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The impossibility of solving a quintic equation

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Abstract

One of the main purposes of algebra is to study algebraic equations and their solutions. This paper will show how it is impossible to solve the general quintic equation by the use of radicals, but also how a soluble quintic equation must have either one real and four complex conjugate roots or five real roots. The paper also gives an account of the history that lead to the solving of the general quadratic, cubic and quartic equations and provides methods for solving those. In those methods it is also shown how in order to solve an equation of degree n, an auxiliary equation of degree $n-1$ needs to be solved as well.

Contents

1 Introduction

Since the dawn of mathematics, people have sought to solve the complex puzzles called equations. In the beginning it was thought that all equations have a general solution, and it was just a matter of mental prowess and logical capacity to find the formula for each and every one of them. Mathematicians all over the world struggled to produce solutions for the general quadratic, cubic and quartic equations, until all the solutions had finally been found. Then a new question arose: How could a quintic equation, or equations of an even higher degree be solved? The answer is that there are no general solutions for such equations, and it all began with the proof of how the quintic is not soluble with radicals.

This paper begins with an account of definitions and theorems that will be used, which are referred to in the text, followed by an account of the history behind the solving of equations up to the quartic. In the following subsections the formulas for all aforementioned equations will be derived. Finally, some history behind the quintic equation and a proof of how the quintic equation is impossible to solve using radicals will be displayed.

Most of the theorems and proofs are taken from ”Lärobok i algebra” by Nagell as well as ”Polynom och ekvationer” by Tambour. Regarding the quintic equation, the reader may find the original proof by Abel as well as the re-printed version in the reference list, in addition to the explained proof taken from Nagell.

I would like to give appreciation to my tutor, Torbjörn Tambour, for continuously being incredibly patient, supportive and helpful on my path to increased knowledge.

2 Concepts and denotations

In this paper, mostly basic algebraic notations such as *polynomials, equations* and *roots* will be used, together with some definitions required to compute the equations of interest. The definitions and theorems have been divided into two parts, where the first will be referred to by mostly the quadratic, cubic and quartic equation and where the second is only relevant for the quintic equation.

2.1 Polynomials, equations and roots

Definition 1.1: A polynomial with r variables x_1, x_2, x_3 is defined by

$$\sum a_{k_1,k_2,...,k_r} x_1^{k_1} x_2^{k_2} ... x_r^{k_r}$$

where a are the polynomials coefficients which are constant complex numbers, and $k_1 = 0, 1, ..., n_1$;... ;$k_r = 0, 1, ..., n_r$. The sum, difference and product of two polynomials is also a polynomial. For polynomials it is also known that the laws of commutativity, associativity and distributivity apply. (Nagell, p. 1)

Definition 1.2: If $p(x)$ is a polynomial, then $p(x) = 0$ is called an *algebraic equation.* (Nagell, p. 21)

Definition 1.3: The roots (or solutions) of an equation are the values of x that satisfies the equation. The existence of said roots are confirmed by an *existential theorem*, which states that for all algebraic equations

$$p(x) = x^n + a_1 x^{n-1} + ... + a_n = 0$$

where $n > 0$ and all coefficients are complex numbers, there is at least one root x. (ibid)

Theorem 1.4: The number of roots of an algebraic equation of degree n has exactly n roots if they are counted with multiplicity. (Nagell, p. 25)(Tambour, 2003)

Theorem 1.5 (The Factor Theorem): The polynomial $p(x)$ has a factor $(x - k)$ if and only if $p(x)$ has a root k such that $p(k) = 0$.

Theorem 1.6: The coefficients of an equation may be expressed as polynomials in the roots. According to the factor theorem, a general equation $x^n + a_1x^{n-1} + ... + a_n = 0$ with roots $x_1, x_2, ..., x_n$ has the factors $(x - x_1), (x - x_2), ..., (x - x_n)$ and it may be factorised accordingly, inserting the polynomial a_n

$$x^n + a_1x^{n-1} + ... + a_n = a_n(x)(x - x_1)(x - x_2)...(x - x_n).$$

Since the degree of the polynomial is n, a_n is a constant. Furthermore, since x^n on the left side has the coefficient 1, a_n must also be equal to 1. Multiplying the factors on the right side and comparing the coefficients on both sides gives the following relations

$$\begin{aligned}
a_1x^{n-1} &= -(x_1 + x_2 + ... + x_n)x^{n-1} \\
x_1 + x_2 + ... + x_n &= -a_1 \\
a_2x^{n-2} &= (x_1x_2 + x_1x_3 + ... + x_{n-1}x_n)x^{n-2} \\
x_1x_2 + x_1x_3 + ... + x_{n-1}x_n &= a_2 \\
&... \\
a_n &= (x_1x_2...x_n)a_n \\
x_1x_2...x_n &= (-1)^n a_n.
\end{aligned}$$

For quadratic and cubic equations the relations are the following

$$\begin{aligned}
\textit{Quadratic:} &\quad x^2 + a_1x + a_2 \\
x_1 + x_2 &= -a_1 \\
x_1x_2 &= a_2
\end{aligned}$$

$$\begin{aligned}
\textit{Cubic:} &\quad x^3 + a_1x^2 + a_2x + a_3 \\
x_1 + x_2 + x_3 &= -a_1 \\
x_1x_2 + x_1x_3 + x_2x_3 &= a_2 \\
x_1x_2x_3 &= -a_3.
\end{aligned}$$

Definition 1.7: The discriminant of a polynomial is a quantity that depends on the polynomials coefficients and determines some properties of the roots, denoted D. Consider the general polynomial $p(x) = x^n + a_1x^{n-1} + ... + a_n$ with the roots $x_1, ..., x_n$. Then the discriminant of p is defined as

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

When $n = 2$ the polynomial is $p(x) = x^2 + a_1x + a_2$ and according to theorem 1.6 the relation between the roots may be expressed as $x_1 + x_2 = -a_1$ and $x_1x_2 = a_2$. Expressing D as a polynomial with the help of a_1, a_2 gives

$$D = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 - 4x_1x_2 = a_1^2 - 4a_2$$

Considering the first parenthesis, it is clear that $D = 0$ if and only if the roots are equal, which means the polynomial has one root with multiplicity 2, which also makes it a square. (Tambour, 2003)

When $n = 3$ the polynomial is $p(x) = x^3 + a_1x^2 + a_2x + a_3$ but may be written $p(x) = x^3 + px + q$, which is the depressed form of the cubic polynomial.[1] The determinant is then

$$D = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = -108(\frac{q^2}{4} + \frac{p^3}{27})$$

Definition 1.8: The nth roots of unity are the complex numbers that satisfy the equation $x^n - 1 = 0$. A root is called primitive if $n > 0$ is the smallest number such that $x^n = 1$. According to Moivre, we may find the primitive roots in the corners of a regular n-sided polygon inscribed in the unit circle. The reason that the roots are located on the unit circle is quite simple, consider what happens when the absolute value is applied to both sides

$$\begin{aligned} |x|^n &= |1| \\ |x| &= 1 \end{aligned}$$

which means that all roots have the absolute value 1.
Applying de Moivres formula, $x^n = \cos(n\phi) + i\sin(n\phi)$, gives the formula for finding said roots, which is $\epsilon_m = \cos(\frac{2\pi m}{n}) + i\sin(\frac{2\pi m}{n})$, where n is the degree of the original equation and $m = 1, 2, ..., n-1$. This could be

