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Perfect squared squares

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Abstract

A perfect squared square is a tilling of a square by smaller squares, all having different sizes. To find such tillings an analogy with electrical networks is used. In particular one has to study Laplacians on graphs. Therefore some important definitions and theorems from graph theory and for electrical networks are introduced, illustrated by several examples. Different types of squared squares are discussed including problems in higher dimensions.

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Contents

1 Introduction

Imagine having a paper with a drawn square in front of you, if someone asks you to divide this square into smaller squares it is easily done by drawing two lines which divide the square in 4 smaller squares all of the same size. What if you get told that no two of the squares could be of the same size? The problem now gets a lot harder and at first you would probably think it is impossible. The problem of tilling a square into smaller squares all of different sizes is solvable and this is what we call a perfect squared square (PSS). This work will explain how it is done and which methods are used to find perfect squared squares.

1.1 History

In 1902 Henry Dudeney, an English puzzler and writer of recreational mathematics, published *"Lady Isabel's Casket"* in the London Magazine. *"Lady Isabel's Casket"* was a puzzle which concerns the problem of dividing a square into squares of different sizes and one rectangle. This is the earliest reference dealing with the problem of tilling a square into squares.[1]

The first perfect squared square was found in 1939 by the German mathematician Roland Sprague, the square tiling contained 55 squares and had a side of 4205. But the problem of finding a perfect squared square was known earlier. Four students at Trinity College, Cambridge named Rowland Leonard Brooks, Cedric Smith, Arthur Harold Stone and William Thomas Tutte studied this between 1936 and 1938. Before starting searching for perfect squared squares they tried to find perfect squared rectangles and it didn't take long before they found some. They tried to find a way to represent squared rectangles by diagrams of different kinds. Smith was the one that introduced a diagram that was a big step forward, therefore called the Smith diagram. The Smith diagram reduced the problem of squaring a rectangle to the theory of electrical networks. Now when they had found a connection between the problem of squaring a rectangle and electrical circuits this could also be used to find perfect squared squares. Using this method the first perfect squared square found was of order 69, [1].

Even though the four students found a way to search for perfect squared squares the method required a lot of computations and it was not an efficient way to find such squares. In the 70s it was possible to let computers do the computations and all of a sudden their method became computationally efficient. During this time a large number of squared squares were found. Today all possible perfect squared squares of orders 21 to 35 are found; it seems like the number of perfect squared squares grows exponentially with the order. It is also known that there are no perfect squared squares of order less than 21, [1].

It is natural to wonder whether this is possible to do in higher dimension? For example in dimension 3, the problem would then be to find a tilling of a cube into smaller cubes all of different sizes. This is not possible and means that the perfect squared square problem is unique. Why this is impossible in three dimensions will be discussed in section 4.7.

2 Graph theory

Before solving the problem of perfect squared squares we need to start with some basic definitions and theorems in graph theory and electrical networks.

Graph theory is the study of graphs, a graph is a mathematical structure built up of vertices and edges and is used to model relations between objects. Unlike most areas in mathematics graph theory has a definite starting place and point, the Swiss mathematician Leonard Euler (1707-1783) published a paper in 1736, which is regarded as the beginning of graph theory. At this time "the seven bridges of Königsberg" was a well-known problem. While solving this problem Euler developed what we today consider to be the fundamental concepts of graph theory.

Definition 1. *Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a directed graph (on V), where V is the set of vertices, or nodes, and E is its set of (directed) edges or arcs. We write G=(V,E) to denote such a graph.*

When we are not interested in direction of edges, we still write G=(V,E). But now E is a subset of unordered pairs of elements of V, and G is called an undirected graph. [2]

Independent of whether $G = (V, E)$ is directed or undirected, we call V the *vertex set* of G and E the *edge set* of G. Throughout this text we will use n and m to denote the number of vertices, $|V| = n$ and the number of edges $|E| = m$. An edge $\{x, y\}$ is said to join the vertices x and y and is denoted by xy, the vertices x and y are called the endvertices of this edge, we say that x and y are adjacent vertices of G and that the vertices x and y are incident with the edge xy. It is possible to assign each edge a number, which we call the *weight* of the edge. Graphs with weighted edges are called *weighted graphs*.

It is convenient to use pictures to describe small graphs, let us look at an example of a graph.

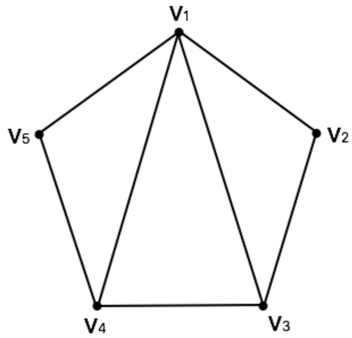


Figure 1: A graph G_1

Example 1. In Figure 1 we see the undirected graph G_1 with vertices v_1, v_2, v_3, v_4, v_5 and edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_3, v_1v_4$ and v_1v_5. The graph is loop-free, that is, no edge connects a vertex to itself.

