SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Unintuitive Infinity

av

Ashkan Ek

2021 - No K10

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Unintuitive Infinity

Ashkan Ek

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Salvador Rodríguez López

2021

Abstract

Even if the concept of infinity is not totally out of reach of human comprehension, the size of it certainly is. Some phenomena only exist within the content of infinity, the only context in which they can make sense. As a result, these phenomena not only may sound absurd, but they might also appear equally impractical. But we will see how some of these unintuitive results that emerge from infinite sets can complement our understanding of what already is within our grasp.

Some infinities are greater than other infinities, and one-to-one correspondence between two infinite sets may or may not be possible. Is there a way to map all the points on a line segment onto all the points in a rectangle or cube? Could a curve cover a surface or volume by passing through all their points?

Some infinite sets may lead to paradoxical results. It would be impossible to duplicate a physical spherical ball that is made up of finite number of atoms; there is no way to magically have two of each atom. But what if this duplication is made possible by having the sphere be made of non-measurable infinite set?

There are also infinite sets with unexpected properties. Out of all the continuous real-valued functions on a compact interval, what portion of them are nowhere differentiable?

In the following chapters we explore the concepts behind these questions, which are only a few out of many rather peculiar possibilities revealed by infinite sets.

Contents

Chapter 1

Introduction

1.1 Hilbert's Hotel

We count using natural numbers, so we call a set countable if its elements have a one-to-one correspondence with the set of natural numbers. There are infinitely many natural numbers, so the counting can take forever, making the set countably infinite.

One illustration of the counter-intuitive property of infinite sets is the thought experiment, Hilbert's hotel. Imagine a hotel with countably infinite number of rooms, all of which are booked up. A countably infinite set of buses arrive, each carrying a countably infinite set of travelers in need of room. At first it may sound strange to ask the questions "could we fit all the new guests in the hotel" and "if so, how?". But by looking at it in the following way they may not sound that unreasonable.

To open room for the new guests, let us label every bus b with p, where p is the $b+1$'th prime number. In other words we are labeling the buses with prime numbers greater than the first prime 2 which we are sparing for a later purpose. Then we label the n'th passenger of the b'th bus, p^n, where n is a natural number. The fundamental theorem of arithmetic states that any integer greater than 1, is either a prime number or can be factored into a unique product of prime numbers. Due to the fundamental theorem of arithmetic everyone outside the hotel is now uniquely labeled. Since we have not used the powers of 2 yet, back in the hotel, we can ask every guest in the room number n, to move to the room 2^n, without anyone landing on any room numbered p^n. Now every new guest can occupy the room number that matches their label. Besides, there are still empty rooms left, such as $1, 6, ...$ and so on.

The Banach-Tarski paradox, which we will go over in a later section, is another manifestation of infinity's peculiarity. However, this kind of reasoning is invalid as soon as "countably infinite" becomes either "uncountable"or "finite".

To be better equipped for encountering more of such unintuitive cases, we take some steps back to lay out some foundation.

1.2 Spaces

Spaces and points do not necessarily need to be geometrical in nature. Spaces seen as a selection of points, can more generally represent a selection of mathematical objects, together with a selection of relationships between those objects. These objects or "points", could be elements of a set, functions on another space, subsets of another space, etc. The relationships between the objects establish the identity and the structure of the spaces. Euclidean spaces, linear spaces, topological spaces and metric spaces, are some of the examples among many others. Two spaces are considered identical when their structures are preserved upon a one-to-one correspondence between their objects. These spaces with the same structure are isomorphic spaces and the correspondences are isomorphisms. [1, p.1]

1.3 Metric space

Before talking about measuring sets we could define a notion of "distance" between two elements, by a function called metric.

Definition 1.1. *(Metric and metric space) A metric space, "d on X", is a pair (X, d), in which the set X is endowed with a distance function d, called metric, that defines a notion of distance between two elements by $d : X \times X \to [0, \infty) \in \mathbb{R}$. [2, p.8] For all $x, y, z \in X$ the metric satisfies the axioms $d(x, y) \leq d(x, z) + d(z, y)$, $d(x, y) = d(y, x)$ and $d(x, y) = 0 \iff x = y$.*

Definition 1.2. *(Norm) The norm f on a vector space V over a field $\mathbb{F}$, is a positive-valued function $f : V \to \mathbb{R}$ that for $\overline{u}, \overline{v} \in V$ and $a \in \mathbb{F}$, satisfies the triangle inequality $f(\overline{u} + \overline{v}) \leq f(\overline{u}) + f(\overline{v})$, the equality $|a| \cdot f(\overline{u}) = f(a\overline{u})$ and that if $f(\overline{u}) = 0$ then $\overline{u}$ is the zero vector. [3, p.28]*

Normed vector spaces are examples of metric spaces. After all norm defines the length of a vector which is the distance between two points and the metric could be defined by $d(\overline{u}, \overline{v}) = \|\overline{u} - \overline{v}\|$.

1.4 Open sets

Definition 1.3. *(Open ball) The set $B_r(a) = \{x \in X : d(x, a) < r\}$ is called a ball with center a and radius r.*

Despite their name, these sets do not have to be round or symmetrical. One of their functions is to help us distinguish between distinct points of a metric

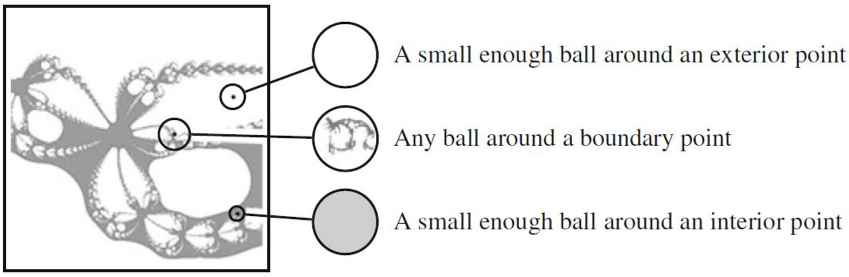

Figure 1.1: Interior, boundary, and exterior

space, by the ability to separate them in disjoint balls. [4, p.16]

Definition 1.4. *(Interior, exterior, boundary, closure) If*

$$\exists r > 0 : B_r(x) \subseteq A \tag{1.1}$$

then we can say x is in the interior of A denoted as A°, while A is a neighbourhood of x. On the other hand if

$$\exists r > 0 : B_r(x) \subseteq X \setminus A \tag{1.2}$$

then x is not in A but in its exterior. Points that are neither interior nor exterior are in the boundary of A denoted as ∂A. The closure of A is defined as $\overline{A} = A^\circ \cup \partial A$. So the complement of closure $(\overline{A})^\complement$ would be the exterior. These are represented in Figure 1.1. [4, p.18]

Definition 1.5. *The set A is open if $A = A^\circ$.*

[4, p.17]

1.5 Topology

Definition 1.6. *(Topological space) A collection of subsets of X is called a topology τ on X, if τ contains $X, \varnothing$, any arbitrary union of the members of τ, as well as any finite intersection of them. Then the elements of τ are called open sets and the ordered pair (X, τ) is called a topological space T. [5, p.188]*

For instance given the set $X = \{1, 2, 3\}$, the collection $\tau = \{\varnothing, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}$ is a topology. But if we remove $\{2\}$ from τ, the intersection of $\{1, 2\}$ and $\{2, 3\}$ is no longer contained in the collection, hence τ is no longer a topology. In (X, τ) any open set containing $p \in X$ is the neighbourhood of p.

Generally speaking the function $\phi : T \to \Delta$, where T and Δ are topological spaces, is continuous if for all $x \in T$ and any neighbourhood V of the point

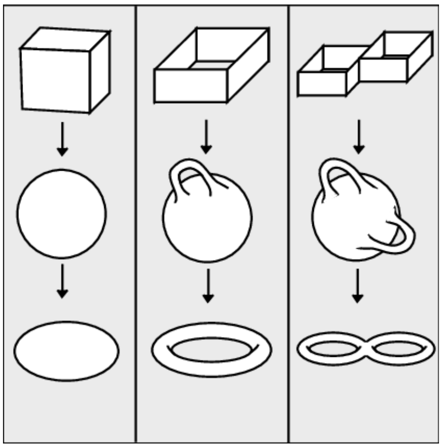

Figure 1.2: Number of holes are a topological invariant.

$\phi(x)$ in Δ, there exists a neighbourhood U of the point x in T so that $\phi(U) \subseteq V$.

Definition 1.7. *(Homeomorphism) A continuous bijection $f : (X_1, \tau_1) \to (X_2, \tau_2)$ which has a continuous inverse $f^{-1} : (X_2, \tau_2) \to (X_1, \tau_1)$ is called a homeomorphism of (X_1, τ_1) to (X_2, τ_2); while (X_1, τ_1) and (X_2, τ_2) are homeomorphic to each other i.e. $(X_1, \tau_1) \sim (X_2, \tau_2)$.*

Topological space is a generalization of the structure of Euclidean space (or sometimes called n-space) defined as a finite-dimensional inner product space over $\mathbb{R}$. In a geometrical sense, topology deals with forms and shapes in a qualitative way, as they are able to be pulled, stretched, bent, distorted and twisted but not teared or glued. One may recognize this as the idea of a mug and a doughnut being topologically equivalent, similar to the columns in the Figure 1.2[6, p.17]. As a result, the notions of homeomorphism and dimension emerge, which we will shortly take a closer look at. [7, p.106]

Topologically speaking T and Δ in the Figure 1.3 [5, p.6] are homeomorphic because we can deform T into Δ by the bijection $\phi : T \to \Delta$ yielding a one-to-one correspondence between the points $a, b, c, ...$ and $a', b', c', ...$ This bijeciton (as well as its inverse) is continuous because if the point $x \in T$ approaches the point $f \in T$, then the corresponding $x' \in \Delta$ approaches the point $f' \in \Delta$, or else we have to tear the shape apart which is topologically unacceptable. [5, p.7]

The idea of space-filling curves contributed to the development of the concept of dimension. Georg Cantor's proposition of there being a mapping from one-dimensional unit interval to two-dimensional unit square, in 1878, made mathematicians to entertain the possibility of this transformation being a homeomorphism, excluding the dimension as a topological property. As we will show, this

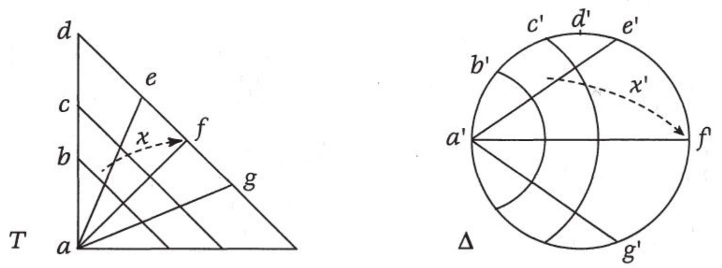

Figure 1.3: Topologically equivalent shapes.

transformation ended up not being a homeomorphism, but the struggles enriched our understanding of dimensions, for instance by the idea that we could define dimensions that lie between the one and two dimensions. [7, p.107]

1.6 The coastline, Measure and Hausdorff dimension

When assigned to the task of measuring the coast of an island, it soon becomes apparent that the precision by which we measure affects the result. One approach would be to approximate the edges of the island by straight line segments of a fixed length δ. The other would be to cover the map of the island with a grid of $\delta \times \delta$ squares (See Figure 1.4[8, p.7]) and count the number of cells containing the coastline. We let $N(\delta)$ be the number of steps required to "walk" the edge or alternatively the number of squares (from the grid) containing all the pieces of the coastline. We can then increase the precision of our measurement by letting $\delta \to 0$. This is almost like increasing the resolution of an image by adding more and smaller pixels in order to capture the intricacies.

The result in the case of the line segments, becomes $L = N(\delta)\delta$. Same with the case of covering squares, except since now we are associating an area to the coastline, we would have δ^2 instead of δ.

If there would be such a thing as a "true" length of the coast, L_0, then we would expect $N(\delta)$ and δ to be inversely proportional so that we would approach L_0 by decreasing δ. However this is not the case. In an attempt to measure the coast of Norway using the grid method, we arrive at the log-log plot in Figure 1.6 [8, p.8] where we study the change of the log of our estimated measure with respect to the log of our units. This linear function shows no sign of approaching a fixed value. Instead by assigning the slope of this graph to $1 - D$ we obtain the function $L(\delta) = a\delta^{1-D}$ where $D \simeq 1.54$. If it had been an ordinary curve, "a" would be L_0 and D would be the dimension of the line, 1.

