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FI-Modules and Church's Theorem

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Abstract

In this master thesis we use the language of FI-modules to prove Church's theorem regarding cohomological stability of configuration spaces with coefficients in a Noetherian ring.

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Contents

1 Introduction

The goal of this thesis is to introduce FI-modules and highlight their connection to the notion of representation stability, which is a phenomenon concerning sequences of representations of symmetric groups. This is done in Chapter 2, where we prove some basic properties of FI-modules as well as some not so basic ones, such as the very handy Noetherian property these objects have. Then in Chapter 3 we will use this connection to prove Church's theorem regarding the cohomological stability of configuration spaces after we have recalled some facts about spectral sequences and homological algebra in general. We assume the reader is familiar with algebraic topology, homological algebra and representation theory. Some basic familiarity with abelian categories and Noetherian rings are also assumed.

We start by reviewing the notion of homological stability. Suppose we are given a sequence of topological spaces $\{X_i\}$ (or in some cases of groups) equipped with maps $\phi_i : X_i \to X_{i+1}$. The idea of homological stability is to see if for all $m \geq 0$ and for some coefficient ring R, the induced maps

$$(\phi_i)_* : H_m(X_i; R) \to H_m(X_{i+1}; R)$$

become isomorphisms for large enough $i = i(m)$. If that is the case we say that the sequence is *homologically stable* (over R). Consider the following example by Arnol'd:

Given a topological space X we can for any positive integer n define the *ordered configuration space of n points* (or simply the ordered n:th configuration space) *in X*,

$$C_n(X) := \{(x_1, ..., x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

This space carries an action of the symmetric group S_n which acts by permuting the coordinates, and the quotient space $B_n(X) := C_n(X)/S_n$ is called the *unordered configuration space of n points in X*. The natural map $C_n(X) \to B_n(X)$ is in fact a covering space projection.

If we let X be the complex plane $\mathbb{C}$ we have many inclusions $C_n(\mathbb{C}) \hookrightarrow C_{n+1}(\mathbb{C})$, for example the map $(z_1, ..., z_n) \mapsto (z_1, ..., z_n, \sup \Re(z_i) + 1)$, where $\Re(z)$ denotes the real part of z, and similarly for the unordered counterparts. Arnol'd showed in [1] that the spaces $B_n(\mathbb{C})$ are homologically stable over $\mathbb{Z}$, i.e. for all $m \geq 0$ the maps

$$H_m(B_n(\mathbb{C}); \mathbb{Z}) \to H_m(B_{n+1}(\mathbb{C}); \mathbb{Z})$$

all eventually become isomorphisms. They also showed that for $n \geq 3$

$$H_i(B_n(\mathbb{C}); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } i = 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

so homological stability holds in this case as well. However for the ordered configuration spaces we have

$$H_1(C_n(\mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}^{n(n-1)/2}$$

so stability fails in this case. There are many more cases where homological stability is known to hold, including sequences of mapping class groups of surfaces.

We can do the same thing for cohomology: given a sequence of spaces $\{X_i\}$ and maps $\psi_i : X_{i+1} \to X_i$ we can ask if for all $m \geq 0$ the maps

$$\psi_i^* : H^m(X_i; R) \to H^m(X_{i+1}; R)$$

become isomorphisms for large enough $i = i(m)$.

We have maps $C_{n+1}(X) \to C_n(X)$ for all n defined by forgetting the last point, and these maps induce homomorphisms

$$H^i(C_n(X)) \to H^i(C_{n+1}(X))$$

between the cohomology groups of the respective spaces. By dualizing the result for homology of $C_n(\mathbb{C})$ by Arnol'd in [1] we see that

$$\dim_{\mathbb{Q}} H^1(C_n(\mathbb{C}); \mathbb{Q}) = \frac{n(n-1)}{2}$$

and since the dimension grows with n the maps $H^1(C_n(\mathbb{C}); \mathbb{Q}) \to H^1(C_{n+1}(\mathbb{C}); \mathbb{Q})$ never become isomorphisms. However, the action of S_n on $C_n(X)$ induces an action on $H^*(C_n(X))$, so for a field k the sequence $\{H^*(C_n(X); k)\}$ is a sequence of S_n-representations. In general, if a group G acts on a space X we get for every $g \in G$ a map $\phi_g : X \to X$, and this gives an action $g \cdot v = \phi_{g^{-1}}^*(v)$ where $v \in H^*(C_n(X))$. The reason for the g^{-1} is to take into account the contravariance of $H^*(-)$.

In [3] the authors introduced the notion of representation stability which is something that applies to *consistent sequences* of S_n-representations.

Definition 1.1. Let $\{V_n\}$ be a sequence of S_n-representations together with linear S_n-equivariant maps $\varphi_n : V_n \to V_{n+1}$. In other words the maps φ_n are such that for all $\sigma \in S_n$ the following diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\varphi_n} & V_{n+1} \\ \downarrow{\sigma} & & \downarrow{\sigma} \\ V_n & \xrightarrow{\varphi_n} & V_{n+1} \end{array}$$

commutes, where the σ acts on V_{n+1} by viewing S_n as a subgroup of S_{n+1} under the standard inclusion. We call a sequence such as this a *consistent sequence*.

The representations V_n and V_{n+1} are representations of different groups, so we cannot ask the maps φ_n to become isomorphisms as representations for large enough n. However, after decomposing into irreducibles we can ask if the powers of these become independent of n. It is known (see for example [2]) that the irreducible S_n-representations is in a 1 to 1 correspondence with partitions of n. A partition of n is a sequence of positive nonzero integers $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_i \geq \lambda_{i+1}$ for $i = 1, ..., k-1$ such that $\lambda_1 + ... + \lambda_k = n$. We write $\lambda \vdash n$ to signify that the sequence λ is a partition of n. If $\lambda \vdash n$, then for any $m \geq n+\lambda_1$ we can define a partition $\lambda(m) \vdash m$ by

$$\lambda(m) := (m-n, \lambda_1, ..., \lambda_k).$$

Let V_λ be the irreducible S^n-representation corresponding to λ. Then we define for any $m \geq n+\lambda_1$ the S_m-representation

$$V(\lambda)_m := V_{\lambda(m)}.$$

Every irreducible representation of S_m is of this form for some unique λ. For example the trivial representation in V_n is $V(0)_n$ in this notation. Given a representation V_n of S_n, write $c_\lambda(V_n)$ for the multiplicity of $V(\lambda)_n$ in V_n.

Definition 1.2. We say that a consistent sequence of finite-dimensional S_n-representations $\{V_n\}$ is *uniformly representation stable with stable range* $n \geq N$ if for all $n \geq N$ the following conditions hold:

1. The maps $\varphi_n : V_n \to V_{n+1}$ are injective.
2. The representation V_{n+1} is spanned by the image $\varphi_n(V_n)$ as an S_{n+1}-module.
3. For each partition λ, the multiplicity $0 \leq c_\lambda(V_n) < \infty$ of the irreducible representation $V(\lambda)_n$ in V_n is independent of n for all $n \geq N$.

The second condition is essentially surjectivity of the maps, and the third condition is called *uniform multiplicity stability.*

In [6] the authors study consistent sequences of S_n-representations by using the language of FI-*modules*. An FI-module over a ring R is a functor from the category FI, whose objects are **F**inite sets and whose morphisms are **I**njections, to the category of R-modules. Such a functor gives rise to a family of R-modules linked together by a family of homomorphisms, and since the endomorphisms of a finite set N in FI can be viewed as the symmetric group $S_{|N|}$, each R-modules comes with an action of this group. Inside FI we have the sets of the form $[n] := \{1, 2, ..., n\}$, so each FI-module gives rise to a consistent sequence in the sense of the definition above whenever R is a field. The purpose of using FI-modules to study consistent sequences is that representation stability corresponds to a finite generation property which is much easier to conceptualize and work with.

We say that an FI-module V is generated by a the set $S \subset \coprod_{n\geq 0} V([n])$ if V is the smallest FI-module containing S, and it is finitely generated if S is finite. For example, let $v \in V([n])$ for some $n > 0$. Then the smallest FI-module containing v is the FI-module $\langle v \rangle$ taking a finite set N to $\langle v \rangle(N) = \text{span}\{f_*(v) \mid f : [n] \hookrightarrow N\}$.

The connection between representation stability of consistent sequences and finite generation of FI-modules lies in the following theorem (Theorem 1.13 in [6]):

Theorem 1.1. *Let V be an* FI*-module over a field k of characteristic* 0*. Then V is finitely generated if and only if the consistent sequence $\{V([n])\}$ of S_n-representations is representation stable.*

In the proof of this theorem we also obtain the result that for a finitely generated FI-module V, the consistent sequence $\{V_n\}$ is *monotone* as a byproduct. The notion of monotonicity was one of the key features of [3].

Definition 1.3. We say that a consistent sequence $\{V_n, \phi_n\}$ of S_n-representations is *monotone for $n \geq N$* if for any $n \geq N$ and for every subspace $W \subset V_n$ isomorphic to $V(\lambda)_n^{\oplus l}$, the S_{n+1}-span of $\phi_n(W)$ contains $V(\lambda)_{n+1}^{\oplus l}$ as a subrepresentation.

Similarly to FI-modules we can define other FI-objects as functors from FI to some category. Consider for example the FI-group $GL_\bullet(R)$ taking $[n]$ to $GL_n(R)$, the group of automorphisms of R^n. Injections $f : [n] \hookrightarrow [m]$ induces maps $f_* : GL_n(R) \to GL_m(R)$ defined by taking a matrix $M = (M_{ij})$ to

$$(f_*M)_{ij} = \begin{cases} M_{ab} & \text{if } i = f(a),\ j = f(b) \\ \delta_{ij} & \text{if } \{i,j\} \not\subseteq f([n]) \end{cases}$$

where δ_{ij} is the Kronecker delta. Similarly we have co-FI-objects defined as functors from FI^{op}. For example, for a fixed topological space X the co-FI-space

$$\mathrm{Conf}_\bullet(X) : \mathrm{FI}^{op} \to \mathrm{Top}$$

where $\mathrm{Conf}_S(X) = \mathrm{Emb}(S, X)$ is the space of embeddings $S \hookrightarrow X$. An injection $f : S \hookrightarrow T$ induces a map $\mathrm{Conf}_T(X) \to \mathrm{Conf}_S(X)$ defined by precomposition with f. For $S = [n]$ we recover our original configuration space $C_n(X)$. In [5] the authors proved the following theorem, which is the focus of this thesis:

Theorem 1.2. *(Church's Theorem) Let R be a Noetherian ring and let M be a connected orientable manifold of dimension ≥ 2 with homotopy type of a finite CW complex. For any $m \geq 0$, the* FI*-module $H^m(\mathrm{Conf}(M); R)$ is finitely generated.*

This relies heavily on a Noetherian property of FI-modules over Noetherian rings which had previously only been proved for fields containing $\mathbb{Q}$, namely that any sub-FI-module of a finitely generated FI-module over a Noetherian ring is itself finitely generated. It also relies on the paper [8] which describes the E_2-page of the Leray spectral sequence associated to the inclusion $C_n(X) \hookrightarrow M^n$.

We here present these proofs along with some further details and explanations. In Section 3 we show that we have an isomorphism $H^m(B_n(M); \mathbb{Q}) \cong H^m(C_n(M); \mathbb{Q})^{S_n}$ induced by the covering space projection, where the right side denotes the S_n-invariant vectors in $H^m(C_n(M); \mathbb{Q})$. We use this combined with Theorem 1.2 to show cohomological stability for the unordered configuration spaces.

The two main sources have been [5] and [6]. We have worked out a lot of the minor results ourselves and we have also added some additional comments and discussions. The major ones, such as Theorem 1.2, are from [5] but are here expanded on a bit more with some details worked out.

2 FI-Modules

2.1 Definitions and properties

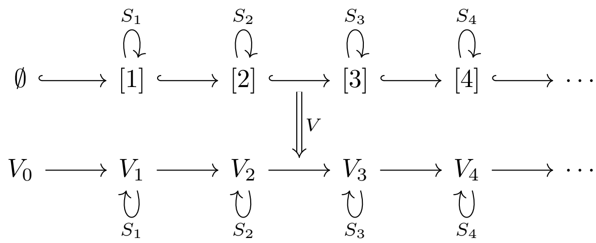
Throughout this thesis, unless otherwise specified let R be a fixed commutative ring. Let FI denote the category whose objects are finite sets (including the empty set) and whose morphisms are the injections.

Definition 2.1 (FI-module)**.** An FI-*module* is a functor $V : \mathrm{FI} \to \mathrm{R\text{-}Mod}$.

For a finite set S we denote $V(S)$ by V_S, and for injections $f : S \hookrightarrow T$ we usually just write $f_* : V_S \to V_T$ for the induced map $V(f)$.

Note that every finite set is isomorphic to a set of the form $[n] := \{1, 2, ..., n\}$ for some $n \geq 0$, where we set $[0] := \emptyset$, and the inclusion of the full subcategory of FI whose objects are sets of this form induces an equivalence of categories. We usually write V_n for $V([n])$, as opposed to $V_{[n]}$. For some purposes it might be more convenient to only look at FI-modules from this subcategory, but sometimes it is not. For example when we take the disjoint union of sets $[n] \sqcup [m]$ we need to choose an isomorphism $[n] \sqcup [m] \cong [n+m]$, but in the case of general finite sets this is not necessary.