Definition 2. *If $G = (V, E)$ is a graph (directed or undirected), then $G_1 = (V_1, E_1)$ is called a subgraph of G if $\emptyset \neq V_1 \subseteq V$ and $E_1 \subseteq E$, where each edge in E_1 is incident with vertices in V_1, [2].*

Definition 3. *A path is a sequence of vertices $v_{n_1}, v_{n_2}, ..., v_{n_l}$ such that there exist edges between any two consecutive vertices, $e_m = v_{n_j}v_{n_{j+1}}$. Then the path is the union of these edges. A cycle is a closed path, that is the first and last vertex of the path is the same. [2]*

Definition 4. *Let $G = (V, E)$ be a loop-free undirected graph. The graph G is called a tree if G is connected and contains no cycles. We denote a tree by T. A disjoint union of trees is called a forest, denoted by F, [2].*

Consider a weighted graph describing all roads from one city to another city. If you want to find the shortest way between the two cities it can be described as the problem of finding a path of minimal weight between the two vertices representing the two cities. Finding a minimum path to connect all vertices is an important problem in network planning, minimal spanning trees are used to solve this problem. We will later on discuss how spanning trees are used to find an upper bound for the size of a perfect squared square.

Definition 5. *A spanning tree T for a connected graph G is a tree which contains all the vertices of G. If $|v| = n$, then a spanning tree contains $n - 1$ edges, [2].*

In general, a graph may have many spanning trees. In weighted graphs one consider the minimal spanning tree a spanning tree with minimal total weight compared to all other spanning trees of the same graph. In real world situations this weight can measure distance, congestion, traffic load or any other value assigned to the edges.

For large graphs it is not always easy or efficient to draw them on paper so we need another way to describe them, which is done using matrices. Let us first go through some important definitions and then look at examples.

Definition 6. *An adjacency matrix A of a graph G, denoted by $A(G) = (a_{ij})$, is the $n \times n$ matrix given by :*

$$a_{ij} = \begin{cases} 1 \; \textit{if } v_i v_j \in E(G) \\ 0 \; \textit{otherwise.} \end{cases}$$

[3]

Definition 7. *The degree or valency of a vertex v is the number of edges that are incident to v; loops are counted twice. We denote the degree of vertex v by $deg(v)$.*

Definition 8. *Let $D = (D_{ij})$ be the diagonal $(n \times n)$ matrix with $D_{ij} = d(v_i)$, the degree of v_i in G. We call D the degree matrix,[3].*

Definition 9. *With A and D as above, the graph's Laplacian L, is defined as $L = D - A$, [3].*

(a) Undirected graph G_2

(b) Directed graph G_2

Figure 2: Undirected and directed graphs

[3]

Example 2. Given the graph G_2 in Figure 2(a) with vertex set $V = \{v_1, v_2, v_3, v_4\}$ the adjacency matrix is given by:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The *adjacency matrix* A shows that vertex 1 is connected to all the other vertices, vertex 2 is connected to vertex 1 and 4, vertex 3 is connected to vertex 1 and 4 and vertex 4 is also connected to all other vertices.

The *degree matrix* D given by the graph G_2 in Figure 2(a) is:

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The *degree matrix* D is determined by the degrees of each vertex. We can see that $d(1) = 3, d(2) = 2, d(3) = 2, d(4) = 3$.

With matrix A and B we can calculate the graphs Laplacian L.

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Definition 10. *An incidence matrix B of a graph G, denoted $B(G) = (b_{ij})$ is the $n \times m$ matrix defined by:*

$$b_{ij} = \begin{cases} 1, \textit{ if } v_i \textit{ is the initial vertex of the edge } e_j, \\ -1. \textit{ if } v_i \textit{ is the terminal vertex of the edge } e_j, \\ 0, \textit{ otherwise.} \end{cases} \quad [3]$$

Theorem 1. *$BB^t = D - A$, where B^t is the transpose of B, [3].*

Proof. $(BB^t)_{ij}$ is given by the sum $\sum_{i=1}^{m} b_{il}b_{jl}$. Let $i = j$ then $(BB^t)_{ii}$ is the degree of the vertex $d(v_i)$. The diagonal element b_{ii} is determined by vector multiplication with the row vector corresponding to vertex v_i in B and the column vector corresponding to vertex v_i in B^t. If the edge e_l and vertex v_i are incident then the l:th element in the vectors is either 1 or -1 and the rest of the elements in the vectors are 0. The vector multiplication gives the sum of squares of all vector elements. Hence this sum will give the number of edges incident with vertex v_i which is exactly the degree of the vertex.

If v_iv_j is an edge then $(BB^t)_{ij} = -1$ and if v_iv_j is not an edge and $i \neq j$ then $(BB^t)_{ij} = 0$. If v_iv_j is an edge then the vectors in the matrices B and B^t corresponding to the vertices v_i and v_j will both have one nonzero element at the same place, this element corresponds to edge e_l. All other elements in the vectors are zero since we are working with graphs without multiple edges. If e_l is the edge from v_i to v_j then $b_{il}b_{jl} = (-1) \cdot 1$ and if the edge is directed in the opposite direction then $b_{il}b_{jl} = 1 \cdot (-1)$. Therefore $(BB^t)_{ij} = -1$ if v_iv_j is an edge.

If there is no edge between two vertices v_i and v_j the vectors corresponding to these vertices have no entry in common and the vector multiplication sums to 0.

It is now clear that BB^t is the same as the diagonal matrix D minus the adjacency matrix A, [3]. $\square$

Example 3. With the graph G_2 from Figure 2(b), here is an example of the matrices B, B^t and BB^t

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B^t} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{BB^t} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

As stated in Theorem 1 we can see that $BB^T = D - A = L$.

