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Curves & Cash: Finding the Optimal Path Using Control Theory

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Abstract

Control theory can be used to solve a large variety of problems and has played a vital role in generating accurate solutions to complex problems. This thesis will explore the history and the mathematics behind *control theory* and to better conceptualize the utilization of *control theory* two practical problems are examined: *the brachistochrone problem* and *Merton's portfolio problem.* Furthermore, to display the benefits of using *control theory*, different methods of solving *the brachistochrone problem* are shown, which provides a more intuitive understanding of the strengths of *control theory.*

Keywords: Hamilton-Jacobi-Bellman equation, HJB, Pontryagin's maximum principle, PMP, the brachistochrone problem, Merton's portfolio problem, optimal control, control theory.

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Contents

1

Introduction

The aim of this thesis is to give the reader an introduction to *control theory* and its applications. This is accomplished partly be describing and deriving the methodological components of control theory as the *Hamilton-Jacobi-Bellman equation* and *Pontryagin's minimum (maximum) principle*, and partly by applying these methods on practical problems.

Control theory, can be applied in various disciplines and in this thesis the classic and famous *brachistochrone problem* together with *Merton's portfolio problem*, will be examined closer and then also solved. The *brachistochrone problem* will be solved using three different methods to show the benefits *control theory* can reap in contrast to other methods.

2

Control Theory

2.1 Background

The word control, (/kən'tɹoʊɫ/), has a unique duality to it as *controlling* can mean two things. Firstly, it may mean, checking that whatever is being observed behave adequately. Secondly however, it can also mean, making sure that whatever is being observed in fact behaves adequately and thus, in a way guaranteeing it. This is in fact also what *control theory* is all about, making sure that a process reaches the desired outcome while minimizing unwanted effects (Fernandez-Cara, 2003). In that sense, *control theory* in its very essence can be dated back to the very antique as Aristotle so eloquently described the phenomenon in "Politics".

"*. . . if every instrument could accomplish its own work, obeying or anticipating the will of others . . . if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves.* (Aristotle, ca. 350 B.C.E./ 1905)"

Since then, the theories surrounding *control theory* have been developing rapidly and new applications are found continuously. The endless possibilities where *control theory* can play a crucial role includes regulation of machines, design of medical and prosthetic devices, as well as a wide array of coordinated activities in the social sphere such as optimization of business operations, control of economic activity by policies, and even control of political decisions by democratic processes. However, before taking a closer look at how *control theory* can be used solving *the brachistrochrone problem* and *Merton' portfolio problem*, the basics of *control theory* are considered.

2.2 Optimal Control Theory

Optimal control is commonly used today as a way of optimizing a large variety of problems in different fields. One example concerning *stochastic optimal control* is its utilization in finance. In finance, *stochastic control theory* can be used to optimize an investors return on assets with regards to the investors preferences and assumptions. This sort of optimization was first introduced by *Robert C. Merton* who stated and solved the famous *"Merton's Portfolio Problem"* (Merton, 1971), which will be discussed in coming sections. Since then, the method has been used by institutional investors such as mutual funds, banks and pension funds and individual investors to optimize their asset allocation.

There are two main approaches to this method: the traditional approach, and the modern approach. The traditional approach uses dynamic programming, pioneered by Bellman, to solve the stochastic problems utlizing the *Hamilton-Jacobi-Bellman equation* (Bellman, 1957). There, the *optimal feedback control* can optimize the Hamiltonian in the *Hamilton-Jacobi-Bellman equation* and thus, a solution can be found. The *Hamilton-Jacobi-Bellman equation* will also be covered further in later sections.

The modern approach on the other hand uses martingales or the duality approach. By using the martingale representation of wealth, the more direct martingale approach can be used to solve the problem. This approach is however more complicated for incomplete markets as it does not conclude a unique solution whereas the traditional method can be used for both complete and incomplete markets. As most economic market are considered as incomplete markets, the traditional method is most likely to be favoured (Fahri and Werning, 2016).

Optimal control problems can take form in many ways and in many scientific fields. In general, the components regarding these sorts of problems are (Kashif, 2016):

State Variable:
The state variable follows a Markovian structure and can only be affected by the control variables. It provides the minimum required information in order to properly describe the problem. For example, in finance, the state variable usually represents the wealth, $R(t)$, by a stochastic differential equation. However, it is possible to consider more than one state variable and these cases often include interest rate, inflation etc.

