SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Johannes Kepler and his Planetary Model

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2021 - No K16

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

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2021

Abstract

This essay sets out to give an account on Johannes Kepler's planetary model. It starts by looking at the Platonic solids, which play a major role in Kepler's theory, and leads us back to the Greek mathematicians and philosophers, especially Euclid and his mathematical text the *Elements*. We look at the proof from the *Elements* of why there are only five Platonic solids and compare that proof to a more modern proof by Euler during the eighteenth century. We then move on to look at the time Kepler lived in to finally conclude how he constructed his planetary model.

Acknowledgement

I would like to thank my supervisor Rikard Bögvad for introducing me to Kepler and for pointing me in the right directions. Thank you John for being encouraging and always believing in me. Thank you Matilda for listening to me and calming me down whenever I get stressed.

Contents

1 Introduction

Why waste words? Geometry, which before the origin of things was coeternal with the divine mind and is God himself (for what could there be in God which would not be God himself?), supplied God with patterns for the creation of the world, and passed over Man along with the image of God; and was not in fact taken in through the eyes.

Johannes Kepler in *Harmonice Mundi* 1619.

This essay sets out to give an account on Johannes Kepler's planetary model presented in *Mysterium Cosmographicum* in 1596 and then in a second edition in 1621. It starts by looking at the Platonic solids, which play a major role in Kepler's theory, and leads us back to the Greek mathematicians and philosophers. We delve into some of the philosophy of Plato and the reason the Platonic solids has been named after him. We then look more closely how the solids are formed which introduces Euclid and his mathematical text the *Elements*. We look at the proof from the *Elements* of why there are only five Platonic solids and compare that proof to a more modern proof by Euler during the eighteens century. We then move on to look at the time Kepler lived in to finally conclude how he constructed his planetary model.

As we move through Kepler's model, we start with what sparked his initial idea and then how he divides the Platonic solids in two classes by stating three of them being primaries and two secondaries due to their mathematical compositions. This divide leads us to how the solids are placed between the planet's orbs and Kepler's reasoning for why the order made sense. He also justifies the placement of the solids by comparing the ratio between the inscribed and the circumscribed sphere of the Platonic solid to Copernicus observed data of the planet's orbs and therefore we look at the calculations of the ratios of the cube, icosahedron and tetrahedron's spheres. We then look at the duality between the cube and octahedron as well as the icosahedron and the dodecahedron to also get the ratios of the octahedron and dodecahedron.

Finally, we conclude this essay by stating that although Kepler's planetary model was wrong it did allow for him to make other scientific discoveries that we use till this day as well as laying the foundation for Isaac Newton's work on the laws of motion.

2 Background

The Platonic solids have been known for thousands of years. When exactly they were discovered is unknown, but stones carved in a similar manner to the regular polyhedral have been found in Scotland dating back to the late Neolithic or early Bronze Age, between 3200 and 1500 BC.

Figure 1: Carved stones found in Scotland believed to be from 2000 BC

What they were used for, or if they were simply the very beginning of abstract mathematical thinking, we do not know [14] but as figures we can assume they have fascinated mankind for a long time. The first known written account of them can be traced back to the Pythagoreans around the sixth century BC and we believe they were the first to study the regular solids [12]. The first proof of why there are only five regular solids can be found in the *Elements* written by Euclid around 300 BC [12]. And though Euclid is the one known and often the one credited for the mathematics in the *Elements* it is believed that he was more of a collector of the thoughts of others. Many historians contend that most of the mathematics in Book XIII of the *Elements*, where we find the regular solids, are due to Theaetetus. Theaetetus was a Greek mathematician and lived around 400 BC. His friend and teacher, the influential philosopher Plato (c. 427-347 BC) wrote about him in two of his dialogues. Plato also wrote about the regular solids in his work *Timaeus*, a philosophical account of how God created the Universe and were he connects the five figures to the elements and the whole creation, which is why they are known as the Platonic solids.

3 The Platonic solids

To understand the Platonic solids we start by looking at the components that form the Platonic solids faces, the polygons.

3.1 Polygons

A polygon, from its Greek meaning 'many', 'much' and 'corner', 'angle' [14], is a two-dimensional polytope that has straight line edges and vertices. We need to have at least three vertices connected by edges to form a closed figure to get a polygon but there is no upper limit in how many vertices a polygon may have. We can therefore create an infinite number of polygons by connecting vertices by edges.

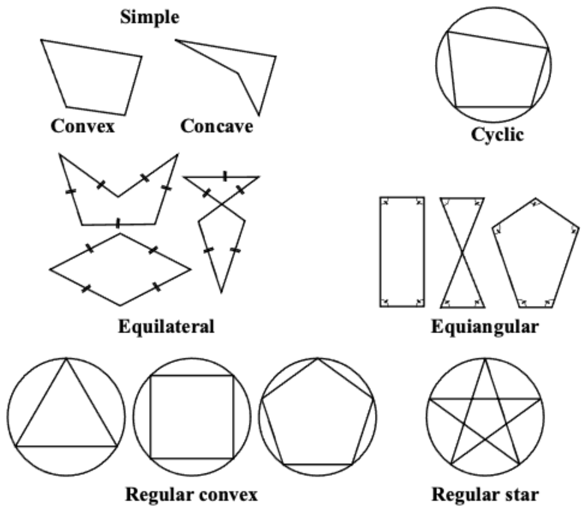

Figure 2: Polygons ordered in different classifications.

As shown in Figure 2 the polygons can be classified in a number of different ways but the one class we are going to have a closer look at is the class of convex regular polygons. Below follows some definitions

Definition 3.1. A polygon is called equilateral if all its edges are the same length.

Definition 3.2. A polygon is called equiangular if all its interior angles are equal.

Definition 3.3. A polygon is regular if it is both equilateral and equiangular.

In other words, a convex polygon is called regular if all its sides have the same length, and all its angles are congruent [14]. From this follows that all interior angles are less than 180°.

3.2 Polyhedra

Now, as we have our building blocks, we can move to three dimensions and our Platonic solids. Using the regular polygons, we have the faces to form a polyhedron. A polyhedron is the three-dimensional equivalent to the polygon and is made up of edges, vertices and the already mentioned faces, see Figure 3 depicting one of the regular polyhedron; the cube with the edges, vertices and faces clearly marked. If all the faces are regular polygons and the regular polygons are all the same, we get a regular polyhedron. To clarify, for a polyhedron to be regular it needs to satisfy the following conditions [12]:

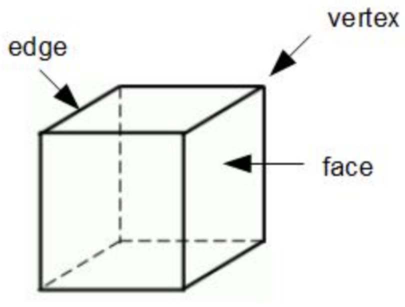

Figure 3: The cube

1. The polyhedron is convex.
2. Every face of the polyhedron is a regular polygon.
3. All faces are congruent (identical).
4. Every vertex is surrounded by the same number of faces.

The figures that meet these conditions are what is known as the Platonic solids. They are five in number, in comparison to the infinite number of polygons [12], and first proved to be only five in Euclid's the *Elements*, which we will return to in Chapter 4.

The five Platonic solids are:

- the regular tetrahedron,
- the regular cube,
- the regular octahedron,
- the regular icosahedron,
- the regular dodecahedron.

They are often referred to, and will be throughout this essay, as only the tetrahedron, cube, octahedron, icosahedron and dodecahedron i.e. without 'regular' preceding them.

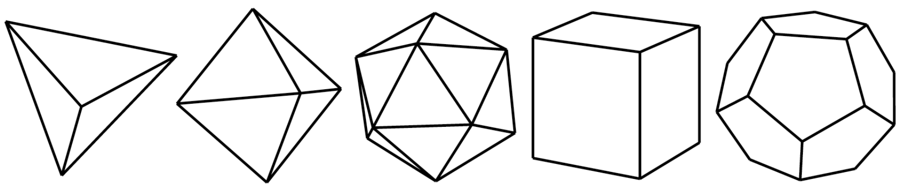

Figure 4: Tetrahedron, Octahedron, Icosahedron, Cube and Dodecahedron

The tetrahedron, the octahedron and the icosahedron are all three formed by the equilateral triangle, four for the tetrahedron, eight assembles to the octahedron, and twenty form the icosahedron. The cube is formed from six squares and the dodecahedron is made from twelve pentagons. Table 3.2 shows the different number of vertices, edges and faces of each solid.

Polyhedron	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Icosahedron	12	30	20
Dodecahedron	20	30	12

Table 3.2

3.3 The history behind the name

So how is it that the regular polyhedrons are referred to as the Platonic solids? The name is derived from the Greek philosopher Plato who, in his work *Timaeus* [11] from around 360 BC, discusses the solids. In the text, the character Timaeus tells the story of the creation, explaining how God desired that all things should be good, so far as this was possible [13]. He made the world in the shape of a sphere, the most perfect shape possible and continues to describe the smallest parts there is in geometrical terms. "The building blocks of matter are the regular polyhedra, which themselves are formed from regular polygons, which in turn are ultimately pieced together from two types of tiny triangles." [13]. He then goes into a detailed account of the tiny triangles and how they are the 'most beautiful' triangles, but we shall not linger on those triangles but instead jump straight to the solid figures; the tetrahedron, the octahedron, the icosahedron, the cube and the dodecahedron.

What then follows in Plato's *Timaeus* is that the Creator makes the elements out of these five solids, who in turn construct the world. Fire was made from the tetrahedron, air from the octahedron, water from the icosahedron and earth was made from the cube. As there are only four elements, he uses the fifth solid, the dodecahedron, for the heavens. Stephenson [13] writes in *The Music of the Heavens: Kepler's Harmonic Harmony* (1994) how this description of the world resonated through the Western world and became firmly established, clear examples can be found within the arts amongst other places. Figure 5 shows the regular polyhedron as printed in Kepler's *Harmonice Mundi* (1619) with the images inside the solids illustrating the elements [10].