We have now to state Kirchhoff's matrix-tree theorem which will be of great importance later when we will be studying squared squares.

Theorem 2 (Kirchhoff's matrix-tree theorem)**.** *If $G(V, E)$ is an undirected graph and L is its graph's Laplacian, then the number N_T of spanning trees contained in G is given as follows.*

1. Choose a vertex v_j and eliminate the $j-th$ row and column from L to get a new matrix $\hat{L}_j$;
2. Compute $N_T = det(\hat{L}_j)$. *[3]*

To prove Kirchhoff's matrix-tree theorem we first need the Binet-Cauchy theorem and a lemma.

Definition 11. *Given a $(n \times m)$ matrix $A = (a_{ij})$ with $n \leq m$ and a n-element subset S of $\{1, 2, ..., m\}$, let A[S] denote the $(n \times n)$-submatrix of A obtained by taking the columns indexed by the elements of S. If we have a $(m \times n)$ matrix $A^t = (a_{ji})$ with $n \leq m$ and a n-element subset S of $\{1, 2, ..., m\}$ then let A^t[S] denote the $(n \times n)$-submatrix of A^t obtained by taking the rows indexed by the elements of S. [4]*

Example 4. Using B and B^t from Example 3 and the subset $S = \{1, 2, 3, 4\}$ we get:

$$\mathbf{B}[\mathbf{S}] = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{B^t}[\mathbf{S}] = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

If we take the transpose of $B[S]$ we can see that $B^t[S] = B[S]^t$, which is true in general. For any matrix A, as in Definition 11, its transpose A^t and a set S we have $A^t[S] = A[S]^t$.

Theorem 3 (Binet-Cauchy theorem). *Let $A = (a_{ij})$ be an $n \times m$ matrix, with $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $B = (b_{ij})$ be an $m \times n$ matrix with $1 \leq i \leq m$ and $1 \leq j \leq n$, thus AB is an $n \times n$ matrix. If $n > m$ then det(AB)=0 and if $n \leq m$, then*

$$det(AB) = \sum_S (det(A[S]))(det(B[S]))$$

where S ranges over all n-element subsets of $\{1, 2, ..., m\}$. [4]

Lemma 1. *Let S be a set of n-1 edges of G. If S does not form the set of edges of a spanning tree, then $det(\hat{B}_j[S]) = 0$. If, on the other hand, S is the set of edges of a spanning tree in G, then $det(\hat{B}_j[S]) = \pm 1$. By $\hat{B}_j$ we mean the matrix B with the j-th row deleted. Later when we write $\hat{B}_j^t$ we mean the matrix B^t with the j-th column deleted. [4]*

Proof. If S does not form a set of edges of a spanning tree then S will contain at least one cycle. Since the edges are directed one edge in the cycle can be described as a linear combination of the other edges. In the matrix the edges correspond to the column vectors and this means that the vectors are linearly dependent. If vectors in a matrix are linearly dependent the determinant of the matrix is zero. This is why the determinant of $\hat{B}_j[S])$ vanishes if S does not form a set of a spanning tree.

If B is the incidence matrix of a graph G, its columns is represented by *G:s* edges and the rows are represented by *G:s* vertices. Therefore every column will contain exactly two non-zero elements (representing the initial vertex and the end vertex of this edge). Also every row contains at least one non-zero element since all vertices are incident to at least one edge. We obtain $\hat{B}_j$ by deleting row j, therefore at least one column in $\hat{B}_j$ will contain only one non-zero element, the other non-zero element is deleted when we delete row j. If S is the set of edges of a spanning tree then $\hat{B}_j[S]$ will also contain at least one column with only one non-zero element, if it does not then the set of edges forms a cycle and the determinant of $\hat{B}_j[S]$ is equal to zero as showed above. When calculating the determinant of a matrix we can add a multiple of a row or column to another without changing the determinant. Hence by adding a multiple of the column with only one non-zero element we can delete all other elements in the same row as the non-zero element. By doing this we get at least one new column with only one non-zero element and the process can be repeated until all columns only contain one non-zero element. Now each row and column contains exactly one non-zero element and with Laplace expansion we can expand the determinant along these non-zero elements. The determinant equals the product of these non-zero elements. In our case all non-zero elements are ± 1, hence the determinant of $\hat{B}_j[S] = \pm 1$ if S forms the set of a spanning tree.

$\square$

Proof - Kirchhoff's matrix-tree theorem. We know that $L = BB^t$, therefore $\hat{L}_j = \hat{B}_j \hat{B}_j^t$. Now by the Binet-Cauchy theorem

$$det(\hat{L}_j) = \sum_S (det(\hat{B}_j[S]))(det(\hat{B}_j{}^t[S])), (1)$$

where S range over all $(n-1)$-element subsets of the set of edges of G. Since in general, $A^t[S] = A[S]^t$ equation (1) can be rewritten:

$$det(\hat{L}_j) = \sum_S (det(\hat{B}_j[S]))(det(\hat{B}_j[S]^t)).$$

Since the determinant of a matrix and the determinant of its transpose coincide we get:

$$det(\hat{L}_j) = \sum_S (det(\hat{B}_j[S]))^2.$$

According to Lemma 1, $det(\hat{B}_j[S]) = \pm 1$ if S forms the set of edges of a spanning tree of G and 0 otherwise. Therefore $(det(\hat{B}_j[S]))^2$ is 1 if S forms the set of edges of a spanning tree of G, and 0 otherwise. Hence the sum is exactly the number of spanning trees in G, as we wanted to prove, [4]. $\square$