Control Processes:
The controls themselves are chosen by the optimizer to solve the optimization problem. Moreover, the control variables take certain values for each instance, t, and those variables that satisfy the constraints put on them are called admissible controls, which usually is represented by, $\mathcal{U}_{ad}$.

The Objective Functional:
The objective can either be to maximize or to minimize a trait over all admissible controls. Examples of this could be maximizing the utility or minimizing the energy of a system.

Furthermore, to get a better sense of what *optimal control theory* really is, a more formal formulation could be beneficial:

Consider the following system,

$$\dot{x} = f(x,u), \quad x(0) = x_0, \quad u \in \mathcal{U}_{ad}, \quad t \in [0,T].$$

From here, to compare the different controls, u, and to evaluate which control in fact is the *optimal control*, a *cost function* which acts as the *objective functional* is introduced as

$$\mathcal{J}(u) = \int_0^t \mathscr{L}(x(t), u(t))dt. \tag{2.2.1}$$

The objective is to minimize the functional, $\mathcal{J}$, which here is denoted as the integral of the lagrangian, $\mathscr{L}$ (the running cost), from 0 to t, by choosing the optimal, u. The "best" control, u, will be considered the *optimal control* and the corresponding curve, $x(u)$, will be considered the *optimal trajectory*.

3

The Brachistochrone Problem

The Brachistochrone problem has a long and interesting history. To begin with, the word itself stems from two greek words: brachisto (/brə'kıs tə/) and chronos (/'krɒn oƱ/). Brachisto is Greek for *shortest* and, chronos is Greek for *time*. Thus, the problem can be referred to as *the shortest time problem*. It is well-known that the solution to the shortest path from *point A* to *point B* (*see figure 1*) is the hypotenuse but what path that yield the shortest time is more complex.

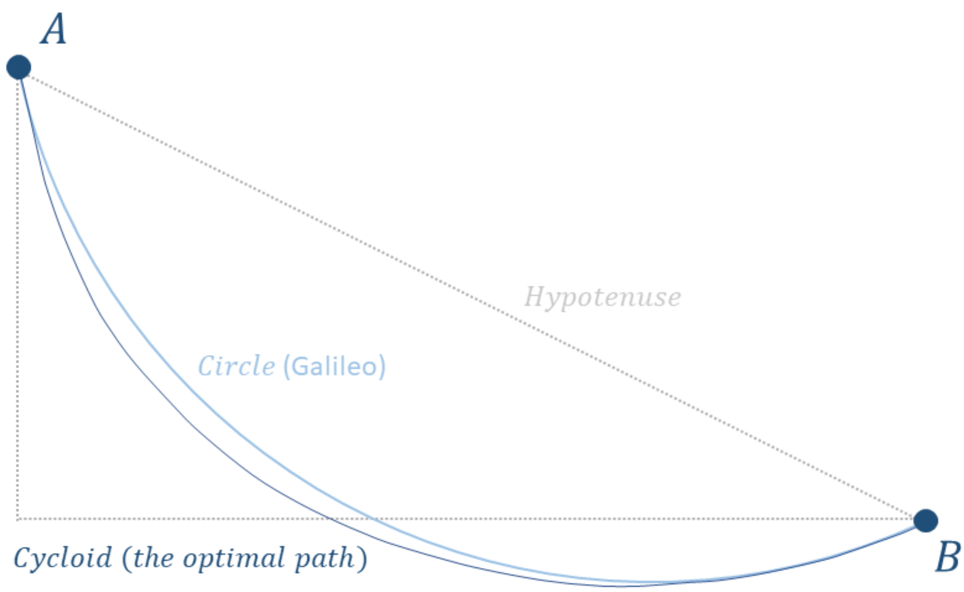

Figure 1: *The brachistochrone problem visualized with Galileo's solution and the optimal path (cycloid).*

The official problem was first posed by Johann Bernoulli in 1696 in the renowned *Acta Eruditorium*. Translated, it is officially formulated as:

"*Given in a vertical plane two points A and B, assign to the moving [body] M, the path AMB, by means of which — descending by its own weight and beginning to be moved [by gravity] from point A — it would arrive at the other point B in the shortest time* (Bernoulli, 1696)."