The key point here is the fact that $\mathrm{End}([n])$ is the symmetric group on n elements S_n, and hence the R-module V_n comes with an S_n-action. Even though we have many injections $[m] \hookrightarrow [n]$ for $m \leq n$, they are all generated by the natural inclusions $\iota_{n,n+1} : [n] \hookrightarrow [n+1]$ together with the action of the symmetric group, so to explicitly define a particular FI-module V it is enough to say where it sends the sets $[n]$, how it acts on $\iota_{n,n+1}$ and how the group action works. We try to illustrate this structure in the following diagram:



Since for any $\sigma \in S_n$ we have $\sigma \circ \iota_{n,n+1} = \iota_{n,n+1} \circ \sigma$, where we view the sigma on the left hand side as an element of $S_n \subset S_{n+1}$, we can think of FI-modules as consistent sequences of S_n-representations. This means that we have R-modules V_n and homomorphisms $f_n : V_n \to V_{n+1}$

$$\cdots \to V_n \stackrel{f_n}{\to} V_{n+1} \stackrel{f_{n+1}}{\to} V_{n+2} \to \cdots$$

such that for all m, n with $n > m$ and all $\sigma \in S_m$ the following diagram commutes

$$\begin{array}{ccc} V_m & \xrightarrow{f} & V_n \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ V_m & \xrightarrow{f} & V_n. \end{array}$$

Here are some examples to get a better idea:

1. A trivial example is $V_n = R$ the trivial representation and every injection to the identity.
2. Any sequence of S_n-representations $\{V_n\}$ where every injection $f : [n] \hookrightarrow [m]$ with $n < m$ gets taken to the zero map.
3. $V_n = R^n$, the canonical permutation representation and maps natural inclusions.
4. $V_n = R[x_1, ..., x_n]$, the polynomial ring with maps natural inclusions.
5. $V_n = H^m(C_n(X); R)$ where X is a fixed topological space and m a positive integer. The maps are induced from the maps $C_{n+1}(X) \to C_n(X)$ that forgets the last point. The precise structure will become clear in Chapter 3.

Given an FI-module V we can construct new FI-modules by post composition with any functor R-Mod $\to$ R-Mod, for example $\otimes$, $\oplus$, $\bigwedge^k$, Sym^k and so on. We can also define the *truncated* FI-module $\tau_N V$

$$\tau_N V_n = \begin{cases} V_n & \text{if } n \geq N \\ 0 & \text{if } n < N \end{cases}$$

where the maps are the same if the domain and codomain are the same as in V.

Not every consistent sequence comes from an FI-module. We have the following condition:

Proposition 2.1. *Let $\{V_n\}$ be a consistent sequence of S_n representations with maps $\phi_n : V_n \to V_{n+1}$. For $m < n$, let $\iota_{m,n} : [m] \hookrightarrow [n]$ be the natural inclusion and let $S_{n-m} \subset S_n = \mathrm{End}([n])$ be the subgroup permuting the last $n - m$ elements, leaving the rest fixed. Then $\{V_n\}$ comes from an* FI-*module with $\phi_n = (\iota_{n,n+1})_*$ if and only if for all $m < n$,*

$$\sigma v = v \quad \textit{for all } \sigma \in S_{n-m} \textit{ and } v \in \mathrm{im}(\iota_{m,n})_*. \qquad (\star)$$

Proof. Consider the following diagram in FI

$$\begin{array}{ccc} [m] & \xhookrightarrow{\iota_{m,n}} & [n] \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ [m] & \xhookrightarrow{\iota_{m,n}} & [n] \end{array}$$

which always commutes if $\tau = \sigma \in S_m \subset S_n$ or if $\sigma = id$ and $\tau \in S_{n-m}$ only permutes the last $n - m$ letters. Hence if V is an FI-module, the corresponding diagram in R-Mod should commute as well. The morphisms in FI are generated by the natural inclusions $\iota_{m,m+1} : [m] \hookrightarrow [m+1]$ and the invertible injections $\sigma : [n] \to [n]$, as in if $f : [m] \hookrightarrow [n]$ is any injection we can write it as the composition $f = \sigma \circ \iota_{n-1,n} \circ \iota_{n-2,n-1} \circ ... \circ \iota_{m,m+1}$ for some $\sigma \in \mathrm{End}([n])$. Consider then the FI-module V defined by

$$\begin{aligned} [n] &\mapsto V_n \\ \iota_{n,n+1} &\mapsto \phi_n \\ \mathrm{End}([n]) \ni \sigma &\mapsto \sigma \end{aligned}$$

as in the statement of the proposition. Then by assumption $(\iota_{m,n})_* = (\iota_{n-1,n} \circ ... \circ \iota_{m,m+1})_* = \phi_{n-1} \circ ... \circ \phi_m$ so we can see that this definition indeed is functorial.

Suppose now that $(\star)$ holds. Then the diagram in R-Mod

$$\begin{array}{ccc} V_m & \xrightarrow{\phi} & V_n \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ V_m & \xrightarrow{\phi} & V_n \end{array}$$

corresponding to the square in FI at the beginning of the proof commutes and $\{V_n\}$ comes from V.

Conversely, if there is some $\tau \in S_{n-m}$ permuting the last $n - m$ letters for which $(\star)$ does not hold, then commutativity fails, so $\{V_n\}$ cannot come from an FI-module. $\square$

With this condition in place we can see that the following consistent sequences does not come from FI-modules:

1. Assume R is a field whose characteristic is not equal to 2, and let $V_n = R$ be the alternating representation with maps the natural inclusion. Let $\sigma \in S_2 \subset S_{n+2}$ be the non-identity, where S_2 is the subgroup leaving the first n letters fixed, and let $v \in V_n = R$. Then $(\iota_{n,n+2})_*(v) = v$ and $\sigma v = -v \neq v$ unless $v = 0$.

2. $V_n = R[S_n]$, the group ring where S_n acts by left multiplication and maps natural inclusions. For example if we look at $1 = e_{id}$, then for $\sigma \in S_{n-m} \subset S_n$ not equal to the identity where S_{n-m} is the subgroup fixing the first m letters, we have $\sigma e_{id} = e_\sigma \neq e_{id}$, where e_σ is the basis element of $R[S_n]$ corresponding to σ. However, the if S_n acts by conjugation we will get an FI-module.

The FI-modules together with natural transformations form a category themselves.

Definition 2.2 (Category of FI-modules)**.** The *category of* FI-*modules*, denoted FI-Mod, is the category whose objects are FI-modules and whose morphisms are natural transformation. In particular, a morphism $F : V \to W$ of FI-modules V and W consists of, for every finite set S in FI, an R-module homomorphisms $F_S : V_S \to W_S$ called the *component of* F *at* S, such that for every injection $f : S \hookrightarrow T$ the following diagram

$$\begin{array}{ccc} V_S & \xrightarrow{F_S} & W_S \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ V_T & \xrightarrow{F_T} & W_T \end{array}$$

commutes.

This category is abelian, with notions like kernel, cokernel, sub-FI-modules and so on, being defined pointwise. This is true in general for functor categories from small categories to abelian categories (see [7]). For example, for a natural transformation $F : V \to W$ between FI-modules, $\ker(F)$ is defined to be the FI-module which assigns for every finite set S, the R-module $\ker(F)_S := \ker(F_S : V_S \to W_S)$, and for every injection $f : S \hookrightarrow T$ the morphism $f_*|_{\ker(F_S)} : \ker(F_S) \to \ker(F_T)$. Note that $f_*|_{\ker(F_S)}$ has image in $\ker(V_T)$ since F is a natural transformation.

Another example is that $F : V \to W$ is surjective (or injective) if and only if the maps $F_S : V_S \to W_S$ are surjective (or injective) for every finite set S. Since every finite set is isomorphic to $[n]$ for some $n \geq 0$ it is enough to verify that it holds for $F_n : V_n \to W_n$ for every $n \geq 0$. This is in general much easier, so this also serves as an example for when the equivalence of categories mentioned above comes in hand.

The category of FI-modules is closed under any (covariant) functorial construction on R-modules, such as direct sums and tensor products, by applying the functors pointwise. For example if V, W are FI-modules then $V \oplus W$ is the FI-module defined by

$$(V \oplus W)_S := V_S \oplus W_S,$$

and $V \otimes W$ is defined by

$$(V \otimes W)_S := V_S \otimes W_S.$$

We now define the notion of finite generation of FI-modules.

Definition 2.3 (Finite generation)**.** We say that an FI-module V is generated by a set $S \subset \coprod_{n \geq 0} V_n$ if V is the smallest sub-FI-module containing S. We say V is *finitely generated* if it is generated by a finite set. If V is generated by a set $S \subset \coprod_{k \geq n \geq 0} V_n$ we say V is *generated in degree* k.

It is clear that finite generation implies generation in some degree, but the reverse is not always true. For example when V_n is not finitely generated.

To get a better grasp of finite generation, it sometimes help to understand it in terms of "free" objects.

Definition 2.4 (Free FI-module)**.** For all $d \geq 0$, let $M(d)$ be the FI-module defined by $M(d) := R \cdot [\mathrm{FI}([d], -)]$, i.e. for each finite set S, $M(d)_S$ is the free R-module on the set of injections $[d] \hookrightarrow S$. We say that an FI-module is *free* if it is isomorphic to a direct sum of FI-modules of this form, $\bigoplus_{i \in I} M(d_i)$.

It is straight forward to see that $M(d)$ is generated by $\mathrm{id}_d \in M(d)_d$:

By the Yoneda lemma, for any FI-module V we have $[M(d), V] \cong V_d$, where the left side denotes the morphisms in the category of FI-modules. For $v \in V_d$, let $F^v : M(d) \to V$ denote the corresponding homomorphism, i.e F^v has components

$$F^v_S : M(d)_S \to V_S, \quad F^v_S(f) = V(f)(v).$$

We can see that $\operatorname{im} F^v$ is the FI-module

$$(\operatorname{im} F^v)_S = \operatorname{im}(F^v_S : M(d)_S \to V_S) = \operatorname{span}\{f_*(v) \mid f : [d] \hookrightarrow S\}.$$

This is the smallest sub-FI-module $W \subset V$ for which $v \in W_d$. In particular let $V = M(d)$. Then we have $[M(d), M(d)] \cong M(d)_d$ and $\operatorname{im} F^{\mathrm{id}_d}$ is the FI-module

$$(\operatorname{im} F^{\mathrm{id}_d})_S = \operatorname{span}\{f_*(\mathrm{id}_d) \mid f : [d] \hookrightarrow S\} = \operatorname{span}\{f \mid f : [d] \hookrightarrow S\} = M(d)_S,$$

so $M(d)$ is finitely generated by the element $\mathrm{id}_d \in M(d)_d$.

Conversely, given $F : M(d) \to V$, let $v_F \in V_d$ denote the image of $\mathrm{id}_d \in M(d)_d$ under F. For any injection $f : [d] \hookrightarrow S$ we have the following commutative diagram:

$$\begin{array}{ccc} M(d)_d & \xrightarrow{F_d} & V_d \\ \downarrow f & & \downarrow f_* \\ M(d)_S & \xrightarrow{F_S} & V_S \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{id}_d & \longmapsto & v_F \\ \downarrow & & \downarrow \\ f & \longmapsto & F_S(f) = f_*(v_F) \end{array}$$

so F is determined by where it sends id_d.

The FI-modules $M(d)$ are projective objects in the category FI-Mod. Indeed, since it is an abelian category, $M(d)$ being projective is equivalent to the condition that $\operatorname{Hom}(M(d), -)$ is an exact functor. Let

$$0 \to U \xrightarrow{F} V \xrightarrow{G} W \to 0$$

be a short exact sequence of FI-modules. Then by assumption

$$0 \to U_d \xrightarrow{F_d} V_d \xrightarrow{G_d} W_d \to 0$$

is exact. Since $\operatorname{Hom}(M(d), X) \cong X_d$ for any FI-module X, by applying the functor $\operatorname{Hom}(M(d), -)$ to the sequence gives us the sequence

$$0 \to U_d \xrightarrow{\operatorname{Hom}(M(d),F)} V_d \xrightarrow{\operatorname{Hom}(M(d),G)} W_d \to 0,$$

and by the above discussion the induced maps are exactly F_d and G_d. For these reasons $M(d)$ are sometimes referred to as *the d:th principle projective* in the literature.

The following characterization of finite generation usually makes things easier to work with:

Proposition 2.2. *Let V be and* FI-*module. Then V is finitely generated if and only if there exists a surjection*

$$\bigoplus_{i=1}^{k} M(d_i) \twoheadrightarrow V$$

for some $d_i \geq 0$. It is generated in degree $\leq d$ if and only if there exists a surjection

$$\bigoplus_{i\in I} M(d_i) \twoheadrightarrow V$$

with all $d_i \leq d$.

Note that this implies that any quotient of a finitely generated FI module is also finitely generated, by considering the composition $\bigoplus M(d) \twoheadrightarrow V \twoheadrightarrow V/W$. It also implies that the direct sum of two finitely generated FI-modules is finitely generated.

