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Which schools would take me?
Truth-telling in School Choice with Admission Probability Uncertainty

av

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Which schools would take me? Truth-telling in School Choice with Admission Probability Uncertainty

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Abstract

In school choice systems all over the world, the *deferred acceptance procedure* is becoming more popular. One of the reasons for this is that students will always have incentives to submit their true preference order under this mechanism. However, Fack et al (2019) showed that this result is restricted to when students do not experience any costs for applying to a large amount of schools. When allowing students to have an application cost, students may "skip the impossible" and not apply to selective schools, despite preferring them to other schools that they apply to. This causes problems for policy makers when estimating students' preferences and when trying to simulate outcomes from alternative admission rules. But when does this hurt the truth-telling assumption? Here, I build a model where students also have uncertainty regarding their admissions probability. I show that students are more likely to "skip the impossible" when they have a low degree of uncertainty of their admission probability, or are underestimating it. When students have a high degree of uncertainty or are overestimating their admission probabilities, the truth-telling assumption is more likely to hold.

1 Introduction

School choice schemes are an increasingly common occurrence in school systems around the world, either in a centralised or decentralised fashion. A decentralised school choice system allows custodians to directly apply to a school by personal contact. However, this will cause problems for both students and schools. Schools will not know if the student already have offers from other schools that they prefer, while students may get offers from a school that is not their first choice with a time limit, which creates the need to strategize between accepting this offer or waiting for a response from their first choice.

Along with other issues, this has caused many school districts to have centralized school choice systems. Here, all students submit rank-ordered lists (ROL) over which schools they prefer. Meanwhile, schools have limited capacities and therefore need admission rules that decide in which order they give priority to students if they get more applicants than they have seats for.

The problem for a policy-maker is to find the best matching for students and schools, given these restrictions. Gale and Shapley (1962) presented a procedure of how to find the best matching for all students that at the same time respects admission rules. Later on, Roth (1982) showed that this procedure also gives students incentives to send in ROL's that corresponds to their true preferences. That is, a student can never get a better application by strategising, such as omitting selective schools from their ROL's as a rejection may worsen their opportunities to get into other, perhaps less selective, schools.

This is important. Strategic incentives give rise to multiple problems as they:

- Lead to less students getting their preferred schools.
- Make it harder for parents to rank schools, which particularly hurts children from disadvantaged backgrounds whose parents often lack a good understanding of the system (de Haan et al., 2015).
- Make it harder for policy makers to know what schools families *truly* prefer. This further complicates the evaluation of which schools are popular and which are in need of intervention.
- Make it harder for policy makers to tweak the school choice system through priority structures and "catchment areas" to make it better, as they do not know if families will augment their ROL's based on their admission probabilities.

However, Roth (1982) only shows that ranking all schools in a true fashion is a weakly dominant strategy, which we call strict truth-telling (STT). Not ranking schools "in the bottom" that you will never apply to as you have a guarantee of getting into a more highly ranked school is not a dominated strategy. We call this weakly truth-telling, as we know that a student will prefer schools in the order they submit their preference lists and they will prefer ranked schools over schools they haven't ranked.

But as Fack et al. (2019) points out, not ranking schools that the students have zero admission probability is not a dominated strategy over being strict truth-telling (STT).

STT may even be a dominated strategy if one incorporates an application cost in students' utility functions, which would correspond to students experiencing ranking many schools as cognitively burdensome or that they are not allowed to rank all schools.

If students only ranks schools that they can get into, which we call their feasible schools, then this matters for policy-makers and researchers as it will make it more difficult to estimate students' preferences. For example, it is likely that researchers underestimate the popularity of selective schools when they assume that all students that do not rank a certain school prefers all of their ranked schools above that one. Furthermore, it makes simulations for changes in admission rules less reliable, as one can expect parents to change their preference lists if their admission probabilities change.

In this thesis, I will develop a model that builds upon this discussion in the school choice literature. More specifically, I wish to investigate the importance of admission probability uncertainty on what incentives parents have to be truth-telling or "skip the impossible".

In section 2, a background will be given regarding school choice algorithms and why some are more popular than others. In section 3, the model from Fack et al will be explained and examples will be given for how uncertainty may affect outcomes in partial equilibria games with simplified notations. In section 4, a formal model with uncertainty will be given and proofs will be given for a new set of propositions. In section 5, I conclude the thesis with a discussion.

2 School choice as a matching problem

Here, a brief background will be given for the school choice problem. First, we will introduce the notion of lattices. Afterwards, the school choice problem will be defined and the properties of stability and strategy-proofness will be explained. Later on, we will describe how these properties are related to the notion of lattices.

2.1 Lattices

Recall that a partially ordered set A is a set taken together with a partially order on it: formally A is a pair $(S, >)$, where S is the "ground set" and $>$ is a "partial order".

Definition 1 *A partially ordered set A is a* ***lattice*** *if:*

(i) For any $x, y \in A$, there exists a member $z_1 \in A$ such that $z_1 \geq x, y$ and $\forall q \in A$ such that $q \neq z_1$ and $q > x, y$ it is true that $z_1 < q$. In this case, z_1 is called the ***join*** *of x and y and is denoted $x \vee y$.*

(ii) For any $x, y \in A$, there exists a member $z_2 \in A$ such that $z_2 \leq x, y$ and $\forall q \in A$ such that $q \neq z_2$ $q < x, y$ it is true that $z_2 > q$. In this case, z_2 is called the ***meet*** *of x and y and is denoted $x \wedge y$.*

Remark 1 *How should one understand this definition? In a lattice, we can find for every pair of elements not only some element that is bigger than both, but even the smallest such element, which is called their join. Similarly, the meet of a pair of elements can be understood.*

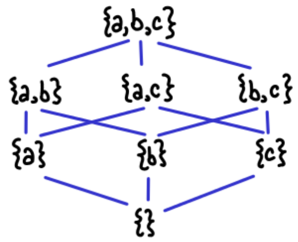

Figure 1: A lattice

Example 1 *The set of pairs of all positive numbers $\mathbb{R}^2_+$ is a lattice, when considered with the natural order. For example, if $x = (3, 1)$ and $y = (2, 6)$, then $z_1 = (\max(3, 2), \max(1, 6)) = (3, 6)$ is the join of x and y and $z_2 = (\min(3, 2), \min(1, 6)) = (2, 1)$ is meet of x and y. Of course, $z_1, z_2 \in \mathbb{R}^2_+$ and this holds true for all $x, y \in \mathbb{R}^2_+$.*

2.2 School choice as a matching problem

We will use the following definition of a school choice problem:

Definition 2 *Formally, a school choice problem consists of five objects:*

(i) *A set of students* $I = \{i_1, \ldots, i_n\}$

(ii) *A set of schools* $S = \{s_1, \ldots, s_m\}$

(iii) *A capacity vector* $q = (q_{s_1}, \ldots, q_{s_m})$

(iv) *A vector of submitted strict student preferences* $P = (P_{i_1}, \ldots, P_{i_n})$*, where* P_{i_1} *is a vector or ordered set with length* $k \leq m$ *of all schools for student* i_1*, with* $P_{i_1,s_q} > P_{i_1,s_r}$*, with* $q \neq r$*, meaning that* $s_q \succ_{i_1} s_r$*.*

(v) *A vector of strict school priorities* $\pi = (\pi_{s_1}, \ldots, \pi_{s_m})$*, where* π_{s_1} *is a vector with length* n *of all students for school* s_1*, with* $\pi_{s_1,i} > \pi_{s_1,j}$ *meaning that* $i \succ_{s_1} j$*.*

Thus, it consists of a finite set of students I and a finite set of schools S. Each student i submits a preference profile P_i over the different schools, while each school s has a capacity in terms of number of seats q_s and ranks students based on strict school priorities π_s.