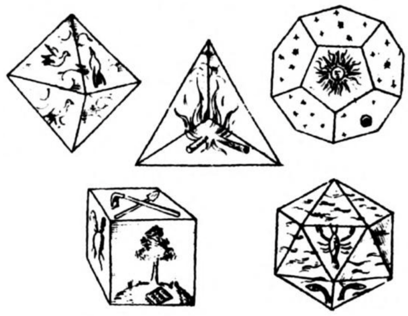

Figure 5: The Platonic solid with their corresponding element.

Though Kepler was familiar with Plato's view of the solids in connection to the elements he never explored it further, as Field puts it "Kepler's main concern was, in any case, always with astronomy and it appears that he never concerned himself directly with the properties of the elements" [3].

4 Only five Platonic solids

In this section we will, as promised, come back to the regular polyhedra and show why there are only five of them. The fascination for the Platonic solids has been huge throughout history with many different proofs constructed as for why there are only five. But let us start by looking at what Kepler worked with and often referred to in his own work, the proof in Euclid's *Elements*.

4.1 Euclid's proof

First a definition from Euclid, he defines a solid angle as follows.

Definition 4.1. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point [4].

Field summarises it more clearly as 'a solid angle is an angle formed by three or more planes intersecting at a common point (the vertex)' [3]. Both of these definitions are a bit simplified in the view of modern mathematics so we will have a closer look at the definition of a solid angle we know of today.

A solid angle is the analogy of a plane angle's one dimensional angle but in three dimensions. Instead of finding the angle in a circle we find it in the sphere. In the same way a plane angle of the full circle gives us 2π radians a solid angle of the full sphere is 4π steradians as a solid angle is measured in steradians, sr. We have that the mid-point of the sphere being the vertex where the lines of a closed curve on the surface of the sphere meet, as shown in Figure 6. The area, a, that

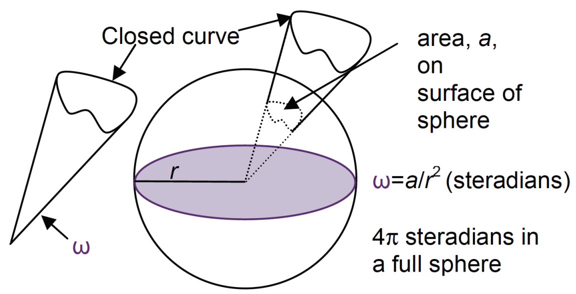

Figure 6: A solid angle projected on the unit sphere.

the curve encompassed divided with the square radius, r of the sphere gives us the solid angle. Let a solid angle be ω, then

$$\omega = \frac{a}{r^2}.$$

To connect this to our solids, let us look at the cube.

Place a cube in a sphere, as shown in Figure 7, with the mid-point of the cube being the same as the mid-point of the sphere. Then the solid angle with the mid-point of the cube and sphere being the vertex is limited by the cubes sides projected onto the sphere. Since the cube is symmetrical and has six sides one side is subtended by one-sixth of the whole sphere giving us $\frac{\text{full sphere}}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$ steradians.

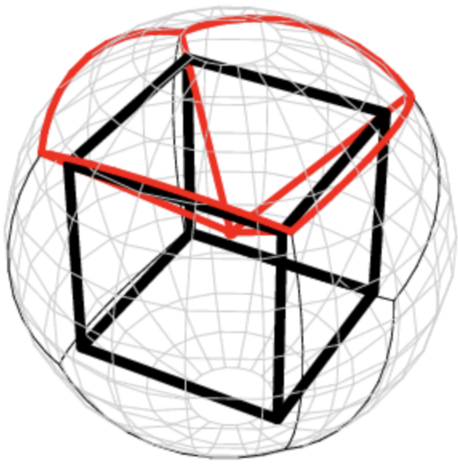

Figure 7: The solid angle of one side of the cube.

Though the example above makes for a very clear image it is not quite the solid angle of the cube we are interested in. The solid angle we want to look at is the one where one of the cube's vertices is the mid-point of the sphere. This makes the diagonal, from said vertex to the opposite vertex, the radius of the sphere. The area on the sphere that we are interested in is thus the area that is limited by the three faces of the cube. This area can be calculated with a complicated formula but let us look at an informal, more intuitive argument. Place a cube, c with seven more cubes of same size to form a bigger cube, C, see Figure 8. Let the same vertex of the above cube be the

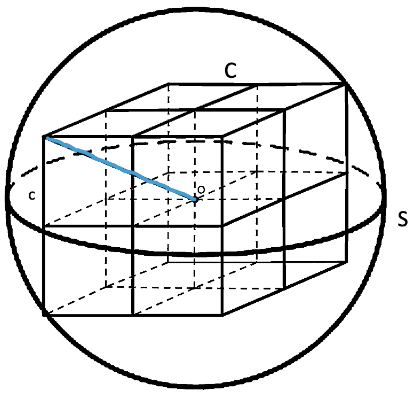

Figure 8: Solid angle from the vertex of a cube.

mid-point, O of cube C having placed the other seven cubes accordingly, making one vertex from each of the eight cubes meet in O. Let the sphere, S circumscribe cube C, and thus having the same mid-point as cube C. The planes corresponding to the faces of the eight smaller cubes then clearly divides the sphere in eight area segments making it easy to see that the solid angle of the vertex in one of the smaller cubes being one-eights of the full sphere, i.e. $\frac{4\pi}{8} = \frac{\pi}{2}$ steradian.

Let us now come back to the proof of why there is only five regular polyhedra. Both Euclid and Kepler structured their works starting with more simpler propositions and proofs to then build up to more complicated ones often having the latter referring back to the former. So for us to understand the proof of the five solids that Euclid gives, and later on Kepler gives an account of, we need to first look at two earlier proposition. Proposition 20 and 21 from Book 11 in the *Elements* [4] both taken from Richard Fitzpatrick's 2007 translation of said book. Let us start with Proposition 20.

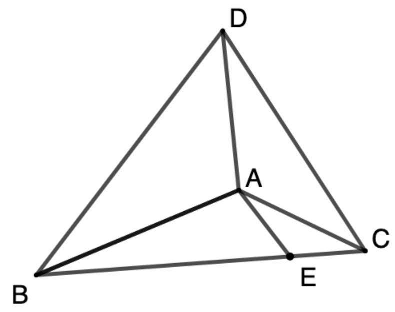

Figure 9: A solid angle

Proposition 4.1. *If a solid angle is contained by three plane angles, then the sum of any two is greater than the remaining one.*

Proof. Let the solid angle at A be contained by the three plane angles BAC, CAD and DAB, see Figure 9. I say that the sum of any two of the angles BAC, CAD and DAB is greater than the remaining one.

If the angles BAC, CAD and DAB are equal to one another, then it is clear that the sum of any two is greater than the remaining one. But, if not, let BAC be greater. In the plane through BA and AC, construct the angle BAE equal to the angle DAB at the point A on the straight line AB. Make AE equal to AD, draw BEC across through the point E cutting the straight lines AB and AC at the points B and C, and join DB and DC.

Now, since DA equals AE, and AB is common, therefore two side are equal to two sides. And the angle DAB equals the angle BAE, therefore the base DB equals the base BE. And, since, from the triangle inequality, the sum of the sides BD and DC is greater than BC, and of these DB was proved equal to BE, therefore the remainder DC is greater than the remainder EC.

Now, since DA equals AE, and AC is common, and the base DC is greater than the base EC, therefore the angle DAC is greater than the angle EAC. But the angle BAE equals the angle DAB, therefore the sum of the angles DAB and DAC is greater than the angle BAC.

Similarly we can prove that the sum of any two of the remaining angles is greater than the remaining one.

Therefore, *if a solid angle is contained by three plane angles, then the sum of any two is greater than the remaining one.* □

This proposition is used in the following proposition, Proposition 21 from Book 11 [4]. We have

Proposition 4.2. *Any solid angle is contained by plane angles whose sum is less than four right angles [4].*

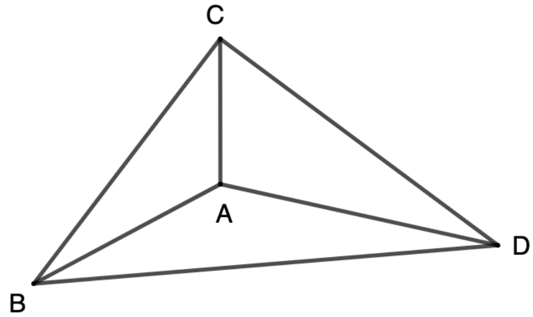

Figure 10: A solid angle

Proof. Let the solid angle A be contained by the plane angles BAC, CAD, and DAB, see Figure 10. I say that the sum of the angles BAC, CAD, and DAB is less than four right-angles.
For let the random points B, C, and D have been taken on each of the straight-lines AB, AC, and AD respectively. And let BC, CD, and DB have been joined. And since the solid angle at B is contained by the three plane angles CBA, ABD, and CBD, the sum of any two is greater than the remaining one, as seen in Proposition 4.1. Thus, the sum of CBA and ABD is greater than CBD. So, for the same reasons, the sum of the angles BCA and ACD is also greater than BCD, and the sum of CDA and ADB is greater than CDB. Thus, the sum of the six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than the sum of the three angles CBD, BCD, and CDB. But, the sum of the three angles CBD, BDC, and BCD is equal to two right-angles. Thus, the sum of the six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than two right-angles. And since the sum of the three angles of each of the triangles ABC, ACD, and ADB is equal to two right-angles, the sum of the nine angles CBA, ACB, BAC, ACD, CDA, CAD, ADB, DBA, and BAD of the three triangles is equal to six right-angles, of which the sum of the six angles ABC, BCA, ACD, CDA, ADB, and DBA is greater than two right-angles. Thus, the sum of the remaining three angles BAC, CAD, and DAB, containing the solid angle, is less than four right-angles.
Thus, any solid angle is contained by plane angles whose sum is less than four right-angles. Which is the very thing it was required to show. □

Note: This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalisation to a solid angle contained by more than three plane angles is straightforward.