[1]See 3.3.1

described as a full rotation, 2π, being divided into n parts, where one primitive root is found after each partly completed rotation. (Nagell, p. 177) If one root x_0 for a binomial equation $x^n - \alpha = 0$ is found, the other roots can be written $\epsilon_m x_0$. (Tambour, 2003)

Definition 1.9: A radical of the nth degree, also called an nth root, may be expressed $\sqrt[n]{c}$. The number n is the exponential of the radical; for example, when $n = 2$ the radical is a square root. If an equation may be solved by operating on its coefficients using the four elementary rules of arithmetic and root operations it is said to be solvable by means of radicals, (Nagell, p. 179)

Theorem 1.10 (The binomial theorem): Any non-negative power of $x+y$ may be expanded to a sum of the form

$$(x+y)^n = x^n + A_1 x^{n-1} y + ... + A_k x^{n-k} y^k + ... + A_{n-k} x y^{n-1} + y^n$$

where

$$A_k = \frac{n(n-1)...(n-k+1)}{1 \cdot 2...(p-1)p}.$$

(Nagell, p. 56)

Definition 1.11: The general form of the different equations that will be mentioned in this paper, where $a_n \neq 0$:

The linear equation:	$a_1 x + a_2 = 0$
The quadratic equation:	$a_1 x^2 + a_2 x + a_3 = 0$
The cubic equation:	$a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$
The quartic equation:	$a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0$
The quintic equation:	$a_1 x^5 + a_2 x^4 + a_3 x^3 + a_4 x^2 + a_5 x + a_6 = 0$

2.2 Functions, number fields and groups

Definition 2.1: A rational function, usually denoted $f(x_1, x_2, ..., x_n)$, of one or more variables is defined as the quotient of two polynomials. If P, P_1, Q, Q_1 are polynomials and $PQ_1 = P_1Q$ then the two rational functions $\frac{P}{Q}$ and $\frac{P_1}{Q_1}$ are equal. Just as for polynomials, the sum, difference and product of two rational functions is another rational function. (Nagell, p. 15)

Definition 2.2: A rational function $f(x_1, x_2, ..., x_n)$ of the n variables $x_1, x_2, ..., x_n$ is called *symmetric* if it does not change when the variables are permuted in any of the $n!$ possible ways. The sum, difference and product of two symmetric functions is once again a symmetric function, but in addition this is also true for the quotient of two symmetric funtions. Furthermore, every symmetric rational function may be written as the quotient of two symmetric polynomials. (Nagell, p. 129)

Definition 2.3: An algebraic number field, or field for short, is usually denoted $\mathbf{K}(\alpha)$ or $\mathbf{\Omega}$, where α is an algebraic number. A field is a set of numbers which may be both complex and real. This set is denoted M, where $M \neq 0$ and if $a, b \in M$ then $a + b, a - b, ab, \frac{a}{b} \in M$. A property of a field is therefore that it does not expand when the four elementary arithmetic operations are applied to numbers within the field. For example, the set containing all rational numbers is a field, called the *rational field*. In fact, all fields contain this set, since $\frac{a}{a} = 1$ exists within all sets where $M \neq 0$ and all rational numbers may be constructed from the number 1 by repeated application of aforementioned operations.

If $\alpha \neq 0$ is an arbitrary number, $\mathbf{K}(\alpha)$ is defined as the smallest field that contains α and is defined

$$\frac{a_0 + a_1\alpha + a_2\alpha^2 + ... + a_m\alpha^m}{b_0 + b_1\alpha + b_2\alpha^2 + ... + b_n\alpha^n}$$

where a_k, b_k are integers and m, n natural numbers or zero. For example, the numbers $-\alpha^2, 1, 2\alpha...$ belongs to the field.

In the same manner $\mathbf{K}(a_1, a_2, ..., a_r)$ is the smallest field constructed from the numbers $a_1, a_2, ..., a_r$, and $\mathbf{K}(\boldsymbol{\Omega}, \xi)$ is the smallest field constructed from all numbers in the field $\boldsymbol{\Omega}$ together with the number ξ. It is said that ξ is adjoined to or an adjunction of $\boldsymbol{\Omega}$. The last example is also denoted $\boldsymbol{\Omega}(\xi)$, or in the case where $\xi \in \boldsymbol{\Omega}$ simply $\boldsymbol{\Omega}$. (Nagell, p. 32)

Theorem 2.4: The radical $\sqrt[n]{\beta}$ is called *irreducible* if the binomial $x^n - \beta$ is irreducible in $\mathbf{K}(\beta)$. The number n is called the relative degree of the radical with respect to $\mathbf{K}$. (Nagell, p. 249)

Theorem 2.5: Let $f(x)$ and $g(x)$ be two polynomials in the field $\mathbf{K}$. If $f(x)$ is irreducible in $\mathbf{K}$ and if $f(x)$ and $g(x)$ has a common root, $f(x)$ is a factor of $g(x)$. (Nagell, p. 220)

Theorem 2.6 (The Schönemann-Eisenstein Theorem): The integer polynomial

$$f(x) = x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

where all coefficients may be divided with the prime number p, but where a_n may not be divided with p^2, is irreducible in the rational field. (Nagell, p. 225)

Theorem 2.7: If ξ is an algebraic number with the relative degree n with respect to $\boldsymbol{\Omega}$, every number α in $\mathbf{K}(\boldsymbol{\Omega}, \xi)$ may be described in one way and one way only on the form

$$\alpha = a_0 + a_1 \xi + a_2 \xi^2 + ... + a_{n-1} \xi^{n-1}$$

where $a_0, a_1, ..., a_{n-1}$ belongs to $\boldsymbol{\Omega}$. (Nagell, p. 234)

Theorem 2.8: Let $R(x_1, x_2, x_3, ...)$ be a rational function of $x_1, x_2, x_3, ...$ with coefficients in $\boldsymbol{\Omega}$, and let $\alpha, \beta, \gamma, ...$ be numbers in $\mathbf{K}(\boldsymbol{\Omega}, \xi)$. If

$$R(\alpha, \beta, \gamma, ...) = 0,$$

then

$$R(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, ...) = 0$$

for all $i = 1, 2, 3, ...$ and where $\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, ...$ denotes the algebraic conjugates to $\alpha, \beta, \gamma, ...$ That is, $\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, ...$ are respectively roots to irreducible polynomials. (Nagell, p. 236)

Theorem 2.9: Let α be an algebraic number with the relative degree n with respect to $\mathbf{\Omega}$ and β an algebraic number with the relative degree q with respect to $\mathbf{K}(\mathbf{\Omega}, \alpha)$. Then the field $\mathbf{K}(\mathbf{\Omega}, \alpha, \beta)$ has the relative degree nq with respect to $\mathbf{\Omega}$. (Nagell, p. 243)