Figure 1.4: Squares in the grid contain the coast

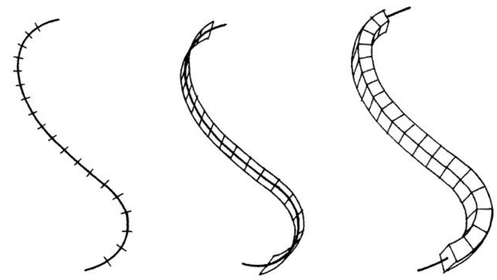

Figure 1.5: Measuring a curve through "covering" of some sort

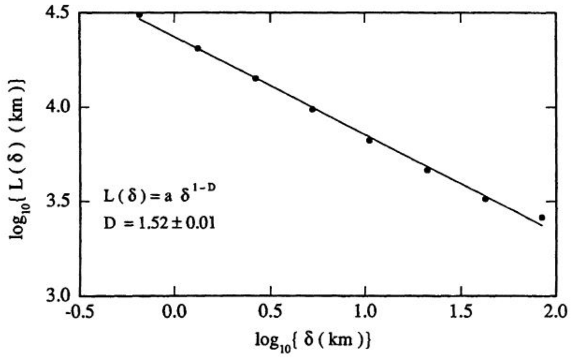

Figure 1.6: How the size δ in the $\delta \times \delta$ squares affects the measured length.

However the coast is no ordinary curve, but a fractal and D is what we call the fractal dimension or Hausdorff dimension. Topologically the coastline remains one-dimensional.

Effectively this method is the same as comparing the number of grid cells that cover an image of the coast, before and after scaling the image up by some factor on the same grid, to then seek the value to which we raise this scaling factor to capture the change in number of cells. A simpler case would be to ask "when the area of a circle quadruples due to scaling its radius by a factor of two, what power of two captures this increase in area by a factor four?" and find the answer to be the dimension of the circle.

Fractal dimensions are not necessarily integers while topological dimensions are. As long as these two dimension types have the same value we would not have a fractal.[8, p.8] The fractal dimension could be considered a measure of fractal's roughness.

If the coast was a set, covering it by a geometrical unit would still be a way of measurement. Covering a set is a recurring method to reveal certain properties. For instance a set that can be covered by a finite number of balls of arbitrary radius, is bounded. In a metric space (M, d) a subset S is bounded if $\forall s, t \in S\ \exists k > 0 : d(s, t) < k$.

We may notice covering a set by a unit of a lower dimension would not help us measure it. For example it is not possible to cover a surface with finite number of line segments to then count how many they are. However we could have covered the coastline with cubes and measured it as volume (See Figure 1.5[8, p.12]). To unify all the outcomes of covering the same body differently, we define the notion of measure.

1.7 Lebesgue measure

Definition 1.8. *(Open box) An open box B in $\mathbb{E}^n$ is a set*

$$B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1, ..., x_n) \in \mathbb{E}^n : \forall 1 \leq i \leq n, x_i \in (a_i, b_i)\} \tag{1.3}$$

with the volume $vol(B) = \prod_{i=1}^{n}(b_i - a_i)$ where $a_i \leq b_i$ and $a_i, b_i \in \mathbb{R}$.

Definition 1.9. *(Outer measure) The outer measure of a set $\Omega \subseteq \mathbb{E}^n$ is defined by*

$$m^*(\Omega) = \inf\{\sum_{j\in J} vol(B_j) : \Omega \subseteq \bigcup_{j\in J}(B_j) \text{ where } B_j \text{ are open boxes}\}. \tag{1.4}$$

where J is countable. [9, p.167]

Definition 1.10. *(Lebesgue measurability) The set $A \subseteq \mathbb{E}^n$ is Lebesgue measurable if for all $B \subseteq \mathbb{E}^n$ we have*

$$m^*(B) = m^*(A \cap B) + m^*(B \setminus A) \tag{1.5}$$

where the Lebesgue measure is $m^*(A) = m(A)$. [9, p.177]

Definition 1.11. *(Countably additivity) The measure m is countably additive if for any countable collection of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$ in $\mathbb{E}^n$, we have*

$$m(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} m(A_i). \tag{1.6}$$

Not all sets are measurable. In Euclidean space $\mathbb{E}^n$, in order to assign a value $m(A) \in [0, \infty]$ to $A \subseteq \mathbb{E}^n$ as its measure, it is required that an n-dimensional hypercube (cubical box) of side x has the measure x^n and m is countably additive. [10, p.4]

Chapter 2

Hilbert Space-filling Curve

2.1 Space-filling

There are many ways to define a curve but for our purposes we may find the following useful.

Definition 2.1. *(Curve) Let* $\mathbb{E}$ *denote Euclidean space, and* $f : \mathbb{E}^m \to \mathbb{E}^n$ *be a function with the domain* D_f *and the range* R_f*. We define the image of* $A \subseteq \mathbb{E}^m$ *under* f *as*

$$f_*(A) = \{f(x) \in R_f | x \in A \cap D_f\}. \tag{2.1}$$

Then for $I = [0, 1]$ *if* $f : I \to E^n$ *is continuous we can call* $f_*(I)$ *a curve. [11, p.4]*

Definition 2.2. *(Space filling) The curve* $f(I)$ *is space-filling if it goes through (contains) all the points in a given* n*-dimensional region with* $n \geq 2$*, having a strictly positive measure. [11, p.5]*

2.2 Mapping

Theorem 2.3. *The unit interval is an uncountably infinite set of points.*

Proof. If we assume that a countable set S contains all the real points on the unit interval, we can always construct a new number N that is not in S, by the following method, also known as Cantor's diagonal argument.

To construct this new N we can imagine the content of S in a column (in any order) where we start from the top, picking the first decimal digit of the first element $s_1 \in S$, add one to it and use it as the first decimal digit of N, then moving on to the next element $s_2 \in S$ and take its second decimal plus one, as the second decimal digit of N and so on. If the digit is nine we use zero instead.

This N contradicts the assumption since $N \notin S$. This insufficiency to exhaust all the points on the interval shows real numbers are uncountable. $\square$

Theorem 2.3 shows that the unit interval has the same cardinality as the real line and prevents the unit interval from having a one-to-one correspondence to $\mathbb{N}$ which is a countably infinite set. In order to have a more organized mapping from the unit line to the unit square we can chop them down into same countable number of smaller pieces, called sub-intervals and sub-squares respectively. Then there could be a one-to-one mapping between the pieces. Chopping the pieces down infinitely many times makes both the unit line and the unit square countably infinite sets, allowing them to have a one-to-one mapping to $\mathbb{N}$ as well as to each other.

Definition 2.4. *(Isometry) An isometry $f : X \to Y$ is a distance preserving bijection, with X and Y being metric spaces with metrics d_X and d_Y, where for all $a, b \in X$ we have $d_X(a, b) = d_Y(f(a), f(b))$.*

Definition 2.5. *(Congruence) The subsets A and B are congruent, if there is an isomerty from one to the other. [12, p.1]*

Let I be the unit interval $[0, 1]$ and Q be, the unit square $[0, 1]^2$, with four corners $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$. Assuming I can be continuously mapped onto Q, then it must also be possible that "after partitioning I into four [to each other] congruent sub-intervals and partitioning Q into four congruent sub-squares, each sub-interval can be mapped continuously onto one of the sub-squares."[11, p.10]. The same argument goes even for the partitions of each partition and can be repeated ad infinitum.

We say every stage of this partition is an iteration labeled with $n \in \mathbb{Z}^+$. As shown in Figure 2.1, "for each positive integer n we partition the interval I into 4^n sub-intervals of length 4^{-n} and the square Q into 4^n sub-squares of side 2^{-n}."[13, p.2]

To refer to a particular sub-interval or sub-square, we need to specify which piece and from what iteration. To do so, let I_{n,k_n} and Q_{n,k_n} each be the partition (a sub-interval and a sub-square respectively) with the index k_n from the n'th iteration. Of course, for every n, our options for k_n would fall within indices $1 \leq k_n \leq 4^n$.

For every iteration n, we define a binary relation between the sets of partitions $I_{n,\{k_n\}_{k_n=1}^{4^n}}$ and $Q_{n,\{k_n\}_{k_n=1}^{4^n}}$. Let this correspondence be f_n, a set of ordered pairs (I_{n,k_n}, Q_{n,k_n}). The mappings

$$f_n : I_{n,\{k_n\}_{k_n=1}^{4^n}} \to \; Q_{n,\{k_n\}_{k_n=1}^{4^n}} \tag{2.2}$$

are onto. Initially it doesn't seem to be a problem to have these correspondences be one-to-one as well. Simply map each I_{n,k_n} to a unique Q_{n,k_n}. Considering the fact that there are more than one way of doing so, we restrict the correspondence by two conditions, adjacency and nesting (Figure 2.2). [13, p.2]

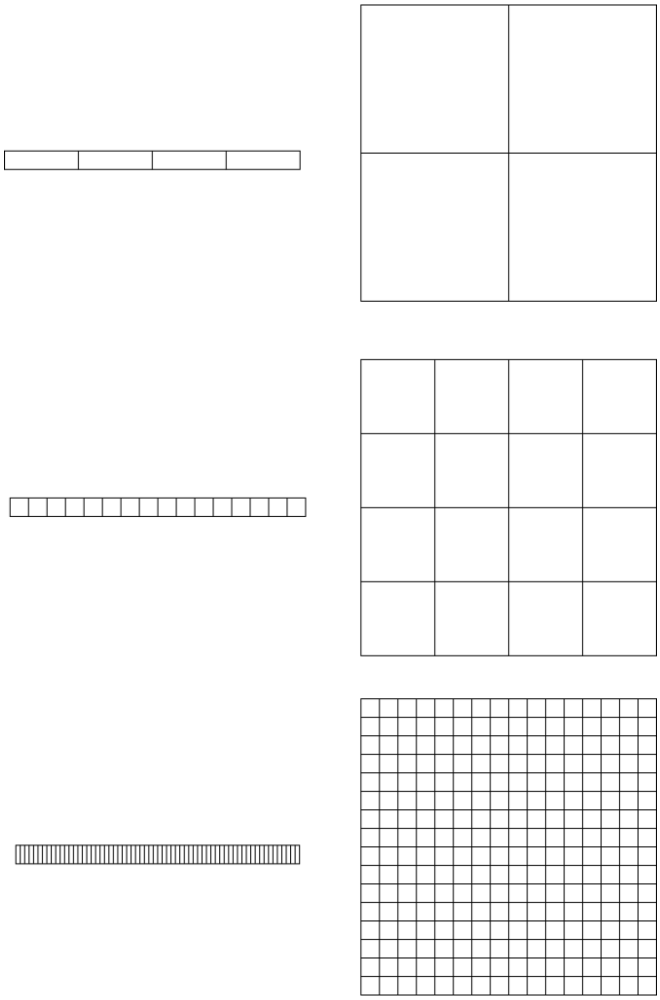

Figure 2.1: Iterations two through four (top to bottom).

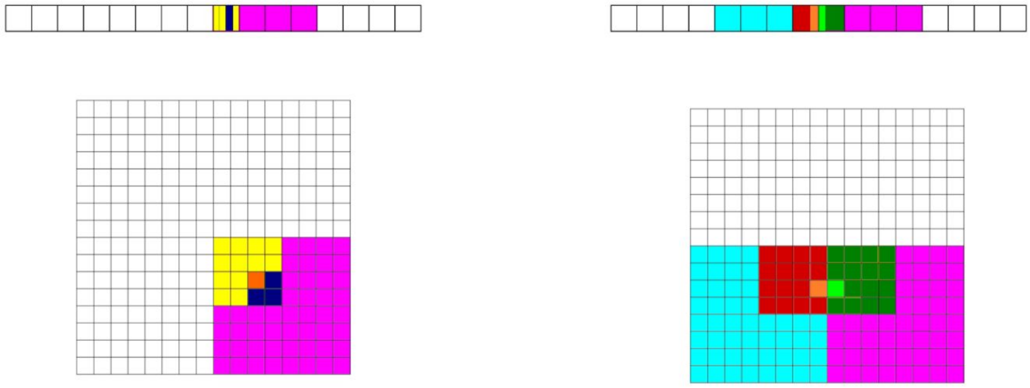

Figure 2.2: Visualization of the nesting (left) and the adjacency (right) conditions.