The problem, however, did not originate with Bernoulli's formulation but can be traced back to *Galileo Galilei*, who in 1638 tried to solve a problem with a simi-lair formulation (Galilei, 1638). Although Galileo came close to finding the optimal path, he did not fully succeed. He posed that the arc of a circle would yield the fastest time, which is quite close (*see figure 1*).

In conjunction with Bernoulli's formulation of the problem, he posed a challenge to other inventive mathematician to solve the problem even though he already had solved it himself. The mathematicians he challenged were given six months to solve the problem but when the deadline was due, no answers to the problem had been submitted. The prominent mathematician *Gottfried Leibniz* famously asked for an extension of the deadline which was approved, setting the deadline approximately one year into the future (Sagan, 2011). Around the same time *Isaac Newton* found the challenge in a letter addressed to himself. *Newton* successfully solved the problem the same night as he received the challenge and then went on to submit his solution anonymously. *Johann Bernoulli* then famously exclaimed that he recognized the mysterious solver, "*tanquam ex ungue leonem*", which more or less means "*as the lion by his claw*" as he was positive that *Newton* was behind the solution just by looking at it (Sagan, 2011).

By the end of the second deadline, five correct solutions had been submitted by *Isaac Newton*, *Jakob Bernoulli* (Johann Bernoulli's brother), *Gottfried Leibniz*, *Ehrenfried Walther von Tschirnhaus* and, *Guillaume de l'Hôpital*, respectively. *Jakob*

Bernoulli's solution to the problem became a steppingstone in developing *calculus of variation* which was later on refined and expanded on by *Leonhard Euler* (Shafer, 2007).

4

Merton's Portfolio Problem

Merton's portfolio problem is a well-known problem which was originally stated by the prolific and Nobel Price-decorated economist and mathematician, *Robert Cox Merton* (Merton, 1969). *Merton* is, among many things, famous for his pioneering work on the *Black-Scholes-Merton model* used in continuous-time finance to price options (Merton, 1973). This problem, however, does not concern options but instead considers an investor who wants to maximize the utility of his or her wealth.

The way the investor can do this is by choosing the optimal proportion of risk-free and risky assets with respect to the investors risk aversion. The risk aversion can be observed using different models. This problem most often, and will in the solution later, considers a *constant relative risk aversion* (CRRA). Moreover, the problem itself is solved using *optimal control* and will in *section 9* be solved using the *Hamilton-Jacobi-Bellman equation.* This problem makes for a good example showing how *control theory* or rather *optimal control theory* can be used. However, the world of investing is more complex than just choosing between two assets rendering the practical usage of the solution debatable, but Merton's work on the problem remains ground-breaking and leaves a staggering amount of development to be made henceforth.

5

Hamilton's Principle and the Euler-Lagrange Equation

5.1 Background

The principle of least action has become one of the most central ideas in many scientific fields. It is not only used in grand theories such as *quantum mechanics* and the *theory of relativ*ity, but it is also used in *control theory* to, for instance, optimize the problems at hand. The first mathematician to define the principle was, *Pierre Louis Maupertius*, who defined the action as (Svensson, 2015):

$$\int v ds, \tag{5.1.1}$$

where v is the conjugate momenta of the generalized coordinates, s. The idea is that the definition can find the "correct way" by minimizing the action. However, Maupertius only sought to apply this method to light, not matter, and derived the formulation by considering *Fermat's principle* which states that light always follows the path that takes the shortest time. At approximately the same time in history *Leonhard Euler* instead formulized the action as:

$$\int mv ds, \tag{5.1.2}$$

where, m, is the weight of the matter, v, the velocity and ds, an infinitesimal distance. Unlike Maupertius principle, Euler therefore took into account that the action can affect matter, thus rendering Euler's definition suitable for application in mechanical systems. None of these are, however, used in a broader sense today. Instead the more modern formulation by Hamilton is used where the integral instead considers the Lagrangian, $\mathscr{L}$, from some time, t_1, to some time, t_2. This formulation is usually denoted by S and thus the action is formulized as:

$$S = \int_{t_1}^{t_2} \mathscr{L}(q, \dot{q}, t)dt, \tag{5.1.3}$$

where $\dot{q} = \frac{dq}{dt}$. *Hamilton's principle* further states that the dynamics of a mechanical system is specified by the condition that the action S has a stationary value, i.e.

$$\delta S = 0. \tag{5.1.4}$$

Hence, saying that it satisfies the condition of least action is somewhat of a misnomer as the action itself is not required to be minimum but rather take the form of a stationary value. Therefore, it would be more accurate calling it the principle of stationary action which is the name the principle often takes in modern literature.