Now, finally, the proof of why there is only five Platonic solids. From Richard Fitzpatrick's translation of the *Elements* [4] we have:

Theorem 4.3. *So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.*

What follows is the proof which have been slightly rewritten here for the purpose of this essay with added figures for clarity.

Proof. From definition 4.1, a solid angle cannot be constructed from two triangles, or indeed from two planes of any sort. The solid angle of the tetrahedron is constructed from three equiangular triangles, that of the octahedron from four equiangular triangles and that of the icosahedron from five equiangular triangles. A solid angle cannot be made from six equilateral and equiangular triangles set up together at one point. For, since the angles of the equilateral triangle are each two-thirds of a right-angle, the sum of the six plane angles containing the solid angle, will be four right angles. That very thing is impossible. For every solid angle is contained by plane angles whose sum is less than four right angles from Proposition 4.2 and shown in Figure 11.

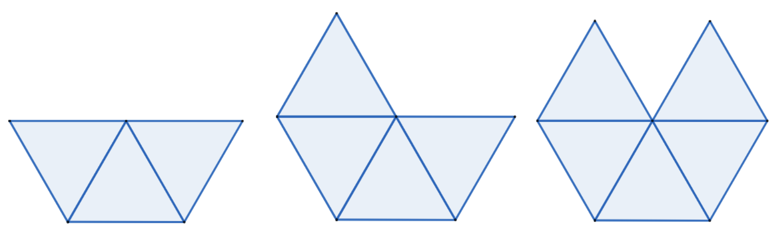

Figure 11: The solid angle of the tetrahedron, octahedron and icosahedron.

For the same reason, a solid angle cannot be constructed from more than six plane angles equal to two-thirds of a right angle either.
The solid angle of the cube is contained by three squares.
A solid angle contained by four, or more, squares is impossible. For, again, the sum of the plane angles containing the solid angle will be four right-angles or greater. See Figure 12.

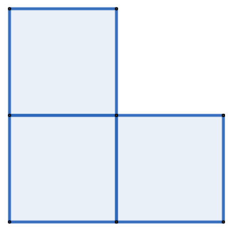

Figure 12: The solid angle of the cube.

The solid angle of a dodecahedron is contained by three equilateral and equiangular pentagons i.e. regular pentagons.
A solid angle contained by four, or more, regular pentagons is impossible. For, the angle of a regular pentagon being one and one-fifth of right-angle, four such angles will therefore be greater in sum than four right-angles. The very thing is impossible. See Figure 13. And, on account of

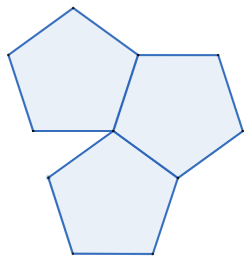

Figure 13: The solid angle of the dodecahedron.

the same absurdity, a solid angle cannot be constructed from any other regular polygonal figure either. Thus, beside the five solid figures, no other solid figure can be constructed by equilateral and equiangular planes. □

4.2 Euler's proof

The above proof by Euclid has been criticised throughout history for not being rigorous enough [2] so to really convince ourselves that they indeed are only five regular polygons we are going to look at the proof Euler found in the eighteenth century. Leonhard Euler (1707-1783) studied the polyhedron and arrived at what is today know as *Euler's polyhedron formula* which states that for any polyhedron, with V vertices, E edges and F faces, they satisfy the relationship [12]

$$V - E + F = 2.$$

Using this formula he proved that, indeed there is five and only five regular polyhedral. This proof has also been criticised mainly due to Euler's unstated assumption that all polyhedra are convex which his definition of polyhedra is not [12]. But for the purpose of this essay let us assume they are indeed convex and let us look at the proof [12].

Theorem 4.4. *There are at most five regular polyhedra.*

Proof. Assume that the polyhedron has V vertices, E edges and F faces. Euler's formula states that

$$V - E + F = 2. \tag{1}$$

Because the polyhedron is regular, each face is a regular polygon with the same number of edges. Define n and m as

$$\begin{aligned} n &= \text{the number of edges on each face,} \\ m &= \text{the number of edges meeting at each vertex.} \end{aligned}$$

The number of edges, n, must be at least three as we have seen from definition 4.1 of a solid angle. By definition the same number of edges meet at each vertex, and hence the same number of faces also meet at each vertex. This number, m must therefore also be at least three.

Counting all edges on each face gives us $F \cdot n$, but because each edge is shared by two faces every edge is counted twice. This gives us the following relationship

$$E = \frac{F \cdot n}{2}.$$

Similarly, each face has n vertices, but when counting all the vertices on every face the vertices are counted m times too many as there are m faces meeting at each vertex. So,

$$V = \frac{F \cdot n}{m}.$$

Substituting E and V in Euler's formula, (1), gives us

$$\frac{Fn}{m} - \frac{Fn}{2} + F = 2$$

and solving for F gives us

$$\begin{aligned} F\left(\frac{n}{m} - \frac{n}{2} + 1\right) &= 2 \\ F\left(\frac{2n - nm + 2m}{2m}\right) &= 2 \\ F &= \frac{4m}{2n - nm + 2m}. \end{aligned}$$

We know that both F and $4m$ are positive. So for the last equation to be true, it must be the case that

$$2n - mn + 2m > 0$$

As stated above, both n and m are greater than three which give us the three inequalities

$$2n - mn + 2m > 0 \tag{2}$$

$$n \geq 3 \tag{3}$$

$$m \geq 3. \tag{4}$$

By rewriting the first inequality, (2), we get

$$\begin{aligned} 2n - mn + 2m > 0 &\iff \\ 2n + 2m > mn &\iff \\ 2n > mn - 2m &\iff \\ 2n > m(n-2) &\iff \\ \frac{2n}{n-2} > m&. \end{aligned}$$

Now adding the third inequality, (4), we have

$$\frac{2n}{n-2} > \ m \geq 3$$

which gives us

$$\begin{aligned} 2n &> 3n - 6 \\ 6 &> n. \end{aligned}$$

Because of symmetry in $2(n+m) > \ mn$ we also get that $m < 6$. We now have the three inequalities

$$\begin{aligned} 2n - mn + 2m > \ & 0 \\ 3 \ \leq & \ n < 6 \\ 3 \ \leq & \ m < 6. \end{aligned}$$

With only the following solutions to the system

$$(n, m) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$$

corresponding to the tetrahedron, octahedron, icosahedron, cube and dodecahedron respectively. □

What Euler does here is, rather than to use geometry and the measures of lengths and angles that Euclid used to draw conclusions about the global nature of the polyhedron from local information, he does the proof purely combinatorial. The relation between the number of vertices, edges and faces is enough to find that there are only five Platonic solids. But let us get back to Kepler who predated Euler by about two centuries and therefore did not have access to this proof but regarded the Euclidean mathematics highly and most likely thought it an adequate proof as he included a summary of it in the *Mysterium Cosmographicum* together with a reference to Euclid [3].

5 The time Kepler lived in

With the decline of the Greek civilisation and the entry of the Middle Ages very few received any formal education in Europe. Only minimal teaching of geometry and arithmetic remained and were mostly taught in monastic schools and Universities from c. 1000 AD.

It was not until the fifteenth century with the European Renaissance that the study of mathematics re-emerged, and the teaching of the Greek mathematicians became the norm again. This could be seen in artwork from that time, as artist such as Piero della Francesca and Leonardo da Vinci, amongst others, masterful demonstration of perspective paining, often having the polyhedra featuring in their artwork. Figure 14 shows Paolo Uccello's, another prominent painter during the

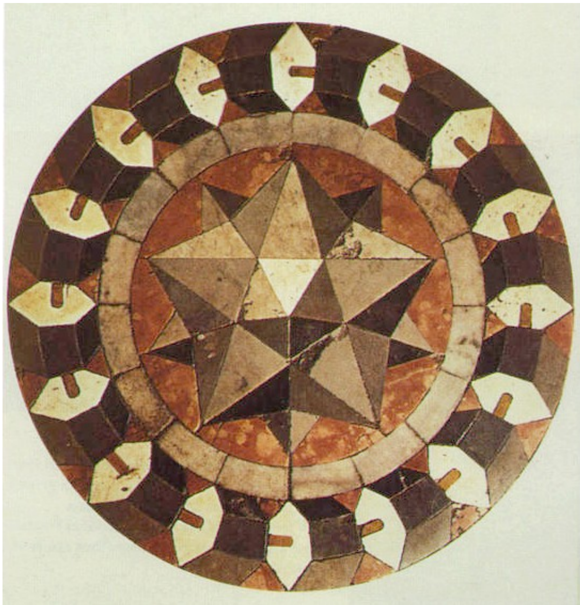

Figure 14: Floor of St Mark's Basilica.

Renaissance, use of polyhedral geometry in his marble mosaic on the floor of St Mark's Basilica in Venice.

By the time Johannes Kepler was born, in 1571, in a small town in what today is Germany, scientific teaching had again become of big interest in Europe. In 1577, a young Kepler at only six saw the *Great Comet of 1577*, which was the start of his fascination of astronomy. But at the time, Europe was deeply religious, having gone through the reformation only recently, dividing the God believers in two, the Catholics and the Lutherans. Kepler had a strong religious faith and confessed to the Lutheran faith [13]. When Kepler, in 1589, enrolled at University, he was set on becoming a theologian but was soon led to the works of Nicolaus Copernicus and astronomy by his mathematical teacher Michael Maestlin (1550-1630) and he left his aspiration of becoming a Lutheran minister behind.