The adjacency condition requires that for all iterations n, if two sub-intervals share a common point, then the function f_n maps the two sub-intervals to two sub-squares that "share a common edge". [14, p.6]

Let us introduce the notations $\{I_{n,k_n}\}_{n=0}^{L}$ and $\{Q_{n,k_n}\}_{n=0}^{L}$ for the sequences of nested closed sub-intervals and sub-squares respectively. These are sets containing a "choice " of piece from each iteration (0 to L) so that each subsequent choice is from the partitions of the previous one. Seen as a sequence of sets, we can say each is a subset of the previous.

The nesting condition requires that for all iterations n, if two elements of a sequence of sub-intervals are nested, then the two corresponding elements of the sequence of sub-squares are also nested. [14, p.7]

In any iteration $n = u > 0$, for every point shared by the adjacent pairs I_{u,k_u} and I_{u,k'_u}, there is another pair in iteration $u+1$ that share the same point. The same point is even shared by two adjacent sub-intervals in iteration $u + 2$, so the point will be shared by a pair of sub-intervals in any iteration $L > u$. Then we can have sequences of adjacent pairs $\{I_{n,k_n}\}_{n=u}^{L}$ and $\{I_{n,k'_n}\}_{n=u}^{L}$, that share the same point. Now if all f_n satisfy the adjacency condition, these sequences correspond to sequences of adjacent sub-squares $\{Q_{n,k_n}\}_{n=u}^{L}$ and $\{Q_{n,k'_n}\}_{n=u}^{L}$. [14, p.6]

In any iteration $n = u > 0$, each sub-interval I_{u,k_u} is nested in a $I_{u-1,k_{u-1}}$. In iteration $u + 1$, the sub-intervals that are nested in I_{u,k_u} are also nested in $I_{u-1,k_{u-1}}$. So in any iteration $L > u - 1$ there are sub-interval nested in $I_{u-1,k_{u-1}}$. If all f_n satisfy the nesting condition, the nested sequence $\{I_{n,k_n}\}_{n=u-1}^{L}$ corresponds to a nested sequence $\{Q_{n,k_n}\}_{n=u-1}^{L}$. [14, p.7]

We apply these for two aims. One is to show continuity, which by Definition 2.1 is a necessity for a curve, and the other is to argue against the bijection.

2.3 Surjection and continuity

Function of each iteration map the pieces in that iteration but we are interested in mapping points from I to Q. Imagining that after infinite iterations pieces are small enough to be points, we can call the function of that particular iteration f that maps individual points.

Theorem 2.6. *"Any correspondence between the sub-intervals and sub-squares that satisfies the adjacency and nesting conditions determines a unique continuous function f which maps I onto Q."[13, p.3]*

Proof. To make sure we are dealing with a function we have to show every element from the domain is being mapped to a single element in the range.

There are two cases for a point $p \in I$. One is if p is not the intersection point of two adjacent intervals. Then every such point p belongs to a unique sequence of closed nested sub-intervals whose lengths approach zero, i.e. for every p, there is a unique sequence $\{I_{n,k_n}\}_{n=0}^{\infty}$ whose all elements contain p.

According to the nesting condition there is a unique corresponding sequence of nested sub-squares whose diameters also approach zero, determining a point $q = (q_x, q_y) \in Q$, contained by all the elements of the sequence $\{Q_{n,k_n}\}_{n=0}^{\infty}$.

Each element of $\{I_{n,k_n}\}_{n=0}^{\infty}$ belongs to an iteration n and is mapped by the correspondence f_n. Despite the uniqueness of each sequence $\{I_{n,k_n}\}_{n=0}^{\infty}$ (due to their k_n values or the point they have in common), all of them are mapped by the same sequence of functions $\{f_n\}_{n=0}^{\infty}$.

Since $\lim\limits_{n\to\infty} \{I_{n,k_n}\} = p$ and $\lim\limits_{n\to\infty} \{Q_{n,k_n}\} = q$ we could have $\lim\limits_{n\to\infty} \{f_n\} = f$ such that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \textit{if } n \geq N \textit{ then } |f_n - f| < \epsilon \tag{2.3}$$

where $f(p) = q$ for all $p \in I$, which reveals that the sequence of functions converges pointwise to $f : I \to Q$. [14, p.10]

The other case for p is if it would be shared by two adjacent sub-intervals I_{u,k_u} and I_{u,k'_u} , making p a common end-point in iteration $n = u$. Then in the next iteration, there would be a unique adjacent pair $I_{u+1,k_{u+1}}$ and $I_{u+1,k'_{u+1}}$ that would still have p in common. These two sub-intervals would also automatically be nested within their predecessors. So adjacency carries over to subsequent iterations. In this case, p belongs to two adjacent sequences $\{I_{n,k_n}\}_{n=u}^{\infty}$ and $\{I_{n,k'_n}\}_{n=u}^{\infty}$. On the other hand, based on adjacency condition, the corresponding $\{Q_{n,k_n}\}_{n=u}^{\infty}$ and $\{Q_{n,k'_n}\}_{n=u}^{\infty}$ share an edge whose length shrinks to zero after infinite iterations. Hence they end up at the same point $q \in Q$. So, yet again the same $\lim\limits_{n\to\infty} \{f_n\} = f$ does the point-mapping.

We have shown that f is a single-valued function, mapping every p to a single q. We can confirm f covers all the points $q \in Q$ by the fact that every q belongs to at least one (not necessarily unique) $\{Q_{n,k_n}\}_{n=0}^{\infty}$ which corresponds to a $\{I_{n,k_n}\}_{n=0}^{\infty}$ that shrinks to a $p \in I$. So f is onto.

To show continuity, suppose $|p_1 - p_2| \leq 4^{-n}$. Then p_1 and p_2 lie either in the same interval or two adjacent ones of the n'th iteration. This carries over to the corresponding images, where $f(p_1) = q_1$ and $f(p_2) = q_2$ are either in the same or two adjacent sub-squares. If we see these two adjacent sub-squares as a $2^{-n} \times 2 \cdot 2^{-n}$ rectangle, the largest distance between two points fitting inside the rectangle would be its diameter. Therefor $\|q_1 - q_2\| \leq \sqrt{5} \cdot 2^{-n}$. In other words, for every rectangle's diameter there exists a sub-interval so that if the distance between two points is less than that sub-interval's length, it is guaranteed that they would fit within the rectangle's diameter. This is the same as the epsilon-delta definition

$$\forall \epsilon > 0 \ \exists n > 0 : \textit{if } |p_1 - p_2| < 4^{-n} \ \textit{then} \ \|f(p_1) - f(p_2)\| < \epsilon \tag{2.4}$$

so f is uniformly continuous. [13, p.3]

□

Locally, I and Q resemble the Euclidean space on which we can apply calculus because they are differentiable. These spaces are called smooth manifolds.

Theorem 2.7. *(Netto) If the bijection f maps an m-dimensional smooth manifold onto an n-dimensional smooth manifold where $m \neq n$, then f is necessarily discontinuous. [11, p.6]*

We have already shown continuity, so Theorem 2.7 eliminates the possibility of bijection. Also in the proof of Theorem 2.6 we have shown the function f to be onto, which means surjection only. This, together with Definition 1.7, confirms that the mapping involved in a space-filling curve, is not a homeomorphism. This surjection has another peculiar property.

Theorem 2.8. *The function f from Theorem 2.6 has infinitely many multiple points; that is, multiple points from the domain will be mapped to the same points.*

Proof. Suppose I_{n,k_n} corresponds to Q_{n,k_n} at the n'th iteration. Then on the next iteration $n+1$ it would be partitioned into four. Let us name these four according to Figure 2.3, along with the corresponding partitions of I_{n,k_n}, as 1 to 4, in left to right order. It is implied that partitions with the same index correspond to each other.

Let P be the center point of the Q_{n,k_n}. There are sequences of nested sub-squares in all four partitions of Q_{n,k_n}, all which shrink to P. However the nested sub-intervals corresponding to these four, particularly the ones lying in

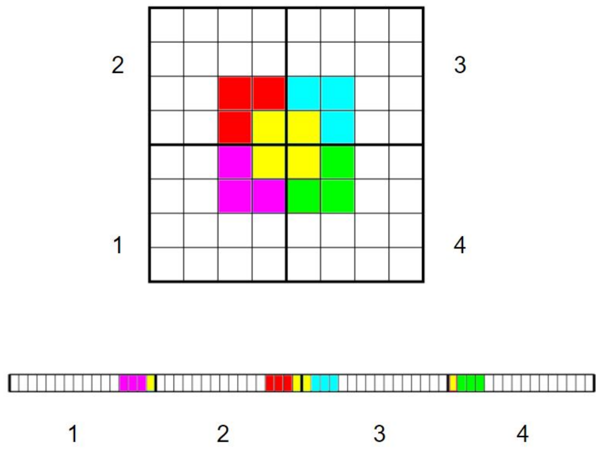

Figure 2.3: The sub-interval I_{n,k_n} and the sub-square Q_{n,k_n} at $(n+1)$'th iteration.

the partitions 1 and 4, lie on the opposite ends of I_{n,k_n}. They have no points in common, which makes them impossible to shrink to the same point. This means that for all P there are at least two distinct points on I that map to P. Infinitely many iterations means infinitely many squares leading to infinitely many such centers. [13, p.3] □

This proof illustrates that if we were to map Q to I, there would be points that have more than one image in I. There not existing any left inverses, is in agreement with the conclusion of Theorem 2.7 about the absence of injection.

2.4 Nowhere differentiable

Theorem 2.9. *The coordinate functions ϕ and ψ, in the function $f(p) = q = (q_1, q_2) = (\phi(p), \psi(p))$ from Theorem 2.6, are nowhere differentiable.*

The following proof is an amalgamation of [13, p.4] and [11, p.12].

Proof. Let $n \geq 2$ so that we can have access to $n-2$ and at least 16 partitions at n. For every $p \in I$ we can choose a $p_n \in I$ so that $|p - p_n| \leq 16 \cdot 4^{-n}$. From $16 \cdot 4^{-n} = 4^{-(n-2)}$ we notice that the points being at most 16 sub-intervals apart means that two iterations prior they were no further than one sub-interval apart. The sub-square that contained $f(p)$ at iteration $n-2$, is currently partitioned into 16 sub-squares of length 2^{-n}. As in Figure 2.4 it is always possible to pick $f(p_n)$ so that they are separated by at least one Q_{n,k_n}, while $\phi(p)$ and $\phi(p_n)$

are separated by at least the length 2^{-n} along one axis. Writing these as one equation gives

$$\frac{|\phi(p) - \phi(p_n)|}{|p - p_n|} \geq \frac{2^{-n}}{16 \cdot 4^{-n}}. \tag{2.5}$$

For the right side of (2.5) we have $\lim_{n\to\infty} \frac{2^n}{16} = \infty$. This makes it impossible for the left hand side to go to zero as p_n gets closer to p. Hence a differentiability condition is violated and ϕ is not differentiable. Analogous argument goes for ψ.

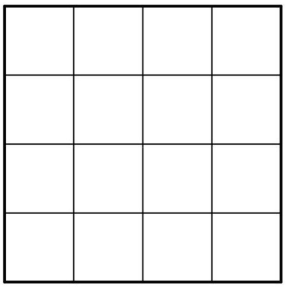

Figure 2.4: It is possible to pick two squares out of 16 such that they are at least a square away from each other.

□

2.5 Construction

The indexing order of I_{n,k_n} is left to right while the Q_{n,k_n} indices adapt according to what is being mapped to them by a correspondence that satisfies both the adjacency and the nesting conditions. But this is still not specific enough and could be done in many ways. Let us always begin from the lower left corner with $f(0) = (0,0)$ and end at the lower right corner of the square with $f(1) = (1,0)$. Having settled these two points, it remains only one way of enumerating the sub-squares such that our conditions are satisfied.

The indexing in between the initial and final squares might not be immediately obvious, but given only these two squares we can follow a general procedure to generate the subsequent iterations. This is illustrated in Figure 2.5 [13, p.7].