Example 5. Using Kirchhoff's matrix-tree theorem and the graph G_2 from Figure 2 we can calculate how many spanning trees it contains. We have already calculated the matrix L in the above example, namely

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{B}\mathbf{B}^\mathbf{T} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

The next step is to choose a vertex v_j and eliminate the j-th row and the j-th column from L. Let's choose vertex 2. Then

$$\hat{\mathbf{L}}_\mathbf{2} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

The number of spanning trees is now the value of the determinant of $\hat{L}_2$ which equals

$\text{Det}(\hat{L}_2) = 3 \cdot ((2 \cdot 3) - (-1 \cdot -1)) - (-1) \cdot ((-1 \cdot 3) - (-1 \cdot -1)) + (-1) \cdot ((-1 \cdot -1) - (-1 \cdot 2)) = 15 - 4 - 3 = 8.$

Figure 3 shows the 8 spanning trees in the graph G_2.

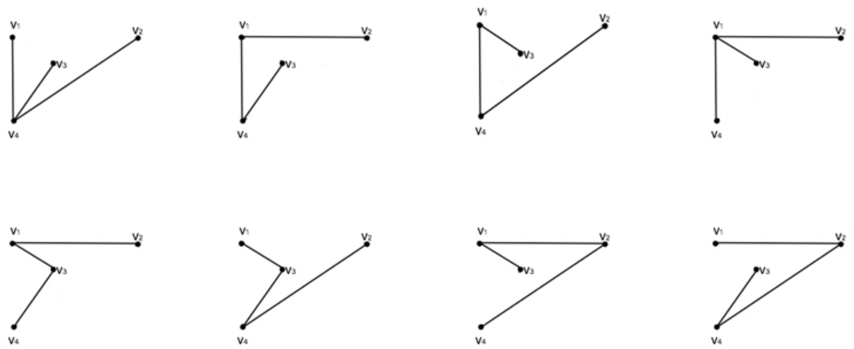


Figure 3: The eight spanning trees of G_2

We are now ready to discuss electrical networks.

3 Electrical networks

An electrical network is an interconnection of electrical components, in our concern it is a model of such an interconnection consisting of electrical elements e.g. currentsources and resistance. We will be studying the current distribution in electrical networks obeying Ohm's law, Kirchhoff's potential law and Kirchhoff's current law. We will see that electrical networks can be described as graphs and study how they can help us to solve the problem of tilling a square.

Definition 12. *A simple electrical network can be regarded as a graph in which each edge e_i has been assigned a real number r_i called its resistance. If a potential difference p_i is applied to the endvertices a_i and b_i, of e_i, then an electrical current w_i in the edge e_i from a_i to b_i will satisfy Ohm's law: $w_i = p_i/r_i$.*

We are interested in solving the problem of perfect squared squares, hence we will restrict our attention to electrical networks corresponding to graphs. In our situation multiple edges are desirable, therefore we will be working with *multigraphs* instead of graphs. The *multigraphs* will be directed and we will by p_i in an edge e_i denote the difference between the potentials of the *initial vertex* and the *endvertex*. Analogously the positive current w_i in the edge e_i is the current flowing in the same directions as the edge. Negative currents mean that the current flows in the direction opposite to the edge orientation.

The electrical current enters the network at one vertex and leaves at another vertex. It is important to understand how the currents and potential differences in the edges affect the total flow in the electrical network. This was essentially done by Gustav Kirchhoff (1824-87), a German physicist who contributed to the fundamental understanding of electrical networks. Two of his most important discoveries are the laws stated below.

Law 1 (Kirchhoff's potential law)**.** *The potential differences along any cycle $x_1, x_2, .., x_k$ sums to 0.*

$$p_{x_1x_2} + p_{x_2x_3} + \cdots + p_{x_{k-1}x_k} + p_{x_kx_1} = 0$$

[3]

Law 2 (Kirchhoff's current law)**.** *The total current outflow from any vertex is 0.*

$$\textstyle\sum_{x_j \sim x_i} w_{x_jx_i} = 0 \ \forall x_i.$$

where $x_j \sim x_i$ means that x_j and x_i are connected with an edge. *[3]*

These two laws allows one to calculate currents in any resistance network.

The networks under consideration have a *source*, s, and a *sink*, t. Currents are only allowed to enter the network at the vertex s and leave the network at the vertex t. If the total current from s to t is w and the potential difference between s and t is p, then according to Ohm's law the total resistance of the network between s and t is given by $r = p/w$.

The total resistance of a network depends on how the resistors are connected. They can be connected in parallel, in series or in any other combination. If we have two resistors r_1 and r_2 and the electrical network is connected in series the total resistance adds together while if it is connected in parallel the total resistance, r, is equal to $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Figure 4 below shows two resistors connected in parallel and series.

(a) Connected in parallel (b) Connected in series

Figure 4: Connected resistors

Example 6. With the electrical network from Figure 4 here is an example on how to calculate the total resistance of the network. In this example each edge have been assigned resistance 1.