The view of the Universe had for over 1400 years been based on Claudius Ptolemy's (c. 100-170 AD) model stating that the Earth was at the centre and all other heavenly bodies circled around. Not until the beginning of the sixteenth century was this notion questioned when Copernicus published, in 1543, the year of his death, *On the Revolutions of the Heavenly Spheres* where he put the Sun in the centre with the planets circling around. Though his theory was mostly rejected by intellectuals of his time and it was not until Kepler, and his polyhedral model half a century later, that the notion of the Sun as the centre of the Universe was established.

6 The construction of Kepler's planetary model

Kepler's first published work was called *Introduction to the cosmographic treatises, containing the cosmographic mystery concerning the remarkable proportion of the heavenly spheres, and concerning the genuine and proper causes of the number, magnitude, and periodic motions of the spheres, demonstrated by means of the five regular geometric solids* or *Mysterium Cosmographicum* [13] as it is generally known of today, and was his first attempt to establish the Copernican system, i.e. moving from a geocentric world view to a heliocentric one. His main concern was trying to answer two questions; 'why are there six planets?' and 'why are the five gaps between them the particular sizes that we can now see them to be?' and even though he was using Copernicus heliocentric view, it is shown in the latter question, the fact that he was focusing on the gaps, rather than the sizes of the orbits, that he was still thinking, partly, in terms of the nested system of spheres shown in the geocentric view [3]. However, it is clear that he placed the, then known, six planets around the Sun in the order of; Mercury, Venus, Earth, Mars, Jupiter and Saturn.

6.1 Great Conjunction

Kepler stated in the preface of the *Mysterium Cosmographicum* that the polyhedral theory came to him when teaching a class, 19 July 1595 [3], on the subject of Great Conjunctions, Field puts it: "It seems that Kepler, like many another, found that having to teach a subject made him learn a great deal about it." A Great Conjunction is when the planets Jupiter and Saturn appear closest together in the sky, and Jupiter "overtakes" Saturn in their orbits. As Jupiter moves, on average, 30° per year and Saturn only 18°, this happens approximately every twenty years. What Kepler could see when using these observations and drawing them in the zodiac, see Figure 15, was

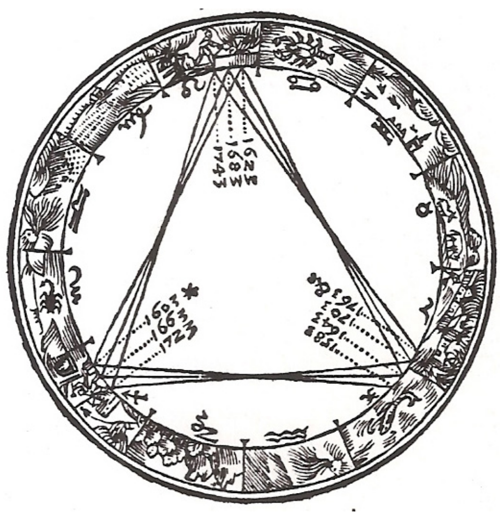

Figure 15: Great conjunctions

that three consecutive Great Conjunctions very nearly made an equilateral triangle, but not quite, hence making the pattern shown in Figure 16 when more than three conjunctions were drawn. What appeared in the figure when it was filled with consecutive conjunctions was that two circles, a bigger one circumscribing the equilateral triangles and a smaller one very clearly distinguished to be inscribed in the equilateral triangles, emerged with the radius of the smaller circle being nearly half that of the outer circle. Kepler knew this because the perpendicular distance from the centre to the side of an equilateral triangle is half the distance from the centre to a vertex [13]. This Kepler connected to the distance between Jupiter and Saturn as he knew the radius of Jupiter's path is about half of Saturn's path. He then went on to compare the radius of the other planets with the hopes of connecting them to other polygons, but his hopes were quickly dashed as he could not find polygons that fit with the ratios and maybe even more importantly, it did not explain why there are six planets, with five gaps when there is an infinite number of polygons. This is

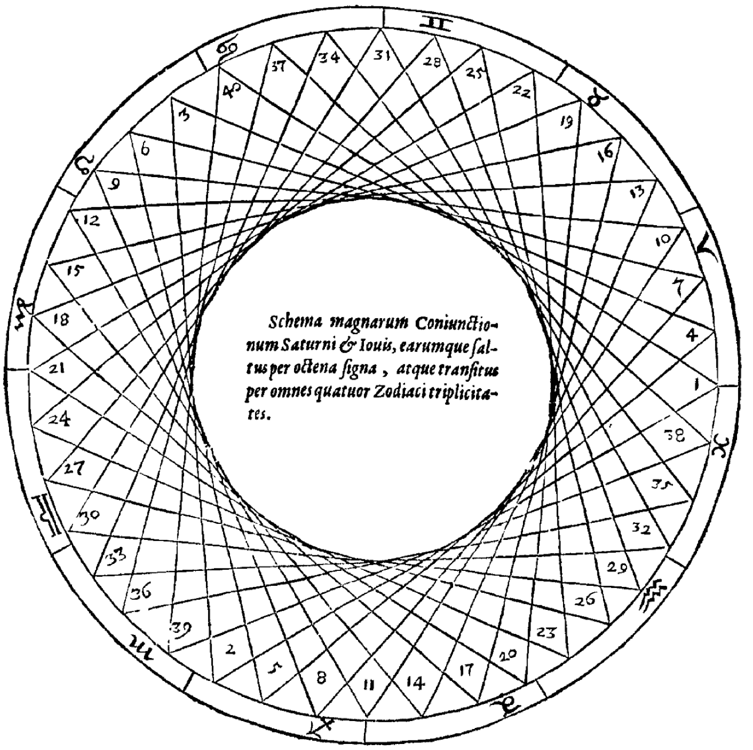

Figure 16: Great conjunctions

when he turned to the three-dimensional figures of the regular polyhedron. "what were polygons, plane figures, doing among the spheres of the heavens? He should use the polyhedra, solid figures, instead." [13]. As the regular polyhedra was well known at the time and also to Kepler, proved as only five in numbers by Euclid in the *Elements*, Kepler could trust that his initial idea was strong, as Field writes in *Kepler's Geometrical cosmology*: "His theory was thus established on a secure, Euclidean, mathematical base."[3].

6.2 Dividing the Platonic solid in two classes

So how did he construct his planetary model with the Platonic solids? He now had exactly what he needed, five geometrical figures; the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron that, although composed of planes, resembled the sphere, the "most perfect of shapes." [13]. All their faces were equidistant from the centre (and hence were tangent to a single inscribed sphere). All their vertices were equidistant from the centre (and defined a sphere that circumscribed the figure). The ratio between the radii of these two spheres was thus well defined, and characteristic, for each of the figures. A sphere's perfection could not be constructed from planes, but it was most nearly approached in these five figures.

What Kepler then did when creating his model was to divide the Platonic solids into two classes, the primary and the secondary. The cube, tetrahedron and dodectahedron being primary and the icosahedron and octahedron being secondary solids. Kepler lists seven reasons for this divide in Chapter III of *Mysterium Cosmographicum* [9], they are

1. The primaries differ from each other in shape of face; the secondaries both have triangular faces.

2. Every one of the primaries has its particular type of face: the cube has the square, the pyramid the triangle, the dodecahedron the pentagon; the secondaries borrow the triangular face from the pyramid.

It could be argued here that number one and two could be made into one point as they both refer to the configuration of the faces and number two is more of a development of number one.

3. All the primaries have a simple vertex, that is, one which is included between three faces; the secondaries combine four or five faces in one solid angle.

This reason refers back to the solid angles in Chapter 4. It is also an illustration of Kepler's opinion that 'the simplest form is the most beautiful' and his belief that God made the Universe as beautiful as possible [3]. See Figure 17 for a visual image of reason number three.

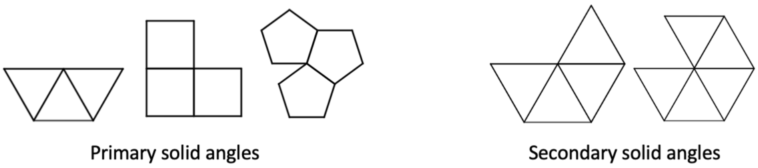

Figure 17: The angles of the primary and secondary solids

4. The primaries owe their origin and properties to no one; the secondaries have got several things from the primaries by borrowing, and are so to speak generated by them.

Kepler shows how to derive the octahedron from the cube and the icosahedron from the dodecahedron in a diagram, see Figure 18. In the figure we also have the tetrahedron who is derived from itself. This is what is today referred to as 'dual polyhedron' and although Kepler did know that it is a mutual relationship [3], i.e. the cube could just as well be derived from the octahedron and the icosahedron from the dodecahedron he still chose to use it as a way of distinguishing between the primary and the secondary solids as that suited his theory better.

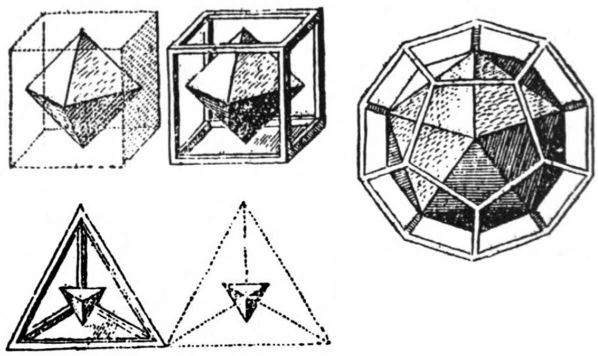

Figure 18: Primary and secondary solids

5. The primaries cannot move harmoniously except on a diameter drawn through the centres of a single or of opposite faces; but the secondaries on a diameter drawn through opposite vertices.
6. It is characteristic of the primaries to stand upright, of the secondaries to balance on a vertex. For if you roll the latter onto their base, or stand the former on a vertex, in either case the onlooker will avert his eyes at the awkwardness of the spectacle.