The following is done in the two-dimensional plane of our square. To obtain the grid of Q_{n+1}, we begin by partitioning Q_n into four, each containing a scaled down versions of Q_n by a factor of $\frac{1}{2}$. In the lower left of Q_{n+1}, Q_n is reflected in the line $y = x$. We add 4^n and $2 \cdot 4^n$ to all the indices of Q_n in the upper left and upper right respectively. Finally for the lower right version we reflect it in the line $x + y = 1$ and add $3 \cdot 4^n$ to the indices. In Figure 2.5 only by looking at

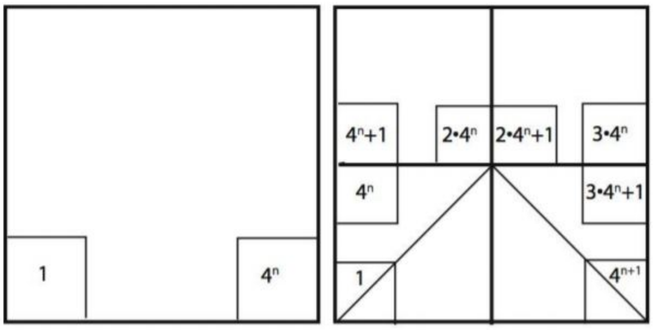

Figure 2.5: The n'th and $n+1$'th partition

the known squares, it is apparent that, if the adjacency and nesting conditions are satisfied in any n, then they are carried over to $n+1$.

This process results in symmetry about the line $x = \frac{1}{2}$. Every point $p \in I$, contained in I_{n,k_n} (which is mapped to Q_{n,k_n}) is symmetrical with its compliment point $1-p \in I$ contained in $I_{n,4^n+1-k_n}$ (which is mapped to $Q_{n,4^n+1-k_n}$). This result in the indices of all mirrored pairs of sub-squares, adding up to 4^n+1. A few are marked in Figure 2.8.

So the unique continuous function f determined by this way of enumerating the squares is the Hilbert space-filling curve, usually denoted by f_h. [13, p.5]

2.6 Approximating polygons

When it comes to the shape of the curve itself, if initially every I_{n,k_n} is mapped to an edge of Q_{n,k_n}, there would only be one choice of edge for each Q_{n,k_n} that satisfies our continuity, adjacency and beginning/end conditions. Every sub-interval is oriented in such way that the point at which I_{n,k_n} exits Q_{n,k_n} would be the same point as where $I_{n,k_{n+1}}$ enters $Q_{n,k_{n+1}}$. See Figure 2.6 [11, p.14].

These polygonal lines are passing through the images of the end points of all sub-intervals $I_{n,\{k_n\}}$. That is, they are connecting the points

$$f_h(0), f_h(\frac{1}{4^n}), f_h(\frac{2}{4^n}), f_h(\frac{3}{4^n}), ..., f_h(\frac{4^n-1}{4^n}), f_h(1) \tag{2.6}$$

at iteration n by the function

$$g_n(p) = 4^n(p - \frac{k}{4^n})f_n(\frac{k+1}{4^n}) - 4^n(p - \frac{k+1}{4^n})f_n(\frac{k}{4^n}) \tag{2.7}$$

for

$$\frac{k}{4^n} \leq p \leq \frac{k+1}{4^n} \quad \text{where} \quad k = 0, 1, 2, ..., 4^n - 1. \tag{2.8}$$

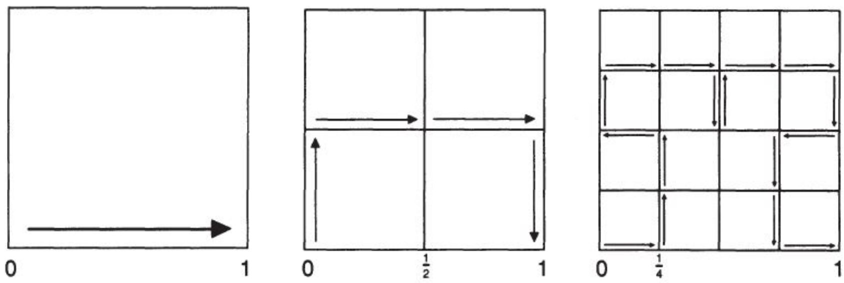

Figure 2.6: Approximating polygons following the unique "path"

We can argue the functions g_n approximate f_h because the difference between the images of these two functions are bounded by the diameter of the sub-square they fall into. So the distance $\sup\{|f_h(p) - g_n(p)|\}$ is always less than or equal to $\frac{\sqrt{2}}{2^n}$. This distance goes to zero as $n \to \infty$ because for all p,

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : \textit{if } n > N \textit{ then } \sup\{|f_h(p) - g_n(p)|\} < \epsilon. \tag{2.9}$$

Thus the sequence $\{g_n\}_{n=0}^{\infty}$ converges uniformly to f_h. [11, p.22]

There is a variety of approximating polygons, all following the same principle. For instance another approximation could be constructed by connecting the midpoints of adjacent sub-squares to achieve Figure 2.7 [13, p.6]

2.7 Applications

Space-filling curves are topologically one-dimensional while their fractal dimension acquires the dimension of the space they "fill". As we mentioned at the end of section 1.6, the difference of these two values leads to these curves being fractals, except here with integers as their fractal dimension.

It comes as no surprise that Hilbert curve can be viewed as a fractal since there is a recursive nature in computing different iterations. [11, p.23] Fractals in general are efficient ways of organizing complicated structures, both in nature and technology.

In many cases, an organ is supplied with necessary substances through vessel systems that reach every point in the volume of the organ. "For example, the kidney houses three interwoven tree-like vessel systems, the arterial, the venous, and the urinary systems", all of which have access to every part of the kidney. (See Figure 2.9 [7]) [7, p.94]

A similar application is in fractal antennae, which use fractals to maximize the effective length of a receiving or transmitting material within a given total surface area or volume. [15, p.3]

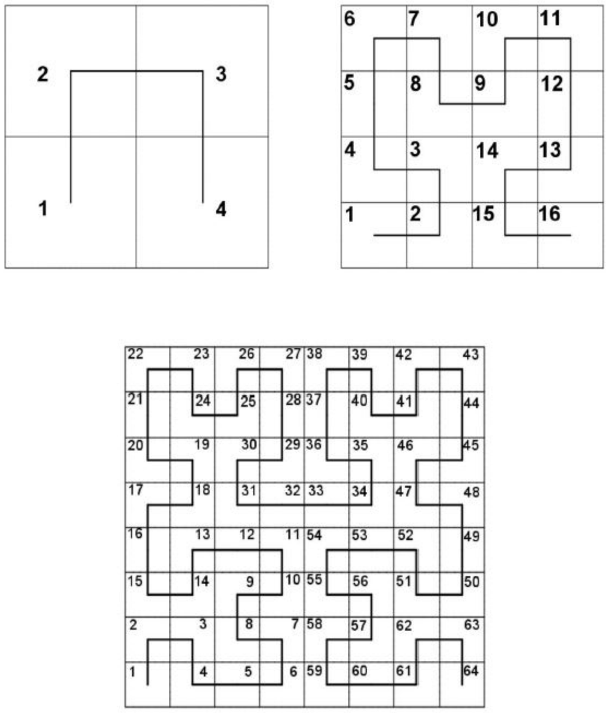

Figure 2.7: Another approximating polygon for the Hilbert curve

The transformation involved in a space-filling curve results in a "substantial compression of the information and retaining some of the spatially associative properties of the space."[16, p.1] Since data comes in a different dimensions, space-filling curves could be a way of posing a linear order to a multi-dimensional data set. [17, p.11]

As we showed in the proof of Theorem 2.8 a pair of points in two-dimensions are not necessarily going to correspond to neighbouring points on the one-dimensional line. However conversely, the images of two points close to each other on I will remain close to each other on Q. In Figure 2.10 any color on the scale keeps its neighbouring colors on the square. This speaks to the ability of Hilbert space filling-curve to preserve locality.

This enables us to turn, for instance, the two-dimensional content of a picture into a single linear sequence for processing purposes, and then back into a picture, without disturbing the pixel structure.

Printers render an array of black and white pixels, which they used to achieve through scanning a given grey scale image line by line or in small square blocks. But putting these parts back together was not flawless and was countered by dithering. Dithering refers to the idea of intentionally applying noise to in-

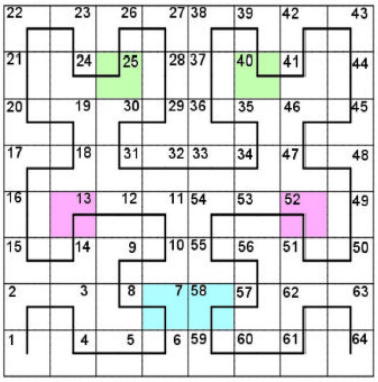

Figure 2.8: The symmetrical indices add up to $4^n + 1$.

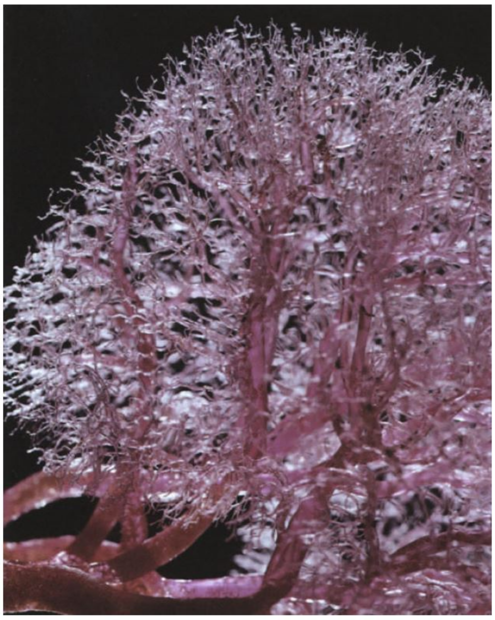

Figure 2.9: Cast of a child's kidney, venous and arterial system

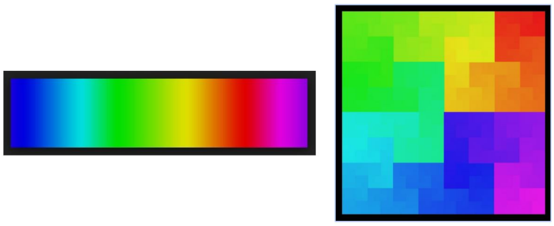

Figure 2.10: Any color on the scale maintains its neighbouring colors on the square.

crease the accuracy, or rather mask the flaws by randomizing the errors. In SIGGRAPH convention of 1991, an image processing application was introduced as a replacement for the old dithering techniques. This newer technique runs a Hilbert curve through all the pixels (a more sophisticated version of the idea in Figure 2.11 [7, p.103]), with the advantage of being free from any directional features that are in need of being rearranged. As a result of this technique, the images produced have an appealing grainy quality to their look. The Figure 2.12 compares the old with the new method using the test image Lena. [7, p.102]

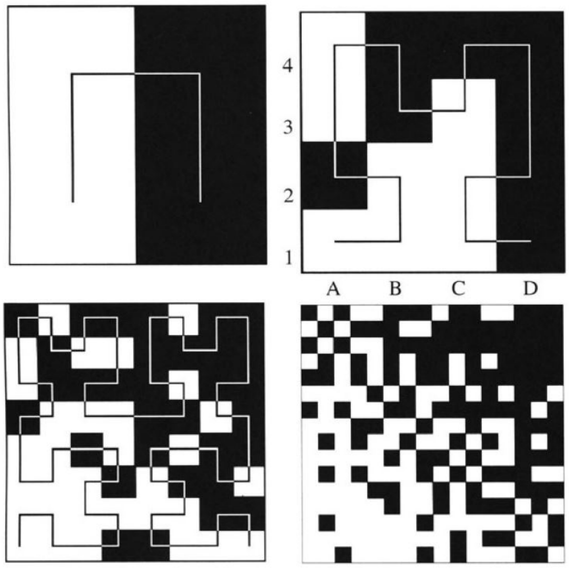

Figure 2.11: Running a Hilbert curve through all the pixels in different resolutions

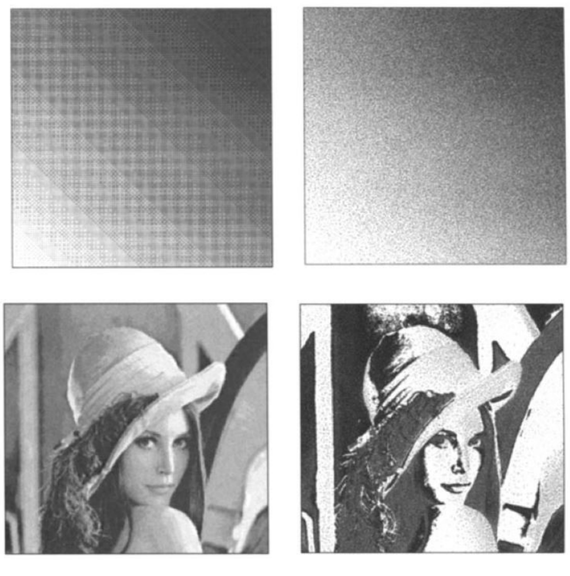

Figure 2.12: Dithering with the Hilbert curve (right) versus traditional (left)

Chapter 3

Banach-Tarski Paradox

Stefan Banach and Alfred Tarski introduced the idea of starting "with any bounded set with nonempty interior and reassemble it into any other such set of any volume". We can think of it as the possibility of chopping up a pea and reassemble it in a way that we end up with a ball as large as the Sun. [10, p.2]

One proof of this seemingly impossible phenomena, is duplicating a sphere in $\mathbb{E}^3$. We split the sphere into finite pieces (sets) and rearrange them into two spheres identical to the original. This seems less impossible considering cases such as the sets of odd and even integers, where their union has the same cardinality as each of them individually, or the guests in the Hilbert Hotel being "duplicated" every time a new bus arrived.