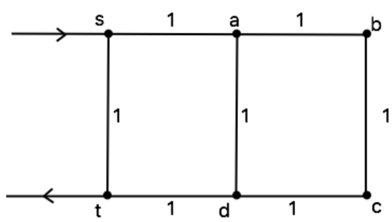


Figure 5: Electrical network

The resistors on ab, bc and cd are connected in series; we know that the total resistance add together, hence we can delete these edges and make a new one with resistance 1+1+1=3.

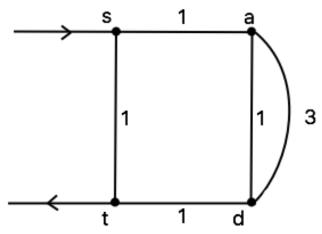


Figure 6: Electrical network

There are now two resistors connected in parallel. The total resistance from a to d is then equal to $\frac{1}{r} = \frac{1}{1} + \frac{1}{3} = \frac{4}{3} \implies r = \frac{3}{4}$.

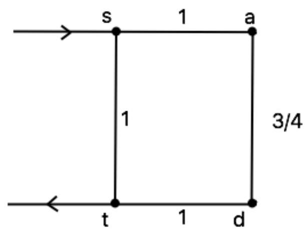


Figure 7: Electrical network

The resistors along sa, ad and dt are connected in series so they add together, the total resistance being $1 + \frac{3}{4} + 1 = \frac{11}{4}$.

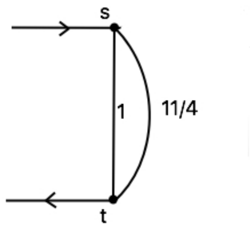


Figure 8: Electrical network

The last step to calculate the total resistance of the network gives us $\frac{1}{r} = \frac{1}{1} + \frac{4}{11} = \frac{15}{11} \implies r = \frac{11}{15}$.

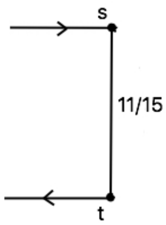


Figure 9: Equivalent electrical network

Theorem 4. *Let G be a resistance network with resistance 1 of every edge. Then the current through each edge ab is given by $W_{ab} = N(s,a,b,t) - N(s,b,a,t)/N$ where N(s, a, b, t) is the number of spanning trees of G in which the (unique) path from s to t contains the edge ab, in this order. Define N(s, b, a, t) analogously and write N for the total number of spanning trees. Then there is a total current of value 1 from s to t satisfying the Kirchhoff laws. [3]*

Proof. To simplify the proof, multiply each current, w_{ab}, with the total number of spanning trees, N. Let $w^{(T)}$ be the current of size 1 along the unique s-t path in T for every spanning tree T and the edge $ab \in E(G)$

$$w_{ab}^{(T)} = \begin{cases} 1, \text{ if } T \text{ has a path } a \cdots ab \cdots t \\ -1, \text{ if } T \text{ has a path } a \cdots ba \cdots t \\ 0, \text{ otherwise.} \end{cases} \quad [3]$$

Then, $N(s,a,b,t) - N(s,b,a,t) = \sum_T w_{ab}^{(T)}$.

Now we need to show that if we send a current of size $\sum_T w_{ab}^{(T)}$ from a to b for every edge ab, we will obtain a total current of size N from s to t satisfying Kirchhoff's laws. Each $w^{(T)}$ is a current of size 1 from s to t satisfying Kirchhoff's current law, therefore the sum of all currents $w^{(T)}$ for every spanning tree is of size N from s to t satisfying Kirchhoff's current law.

The last part of the proof is to show that Kirchhoff's potential law is satisfied. Since we are working with networks where all edges have the same resistance, the potential law claims that the total current in a cycle is zero.

We will slightly reformulate the definition of $N(s,a,b,t)$ to show this. If we have a spanning forest F in G we will call it a *thicket* if it has exactly two components, F_s and F_t such that s is in F_s and t is in F_t. Now $N(s,a,b,t)$ is the number of thickets $F = F_s \cup F_t$ for which $a \in F_s$ and $b \in F_t$. $N(s,b,a,t)$ is the number of thickets $F = F_s \cup F_t$ for which $b \in F_s$ and $a \in F_t$. The contribution of a thicket F to the total current in a cycle is the number of cycle edges from F_s to F_t minus the number of cycle edges from F_t to F_s, since there are as many cycle edges from F_s to F_t as there are cycle edges from F_t to F_s the contribution of a thicket F to the total current in a cycle is 0, and Kirchhoff's potential law is satisfied, [3]. $\square$

Theorem 4 will be useful in the search for perfect squared squares.

Let us now use the theory of graphs and electrical networks that have been introduced to solve the actual problem of tilling a square.

4 Squared Squares

With some understanding of graphs and electrical networks it is now time to look closer into what a perfect squared square is and how it can be turned into an electrical network.

There are 4 different kinds of squared squared, Simple Perfect Squared Squares namely (SPSS), Simple Imperfect Squared Squares called (SISS), Compound Perfect Squared Squared (CPSS) and Mrs Perkin's Quilts.

Definition 13. *Squared squares are called simple if they do not contain a smaller squared square and compound otherwise. They are called perfect if the squares of the tilling are all of different sizes and imperfect if they are not. Mrs Perkins Quilt include SPSS, SISS, CPSS and also squares that are imperfect and compound.*

The order of a Perfect Squared Square is the number of squares in the tilling and the size of a Perfect Squared Square is the side length of the outer square.