Both reason five and six refer to what we today would call rotational symmetry. The primary solids have a higher order of rotational symmetry about an axis through the centre of a face, while the secondary figures have a higher order of rotational symmetry about an axis through an angle. The tetrahedron answers to both since every vertex is opposite a face.

7. Add finally that the primaries are three, the perfect number, the secondaries two, an imperfect number; and that the former have all types of vertex, the cube a right angle, the tetrahedron acute, and the dodecahedron obtuse, but the latter both employ a single type of angle, the obtuse. In fact, in the case of the octahedron all three types of angle occur: the obtuse at the junction of the faces; a right angle between two edges running from opposite vertices; whereas the actual solid angle is acute.

A lot of Kepler's reasoning in *Mysterium Cosmographicum* about the order and division of the solids can be traced to his religious faith from statements such as "For I think that from the love of God for Man a great many of the causes of the features in the universe can be deduced" in Chapter IV [9] showing that the theory Kepler was 'finding' was in fact the intentions of God when creating the Universe.

6.3 Placement of the Platonic solids

Having decided on the divide of the solids next came the job of placing them between the planets. Kepler started by asserting that nothing made more sense than that the Earth, "the pinnacle and pattern of the whole universe," [9] would be the heavenly body to differentiate between the primary and the secondary and hence placing the primary solids, in the three gaps outside of the Earth and the secondary solids in the two gaps inside. He then placed the cube in between the orbits of Saturn and Jupiter, due to it being the most perfect of the solids. This interval was closest to the fixed stars, most dignified part of the world aside from the Earth, so it made sense to place the primary cube in that interval. The next solid to follow was the tetrahedron, who "almost dared to contend with the cube for the chief place" [9].

On the other side of Earth was then the two last orbits, the one between Earth and Venus and the one between Venus and Mercury. One could imagine that what would follow was the octahedron as Kepler considered it to be in higher regard stating "that the octahedron takes precedence over the icosahedron" [9] due to the octahedron having been derived from the cube compared to the icosahedron that was only derived from the dodecahedron. Similarities within the two pairs can be seen in Table 3.2 in the number of faces and vertices of the solids further establishing that the octahedron taking after the cube in being the superior of its class. But Kepler argued that the two classes of the five solids differed and so their heads face outwards towards different directions in the universe and hence placing the icosahedron closest to the Earth and the octahedron, as the head of its class, between Venus and Mercury made the most sense. Figure 19 shows Kepler's model of the spheres representing the orbits and the polyhedra between them as pictured in *Mysterium Cosmographicum.*

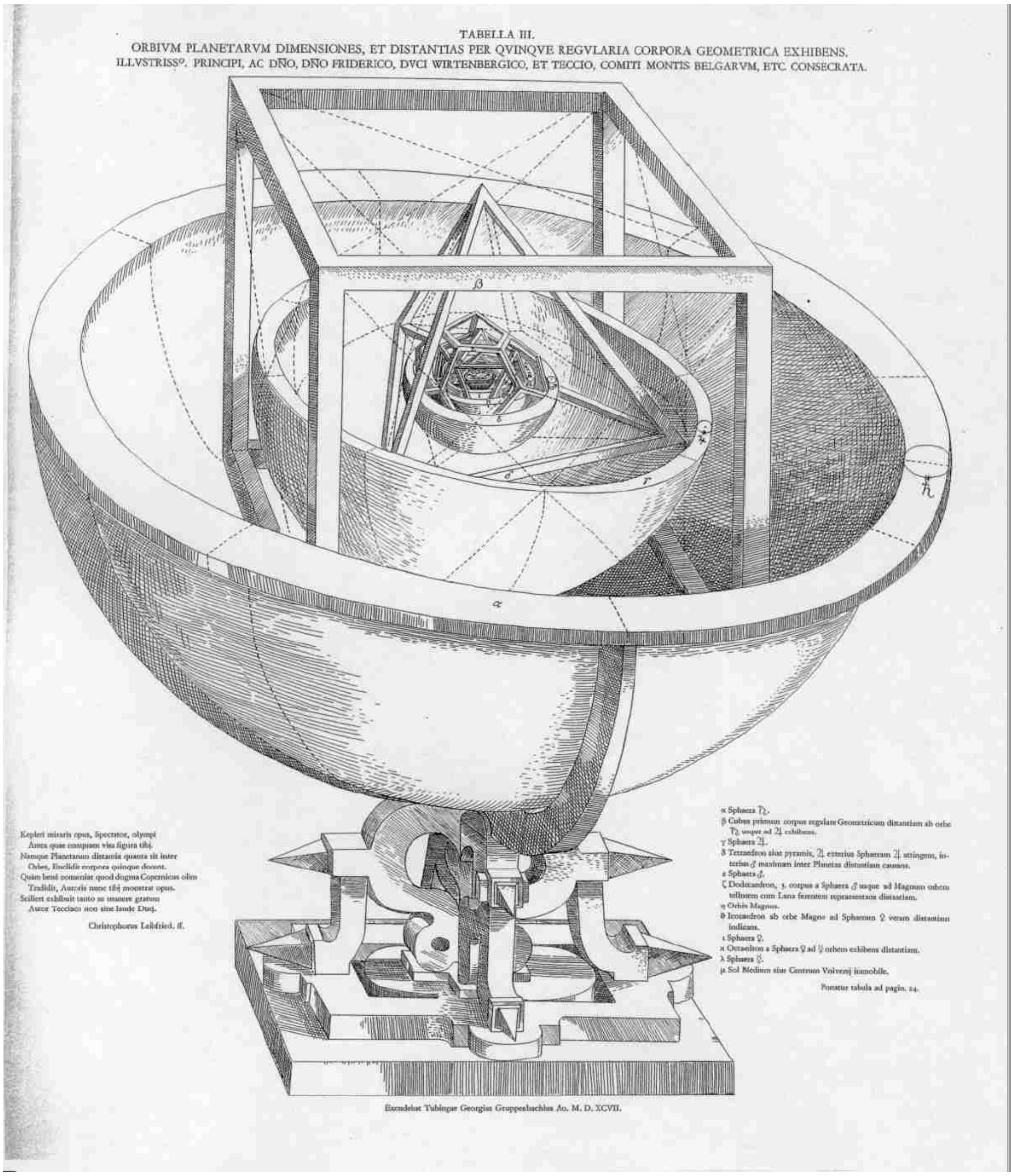

Figure 19: Kepler's planetary model

6.4 The ratio between the inscribed and the circumscribe sphere of the Platonic solids

But Kepler did not just base his theory on purely theoretical hypothesise, he also showed that the above order was indeed the way to place the Platonic solids between the planets orbs by comparing the ratios of the inscribed to the circumscribed sphere to Copernicus observational data of the orbs.

In a similar way to what we have already seen with the polygon above, Kepler wanted to compare the ratios between the inscribed sphere and the circumscribed sphere in each of the solids to the ratios of the six planets orbits. So let us look at the calculations comparing the two spheres starting with the cube.

The following calculation of the radii of the inscribed as well as the circumscribed spheres of the cube is found in Field's book *Kepler's Geometrical Cosmology.*

Proof. Let $ABCDEFGH$ be a cube, with centre O, as shown in Figure 20. Let the centre of the face $ABCD$ be P and let the side of the cube be $2a$.

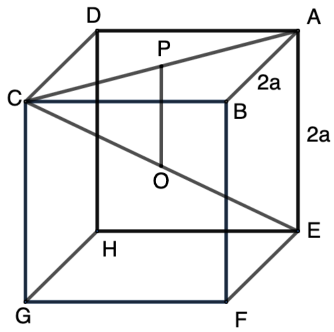

Figure 20: Cube

Since O is the common centre of the circumscribe sphere and the inscribed sphere the radii of the spheres are clearly OC and OP respectively. Since P and O are the mid-points of CA and CE respectively, it is clear that the triangles CPO and CAE are similar their corresponding sides being in the ration 1 : 2.
Therefore

$$\begin{aligned} PO &= \frac{1}{2}AE \\ &= a. \end{aligned}$$

In $\triangle ABC$, see Figure 21 (the cube seen from above), by Pythagoras' Theorem we have

$$\begin{aligned} CA^2 &= AB^2 + CB^2 \\ &= (2a)^2 + (2a)^2 \\ &= 8a^2. \end{aligned}$$

In $\triangle ACE$, see Figure 22, $\angle EAC$, is a right angle, since AE, being a side of the cube, is perpendicular to the face $ABCD$ and therefore to the line AC. Therefore, by Pythagoras' Theorem,

$$\begin{aligned} EC^2 &= AE^2 + CA^2 \\ &= 4a^2 + 8a^2 \\ &= 12a^2 \\ EC &= 2\sqrt{3}a. \end{aligned}$$

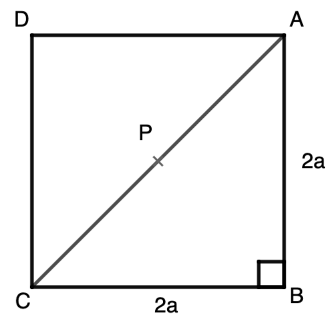

Figure 21: The cube seen from above

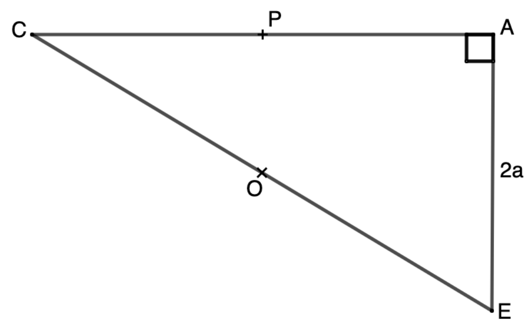

Figure 22: The triangle found inside the cube

As O is the mid-point of EC we therefore have

$$\begin{aligned} OC &= \frac{1}{2}EC \\ &= \sqrt{3}a. \end{aligned}$$

Therefore the ratio of the radius of the inscribed sphere to the radius of the circumscribe sphere is $1 : \sqrt{3}$. □

Below follows an account of how to calculate the radii of the spheres for the icosahedron. But compared to Fields proof this is based on analytical geometry which Kepler would not have at his disposal as it was not invented until 1637 by René Descartes (1596-1650) and Pierre de Fermat (1601-1665) [7].