3.1 Groups

Definition 3.1. *(Group) A group G is a non-empty set endowed with a binary operation denoted by juxtaposition that obeys the axioms $(ab)c = a(bc)$ for all $a, b, c \in G$ (associativity), $\exists \mathbb{1} \in G : \mathbb{1}a = a\mathbb{1} = a$ (identity) and $\forall a \in G\ \exists a^{-1} \in G : a^{-1}a = aa^{-1} = \mathbb{1}$ (inverse). [18, p.20]*

A group G acting on a set X entails that each $g \in G$ is a bijection $g : X \to X$ so that $g(h(x)) = (gh)(x)$ and $\mathbb{1}(x) = x$, where $x \in X$, $g, h \in G$ and $\mathbb{1}$ is the identity of G.

3.2 Non-measurable

Lebesgue measurability (Definition 1.10) is the ability to assign a size to a set, whether it is length in one-dimension, area in two-dimensions or volume in three-dimensions. Then the properties that follow are that "the measure of any unit cube should be 1", that "for disjoint A and B, $m(A \cup B) = m(A) + m(B)$ should hold", that "if $A \subseteq B$ then it should be the case that $m(A) \leq m(B)$" and that "after shifting A by a vector x, it still should hold that $m(A+x) = m(A)$". Without these requirements being fulfilled Lebesgue measurability becomes meaningless and the size of sets are no longer guaranteed to be conserved. [9, p.163]

In set theory, one axiom necessary for constructing non-measurable sets is the axiom of choice which asserts the existence of a "choice function" that makes selections from each member of a family of non-empty sets. Even with infinite number of decisions, this function will assign a choice to each. This could be applied when pairing elements as in Theorem 3.2.

Theorem 3.2. *For any sets A, B and a binary relation $P \subseteq A \times B$, if $\forall x \in A, \exists y \in B : P(x, y)$, then $\exists f : A \to B : \forall x \in A, P(x, f(x))$. [10, p.2]*

A quick construction of a non-measurable set using the axiom of choice is the following. Let S be the uncountable set of points on a circle and G be the countable group of rotations by rational multiples of π. An orbit is the collection of various rotated versions of a point in S. So S could be broken down into uncountable orbits under G. Here axiom of choice provides an uncountable choice set X made of choices, one from each orbit. Now the countable and disjoint rotations of X by G, that we call X_i could also make up the whole S. Whether X has zero measure or a positive measure, the measure of the circle by $\sum\limits_{i=1}^{\infty} m(X_i)$ (from Definition 1.11) would be either zero or infinite, none of which is consistent with the size of the circumference. Measurability of the choice set is lost by the violation of the countable additivity condition.

The "duplication" is made possible by the fact that the pieces that make up the sphere we are going to work with are not Lebesgue measurable to begin with, so it does not have a volume associated with it. Instead, the sphere is made of a set of choices, allowing it to defy geometric intuition.

3.3 Dissection

Definition 3.3. *(Congruence by dissection) Two polygons A and B are congruent by dissection, if A and B could be partitioned (boundaries aside) into $A_1, ..., A_n$ and $B_1, ..., B_n$ so that $\exists f_i : f_i(A_i) = B_i$ where f_i is an isometry (meaning $A_i \cong B_i$).*

The Figure 3.1 [10, p.3] illustrates a geometrical proof of Pythagorean Theorem using the method of dissection and reassembly. In this method all we do is to cut a polygon into pieces and rearrange them into a new one, so the area is unchanged. These shapes are said to be congruent by dissection. This indicates a correspondence between pairs of pieces. They differ only by a sequence of translations, rotations, and/or reflections.

Theorem 3.4. *Two polygons are congruent by dissection, provided they have*

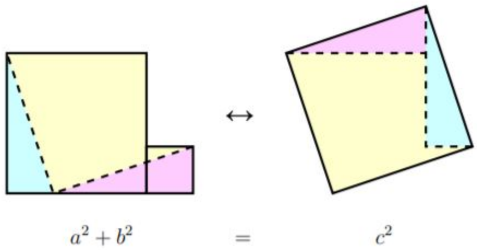

Figure 3.1: Dissection and reassembly

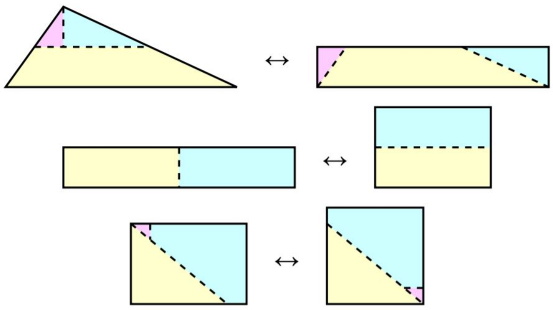

Figure 3.2: Any polygon can be turned into a square through dissection and reassembly

the same area.

Proof. Congruence is an equivalence relation. To show the transitivity property, consider a polygon P that can be dissected by a set of cuts G_1 to then be reassembled into another polygon Q, which dissected by another set of cuts G_2 can be reassembled into yet another polygon R. So if we were to perform both G_1 and G_2 on Q, it would give us the pieces required to build R directly.

If we dissect any polygon into triangular pieces, then, like Figure 3.2 [10, p.4] through dissection and reassembly we can turn each triangle into rectangles, each rectangle into squares and all of these squares into a single square like in Figure 3.1.

Now for two given polygons of the same area, two sets of cuts are required to turn each of them into a square of the same area. By performing these two sets of cuts simultaneously on both, they can be reassembled into each other. [10, p.3] □

This proof shows that to redistribute the same (amount of) area over another shape is a matter of isometric transformations of some partitions.

3.4 Equidecomposability

Similar "reassembling"s to the ones in the previous section could be performed by group actions. This could be done by for example a so called isometry group, the group of all the isometric bijections from a metric space onto itself, under function composition.

Definition 3.5. *(Equidecomposability) Let the group G act on the set X. We say A and $B \subseteq X$ are G-equidecomposable, if A and B could be partitioned into $A_1, ..., A_n$ and $B_1, ..., B_n$ so that $\exists g_i \in G : g_i(A_i) = B_i$ (meaning $A_i \cong B_i$). [12, p.3]*

The reason in the Definition 3.3, we ignored boundaries was that in the case of polygons, line-segments have measure zero in $\mathbb{E}^2$ (have no area). Equidecomposability is a case of congruence by dissection where we are not indifferent towards the boundaries. [10, p.4]

Two G-equidecomposable sets can be partitioned into the same finite number of pieces, so that corresponding pairs are related by G and G can reassemble each set into the other.

3.5 Broken circle

Theorem 3.6. *A broken circle $S^1 \setminus \{p\}$ is G-equidecomposable to S^1, where G is the isometry group of $\mathbb{E}^2$.*

Proof. We choose the circle to be the unit circle. Let the missing point of the broken unit circle $S^1 \setminus \{p\}$ in complex notation be $p = e^{i0}$. We define a set $A = \{e^{in} : n \in \mathbb{N}\}$ whose elements are unique since rotations by one radian never coincide due to 2π being irrational. The circle is made of uncountably infinite points (not unlike Theorem 2.3), so beside the countably infinite points in A, there still exists a set of points $B = (S^1 \setminus \{p\}) \setminus A$. Let B remain fixed while rotating A one radian clockwise (Figure 3.3). The place of the missing p is now filled by what previously was e^{i1}. Every point is now covered either by A or B, so the circle is now S^1.[12, p.3] $\square$

This is the same treatment as Hilbert's Hotel in a circular context. If the first room is empty, to fill it, all the guests move one room back.

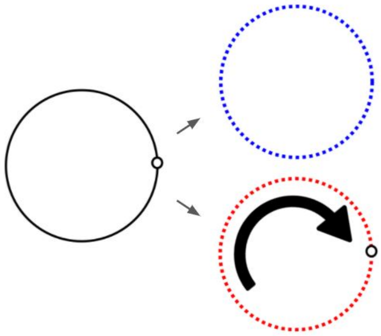

Figure 3.3: The set B (blue) is fixed while the set A (red) is rotated.

3.6 Paradoxical sets

Definition 3.7. *(Paradoxical set) Let the group G act on the set X. We say $E \subseteq X$ is G-paradoxical, if there exist $g_1, ..., g_m$ and $h_1, ..., h_n \in G$, as well as pairwise disjoint $A_1, ..., A_m$ and $B_1, ..., B_n \subseteq E$, such that $\cup g_i A_i = \cup h_j B_j = E$. [12, p.2]*

The spherical ball in the Banach–Tarski paradox is also made of a paradoxical set, meaning it has a paradoxical decomposition. This entails that it can be partitioned into two disjoint subsets, so that each can be mapped to the entirety of the original set (of which they are part of), by a finite number of functions. These sets can be created by an appropriate group.

Theorem 3.8. *The circle S^1 is SO_2-paradoxical with countable number of pieces, where SO_2 is the rotation group whose elements are two-dimensional rotations.*

Proof. With the group of rotations by angles that are rational multiples of 2π radians about the origin, we can define an equivalence relation where two points on S^1 are equivalent if one is achievable through such rotation of the other. Since equivalence classes partition the underlying set we can identify S^1 with SO_2. Since rational numbers are countable, the set of rotations by rational multiples of 2π radians, denoted by $\{\rho_i : i = 1, 2, ...\}$, is countable. Using the axiom of choice, let M be a choice set for the equivalence classes of this relation on S^1. Now rotations of this set $\{M_i : i = 1, 2, ...\}$, where $M_i = \rho_i(M)$, also partition S^1 into countable pieces. Any two M_i are congruent by rotation which yields that the sets in $\{M_i : \textit{with odd } i\}$ each can be individually rotated to attain the whole circle $\bigcup\limits_i M_i$. Same goes for $\{M_i : \textit{with even } i\}$. These are visualized in Figure 3.4 [12, p.3] □

If we assume that in the proof of Theorem 3.8, $A = \bigcup\limits_{i \textit{ even}} M_i$ and $B = \bigcup\limits_{i \textit{ odd}} M_i$

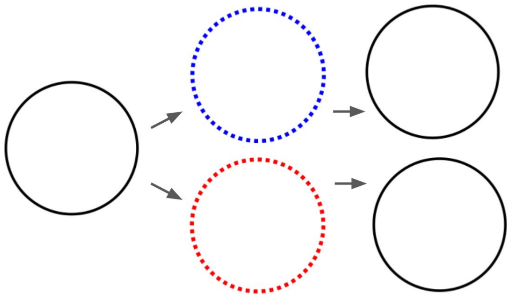

Figure 3.4: Paradoxical decomposition of the circle into two piece, each individually equidecomposable to the circle itself.

are measurable while $m(S^1) = 1$, by Definition 1.11, we can arrive at the contradiction

$$1 = m(S^1) = m(A \cup B) = m(A) + m(B) \geq m(S^1) + m(S^1) = 2. \tag{3.1}$$

This shows that there is no countably additive rotation-invariant finite measure defined for the subsets of S^1, depriving them from being Lebesgue measurable. This can be extended to certain sets in $\mathbb{E}^n$ being non-measurable.

In the case we will be looking at the unit ball in $\mathbb{E}^3$ is paradoxical with respect to the group of isometries of $\mathbb{E}^3$, that is the set of distance preserving bijections from $\mathbb{E}^3$ to itself.

3.7 Paradoxical groups

A free group F over a generating set X is a group of all the sequences (called words), that use the elements of X (called the generators of F) and their inverses as "letters" to build these words under composition.