4.1 Smith's diagrams

As mentioned in section 1.1 Cederic Smith came up with an idea to describe a perfect squared square as an electrical network. Each horizontal line is considered as a vertex. Two such vertices are connected if and only if there is a square between the horizontal lines. The current flowing in each edge is the same as the height of the corresponding square. The topside of the perfect squared square corresponds to the source and the bottom side corresponds to the sink, [5].

We illustrate this in Figure 10 where a perfect squared rectangle of order 9 is used. The same idea could be used for perfect squared squares but since the lowest ordered squared square is of order 21 it is easier to illustrate the idea with a squared rectangle.

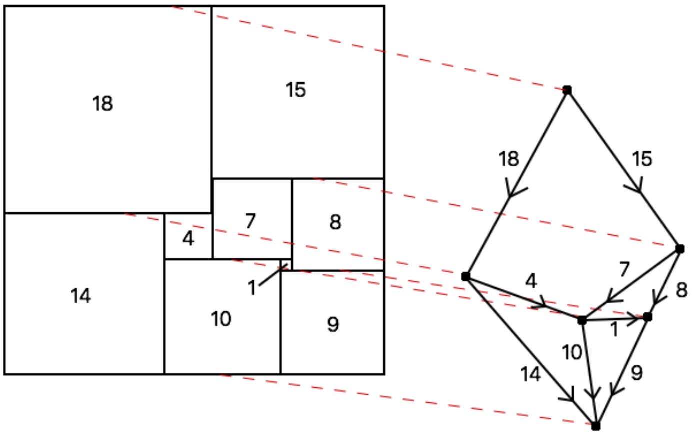


Figure 10: Smith diagram

4.2 Physics entries into the game

We now know that it is possible to describe a perfect squared square as an electrical network but we haven't discussed why it works yet. This section will describe why it is possible to do this and how it is possible to figure out whether an electrical network corresponds to a perfect squared square or not.

Cut a square out of a sheet of a material with low conductivity and put rods made of a material with high conductivity at top and bottom of the square. If we make sure that the rod at the top is at n volts, where n is the side length of the square and the rod at the bottom is kept at 0 volt there will be a uniform current flowing from top to bottom. The potential at a point of the square will only depend on the distance to the top rod, in fact: the potential at a point x where x is the distance to the top rod will be (n-x) volts. The current will be flowing from top to bottom, not across the square, therefore the current will not change if we put rods on the horizontal sides and cut narrow slits along the vertical sides of the squares in the tilling, [3].

Now since we have used rods with high conductivity the points of each rod have been shortened so they can be identified. By doing this we concentrate the electrical current to only flow in the rods. Thus the whole square behaves like a plane network as an electric conductor (the Smith diagram corresponding to the PSS). The conductance of an edge is equal to the conductance of the corresponding square from top to bottom, [3].

The conductance of a square from top to bottom is proportional to the horizontal side and the resistance is proportional to the vertical side, thus all squares have the same resistance and therefore all edges in the network have unit resistance, [3].

The potential drop in an edge equals the side length of the corresponding square, and the resistance of the whole system is the ratio of the vertical side of the outer square to the horizontal side. Since we have a square the resistance of the whole network is 1, [3].

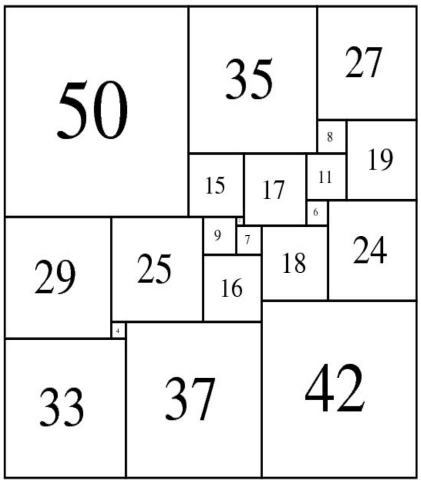
It is clear that a squared square can be turned into an electrical network, although if we find a network we may not find a squared square. Nevertheless it is enough to look for networks to find squared squares. After finding a network we can check if it corresponds to a squared square or not. What type of electrical networks are we looking for?

If we turn a connected planar graph G into an electrical network by giving each edge resistance 1 and the total resistance from the source to the sink is also 1 then the network may corresponds to a squared square. If the potential differences in every edge are all distinct then all squares in the tilling are of different sizes and we have found a perfect squared square. [3]

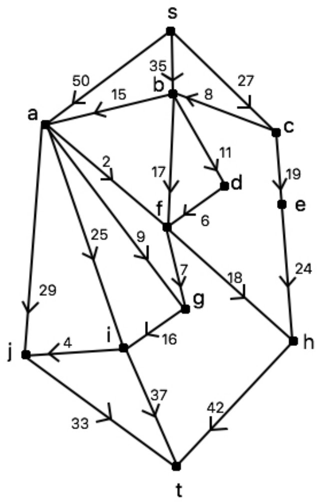
This gives us an effective way to search for perfect squared squares by looking at graphs corresponding to electrical networks.