We start by showing how to create an icosahedron in the three-dimensional space.

Definition 6.1. Let ϕ be the golden ratio, $\phi = \frac{\sqrt{5}+1}{2}$, i.e. the positive solution to the equation $\phi^2 = 1 + \phi$.

Theorem 6.1. *An icosahedron is obtained from the vertices in the 12 points* $(0, \pm 1, \pm\phi)$, $(\pm\phi, 0, \pm 1)$ *and* $(\pm 1, \pm\phi, 0)$*, that create 20 equilateral triangels with side 2.*
Therefore, it is a Platonic solid.

Proof. We have that the distance between two points $p = (a, b, c)$ and $q = (d, e, f)$ in a three-dimensional space is obtained by

$$|p - q| = |(a - d, b - e, c - f)| = \sqrt{(a - d)^2 + (b - e)^2 + (c - f)^2}. \tag{5}$$

Place the points with the corresponding vertices as follows, see Figure 23.

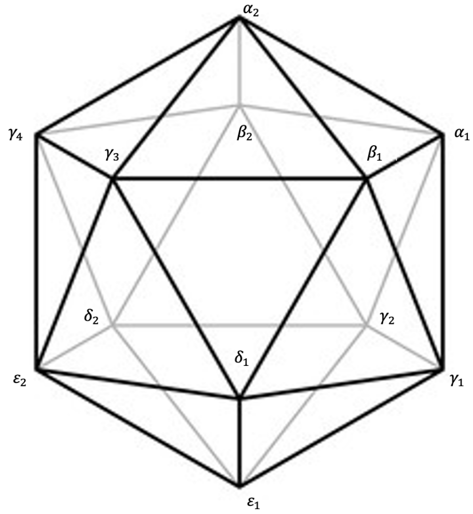

Figure 23: Icosahedron

$$\begin{array}{rrr} \alpha_1 = (0,-1,\phi) & \gamma_1 = (-1,-\phi,0) & \delta_1 = (-\phi,0,-1) \\ \alpha_2 = (0,1,\phi) & \gamma_2 = (1,-\phi,0) & \delta_2 = (\phi,0,-1) \\ \beta_1 = (-\phi,0,1) & \gamma_3 = (-1,\phi,0) & \epsilon_1 = (0,-1,-\phi) \\ \beta_2 = (\phi,0,1) & \gamma_4 = (1,\phi,0) & \epsilon_2 = (0,1,-\phi). \end{array}$$

The length of each edge is calculated using (5). We have

$$\begin{aligned} |\beta_1 - \delta_1| &= |(-\phi - (-\phi), 0 - 0, 1 - (-1))| = \sqrt{0 + 0 + 2^2} = \sqrt{4} = 2, \\ |\delta_1 - \gamma_3| &= |(-\phi - (-1), 0 - \phi, -1 - 0)| = \sqrt{(1-\phi)^2 + (-\phi)^2 + (-1)^2} \\ &= \sqrt{1 - 2\phi + \phi^2 + \phi^2 + 1} = \sqrt{2 + 2(\phi^2 - \phi)} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2 \end{aligned}$$

and

$$\begin{aligned} |\gamma_3 - \beta_1| &= |(-1 - (-\phi), \phi - 0, 0 - 1)| = \sqrt{(\phi - 1)^2 + \phi^2 + 1} \\ &= \sqrt{\phi^2 - 2\phi + 1 + \phi^2 + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2 \end{aligned}$$

which are the three sides of the triangle $\triangle\beta_1\delta_1\gamma_3$, in Figure 23. We can see that all three sides have the same length which was required to build one face of the icosahedron. This can be shown for all edges and the calculations for them can be found in the Appendix. We have that the distance from a point, $p = (a, b, c)$ in the three-dimensional space to origo is obtained from

$$|(a, b, c)| = \sqrt{a^2 + b^2 + c^2}. \tag{6}$$

Therefore, all vertices have the distance

$$\sqrt{\phi^2 + 1^2} = \sqrt{\phi + 2} = \sqrt{\frac{\sqrt{5}+1}{2} + 2} \tag{7}$$

to origo and we have a Platonic solid, the icosahedron. □

Now till the inscribed and circumscribed spheres of the icosahedron.

The circumscribed sphere passes through all vertices of the icosahedron and shares the same midpoint. Therefore the radius of the circumscribe sphere is the same as the distance from one vertex to the centre of the solid. As calculated above in (7), we have that the radius of the circumscribe sphere of the icosahedron is $\sqrt{\frac{\sqrt{5}+1}{2} + 2}$.

The inscribed sphere is tangent to the mid-point T_i of the equilateral triangle that is the face of the icosahedron. Therefore we get that

$$\begin{aligned} OT_i =& \frac{1}{3}((\phi, 0, 1), (0, 1, \phi), (1, \phi, 0)) \\ =& \frac{1}{3}(\phi + 0 + 1, 0 + 1 + \phi, 1 + \phi + 0) \\ =& \frac{\phi + 1}{3}(1, 1, 1) \end{aligned}$$

and

$$|OT_i| = \frac{\phi + 1}{\sqrt{3}} = \frac{\frac{\sqrt{5}+1}{2} + 1}{\sqrt{3}},$$

where $|OT_i|$ is the radius of the inscribed sphere.

We therefore have the ratio of the inscribed sphere to the circumscribed sphere being

$$\frac{\frac{\sqrt{5}+1}{2} + 1}{\sqrt{3}} : \sqrt{\frac{\sqrt{5}+1}{2} + 2} \tag{8}$$

which can be simplified to $1 : \sqrt{15 - 6\sqrt{5}}$.

Let us now do the same for the tetrahedron.

Theorem 6.2. *A tetrahedron is obtained from the vertices in the four points* $\alpha = (1, 0, -\frac{1}{\sqrt{2}})$, $\beta = (-1, 0, -\frac{1}{\sqrt{2}})$, $\gamma = (0, 1, \frac{1}{\sqrt{2}})$ *and* $\delta = (0, -1, \frac{1}{\sqrt{2}})$, *that create four equilateral triangles with the side 2.*

Proof. Place the four points with the corresponding vertices as seen in Figure 24. The length of

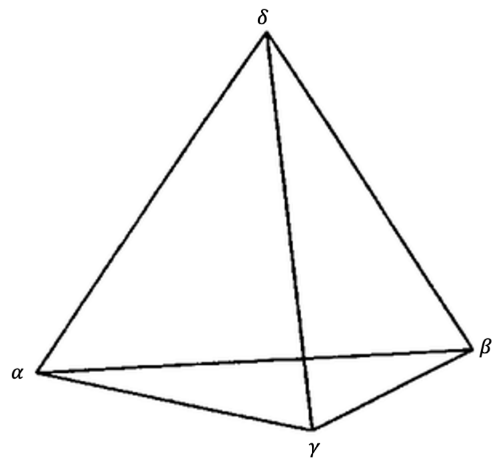

Figure 24: Tetrahedron

each edge is calculated using (5) in the same way as for the icosahedron. We have that

$$
\begin{aligned}
|\alpha-\beta| &= |(1-(-1), 0-0, -\frac{1}{\sqrt{2}}-(-\frac{1}{\sqrt{2}}))| = \sqrt{2^2} = 2 \\
|\alpha-\gamma| &= |(1-0, 0-1, -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})| = \sqrt{1^2+(-1)^2+(-\frac{2}{\sqrt{2}})^2} = \sqrt{2+\frac{4}{2}} = \sqrt{4} = 2 \\
|\alpha-\delta| &= |(1-0, 0-(-1), -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})| = \sqrt{1^2+1^2+(-\frac{2}{\sqrt{2}})^2} = 2 \\
|\beta-\gamma| &= |(-1-0, 0-1, -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})| = \sqrt{(-1)^2+(-1)^2+(-\frac{2}{\sqrt{2}})^2} = 2 \\
|\beta-\delta| &= |-1-0, 0-(-1), -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})| = \sqrt{(-1)^2+1^2+(-\frac{2}{\sqrt{2}})^2} = 2 \\
|\gamma-\delta| &= |0-0, 1-(-1), \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}})| = \sqrt{2^2} = 2.
\end{aligned}
$$

We can see that all edges forming the tetrahedron are of length 2. From (6), in the same way as with the icosahedron we have that the distance from each vertex to origo, or the mid-point is

$$
\sqrt{(\pm 1)^2+(\pm\frac{1}{\sqrt{2}})^2} = \sqrt{1+\frac{1}{2}} = \sqrt{\frac{3}{2}}. \tag{9}
$$

And thus we have a Platonic solid, the tetrahedron. □

The circumscribe sphere of the tetrahedron passes through all vertices, in the same manner as all the other Platonic solids, and therefore the radius of said sphere is $\sqrt{\frac{3}{2}}$ from (9).

The inscribed sphere is tangent to the mid-point T_t of the equilateral triangle that is the face of the tetrahedron. Therefore we get that

$$
\begin{aligned}
OT_t &= \frac{1}{3}((1,0,-\frac{1}{\sqrt{2}}),(-1,0,-\frac{1}{\sqrt{2}}),(0,1,\frac{1}{\sqrt{2}})) \\
&= \frac{1}{3}(1+(-1)+0, 0+0+1, -\frac{1}{\sqrt{2}}+(-\frac{1}{\sqrt{2}})+\frac{1}{\sqrt{2}}) \\
&= \frac{1}{3}(0,1,-\frac{1}{\sqrt{2}})
\end{aligned}
$$

and

$$
\begin{aligned}
|OT_t| &= \sqrt{(\frac{1}{3})^2+(-\frac{1}{3\sqrt{2}})^2} \\
&= \sqrt{\frac{1}{9}+\frac{1}{18}} \\
&= \frac{\sqrt{3}}{3\sqrt{2}},
\end{aligned}
$$

where $|OT_t|$ is the radius of the inscribed sphere.