These words are reduced to their shortest forms, devoid of any other identical operations i.e. if there are several sequences leading to the same result, we only include one with the least number of letters.

For example, if the generators are $\uparrow$ and $\rightarrow$, each representing a step in the direction of the arrow on a surface, the letters would in addition include the inverses of these two, $\downarrow$ and $\leftarrow$ respectively where a sequence of these four directions would give an address (a word). In the address $(\uparrow\leftarrow\downarrow\rightarrow)\uparrow$ the term in the parenthesis is redundant so the whole word could be reduced to just $\uparrow$.

If the generators of F are rotations, then the words in F are sequences composed of these rotations and their inverses, where every word is a unique rotation.

Definition 3.9. *(Paradoxical group) A paradoxical group is a group that is equidecomposable with two disjoint copies of itself, under its own group action.*

Theorem 3.10. *A free group on two generators is a paradoxical group. [10, p,6]*

Proof. If we call the two generators of the free group F, a and b, then F could be described by the decomposition

$$F = \{\mathbb{1}\} \cup W_a \cup W_b \cup W_{a^{-1}} \cup W_{b^{-1}} \tag{3.2}$$

where W_i is the set of all the words starting with the letter i. However the rest of the letters after the first ones, are identical for all W_i, because every time we remove the first letter from each W_i, we are left with all the possible permutations for the remaining letters. Except, since we assume the words to be reduced we would never put the letters i and i^{-1} next to each other, so for example removing the initial i will also make the initial i^{-1} impossible. This gives rise to a paradoxical decomposition

$$F = W_a \cup aW_{a^{-1}} = W_b \cup bW_{b^{-1}}. \tag{3.3}$$

□

Theorem 3.11. *There exists two independent rotations S and T of the unit sphere in $\mathbb{E}^3$ that generate a free group F, where no non-trivial word f on the symbols S, T, S^{-1}, T^{-1} would be equivalent to the identity rotation i.e. every rotation lands on a unique spot.*

[10, p.7]

Proof. We choose the sphere to be the unit sphere and the rotations to be around two perpendicular axes, by the angle $\arccos(\frac{1}{3})$. These two rotations work in the similar way as the generators of the free group in Theorem 3.10. Seeing the sphere as the union of concentric shells, points will only traverse the shells they belong to as the result of these rotations. Therefor we can only concern ourselves with the outer shell, assuming the rest will follow the same (needless to say the center point is not affected by any rotations). We let the rotation matrices

$$S = \begin{bmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} \tag{3.4}$$

rotate the point $(1, 0, 0)$. This point is fixed by T so the sequences start with S. The results acquired by these rotations are points that have the form $\frac{(a,\sqrt{b},c)}{3^n}$ where $a, b, c \in \mathbb{Z}$, $0 < b$, $b \not\equiv 0(mod3)$ and n is the length of the word. So no rotation ends up back to $(1, 0, 0)$. [12, p.4] □

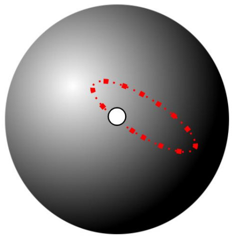

Figure 3.5: Same treatment as in broken circle but in three dimensions.

Similar to the circle in the proof of Theorem 3.6, our sphere is made of uncountably infinite points, so countably infinite rotations of $(1, 0, 0)$ will not cover the whole sphere.

3.8 Construction

3.8.1 The center

If we show a spherical shell is equidecomposable with two other copies of itself under the isometry group of $\mathbb{E}^3$, this quality then can be extended to the unit ball B^3 seen as a collection of concentric shells. This however does not cover the center $(0, 0, 0)$ which we temporarily disregard. The reason center is not an issue is the following.

Lemma 3.12. *The B^3 is equidecomposable with $B^3 \setminus \{(0, 0, 0)\}$.*

Proof. Imagine the broken circle from Theorem 3.6 contained within B^3 in any orientation as long as the missing point is $(0, 0, 0)$. Let ρ be a similar one-radian rotation as in the proof of Theorem 3.6 but in three-dimensions (see Figure 3.5). The set of distinct points $\{0, \rho 0, \rho^2 0, ...\}$ can be rotated by another ρ into $\{\rho 0, \rho^2 0, \rho^3 0, ..\}$ where the position of the center is now filled. This shows equidecomposability under ρ. [12, p.4] $\square$

3.8.2 The fixed points

Besides the center, there are other problematic points, namely the ones through which the rotation axis of each word pass, if we were to perform the total rotation of each sequence in one step. These points stay fixed by non-trivial words in the free group F that we constructed in the last section. Fixed points in Figure 3.6 are visualized as the red points. An argument similar to to center point is made for these points as well.

Lemma 3.13. *Let D be the set of fixed points by some non-trivial element of F on the spherical shell S^2. Then S^2 is equidecomposable with $S^2 \setminus D$. [12, p.4]*

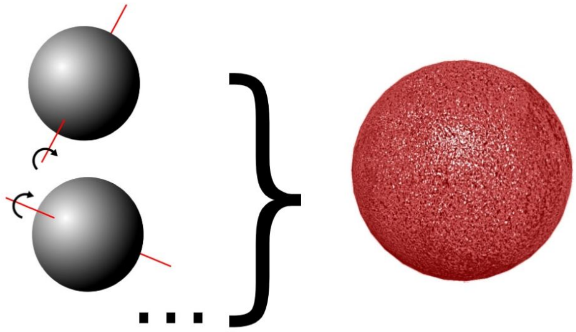

Figure 3.6: Axis of every rotation passes through two points fixed by that rotation.

Proof. The number of words are countable, so there are countably many fixed points (two for each rotation). This is why there could exist a rotation axis through the center that does not contain any points in D. Around this axis, there exist a rotation ρ, similar to the one in the proof of Lemma 3.12, so that the disjoint sets $D, \rho D, \rho^2 D, \rho^3 D, ...$ after another rotation ρ, can turn into $\rho D, \rho^2 D, \rho^3 D, ...$ from which D is absent. So $S^2 \setminus D$ (visualized as the blue points in Figure 3.7) is a rotation away from completing S^2. [12, p.5] □

3.8.3 The partitioning of the rest

Rotations of the remaining points, $S^2 \setminus D$, have distinct images, because if for $f_i \in F$ and $x \in (S^2 \setminus D)$ we had $f_1 x = f_2 x$ then x would be a fixed point by the word $(f_2^{-1} f_1)x = x$, and would have belonged to D.

All the points in $S^2 \setminus D$ can be partitioned into orbits consisting of all the points that can be rotated into one another. The orbit of a point $x \in S^2 \setminus D$ is the equivalence class $Fx = \{fx : f \in F\}$, under the relation of being a rotation of each other; that is $x \sim y \Leftrightarrow y \in Fx$.

According to the axiom of choice we can build a set M, consisting of one element from each of these orbits. Every point $x \in (S^2 \setminus D)$ is represented by a unique fm where $f \in F$ and $m \in M$, so the rotations $\{f_i^n M : n \in \mathbb{N}\}$ also partition $(S^2 \setminus D) = FM$.

Following the Theorem 3.10, F can be partition into two subsets F_1 and F_2, both of which being F-equidecomposable with F. So we want to perform on fM, an equivalent action to ones taken on each f, in order to similarly partition $(S^2 \setminus D) = FM$ into $F_1 M$ and $F_2 M$, represented as green and purple dots in Figure 3.8.

Let $\phi_i : F_i \to F$ $(i = 1, 2)$ be the F-equidecompositions that map $f \mapsto \phi_i f$.

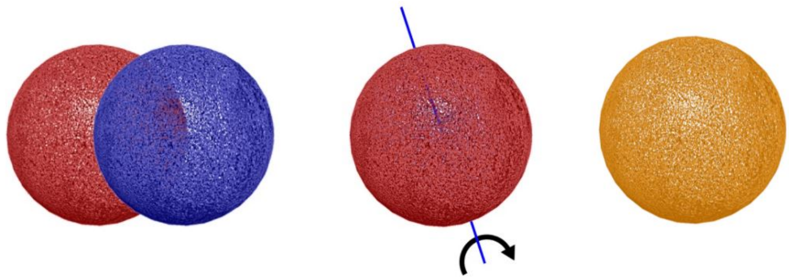

Figure 3.7: The set of one radian rotations of the fixed points around the new axis (the orange points) are contained in the blue points. With the red points missing we can just rotate the orange points by one radian in the opposite direction to obtain the whole shell.

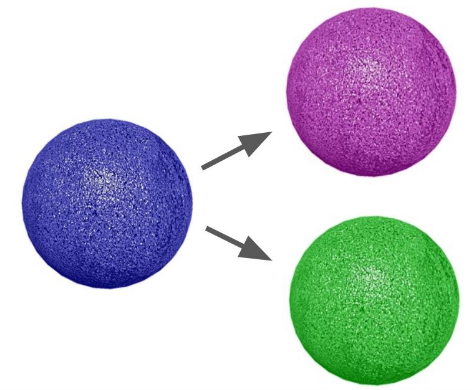

Figure 3.8: Paradoxical decomposition of the blue points. Each piece is individually equidecomposable to the blue piece itself, similar to the result of Theorem 3.10.

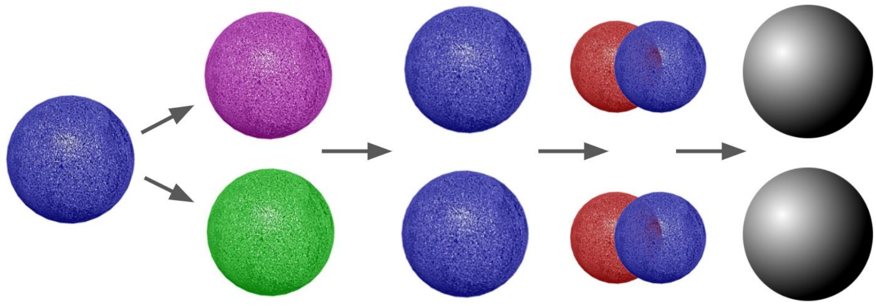

Figure 3.9: Obtaining two identical unit spheres by going through the steps in reverse.

If a piece $A_{ik} \subseteq F_i$ is mapped by ϕ_{ik} to $B_{ik} \subseteq F$, then corresponding to this, there would be a piece $\overline{\phi}_{ik}$ that maps $A_{ik}M$ to $B_{ik}M$. So generally there exist equidecompositions $\overline{\phi}_i : F_iM \to FM$ that map $fm \mapsto (\phi_i f)m$.

Now we have successfully partitioned $S^2 \setminus D$ into two sets, both being equidecomposible with $S^2 \setminus D$ itself. [10, p.8]

3.8.4 Putting it together

Now we can go through the steps in reverse as shown in Figure 3.9. The two copies of $S^2 \setminus D$, based on Lemma 3.13, can be reassembled into two spherical shells S^2, identical to the original, so S^2 is equidecomposible with two copies of itself.

Applying this, using the same rotations, simultaneously for all spherical shells of radii $0 \leq r \leq 1$, leads to two copies of $B^3 \setminus \{0\}$ equidecomposible with $B^3 \setminus \{0\}$, which due to Lemma 3.12 can extend to two B^3 equidecomposible with the original.

This paradox is only achievable in three or higher dimensions, since the isometry groups in lower than three dimensions do not include a free subgroup generated by two appropriate rotations. [12, p.5]

3.9 What does the paradox show?

This paradox is not a direct contradiction or a proof of falsehood, but merely a consequence of unituitive nature of sets and the way the axiom of choice is defined. As of now, Kurt Gödel's idea that "no axiom system can completely and consistently decide the truth of all propositions about set" is generally accepted.