Theorem 2.10: An algebraic equation is said to be solvable by means of radicals with respect to the field $\mathbf{\Omega}$ or metacyclic with respect to $\mathbf{\Omega}$ if all its roots are possible to write with radicals with respect to $\mathbf{\Omega}$. (Nagell, p. 248)

Theorem 2.11: Every nth root of unity[2] may be presented with irreducible radicals with respect to the rational field.[3] (Nagell, p. 255)

Theorem 2.12: Let $f(x)$ be a polynomial in an arbitrary field $\mathbf{\Omega}$ which is irreducible in $\mathbf{\Omega}$ and has the prime number degree p. Let ϱ be an algebraic number with the relative degree q with respect to $\mathbf{\Omega}$. If $f(x)$ is reducible in $\mathbf{K}(\mathbf{\Omega}, \varrho)$ and if q is a prime number, then $p = q$. (Nagell, p. 259)

[2]See definition 1.8
[3]See definition 2.3

3 History of equations and how to solve them

To understand how the proof of the impossibility to solve quintic equations came to be, we must first understand the discoveries of the solutions of lower grade equations.
The exclusion of the linear equation is intentional, since the solution of it is quite trivial.

3.1 The founders

It is difficult to accredit one or more specific individuals to the solution of the quadratic equation, since it dates back incredibly far in time. Evidence has been found that the Babylonians in 3879 BC had methods for solving quadratic equations, using area and length. (Friberg, 2009) However, it was not until 1637 that the formula for solving second degree equations as we know it today was published by René Descartes in *La Géométrie*. (Serfati, p. 4)

As for cubic equations, the foundation was laid by the Greeks when they invented conic sections in 429 B.C., but it was not until around 300 A.D. that Diophantus of Alexandria succeeded in solving one single cubic equation using this method. In 1505, Scipione del Ferro (also known as *Dal Ferro* or *Scipo Ferro*, 1465-1526) told his pupil Antonio Fiore (also known as *Floridas*) of the solution to $x^3 + mx = n$, which would be the first step in solving a general cubic equation. However, it was Nicolo of Brescia (also known as *Tartaglia*, 1499/1500-1557) who first found a method for solving $x^3 + px^2 = q$. This method was not perfected and only worked for specific cases, but in 1541 he also found a way to solve the equation by first reducing it to the form $x^3 + mx = n$, which worked for all cubic equations. This discovery was shared with Cardano (also known as *Cardan*, 1501-1576) who had to promise to keep it secret. However, when Cardano's pupil Ferrari (1522-1565) managed to discover the solution of the quartic equation based on Tartaglia's work, Cardano realised that they had to publish the cubic solution in order to publish the quartic. This was because of the necessity to calculate a cubic equation to solve a quartic, which meant that if Cardano published the solution of the quartic it would not be complete without the solution of the cubic. Ferrari, who realised the importance of publishing his findings, succeded in finding Ferro's old formula, which Cardano could use to justify the publishing of the general formula of both the cubic and quartic equations. Both solutions were published in 1545 in a book called *Artis Magnae, Sive de Regulis Algebraicis Liber Unus*, commonly called *Ars Magna* or *The Great Art*. Even though Cardano attributed the solution to the cubic equation to Ferro and Tartaglia, it was published by him and thus the solution was viewed as his. To this day, the general formula, which was founded by Ferro and completed by Tartaglia, is called "Cardano's solution". This in contradiction to the general solution to the quartic equation, which was attributed to Ferrari and was named "Ferrari's solution". (Guilbeau, 1930) (Tambour, 2003) (Gårding, p. 7)

Many others have since proposed other formulas for both the cubic and quartic equations. For example, Euler (1707-1783) and Lagrange (1736-1813) developed their own formulas for solving both equations, but they are of course all based on the work of Ferro and Tartaglia, so substitution and solving an auxiliary equation of a lower degree are vital aspects of them. (Zhao, 2019)

3.2 The quadratic equation

The general form of a quadratic equation is $a_1x^2 + a_2x + a_3 = 0$, where a_n are complex numbers and $a_n \neq 0$. If $a_1 \neq 0$, the equation can be divided with a, which gives the following

$$x^2 + \frac{a_2}{a_1}x + \frac{a_3}{a_1} = 0.$$

Substituting $p = \frac{a_2}{a_1}$ and $q = \frac{a_3}{a_1}$ gives the (hopefully) familiar equation

$$x^2 + px + q = 0.$$

The general formula for solving this equation is called the quadratic formula, and is derived from completing the square in the equation. Start with subtracting q from both sides, which gives

$$x^2 + px = -q. \tag{1}$$

To make the left side into a complete square on the form $(x + \alpha)^2$, it must be modified. Expanding the square gives

$$(x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2. \tag{2}$$

If the left side in (1) and the right side in (2) are compared, it is obvious that $\alpha = \frac{p}{2}$, and that $\alpha^2 = (\frac{p}{2})^2$ must be added to (1) for them to be equal. This gives

$$x^2 + px + (\frac{p}{2})^2 = (\frac{p}{2})^2 - q. \tag{3}$$

But, the reason the square was completed was so that the left side could be simplified, which gives

$$\begin{aligned} x^2 + px + (\tfrac{p}{2})^2 &= (x + \tfrac{p}{2})^2 \\ (x + \tfrac{p}{2})^2 &= (\tfrac{p}{2})^2 - q \end{aligned}$$

Worth mentioning is that the right side in the equation is $\frac{D}{4}$, according to definition 1.7, where D determines how many real solutions the equation has. If $D = 0$ the equation has the real solution $x = -\frac{p}{2}$ with multiplicity 2, meaning that the polynomial in the left side of the equation is square.

Now it is possible to take the square root of both sides in (3), which results in the famous quadratic formula

$$\begin{aligned}\sqrt{(x + (\tfrac{p}{2}))^2} &= \sqrt{(\tfrac{p}{2})^2 - q} \\ x + \tfrac{p}{2} &= \sqrt{(\tfrac{p}{2})^2 - q} \\ x &= -\tfrac{p}{2} \pm \sqrt{(\tfrac{p}{2})^2 - q}.\end{aligned}$$

Note that since $\frac{D}{4}$ is now under a square root it is clear that if $\frac{D}{4} > 0$ the equation has two real solutions, and if $\frac{D}{4} < 0$ it has two conjugate complex solutions.

3.3 The cubic equation

The general form of the cubic equation is $a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, where a_n are complex numbers and $a_n \neq 0$. To solve a cubic equation, it is first reduced to its depressed form $x^3 + px + q = 0$, without quadratic term. After finding one root by the use of substitutions, the primitive cube roots are used to find the remaining roots.