We therefore have the ratio of the inscribed sphere to the circumscribed sphere being

$$
\frac{\sqrt{3}}{3\sqrt{2}} : \frac{\sqrt{3}}{\sqrt{2}}
$$

which can be simplified to $1:3$.

The same method can be used to attain the ratios of the octahedron and the dodecahedron but let us come back to the argument of duality that Kepler uses as a reason for dividing the solids into primaries and secondaries. To explain this we start with a definition.

Definition 6.2. The dual of a body K in the three-dimensional space is

$$K^* = \{(x, y, z) : ((x, y, z,) \cdot (x', y', z') \leq 1\}$$

for all $(x', y', z') \in K$.

From Jean Gallier'S *Notes on Convex Sets, Polytopes, Polyhedra, Combinatorial Topology, Voronoi Diagrams and Delaunay Triangulations* [5] we know that the dual K^* of a convex polyhedron K that contains O in its interior is a convex polyhedron and $K^{**} = K$. We also have that if $K \subset L$ then $L^* \subset K^*$. And lastly, if $K(R)$ is a ball centred at the origin with the radius R then $K(R)^* = K(\frac{1}{R})$.

Now to the regular polyhedrons.

Let C be a regular polyhedron placed symmetrically around O and assume that the circumscribed sphere has radius R and the inscribed sphere has radius 1. Then

$$K(1) \subset C \subset K(R)$$
$$\Leftrightarrow$$
$$K(\frac{1}{R}) \subset C^* \subset K(1).$$

So from this equivalence we conclude that if $K(R)$ is the minimal ball containing C, i.e. the circumscribed sphere has radius R, then $K(\frac{1}{R})$ is the maximal ball contained in C^*, i.e. the inscribed sphere has radius $\frac{1}{R}$. Similarly, if the inscribed sphere of C has radius 1, then the circumscribed sphere of C^* has radius 1.

Furthermore, one can show (by a more technical argument, using rotational symmetry of the polyhedra) that C^* is the polyhedron that has vertices that are the midpoints of the faces of C.

Finally, since we now have the circumscribe and the inscribed sphere of the dual polyhedron as well as the original polyhedron, we have equality between the quotients

$$\frac{1}{(\frac{1}{R})} = \frac{R}{1}.$$

What we can take from the duality is that the two solids in the pair of duals, or marriage as Kepler refers it to in *Harmonic Mundi* [10], have the same ratio between the circumscribe sphere and the inscribe sphere. So, as we already know the ratio for the cubes sphere to be $1 : \sqrt{3}$ we can conclude that that is also the ratio for the octahedrons spheres. And similarly, we know the ratio for the spheres of the icosahedrons to be $1 : \sqrt{15 - 6\sqrt{5}}$ and therefore that is also the ratio for the dodecahedrons spheres.

Let us now come back to Table 3.2 from Chapter 3 showing the vertices, faces and edges of each Platonic solid. Here shown again in Table 6.4.

Polyhedron	Vertices	Faces	Edges
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Icosahedron	12	20	30
Dodecahedron	20	12	30

Table 6.4

Looking at the pairs we can see that the number of vertices of one in the pair corresponds to the number of faces of the other. For example, with the cube and the octahedron; the cube has eight vertices, the same amount as the octahedron has faces and the six faces of the cube corresponds to the octahedrons six vertices. Same can be observed with the icosahedron and dodecahedron confirming that we get one solid in the pair by placing vertices at the mid-points of the other solid and connecting them.

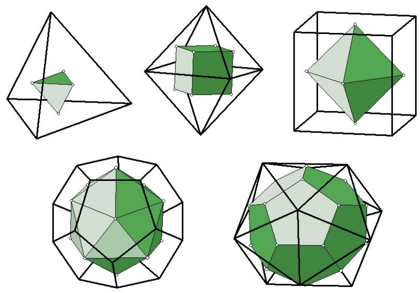

Figure 25: Dual Platonic solids

Figure 25 shows the dual relationships, the cube is dual to the octahedron, the dodecahedron is dual to the icosahedron and the tetrahedron is its own dual.

6.5 Calculated ratios vs observed data

Kepler used Copernicus measurements of the different orbs of the planets when comparing them to the ratios of the inscribed sphere to the circumscribed sphere. Figure 26 shows a table of how Kepler presented his results, here translated by Field [3]. To be able to compare the numbers,

						Book V of Copernicus	
If the inner radius for the sphere of		is 1000. The outer radius should be for the sphere of			And is, according to Copernicus,		
	♄		Jupiter	577		635	ch. 9
	♃		Mars	333		333	ch. 14
	♂		Earth	795		757	ch. 19
	Earth		Venus	795		794	ch. 21 & 22
	♀		Mercury	577 or 707		723	ch. 27

Figure 26: Table comparing the ratios of the Platonic solids to Copernicus' observed data.

Kepler presented his results from the Platonic solids rounding them off to three significant figures, i.e. the ratio of the tetrahedron, 1 : 3 being 333 : 1000, the ratio of both the cube and the octahedron, $1 : \sqrt{3}$ being 577 : 1000 and the ratio of the dodecahedron and icosahedron, $1 : \sqrt{15 - 6\sqrt{5}}$ being 795 : 1000. Field has compiled another table, seen in Figure 27 that more clearly show the relationships between the solids and the different planets orbs.

Table 3.3 *Orbs and polyhedra* (*Myst. Cosm.* Ch. XIV)

planet	outer radius of lower orb if inner radius of upper one is 1000 from polyhedron 'th'	from Cop. 'obs'
♃	577	635
♂	333	333
⊕	795	757
♀	795	794
☿	577	723
	707 (mid)	

Figure 27: Relationship between the solids and the different planets orbs.

The symbols in the figures being

♄ = Saturn	♁ = Earth
♃ = Jupiter	♀ = Venus
♂ = Mars	☿ = Mercury
'th' = theoretical	'obs' = observational.

As we can see in the table the calculated polyhedra ratios does not align exactly with the observed values from Copernicus. Kepler was aware of this but still thought his theory was strong enough as theoretical and observed figures may differ. Kepler's comment on it was "Notice that corresponding numbers are close to each other, and indeed in the cases of Mars and Venus, the same. Indeed in the cases of the Earth and Mercury they are not very different: only in the case of Jupiter is there an undue discrepancy, which however at such a great distance should surprise nobody." showing that he did not think the discrepancies were too big to dismiss his theory.

6.6 Then what

Kepler did try though, using Tycho Brahe, (1546-1601), a danish astronomer's significant observational data, to get a more accurate calculation of the radii of the orbs. This still did not turn out to yield any closer accuracy, so he then tried to modify his theory to accommodate the variations. He introduced in *Harmonice Mundi* (1619), a modification of his theory that involved musical harmonies.

The model Kepler presented in *Mysterium Cosmographicum* in 1596 was his very first piece of work, and though he did later on come to criticise many parts of it he never abandoned the main theory; to use the Platonic solids as proportions amongst the planetary spheres and as the answer to the question 'why are there six planets?'. When he published the second edition of said book in 1621, he added notes on almost all parts of why or how he had changed his view, but the solids remained as the main reason as why God had created the universe in that way. As we know today, not least from the discovery of Uranus and Neptune, his planetary model does not hold, but it did allow him to find other theories that do hold till this day. For example, his three laws of planetary motion that are today known as *Kepler's laws* [7], has a direct path to the fundamental work of Isaac Newton (1643-1727) on the laws of motion.

Kepler's strong belief in God, and reasoning such as "Geometry is unique and eternal, a reflection of the mind of God. That men are able to participate in it is one of the reasons why man is an image of God." from a letter sent by Kepler to Herwart von Hohenburg in 1599, have made many questioning if Kepler could indeed be thought of as a scientist [7]. That he most definitely can, as even though he might have come from what we would today view as a questionable scientific foundation he was responsible for the most important astronomical discoveries of his time regardless of the purpose for his quest of knowledge.

7 Appendix

Below follow the calculations of the lengths of all the edges of the icosahedron from Chapter 6.4. Figure 28 show the same icosahedron as in Figure 23 with the coordinates appointed to the vertices.

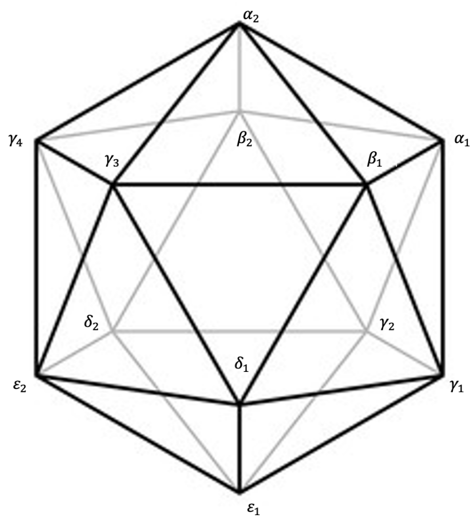

Figure 28: Icosahedron

$$\begin{array}{ccc} \alpha_1 = (0,-1,\phi) & \gamma_1 = (-1,-\phi,0) & \delta_1 = (-\phi,0,-1) \\ \alpha_2 = (0,1,\phi) & \gamma_2 = (1,-\phi,0) & \delta_2 = (\phi,0,-1) \\ \beta_1 = (-\phi,0,1) & \gamma_3 = (-1,\phi,0) & \epsilon_1 = (0,-1,-\phi) \\ \beta_2 = (\phi,0,1) & \gamma_4 = (1,\phi,0) & \epsilon_2 = (0,1,-\phi). \end{array}$$