Remark 2 *Only considering strict student preferences means that for every student* $i \in I$*, we can extract an order of any two schools* $s_1, s_2 \in S$*. We know that* $s_1 = P_{i,k_1}$ *for some* $k_1 \leq m$ *and that* $s_2 = P_{i,k_2}$ *for some* $k_2 \leq m$*. Student* i *prefers* s_1 *over* s_2 *if* $k_1 < k_2$*. We then use the notation that* $s_1 \succ_i s_2$*. Similarly for strict priorities, for every school* $s \in S$*, we can extract an order of any two students* $i_1, i_2 \in I$*, and in the same way as for students, it will be true that* i_1 *has a higher priority to* s *than* i_2*,* $i_1 \succ_s i_2$*, or that* i_2 *has a higher priority to* s *than* i_1*,* $i_2 \succ_s i_1$*.*

Definition 3 *Let* I *be a set of students and* S *a set of schools. We do not restrict any matching (no student is disallowed at any school). Formally, a school choice matching is then a function* f *that maps students to schools such that:*

$$f : I \to S \text{ such that } |f^{-1}(s)| \leq q_s \text{ for all } s \in S. \tag{1}$$

´ Here, $f^{-1}(s) = \{i \in I | f(i) = s\}$ *denotes the pre-image of* S *under* f*.*

Thus, a school choice matching is a mapping from the set of students to the set of schools. Furthermore, the number of students mapped to a certain school may not exceed the school's capacity.

Remark 3 *We restrict ourselves so that all students need to get a seat, i.e.* f *is defined on the whole of* I*. However, the function is neither injective (multiple students can be placed at the same school) nor surjective (not all schools must get placed students).*

Furthermore, for there to exist a matching, a restriction to the school choice problem is that the total number of capacities is not less than the number of seats:

$$\sum_{i=1}^{m} q_m \geq |I| \tag{2}$$

Example 2 *Suppose we have three schools A, B, C with capacities $q_s = 1$ for all $s = A, B, C$ and three students i, j, k. Students' preferences, P, are:*

$$P_i\text{: } A \succ_i B \succ_i C$$

$$P_j\text{: } B \succ_j C \succ_j A$$

$$P_k\text{: } C \succ_k A \succ_k B$$

In turn, schools have strict priorities π:

$$\pi_A\text{: } j \succ_A k \succ_A i$$

$$\pi_B\text{: } k \succ_B i \succ_B j$$

$$\pi_C\text{: } i \succ_C j \succ_C k$$

Every school can be matched with three different students. Thus there are $3! = 6$ possible sets of matchings between the three schools and three students:

$$f_1 = \begin{pmatrix} A & B & C \\ i & j & k \end{pmatrix}$$

$$f_2 = \begin{pmatrix} A & B & C \\ i & k & j \end{pmatrix}$$

$$f_3 = \begin{pmatrix} A & B & C \\ j & i & k \end{pmatrix}$$

$$f_4 = \begin{pmatrix} A & B & C \\ k & i & j \end{pmatrix}$$

$$f_5 = \begin{pmatrix} A & B & C \\ j & k & i \end{pmatrix}$$

$$f_6 = \begin{pmatrix} A & B & C \\ k & j & i \end{pmatrix}$$

2.3 Stability and optimality in school choice

In a school choice system, there are mainly three properties that of interest for market designers when comparing placement algorithms: efficiency (students getting their most preferred seat), stability (priorities are respected), and strategy-proofness (a student can not "game the system"). In this section, we will discuss the former two.

The first property concerns *stability*. In a school choice matching, it is undesirable if a student gets rejected from a school that at the same time admits another student with lower priority. We use the following definition for matchings where this occurs, with inspiration from Gale and Shapley (1962):

Definition 4 *Given a matching of applicants to schools, the matching f is* ***unstable*** *if there are two applicants α and β who are assigned to schools A and B, respectively, although β prefers A to B and β has a higher priority to A than α.*

Remark 4 *Building upon* ***Remark 2****, a more formal definition is that a matching f is unstable if $\exists \alpha, \beta \in I, A, B \in S$, such that $f(\alpha) = A, f(\beta) = B$, while $A \succ_\beta B$ and $\beta \succ_A \alpha$.*

We use our previous example to illustrate the stability notion.

Example 3 *Recall that we have three schools A, B, C with capacities $q_s = 1$ for all $s = A, B, C$ and three students i, j, k. Students' preferences, P, are:*

$$P_i\text{: } A \succ_i B \succ_i C$$

$$P_j\text{: } B \succ_j C \succ_j A$$

$$P_k\text{: } C \succ_k A \succ_k B$$

In turn, schools have strict priorities π:

$$\pi_A\text{: } j \succ_A k \succ_A i$$

$$\pi_B\text{: } k \succ_B i \succ_B j$$

$$\pi_C\text{: } i \succ_C j \succ_C k$$

As mentioned, there are six possible matching between the three schools and three students. However, only three of these are stable. The first stable matching is to give all students their first choice, i.e. f_1. The second stable matching is to give all schools their most preferred student, f_5. The third stable matching is to give every student their second choice, f_4.

Examining f_2, we can see that this matching is not stable, as $f(i) = A, f(k) = B$, while $A \succ_k B$ and $k \succ_A i$.

For future discussion, also note that the assignment that is the best for all students is the worst for all schools, and vice versa.

Gale and Shapley (1962) shows that for any school choice setting, it will always be possible to find a stable outcome. They show that such a matching always exist by introducing a *school choice matching mechanism* that always finds a stable allocation. After formally defining such a mechanism, we give a version of this proof below.

Definition 5 *Fix a set of students I and a set of schools S with capacities $(q_s)_{s \in S}$ with priority orders $(\pi_s)_{s \in S}$. A matching mechanism δ is a function from the set of school choice problems to the set of matchings:*

$$\delta : (I, S, q, P, \pi) \mapsto f \tag{3}$$

Remark 5 *Some clarifications are in order to explain the difference between a matching and a matching mechanism. A matching denotes the specific matching of a school choice problem, i.e. which students are matched with which schools. In contrast, a matching mechanism is a function that, given a certain school choice problem, generates a matching. In practice, the later often takes the form of a systematic algorithm.*

Theorem 1 *For every school choice problem, there exists a stable matching.*
Proof We can prove this theorem by presenting a procedure that for every set of students and schools will find a stable matching. The school matching mechanism presented in Gale and Shapley (1962) for the "marriage problem" of men and women wanting to be matched through the **deferred acceptance** (DA) procedure has been developed further by Abdulkadiroglu and Sönmez (2003) for a school choice setting. It goes as follows:

Step 1: Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

In general, at

Step k: Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

The algorithm terminates when no student proposal is rejected and each student is assigned her final tentative assignment, or when no student have any more schools to apply to.