3.3.1 The depressed cubic

Constructing a depressed cubic is done by first dividing the equation with a_1 and then substituting. Since $a_n \neq 0$ it is indeed possible to divide the general equation with a_1, which gives the equation

$$x^3 + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x + \frac{a_4}{a_1} = 0$$

in which the substitution $x = t - \frac{a_2}{3a_1}$ is made, after which the equation is simplified.

$$\begin{aligned}
(t - \tfrac{a_2}{3a_1})^3 + \tfrac{a_2}{a_1}(t - \tfrac{a_2}{3a_1})^2 + \tfrac{a_3}{a_1}(t - \tfrac{a_2}{3a_1}) + \tfrac{a_4}{a_1} &= 0 \\
t^3 - \tfrac{t^2a_2}{a_1} + \tfrac{ta_2^2}{3a_1^2} - \tfrac{a_2^3}{27a_1^3} + \tfrac{a_2}{a_1}(t^2 - \tfrac{2ta_2}{3a_1} + \tfrac{a_2^2}{9a_1^2}) + \tfrac{a_3}{a_1}(t - \tfrac{a_2}{3a_1}) + \tfrac{a_4}{a_1} &= 0 \\
t^3 + t(\tfrac{a_3}{a_1} - \tfrac{a_2^2}{3a_1^2}) + \tfrac{2a_2^3}{27a_1^3} - \tfrac{a_2a_3}{3a_1^2} + \tfrac{a_4}{a_1} &= 0
\end{aligned}$$

Substituting $p = \frac{a_3}{a_1} - \frac{a_2^2}{3a_1^2}$ and $q = \frac{2a_2^3}{27a_1^3} - \frac{a_2a_3}{3a_1^2} + \frac{a_4}{a_1}$ gives the depressed cubic

$$t^3 + pt + q = 0 \tag{4}$$

3.3.2 Primitive cube roots of unity

Finding the primitive cube roots of unity is according to definition 1.8 done by computing $\epsilon_m = \cos(\frac{2\pi m}{n}) + i\sin(\frac{2\pi m}{n})$ for $m = 1, 2$ and $n = 3$. This gives

$$
\begin{aligned}
\epsilon_1 &= \cos(\tfrac{2\pi}{3}) + i\sin(\tfrac{2\pi}{3}) \\
&= \tfrac{-1+i\sqrt{3}}{2} \\
\epsilon_2 &= \cos(\tfrac{2\pi 2}{3}) + i\sin(\tfrac{2\pi 2}{3}) \\
&= \tfrac{-1-i\sqrt{3}}{2}.
\end{aligned}
$$

3.3.3 Solving the depressed cubic

Solving the depressed cubic using Cardano's solution starts with introducing another substitution, namely $t = u + v$. This transforms (4) into

$$u^3 + 3u^2v + 3uv^2 + v^3 + (u+v)p + q = 0$$

$$u^3 + v^3 + (3uv + p)(u+v) + q = 0$$

which has the solution $u+v$ if $u^3 + v^3 = -q$ and $uv = -\frac{p}{3}$. Viewing the form of these conditions reminds of the relation between roots and coefficients in a quadratic equation, where if x_1, x_2 are roots to a quadratic equation, and a_1, a_2 are its coefficients, $x_1 + x_2 = -a_1$ and $x_1x_2 = a_2$ (definition 1.6). Thus, an equation with the roots u^3 and v^3 can be constructed in the following manner

$$
\begin{aligned}
(y - u^3)(y - v^3) &= 0 \\
y^2 - (u^3 + v^3)y + (uv)^3 &= 0
\end{aligned}
$$

where $u^3 + v^3$ and uv can be replaced according to the aforementioned relations, which gives

$$y^2 + qy - \frac{p^3}{27} = 0.$$

Solving for y gives

$$y = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \tag{5}$$

The expression under the square root might be familiar. It is the disriminant for a cubic polynomial divided with -108. Let this expression be denoted Δ, then $D = -108\Delta$. When $D > 0$ or $D = 0$ the equation gives three real roots, and when $D < 0$ it instead gives one real root and two complex roots. (Tambour, 2003)

Since u and v are symmetric variables, (5) gives

$$\begin{aligned} u_1 &= \sqrt[3]{-\tfrac{q}{2} + \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \\ v_1 &= \sqrt[3]{-\tfrac{q}{2} - \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}}. \end{aligned}$$

Here the primitive cube roots are used to find the other possible solutions

$$\begin{aligned} u_2 &= \epsilon_1 u_1 &= \tfrac{-1+i\sqrt{3}}{2}\sqrt[3]{-\tfrac{q}{2} + \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \\ u_3 &= \epsilon_2 u_2 &= \tfrac{-1-i\sqrt{3}}{2}\sqrt[3]{-\tfrac{q}{2} + \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \\ v_2 &= \epsilon_1 v_1 &= \tfrac{-1+i\sqrt{3}}{2}\sqrt[3]{-\tfrac{q}{2} - \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \\ v_3 &= \epsilon_2 v_1 &= \tfrac{-1-i\sqrt{3}}{2}\sqrt[3]{-\tfrac{q}{2} - \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \end{aligned}$$

Since $uv = -\frac{p}{3}$, not all combinations of these solutions are valid. This means that some calculations are necessary to find the actual solutions

$$\begin{aligned} u_1 v_1 &= \sqrt[3]{-\tfrac{q}{2} + \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}}\sqrt[3]{-\tfrac{q}{2} - \sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}}} \\ &= \sqrt[3]{\tfrac{q^2}{4} - (\sqrt{\tfrac{q^2}{4} + \tfrac{p^3}{27}})^2} \\ &= \sqrt[3]{-\tfrac{p^3}{27}} = -\tfrac{p}{3} \end{aligned}$$

In the same manner

$$
\begin{aligned}
u_1v_2 &= u_1v_1\epsilon_1 &&= \tfrac{p(1-i\sqrt{3})}{6} \\
u_1v_3 &= u_1v_1\epsilon_2 &&= \tfrac{p(1+i\sqrt{3})}{6} \\
u_2v_2 &= u_1v_2(\epsilon_1)^2 &&= \tfrac{2p(i\sqrt{3}+1)}{12} \\
u_2v_3 &= u_1v_1\epsilon_1\epsilon_2 &&= -\tfrac{p}{3} \\
u_3v_2 &= u_1v_1\epsilon_2\epsilon_1 &&= u_2v_3 = -\tfrac{p}{3} \\
u_3v_3 &= u_1v_1(\epsilon_2)^2 &&= \tfrac{p(-i\sqrt{3})}{3}.
\end{aligned}
$$

Remembering that $t = u+v$ and $x = t-\frac{b}{3a}$, the solution to the cubic equation can now be constructed

$$
\begin{aligned}
x_1 &= u_1 + v_1 - \tfrac{b}{3a} \\
x_2 &= u_2 + v_3 - \tfrac{b}{3a} \\
x_3 &= u_3 + v_2 - \tfrac{b}{3a}.
\end{aligned}
$$

With this in mind, if the cubic equation lacks a quadratic term, $-\frac{b}{3a}$ will be excluded from the solutions since the process of depressing the equation does not have to be computed and the coefficients p and q can be directly extracted from the equation.

3.4 The quartic equation

To solve the quartic equation, a variation of Ferrari's solution will be used, which utilises Cardano's solution. Ferrari originally solved the quartic equation by first reducing it to depressed form, without cubic term, and then made use of an auxiliary variable to write the equation as two squares. It is however possible to solve it in the same manner as Ferrari without reducing it first, which will be displayed here. An example of how to solve a quartic equation without a cubic term can be found after the general formula, and the following section will show how indeed all quartic equations can be reduced to such an equation.