5.2 The Euler-Lagrange Equation

Lemma 5.1 (Fundamental Lemma of Calculus of Variations)**.** *If a continuous function f on an open interval (a, b) satisfies the equality*

$$\int_a^b f(x)g(x)dx = 0 \tag{5.2.1}$$

for all compactly supported smooth functions g on (a, b), then f is equal to 0 *as well.*

Theorem 5.2. *For a curve $q(t)$, in $C^2_{[t_1,t_2]}$, that minimizes the functional*

$$\mathcal{J}[q(t)] = \int_{t_1}^{t_2} \mathscr{L}(q, \dot{q}, t)dt \tag{5.2.2}$$

the following differential equation must be satisfied:

$$\frac{\partial \mathscr{L}}{\partial t} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = 0 \tag{5.2.3}$$

which is what is called the Euler-Lagrange equation*.*

Proof. Presume that $q(t)$ is a curve which minimizes the functional, $\mathcal{J}$, i.e. for any other permissible curve, $r(t)$, $\mathcal{J}[q(t)] \leq \mathcal{J}[r(t)]$. The idea from here is to construct a function of a real variable such as $\Xi(\epsilon)$, which has the following properties:

(i) $\Xi(\epsilon)$ is a differentiable function near, $\epsilon = 0$.

(ii) $\Xi(0)$ is a local minimum for Ξ.

After the construction of Ξ, (ii) will be shown to imply that the *Euler-Lagrange equation* must be satisfied. To begin with, let ϵ be a small real number and consider the construction of variation of $q(t)$ as

$$q_\epsilon(t) = q(t) + \epsilon\xi(t), \tag{5.2.4}$$

where $\xi(t) \in C^2_{[t_1,t_2]}$ and $\xi(t_1) = \xi(t_2) = 0$. Now, the function Ξ can be defined as

$$\Xi(\epsilon) = \mathcal{J}[q_\epsilon(t)]. \tag{5.2.5}$$

Since $q(t)$ minimizes $\mathcal{J}[q(t)]$, it follows that $\Xi(\epsilon)$ is minimized by 0. Furthermore, as $\Xi(0)$ is a minimum for Ξ, $\Xi'(0) = 0$, must be true. Now, Ξ can be differentiated by using *Leibniz rule* (i.e. $\frac{d}{dx}\left(\int_a^b f(x,t)dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)dt$, where a and b are constants) accordingly:

$$\begin{aligned} \frac{d}{d\epsilon}\left(\Xi(\epsilon)\right) &= \frac{d}{d\epsilon}\int_{t_1}^{t_2} \mathscr{L}(t, q_\epsilon, \dot{q}_\epsilon)dt \\ &= \int_{t_1}^{t_2} \frac{\partial}{\partial \epsilon}\mathscr{L}(t, q_\epsilon, \dot{q}_\epsilon)dt. \end{aligned} \tag{5.2.6}$$

Then, by applying the chain rule to $\frac{\partial}{\partial \epsilon}\mathscr{L}(t, q_\epsilon, \dot{q}_\epsilon)dt$ from (5.2.5) and denoting $\mathscr{L}(t, q_\epsilon, \dot{q}_\epsilon)$ as $\mathscr{L}_\epsilon$, the following is obtained:

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon}\mathscr{L}_\epsilon(t, q_\epsilon, \dot{q}_\epsilon) &= \frac{\partial \mathscr{L}_\epsilon}{\partial t}\frac{dt}{d\epsilon} + \frac{\partial \mathscr{L}_\epsilon}{\partial q_\epsilon}\frac{dq_\epsilon}{d\epsilon} + \frac{\partial \mathscr{L}_\epsilon}{\partial \dot{q}_\epsilon}\frac{d\dot{q}_\epsilon}{d\epsilon} \\
&= \frac{\partial \mathscr{L}_\epsilon}{\partial q_\epsilon}\frac{dq_\epsilon}{d\epsilon} + \frac{\partial \mathscr{L}_\epsilon}{\partial \dot{q}_\epsilon}\frac{d\dot{q}_\epsilon}{d\epsilon} \\
&= \frac{\partial \mathscr{L}_\epsilon}{\partial q_\epsilon}\xi(t) + \frac{\partial \mathscr{L}_\epsilon}{\partial \dot{q}_\epsilon}\dot{\xi}(t)
\end{aligned}
\tag{5.2.7}
$$