Now, we need to show that this procedure always finds a stable matching.

Let us suppose, for contradiction, that a matching produced by the procedure is unstable. Here, we use **Definition 4** and note that it will then be true that there are at least one pair of applicants $\alpha, \beta \in I$ that where assigned to schools A and B, respectively. Meanwhile, $A \succ_\beta B$ and $\beta \succ_A \alpha$.

However, this would mean that β proposed to school A in a previous stage k than when he proposed to his placement B and was rejected by A (as $A \succ_\beta B$). This can only happen if a set of students $K \subset I$ such that $|K| = q_A$, where $\forall i \in K$ it is true that $i \succ_A \beta$, was already tentatively assigned at the school. But then α does not get assigned to A. Indeed, this would involve one of the two possible scenarious.

Let us assume that $\alpha \in K$. This would mean that $\alpha \succ_A \beta$, which is contradictory to our previous statement that $\beta \succ_A \alpha$.

Now, consider the possibility that $\alpha \notin K$. This would mean that at a later stage $l > k, \alpha$ was tentatively assigned to school A. This would mean that for some $i \in K$, it is true that

$\alpha \succ_A i$. But since $\forall i \in K$ it is true that $i \succ_A \beta, this would mean that \alpha \succ_A \beta$, which is contradictory to our previous statement that $\beta \succ_A \alpha$.

■

However, there may exist multiple stable solutions. One might then wonder: which of these stable outcomes should be considered the best? In our previous example, we can note that the best matching for the students' perspectives would be the first matching. This is a optimal stable outcome, where every applicant is at least as well off under it as under any other stable matching. We define it formally below.

Definition 6 *Let I denote a set of students, S a set of schools, $(P)_{i\in I}$ a vector of submitted preferences and A the set of stable matchings. A stable matching $f \in A$ is called **stable optimal** if for every student $i \in I$, it is true that $f(i) \succsim_i f'(i)$ for all $f' \in A$ s.t. $f' \neq f$.*

Remark 6 *The notation $\succsim$ denotes that there is either a strict preference or indifference between the two objects. As students have strict preferences over schools, this means that a student will under any other matchings than a stable optimal optimal one either be matched with a less preferred school, or to the same school.*

Remark 7 *Note that finding the optimal stable outcome is an optimization problem, where schools' priorities and capacities are boundary conditions.*

But how can we be sure that such an optimal stable assignment exists for any given school choice setting? Gale and Shapley (1962) go on to prove another proposition, which we omit proving here.

Theorem 2 *Every applicant is at least as well off under the assignment given by the deferred acceptance matching mechanism as he would be under any other stable assignment.*

Furthermore, they show that not only does the algorithm identify *a* stable optimal matching, but that there is only one stable optimal outcome for any given school choice problem. Thus, the algorithm identifies *the* stable optimal matching for any given school choice problem.

2.4 Strategy-proofness in school choice

Roth (1982) goes on to show that not only does the deferred acceptance procedure find the unique optimal stable outcome, but it also gives every student incentives to reveal their true preferences. To understand this result, we introduce the following definitions, where we now make a difference between a student's *reported* and *true* preferences :

Definition 7 *Given a set of schools S, a utility profile u is an order on S. We denote this order $\succ$. We denote the set of all utility profiles by $\mathcal{U}$ if the set S is clear from the context.*

Definition 8 *Let S be a set of schools. A reported preference profile P is an order for a fixed subset of S. We denote this order $\succ^P$. $\mathcal{P}$ is the set of all preference profiles in a given* *school choice setting.*

Definition 9 *Let S be a set of schools, $\mathcal{U}$ the set of utility profiles on S and $\mathcal{P}$ the set of preference lists on S. Then a strategy is a function $\sigma : \mathcal{U} \to \mathcal{P}$.*

Definition 10 *The identity function $\sigma^{id} : \mathcal{U} \mapsto \mathcal{P}$ is called the strict truth-telling strategy.*

Example 4 *Consider a student i and the set of schools $S = \{s_1, s_2, s_3\}$ with utility profile $s_2 \succ_i s_1 \succ_i s_3$. This gives us $u_1 = (s_2, s_1, s_3)$. The only strict truth-telling strategy is then submitting the preference profile $P_i = (s_2, s_1, s_3)$.*

Definition 11 *Fix a set of students I, a set of schools S with capacities $(q_s)_{s \in S}$ and their priority orders $(\pi_s)_{s \in S}$. Let $i \in I$ denote by $\mathcal{U}$ the set of utility profiles on S and fix a matching mechanism δ for $I, S, (q_s)_{s \in S}, (\pi)_{s \in S}$.*

Then a strategy σ is called weakly dominant for student i and matching mechanism δ if for all list of student preferences $(P_j)_{j \neq i}$, all strategies $\sigma' \neq \sigma$ and all utility profiles $u \in \mathcal{U}$, we have that $\delta(I, S, q, P^{\sigma(u)}, \pi)$ assigns i to a school that is at least as preferred by i than $\delta(I, S, q, P^{\sigma'(u)}, \pi)$. Here, $P^{\sigma(u)}$ denotes the list of preferences P_j for $j \neq i$ and $\sigma(u)$ for i.

Remark 8 *Consider a specific student i for which $u_i = P_i$. A strict truth-telling strategy is weakly dominant if changing P_i would not be a better allocation.*

Next, we want to define what it means for an algorithm to be strategy-proof. Strategy-proofness simply means that all students have incentives to be truth-telling. A student will under an algorithm that is not strategy-proof have incentives to augment their preference list based on how other students behave. This is the case in many Swedish school districts today as it may hurt their chances to get into a school if they do not rank it as their first choice. This creates incentives for students to only rank schools that they are sure they will get into. We show such an algorithm in the example below.

Definition 12 *A matching mechanism δ is strategy-proof if for every school choice problem $(I, S, q, P^{\sigma(u)}, \pi)$, the strict truth-telling strategy is weakly dominant for all students.*

Roth (1982) goes on to show that the deferred acceptance student optimal stable mechanism is strategy-proof. We omit showing this here, but will go on to prove this property later from another perspective for **Theorem 3**. The intuition behind this is simple. The DA mechanism lets student propose to schools in the order of their preference profile. The only way for a student to be rejected to a school is thus that other students have higher priority to that school. Changing the order of schools or omitting ranking some schools will thus never lead to a student being assigned to a school they previously were rejected to.

In the following example, we show how the deferred acceptance procedure works and that the stability and strategy-proofness properties of the procedure are not trivial.