3.4.1 The depressed quartic

When the cubic equation was depressed, the equation was divided by a_1 and then a substitution was made. The same method will be applied here. Dividing the general quartic equation with a_1 gives

$$x^4 + \frac{a_2}{a_1}x^3 + \frac{a_3}{a_1}x^2 + \frac{a_4}{a_1}x + \frac{a_5}{a_1} = 0$$

in which the substitution $t = x - \frac{a_2}{4a_1}$ is made, after which the equation is simplified and calculated by using theorem 1.10 (the binomial theorem)

$$\begin{aligned} (t - \tfrac{a_2}{4a_1})^4 + \tfrac{a_2}{a_1}(t - \tfrac{a_2}{4a_1})^3 + \tfrac{a_3}{a_1}(t - \tfrac{a_2}{4a_1})^2 + \tfrac{a_4}{a_1}(t - \tfrac{a_2}{4a_1}) + \tfrac{a_5}{a_1} &= 0 \\ t^4 + t^2(\tfrac{a_2^2+8a_1a_3}{8a_1^2}) + t(\tfrac{a_2^3-4a_1a_2a_3+8a_1^2a_4}{8a_1^3}) - \tfrac{3a_2^4+4a_1a_2^2a_3-4^2a_1^2a_2a_4+4^3a_1^3a_3}{4^4a_1^4} &= 0. \end{aligned}$$

Substituting $p = \frac{a_2^2+8a_1a_3}{8a_1^2}$, $q = \frac{a_2^3-4a_1a_2a_3+8a_1^2a_4}{8a_1^3}$ and $r = -\frac{3a_2^4+4a_1a_2^2a_3-4^2a_1^2a_2a_4+4^3a_1^3a_3}{4^4a_1^4}$ gives the depressed quartic

$$t^4 + pt^2 + qt + r = 0. \tag{6}$$

3.4.2 Solving the general quartic

Consider the general quartic, $a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$. If new coefficients are introduced, $a = \frac{a_2}{a_1}$, $b = \frac{a_3}{a_1}$, $c = \frac{a_4}{a_1}$ and $d = \frac{a_5}{a_1}$, it can be written as

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{7}$$

$$\begin{aligned} x^4 + ax^3 &= -bx^2 - cx - d \\ x^2(x^2 + ax) &= -bx^2 - cx - d. \end{aligned}$$

The intention is now to be able to write both sides as squares, since this would allow for the square root to be applied to both sides. This starts with completing the square on the left side, which entails adding $\frac{x^2a^2}{4}$ to both sides. This gives

$$\begin{aligned} x^2(x + \tfrac{a}{2})^2 &= -bx^2 - cx - d + \tfrac{a^2x^2}{4} \\ (x^2 + \tfrac{ax}{2})^2 &= (\tfrac{a^2}{4} - b)x^2 - cx - d. \end{aligned}$$

The next step is truly magical. An auxiliary variable t is introduced, with the motivation that it may be chosen in a way that allows the right hand side of the equation to be written as a square. To keep the left side as a square, the expression $(x^2 + \frac{ax}{2})t + \frac{t^2}{4}$ is added, which gives

$$(x^2 + \frac{ax}{2} + \frac{t}{2})^2 = (\frac{a^2}{4} - b + t)x^2 + (\frac{at}{2} - c)x + \frac{t^2}{4} - d \tag{8}$$

The right side can be written as a square according to definition 1.7 if the discriminant $D = p^2 - 4q$ for the quadratic equation is 0. Dividing the right hand side with the coefficient for x^2 gives

$$x^2 + (\frac{2at - 4c}{a^2 - 4b + 4t})x + \frac{t^2 - 4d}{a^2 - 4b + 4t},$$

where $p = \frac{2at-4c}{a^2-4b+4t}$ and $q = \frac{t^2-4d}{a^2-4b+4t}$. Calculating the discriminant gives

$$t^3 - bt^2 + (ac - 4d)t - a^2d + 4bd - c^2 = 0.$$

This is a cubic equation and the solutions to it are calculated according to Cardano's solution.[4] Substituting $t = y + \frac{b}{3}$ gives

$$y^3 + (ac - \frac{b^2}{3} - 4d)y + \frac{abc}{3} - a^2d - \frac{2b^3}{27} + \frac{8bd}{3} - c^2 = 0$$

[4]See chapter 3.3

and

$$\begin{aligned} p_c &= ac - \tfrac{b^2}{3} - 4d \\ q_c &= \tfrac{abc}{3} - a^2d - \tfrac{2b^3}{27} + \tfrac{8bd}{3} - c^2. \end{aligned}$$

Applying Cardano's formula gives

$$\begin{aligned} u &= \sqrt[3]{-\tfrac{q_c}{2} + \sqrt{\tfrac{q_c^2}{4} + \tfrac{p_c^3}{27}}} \\ v &= \sqrt[3]{-\tfrac{q_c}{2} - \sqrt{\tfrac{q_c^2}{4} + \tfrac{p_c^3}{27}}} \\ y &= u + v. \end{aligned}$$

If it is now, according to Cardano's solution, assumed that u_0 and v_0 are roots for u and v and $u_0v_0 = -\frac{p}{3}$, the following are solutions for t:

$$\begin{aligned} t_1 &= u_0 + v_0 + \tfrac{b}{3} \\ t_2 &= \omega u_0 + \omega^2 v_0 + \tfrac{b}{3} \\ t_3 &= \omega^2 u_0 + \omega v_0 + \tfrac{b}{3} \end{aligned}$$

With the solutions to the auxiliary variable t, it is now possible to write the right side in (8) as a square on the general form $(\alpha x + \beta)^2$. Expanding the general form and comparing to the right side in (8) gives

$$\begin{aligned} \alpha^2x^2 + 2\alpha\beta x + \beta^2 &= (\tfrac{a^2}{4} - b + t)x^2 + (\tfrac{at}{2} - c)x + \tfrac{t^2}{4} - d \\ \alpha^2 &= \tfrac{a^2}{4} - b + t \\ \beta^2 &= \tfrac{t^2}{4} - d \\ \alpha\beta &= \tfrac{at}{4} - \tfrac{c}{2} \end{aligned}$$

Re-writing (8) with the completed square on the right side gives

$$(x^2 + \frac{ax}{2} + \frac{t}{2})^2 = (\alpha x + \beta)^2$$

with the solutions

$$x^2 + \frac{ax}{2} + \frac{t}{2} = \alpha x + \beta \tag{9}$$

and

$$x^2 + \frac{ax}{2} + \frac{t}{2} = -\alpha x - \beta. \tag{10}$$

The first two solutions are given by solving (9), and the third and fourth by (10)

$$\begin{aligned} x_{1,2} &= \frac{\alpha-\frac{a}{2}\pm\sqrt{(\alpha-\frac{a}{2})^2-4(\frac{t}{2}-\beta)}}{2} \\ x_{3,4} &= \frac{-\alpha-\frac{a}{2}\pm\sqrt{(\alpha-\frac{a}{2})^2-4(\frac{t}{2}+\beta)}}{2} \end{aligned}$$

Note that according to theorem 4 the number of roots of the quartic equation is exactly equal to 4. This means that even if t has 3 possible solutions with 2 different α and β each, the solutions to x are equal whichever t is chosen, which means that either can be used to give the same solutions to x.