Proof. Suppose first that V is finitely generated by $S = \{v_1, ..., v_n\}$. By the Yoneda lemma this gives rise to a map

$$F := \bigoplus F^{v_i} : \bigoplus_{i\in I} M(d_i) \to V,$$

and $\mathrm{im}(F)$ is the smallest FI-module containing S. Hence $\mathrm{im}(F) = V$ and so F is surjective. The argument for when V is generated in degree $\leq d$ is similar.

For the converse, suppose there is a surjection

$$F : M := \bigoplus_{i=1}^{n} M(d_i) \twoheadrightarrow V.$$

By the Yoneda lemma $M(d_i)$ is finitely generated by $\mathrm{id}_{[d_i]}$, so M is finitely generated as well. Let $e_i := F_{d_i}(\mathrm{id}_{d_i}) \in V_{d_i}$. Then since F is surjective V is the smallest FI-module containing $\{e_1, ..., e_n\}$, so V is finitely generated. If we instead have a surjection

$$F : M := \bigoplus_{i\in I} M(d_i) \twoheadrightarrow V$$

where all $d_i \leq d$ for some $d \geq 0$, then the same argument gives us that V is the smallest FI-module containing a set $\{e_i \mid i \in I\}$ where all $e_i \leq d$, so V is generated in degree $\leq d$ in this case.

□

We can use this characterization to prove the following quick proposition, which we will need to use later.

Proposition 2.3. *Let V, W be finitely generated* FI-*modules. Then $V \otimes W$ is finitely generated as well. If V is generated in degree $\leq d_1$ and W is generated in degree $\leq d_2$, then $V \otimes W$ is generated in degree $\leq d_1 + d_2$.*

Proof. Assume first that V and W is generated in degree $\leq d_1$ and $\leq d_2$ respectively.

By Proposition 2.2 we have two surjections

$$\begin{aligned} F : &M_1 = \bigoplus_{i \in I} M(d_i) \twoheadrightarrow V \\ G : &M_2 = \bigoplus_{j \in J} M(d_j) \twoheadrightarrow W \end{aligned}$$

where $d_i \leq d_1$ and $d_j \leq d_2$ for all $i \in I$ and $j \in J$. The map $F \otimes G : M_1 \otimes M_2 \to V \otimes W$ defined by

$$(F \otimes G)_S : (M_1 \otimes M_2)_S \to V_S \otimes W_S$$

is then also surjective since $(F \otimes G)_S$ is surjective for every finite set S. Hence it is enough to show that $M_1 \otimes M_2$ is generated in degree $\leq d_1 + d_2$. Furthermore, since for every finite set S we have,

$$\bigoplus_i M(d_i)_S \otimes \bigoplus_j M(d_j)_S \cong \bigoplus_{i,j} (M(d_i)_S \otimes M(d_j)_S),$$

it follows that the FI-modules $M_1 \otimes M_2$ and $\bigoplus_{i,j}(M(d_i) \otimes M(d_j))$ are isomorphic, so to prove the proposition it is enough to show that $U := M(d_i) \otimes M(d_j)$ is generated in degree $\leq d_i + d_j$.

For each finite set S the R-module U_S is finitely generated and a basis consists of pairs of injections $f \otimes g$ where $f : [d_i] \hookrightarrow S$ and $g : [d_j] \hookrightarrow S$. For any such basis element, consider the set $T := \operatorname{im} f \cup \operatorname{im} g$. Then $f \otimes g$ is contained in the image of U_T under the the action of the morphisms in FI-Mod, and since $|T| \leq d_i + d_j$ we have that U is generated in degree $\leq d_1 + d_2$. Since each U_S is finitely generated we also get that $V \otimes W$ is finitely generated if both V and W are.

□

Another proposition we will make use of is the following:

Proposition 2.4. *Let*

$$0 \to U \xrightarrow{F} V \xrightarrow{G} W \to 0$$

be a short exact sequence of FI*-modules. Then if U and W are finitely generated then so is V.*

Proof. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus M(d_i) & \hookrightarrow & \bigoplus M(d_i) \oplus \bigoplus M(d_j) & \twoheadrightarrow & M(d_j) & \twoheadrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & U & \xhookrightarrow{F} & V & \xrightarrow{G} & W & \longrightarrow & 0 \end{array}$$

for some d_i, d_j, where ϕ exists since $M(d)$ is projective, $G \circ \phi = \tau$ and where both rows are exact. We can then defined a map

$$\psi : \bigoplus M(d_i) \oplus \bigoplus M(d_j) \to V$$

by $\psi_n(\xi, \zeta) = i_n(\xi) + \phi_n(\zeta)$ for every $n \geq 0$, where $i = F \circ \sigma$ and $\xi \in \bigoplus M(d_i)_n$, $\zeta \in \bigoplus M(d_j)_n$. This is indeed a map of FI-modules since for every $n, m \geq 0$ the following diagram

$$\begin{array}{ccc} \bigoplus M(d_i)_n \oplus \bigoplus M(d_j)_n & \xrightarrow{f_*} & \bigoplus M(d_i)_m \oplus \bigoplus M(d_j)_m \\ \downarrow{\psi_n} & & \downarrow{\psi_m} \\ V_n & \xrightarrow{f_*} & V_m \end{array}$$

commutes, where $f : [n] \hookrightarrow [m]$ is an injection. We can see this since $f_*(\psi_n(\xi, \zeta)) = f_*(i_n(\xi) + \phi_n(\zeta)) = f_*(i_n(\xi)) + f_*(\phi_n(\zeta))$ since f_* is an R-module homomorphism, and $\psi_m(f_*(\xi, \zeta)) = \psi_m(f_*(\xi), f_*(\zeta)) = i_m(f_*(\xi)) + \phi_m(f_*(\zeta))$. Commutativity now follows since both i and ϕ are natural transformations as well, so $f_*(i_n(\xi) = i_m(f_*(\xi))$ and $f_*(\phi_n(\zeta)) = \phi_m(f_*(\zeta))$. To see that ψ is surjective we can go back to the first diagram and apply the Snake lemma, which holds in any abelian category and which gives us an exact sequence

$$0 = \text{coker}(\sigma) \to \text{coker}(\psi) \to \text{coker}(\tau) = 0$$

since σ, τ are surjective, and hence ψ is as well. The proposition follows by Proposition 2.2. □

We can now start to take steps towards the proof of the Noetherian property by defining two functors we will make use of and prove some properties they have. The first one being the functor $H_0(-)$.

Definition 2.5. Let $H_0(-)$: FI-Mod $\to$ FI-Mod be the functor defined by taking an FI-module V to the FI-module $H_0(V)$ which is defined by

$$H_0(V)_S = V_S / \langle \text{im}(f_* : V_T \to V_S) \mid f : T \hookrightarrow S,\ |T| < |S| \rangle.$$

To see how H_0 acts on morphisms, let $F : V \to W$ be a natural transformation between two FI-modules V, W and let $x \in \text{im}(f_*)$ for some injection $f : T \hookrightarrow S$ with $|T| < |S|$. Since the following diagram commutes by naturality of F

$$\begin{array}{ccc} V_T & \xrightarrow{f_*} & V_S \\ \downarrow{F_T} & & \downarrow{F_S} \\ W_T & \xrightarrow{f_*} & W_S \end{array}$$

we get that $F_S(x) = f_*(F_T(x'))$ for some $x' \in V_T$, so $F_S(x)$ is in the image of $f_* : W_T \to W_S$, and hence F_S descends to a map of the quotients

$$H_0(F)_S : H_0(V)_S \to H_0(W)_S,$$

and so we get an induced morphism $H_0(F) : H_0(V) \to H_0(W)$.

In other words we can say that $H_0(V)$ is the largest quotient of V such that for all $f : T \hookrightarrow S$ with $|T| < |S|$, the induced map $f_* : H_0(V)_T \to H_0(V)_S$ is the zero map. Therefore we can think of H_0 as a functor $H_0(-)$: FI-Mod $\to$ FB-Mod, where FB is the category of finite sets and bijections. Any FB-module can be viewed as an FI-module where the maps induced from the injections which are not also surjective, are the zero maps. This gives us an inclusion of categories i : FB-Mod $\hookrightarrow$ FI-Mod, and in fact H_0 is left adjoint to i. Let V be an FI-module and let B be an FB-module. Then we have for any $F \in \text{FI-Mod}(V, i(B))$ and any injection $f : T \hookrightarrow S$ for finite sets T, S with $|T| < |S|$ the following commutative diagram

$$\begin{array}{ccc} V_T & \xrightarrow{F_T} & i(B)_T \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle 0} \\ V_S & \xrightarrow{F_S} & i(B)_S \end{array}$$

so the composition $F_S \circ f_*$ is the zero map. Hence F_S is the same as a map $H_0(V)_S \to B_S$ and we can see that we indeed get a natural bijection

$$\text{FB-Mod}(H_0(V), B) \cong \text{FI-Mod}(V, i(B)).$$

We will see later on in Chapter 3, Theorem 3.2 that this implies that $H_0(-)$ is right exact.

We can compute $H_0(M(d))$. Since $M(d)$ is generated by $\text{id}_d \in M(d)_d$ we get that $M(d)_S = 0$ if $|S| < d$, and we also get that if $|S| > d$ every element of $M(d)_S$ is in $\text{span}\{f_*(\text{id}_d) \mid f : [d] \hookrightarrow S\}$. Therefore we have that

$$H_0(M(d))_S = \begin{cases} 0 & \text{if } |S| < d \\ 0 & \text{if } |S| > d \\ M(d)_d & \text{if } |S| = d. \end{cases}$$

Proposition 2.5. *Let V, W be* FI-*modules.*

1. *If $H_0(V) = 0$, then $V = 0$.*
2. *A homomorphism $F : V \to W$ is surjective if and only if $H_0(F) : H_0(V) \to H_0(W)$ is surjective.*

Proof. For the first claim, suppose for a contradiction that $V \neq 0$. Let $N := \inf\{n \in \mathbb{N} \mid V_n \neq 0\}$. Then for every injection $f : T \hookrightarrow [N]$ with $|T| < N$ we get the induced map $f_* : V_T = 0 \to V_N$, so the quotient defining $H_0(V)_N$ is the quotient by the zero module. Hence $H_0(V)_N = V_N \neq 0$.

For the second claim, if we suppose $F : V \to W$ is surjective, since $H_0(-)$ is right exact $H_0(F)$ is surjective as well. For the converse, right exactness implies that $0 = \text{coker}(H_0(F)) = H_0(\text{coker}(F))$. Applying the first claim we conclude that $\text{coker}(F) = 0$, and hence F is surjective as well. □

Proposition 2.6. *Let V be an* FI-*module.*

1. *In each of the following rows, the conditions* $(a), (b)$ *and* (c) *are equivalent:*

(a) *V is finitely generated*	(b) *$H_0(V)$ is finitely generated*	(c) $\bigoplus_{n=0}^{\infty} H_0(V)_n$ *is f.g.*
(a) *V is gen. in deg.* $\leq d$	(b) *$H_0(V)$ is gen. in deg.* $\leq d$	(c) $H_0(V)_n = 0$ *for all* $n > d$
(a) *V is gen. in finite deg.*	(b) *$H_0(V)$ is gen. in finite deg.*	(c) $H_0(V)_n = 0$ *for* $n >> 0$.

Note that the condition in (c) *is a statement about R-modules as opposed to* FI*-modules, as in* (a) *and* (b)*.*

2. *Assume that V_n is a finitely generated R-module for all $n \geq 0$. Then V is finitely generated if and only if V is generated in finite degree.*

Proof. 1. Note that each condition in the third row is just stating that the corresponding condition in the second row is true for some $d \in \mathbb{N}$, so the equivalence of the third row follows from the equivalence of the second.

$(a) \Rightarrow (b)$: If V is finitely generated or generated in degree $\leq d$, $H_0(V)$ is as well since it is a quotient of V.

$(b) \Rightarrow (c)$: Let $M := \bigoplus_{i \in I} M(d_i)$. By Proposition 2.2 we have a surjection $F : M \twoheadrightarrow H_0(V)$, and this map factors through $H_0(M)$. We now observe that $H_0(M) = H_0(\bigoplus M(d_i)) = \bigoplus H_0(M(d_i))$ since $H_0(-)$ is right exact, and we computed $H_0(M(d_i))$ earlier so we can see that for any finite set S,

$$H_0(M(d_i))_S = \begin{cases} M(d_i)_S & \text{if } |S| = d_i \\ 0 & \text{otherwise.} \end{cases}$$

If $H_0(V)$ is finitely generated, we may assume the index set I is finite, so $\bigoplus_{n=1}^{\infty} H_0(M)_n$ is a free R-module of rank $\sum_{i=1}^{k} d_i!$ for some k, and in particular the module is finitely generated, so $(b) \Rightarrow (c)$ in the first row.

If $H_0(V)$ is generated in degree $\leq d$ we can assume $d_i \leq d$ for all $i \in I$. In this case $H_0(M)_n = 0$ for all $n > d$, so the same is true for $H_0(V)_n$. Hence $(b) \Rightarrow (c)$ in the second row as well.