[10, p.1] Also alternative forms of "choice" have been proposed to avoid similar paradoxical results, but the axiom of choice is essentially agreed upon. [10, p.10]

Chapter 4

Banach-Mazurkiewicz Theorem

4.1 Some definitions

Definition 4.1. *We define $C_{\mathbb{R}}(J)$ as the normed vector space of continuous real-valued functions $f : J \to \mathbb{R}$, on the compact interval $J \subset \mathbb{R}$, where the norm is the supremum norm defined by $\|f\|_{\infty} = \sup\{|f(x)| : x \in J\}$. [2, p.8] The metric derived from the supremum norm is defined by $d_{\infty}(f, g) = \|f - g\|_{\infty}$.*

Definition 4.2. *(Cauchy sequence) A Cauchy sequence in a metric space is one in which the distances between two elements x_m and x_n go to 0 as $n, m \to \infty$. In other words $\forall \epsilon > 0, \exists N : n, m \geq N \Rightarrow d(x_m, x_n) < \epsilon$. [4, p.38]*

Definition 4.3. *(Complete set) A metric space M is complete when every Cauchy sequence in it converges to an element that is in M. [4, p.40] Completeness entails closeness. [4, p.45]*

4.2 Approximation of sets and dense sets

In a metric space X (Definition 1.6), the union of all the subsets of a set that are open in X, is the interiors (Definition 1.4) of that set. An open set A in X, consists of only the interior, in which every point is surrounded by other points in A (as opposed to its boundary), so A can contain balls centered around each point. In fact, a set A is open if and only if A is a union of open balls. [4, p.19] That is

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} B_{r(x)}(x) \subseteq A. \tag{4.1}$$

Any open set A can be contained in a closed set, the smallest of which is the

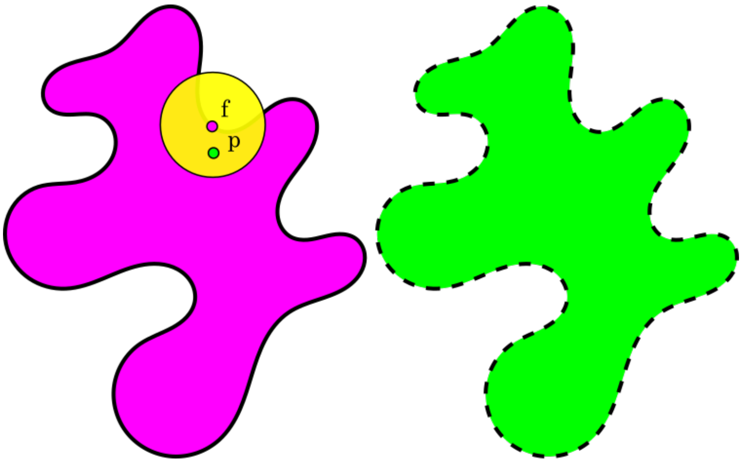

Figure 4.1: Polynomials (green) and the closure of the polynomials (purple). A p can be found in any ball of any radius centered around any f.

closure of A. [4, p.23] In other words, closure contains both the points in the set and their limits.

Sometimes we prefer the elements of a set X to be approximated by the elements of a subset $A \subseteq X$. Then the requirement for this in the presence of a metric, is that for any ball with any center $x \in X$ and an arbitrarily small radius $\epsilon > 0$, there exist an $a \in A$ inside of the ball. In the context of a metric space it follows $d(x, a) < \epsilon$. [4, p.25] An example of this, is the Theorem 4.4.

Theorem 4.4. *(Weierstrass Approximation Theorem) Let $P \subseteq C_{\mathbb{R}}[a, b]$ be the set all polynomials with real coefficients. Then we have $\overline{P} = C_{\mathbb{R}}[a, b]$. This means for every $f \in C_{\mathbb{R}}[a, b]$ and any given $\epsilon > 0$, there exists a $p \in P$, such that for all $x \in [a, b]$ we have $|f(x) - p(x)| < \epsilon$. [2, p.8]*

The idea of the Theorem 4.4 is visualized in Figure 4.1. A function $f \in C_{\mathbb{R}}[a, b]$ could be uniformly approximated by a sequence $\{p_n\}_{n=1}^{\infty}$, where $\|f - p_n\|_{\infty} \to 0$ as $n \to \infty$. [19, p.236] This is an expression in terms of the Definition 4.1, where the sequence $\{p_n\}_{n=1}^{\infty}$ converges uniformly to f, if and only if $\lim\limits_{n \to \infty} d_{\infty}(f, p_n) = 0$.

Definition 4.5. *(Dense and nowhere dense) A subset $A \subseteq X$ is dense in X, if $\overline{A} = X$, hence $\overline{A}$ contains all the balls that cover X. Conversely A would be nowhere dense in X if $(\overline{A})^{\circ} = \varnothing$. [2, p.8]*

For example, both rational and irrational numbers are dense in real numbers, because any real number is either in their set or in the set of their limit points. We can show this by proving there is a $q \in \mathbb{Q}$ in any neighbourhood of any $x \in \mathbb{R}$. This on the real line would look like $x < q < x + \epsilon$ for any $\epsilon \neq 0$, which also could be expressed in the following theorem.

Theorem 4.6. *We can find a rational number in between any two real numbers $x, y \in \mathbb{R}$ on the real line.*

Proof. Let $0 < x < y$. Then $0 < y - x$. By the Archimedean property of $\mathbb{R}$, which states

$$\forall a, b \in \mathbb{R} \textit{ where } a, b > 0, \exists n \in \mathbb{N} \textit{ where } n > 0 : na > b \tag{4.2}$$

we can obtain $1 < n(y - x)$ which gives $1 + nx < ny$. We know nx must lie between two integers z and $z - 1$. Now we have

$$z - 1 < nx < z < 1 + nx < ny \tag{4.3}$$

which leads to $x < \frac{z}{n} < y$ where $\frac{z}{n} \in \mathbb{Q}$. $\square$

On the contrary finding an integer between any two integers is not always possible and they are in fact nowhere dense in real numbers.

Definition 4.7. *(Meager) A subset $A \subseteq T$ is meager (or first category) in the topological space T if it is a union of countably many nowhere dense sets i.e. we have $A = \bigcup_{k=1}^{\infty} A_k$ where A_k are nowhere dense in T. [2, p.19]*

The sets that constitute a meager set do not have to be closed but all meager sets are contained in a union of closed nowhere dense sets. Figure 4.2 is a visualization of a meager set. Baire's category theorem is a showcase of this "meagreness" manifesting itself as the inadequacy to "cover" a set of the type defined in Definition 4.3. The axiom of choice is used for the proof of this theorem.

4.3 Baire's category theorem

Theorem 4.8. *A non-empty, complete metric space cannot be covered by a countable number of nowhere-dense sets i.e. it is non-meager in itself. [4, p.46]*

Proof. Let us assume our complete metric space is $X = \bigcup_{n=1}^{\infty} A_n$ where A_n are nowhere dense in X. We want to disproof this by constructing a nested sequence of balls $\{B_{r_n}(x_n)\}_{n=1}^{\infty}$ of radii r_n, whose centers x_n form a non-convergent Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X; more precisely, we want to show that despite the nestedness of the balls, this sequence does not converge to any point on X.

None of A_n contain any balls. In each step n, we want to make sure $B_{r_n}(x_n)$ is in the exterior of A_n. So to start off we place the first ball in the exterior of A_1; that is $B_{r_1}(x_1) \subseteq (\overline{A}_1)^{\complement}$. To maintain nestedness the next ball will be $B_{r_2}(x_2) \subseteq (\overline{A}_2)^{\complement} \cap B_{r_1}(x_1)$. Every subsequent radius is chosen small enough to fit this requirement. This is visualised in Figure 4.3 [4, p.46].

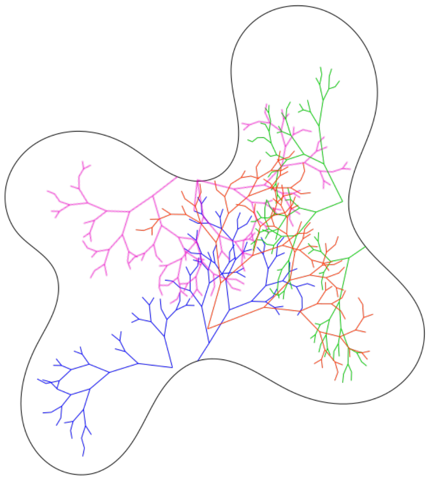

Figure 4.2: A meager set as a union of countably many nowhere dense sets, each represented with a different color.

Our choice of where to center a ball at each stage, invokes the use of the axiom of choice. Repeating this procedure, generates our Cauchy sequence of points

$$x_{n+1} \in B_{r_{n+1}}(x_{n+1}) \subseteq (\overline{A}_{n+1})^{\complement} \cap B_{r_n}(x_n). \tag{4.4}$$

For all n, it is true that for $m > n$, $x_m \in B_{r_m}(x_m) \subseteq B_{r_n}(x_n)$. So assuming $\lim_{n\to\infty}\{x_n\} = x$ then even $\lim_{m\to\infty}\{x_m\} = x$, thus $x \in B_{r_m}(x_m) \subseteq B_{r_n}(x_n)$. Another way of expressing this is

$$x \in \bigcap_n B_{r_n}(x_n) \subseteq \bigcap_n (\overline{A}_n)^{\complement} = (\bigcup_n \overline{A}_n)^{\complement} \subseteq (\bigcup_n A_n)^{\complement} = X^{\complement}. \tag{4.5}$$

This means that our Cauchy sequence does not converge to a point in X, which precludes the completeness. [4, p.46] □

Baire's category theorem leads to yet another unintuitive consequence regarding an infinite set, namely the fact that the nowhere differentiable functions make up the majority of the space $C_{\mathbb{R}}(J)$ (from Definition 4.1).

4.4 Banach-Mazurkiewicz theorem

Among the first continuous nowhere differentiable functions was Weierstrass function, published by Paul du Bois-Reymond in 1875, originally presented by

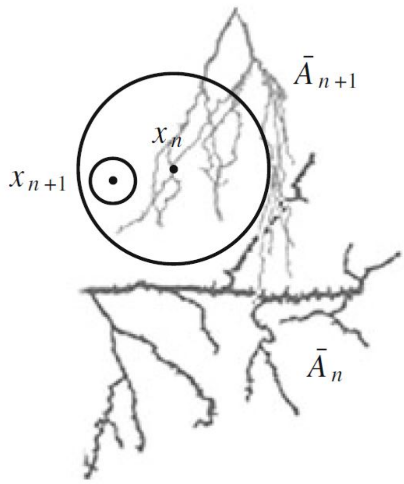

Figure 4.3: Procedure for generating a Cauchy sequence

Karl Weierstrass in 1872. [20, p.20] Weierstrass function is defined by

$$W(x) = \sum_{k=0}^{\infty} a^k \cos b^k \pi x, \tag{4.6}$$

with $0 < a < 1$, $ab < 1 + \frac{3\pi}{2}$ and an odd integer $b > 1$. The plot of this function (Figure 4.4) is a self-similar "zigzag" fractal. This resembles the Hilbert curve in the sense that both are continuous nowhere differentiable fractals (see Theorem 2.9).

One application of Baire's category theorem is to show how ubiquitous these pathological properties are when it comes to real-valued functions on a normed vector space. [2, p.20] It turns out that it is typical of continuous functions on an interval to be nowhere differentiable. [21, p.108]

We can utilize the Baire's theorem to show the "non-meagerness" of nowhere differentiable functions, within the class $C_{\mathbb{R}}(J)$.

Theorem 4.9. *(Banach-Mazurkiewicz Theorem) Let $C_{\mathbb{R}}(J)$ be endowed with a supremum norm. Then the subset of nowhere differentiable functions $A \subset C_{\mathbb{R}}(J)$ is non-meager in $C_{\mathbb{R}}(J)$, meanwhile $A^{\complement}$ is meager in $C_{\mathbb{R}}(J)$.*

The following proof is an amalgamation of [19, p.330], [2, p.22], [22, p.6] and [23, p.2].

Proof. If proven for $J = [0, 1]$, it can then be extended to any J. According to Definition 4.7 we want to show that in contrast to A, $A^{\complement}$ can be a union of

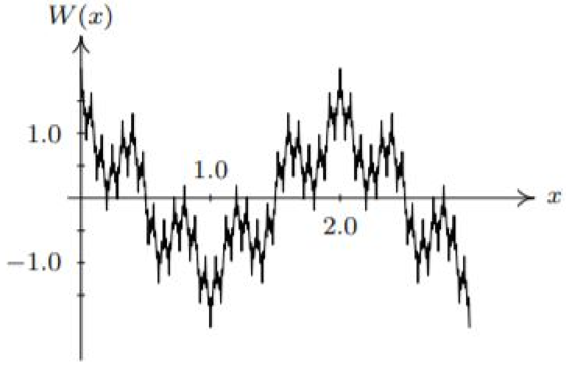

Figure 4.4: Weierstrass function with $a = 0.5$ and $b = 5$

countably many closed nowhere dense sets, which would prove it being meager in $C_{\mathbb{R}}(J)$ by Theorem 4.8.