From here it is possible to rearrange (5.2.5) as

$$
\frac{d}{d\epsilon}\Xi(\epsilon) = \int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}_\epsilon}{\partial q_\epsilon}\xi(t) + \frac{\partial \mathscr{L}_\epsilon}{\partial \dot{q}_\epsilon}\dot{\xi}(t) \right) dt. \tag{5.2.8}
$$

Evaluating the equation at $\epsilon = 0$ entail $q_\epsilon = q$ and $\mathscr{L}_\epsilon = \mathscr{L}$ yields

$$
\frac{d}{d\epsilon}\Xi(\epsilon)\bigg|_{\epsilon=0} = \int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial q}\xi(t) + \frac{\partial \mathscr{L}}{\partial \dot{q}}\dot{\xi}(t) \right) dt = 0. \tag{5.2.9}
$$

Next, integrating the 2$^{\text{nd}}$ term of the integrand gives

$$
\int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \right) \xi(t)dt + \left[\xi(t)\frac{\mathscr{L}}{\partial \dot{q}} \right]_{t_1}^{t_2} = 0, \tag{5.2.10}
$$

which by applying the boundary condition, $\xi(t_1) = \xi(t_2) = 0$, yields

$$
\int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \right) \xi(t)dt = 0 \tag{5.2.11}
$$

From here, by applying lemma 5.1 (*the fundamental lemma of calculus of variations*), *the Euler-Lagrange equation* is given

$$
\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) = 0, \tag{5.2.12}
$$

ending the proof (Fischer, 1999).

□

5.3 Deriving the Euler-Lagrange Equation from Hamilton's Principle

Recall the formulization of *Hamilton's principle* of least action. To avoid unnecessary technicality, we assume that $\mathscr{L}$ and q are sufficiently smooth.

$$S = \int_{t_1}^{t_2} \mathscr{L}(q, \dot{q}, t)dt. \tag{5.3.1}$$

Assume that the state the particle occupies is fixed at time t_1 and t_2. Then further assume that q, and $\dot{q}$ may vary in-between t_1 and t_2. In that case the particle can follow different trajectories and thus generate different values for S. Denoting these variations as δq, $\delta\dot{q}$ and δS, it is possible to write:

$$\delta S = \int_{t_1}^{t_2} \mathscr{L}(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_1}^{t^2} \mathscr{L}(q, \dot{q}, t)dt. \tag{5.3.2}$$

According to *Hamilton's principle* the motion of the particle is, $\delta S = 0$, to the 1^{st} order in variation considering, δq and $\delta\dot{q}$. In other words, the motion of the particle is not affected by small variations. Next, by using *Taylor's theorem* it is possible to write the 1^{st} order approximation as:

$$\mathscr{L}(q + \delta q, \dot{q} + \delta\dot{q}, t) \approx \mathscr{L}(q, \dot{q}, t) + \frac{\partial \mathscr{L}}{\partial q}\delta q + \frac{\partial \mathscr{L}}{\partial \dot{q}}\delta\dot{q}. \tag{5.3.3}$$

Accordingly, it is possible to rewrite (5.3.2) as

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial q}\delta q + \frac{\partial \mathscr{L}}{\partial \dot{q}}\delta\dot{q} \right). \tag{5.3.4}$$

Thus, as $\dot{q} = \frac{dq}{dt}$ and, $\delta\dot{q} = \frac{d(\delta q)}{dt}$, the 2^{nd} term in the integral can be rewritten as

$$\int_{t_1}^{t_2} \frac{\partial \mathscr{L}}{\partial \dot{q}}\delta\dot{q} = \int_{t_1}^{t_2} \frac{\partial \mathscr{L}}{\partial \dot{q}} \frac{d(\delta q)}{dt} dt. \tag{5.3.5}$$

By integrating by parts, the term can be rewritten as

$$\frac{\partial \mathscr{L}}{\partial \dot{q}}\delta q\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} \cdot \delta q dt. \tag{5.3.6}$$

Furthermore, as the initial and final state are fixed, δq can be removed at both states making it possible to rewrite (5.3.4) as

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \right) \delta q dt = 0. \tag{5.3.7}$$

As δq remains arbitrary the terms inside the brackets must be equal to each other in order to satisfy the equality (lemma 5.1). Thus,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) - \frac{\partial \mathscr{L}}{\partial q} = 0, \tag{5.3.8}$$

must be true, which is the formulation commonly known as the *Euler-Lagrange equation*.