$(c) \Rightarrow (a)$: Assume $\bigoplus_{n=0}^{\infty} H_0(V)_n$ is finitely generated and let $\{v_i\}_{i \in I} \subset \coprod_n H_0(V)_n$, where I is finite, be a generating set. We want to define a surjection $\pi : M = \bigoplus M(d_i) \twoheadrightarrow V$.

Pick $d_i \in \mathbb{N}$ such that $v_i \in H_0(V)_{d_i}$. We define $\pi : M \twoheadrightarrow V$ by sending $\mathrm{id}_{[d_i]} \in M(d_i)_{d_i}$ to any element of V_{d_i} lifting v_i. So the map $H_0(\pi) : H_0(M) \to H_0(V)$ sends $\mathrm{id}_{[d_i]} \in H_0(M(d_i))_{d_i}$ to $v_i \in H_0(V)_{d_i}$, and since $H_0(V)_d$ is generated by the elements v_i for which $d_i = d$ we get that $H_0(\pi)_d : H_0(M)_d \to H_0(V)_d$ is surjective for every d, hence $H_0(\pi)$ is surjective, and by Lemma 2.5, π is surjective as well. Since I is finite, the surjection $\pi : M \twoheadrightarrow V$ shows that V is finitely generated as well, and so $(c) \Rightarrow (a)$ in the first row.

If we assume $H_0(V)_n = 0$ for $n > d$ we can assume that $d_i \leq d$ for all $i \in I$ (I not necessarily finite), so the surjection π gives us that V is generated in degree $\leq d$. Hence $(c) \Rightarrow (a)$ in the second row as well.

2. Firstly, if V is finitely generated it is also generated in finite degree.

   For the converse, by the equivalence of the third row in (1) we get that $H_0(V)_n = 0$ for n large enough, and hence

$$\bigoplus_{n=0}^{\infty} H_0(V)_n = \bigoplus_{n=0}^{k} H_0(V)_n$$

   for some $k < \infty$. Since V_n is a finitely generated R-module for each $n \geq 0$ the same is true for $H_0(V)_n$ since it is a quotient of V_n. This implies that the sum $\bigoplus_{n=0}^{\infty} H_0(V)_n$ is a finite sum of finitely generated R-modules and hence it is finitely generated so the equivalence of the first row gives us that V is a finitely generated FI-module.

□

The second functor we need is called ***shift functor***. Let $\sqcup : \text{Sets} \times \text{Sets} \to \text{Sets}$ be the disjoint union functor on sets. Since $f \sqcup g : S \sqcup S' \to T \sqcup T'$ is injective if $f : S \hookrightarrow T$ and $g : S' \hookrightarrow T'$ are both injective, this functor restricts to a functor $\sqcup : \text{FI} \times \text{FI} \to \text{FI}$.

Definition 2.6. For $a \geq 0$, let $[-a]$ denote the set $\{-1, ..., -a\}$, and let Ξ_a be the functor

$$\Xi_a : \text{FI} \to \text{FI}, \quad \Xi_a(S) := S \sqcup [-a].$$

If $f : S \hookrightarrow T$ is an injection, $\Xi_a(f)$ is the map $f \sqcup \text{id}_{[-a]} : S \sqcup [-a] \hookrightarrow T \sqcup [-a]$.

Let $i_{-a} : [-a] \hookrightarrow [-(a+1)]$ denote the natural inclusion.

Given an FI-module V and an integer $a \geq 1$, let $S_{+a} : \text{FI-Mod} \to \text{FI-Mod}$ be the functor defined by precomposition by Ξ_a. That is,

$$S_{+a} : \text{FI-Mod} \to \text{FI-Mod}, \quad S_{+a}(V) := V \circ \Xi_a : \text{FI} \xrightarrow{\Xi_a} \text{FI} \xrightarrow{V} R\text{-Mod}.$$

The functor S_{+a} is called a ***positive shift functor***.

Since kernels and cokernels are computed pointwise, this is an exact functor. For example if

$$0 \to U \xrightarrow{F} V \xrightarrow{G} W \to 0$$

is an exact sequence of FI-modules we have for every finite set T, $\ker(G)_S = \ker(G_S : V_S \to W_S) = \text{im}(F_S : U_S \to V_S) = (\text{im}\, F)_S$, and in particular the same thing holds for $S \sqcup [-a]$, and hence the same thing holds for $S_{+a}(F)_S = F_{S \sqcup [-a]}$ and $S_{+a}(G)_S = G_{S \sqcup [-a]}$.

Given an FI-module V we could ask ourselves what the difference between the S_{n+a}-representation V_{n+a} and the S_n-representation $S_{+a}(V)_n$ is. Given $\sigma \in \text{End}([n])$ we have $S_{+a}(\sigma) = (\sigma \sqcup \text{id}_{[a]})_* : V_{[n] \sqcup [-a]} \cong V_{n+a} \to V_{n+a}$. In other words S_n acts as on V_{n+a} under the image of the natural inclusion $S_n \hookrightarrow S_{n+a}$, and we have an isomorphism of representations

$$S_{+a}(V)_n \cong \operatorname{Res}_{S_n}^{S_{n+a}} V_{n+a}.$$

The point of the shift functor is to apply this restriction in such a way that result still forms an FI-module. Note that the choice of set for $[-a]$ is irrelevant, any set of cardinality a would do.

Definition 2.7. Let T be a finite set. The natural inclusion of T into $\Xi_a(T) = T \sqcup [-a]$ induces a natural transformation $\mathrm{id}_{\mathrm{FI}} \Longrightarrow \Xi_a$, so for any FI-module V this gives us a homomorphism of FI-modules $X_a : V \to S_{+a}(V)$. Explicitly, for every finite set T, X_a has components induced from the natural inclusion $T \hookrightarrow T \sqcup [-a]$:

$$X_a : V_T \to V_{T\sqcup[-a]} = S_{+a}(V)_T.$$

We also have that the natural inclusion $\mathrm{id} \sqcup i_{-a} : T \sqcup [-a] \hookrightarrow T \sqcup [-(a+1)]$ induces a homomorphism

$$Y_a : S_{+a}(V) \to S_{+(a+1)}(V),$$

satisfying $X_{a+1} = Y_a \circ X_a : V \to S_{+(a+1)}(V)$.

If V, W are FI-modules, we write $V \sim W$ if $S_{+a}(V) \cong S_{+a}(W)$ for some $a \geq 0$. This notation is mostly used as $V \sim 0$, which means that $V_n = 0$ for sufficiently large n.

2.2 The Noetherian property

We can now prove the following theorem, which will be essential for the proof of Theorem 1.2:

Theorem 2.7. *Every sub-*FI*-module of a finitely generated* FI*-module over a Noetherian ring R is finitely generated.*

We say that finitely generated FI-modules over Noetherian rings are Noetherian. Some of properties of Noetherian rings carry over to corresponding versions for Noetherian FI-modules. Consider the following proposition.

Proposition 2.8. *Let*

$$0 \to U \overset{F}{\to} V \overset{G}{\to} W \to 0$$

be a short exact sequence of FI*-modules. Then V is Noetherian if and only if U and W are Noetherian.*

Proof. Suppose first that V is Noetherian and let $U' \subset U$ and $W' \subset W$ be sub-FI-modules. By Proposition 2.2 we have two surjections

$$\phi : M_1 := \bigoplus_{i=1}^{n} M(d_i) \twoheadrightarrow F(U') \subset V$$
$$\psi : M_2 := \bigoplus_{j=1}^{m} M(d_j) \twoheadrightarrow G^{-1}(W') \subset V.$$

Since F is injective we can define a map $F^{-1} : F(U') \to U'$ and composing with ϕ gives us a surjection $M_1 \twoheadrightarrow U'$ so U' is finitely generated. Similarly we can compose ψ and G to get a surjection $M_2 \twoheadrightarrow W'$.

Conversely, suppose U and W are Noetherian and let $V' \subset V$ be a sub-FI-module. We then have the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{F} & V & \xrightarrow{G} & W & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F^{-1}(V') & \longrightarrow & V' & \longrightarrow & G(V') & \longrightarrow & 0 \end{array}$$

and applying Proposition 2.4 to the sequence below gives us the desired result. $\square$

Corollary 2.9. *If V and W are Noetherian* FI*-modules if and only if the direct sum $V \oplus W$ is Noetherian.*

Proof. This follows from the previous proposition by applying it to the short exact sequence

$$0 \to V \to V \oplus W \to W \to 0.$$

$\square$

We shall break down the proof of 2.7 into several steps. First we investigate how the positive shift functors behave when applied to the FI-modules $M(d)$.

Proposition 2.10. *For any $a, d \geq 0$, there is a natural decomposition*

$$S_{+a}(M(d)) = M(d) \oplus Q_a,$$

where Q_a is a free FI*-module finitely generated in degree $\leq d-1$.*

Proof. Let S be a finite set. The maps $\mathrm{FI}([d], S)$ form basis for $M(d)_S$, so the maps $\mathrm{FI}([d], S \sqcup [-a])$ form basis for $S_{+a}(M(d))_S$. Let $f : [d] \hookrightarrow S \sqcup [-a]$ be an injection and consider the subset $T = f^{-1}([-a]) \subset [d]$ as well as the restriction $f|_T : T \hookrightarrow [-a]$. Given another injection $g : S \hookrightarrow S'$, the map $g_* : S_{+a}(M(d))_S \to S_{+a}(M(d))_{S'}$ is induced by the composition

$$g_* f = (g \sqcup \mathrm{id}_{[-a]}) \circ f.$$

Note that $g_* f^{-1}([-a]) = f^{-1}([-a]) = T$ and $g_* f|_T = f|_T$, so neither the subset T nor $f|_T$ are changed by g_*, and arranging the basis of $S_{+a}(M(d))_S$ according to these two factors gives us a decomposition of $S_{+a}(M(d))$ as a direct sum of FI-modules.

Fix a subset $T \subset [d]$ and an injection $h : T \hookrightarrow [-a]$. Let $M^{T,h} \subset S_{+a}(M(d))$ denote the sub-FI-module spanned by the injections f satisfying $f^{-1}([-a]) = T$ and $f|_T = h$. These injections are distinguished by the restrictions $f|_{[d]\setminus T}$, and we have $g_* f|_{[d]\setminus T} = g \circ f|_{[d]\setminus T}$. For any finite set S, the summand of $S_{+a}(M(d))_S$ corresponding to T and h can be viewed as being generated by the pairs (f, g)

where f is an injection from $M_S^{T,h}$ and g is an injection $g : T \hookrightarrow [-a]$. We thus get a composition

$$S_{+a}(M(d))_S = \bigoplus_{T \subset [d]} M_S^{T,h} \otimes_R R[\mathrm{FI}(T, [-a])].$$

We can now choose a bijection $[d] \setminus T \cong [d-|T|]$, which gives us an isomorphism $M^{T,h} \cong M(d-|T|)$, and thus we get a decomposition

$$S_{+a}(M(d)) = \bigoplus_{T \subset [d]} M(d-|T|) \otimes_R R[\mathrm{FI}(T, [-a])].$$

Moreover this decomposition is natural up to choice of bijection $[d] \setminus T \cong [d-|T|]$. Isolating the summand with $T = \emptyset$, which is isomorphic to $M(d)$, we get the desired result. $\square$

Corollary 2.11. *If V is generated in degree $\leq d$, then $S_{+a}(V)$ is generated in degree $\leq d$. Conversely, if $S_{+a}(V)$ is generated in degree $\leq d$, then V is generated in degree $\leq d+a$.*

Proof. For the first claim, we have a surjection $\bigoplus_{i \in I} M(d_i) \twoheadrightarrow V$ where $d_i \leq d$ for all $i \in I$. Since $S_{+a}(V)$ is exact we get a surjection $S_{+a}(\bigoplus M(d_i)) = \bigoplus M(d_i) \oplus Q_a^i \twoheadrightarrow S_{+a}(V)$. Since Q_a^i is generated in degree $\leq d-1$ we have a surjection for every $i \in I$, $\bigoplus_{j \in J_i} M(d_j) \twoheadrightarrow Q_a^i$ where $d_j \leq d-1$ for all j. Combining these we get a surjection

$$\bigoplus_{i \in I \cup (\bigcup_{i \in I} J_i)} M(d_i') \twoheadrightarrow \bigoplus M(d_i) \oplus Q_a^i \twoheadrightarrow S_{+a}(V),$$

with every $d_i' \leq d$, and the claim follows.