3.4.3 Example of depressed quartic

To show how Ferrari's solution may be used as it was intended, an example of a quartic equation without a cubic term will be calculated. This will show how the formula does not depend on whether an equation is reduced to depressed form or not before the roots are calculated.

The equation which will be solved is $x^4 - 51x^2 - 10x + 600 = 0$. Comparing this equation to the one for the general quartic in (7) yields that $a = 0, b = -51, c = -10$ and $d = 600$, which shows that it is exactly the same equation. This means that it indeed is not necessary to depress the quartic equation before calculating it, and the solution of the equation may be calculated according to the general solution. Just like before, the equation is rewritten and an auxiliary variable t is introduced to produce squares.

$$\begin{aligned} x^4 &= 51x^2 + 10x - 600 \\ (x^2 + \tfrac{t}{2})^2 &= x^4 + x^2t + \tfrac{t^2}{4} \\ &= 51x^2 + 10x - 600 + x^2t + \tfrac{t^2}{4} \end{aligned}$$

Factoring the right side of the equation gives

$$(51 + t)(x^2 + \frac{10}{51 + t}x + \frac{t^2 - 2400}{51 + t})$$

and calculating for which t the discriminant for the quadratic equation is 0 gives

$$t^3 + 51t^2 - 2400t - 122500 = 0.$$

Here the formula found in the general solution for p_c and q_c is used to construct Cardano's formula

$$\begin{aligned} p_c &= ac - \frac{b^2}{3} - 4d \\ &= -\frac{-51^2}{3} - 4 \cdot 600 \\ &= -3267 \end{aligned}$$

$$\begin{aligned} q_c &= \frac{abc}{3} - a^2 d - \frac{2b^3}{27} + \frac{8bd}{3} - c^2 \\ &= -\frac{2(-51)^3}{27} + \frac{8(-51)600}{3} - (-10)^2 \\ &= -71874 \end{aligned}$$

$$\begin{aligned} u &= \sqrt[3]{-\frac{q_c}{2} + \sqrt{\frac{q_c^2}{4} + \frac{p_c^3}{27}}} = 33 \\ v &= \sqrt[3]{-\frac{q_c}{2} - \sqrt{\frac{q_c^2}{4} + \frac{p_c^3}{27}}} = 33 \\ y &= u + v = 66 \\ t &= y + \frac{b}{3} = 49. \end{aligned}$$

Now α and β may be calculated using Ferrari's solution

$$\begin{aligned} \alpha^2 &= \frac{a^2}{4} - b + t = 100 \\ \alpha &= \pm 10 \\ \alpha\beta &= \frac{at}{4} - \frac{c}{2} \\ \beta &= \pm\frac{1}{2}. \end{aligned}$$

For simplicity $\alpha = 10$ and $\beta = \frac{1}{2}$ is chosen, since Ferrari's formula will give the same solutions whichever pair of α and β is used. Finally, to find the solutions to the equation, Ferrari's formula is applied with the chosen values of α and β.

$$\begin{aligned} x_{1,2} &= \frac{10-0 \pm \sqrt{(10-0)^2 - 4(\frac{49}{2} - \frac{1}{2})}}{2} \\ &= \frac{10 \pm 2}{2} \\ x_{3,4} &= \frac{-10-0 \pm \sqrt{(-10-0)^2 - 4(\frac{49}{2} + \frac{1}{2}}}{2} \\ &= -5 \end{aligned}$$

Thus, the equation has the solutions $x_1 = 6, x_2 = 4$ and $x_{3,4} = -5$.

4 The quintic equation

4.1 Erland Samuel Bring

Bring (1736-1798) was a Swede based in Lund, who worked as a lawyer and then as a notary after which he became a professor in history. (Gårding, p.7) Given his background, it is most surprising that he also is a famous mathematician, which is mostly due to the dissertation he wrote in 1786 regarding how equations may be transformed, under the name "Meletemata quaedam mathematica circa transformationem aequationum algebraicarum". This dissertation proved how reducing a general quintic equation to the form of $y^5 + py + q = 0$ is indeed possible. His work is most likely based on Tschirnhausen's[5] (1651-1708) earlier calculations, but this is not noted in Bring's work. (Gårding, p.8) Bring also contributed with the Bring radical, which of a real number α is the unique real root to the polynomial $x^5+x+\alpha$. George Jerrard (1804-1863) later realised it was possible to use the Bring radical to solve some quintic equations.

4.2 Niels Henrik Abel

In 1824 Abel showed that the quintic equation is impossible to solve using only algebraic operations, that is by means of radicals. He had to finance the printing of his work by himself, which made the proof relatively short and not quite finished (Nagell, p. 247), but in 1826 a more thorough version was published in "Journal für die reine und angewandte Mathematik", also called Crelle's journal. (Crelle, 1824) He is accredited to be the first person who showed this impossibility, but shortly thereafter Galois (1811-1832) showed the same thing using group theory.

[5] An account of Tschirnhaus transformations may be found in Nagell, p. 206

4.3 The proof

This proof is taken from Nagell and follows Abel in the sense that it uses mostly basic properties of fields and polynomials. Nagell starts with expanding the rational field with the help of radicals, after which properties of factors in polynomials are considered in order to construct a linear system of equations, solving for possible solutions of the quintic equation.

Start with the field consisting of the set of all rational numbers, here denoted $\mathbf{\Omega_0}$, and let $f(x)$ be an irreducible polynomial of the fifth degree in $\mathbf{\Omega_0}$. The equation $f(x) = 0$ can be expressed as the general quintic equation with 1 as the coefficient for x^5 in the following manner

$$x^5 + \frac{a_2}{a_1}x^4 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^2 + \frac{a_5}{a_1}x + \frac{a_6}{a_1} = 0.$$

Assume that this equation is solvable with radicals with respect to $\mathbf{\Omega_0}$. This means that if $\mathbf{\Omega_0}$ is expanded with adjunctions of radicals, $f(x)$ becomes reducible in the expanded field. A polynomial may exist in two forms, its original form and its reduced form

$$\begin{aligned} f(x) &= x^n + a_1x^{n-1} + ...a_{n-1}x + a_n \\ f(x) &= (x - x_1)(x - x_2)...(x - x_n) \end{aligned}$$

where $x_1, ..., x_n$ are the roots of the polynomial. Since the polynomial is irreducible in $\mathbf{\Omega_0}$, it means that the roots of the polynomial exist outside $\mathbf{\Omega_0}$, which is why the field must be expanded. Then it is also logical that the radicals that are adjoined to $\mathbf{\Omega_0}$ in order to make $f(x)$ reducible must aid in constructing the roots of the polynomial, which will be shown below.