Example 5 *Immediate Assignment Mechanism vs Deferred Acceptance Mechanism*

The Immediate Assignment Mechanism δ_{IA}, also often referred to as the Boston Assignment Mechanism, became recognised after being implemented in the centralised school choice for Boston's public schools in 1999. Today, different variations of the mechanism are likely the most commonly used in school choice systems. It works as follows (Pathak, 2011):

Step 1: Only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

In general, at

Step k: Consider the remaining students. Only the k^{th} choices of the students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her k^{th} choice.

The major problem with the Immediate Assignment Mechanism it does not produce stable outcomes and that it is not strategy-proof. This can be illustrated by the following example:

Suppose we have three schools A, B, C with capacities $q_s = 1$ for all $s = A, B, C$ and three students i, j, k. Students' preferences, P, are:

$$P_i\text{: } B \succ_i A \succ_i C$$

$$P_j\text{: } A \succ_j B \succ_j C$$

$$P_k\text{: } A \succ_k B \succ_k C$$

In turn, schools have strict priorities π:

$$\pi_A\text{: } i \succ_A k \succ_A j$$

$$\pi_B\text{: } j \succ_B i \succ_B k$$

$$\pi_C\text{: } k \succ_C i \succ_C j$$

How the matching will work is shown in Table 1 below, where $\boxed{i}$ represents that student i was given a spot at the school, while the absence of a box represents that the student was not assigned a spot.

Table 1: Truth-telling in Immediate Assignment Mechanism

Round	A	B	C
Round 1	$j, \boxed{k}$	$\boxed{i}$	
Round 2	$\boxed{k}$	$j, \boxed{i}$	
Round 3	$\boxed{k}$	$\boxed{i}$	$\boxed{j}$

When all students report their true preferences, students i and k will be assigned to their first choices in the first round, while j is rejected. In the second round, j will again be rejected as the school have no seats left. Finally, in the third round, j will be assigned to C. This will give us the following matching under the Immediate Assignment Mechanism when all students are truth-telling:

$$f_{\delta_{IA}} = \begin{pmatrix} A & B & C \\ k & i & j \end{pmatrix}$$

We can see that the initial outcome $f_{Immediate}$ is unstable, as student j prefers school B to her assignment C while she has higher priority than i, who "got her seat".

Furthermore, the algorithm is not strategy-proof. We show that this by noting the scenario where j considers the fact that she has a high priority at school B and do not want to risk ending up at school C. This would cause her to instead report her preferences as $B \succ'_j A \succ'_j C$. In turn, the matching would be different, as shown in Table 2 below.

Table 2: Strategic Incentives in Immediate Assignment Mechanism

Round	A	B	C
Round 1	$\boxed{k}$	i, $\boxed{j}$	
Round 2	i, $\boxed{k}$	$\boxed{j}$	
Round 3	$\boxed{k}$	$\boxed{j}$	$\boxed{i}$

Now, j and k would be accepted at B and A, respectively, in the first round, while i would be rejected as j has a higher priority to B than i does. This would give us the following matching under the Immediate Assignment Mechanism, when j submits a false preference list:

$$f'_{\delta_{IA}} = \begin{pmatrix} A & B & C \\ k & j & i \end{pmatrix}$$

As seen, submitting a false preference list was a dominant strategy over submitting a true preference list for j and, thus, the Immediate Assignment Mechanism is not strategy-proof.

*In contrast, under DA (described in the proof for **Theorem 1**) each student propose as shown in Table 3 below.*

Table 3: Truth-telling in Deferred Acceptance mechanism

Round	A	B	C
Round 1	j, $\boxed{k}$	$\boxed{i}$	
Round 2	$\boxed{k}$	i, $\boxed{j}$	
Round 3	$\boxed{i}$, k	$\boxed{j}$	
Round 4	$\boxed{i}$	$\boxed{j}$, k	
Round 5	$\boxed{i}$	$\boxed{j}$	$\boxed{k}$

When students propose to their first choice in the first round, i and k will be tentatively assigned to B and A, respectively, as A will reject j as $j \succ_A k$. In the second round, j will propose to her second choice B and will be assigned to that school as $j \succ_B i$. In the third round, student i will, after losing her tentatively assigned seat at B, instead apply to her second choice A, where she will be assigned as $i \succ_A k$. In the fourth round, k will apply to his second choice B, but be rejected as $j \succ_B k$. Finally, in the fifth round, k will apply and be assigned to his third choice C. This will give us the following matching:

$$f_{\delta_{DA}} = \begin{pmatrix} A & B & C \\ i & j & k \end{pmatrix}$$

Under this mechanism, the outcome is always stable. In addition, note that a student will never have incentives to submit a false preference list under DA, as it does not matter in which particular round she proposes to a school. This can be seen in j getting accepted to B, despite being truthful in ranking A as her first choice, in contrast to the outcome from the Immediate Assignment Mechanism. Both of these properties, stability and strategy-proofness, will always hold under DA.

2.5 Lattice sets and school choice

Why is it that there always exists a unique optimal stable matching? And why is it that the procedure that always finds this matching is also strategy-proof? Although it is not explicitly explained in Gale and Shapley (1962) and Roth (1982), the answer to this question comes from order theory.

It can be shown that the set of stable outcomes for a certain school choice problem (I, S, q, P, π), with respect to the order of student preferences P $(\succ_i)_{i \in I}$, is a lattice. Therefore, it is always possible to find the assignment that is the best for all students (and worst for all schools), as well as the assignment that is the best for all schools (and worst for all students). This is because the outcome that is the most preferred by all students is in "the corner" of the lattice, while the one that is least preferred by all students is in the opposite corner. We will illustrate this in the following proposition and with a new proof, although this property has already been proved by e.g. Shapley and Shubik (1971).

Proposition 1 *The set of all stable matchings for any school choice setting is a lattice.*
Proof We refer to set of all stable matchings as $A \subset \mathcal{F}$. For this proof, we need to show that for any two stable matchings $x, y \in A$, it is true that their join z_1 and meet z_2 are also stable matchings, i.e. that $z_1, z_2 \in A$. For this to be true, the meet and join may not (i) exhibit any cases of unstability and (ii) respect capacity constraints. We will here show that this is true for their join and the proof for their meet follows the same logic.

Stability

For any two stable outcomes $x, y \in A$ with respect to their preferences $\succ$, their join z_1 will be the outcome that is weakly preferred by all students compared to the outcomes in x and y. That is, for each student i, it is true that if $f^x(i) \succsim_i f^y(i)$, then $f^x(i) = f^{z_1}(i)$.

Before continuing, note from our previous definitions that all students have strict preferences over all schools, so it can never be the case that a student i is indifferent between two allocations if they correspond to two different schools.

Let us suppose that z_1 is unstable, i.e. that there are two applicants $\alpha, \beta \in I$ and two schools $A, B \in S$ where $f^{z_1}(\alpha) = A$ and $f^{z_1}(\beta) = B$, while $A \succ_\beta B$ and $\beta \succ_A \alpha$ A to B and β has a higher priority to A than α. This will lead to three possible scenarios, where showing stability is trivial for the first two scenarios.