For the converse we use Proposition 2.6 which says that $S_{+a}(V)$ is generated in degree $\leq d$ if and only if $H_0(S_{+a}(V))_n = 0$ whenever $n > d$. Recall now that for every finite set S, the R-module $H_0(S_{+a}(V))_S$ is defined to be the quotient of $S_{+a}(V)_S = V_{S \sqcup [-a]}$ by

$$\langle \mathrm{im}(f \sqcup \mathrm{id}_{[-a]})_* : V_{T \sqcup [-a]} \to V_{S \sqcup [-a]} \mid f : T \hookrightarrow S,\ |T| < |S| \rangle,$$

and $H_0(V)_{S \sqcup [-a]}$ is defined to be the quotient of $V_{S \sqcup [-a]}$ by

$$\langle \mathrm{im}\, g_* : V_{T'} \to V_{S \sqcup [-a]} \mid g : T' \hookrightarrow S \sqcup [-a],\ |T'| < |S| + a \rangle.$$

Since the former is contained in the latter, we get that $H_0(V)_{S \sqcup [-a]}$ is a quotient of $H_0(S_{+a}(V))_S$, and in particular we have a surjection $H_0(S_{+a}(V))_S \twoheadrightarrow H_0(V)_{S \sqcup [-a]}$ for every finite set S. By assumption, this surjection gives us that $H_0(V)_{n+a} = 0$ whenever $n > d$. Using Proposition 2.6 once again we then get that V is generated in degree $\leq d+a$. $\square$

Definition 2.8. Let $\pi_a : S_{+a}(M(d)) \twoheadrightarrow M(d)$ be the projection determined by

$$S_{+a}(M(d)) = M(d) \oplus Q_a \twoheadrightarrow M(d)$$

in Proposition 2.10. More concretely, a basis for $S_{+a}(M(d))_S$ consists of injections $[d] \hookrightarrow S \sqcup [-a]$, and the projection simply sends any injection with image not contained in S to 0.

If we look at $M(d)_n$ for some $n \geq d$, we can split up the injections $[d] \hookrightarrow [n]$ according to their image. Each d-element subset of $[n]$ gives us a summand of $M(d)_n$ isomorphic to $M(d)_d$, yielding a decomposition of R-modules

$$M(d)_n \cong M(d)_d^{\oplus \binom{n}{d}}.$$

In degree d, the projection π_a gives us a map $S_{+(n-d)}(M(d))_d \cong M(d)_n \to M(d)_d$. This is the same as the projection onto a single factor in the decomposition above, so we can see that it is related to the projection π_a.

We can now prove the Noetherian property of FI-modules.

Proof of Theorem 2.7. [5]. We are going to prove by induction on $d \in \mathbb{N}$ that if V is an FI-module, finitely generated in degree $\leq d$, then any sub-FI-module $W \subset V$ is finitely generated. For such an FI-module we have a surjection

$$F : \bigoplus_{i=1}^{k} M(d_i) \twoheadrightarrow V$$

with all $d_i \leq d$. If the Noetherian property holds for $\bigoplus_{i=1}^{k} M(d_i)$, it also holds for V by considering $F^{-1}(W)$, where $W \subset V$. Hence it is enough to prove the theorem for $V = \bigoplus_{i=1}^{k} M(d_i)$. Since the Noetherian property is also preserved under direct sums, it is enough to prove it for $V = M(d_i)$, and by induction it suffices to prove it for $V = M(d)$.

(Reduction to W^a.) Fix a submodule W of $M(d)$. For each $n \in \mathbb{N}$, $M(d)_n$ is a finitely generated R-module. Since R is Noetherian, the submodule W_n is also finitely generated. Using Lemma 2.6 part 2 we get that it is enough to prove that W is generated in finite degree. By Corollary 2.11 it suffices to prove $S_{+a}(W)$ is finitely generated for some $a \geq 0$. Using the decomposition in Proposition 2.10 we get a short exact sequence

$$0 \to Q_a \to S_{+a}(M(d)) \xrightarrow{\pi_a} M(d) \to 0$$

for any $a \geq 0$. Since $S_{+a}(-)$ is exact we can think of $S_{+a}(W)$ as a sub-FI-module of $S_{+a}(M(d))$. This induces a short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_a \cap S_{+a}(W) & \longrightarrow & S_{+a}(W) & \longrightarrow & \pi_a(S_{+a}(W)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q_a & \longrightarrow & S_{+a}(M(d)) & \xrightarrow{\pi_a} & M(d) & \longrightarrow & 0
\end{array}$$

in the top row. Let us denote $\pi_a(S_{+a}(W))$ as W^a.

We know that Q_a is finitely generated in degree $\leq d-1$ by Proposition 2.10, so applying the induction hypothesis gives us that $Q_a \cap S_{+a}(W)$ is finitely generated for any $a \geq 0$. Thus, to prove that $S_{+a}(W)$ is finitely generated, it suffices to prove that W^a is finitely generated. We will do this by showing that there exists some $N \geq 0$ such that W^N is finitely generated in degree $\leq d$. The first step is to show that a certain sub-FI-module $W^\infty \subset M(d)$ is finitely generated in degree d.

(Showing W^∞ is generated by W_d^∞.) The map $Y_a : S_{+a}(M(d)) \to S_{+(a+1)}(M(d))$ from Definition 2.7 satisfies $\pi_{a+1} \circ Y_a = \pi_a$, and we also have that $Y_a(S_{+a}(W)) \subset S_{+(a+1)}(W)$. From this it follows that $W^a \subset W^{a+1}$. Let W^∞ denote the sub-FI-module $\bigcup_a W^a \subset M(d)$.

An element

$$x = \sum_{f:[d]\hookrightarrow T} r_f f \in M(d)$$

lies in W^a if and only if there is an element

$$w = \sum_{g:[d]\hookrightarrow T\sqcup[-a]} r'_g g \in W_{T\sqcup[-a]} \subset M(d)_{T\sqcup[-a]}$$

such that $r'_g = r_g$ whenever $\operatorname{im} g \subset T$. The element $x \in M(d)_T$ lies in W^∞ if the above is true for some $a \geq 0$.

For each $a \geq 0$, let U^a be the smallest sub-FI-modules of W^a containing W_d^a. We will show that for any $a \geq 0$ and any $n \leq a+d$ we have

$$W_n^{a+d-n} \subset U_n^a \subset M(d)_n.$$

Given $x \in W^{a+d-n}$, let $x = \sum_{f:[d]\hookrightarrow[n]} r_f f$ as above, and for each $S \subset [n]$ with $|S| = d$, let x_S denote

$$x_S := \sum_{\operatorname{im} f = S} r_f f \in M(d)_S.$$

We have that $x = \sum_S i_S(x_S)$ where $i_S : S \hookrightarrow [n]$ is the natural inclusion.

Since $x \in W^{a+d-n}$ there exists some $w \in W_{[n]\sqcup[-(a+d-n)]}$ such that writing

$$w = \sum_{g:[d]\hookrightarrow[n]\sqcup[-(a+d-n)]} r'_g g$$

as above, we have $r'_g = r_g$ for all g with $\operatorname{im} g \subset [n]$. But then it is also true that $r'_g = r_g$ for all g with $\operatorname{im} g = S$ with S as above, so choosing a bijection

$$([n] \setminus S) \sqcup [-(a+d-n)] \cong [-a],$$

we can think of w as an element of $W_{S\sqcup[-a]}$, so $x_S \in W_S^a$.

Since $|S| = d$ we have that $U_S^a = W_S^a$. Since $x = \sum_S i_S(x_S)$ we can conclude that $x \in U^a$, and because this holds for all $x \in W_n^{a+d-n}$ we can see that W_n^{a+d-n}

is contained in U^a as was the claim above. Passing to the limit as $a \to \infty$ and setting $U^\infty := \bigcup_a U^a$ we see that W_n^∞ is contained in U^∞ for all $n \in \mathbb{N}$, but since U^∞ is contained in W^∞ by definition this gives us that $U^\infty = W^\infty$. In other words, W^∞ is generated by W_d^∞ as claimed.

(Finding N such that W^N is generated in degree $\leq d$.) Since $W_d^\infty \subset M(d)_d \cong R[S_d]$, it is finitely generated as an R-module, so W^∞ is finitely generated in degree $\leq d$. Consider the following chain of submodules of $M(d)_d$:

$$W_d = W_d^0 \subset W_d^1 \subset ... \subset W_d^\infty = \bigcup_a W_d^a.$$

Since $M(d)_d$ is a finitely generated R-module and R is Noetherian, there has to be some N such that $W_d^N = W_d^\infty$. Since W^∞ is generated by W_d^∞ it follows that $W^\infty = W^N$, and thus W^N is finitely generated in degree $\leq d$ as claimed, and the theorem follows. □

We will end this section by proving that for a finitely generated FI-module V, the dimension of V_n is eventually given by a polynomial in n.

Definition 2.9. Let V be an FI-module. The *torsion submodule* of V, denoted $T(V)$, consists of those $v \in V_S$ for which $f_*(v) = 0$ for some injection $f : S \hookrightarrow T$. We say that V is *torsion free* if $T(V) = 0$.

Let V be an FI-module and let $v \in V_n$ be such that $f_*(v) = 0$ for some injection $f : [n] \hookrightarrow [m]$. Let $\iota_n : [n] \hookrightarrow [n] \sqcup [-(m-n)] \cong [m]$ be the natural inclusion and recall that $f = \sigma \circ \iota_n$ for some $\sigma \in S_m$. Hence $0 = f_*(v) = \sigma(\iota_n)_*(v)$ which give us that $(\iota_n)_*(v) = 0$, i.e. $v \in \ker(X_{m-n} : V_n \to S_{+(m-n)}(V)_n)$ where $X_a : V \to S_{+a}(V)$ is the FI-module homomorphism with components induced from ι_S we defined earlier. Conversely, if $v \in \ker(X_a : V_S \to S_{+a}(V)_S)$ for some $a \geq 0$, then clearly v is in $T(V)$, and hence we can write

$$T(V) := \bigcup_{a \geq 0} \ker(X_a : V \to S_{+a}(V)).$$

Lemma 2.12. *If V is a finitely generated* FI*-module over a Noetherian ring, then $T(V) \sim 0$, i.e. $T(V)_n = 0$ for all n sufficiently large.*

Proof. By the Noetherian property 2.7, the sub-FI-module $T(V)$ is finitely generated. Let $v_1, ..., v_k$, with $v_i \in V_{n_i}$ be the generators, so for every finite set S, $T(V)_S$ is spanned by $\bigcup_i \{f_*(v_i) \mid f : [d_i] \hookrightarrow S\}$. For every $i = 1, ..., k$, by definition there exists some a_i such that $v_i \in \ker(X_{a_i} : V \to S_{+a_i}(V))$. Set $M_i := d_i + a_i$. Then for any $f : [n_i] \hookrightarrow S$ with $|S| \geq M_i$ we have $f_*(v_i) = 0$. Now let $M := \max\{M_i\}$. Then $f_*(v_i) = 0$ for any i and for any $f : [n_i] \hookrightarrow S$ with $|S| \geq M$. Since these elements generate $T(V)_S$ we see that $T(V)_S = 0$ whenever $|S| \geq M$, and hence $T(V) \sim 0$. □

Theorem 2.13. *Let k be a field, and let V be an* FI*-module over k, finitely generated in degree $\leq d$. Then there exists an integer-valued polynomial $p(x) \in \mathbb{Q}[x]$ with $\deg p(x) \leq d$ such that for all sufficiently large n,*

$$\dim_k V_n = p(n).$$

Proof. ([5], p.18). Firstly, by Lemma 2.12, the torsion free quotient $V' := V/T(V)$ satisfies $\dim_k V'_n = \dim_k V_n$ for n sufficiently large, and since V' is a quotient of V it is also generated in degree $\leq d$. Therefore we may assume V is torsion free. By definition,

$$\bigcup_{a \geq 0} \ker(X_a : V \to S_{+a}(V)) = 0,$$

so for all $a \geq 0$ the map X_a is injective. Let $DV := \operatorname{coker}(X_1 : V \to S_{+a}(V))$.

We will proceed by induction on d. We take $d = -1$ as our base case, where we say V is generated in degree ≤ -1 if $V = 0$, and that a polynomial has degree -1 if it vanishes.

We show that DV is finitely generated in degree $\leq d - 1$. If $V = M(n)$ for some $n \leq d$, then by Proposition 2.10 $DV = Q_1$ is finitely generated in degree $\leq n - 1$. The positive shift functors S_{+a} preserve direct sums. If we have two FI-modules V, W we have $S_{+a}(V \oplus W)_n = (V \oplus W)_{n+a} = V_{n+a} \oplus W_{n+a} = S_{+a}(V)_n \oplus S_{+a}(W)_n$. Since V is finitely generated in degree $\leq d$ we have a surjection

$$M := \bigoplus_{i=1}^{k} M(d_i) \twoheadrightarrow V$$

where $d_i \leq d$. Then by Proposition 2.10 we have

$$S_{+1}(M) = \bigoplus_{i=1}^{k} S_{+1}(M(d_i)) = \bigoplus_{i=1}^{k} M(d_i) \oplus Q_{1,i},$$

so $DM = \bigoplus_{i=1}^{k} Q_{1,i}$ where $Q_{1,i}$ is finitely generated in degree $\leq d_i - 1$, and therefore DM is finitely generated in degree $\leq d-1$. Since S_{+a} is exact we have a surjection $S_{+1}(M) \twoheadrightarrow S_{+1}(V)$, so this induces a surjection on the quotients $DM \twoheadrightarrow DV$, and hence DV is finitely generated in degree $\leq d - 1$.