We can construct countable number of sets $\{E_n\}_{n\in\mathbb{N}}$ of the form

$$E_n = \{f \in C_{\mathbb{R}}[0,1] \,|\, \exists x \in [0, 1 - \frac{1}{n}] : \forall h \in (0, 1-x), \left|\frac{f(x+h) - f(x)}{h}\right| \leq n\}. \tag{4.7}$$

These E_n contain functions possessing a finite right-hand derivative in at least one point $x \in [0, 1 - \frac{1}{n}]$. Any function differentiable at least in one point in our interval belongs to an E_n. Equally valid would be to use a similar sequence but with left-hand derivatives instead. No matter which we use, $\bigcup\limits_{n}^{\infty} E_n = A^{\complement}$.

To prove that the sets E_n are closed in $C_{\mathbb{R}}[0,1]$, we proceed the following. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in E_n that converges uniformly to f i.e.

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : \textit{if } i > N \textit{ then } d_{\infty}(f_i.f) < \epsilon. \tag{4.8}$$

For every i there is an $x_i \in [0, 1 - \frac{1}{n}]$ so that

$$\left|\frac{f_i(x_i + h) - f_i(x_i)}{h}\right| \leq n. \tag{4.9}$$

So we can say there is a bounded sequence $\{x_i\}_{i=1}^{\infty}$ corresponding to $\{f_i\}_{i=1}^{\infty}$. By the Bolzano – Weierstrass theorem, every bounded sequence in $\mathbb{E}^n$ has a convergent sub-sequence, so there must be a sub-sequence of $\{x_i\}_{i=1}^{\infty}$, say $\{x_{i_k}\}$, that converges to some $x \in [0, 1 - \frac{1}{n}]$. For convenience we relabel $\{x_{i_k}\}$ into $\{x_i\}$ as well as $\{f_{i_k}\}$ into $\{f_i\}$.

Then for $h \in (0, 1 - x_i)$ and sufficiently large i we can have

$$\begin{aligned}
&|f(x+h) - f(x)| \leq \\
&|f(x+h) - f(x_i+h)| + |f(x_i+h) - f_i(x_i+h)| + |f_i(x_i+h) - f_i(x_i)| + \\
&|f_i(x_i) - f(x_i)| + |f(x_i) - f(x)| \leq \\
&|f(x+h) - f(x_i+h)| + d_\infty(f, f_i) + nh + d_\infty(f_i, f) + |f(x_i) - f(x)|.
\end{aligned} \tag{4.10}$$

By letting $i \to \infty$, (4.10) becomes

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq n \tag{4.11}$$

which shows $f \in E_n$, indicating E_n is closed.

Now it remains to show that all E_n are nowhere-dense. For a fixed n, showing that E_n cannot contain any open balls in $C_\mathbb{R}[0, 1]$ would be enough evidence for all of them being nowhere dense (see Definition 4.5). To do this, the goal is to show that there always exists a function $g \notin E_n$ in the vicinity of any f such that $d_\infty(f, g) < \epsilon$ for all $\epsilon > 0$.

Since f, by the Theorem 4.4 could be uniformly approximated by polynomials and polynomials are smooth functions, we can assume f is smooth and thus its derivative is bounded by some $M \geq f'(x)$.

Theorem 4.10. *(Heine–Cantor) Every $f \in C_\mathbb{R}(J)$ is uniformly continuous on J, that is*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1 \text{ and } \forall x_2 \in J : |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\epsilon}{4}. \tag{4.12}$$

[19, p.157]

Given the ϵ and δ from Theorem 4.10 we can choose a value $s > 0$ so that

$$s < \min\{\delta, \frac{\epsilon}{2(M+n)}\} \tag{4.13}$$

which we use to partition our interval by the points $0 = t_1 < t_2 < ... < t_k = 1$ into $k - 1$ pieces where

$$\max_{1<i<k-1} (t_{i+1} - t_i) < s. \tag{4.14}$$

A consequence of Theorem 4.10 is that every open ball in $C_\mathbb{R}(J)$, will contain a function g from the set of continuous piecewise linear (also known as polygonal) functions L. So similar to Theorem 4.4, L is dense in $C_\mathbb{R}(J)$.

Because a function $g \in L$ can be chosen arbitrarily close to every f by the metric d_∞, it is also possible to approximate every f by a sequence $\{g_\alpha\}_{\alpha=1}^\infty$. The idea

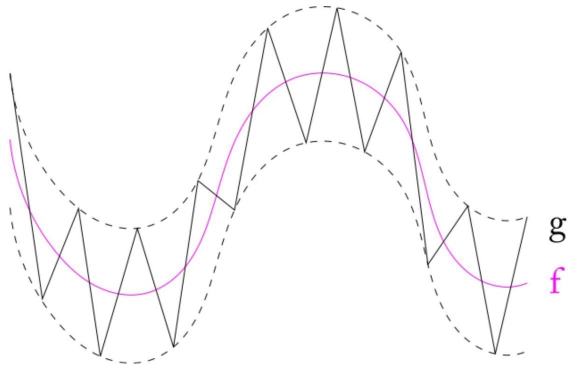

Figure 4.5: Approximating every function with a zigzag piecewise linear one.

here is to construct such g that is not differentiable on the points $\{t_1, ..., t_{k-1}\}$ but is linear in between them. We could define g as

$$g = \begin{cases} f(t_i) + (-1)^i \frac{\epsilon}{4} & \text{for } t_i \\ \frac{t_{i+1}-t}{t_{i+1}-t_i} g(t_i) + \frac{t-t_i}{t_{i+1}-t_i} g(t_{i+1}) & \text{for } t_i < t < t_{i+1} \end{cases} \tag{4.15}$$

which would look something like Image 4.5.

Having (4.15), for $0 \leq i \leq k-1$ and $t \in [t_i, t_{i+1}]$ we can write

$$g(t) - f(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i}(g(t_i) - f(t)) + \frac{t - t_i}{t_{i+1} - t_i}(g(t_{i+1}) - f(t)). \tag{4.16}$$

Now based on (4.14) and (4.12) we could argue

$$|g(t) - f(t)| \leq |f(t_i) - f(t)| + |f(t_{i+1}) - f(t)| + \frac{\epsilon}{2} < \epsilon \tag{4.17}$$

which leads to $d_\infty(f, g) < \epsilon$. On the other hand for any $t, \xi_i \in (t_i, t_{i+1})$

$$g'(t) = \frac{f(t_{i+1}) - f(t_i) + (-1)^{i+1}\frac{\epsilon}{2}}{t_{i+1} - t_i} = f'(\xi_i) + \frac{(-1)^{i+1}\frac{\epsilon}{2}}{t_{i+1} - t_i}. \tag{4.18}$$

Our choice of h according to (4.13) leads to

$$|g'(t)| = \left| \frac{(-1)^{i+1}\frac{\epsilon}{2}}{t_{i+1} - t_i} + f'(\xi_i) \right| \geq \frac{\epsilon}{2(t_{i+1} - t_i)} - |f'(\xi_i)| \geq \frac{\epsilon}{2s} - M > n. \tag{4.19}$$

It becomes apparent that the slope of this function at any point is conveniently steep enough to not fit the definition of E_n containing f (see (4.7)). The accuracy of f's approximation by this saw-tooth function g, improves as the number of line segments go to infinity and $s \to 0$. So as it appears in (4.19), the derivative blows up as g approaches f. This is the same property present in the Weierstrass function. So effectively every f can be approximated by a nowhere differentiable function. As a result we can conclude E_n has an empty interior and is nowhere dense in $C_{\mathbb{R}}[0, 1]$.

□

There are alternative proofs for which the reader is referred to [21, p.109], [19, p.330] and [2, p.22].

Chapter 5

Conclusion

There are several commonalities and recurring themes in the content of preceding chapters both conceptually and methodically. We started by the general idea of spaces, environments that sets inhabit and how the underlying structures governing these spaces determine their properties. We saw how some operations (mappings or group actions) on sets took the form of manipulations of geometrical objects where the notions such as measure and dimension became relevant. Very soon infinity was inevitable which we encountered whether while working with infinite sets or in the form of infinite magnification of fractal structures leading to infinite repetition of the same patterns.

Space filling curves are indeed an extraordinary case of fractals both in terms of their mapping and their dimension. The idea of "covering" sets helped us to measure a coast line as a fractal as well as being used as a tool to define dense sets used in Banach-Mazurkiewicz theorem. On the other hand we used non-measurable sets to arrive at remarkable result of the Banach-Tarski Paradox, which required a function that would make infinitely many choices.

The reason we may consider many of these results unintuitive is due to our inability to fully comprehend concepts such as infinity. But in a mathematical context all these impossibilities become possible when we have the tools to capture the behavior of infinity at least in a hypothetical sense.

Bibliography

[1] Tsirelson B. Spaces in mathematics. WikiJournal of Science. 2018;1. https://upload.wikimedia.org/wikiversity/en/c/cd/Spaces_in_mathematics.pdf.

[2] Ashraf P. Pathological functions and the Baire category theorem; 2017. Uppsala Universityhttps://uu.diva-portal.org/smash/get/diva2:1104162/FULLTEXT01.pdf.

[3] Pugh CC. Real Mathematical Analysis. New York: Springer; 2002.

[4] Muscat J. Functional Analysis. Springer International Publishing Switzerland; 2014. University of Malta.

[5] Buskes G, van Rooij A. Topological Spaces From Distance to Neighborhood. Springer Science+Business Media New York; 1997. University of Malta.

[6] Lachi'eze-Rey M, Luminet JP. Cosmic Topology. Cornell University archive. 1996;https://arxiv.org/abs/gr-qc/9605010.

[7] Peitgen HO, Jurgens H, Saupe D. Chaos and Fractals. Springer Science+Business Media New York; 1992. Universităt Bremen.

[8] Feder J. FRACTALS. Springer Science+Business Media New York; 1988. University of Oslo.

[9] Tao T. Analysis II. vol. 38. 3rd ed. Springer Science+Business Media Singapore; 2015. University of California, Los Angeles.

[10] Kaseorg A. The Banach-Tarski Paradox; 2007. Massachusetts Institute of Technology. http://web.mit.edu/andersk/Public/banach-tarski.pdf.

[11] Sagan H. Space-Filling Curves. New York: Springer Science+Business Media, LLC; 1994.

[12] Wu A. The Banach–Tarski paradox; 2008. University of Chicago: Department of mathematics. http://www.math.uchicago.edu/~may/VIGRE/VIGRE2008/REUPapers/Wu.pdf.

[13] Rose NJ. Hilbert-Type Space-Filling Curves; 2015. North Carolina State University. https://researchgate.net/publication/265074953.

[14] Tarver T. Hilbert's space-filling curve. Asian journal of mathematics and applications. 2015 7;2014. Bethune-Cookman University, https://www.researchgate.net/publication/279203510.

[15] Kaloliya V, Koshti S, Mistry A. Design and Implementation of Fractal Antenna. Institute of Technology, Nirma University; 2015. Https://www.researchgate.net/publication/276355483.

[16] Skubalska-Rafajlowicz E. Applications of the space - Filling curves with data driven measure - Preserving property. Nonlinear Analysis, Theory, Methods and Applications. 1997 4;30. Wroclaw University of Science and Technology, https://www.researchgate.net/publication/242989331.

[17] Valgaerts L. Space-Filling Curves An Introduction; 2005. Department of Informatics Technical University Munich. http://www.mayr.informatik.tu-muenchen.de/konferenzen/Jass05/courses/2/Valgaerts/Valgaerts_paper.pdf.

[18] Roman S. Fundamentals of Group Theory. Springer Science+Business Media, Birkhäuser Boston; 2012.

[19] Montesinos V, Zizler P, Zizler V. An Introduction to Modern Analysis. Springer International Publishing Switzerland; 2015. Universitat Politècnica de València.

[20] Thim J. Continuous Nowhere Differentiable Functions; 2003. Luleå University of Technology https://pdfs.semanticscholar.org/8cfb/8ff14cc7ab6e6010f363355ff5ddf08f11a7.pdf.

[21] Jarnicki M, Pflug P. Continuous Nowhere Differentiable Functions. Springer International Publishing Switzerland; 2015. Jagiellonian University.

[22] Kesavan S. Continuous functions that are nowhere differentiable. Popular Articles by S Kesavan, The Institute of Mathematical Sciences;https://www.imsc.res.in/~kesh/nowhere.pdf.

[23] Green BJ. The Baire Category Theorem. Ben Green's website, University of Oxford. 2009;http://people.maths.ox.ac.uk/greenbj/papers/baire-category.pdf.