In the first scenario, both students have the same school in both matchings, i.e. $f^x(\alpha) = f^y(\alpha)$ and $f^x(\beta) = f^y(\beta)$. Then f^{z_1} will of course also be stable for these two students, as f^x, f^y are stable outcomes.

In the second scenario, only one of the two students have the same school in the matchings. Let us consider the case of $f^x \succ_\alpha f^y$ and $f^y(\beta) = f^x(\beta)$. This would mean that $(f^{z_1}(\alpha), f^{z_1}(\beta)) = (f^x(\alpha), f^x(\beta))$, which we know is stable. The proof for other variations of the scenario follows the same logic.

In the third scenario, both students are admitted to different schools in the two matchings. Let us examine the case of $f^x \succ_\alpha f^y$ and $f^y \succ_\beta f^x$. Then it would be true that $(f^{z_1}(\alpha, f^{z_1}(\beta)) = (f^x(\alpha), f^y(\beta))$. Let us suppose that this would lead to instability, e.g. that $f^x(\alpha) \succ_\beta f^y(\beta)$ and $\beta \succ_{f^x(\alpha)} \alpha$.

As we know that f^x is stable, this means either that $f^x(\beta) \succ_\beta f^x(\alpha)$ or that $\alpha \succ_{f^x(\alpha)} \beta$. From our previous statement, we have assumed that $\beta \succ_{f^x(\alpha)} \alpha$. This would mean that $f^x(\beta) \succ_\beta f^x(\alpha)$. Meanwhile, we know that $f^y(\beta) \succ_\beta f^x(\beta)$. But this would mean that $f^y(\beta) \succ_\beta f^x(\alpha)$, which is a contradiction from our previous statement.

Showing the corresponding variations of the third scenario follows the same logic. Thus, we have shown that the join does not exhibit unstability.

Capacity constraints

Furthermore, one needs to show that z_1 always respects each school's capacity restriction, as $A \in \mathcal{F}$ and $\forall f \in \mathcal{F}$ that follows the previous definition.

We know that (i) $x, y \in A$ is stable and respects capacities and (ii) z_1 is stable. For z_1, it is arbitrary that it will respect capacities in the first case described above, i.e. that all students get the same matching in x, y. What we now need to show is that the occurrence of some students preferring x and some preferring y will not lead to z_1 not respecting capacity constraints.

Let us consider a scenario with school $A \in S$ and that students $i_1, ..., i_q$, where q equals the capacity of school A, were placed at school B in f^y and all of them weakly prefers their placement at school B over their potentially different placement in f^x. Meanwhile, suggest that at least one student i_k, for which $f^x(i_k) = B \succ_{i_k} f^y(i_k) \neq B$.

In that case, placing i_k in z_1 at school A in z_1 may lead to a capacity violation. However, this would also mean that all students placed at school A in f^y has a higher priority than

i_k, as we know that f^y is stable. This, in turn, would suggest that at least one student was placed at a school they prefer over A in f^x, as i_k were placed at A in f^x and f^x is stable. But this contradicts our previous statement that all students placed at A in f^y weakly prefer this placement over their placement in f^x.

It follows that the set A is a lattice, which is what we wanted to show. ■

The answer around strategy-proofness also lies in the fact that the set of stable outcomes is a lattice. As we have already find the matching which is optimal for every student for all stable outcomes, in the "corner" of the lattice, there is no change in preference lists that can make any student better off.

Theorem 3 *The deferred acceptance student optimal stable mechanism is strategy-proof.*
Proof We know that the student-proposing deferred acceptance procedure finds the optimal stable outcome for all students. Let us examine the case when all students are truth-telling, i.e. that they submit $u = P$. This will lead to all students getting the most preferred school that they could have gotten, given capacity constraints and stability requirements.

Suppose that a student i does not submit their true preferences and that this leads to that the final matching $f(i^{P'}) \neq f(i^{P^u})$ and that $f(i^{P'}) \succ_i f(i^{P^u})$. But since the deferred acceptance procedure did not place the student at $f(i^{P'})$ in the truth-telling scenario, it must be the case that there are $k = q_{f(i^{P'})}$ students that have higher priority to $f(i^{P'})$ than i and that all of them prefer that school over any other school they could have gotten into. Thus, this would mean that the matching of $f(i^{P'})$ is unstable. However, we have already shown that the deferred acceptance procedure is always stable, which creates a contradiction. ■

3 A model of "skipping the impossible"

Because of the properties discussed previously, it has become a commonly used assumption that students are strict truth-telling in their rank-ordered lists (ROL's) under DA (see e.g. Abdulkadiroğlu et al. (2017)). Being strict truth-telling implies that every student ranks every school in accordance with their preferences.

However, it is quite seldom that students rank all available schools. For example, if a student believes they have a very high probability of being admitted to their first three choices, they may leave out ranking other schools. We refer to this as *weak truth-telling* (WTT), which may be contrasted with the definition of being strict truth-telling. For the next definition, we identify preferences with lists to simplify notations.

Definition 13 *Fix a set of schools S and denote by $\mathcal{U}$ the utility profiles on S and by P the preference lists on S. A strategy σ os weakly truth-telling if for all $u = (s_1, ..., s_n) \in \mathcal{U}$, there is $k \in \{1, ..., n\}$ such that $\sigma(u) = (s_1, ..., s_k)$. A strategy is strict truth-telling if $k = n$.*

Fack et al. (2019) makes the point that WTT may not be a reasonable assumption either. Students seem to feel it cognitively burdensome to rank all schools they are interested in. This creates the possibility of "skipping the impossible" and not rank selective schools where their probability of admission is small.

A more reasonable assumption would, according to Fack et al. (2019) therefore to be that students only rank schools that one can be to admitted to in their true order, based on a probability analysis of the actual outcome of the school choice. We call this behaviour *feasible truth-telling*, while Fack et al. (2019) refers to it as "stability". Because this is easy to confuse with the stability property of the DA, we use a slightly different term. We define the relevant terms below.

Definition 14 *Given a realized matching of schools, school $s^* \in S$ is ex post **feasible** for student i if i's priority to school s^* is greater than the priority index of the student with the lowest priority index of all the ex-post admitted students to school s^* through the matching f. This matching follows our previous definitions. Formally, we say that in the partially rank-ordered set $(f^{-1}(s^*), \succ_{s^*})$, it is true that $i \succ_{s^*} \min(f^{-1}(s^*))$. We refer to i's feasible schools as F_i.*

Definition 15 *We have $s_1, s_2 \in S$ and we refer to i's feasible schools as $F_i \subset S$. Student i's strategy σ is **feasible truth-telling** (FTT) if $\forall s \in F_i$, there are no two schools $s_1, s_2 \in F_i$ where $s_1 \succ_i s_2$ and where $s_1 \notin P_i$, while $s_2 \in P_i$, or that $s_2 \succ_i^P s_1$ in P_i.*

Definition 16 *Student i's strategy σ_i involves **skipping the impossible** if for $s_1, s_2 \in S$, it is true that $s_2 \succ_i s_1$, while it is also true $s_1 \in P_i$ and $s_2 \notin P_i$, as well as $s_1 \in F_i$ and $s_2 \notin F_i$.*

We will use the following example from Fack et al. (2019) to illustrate in what way WTT and FTT differs from STT.