By induction we can conclude that $\dim_k DV_n$ is eventually a polynomial of degree at most $d - 1$. Since we are working over a field k we have

$$p(n) = \dim_k DV_n = \dim_k \operatorname{coker}(X_{+1})_n = \dim_k S_{+1}(V)_n - \dim_k V_n.$$

If we write $\phi(n) := \dim_k V_n$ we then get

$$p(n) = \phi(n + 1) - \phi(n),$$

so since $p(n)$ is eventually a polynomial of degree at most $d-1$, $\phi(n+1)-\phi(n)$ is also eventually a polynomial of degree at most $d-1$, and hence $\phi(n)$ is eventually a polynomial of degree at most d. □

2.3 Graded FI-modules

From this point, when we say *graded* FI-*module* we really mean FI-graded module, i.e. a functor from FI to the category of $\mathbb{N}$-graded R-modules. If V is such a module, for each $i \geq 0$ let V_S^i denote the the piece of V_S in grading i. Restricting V to V^i yields the FI-module V^i, and V can be seen as the collection of these, i.e. $V = \{V^i\}_{i \in \mathbb{N}}$.

Definition 2.10. Let V be a graded FI-module. We say that V is of *finite type* if each FI-module V^i is a finitely generated FI-module.

The tensor product of two graded R-modules M, N is defined as

$$(M \otimes N)^i = \bigoplus_{a+b=i} M^a \otimes N^b.$$

For graded FI-modules, the tensor product is defined by applying this definition pointwise.

Proposition 2.14. *Let V, W be graded* FI*-modules. If V and W is of finite type, then so is $V \otimes W$.*

Proof. For each $a, b \geq 0$, $V^a \otimes W^b$ is finitely generated by Proposition 2.3. For each $i \geq 0$, $(V \otimes W)^i$ is a direct sum of finitely many such summands, and hence is finitely generated. □

Let M be an R-module and S a set. Let $M^{\otimes S} = \bigotimes_{s \in S} M$. If $F : M \to N$ is a homomorphism and $f : S \to T$ is a *bijection*, then we let $F^{\otimes f} : M^{\otimes S} \to N^{\otimes T}$ be the map that takes the factor labeled by $s \in S$ to the one labeled by $f(s) \in T$ via the map F.

Definition 2.11. Given an FI-module V equipped with an injection $M(0) \hookrightarrow V$, define $V^{\otimes \bullet}$ to be the FI-module $(V^{\otimes \bullet})_S = (V_S)^{\otimes S}$. If $f : S \hookrightarrow T$ is an injection it acts by

$$f_* : (V_S)^{\otimes S} \xrightarrow{(f_*)^{\otimes f}} (V_T)^{\otimes f(S)} \hookrightarrow (V_T)^{\otimes T},$$

where the last map composes the isomorphism $(V_T)^{\otimes f(S)} \cong (V_T)^{\otimes f(S)} \otimes R^{\otimes T \setminus f(S)}$ with the inclusions $R \cong M(0)_T \hookrightarrow V_T$.

Remark. *To explain the role of the map $M(0) \to V$ in the definition above, we need to mention* symmetric monoidal categories.

A monoidal category *is a category $\mathcal{C}$ equipped with the following data:*

1. *A* bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, *usually called* tensor product.
2. *An identity object* 1.
3. *Three natural isomorphisms α, β, γ with components $\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, $\beta_A : A \otimes 1 \cong A$, $\gamma_A : 1 \otimes A \cong A$, such that the following diagrams*

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A \otimes B) \otimes C) \otimes D \\ \downarrow{\scriptstyle id_A \otimes \alpha_{B,C,D}} & & & & \uparrow{\scriptstyle \alpha_{A,B,C} \otimes id_D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow{\qquad \alpha_{A,B\otimes C,D} \qquad} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

$$\begin{tikzcd} A \otimes (1 \otimes B) \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "id_A \otimes \gamma_B"'] & & (A \otimes 1) \otimes B \arrow[dl, "\beta_A \otimes id_B"] \\ & A \otimes B & \end{tikzcd}$$

commute for all A, B, C, D.

Such category is symmetric monoidal *if it in addition is equipped by a natural isomorphism $\sigma_{A,B} : A \otimes B \cong B \otimes A$ for all A, B, making the following diagrams*

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\sigma_{A,B} \otimes id_C"] \arrow[d, "\alpha_{A,B,C}"] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\ A \otimes (B \otimes C) \arrow[d, "\sigma_{A,B \otimes C}"] & B \otimes (A \otimes C) \arrow[d, "id_B \otimes \sigma_{A,C}"] \\ (B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"] & B \otimes (C \otimes A) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes 1 \arrow[r, "\sigma_{A,1}"] \arrow[dr, "\beta_A"] & 1 \otimes A \arrow[d, "\gamma_A"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \arrow[dr, "\sigma_{B,A}"] & \\ A \otimes B \arrow[u, "\sigma_{A,B}"] \arrow[r, "id_{A \otimes B}"] & A \otimes B \end{tikzcd}$$

commute for all A, B, C.

The category of FI*-modules together with the tensor product makes form a symmetric monoidal category where $M(0)$ plays the role of the unit* 1*. If X is an object in any such category $\mathcal{C}$ where there is a morphism $f : 1 \to X$, we can define an* FI*-object which sends a finite set S to $X^{\otimes S}$ and the natural inclusions $\iota_{n,n+1}$ to*

$$X^{\otimes n} \otimes f : X^{\otimes n} \cong X^{\otimes n} \otimes 1 \overset{id_{X^{\otimes n}} \otimes f}{\longrightarrow} X^{\otimes n} \otimes X \cong X^{\otimes n+1},$$

and the morphisms in $\mathrm{End}([n])$ *to the corresponding action in $\mathcal{C}$.*

If we then have a morphism $M(0) \to V$ of FI*-modules we can define an* FI*-object in the category of* FI*-modules, i.e. an* FI*-*FI*-module, which sends a finite set S to $V^{\otimes S}$ which is the* FI*-module which sends a finite set T to $(V_T)^{\otimes S}$. We can view this as functor* $\mathrm{FI} \times \mathrm{FI} \to R$*-Mod, i.e. a* bi-FI-module*, which sends (S, T) to $(V_T)^{\otimes S}$, and by precomposing with the diagonal functor* $\mathrm{FI} \to \mathrm{FI} \times \mathrm{FI}$ *sending S to (S, S) we obtain the* FI*-module $V^{\otimes \bullet}$ defined above.*

If V is a graded FI-module, the individual pieces $(V_S)^{\otimes S}$ are graded R-modules. We then consider $M(0)$ to be concentrated in degree 0, and we require the map $M(0) \hookrightarrow V$ to preserve grading. Then the grading is also preserved by the map $f_* : (V_S)^{\otimes S} \to (V_T)^{\otimes T}$, so in this case $V^{\otimes \bullet}$ is a graded FI-module.

Proposition 2.15. *Let V be a graded* FI*-module with $V^0 \cong M(0)$. If V is of finite type, then so is the* FI*-module $V^{\otimes \bullet}$.*

Proof ([6], p.34). Let M be a graded FI-module which is free in each degree. This means that there exists an index set L such that for each $l \in L$ there exists numbers $m_l, i_l \in \mathbb{N}$ such that we have $M^i \cong \bigoplus_{i_l=i} M(m_l)$. Every graded FI-module comes with a surjection $M \twoheadrightarrow V$ from such an M. If V is of finite type we can assume M is as well, i.e. the set $\{l \in L \mid i_l = j\}$ is finite for any given j. Since we have assumed $V^0 \cong M(0)$ we can assume there is a unique $l_0 \in L$ with $i_{l_0} = 0$, and it satisfies $m_{l_0} = 0$. Since a surjection $M \twoheadrightarrow V$ induces a surjection $M^{\otimes\bullet} \twoheadrightarrow V^{\otimes\bullet}$, we only need to prove that $M^{\otimes\bullet}$ is of finite type.

A basis for $M_S^{\otimes\bullet} \cong (M_S)^{\otimes S}$ is given by a choice of indices $\eta : S \to L$, and for each $s \in S$ an injection $g_s : [m_{\eta(s)}] \hookrightarrow S$. For such a basis element, the multiset $\eta(S)$ can be written uniquely for some $j \leq |S|$ as $\{l_1, ..., l_j\} \cup \{l_0, ..., l_0\}$ with $l_1, ..., l_j \in L \setminus \{l_0\}$. Any map $f_* : M_S^{\otimes\bullet} \to M_T^{\otimes\bullet}$ induced from an injection $f : S \hookrightarrow T$ is going to be basis preserving, and every basis element is going to be taken to a basis element determining the same multiset $\underline{l} := \{l_1, ..., l_j\}$, so $M^{\otimes\bullet}$ splits as a direct sum

$$M^{\otimes\bullet} = \bigoplus_{\underline{l}} M_{\underline{l}}^{\otimes\bullet}$$

indexed by such multisets. We can now show that every such summand is finitely generated.

We fix some $\underline{l}$. For any finite set T, a basis for $(M_{\underline{l}}^{\otimes\bullet})_T$ is determined by

$$\{(\eta : T \to L, g_t : [m_{\eta(t)}] \hookrightarrow T) \mid \eta(T) = \{l_1, ..., l_j\} \cup \{l_0, ..., l_0\}\}.$$

If $j + m_{l_1} + ... + m_{l_j} < |T|$, then there must be some $t \in T$ with $\eta(t) = l_0$, such that $t \notin im(g_{t'})$ for all $t' \in T$. Let $S = T \setminus \{t\}$, so $im(g_s) \subset S$ for all $s \in S$- The map f_* induced by the inclusion $f : S \hookrightarrow T$ is going to send the basis element $(\eta|_S, g_s)$ of $(M_{\underline{l}}^{\otimes\bullet})_S$ to the basis element (η, g_t) of $(M_{\underline{l}}^{\otimes\bullet})_T$, so the FI-module $(M_{\underline{l}}^{\otimes\bullet})_T$ is generated in degree $\leq j + m_{l_1} + ... + m_{l_j}$. It follows that $M_{\underline{l}}^{\otimes\bullet}$ is finitely generated, since the basis is finite for each finite set T.

Each summand $M_{\underline{l}}^{\otimes\bullet}$ only contributes to $(M^{\otimes\bullet})^i$ in grading $i = i_{l_1} + ... + i_{l_j}$. Since $\{l \in L \mid l_i = j\}$ is finite for each j, if we fix $i \in \mathbb{N}$, there are only finitely many $\underline{l}$ with $i = i_{l_1} + ... + i_{l_j}$. Hence for each $i \in \mathbb{N}$ the FI-module $(M^{\otimes\bullet})^i$ is a finite direct sum of finitely generated FI-modules $M_{\underline{l}}^{\otimes\bullet}$, so the graded FI-module $M^{\otimes\bullet}$ is of finite type. □

3 Church's Theorem

Before we can prove Theorem 1.2 we need to take a small detour and talk about a spectral sequences, and in particular one called the Leray spectral sequence.

Definition 3.1. A *cohomology spectral sequence*, or sometimes just *spectral sequence*,starting at E_a in an abelian category $\mathcal{A}$ consists of the following:

1. A family of objects $\{E_r^{pq}\}$ of $\mathcal{A}$ defined for all integers p, q, where $r \geq a$.

2. Maps $d_r^{pq} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that are differentials in the sense that $d^2 = 0$ when the composition is defined.

3. Isomorphisms
$$E_{r+1}^{pq} \cong \ker(d_r^{pq}) / \operatorname{im}(d_r^{p-r,q+r-1}).$$

The *total degree* of the term E_r^{pq} is $p+q$. We say a spectral sequence is *bounded* if for each n there are only finitely many nonzero terms of total degree n in E_a^{**}. If this is the case, then for each p, q there is an r_0 such that $E_r^{pq} = E_{r+1}^{pq}$ for all $r \geq r_0$, since eventually the target of each differential will be 0. We write E_∞^{pq} for the stable value of E_r^{pq}.

A bounded spectral sequence *converges* to H^* if there is a family of objects H^n of $\mathcal{A}$, each having a finite filtration

$$0 = F^t H^n \subset \ldots \subset F^{p+1} H^n \subset F^p H^n \subset \ldots \subset F^s H^n = H^n,$$

such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$.

If a spectral sequence is first quadrant ($E_a^{pq} = 0$ unless $p \geq 0$ and $q \geq 0$) and converges to H^*, then each H^n has a filtration of length $n+1$

$$0 = F^{n+1} H^n \subset F^n H^n \subset \ldots \subset F^1 H^n \subset F^0 H^n = H^n.$$

A spectral sequence starting at a converging to H^* is usually written as

$$E_a^{pq} \Longrightarrow H^{p+q}.$$

A spectral sequence is said to *collapse at E_r* ($r \geq 2$) if there is exactly one nonzero row or column in the lattice $\{E_r^{pq}\}$. If a collapsing spectral sequence converges to H^* it is easy to read of what the objects H^n look like. It will be the unique nonzero E_r^{pq} with $p+q = n$.

3.1 Leray spectral sequence

Proposition 3.1. *A sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in an abelian category $\mathcal{A}$ is exact if for every object M in $\mathcal{A}$ the sequence

$$\operatorname{Hom}(M, A) \xrightarrow{f_*} \operatorname{Hom}(M, B) \xrightarrow{g_*} \operatorname{Hom}(M, C)$$

is exact.