6

Pontryagin's Minimum (Maximum) Principle

6.1 Background

The maximum (minimum) principle was first discovered in the mid-1950's at the Steklov Mathematical Institute, Academy of Sciences of the USSR. The researcher who was credited the discovery, or rather, the formulation, was *Lev Semenovich Pontryagin*. Pontryagin and some of his younger students were recognized all over the world for their impressive scientific achievement and the formulation would come to be pivotal for the subject of optimal control (Gamkrelidze, 2019).

The principle is used in *optimal control theory* to find the *optimal control* for taking a dynamical system from one state to another. *Pontryagin's maximum principle*, in its essence state that it is, with a maximum condition of the control Hamiltonian, necessary for any *optimal control*, with the *optimal state trajectory* to solve the Hamiltonian system. The principle was first used to maximize the terminal speed of rockets and has since been used in a wide variety of domains (Fuller, 1963). Worth noting is that the principle can be used for both maximization and minimization which solely depends on whether the *objective functional* is positive or negative.

6.2 Formulation

To begin with, to use the *fundamental lemma of calculus of variation* as $\delta S = 0$, where dynamic situations are considered, $\mathcal{J}$ (the objective functional), has to be rewritten. Impose, $\rho_1(t), ..., \rho_n(t)$. Then define

$$\begin{aligned}\mathcal{J}_a(u) =&\Psi(x(t_f)) + \int_0^{t_f} \mathscr{L}(x(t), u(t))dt + \int_0^{t_f} \rho^T(f - \dot{x})dt \\ =&\Psi(x(t_f)) + \int_0^{t_f} (\mathscr{L} + \rho^T f)dt - \left[\rho^T x\right]_0^{t_f} + \int_0^{t_f} (\dot{\rho}^T x)dt. \qquad (6.2.1)\end{aligned}$$

(by partial integration)

From here, two situations will be considered, firstly when u is not constrained and, secondly when u is constrained.

(i) *u is not constrained:*

Impose,

$$\mathcal{H}(x, \rho, u) = \mathscr{L}(x, u) + \rho^T f(x, u), \qquad (6.2.2)$$

where $\mathcal{H}$ is the associated hamiltonian, $\mathscr{L}$ the adjointed lagrangian, and ρ is a time-varying vector. This permits the following:

$$
\begin{aligned}
\mathcal{J}_a(u+\delta u) &= \Psi((x+\delta x)(t_f)) - \Big[\rho^T(x+\delta x)\Big]_0^{t_f} \\
&\quad + \int_0^{t_f} \Big(\mathcal{H}(x+\delta x, \rho, u+\delta u) + \dot{\rho}^T(x+\delta x)\Big)\,dt - \Psi(x(t_f)) \\
&\quad + \Big[\rho^T x\Big]_0^{tf} - \int_0^{t_f} \Big(\mathcal{H}(x,\rho,u) + \dot{\rho}^T x\Big)\,dt \\
&= \Psi((x+\delta x)(t_f)) - \Psi(x(t_f)) - \Big[\rho^T \delta x\Big]_0^{t_f} \qquad (6.2.3) \\
&\quad + \int_0^{t_f} (\mathcal{H}(x+\delta x, \rho, u+\delta u) - \mathcal{H}(x,\rho,u))dt + \int_0^{t_f} \dot{\rho}^T \delta x dt \\
&\approx \Psi_x(x(t_f))\delta x - \rho^T(t_f)\partial x(t_f) + \rho^T(0)\delta x(0) \\
&\quad + \int_0^{t_f} (\mathcal{H}_x \delta x + \mathcal{H}_u \delta u + \dot{\rho}^T \delta x)dt.
\end{aligned}
$$

From here it is possible to choose, $\dot{\rho}^T = -\mathcal{H}_x$ and, $\rho^T(t_f) = Q_x(x(t_f))$, as long as $x(t_f)$ is free. Then,

$$\delta \mathcal{J}_a = \int_0^{t_f} \mathcal{H}_u \delta u dt. \qquad (6.2.4)$$

If, u^*, is optimal, then $\delta \mathcal{J}_a = 0$, which entail that, $\mathcal{H}_u = 0$.