Example 6 *Consider a finite economy that has two students* (i_1, i_2) *and three one-seat schools* $S = \{s_1, s_2, s_3\}$. *Furthermore,* $\forall s \in S, i_1 \succ_s i_2$; *student* i_1*'s true preference order* u_{i_1} *is* $s_1 \succ_{i_1} s_2 \succ_{i_1} s_3$, *while for* u_{i_2}, $s_2 \succ_{i_2} s_1 \succ_{i_2} s_3$. *There are multiple strategies that are weakly dominant in addition to STT, stemming from two sources: "irrelevance at the bottom" and "skipping the impossible". Both arise when some admission probabilities are zero.*

For i_1, *the bottom part of her submitted* P_i *is irrelevant as long as* s_1 *is top-ranked. In fact, any* P_{i_1} *(*s_1*, s', s"), for* $s', s'' \in \{s_2, s_3\} \cup \{\}$ *is weakly dominant for* i_1, *as she is always accepted by* s_1.

For student i_2, *"skipping the impossible" comes into play, as there is no* P' *where* $s_1 \in P'$ *that strictly dominates all other* $P \in \mathcal{P}$, *where* $\mathcal{P}$ *refers to all possible submitted ROL's. This is true because* s_1 *is always taken by* i_1 *in any equilibrium. Furthermore, whether or not she omit ranking* s_3 *does not change her outcome, as she will always be admitted to* s_2. *This is the "irrelevance at the bottom" source.*

3.1 The formal model

Fack et al. (2019) builds up the following model:

An economy is defined as a finite set of schools, $\mathcal{S} \equiv \{1, ..., S\}$, where $S \in \mathbb{N}$, and a set of students I. Student i has a type $\theta_i = (u_i, e_i) \in \Theta \equiv [0,1]^S \times [0,1]^S$, where $u_i = (u_{i,1}, ..., u_{i,S}) \in [0,1]^S$ is a vector of utility scores assigned to each school, and $e_i = (e_{i,1}, ..., e_{i,S}) \in [0,1]^S$ is a vector of priority indices at schools, where a student with a higher index has priority to a school. The utilities and priorities for every school are strict, i.e. $u_{i,s} \neq u_{i,s'}$ if $s \neq s'$ and $e_{i,s} \neq e_{i',s}$ if $i \neq i'$. Students are matched with schools through a centralised mechanism.

Note that these notations differ from our previous notations. Previously, our utility profiles and priority indices where of ordinal nature, i.e. orders on the set of schools. Here, they are of cardinal nature and takes the form a of a real number between 0 and 1.

They go on to describe the continuum economy with a unit mass of students. This is denoted by $E = \{G, q, C\}$, where G is an atomless probability measure over Θ representing the distribution of student types (no students are indifferent in either utilities or priority indices), $q = (q_1, ..., q_S)$ are masses of capacities at each school, and C is an application cost, which we will define later.

They go on to describe how a random finite economy of size I can be constructed from the continuum economy and how this random finite economy is realized. We omit doing this here, as we are mostly interested in the construction of every student's utility function. We describe further the motivation for this model in the next section.

First, they describe a student i's admission outcome at school s, with students' submitted preference orders $\mathcal{P}$, as:

$$\begin{aligned}
&a_{i,s} : \mathcal{P} \times [0,1]^S \to \{0,1\} \text{ s.t.} \\
&a_{i,s}(P,e) = \begin{cases} \mathbf{1}(i \text{ is rejected by } P_i^1, ..., P_i^{k+1} | P_i, e_i; P_{-i}, e_{-i}) \text{ if } s \in P_i \\ 0 \text{ otherwise .} \end{cases}
\end{aligned} \tag{4}$$

Here, $\mathbf{1}(\cdot|P_i, e_i; P_{-i}, e_{-i})$ is an indicator function. Further, P_{-i} denotes the preference lists for all students $j \in I$ such that $j \neq i$. Due to the centralised mechanism, a student can receive at most one offer, which gives us:

$$\sum_{s=1}^{S} a_{i,s}(P_i, e_i; P_{-1}, e_{-1}) = 0 \text{ or } 1. \tag{5}$$

For student i, P_{-i} and e_{-i} are unknown at the time of submitting her preference profile P_i, so i takes into account the uncertainty when choosing action. A pure strategy is $\sigma : \Theta \mapsto \mathcal{P}$.

Given σ, the admission probabilities are $\int a_{i,s}(\sigma(\theta_i), e_i; \sigma_{-i}(\theta_{-i}), e_{-i}) dG(\theta_{-i})$ for all i and s, where $\sigma_{-i}(\theta_{-i}) \equiv \{\sigma(\theta_j)\}_{j \neq i}$.

Each student also incurs an application cost when ranking schools:

Definition 17 *An application cost is a function that takes the (integer) length of an object and gives a real number:*

$$C : \mathbb{N} \mapsto [0, \infty] \tag{6}$$

This function is monotonically increasing: if $|P| > |P'|$, then $C(|P|) > C(|P|)$. Thus, adding another element to a preference list always incurs a cost. If there is a restriction on students that they may not rank more than x schools, then for all $k \in \mathbb{N}$ s.t. $k > x, C(k) = \infty$.

Fack et al. (2019) then considers a type-symmetric equilibium σ^* in pure strategies such that σ^* solves the following maximization problems for every θ_i:

$$\sigma^*(\theta_i) \in \underset{\sigma(\theta_i) \in \mathcal{P}}{\arg\max} \left\{ \sum_{s \in \mathcal{S}} u_{i,s} \int a_s \left(\sigma(\theta_i), e_i; \sigma^*_{-i}(\theta_{-i}), e_{-i} \right) d(G(\theta_{-i})) - C\left(|\sigma(\theta_i)| \right) \right\} \tag{7}$$

Remark 9 arg max *denotes the set of preferences for which the student's utility function attains the largest value (if it exists).*

Describing why this optimization is feasible and the integration can be solved is described further in Fack et al. (2019). Since we are focusing on what happens *after* the admission probability has been calculated, we will omit doing this here.

Here, σ^* is the equilibrium outcome of σ, the choice of P_i for a particular student. Note that this may differ from their actual preference order u, due to the application cost. Let us use the notation $\xi_{i,s} = \int a_s \left(\sigma\left(\theta_i\right), e_i; \sigma^*_{-i}\left(\theta_{-i}\right), e_{-i}\right) d(G\left(\theta_{-i}\right))$ for student i's admission probability to school s. We can then understand the model in the following way:

$$\sigma^*\left(\theta_i\right) \in \underset{\sigma(\theta_i)\in\mathcal{P}}{\arg\max} \left\{ \sum_{s\in\mathcal{S}} u_{i,s} \cdot \xi_{i,s} - C\left(\left|\sigma\left(\theta_i\right)\right|\right) \right\} \tag{8}$$

The utility for each ROL for a single student is thus based on taking the utility for each school in the list multiplied by the admission probability of that school, and finally adding a cost based on the length of the list.