$$\begin{aligned}
|\epsilon_1 - \delta_1| &= |(0-(-\phi), -1-0, -\phi-(-1))| = \sqrt{\phi^2 + (-1)^2 + (-\phi+1)^2} \\
&= \sqrt{\phi^2 + 1 + \phi^2 - 2\phi + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\epsilon_1 - \gamma_1| &= |(0-(-1), -1-(-\phi), -\phi-0)| = \sqrt{1^2 + (-1+\phi)^2 + (-\phi)^2} \\
&= \sqrt{1 + 1 - 2\phi + \phi^2 + \phi^2} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\epsilon_1 - \gamma_2| &= |(0-1, -1-(-\phi), -\phi-0)| = \sqrt{(-1)^2 + (-1+\phi)^2 + (-\phi)^2} \\
&= \sqrt{1 + 1 - 2\phi + \phi^2 + \phi^2} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\epsilon_1 - \delta_2| &= |(0-\phi, -1-0, -\phi-(-1))| = \sqrt{(-\phi)^2 + (-1)^2 + (-\phi+1)^2} \\
&= \sqrt{\phi^2 + 1 + \phi^2 - 2\phi + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\epsilon_1 - \epsilon_2| &= |(0-0, -1-1, -\phi-(-\phi))| = \sqrt{0 + (-2)^2 + 0} = \sqrt{4} = 2, \\
|\delta_1 - \gamma_1| &= |(-\phi-(-1), 0-(-\phi), -1-0,)| = \sqrt{(-\phi+1)^2 + \phi^2 + (-1)^2} \\
&= \sqrt{\phi^2 - 2\phi + 1 + \phi^2 + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\gamma_1 - \gamma_2| &= |(-1-1, -\phi-(-\phi), 0-0)| = \sqrt{(-2)^2 + 0 + 0} = \sqrt{4} = 2, \\
|\gamma_2 - \delta_2| &= |(1-\phi, -\phi-0, 0-(-1)| = \sqrt{(1-\phi)^2 + (-\phi)^2 + 1^2} \\
&= \sqrt{1 - 2\phi + \phi^2 + \phi^2 + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2,
\end{aligned}$$

$$
\begin{aligned}
|\delta_2-\epsilon_2| &= |(\phi-0,0-1,-1-(-\phi)| = \sqrt{\phi^2+(-1)^2+(-1+\phi)^2}\\
&= \sqrt{\phi^2+1+1-2\phi+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\epsilon_2-\delta_1| &= |(0-(-\phi),1-0,-\phi-(-1))| = \sqrt{\phi^2+1^2+(-\phi+1)^2}\\
&= \sqrt{\phi^2+1+\phi^2-2\phi+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\gamma_1-\beta_1| &= |(-1-(-\phi),-\phi-0,0-1)| = \sqrt{(-1+\phi)^2+(-\phi)^2+(-1)^2}\\
&= \sqrt{1-2\phi+\phi^2+\phi^2+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\gamma_1-\alpha_1| &= |(-1-0,(-\phi)-(-1),0-\phi)| = \sqrt{(-1)^2+(-\phi+1)^2+(-\phi)^2}\\
&= \sqrt{1+1-2\phi+\phi^2+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\gamma_2-\alpha_1| &= |(1-0,(-\phi)-(-1),0-\phi)| = \sqrt{1^2+(-\phi+1)^2+(-\phi)^2}\\
&= \sqrt{1+1-2\phi+\phi^2+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\gamma_2-\beta_2| &= |(1-\phi,(-\phi)-0,0-1)| = \sqrt{(1-\phi)^2+(-\phi)^2+(-1)^2}\\
&= \sqrt{1-2\phi+\phi^2+\phi^2+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\delta_2-\beta_2| &= |(\phi-\phi,0-0,-1-1)| = \sqrt{0+0+(-2)^2} = \sqrt{4}=2,\\
|\delta_2-\gamma_4| &= |(\phi-1,0-\phi,-1-0,)| = \sqrt{(\phi-1)^2+(-\phi)^2+(-1)^2}\\
&= \sqrt{\phi^2-2\phi+1+\phi^2+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\epsilon_2-\gamma_4| &= |(0-1,1-\phi,-\phi-0)| = \sqrt{(-1)^2+(1-\phi)^2+(-\phi)^2}\\
&= \sqrt{1+1-2\phi+\phi^2+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\epsilon_2-\gamma_3| &= |(0-(-1),1-\phi,-\phi-0)| = \sqrt{1^2+(1-\phi)^2+(-\phi)^2}\\
&= \sqrt{1+1-2\phi+\phi^2+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\beta_1-\delta_1| &= |(-\phi-(-\phi),0-0,1-(-1))| = \sqrt{0+0+2^2} = \sqrt{4}=2,\\
|\delta_1-\gamma_3| &= |(-\phi-(-1),0-\phi,-1-0)| = \sqrt{(1-\phi)^2+(-\phi)^2+(-1)^2}\\
&= \sqrt{1-2\phi+\phi^2+\phi^2+1} = \sqrt{2+2(\phi^2-\phi)} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2\\
|\gamma_3-\beta_1| &= |(-1-(-\phi),\phi-0,0-1)| = \sqrt{(\phi-1)^2+\phi^2+1}\\
&= \sqrt{\phi^2-2\phi+1+\phi^2+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2\\
|\beta_1-\alpha_1| &= |(-\phi-0,0-(-1),1-\phi)| = \sqrt{(-\phi)^2+1^2+(1-\phi)^2}\\
&= \sqrt{\phi^2+1+1-2\phi+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\alpha_1-\beta_2| &= |(0-\phi,-1-0,\phi-1)| = \sqrt{(-\phi)^2+(-1)^2+(\phi-1)^2}\\
&= \sqrt{\phi^2+1+1-2\phi+\phi^2} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\beta_2-\gamma_4| &= |(\phi-1,0-\phi,1-0,)| = \sqrt{(\phi-1)^2+(-\phi)^2+1^2}\\
&= \sqrt{\phi^2-2\phi+1+\phi^2+1} = \sqrt{2(\phi^2-\phi)+2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2}=2,\\
|\gamma_4-\gamma_3| &= |(-1-1,\phi-\phi,0-0)| = \sqrt{(-2)^2+0+0} = \sqrt{4}=2,
\end{aligned}
$$

$$
\begin{aligned}
|\alpha_2 - \beta_1| &= |(0-(-\phi), 1-0, \phi-1)| = \sqrt{\phi^2 + 1^2 + (\phi-1)^2} \\
&= \sqrt{\phi^2 + 1 + \phi^2 - 2\phi + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\alpha_2 - \alpha_1| &= |(0-0, 1-(-1), \phi-\phi)| = \sqrt{0 + 2^2 + 0} = \sqrt{4} = 2, \\
|\alpha_2 - \beta_2| &= |(0-\phi, 1-0, \phi-1)| = \sqrt{(-\phi)^2 + 1^2 + (\phi-1)^2} \\
&= \sqrt{\phi^2 + 1 + \phi^2 - 2\phi + 1} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\alpha_2 - \gamma_4| &= |(0-1, 1-\phi, \phi-0)| = \sqrt{(-1)^2 + (1-\phi)^2 + \phi^2} \\
&= \sqrt{1 + 1 - 2\phi + \phi^2 + \phi^2} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2, \\
|\alpha_2 - \gamma_3| &= |(0-(-1), 1-\phi, \phi-0)| = \sqrt{1^2 + (1-\phi)^2 + \phi^2} \\
&= \sqrt{1 + 1 - 2\phi + \phi^2 + \phi^2} = \sqrt{2(\phi^2 - \phi) + 2} \underset{\phi^2-\phi=1}{=} \sqrt{2+2} = 2.
\end{aligned}
$$

8 List of figures

Figure 1 is collected from http://www.georgehart.com/virtual-polyhedra/neolithic.html and accessed 2021-04-25.

Figure 2 is made by Salix alba at English Wikipedia, CC BY-SA 3.0, collected from https://commons.wikimedia.org/w/index.php?curid=20677030 and accessed 2021-04-08.

Figure 3 is by Study.com, collected from https://study.com/academy/lesson/counting-faces-edges-vertices-c html and accessed on 2021-04-08

Figure 4 is collected from Richeson, D. S. (2012), page 47 [12]. I have removed the top half of the image.

Figure 5 is collected from Kepler, J. (1981), page 111 [9]. I have removed parts of the image to only leave the Platonic solid with the elements drawn in them.

Figure 6 is collected from Arecchi, A. V., Koshel, R. J. and Messadi, T. (2007), page 2 [1].

Figure 7 is collected from https://mathworld.wolfram.com/SolidAngle.html and accessed on 2021-04-23.

Figure 8, **Figure 9**, **Figure 10**, **Figure 11**, **Figure 12** and **Figure 13** are drawn by me in the online tool GeoGebra.

Figure 14 by Paolo Uccello, collected from http://www.georgehart.com/virtual-polyhedra/uccello.html and accessed on 2021-04-26.

Figure 15 by Johannes Kepler in *De Stella Nova* (1606), page 25 [8]. Collected from https://commons.wikimedia.org/wiki/File:Keplers_trigon.jpg and accessed 2021-04-29.

Figure 16 by Johannes Kepler in *Mysterium Cosmographicum* (1596), page 12 [6]. Collected from the e-book version of Stephenson, B. book, page 77 [13].

Figure 17 is drawn by me in the online tool GeoGebra.

Figure 18 by Johannes Kepler, collected from Kepler, J. (1997) [9].

Figure 19 by Johannes Kepler in *Mysterium Cosmographicum* (1596), between page 26 and 27 [6].

Figure 20, **Figure 21** and **Figure 22** are drawn by me in the online tool GeoGebra, modelled on the same images in Field, J. V. book *Kepler's geometrical cosmology*, page 195 and 196.

Figure 23 and **Figure 24** are drawn by me in the online tool Geogebra.

Figure 25 is collected from https://slideplayer.com/slide/8141520/ and accessed 2021-05-12

Figure 26 and **Figure 27** are collected from Field, J. V. (1988), page 65 and 66 respectively.

Figure 28 is the same image as Figure 23.

9 Bibliography

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