Proof. First, let $M = A$. Then we get that $g \circ f = g_* \circ f_*(\mathrm{Id}_A) = 0$. Now if we let $M = \ker(g)$ we get that the inclusion $\iota : \ker(g) \hookrightarrow B$ satisfies $g_*(\iota) = g \circ \iota = 0$, so there exists some $\varphi \in \operatorname{Hom}(M, A)$ with $\iota = f_*(\varphi) = f \circ \varphi$. This gives us that $\ker(g) = \operatorname{im}(\iota) \subset \operatorname{im}(f)$, and hence $\ker(g) = \operatorname{im}(f)$. □

Theorem 3.2. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be a pair of additive adjoint functors between abelian categories $\mathcal{A}$ and $\mathcal{B}$, i.e. there exists a natural isomorphism*

$$\phi : \mathrm{Hom}_{\mathcal{B}}(F(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, G(B)),$$

for all A, B. Then F is right exact and G is left exact.

Proof. We will prove this by first proving every right adjoint G is left exact, and from this we get that $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$, which is right adjoint, is left exact, and so F is right exact.

Suppose now that

$$0 \to B' \to B \to B'' \to 0$$

is exact in $\mathcal{B}$. By naturality of ϕ there is a commutative diagram for every object A of $\mathcal{A}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(A), B') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(A), B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(A), B'') \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, G(B')) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, G(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, G(B'')). \end{array}$$

The top row is exact since $\mathrm{Hom}(F(A), -)$ is a left exact functor, so the bottom row is also exact. By Proposition 3.1 the following sequence must be exact:

$$0 \to G(B') \to G(B) \to G(B''),$$

which shows G is left exact. $\square$

Definition 3.2. Let X be a topological space and let $\mathcal{A}$ be an abelian category. A *presheaf* $\mathcal{F}$ on X with values in $\mathcal{A}$ is a contravariant functor from the poset $\mathcal{U}$ of open sets of X to $\mathcal{A}$, such that $\mathcal{F}(\emptyset) = \{0\}$.

A *sheaf* on X with values in $\mathcal{A}$ is a presheaf $\mathcal{F}$ such that the following axiom holds:

Let $\{U_i\}$ be an open cover of an open set $U \subset X$. If $\{f_i \in \mathcal{F}(U_i)\}$ are such that each f_i and f_j agree on $\mathcal{F}(U_i \cap U_j)$, then there is a unique $f \in \mathcal{F}(U)$ that maps to every f_i under $\mathcal{F}(U) \to \mathcal{F}(U_i)$.

The collections of presheaves $\mathrm{Presh}_{\mathcal{A}}(X)$ and of sheaves $\mathrm{Sh}_{\mathcal{A}}(X)$(with values in $\mathcal{A}$), together with natural transformations both form abelian categories.

Definition 3.3. Let X be a topological space and let $\mathrm{Sh}_{Ab}(X)$ be the category on X with values in abelian groups. Let $\mathcal{F} \in \mathrm{Sh}_{Ab}(X)$ be a sheaf. A *stalk of* $\mathcal{F}$ at $x \in X$ is defined to be the abelian group

$$\mathcal{F}_x := \varinjlim \{\mathcal{F}(U) \mid x \in U\}.$$

There is a way to get a sheaf from a presheaf called *sheafification*.

Proposition 3.3. *Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{S}(\mathcal{F})$ and a morphism $\theta : \mathcal{F} \to \mathcal{S}(\mathcal{F})$ with the property that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\psi : \mathcal{S}(\mathcal{F}) \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Furthermore the pair $(\mathcal{S}(\mathcal{F}), \theta)$ is unique up to unique isomorphism. The sheaf $\mathcal{S}(\mathcal{F})$ is called the* sheaf associated to the presheaf $\mathcal{F}$ *or the* sheafification of $\mathcal{F}$.

Proof. See [9]. □

The sheafification functor $\mathcal{S} : \mathrm{Presh}(X) \to \mathrm{Sh}(X)$ is left adjoint to the inclusion $\iota : \mathrm{Sh}(X) \to \mathrm{Presh}(X)$, so by Theorem 3.2 $\mathcal{S}$ is right exact.

Theorem 3.4. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories such that for any injective object I of $\mathcal{A}$, $F(I)$ is a G-acyclic object of $\mathcal{B}$, i.e. $R^iG(F(I)) = 0$ for all $i > 0$, where R^iG is the right derived functors of G. If $\mathcal{B}$ has enough injectives, then there is a first-quadrant spectral sequence for each object A of $\mathcal{A}$:*

$$E_2^{pq} = (R^pG \circ R^qF)(A) \Longrightarrow R^{p+q}(G \circ F)(A).$$

This is what is called the *Grothendieck spectral sequence*.

Proof. See section 5.8 in [7]. □

Let X, Y be topological spaces and let $\mathcal{A} = \mathrm{Sh}_{Ab}(X)$, $\mathcal{B} = \mathrm{Sh}_{Ab}(Y)$ be the category of sheaves of abelian groups on X and Y respectively. Let $\mathcal{C} = \mathrm{Ab}$ be the category of abelian groups. For any continuous function $f : X \to Y$ we have a direct image functor

$$f_* : \mathrm{Sh}_{Ab}(X) \to \mathrm{Sh}_{Ab}(Y)$$

defined by $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. The functor f_* is right adjoint to the inverse image sheaf $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ defined to be the sheafification of the presheaf

$$U \mapsto \varinjlim \{\mathcal{G}(V) \mid f(U) \subset V\},$$

where U is open in X, V is open in Y and $\mathcal{G}$ is a sheaf on Y (exercise 2.6.2). By Theorem 3.2, f_* is left exact. The following lemma shows that a functor that is right adjoint to an exact functor preserves injective objects:

Lemma 3.5. *Suppose $R : \mathcal{B} \to \mathcal{A}$ is an additive functor that is right adjoin to an exact functor $L : \mathcal{A} \to \mathcal{B}$. If I is an injective object in $\mathcal{B}$, then $R(I)$ is an injective object in $\mathcal{A}$.*

Proof. We can prove this by showing $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Let $f : A \to A'$ be an injection in $\mathcal{A}$. Since R, L are adjoint the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(L(A'), I) & \xrightarrow{Lf_*} & \mathrm{Hom}_{\mathcal{A}}(L(A), I) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_{\mathcal{A}}(A', R(I)) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathcal{A}}(A, R(I))
\end{array}$$

commutes. Since L is exact, $Lf : L(A) \to L(A')$ is injective, and since I is an injective object, for every $\varphi : L(A) \to I$ there exists a ψ such that the following diagram

$$\begin{array}{ccc} L(A) & \xhookrightarrow{Lf} & L(A') \\ \downarrow \varphi & \swarrow \psi & \\ I & & \end{array}$$

commutes, and so the map Lf_* is surjective. So we get that f_* is surjective as well, and hence $R(I)$ is an injective object. □

We also have the global section functor $\Gamma_X : \mathrm{Sh}_{Ab}(X) \to \mathrm{Ab}$ defined by

$$\Gamma_X(\mathcal{F}) = \mathcal{F}(X)$$

and similarly for Γ_Y. The right derived functors of the global section functor defines a cohomology theory called sheaf cohomology. We usually write

$$\mathcal{R}^i(\Gamma_X)(\mathcal{F}) := H^i(X; \mathcal{F}).$$

Since $\Gamma_Y \circ f_*(\mathcal{F}) = \Gamma_Y(\mathcal{F}(f^{-1}(-))) = \mathcal{F}(f^{-1}(Y)) = \mathcal{F}(X) = \Gamma_X$ and since Γ_Y and f_* satisfies the conditions for the Grothendieck spectral sequence and since $\mathrm{Sh}_{Ab}(X)$ and $\mathrm{Sh}_{Ab}(Y)$ has enough injectives, we get the following spectral sequence:

$$E_2^{pq} = H^p(Y, \mathcal{R}^q f_* \mathcal{F}) \Longrightarrow H^{p+q}(X, \mathcal{F}).$$

This special case is usually called the *Leray spectral sequence*.

3.2 Transfer homomorphism

The purpose of this section is to show that there is a relation between the spaces E and X whenever $\pi : E \to X$ is a finite sheeted covering space defined by an action of a group G on E. In particular we show that there is an isomorphism between the cohomology of X and the G-invariant elements of the cohomology group of E whenever the coefficients are in a field containing $\mathbb{Q}$.

Definition 3.4. Let G be a group and let A be an abelian group together with a left G-action. We call such A a *G-module*. The collection of G-modules together with G-equivariant maps form a category G-Mod.

For a G-module A, let A^G denote the subgroup of A

$$A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}.$$

We call this subgroup the *invariant subgroup*. We also have the *coinvariants* of A:

$$A_G = A / \langle \{(ga - a) \mid g \in G, \ a \in A\} \rangle$$

the quotient of A by the submodule generated by the elements $(ga - a) \in A$.

Remark. *The subgroup A^G is the maximal trivial submodule of A. If B is another trivial submodule such that $A^G \subset B$, then for $b \in B$ and $g \in G$ we have $gb = b$, so by construction $b \in A^G$ and hence $A^G = B$. The assignment $A \mapsto A^G$ gives us a functor*

$$-^G : G\text{-}Mod \to \mathrm{Ab}$$

and this functor is in fact right adjoint to the functor

$$T : \mathrm{Ab} \to G\text{-}Mod$$

which takes an abelian group and considers it as a trivial G-module. To see this, consider a map $f \in \mathrm{Hom}_{G\text{-}Mod}(TX, Y)$. We have the following commutative diagram

$$\begin{array}{ccc} TX & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G} \\ TX & \xrightarrow{f} & Y \end{array}$$

and since G acts trivially on TX we have that for all $g \in G$ and $x \in TX$, $gf(x) = f(gx) = f(x)$, so this is exactly the same as a group homomorphism φ from X to a G-invariant subgroup of Y and by maximality of Y^G the map φ extends to a homomorphism $X \to Y^G$. Hence we have a bijection

$$\mathrm{Hom}_{G\text{-}Mod}(TX, Y) \cong \mathrm{Hom}_{\mathrm{Ab}}(X, Y^G),$$

natural in X and Y. By Lemma 3.2 $-^G$ is left exact.

Similarly, $A_G = A/\Gamma$ is the largest trivial quotient of A. If $A/\tilde{\Gamma}$ is another such quotient, then for all $g \in G$ and $\bar{a} \in A/\tilde{\Gamma}$ we have $g\bar{a} = \bar{a}$ so $0 = g\bar{a} - \bar{a} \in \tilde{\Gamma}$ and hence $\Gamma \subset \tilde{\Gamma}$ which gives us $A/\tilde{\Gamma} \subset A_G$. From this we can conclude that

$$\mathrm{Hom}_{Ab}(X_G, Y) \cong \mathrm{Hom}_{G\text{-}Mod}(X, TY),$$

since a map $f \in \mathrm{Hom}_{G\text{-}Mod}(X, TY)$ is the same as a group homomorphism $X \to Y$ such that $f(gx) = f(x)$, i.e. $0 = f(gx) - f(x) = f(gx - x)$ which is the same as a group homomorphism $X_G \to Y$ and by universality of X_G we have a natural bijection. From this we get that $-_G$ is right exact.

We call the right derived functors $\mathcal{R}^*(-^G)(A)$ the group cohomology *of G with coefficients in A, written $H^*(G; A)$. Similarly, the left derived functors of $-_G$ are called the* group homology *of G.*

Let $p \colon E \to X$ be an n-sheeted covering space for some $n < \infty$. This induces a map of on singular chains $p' : C_i(E) \to C_i(X)$ for all $i \geq 0$. We also have homomorphism $q \colon C_i(X) \to C_i(E)$ for every $i \geq 0$ defined by taking a singular simplex $\sigma : \Delta^i \to X$ to the sum of the n distinct lifts $\tilde{\sigma} : \Delta^i \to E$. The map q is clearly a chain map, so it induces a homomorphisms

$$q^* \colon H^*(E; G) \to H^*(X; G)$$

for any group G, called *transfer homomorphisms*.

Consider the composition $p' \circ q : C_i(X) \to C_i(X)$. We have

$$p' \circ q(\sigma) = p'\big(\sum_{i=1}^{n} \tilde{\sigma}\big) = \sum_{i=1}^{n} p'(\tilde{\sigma}) = n \cdot \sigma.$$

Hence the induced map $(p \circ q)^* = q^* \circ p^*$ is multiplication by n. If we now look at the kernel of p^* we see that $p^*(x) = 0$ implies $q^* \circ p^*(x) = nx = 0$, so $\ker(p^* \colon H^*(X; G) \to H^*(E; G))$ consists of torsion elements whose order divide n. We now prove the following proposition:

Proposition 3.6. *Let $p : E \to X$ be an n-sheeted covering space projection defined by an action of G on E. If k is a field whose characteristic is 0 or a prime of order not dividing n, then the map $p^* : H^*(X; k) \to H^*(E; k)$ is injective and the image is $H^*(E; k)^G$, the subgroup consisting of classes α such that $g^*\alpha = \alpha$ for all $g \in G$.*

Proof. We described the kernel of p^* above, and by the assumptions on k the only such element is 0, and hence p^* is injective.