Assume that the radicals which are adjoined to $\mathbf{\Omega_0}$ have exponentials that are prime numbers. First the 5th root of unity, $\epsilon^{(5)}$, will be adjoined to the field. This will allow for remaining roots of the binomial equation $x^5 - a = 0$ to be calculated when the first root is found. When adjoining a complex number to a field, its complex conjugate must also be adjoined, because the adjunction of a complex number might not cause the adjunction of its conjugate. Obviously, if the adjunction of the complex number causes its conjugate to be adjoined, the additional adjunction is unnecessary. However, in the case when the first 5th root of unity is adjoined, all other 5th roots of unity will be adjoined as well, since $\epsilon_k = \epsilon^k$.

Let $\mathbf{\Omega_1}$ be the field that is constructed when $\epsilon^{(5)}$ is adjoined to $\mathbf{\Omega_0}$, which according to definition 1.8 is

$$\epsilon^{(5)} = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}) = \frac{1}{4}(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}).$$

The polynomial $f(x)$ is not reducible in $\mathbf{\Omega_1}$. According to theorem 2.12, in order for $f(x)$ to be reducible in $\mathbf{\Omega_1}$, the adjunction that constructed $\mathbf{\Omega_1}$ needs to have the relative degree 5. Since this adjunction is supposed to make the construction of the roots to the quintic equation $f(x)$ possible, the adjunction must be a fifth root, and $\epsilon^{(5)}$ only has square roots. This means that $f(x)$ is indeed irreducible in $\mathbf{\Omega_1}$.

Now let ϱ be the second radical that is adjoined to the field and also the one that makes $f(x)$ reducible. As mentioned earlier, if the adjunction ϱ yields that $f(x)$ becomes reducible, ϱ must according to theorem 2.12 have the relative degree 5 and thus be on the form $\varrho = \sqrt[5]{\eta}$. The number η is in $\mathbf{\Omega_1}$ because the adjunction of ϱ is made so that $f(x)$ becomes reducible, then ϱ must be an operation on an algebraic number that is already in the field. It is also possible to write ϱ on the binomial form $\varrho^5 - \eta = 0$, which means that according to theorem 2.4 it is irreducible in $\mathbf{\Omega_1}$, since η belongs to the field.

Let $\mathbf{\Omega_2}$ be the field in which $f(x)$ is reducible, consequently constructed by the adjunction ϱ, so that $\mathbf{\Omega_2}= \mathbf{K}(\mathbf{\Omega_1}, \varrho)$. Since $f(x)$ now is reducible, it has at least one polynomial factor. Let this factor be $g(x)$, and let $g(x)$ be an irreducible polynomial in $\mathbf{\Omega_2}$. According to theorem 2.7, $g(x)$ may be written as $g(x, \varrho)$, since the coefficients of g belongs to $\mathbf{\Omega_2}$. Then $g(x)$ is a polynomial in the variables x, ϱ with coefficients in $\mathbf{\Omega_1}$. This is due to theorem 2.7 which states that there is only one way to present all numbers α in $\mathbf{K}(\mathbf{\Omega_1}, \varrho)$, or the coefficients for $g(x)$, which is

$$\alpha = \epsilon_1 + \epsilon_2\varrho + \epsilon_3\varrho^2 + ... + \epsilon_5\varrho^4$$

Assume that the variable with the highest degree in $g(x, \varrho)$ has the coefficient 1. Now, according to theorem 2.8, since the polynomial $f(x)$ has the factor

$g(x, \varrho)$ it must have the factors

$$g(x, \varrho),\ g(x, \epsilon\varrho),\ g(x, \epsilon_2\varrho),\ g(x, \epsilon_3\varrho),\ g(x, \epsilon_4\varrho). \tag{11}$$

These are all irreducible in $\mathbf{\Omega_2}$ since $g(x, \varrho)$ is irreducible. This is because of theorem 2.8 and how the factors $\epsilon_k\varrho$ in the polynomials are algebraic conjugates. Furthermore, the polynomials in (11) are irreducible which means that they do not have a common factor. If they did, the factor would be the same as the polynomial, which would mean that the polynomials would coincide.

The polynomial

$$F(x) = g(x, \varrho)g(x, \epsilon\varrho)g(x, \epsilon_2\varrho)g(x, \epsilon_3\varrho)g(x, \epsilon_4\varrho)$$

belongs to $\mathbf{\Omega_1}$, since the multiplication of the polynomials on the right hand side will produce a polynomial in x with coefficients that are polynomials and symmetric in $\varrho, \epsilon\varrho, ..., \epsilon_4\varrho$. Since they are symmetric, they belong to $\mathbf{\Omega_1}$, which means that their product also belongs to $\mathbf{\Omega_1}$. Furthermore, $F(x)$ is according to theorem 2.5 divisible with $f(x)$. But, as was shown earlier, all the factors in $F(x)$ are factors in $f(x)$, which means that $F(x)$ is a power of $f(x)$. Since none of the factors are the same, $F(x)$ must be the first power of $f(x)$, which means that $F(x) = f(x)$ and that the factors are linear. If now ξ_i denotes the roots of the equation $f(x)$, the following linear system of equations may be constructed according to theorem 2.6

$$\begin{cases} \xi_1 &= \alpha_0 + \alpha_1\varrho + \alpha_2\varrho^2 + \alpha_3\varrho^3 + \alpha_4\varrho^4 \\ \xi_2 &= \alpha_0 + \alpha_1\epsilon\varrho + \alpha_2\epsilon_2\varrho^2 + \alpha_3\epsilon_3\varrho^3 + \alpha_4\epsilon_4\varrho^4 \\ \xi_3 &= \alpha_0 + \alpha_1\epsilon_2\varrho + \alpha_2\epsilon_4\varrho^2 + \alpha_3\epsilon\varrho^3 + \alpha_4\epsilon_3\varrho^4 \\ \xi_4 &= \alpha_0 + \alpha_1\epsilon_3\varrho + \alpha_2\epsilon\varrho^2 + \alpha_3\epsilon_4\varrho^3 + \alpha_4\epsilon_2\varrho^4 \\ \xi_5 &= \alpha_0 + \alpha_1\epsilon_4\varrho + \alpha_2\epsilon_3\varrho^2 + \alpha_3\epsilon_2\varrho^3 + \alpha_4\epsilon\varrho^4 \end{cases} \tag{12}$$

where the coefficients $\alpha_0, ..., \alpha_4$ belong to $\mathbf{\Omega_1}$. The roots ξ_i are polynomials in $\mathbf{\Omega_2}$ and are constructed with variables up to ϱ^4 since $\varrho^5 = \eta$.

The polynomial $f(x)$ has real coefficients, since it is defined in the real field, and since it is a polynomial of an odd degree it has at least one real root.[6]

[6]Can be proven with the help of the intermediate value theorem, but is not shown here

It is known that $\varrho = \sqrt[5]{\eta}$, which means that η may be both real and complex for ϱ to produce the roots of $f(x)$. It is necessary to divide further inquisitions into two different cases.