3.2 Motivation for the model

Some explanations are in order, after this brief summary of the model in Fack et al. (2019). Let us begin with the economy description.

The economy is built up in a such a way that one begins with a vector space that is infinite and uncountable, filled with all possible student types. From this population of student types, one can draw a finite sample of student types (such as a single school district) in a mathematically convenient way, based on the "Law of Large Numbers":

Theorem 4 *Assume that $X_1, X_2, \ldots$ are independent with means $\mu_1, \mu_2, \ldots$ and variances $\sigma_1^2, \sigma_2^2, \ldots$ such that $\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$. Then*

$$\frac{X_1 + X_2 + \cdots + X_n - (\mu_1 + \mu_2 + \cdots + \mu_n)}{n} \xrightarrow{a.s.} 0$$

We omit proving this theorem, but the implication of it for our case is that if the economy of interest and the sample is large enough, one can assume that the mean of the sample will converge to the mean of the population.

The optimization problem for each student can be understood in the following way. $\mathcal{P}$ consists of all possible ROL's that i can submit. Note that this notation differs somewhat from our previous setting. To choose which one, i needs to put a utility score at all possible ROL's, and will then choose the one with the highest value.

The admission probability estimate is quite interesting in this model. In the rational expectations paradigm in economics, every agent in an economy is assumed to take all available information into account when making decisions. This may not actually be how student behave, but it may be true that agents behave *as if* they are. In line with this way of modelling, Fack et al. (2019) assumes that each student i knows the distribution of Θ, from which other students also will be drawn, and creates an integral from which the can estimate their probability of admission. Although the students may actually be getting their admission probability estimates through examining previous years' cutoffs for every school, this behaviour could be identical to what the model expect students to be doing.

3.3 Implications of application cost

In the previous examples, both students will omit schools from their optimal ROL if there are any application cost. For example, the optimal ROL in a setting with any application cost would be (s_1) for student i_1. Meanwhile, it will always be optimal for student i_2 to omit from ranking s_1 in any setting where i_2's admission probability is zero, while its application cost is non-zero. We can show this in the following way.

Example 7 *A student i is making her decision on which ROL to submit in the school choice process. There are four schools, for which she has the following ordinal utility:*

$$s_3 \succ_i s_1 \succ_i s_2 \succ_i s_4$$

She has the following cardinal utility scores for admission to the schools:

$u_{i,s_1} = 0,6, \xi_{i,s_1} = 0,5$ $u_{i,s_2} = 0,3, \xi_{i,s_2} = 1$ $u_{i,s_3} = 1, \xi_{i,s_3} = 0,05$ $u_{i,s_4} = 0,1, \xi_{i,s_4} = 0,7$

Let us use the simplified notation for the previous model to determine which ROL the student will submit:

$$\sigma^* (\theta_i) \in \underset{\sigma(\theta_i)\in\mathcal{P}}{\arg\max} \left\{ \sum_{s\in\mathcal{S}} u_{i,s} \cdot \xi_{i,s} - C \left(|\sigma (\theta_i)|\right) \right\}$$

We will set $\frac{\mathrm{d}|P_i|}{\mathrm{d}C} = \frac{1}{10}$. *That is, for each added school to the ROL, i will experience a negative utility of 0,1.*

It is straight-forward that i will not submit a ranking list that includes s_4, as she is guaranteed a seat at s_2 that is preferable over s_4. This is where "irrelevance at the bottom" comes in.

We can also show that student i will skip ranking her favourite school s_3 in this case by computing utility scores for two competing ROL's:

$$u(\{s_1, s_2\}) = 0,6 \cdot 0,5 + 0,3 \cdot 1 - 0,1 \cdot 2 = 0,4$$

$$u(\{s_3, s_1, s_2\}) = 1 \cdot 0,05 + 0,6 \cdot 0,5 + 0,3 \cdot 1 - 0,1 \cdot 3 = 0,35$$

Thus, i will "skip the impossible" and submit $P_i = \{s_1, s_2\}$.

Based on their model, Fack et al. (2019) proves that in any system of non-zero application costs, at least one student will skip ranking a preferable school. This goes against the famous result from Roth (1982) (which was an important component for his Nobel Prize), from which researchers have assumed that students will be strict truth-telling in systems with the deferred acceptance procedure.

This has important implications in terms of researchers and policy-makers ability to analyse student preferences and estimate the effects of changes in the school choice system. When analysing student preferences, econometric models assume that omitted schools are less preferred than ranked schools. It will also make it harder for local policy-makers to understand which schools are popular and perhaps should be expanded. Furthermore it makes it harder to simulate the effects of changes in admission rules or school capacities on proximity, efficiency and segregation. If student rank based on their admission probabilities and changes in the systems affect these probabilities, one might expect behavioural changes in what preference lists students submit.

To better understand when to expect that student's will "skip the impossible", we will build upon the discussion in Fack et al. (2019) and prove yet another proposition. When policy-makers are making estimates on student preferences and likely outcomes from admission rules changes, the "skipping the impossible" event is far more challenging than the "irrelevance at the bottom", which is why we put our focus here.

We can show that students will "skip the impossible" when the marginal expected utility from ranking the school is smaller than the marginal cost of making ones ROL longer. We formalise this in the following proposition.

Proposition 2 *Student i will "skip the impossible" when the expected utility from adding a school s to the ROL is lower than the marginal cost of lengthening the ROL:*

$$u_{i,s} \cdot \xi_{i,s} < \frac{C(|P_i|)}{C(|P_i| - 1)} - 1 \tag{9}$$

Proof This probability straightly follows from our utility function for student i:

$$\sigma^* (\theta_i) \in \underset{\sigma(\theta_i) \in \mathcal{P}}{\arg\max} \left\{ \textstyle\sum_{s \in \mathcal{S}} u_{i,s} \cdot \xi_{i,s} - C\left(|\sigma\left(\theta_i\right)|\right) \right\}$$

Doing this maximization, it is straight-forward to show that any addition to a ROL involves the expected positive utility from ranking the school minus the cognitive burden of ranking another school, i.e. $u_{i,s} \cdot \xi_{i,s} - \frac{C(|P_i|)}{C(|P_i|-1)} - 1$. A student i will then only add the school if the expected utility from being admitted to the school $u_{i,s} \cdot \xi_{i,s}$ is larger than the cognitive burden of lenghtening the preference list $\frac{C(|P_i|)}{C(|P_i|-1)} - 1$, i.e. when $u_{i,s} \cdot \xi_{i,s} - \frac{C(|P_i|)}{C(|P_i|-1)} - 1 > 0$. From here, it is arbitrary to get to our equation 9. Thus, the proposition is proved. ■

4 A model of admission probability uncertainty

We will now build a simplified model for an individual student which they use when determining on which ROL they will send to the admission office, based on the general equilibrium model of Fack et al. (2019).