For the second statement, note that the composition $q \circ p'$ sends a singular simplex $\tilde{\sigma} : \Delta^i \to E$ to

$$\tilde{\sigma} \overset{p'}{\mapsto} \sigma \overset{q}{\mapsto} \sum_{g \in G} g\tilde{\sigma},$$

the sum of all its images under the action of G. Hence the induced map $p^* \circ q^*$ is defined by

$$p^* \circ q^*(\alpha) = \sum_{g \in G} g^*\alpha.$$

If $\alpha \in H^*(E; k)$ is fixed under the G action, then $p^* \circ q^*(\alpha) = n\alpha$, so since k has characteristic 0 or a prime not dividing n we have

$$\alpha = p^* \circ q^*(\alpha/n)$$

in this particular case, and thus $\alpha \in \mathrm{im}(p^*)$.

For the converse, note that since $p \circ g(x) = p(x)$ for all $x \in E$ and $g \in G$, and hence the induced map is $g^* \circ p^* = p^*$, so $\mathrm{im}(p^*) \subset H^*(E; k)^G$. Thus $\mathrm{im}(p^*) = H^*(E; k)^G$. $\square$

In particular, for the covering space projection of the configuration spaces of some topological space X, $C_n(X) \to B_n(X)$ defined by the action of the symmetric group S_n gives us an isomorphism $H^*(C_n(X); k)^{S_n} \cong H^*(B_n(X); k)$ whenever k is as in the above proposition. We will mainly care about the case when $k = \mathbb{Q}$ or some field containing $\mathbb{Q}$.

3.3 Noetherian approach

Similarly to FI-modules, we have the notion of an FI-space which is a functor FI $\to$ Top. We can do the same thing with other categories too, for example FI-groups. When we talk about co-FI objects we mean a contravariant functor from FI to some category. For example a co-FI-space is a functor $\mathrm{FI}^{op} \to \mathrm{Top}$.

Definition 3.5. Let $M^\bullet$ be the co-FI-space defined by $M^S = \mathrm{Maps}(S, M)$, the space of continuous functions $\varphi : S \to M$. For an injection $f : T \to S$ we get a map $f^* : M^S \to M^T$ defined by precomposition $\varphi \circ f$.

Definition 3.6. Let $\mathrm{Conf}(M)$ be the co-FI-space defined by $\mathrm{Conf}_S(M) = \mathrm{Inj}(S, M)$, the space of injections $\varphi : S \to M$.

There is a natural inclusion of co-FI-spaces $i : \mathrm{Conf}(M) \hookrightarrow M^\bullet$. Another FI-object we will make use of is a co-FI-chain complex.

Definition 3.7. Let $\mathbf{C}_*$ be a chain complex equipped with a surjection $s : \mathbf{C}_* \twoheadrightarrow \mathbf{R}_*$ where

$$\mathbf{R}_i = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We define the co-FI-chain complex $\mathbf{C}_*^{\otimes\bullet}$ by $(\mathbf{C}_*^{\otimes\bullet})_{[n]} = \mathbf{C}_*^{\otimes n}$. For an injection $f : [n] \hookrightarrow [m]$, the induced map $f^* : \mathbf{C}_*^{\otimes m} \to \mathbf{C}_*^{\otimes n}$ projects each factor lying in $[m] \setminus f([n])$ onto $\mathbf{R}_*$ and acts on the rest by permuting them according to f^{-1}. Sometimes a sign is also introduced depending on grading.

For example in degree 2, if $n = 2$ and $m = 3$ and $f : [2] \hookrightarrow [3]$ is defined by $f(1) = 3,\ f(2) = 1$ we have the following case:

$$\begin{aligned}
\mathbf{C}_2^{\otimes 3} =& C_2 \otimes C_0 \otimes C_0 \\
&\oplus C_0 \otimes C_2 \otimes C_0 \\
&\oplus C_0 \otimes C_0 \otimes C_2 \\
&\oplus C_1 \otimes C_1 \otimes C_0 \\
&\oplus C_0 \otimes C_1 \otimes C_1 \\
&\oplus C_1 \otimes C_0 \otimes C_1 \\
\mathbf{C}_2^{\otimes 2} =& C_2 \otimes C_0 \\
&\oplus C_0 \otimes C_2 \\
&\oplus C_1 \otimes C_1.
\end{aligned}$$

Since $[3] \setminus f([2]) = \{2\}$, the middle row of $\mathbf{C}_2^{\otimes 3}$ first gets mapped to $\mathbf{R}_*$, i.e. to

0 unless the entry is C_0 in which case it gets mapped to R:

$$\begin{aligned}\mathbf{C}_2^{\otimes 3} \to \bigoplus_{a+b+c=2} C_a \otimes s(C_b) \otimes C_c =& C_2 \otimes R \otimes C_0 \\ &\oplus C_0 \otimes 0 \otimes C_0 \\ &\oplus C_0 \otimes R \otimes C_2 \\ &\oplus C_1 \otimes 0 \otimes C_0 \\ &\oplus C_0 \otimes 0 \otimes C_1 \\ &\oplus C_1 \otimes R \otimes C_1 \cong \mathbf{C}_2^{\otimes 2}.\end{aligned}$$

Each factor then gets permuted according to $3 \mapsto 1,\ 1 \mapsto 2$.

The main difference in proving Theorem 1.2 when R is a field as opposed to a general Noetherian ring lies in the following lemma:

Lemma 3.7. *Let M be a connected topological space with the homotopy type of a CW complex with finitely many cells in each dimension. Then for all $m \geq 0$, the* FI*-module $H^m(M^\bullet; R)$ is generated in finite degree.*

When R is a field we get that $H^*(M^S; R) \cong H^*(M; R)^{\otimes S}$ by the Künneth theorem (see for example [11]), so we can guess that $H^*(M^\bullet; R) \cong H^*(M; R)^{\otimes \bullet}$ as FI-modules. The problem is that the Künneth isomorphism depends on an ordering of the set S, so in the setting of FI-modules, when these factors get permuted a sign might get introduced depending on the grading. However, if we view $H^*(M; R)$ as a constant graded FI-module V, the assumption that M is connected gives us that $H^0(M; R) = R$ so we have an isomorphism $V^0 \cong M(0)$, and since we assume M has the homotopy type of a CW complex with finitely many cells in each dimension, V is of finite type. Now we can apply Proposition 2.15 and get our result. For general Noetherian rings things are a bit more complicated.

Proof of Lemma 3.7 ([5], p.29). Since M is connected, and since M is homotopy equivalent to a CW complex, we may assume that this CW complex has exactly one 0-cell. We let $\mathbf{C}_*$ denote the cellular chain complex over R corresponding to the CW complex. In degree n this is the free R-module generated by the n-cells of the CW complex. In particular, this chain complex consists of projective finitely generated R-modules and will be bounded below with $\mathbf{C}_0 = R$.

Since $\mathbf{C}_*$ is quasi isomorphic to $\mathcal{C}_*(M)$ (see for example Chapter 4 in [10]) we have that $\mathbf{C}_*^{\otimes \bullet}$ is quasi isomorphic to $\mathcal{C}_*(M)^{\otimes \bullet}$. This is because both complexes consists of free R-modules and the tensor product of free R-modules is free, so in each degree we have a direct sum of copies of R and induced map of the quasi isomorphism is again a quasi isomorphism under these conditions.

By the Eilenberg-Zilber theorem (see for example [7] Chapter 5.8) we have a quasi isomorphism $\mathcal{C}_*(M)^{\otimes \bullet} \simeq \mathcal{C}_*(M^\bullet)$, so $\mathbf{C}_*^{\otimes \bullet} \simeq \mathcal{C}_*(M^\bullet)$. Thus to compute $H^*(M^\bullet; R)$ we can compute the cohomology of the chain complex $\mathrm{Hom}((\mathbf{C}_*)^{\otimes \bullet}, R)$. This complex consists of projective finitely generated R-modules, and it can now be viewed as an FI-chain complex since both functors $(\mathbf{C}_*)^{\otimes \bullet}$ and $\mathrm{Hom}(-, R)$ are contra variant. We denote the component in degree m by $\mathrm{Hom}((\mathbf{C}_*)^{\otimes \bullet}, R)^m$, so for all n we have

$$\mathrm{Hom}((\mathbf{C}_*)^{\otimes n}, R)^m := \bigoplus_{m_1+...+m_n=m} \mathrm{Hom}(\mathbf{C}_{m_1} \otimes ... \otimes \mathbf{C}_{m_n}, R).$$

When $n > m$, each such component must have $m_i = 0$ for at least one i, and hence it lies in the image of f_* for some $f : [n-1] \hookrightarrow [n]$. Therefore $\mathrm{Hom}((\mathbf{C}_*)^{\otimes \bullet}, R)^m$ is finitely generated in degree m, and since $H^m(M^\bullet)$ is a subquotient of this FI-module it is also finitely generated by Theorem 2.7. □

Now we use this to prove Theorem 1.2.

Proof of Theorem 1.2 ([5] p.29). The inclusion of co-FI-spaces $i : \mathrm{Conf}(M) \hookrightarrow M^\bullet$ results in a Leray spectral sequence of FI-modules

$$E_2^{p,q} = H^p(M^\bullet; \mathcal{R}^q i_*(\underline{R})) \Longrightarrow H^{p+q}(\mathrm{Conf}(M); R).$$

We start by showing that the $E_2^{p,q}$ is finitely generated for all $p, q \geq 0$. In [8, Theorem 1], Totaro describes the E_2 page of this spectral sequence. In particular, he shows that $E_2^{*,*}$ is generated by two subalgebras: $E_2^{*,0} = H^*(M^\bullet; R)$, which is finitely generated by Lemma 3.7, and $E_2^{0,*}$. He also shows that $E_2^{0,*}$ is generated by $E_2^{0,d-1}$ which is generated in degree 2.

Since this spectral sequence is first quadrant, for any given $p, q \geq 0$ there are only finitely many terms on each axis which can multiply to $E_2^{p,q}$. Each entry on the E_2 page is thus a finite direct sum of finite tensor products of finitely generated FI-modules, and such a tensor product is finitely generated by Proposition 2.3. It follows that $E_2^{p,q}$ is finitely generated.

Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, it is also finitely generated for each $p, q \geq 0$ by Theorem 2.7. For every $m \geq 0$, the FI-module $H^m(\mathrm{Conf}(M); R)$ has a finite length filtration

$$0 = F^{m+1}H^m \subset F^m H^m \subset ... \subset F^1 H^m \subset F^0 H^m = H^m(\mathrm{Conf}(M); R)$$

with $E_\infty^{pq} \cong F^p H^{p+q}/F^{p+1}H^{p+q}$. In particular we have

$$E_\infty^{m,0} \cong F^m H^m / F^{m+1} H^m = F^m H^m$$
$$E_\infty^{0,m} \cong F^0 H^m / F^1 H^m = H^m / F^1 H^m.$$

To show that $H^m(\mathrm{Conf}(M); R)$ is finitely generated, consider first the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^m H^m & \longrightarrow & F^{m-1} H^m & \longrightarrow & F^{m-1} H^m / F^m H^m & \longrightarrow & 0. \\ & & \| & & & & \| & & \\ & & E_\infty^{m,0} & & & & E_\infty^{m-1,1} & & \end{array}$$

Since E_∞^{pq} is finitely generated for each $p, q \geq 0$, the term $F^{m-1}H^m$ in the middle is also finitely generated by Proposition 2.4. Inductively, for every $0 \leq j < m$ we get a short exact sequence

$$0 \longrightarrow F^{m-j}H^m \longrightarrow F^{m-(j+1)}H^m \longrightarrow E_\infty^{m-(j+1)} \longrightarrow 0$$

where both terms on either side of the middle term is finitely generated. For $j = m-1$ the middle term is $F^0H^m = H^m(\mathrm{Conf}(M); R)$, so applying Proposition 2.4 give us the desired result. □

By applying Theorem 2.13 we get the following corollary:

Corollary 3.8. *Let k be a field, and let M be a connected orientable manifold of dimension ≥ 2 with the homotopy type of a CW complex. Then for any $m \geq 0$ there exists a polynomial $p(x) \in \mathbb{Q}[x]$ (depending on M, m and k) such that for n sufficiently large*

$$\dim_k H^m(\mathrm{Conf}_n(M); k) = p(n).$$

We also have the following corollary about the cohomology of the unordered configuration spaces $B_n(M)$:

Corollary 3.9. *Let M be a connected orientable manifold of dimension ≥ 2 with the homotopy type of a CW complex. Then the sequence $\{H^m(B_n(M); \mathbb{Q})\}$ is homologically stable.*

Proof. The sequence $\{H^m(C_n(M); \mathbb{Q})\}$ is representation stable and in particular the multiplicity of the trivial representation in $H^m(C_n(M); \mathbb{Q})$ eventually stabilizes. Since the trivial representation is $H^m(C_n(M); \mathbb{Q})^{S_n} \cong H^m(B_n(M); \mathbb{Q})$, the dimension of $H^m(B_n(M); \mathbb{Q})$ eventually stabilizes and hence the sequence $\{H^m(B_n(M); \mathbb{Q})\}$ is homologically stable.

□

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