Example 7. Let's look at the Smith diagram for the Simple Perfect Squared Square of order 21, see Figure 11. We see that the potential differences along any cycle sums to 0 in accordance with Law 1 (Kirchhoff's potential law). E.g consider the cycle $bcehfdb$, the potential differences along this cycle is $-8+19+24-18-6-11=0$ or the cycle $bfab$ with the potential differences $17-2-15=0$.

We also see that the total current outflow from any point is zero in accordance with Law 2 (Kirchhoff's current law). E.g take the vertex g, the current flowing into g is $7+9=16$ but there is also a current flowing out from g of the same size, 16, hence the total current outflow from g is zero.



(a) Simple Perfect Squared Square of order 21



(b) Smith Diagram

Figure 11: Simple Perfect Squared Square with its Smith diagram

[3]

4.3 Rational and irrational perfect squared squares

The following question is natural: Are all squares in a tilling always of rational size? We have already discussed that a perfect squared square can be described as an electrical network with resistance 1. The current flowing in the network is given by Kirchhoff's laws, to calculate it we only use addition, subtraction, multiplication, and division. Rational numbers are closed under these operations so if we choose one square, or equally the current flowing in the edge corresponding to the square in the network, to be a rational number then all other squares will be of rational size. We can change the scale of a perfect squared square by an irrational number but then all squares in the tilling will be of irrational size. This concludes that there are no perfect squared squares with both rational and irrational sized squares in the same tilling.

4.4 Bouwkamp code

To describe squared squares and squared rectangles we use the so-called Bouwcamp code. In this notation we use brackets to group squares. Starting from the upper-left corner the first bracket will contain all the squares connected to the upper edge of the squared square from left to right. For the second bracket, we start with the highest and leftmost square that haven't already been used, followed by adjacent squares with flush tops. The squares are then grouped in the same way, and the groups are sequentially placed in the highest and leftmost possible slot. For example, the Bouwcamp code for the 21-square illustrated in Figure 12 is: $[50, 35, 27], [8, 19], [15, 17, 11], [6, 24], [29, 25, 9, 2], [7, 18], [16], [42], [4, 37], [33]$. [6]

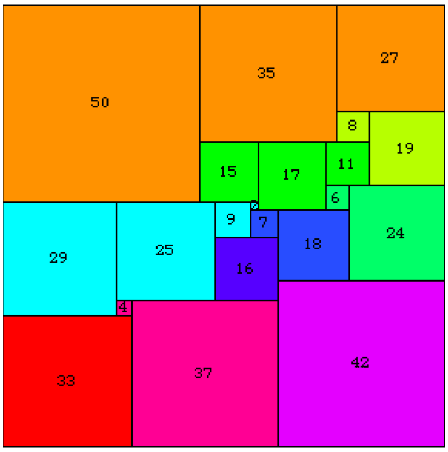
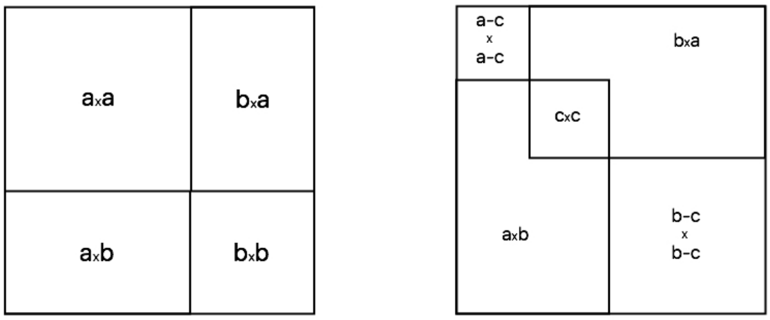


Figure 12: Bouwcamp code [6]

4.5 Constructing a SPSS from a SPSR

When Brooks, Smith, Stone and Tutte first started studying perfect squared squares they found out that it was much easier to find perfect squared rectangles. Additionally, they realised that it was possible to construct a perfect squared square out of two perfect squared rectangles of the same size and with none (or one) inner squares in common. Namely, if we have two perfect squared rectangles of the same size and with no squares in common we can construct a simple perfect squared square by placing one of the rectangles in the lower-left corner and take the other rectangle, rotate it 90 degrees and place it in the upper-right corner. These two rectangles together with two squares (in the upper-left corner and lower-right corner) will form a simple perfect squared square. This is illustrated in Figure 13(a). If we have two squared rectangles of the same size, but with one common corner square we can construct a simple perfect squared square by letting the rectangles overlap in the common corner square and place two squares in the upper-left corner and lower-right corner. This is illustrated in Figure 13(b). [5]



(a) Perfect squared square 1 (b) Perfect squared square 2

Figure 13: Construction of perfect squares

4.6 Special PSS

4.6.1 Lowest order SPSS

Finding the lowest order of a Simple Perfect Squared Square was an unsolved problem for a long time, but on the 25th of April 1978 A.J.W Duijevestin published an article claiming that he had found the lowest order SPSS; this square was of order 21. Before this the lowest order SPSS known was of order 25 but with help of the DEC-10 computer Duijevestin checked all graphs that could represent a perfect squared square of order less than 21 and none was found. He also found out that there is only one SPSS of order 21, see Figure 11(a). [8]

4.6.2 Smallest size PSS

A common misconception is that the lowest order SPSS is also the smallest size PSS, but this is not the case. The lowest order SPSS is of size 112, but there are actually 3 different PSS of size 110 (2 of order 22 and one of order 23) and this is the smallest size of a perfect squared square.

