Orthoscheme

One of the qualities such a tetrahedron possesses is of extra interest to us, namely that a cube can be decomposed into six congruent orthoschemes.



A cube dissected into six orthoschemes (Wikipedia)

Hence, it is trivial to identify the dihedral angles of an orthoscheme $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6})$ as rational multiples of π.

For the next step, note that the volume of T_2 (the tetrahedron from example two) and the volume of T_3 are equal as they have congruent bases and are of equal height. If we can now show that these tetrahedra are neither equidecomposable nor equicomplementable, Hilbert's third problem will be solved.

Once again we apply our knowledge of Bricard's Condition in order to verify whether or not T_2 and T_3 are equidecomposable or equicomplementable. The values of the two tetrahedra's dihedral angles are inserted into the equation.

$$m_1\frac{\pi}{2} + m_2\frac{\pi}{4} + m_3\frac{\pi}{6} = n_1 \arccos(\frac{1}{\sqrt{3}}) + n_2\frac{\pi}{2} + k\pi$$

The equation is then rearranged in order to solve for k.

$$k = \frac{1}{2}m_1 + \frac{1}{4}m_2 + \frac{1}{6}m_3 - \frac{1}{2}n_2 - \frac{1}{\pi}n_1 \arccos(\frac{1}{\sqrt{3}})$$

In view of the irrationality arguments used in examples one and two, it is now evident that Bricard's Condition cannot hold for T_2 and T_3 nor for T_1 and T_3.
Hilbert's third problem is therefore solved as T_2 and T_3 are of equal volume but are neither equidecomposable nor equicomplementable.

5 References

Aigner, M. & Ziegler, G. (2018): *Proofs from the book - Sixth Edition.* Springer.

Benko, D. (2007): *A new approach to Hilbert's third problem.* The American Mathematical Monthly,8, 665-676.

Boltjanskii, V.G. (1978): *Hilbert's third problem.* Winston.

Cromwell, P.R. (1997): *Polyhedra.* Cambridge Univ. Press.

Dupont, J.L. (2001): *Scissors congruences, group homology and characteristic classes.* World Scientific.

Hartshorne, R. (2000): *Geometry, Euclid and beyond.* Springer.

Karlqvist, A. (2003): *Århundradets matematik - Hilbert och hans problem.* B.Östlings Bokförlag.

Newson, M.W. (1902): *Mathematical Problems.* Bulletin of the American Mathematical Society,8, 437-479.

Sah, C. (1979): *Hilbert's third problem: scissors congruence.* Pitman Advanced Publ. Program.

Sjöberg, B. (1995): *Från Euklides till Hilbert: historien om matematikens utveckling under tvåtusen år.* Åbo Akademis förlag.

Wikipedia.https://en.wikipedia.org/wiki/Schlafliorthoscheme