Case 1: The number η is real. Since $f(x)$ has real coefficients, at least one ξ_i must be a real root. The roots $\xi_{2,3,4,5}$ contain algebraic conjugates of ϵ, which means that they have complex components. It is then logical to assume that the real root is ξ_1. Now, it is known that ϵ and all its conjugations belongs to $\mathbf{\Omega_1}$, which means that it can be assumed that ϱ must be the real root to $x^n = \eta$. Then, for ξ_1 to be real, $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ must be real. If they were complex, their complex conjugates would also belong to $\mathbf{\Omega_1}$. If α_k' denotes the complex conjugate to α_k, then

$$\xi_1 = \alpha_0' + \alpha_1'\varrho + \alpha_2'\varrho^2 + \alpha_3'\varrho^3 + \alpha_4'\varrho^4$$

and comparing with ξ_1 from (12)

$$\alpha_0 + \alpha_1\varrho + \alpha_2\varrho^2 + \alpha_3\varrho^3 + \alpha_4\varrho^4 = \alpha_0' + \alpha_1'\varrho + \alpha_2'\varrho^2 + \alpha_3'\varrho^3 + \alpha_4'\varrho^4$$
$$(\alpha_0 - \alpha_0') + (\alpha_1 - \alpha_1')\varrho + (\alpha_2 - \alpha_2')\varrho^2 + (\alpha_3 - \alpha_3')\varrho^3 + (\alpha_4 - \alpha_4')\varrho^4 = 0.$$

This kind of equation may be solved in two ways. Either solve for the radical ϱ or the trivial solution, where all coefficients and constants are zero. According to theorem 2.4, the binomial $\varrho^5 - \eta$ is irreducible in $\mathbf{\Omega_1}$, which means that it is impossible to solve for ϱ. This means that the only possible solution is the trivial, where

$$(\alpha_n - \alpha_n') = 0, \;\; n = 0, 1, 2, 3, 4.$$

The only way for this to be true is if a_n is real, since the conjugate of a real number is the same real number. Therefore a_n is real, which makes the root ξ_1 real.

Consider definition 1.8, which says that the nth roots of unity may be found in the corners of a regular n-sided polygon inscribed in the unit circle. This means that ϵ and ϵ_4 as well as ϵ_2 and ϵ_3 are complex conjugates. From (12) it is now clear that ξ_2 and ξ_5 as well as ξ_3 and ξ_4 must be complex conjugates. The polynomial $f(x)$ then has one real and four complex roots.

Case 2: The number η is complex, with η' as complex conjugate. The equation $x^5 = \eta$ then has the complex roots $\varrho, \varrho_1, \varrho_2, \varrho_3, \varrho_4$ and the complex conjugate equation $x^5 = \eta'$ consequently has the complex conjugate roots $\varrho', \varrho'_1, \varrho'_2, \varrho'_3, \varrho'_4$. Then

$$\varrho\varrho' = \varrho_1\varrho'_1 = \varrho_2\varrho'_2 = \varrho_3\varrho'_3 = \varrho_4\varrho'_4 = \sqrt[5]{\eta\eta'} = R$$

where R is a real number. If now the adjunction R results in $f(x)$ becoming reducible, the adjunction ϱ becomes unnecessary, and it is possible to continue with case 1 but where the real number R replaces η.

If $f(x)$ does not become reducible from the adjunction R, ϱ must be adjoined, which means that also $\varrho' = \frac{R}{\varrho}$ is adjoined. If ξ_1 is real and the conjugate of a real number is the same real number, then

$$\alpha_0 + \alpha_1\varrho + \alpha_2\varrho^2 + \alpha_3\varrho^3 + \alpha_4\varrho^4 = \alpha'_0 + \alpha'_1\varrho' + \alpha'_2\varrho'^2 + \alpha'_3\varrho'^3 + \alpha'_4\varrho'^4$$

where α'_i is the complex conjugate to α_i. According to theorem 2.7, this equation must be true even if ϱ is replaced with any of $\varrho_1, \varrho_2, \varrho_3, \varrho_4$, which gives the following equation

$$\alpha_0 + \alpha_1\varrho_i + \alpha_2\varrho_i^2 + \alpha_3\varrho_i^3 + \alpha_4\varrho_i^4 = \alpha'_0 + \alpha'_1\varrho'_i + \alpha'_2\varrho_i'^2 + \alpha'_3\varrho_i'^3 + \alpha'_4\varrho_i'^4$$

where $i = 1, 2, 3, 4$. This means that all roots ξ_j, $j = 1, 2, 3, 4, 5$ are real.

This results in the following theorem:

Theorem 2.13: An algebraic equation of the fifth degree which belongs to, is irreducible in and is metacyclic with respect to the field $\mathbf{\Omega}$ either has one real and four imaginary roots or five real roots. (Nagell, p. 263)

According to definition 2.8, the equations which are not metacyclic with respect to $\mathbf{\Omega}$ are irrelevant, since they cannot be solved with radicals with respect to $\mathbf{\Omega}$. According to theorem 2.6, a polynomial which has coefficients divisible with a prime number is irreducible in the rational field. One such polynomial may be constructed by letting $q \geq 2$ be a prime number and the polynomial's corresponding equation be on the form

$$f(x) = x^5 - 2qx - q = 0. \tag{13}$$

Now the derivative is used to analyse how the function behaves. The following equation is given

$$f'(x) = 5x^4 - 2q = 0$$

and the roots may be calculated

$$\begin{aligned}
5x^4 &= 2q \\
x^4 &= \tfrac{2q}{5} \\
x_1 &= \sqrt[4]{\tfrac{2q}{5}} \\
x_2 &= -\sqrt[4]{\tfrac{2q}{5}}.
\end{aligned}$$

Note that x_1 and x_2 are the real roots of the equation. The original equation is then calculated with the given values for x

$$\begin{aligned}
f(x_1) &= \sqrt[4]{\tfrac{2q}{5}}^5 - 2q\sqrt[4]{\tfrac{2q}{5}} - q \\
&= -\tfrac{8q}{5}\sqrt[4]{\tfrac{2q}{5}} - q \qquad < 0 \\
\\
f(x_2) &= -\sqrt[4]{\tfrac{2q}{5}}^5 - 2q(-\sqrt[4]{\tfrac{2q}{5}}) - q \\
&= \tfrac{8q}{5}\sqrt[4]{\tfrac{2q}{5}} - q \qquad > 0
\end{aligned}$$

for $q \geq 2$. Consequently, three of the roots must be real.[7] However, this means that the equation is not solvable by the means of radicals, according to theorem 2.13. Thus, the only quintic equations that may be solved by the means of radicals indeed has either one real and four imaginary roots or five real roots. Therefore, it has been proven that the general quintic equation is unsolvable by the means of radicals.

[7] Intermediate value theorem

5 Aftermath

In this paper I have made an effort to stay as close to the original solutions to the equations as possible. Today, there are a multitude of different ways to solve equations up to the quartic, and it is usually the form of the equation that decides which route to take. The same goes for showing how the quintic equation is impossible to solve with radicals. Another person who showed the same thing as Abel in the same decade, is Galois. He wrote an article in 1830 called "On the condition that an equation be soluble by radicals", which not only considered the quintic equation, but all equations of higher degrees. From his work group theory and especially Galois theory sprung forth, which is the study of certain groups that may be associated with equations. (O'Connor & Robertson, 1996) Many have since expanded upon the work of Abel and Galois, and group theory is now an important element in higher algebra, and especially in equation theory.

6 References

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[8]Can be previewed at `https://books.google.se/books?id=UdGBy8iLpocC`