We know that a perfect squared square of higher order can have a smaller size than a perfect squared square of lower order. So how can we be sure that 110 is the smallest size for a PSS? Is it not possible that there is a PSS of high order that still haven't been found with a smaller size than 110? It turns out that it is impossible. There are several proofs that 110 is smallest size for a PSS. See below. [7]

Theorem 5. *The smallest sized PSS is of size 110.*

Proof. To get a PSS of the smallest size we need to minimize the size of the squares in the dissection, minimize the difference of the sizes between any two squares and make sure that every square in the tilling is of different size.

If we use consecutive integers starting from 1 to describe the squares in the tilling it will satisfy the conditions above. The side of the PSS can not be smaller than the square root of the sum of the consecutive integers. Denote the side of the PSS by S. Using the sum of squares formula and letting it be equal to the total area of the PSS we get:

$$\sum_{k=1}^{n} k^2 = \tfrac{n\cdot(n+1)\cdot(2n+1)}{6} = S^2.$$

This gives us an upper bound for the size of PSS by order n. Set n=33 and we get :

$$\sum_{k=1}^{33} k^2 = \tfrac{33\cdot(33+1)\cdot(233+1)}{6} = 12529 = S^2$$

Hence $S \approx 111.93$ and a PSS with side length less than 110 can't have more than 32 squares.

Since we know that there are no PSS of order less than 21 we also know that if there is a PSS of smaller size than 110 it has to have the order 21 to 32. In September 2013 all PSSs of order 21 to 32 were completely enumerated by Lorenz Milla and Stuart Andersson and none of their squares are of a smaller size than the three PSS of size 110. Therefore it is now proved that these three squares are the smallest possible PSSs. [7]

□

4.6.3 Largest size PSS by order

As the order of PSS increases so does the element sizes and the possible sizes of the PSS. The graph from which PSSs can be derived contains as many edges as there are squares in the tilling of the PSS. This means that at the same time the order of a PSS increases so does the number of edges in the graph - and also the number of spanning trees of the graph. There are finitely many PSSs in a given order and therefore there will always be a largest size PSS in a given order.

It is shown that an upper bound for PSS size by order can be obtained from the number of spanning trees the graph holds. Theorem 2 (Kirchhoff's matrix-tree theorem) from section 2 gives us an efficient way to calculate the number of spanning trees in a graph. Hence we get an upper bound for a PSS by finding the number of spanning trees in the corresponding graph. [7]

4.7 Cubed Cubes

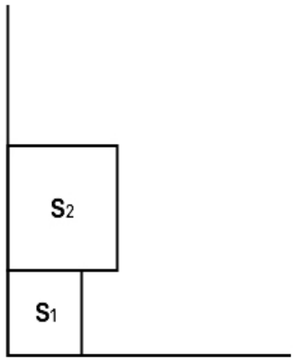
The problem of cubing the cube is the natural analog of squaring the square in three dimensions. Given the cube C, we want to divide it into finitely many smaller cubes, with no cubes of the same volume.

Unlike the problem of squaring a square there is no solution to this problem, a perfect cubed cube does not exist.

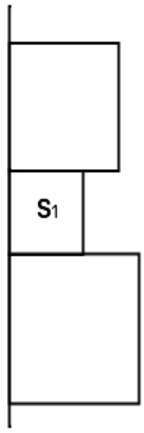
To prove that there are no cubed cubes we first need to understand that the smallest square in a squared square can not be an edge square.

Assume we have a corner square of arbitrary size in the lower-left corner. Let us call it s_1, put a bigger square on top of it and call this square s_2. Since s_2's side is longer than s_1's it will form a space to the right of s_1 that need to be filled with a square smaller than s_1, hence the smallest square can not be a corner square. See Figure 14(a).

Now name a square that lies on the left edge of the outer square but not in the corner s_1. If s_1 were to be the smallest square in the dissection then the space above and below s_1 need to be filled with bigger squares, but this will form a space to the right of s_1 which need to be filled with a smaller square, hence the smallest squared square can not be an edge square. See Figure 14(b).



(a) Corner square



(b) Edge square

Figure 14: Simple Perfect Squared Square with Smith diagram

[3]

Proof. Claim: The smallest square in any perfect dissection of a square can not be an edge square.

Suppose that there is a perfect dissection of a cube C into smaller cubes. Then the base of the cube need to be a perfect squared square S. Let s_1 be the smallest square of S. The smallest cube c_1 of the first layer lies on s_1 which is surrounded by larger squares and therefore c_1 will be surrounded by higher cubes. The upper face of the cube c_1 is divided into a perfect squared square by the cubes which rest on it. Now let s_2 be the smallest square in this dissection with the corresponding cube c_2, by the claim this is surrounded on all 4 sides by squares which are larger than s_2 and therefor cubes higher than c_2.

This builds a infinite sequence of squares $s_1, s_2, s_3, ...$ and cubes $c_1, c_2, c_3, ...$ Hence the number of cubes is infinite and contradicts our original supposition that there is a perfect dissection of a cube C.

More generally, there is no solution to the problem in any dimension higher than 2. If there were to exist a perfect dissection of a 4-dimensional hypercube then its 'base' would be a perfect cubed cube, but we know this is impossible. [5]

□

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