4.1 Why uncertainty matters

Every student has a set of all possible ROL's $\mathcal{P}$. To determine which ROL they will submit, each student will calculate an expected utility for each ROL. We will assume that three factors will be accounted for: (i) the utility of being admitted to the school, (ii) the probability of being admitted, (iii) an application cost based on the length of the ROL.

We will build on the previous discussion and build in a caveat to when students are expected to skip the impossible. For example, if students have biases in their admission probability, it may be costly to omit their preferences. We show this in the following example.

Example 8 *Let us use our previous example, but now, student i (incorrectly) believes that their probability of getting in to school s_3 is higher than it actually is:*

$u_{i,s_1} = 0,6, \xi_{i,s_1} = 0,5$ $u_{i,s_2} = 0,3, \xi_{i,s_2} = 1$ $u_{i,s_3} = 1, \xi_{i,s_3} = 0,15$ $u_{i,s_4} = 0,1, \xi_{i,s_4} = 0,7$

We will now get the following utility measures:

$$u(\{s_1, s_2\}) = 0,6 \cdot 0,5 + 0,3 \cdot 1 - 0,1 \cdot 2 = 0,4$$

$$u(\{s_3, s_1, s_2\}) = 1 \cdot 0,15 + 0,6 \cdot 0,5 + 0,3 \cdot 1 - 0,1 \cdot 3 = 0,45$$

Thus, i will not "skip the impossible" and submit $P_i = \{s_3, s_1, s_2\}$.

The following example shows that a bias in estimates will affect the student's choice of submitting ROL's. In the same way, uncertainty along with risk-aversion may also affect student's decision-making. In the following sector, we will formalise a model that incorporates both of these concerns. Down below, we will define an admission uncertainty function.

Definition 18 *For a student $i \in I$, school $s \in S$, admission probability $\xi_{i,s} \in [0,1]$ and utility $u_{i,s} \in U$, we define an admission uncertainty function Φ :*

$$\Phi : [0,1] \times [0,1] \rightarrow [0,1] \tag{10}$$

, i.e. $\Phi(u_{i,s}, e_{i,s}) \in [0,1]$.

4.2 The model

We build on the model from Fack et al. (2019) but incorporate our admission uncertainty function in the following equation:

$$\sigma^*(\theta_i) \in \underset{\sigma(\theta_i)\in\mathcal{P}}{\arg\max} \left\{ \sum_{s\in\mathcal{S}} \Phi_i(u_{i,s}, \xi_{i,s}) - C\left(|\sigma(\theta_i)|\right) \right\} \tag{11}$$

With this model, we can show the following proposition on how the relationship of application cost and uncertainty in admission probability affects a student's propensity to "skip the impossible".

Proposition 3 *Student i will only "skip the impossible" when expected utility from adding it to the ROL is lower than the marginal cost of lengthening the ROL if their estimation function follows the following restriction:*

$$\Phi_i(u_{i,s}, \xi_{i,s}) < \frac{C(|P_i|}{C(|P_i| - 1)} - 1 \tag{12}$$

Proof This proposition follows from using the proof of **Proposition 2** for the above equation. ■

It is simple to see that there will now be a larger ambiguity as to when student i will "skip the impossible". When students are overestimating their admission probabilities or are uncertain (and experience risk-adverseness), this will affect their strategies when mapping their utility orders. Furthermore, note that student's admission uncertainty functions are individual. Thus, some students may have a function that very well maps onto the function in Fack et al. (2019) and correctly estimate their admission probability and be fairly certain with their estimates, while other students are less productive in that endeavour.

Our previous examples of a student optimising her utility function in order to determine which ROL she should submit have shown the differences between correctly estimating admission probabilities (**Example 7**) and one that overestimate their probabilities (**Example 8**). These cases can also be transferred to better understanding student's with uncertainty and risk-adverseness or other properties that may affect their submission behaviour.

5 Conclusion

In this thesis, we have discussed the school choice problem from a theoretical perspective. From our discussion of lattices and stable outcomes, it is simpler to understand why the student-proposing deferred acceptance procedure can find the student optimal stable outcome (which is in the corner of the lattice) and why the procedure also exhibit the strategy-proofness property.

However, these results stem from research that simply assume that students does not find it cognitively burdensome to rank schools. This may not map onto reality very well: few students apply to all available university programs that they can find. Sometimes, administrators actively restrict students from ranking more than x schools, which is equivalent to an infinitely large marginal cost of ranking more schools. Therefore, it is convenient to expand upon our previous framework to better understand the implications of application costs to how to understand student ranking behaviour.

With that said, the framework put forward in Fack et al. (2019) implicitly assumes that students do not find it cognitively burdensome to estimate their admission probabilities when determining when to omit ranking schools. A more generalised framework may then be needed to incorporate that students may have different levels of understanding of their admission probabilities. This can both be on an individual level or based on their school choice setting. For example, students applying to upper secondary schools with admission cut-off history based on grades are more likely to "skip the impossible" than students involved in a school choice to elementary school with more complex admission rules, i.e. the case in Sweden.

There are two important endeavours for future research: one theoretical and one empirical. Theoretically, it is worth expanding upon the framework from Fack et al. (2019) with an embedded admission uncertainty to better understand if the propositions they make about the general equilibrium effects of application costs still holds true. Furthermore, future empirical research may indicate when weak truth-telling and feasible truth-telling, respectively, is the most appropriate assumption to make.

References

Abdulkadiroglu, A. and Sönmez, T. (2003), 'School Choice: A Mechanism Design Approach', *American Economic Review* **93**(3), 729–747.

Abdulkadiroğlu, A., Agarwal, N. and Pathak, P. A. (2017), 'The Welfare Effects of Coordinated Assignment: Evidence from the New York City High School Match', *American Economic Review* **107**(12), 3635–89.

de Haan, M., Gautier, P. A., Oosterbeek, H. and van der Klaauw, B. (2015), The Performance of School Assignment Mechanisms in Practice, Discussion Paper 9118.

Fack, G., Grenet, J. and He, Y. (2019), 'Beyond truth-telling: Preference estimation with centralized school choice and college admissions', *American Economic Review* **109**(4), 1486–1529.

Gale, D. and Shapley, L. S. (1962), 'College Admissions and the Stability of Marriage', *The American Mathematical Monthly* **69**(1), 9–15.

Pathak, P. A. (2011), 'The Mechanism Design Approach to Student Assignment', *Annual Review of Economics* **3**(1), 513–536.

Roth, A. E. (1982), 'The Economics of Matching: Stability and Incentives', *Mathematics of Operations Research* **7**(4), 617–628.

Shapley, L. S. and Shubik, M. (1971), *The Assignment Game: I, The Core*, RAND Corporation, Santa Monica, CA.