Theorem 6.1 (Pontryagin's minimum principle)**.** *If (x, u), is an optimal solution, considering a free final state, then $\rho \not\equiv 0$, exists such that:*

$$
\begin{aligned}
&(i) \quad \dot{x} = (f(x,u), x(t)) = x_0 \qquad (\textit{where } x(t_f) \textit{ is free}) \\
&(ii) \quad \dot{\rho}^T = -\mathcal{H}_x = \mathscr{L}_x(x,u) + \rho^T f_x(x,u), \quad \rho(t_f) = Q(x(t_f)) \qquad (6.2.5) \\
&(iii) \quad \mathcal{H}_u = 0 \leftrightarrow \mathscr{L}_u(x,u) + \rho^T f_u(x,u) = 0.
\end{aligned}
$$

If instead, the state final state is considered as fixed, then, (x, u) is an optimal solution when, $(\dot{\rho}(t), \rho_0 \not\equiv 0)$, with, ρ_0 being a non-negative constant. This instead results in:

$$
\begin{aligned}
&(i) \quad \dot{x} = f(x,u), \quad x(0) = x_0, \quad x(t_f) = \overline{x} \ \textit{(fixed)} \\
&(ii) \quad \dot{\rho}^T = -\mathcal{H}_x \\
&(iii) \quad \mathcal{H}_u = 0
\end{aligned} \tag{6.2.6}
$$

where, $\mathcal{H}(x,\rho,u) = \rho_0 \mathscr{L}(x,u) + \rho^T f(x,u)$.

Comment:

Note that ρ_0 is needed to counteract inconsistency as, $x(t_f)$ is unable to reach, $\overline{x}$, see example 6.2.

Example 6.2. (*ill-conditioned problem)*
With, $\dot{x}(t) = u^2$, $x(0) = 0$ and $x(1) = 0$ consider

$$
\mathcal{J} = \int_0^1 u(t)dt. \tag{6.2.7}
$$

Here, it is easy to see that $u = 0$ is a solution but with, $\rho_0 = 1$, Pontryagin's minimum principle will not work. However, $\lambda_0 = 0$ is fine.

(ii) *u is constrained:*
What separates the constrained form and the non-constrained form is by and large that it is no longer possible to assume, $\delta S = 0$, but instead, $\delta S \geq 0$ has to be assumed. This leads to a modification in the derivation as:

$$
\begin{aligned}
&\int_0^{t_f} \left(\mathcal{H}(x+\delta x, \rho, u+\delta u) - \mathcal{H}(x,\rho,u)\right) dt \\
&= \int_0^{t_f} \left(\mathcal{H}(x+\delta x, \rho, u+\delta u) - \mathcal{H}(x,\rho,u+\delta u)\right) dt \\
&\quad + \int_0^{t_f} \left(\mathcal{H}(x,\rho,u+\delta u) - \mathcal{H}(x,\rho,u)\right) dt \\
&\approx \int_0^{t_f} \mathcal{H}_x \delta x dt + \int_0^{t_f} \left(\mathcal{H}(x,\rho,u+\delta u) - \mathcal{H}(x,\rho,u)\right) dt.
\end{aligned} \tag{6.2.8}
$$

Just as before,

$$\delta \mathcal{J}_a = \int_0^{t_f} \left(\mathcal{H}(x, \rho, u + \delta u) - \mathcal{H}(x, \rho, u) \right) dt, \tag{6.2.9}$$

which is non-negative. This now yield

$$\begin{aligned} &\mathcal{H}(x, \rho, u + \delta u) - \mathcal{H}(x, \rho, u) \geq 0 \\ &(\forall\ \delta u,\ \textit{such that}\ u + \delta u \in \mathcal{U}_{ad}\ \textit{and}\ , \forall\ t \in [0, t_f]). \end{aligned} \tag{6.2.10}$$

This frames *Pontryagin's minimum principle* as:

If, (x, u), is an optimal solution for, $u \in \mathcal{U}_{ad}$, then, $\rho(t) \not\equiv 0$, such that the conditions for *Pontryagin's minimum principle* can be written as

$$\begin{aligned} \dot{x} =& f(x, u), \quad (x(t_0) = x_0\ \textit{fixed}) \\ \dot{\rho}^T =& -\mathcal{H}, \quad (\rho^T(t_f) = Q_x(x(t_f))) \\ \mathcal{H}(x, \rho, u) \geq& \mathcal{H}(x, \rho, u), \quad (\forall\ u \in \mathcal{U}_{ad}, 0 \leq t \leq t_f) \end{aligned} \tag{6.2.11}$$

or

$$\begin{aligned} \dot{x} =& f(x, u), \quad (x(t_0) = x_0, x(t_f) = \overline{x}\ ,\ \textit{fixed}) \\ \dot{\rho}^T =& -\mathcal{H} \\ \mathcal{H}(x, \rho, u) \geq& (x, \rho, u), \quad (\forall\ u \in \mathcal{U}_{ad}, 0 \leq t \leq t_f). \end{aligned} \tag{6.2.12}$$

Comments:

(i) In this formulation, $t_0 = 0$ is assumed but it is not necessary.

(ii) The *Euler-Lagrange equation* discussed earlier is a special case of *Pontryagin's minimum principle* which is given by assuming that, $u = \dot{q}$. Then, the following is given

$$\min \int_{t_0}^{t_1} \mathscr{L}(q, \dot{q}) dt \tag{6.2.13}$$

with, $\dot{q}$, and, $(((q(t_0), q(t_1))$ *fixed*). Then,

$$\mathcal{H}(q, \rho, u) = \mathscr{L}(q, u) + \rho^T u, \tag{6.2.14}$$

gives

$$0 = \mathcal{H}_u = \mathscr{L}_u(q, u) + \rho^T \tag{6.2.15}$$

and

$$\dot{\rho}^T = -\mathcal{H}_q = -\mathscr{L}_q(q, u) \tag{6.2.16}$$

which in turn yield

$$-\dot{\rho}^T = \frac{d}{dt}\left(\mathscr{L}_u(q, u)\right) = \mathscr{L}(q, u) \leftrightarrow \frac{d}{dt}\left(\mathscr{L}_u(q, \dot{q})\right) + \mathscr{L}_q(q, \dot{q}) = 0, \tag{6.2.17}$$

which indeed is the *Euler-Lagrange equation.*

(iii) The Hamiltonian is constant along the trajectories, $\dot{x} = f(x, u)$.

7

The Hamilton-Jacobi-Bellman Equation

7.1 Background

The *Hamilton-Jacobi-Bellman equation* has a unique and complex history as many prolific mathematicians' individual work came to create one of the most important equations in *optimal control theory*. The foundation for the equation was laid out by the two famous mathematicians *Pierre de Fermat* and *Christiaan Huygens*, who researched the properties of geometrical optics. Deriving from their research Hamilton formed what is called "Hamiltonian Dynamics/Hamiltonian mechanics" which is integral in the formulation of the *Hamilton-Jacobi-Bellman equation*.

Further additions to this theory came from *Carl Gustav Jacob Jacobi* who sharpened the theory and made significant additions to it. The resulting theory, called the Hamilton-Jacobi equation, is used in a multitude of ways. Examples of this includes modern PDE theory and wave-particle duality (Bahram, 2020).

The third and last contribution to the theory was developed by the mathematician *Richard Bellman*. Bellman pioneered the concept of *dynamic programming* which is an optimization method that since has found applications in many fields such as aerospace engineering and financial economics (Ross, 1995). The method itself is built on simplifying complex problems by breaking the problem down recursively and thus, creating simpler sub-problems. This can for example be used to minimize cost functions over continuous time intervals. However, it was never Bellman's intention to continue building the theory created by Hamilton and Jacobi. Bellman did his research on his own and it was first later when the mathematician *Rudolf Kálmán* realised the natural connection between the works of Bellman, Hamilton and Jacobi and formed the equation that is called the *Hamilton-Jacobi-Bellman equation* (Kálmán, 1963).

7.2 Formulation

The idea behind Bellman's *dynamic programming* is to embed the *optimal control* problem in a larger class of problems. The problem at hand is

$$\min \mathcal{J}(u) = \int_0^{t_f} \mathscr{L}(x(t), u(t))dt + \Psi(x(t_f)) \tag{7.2.1}$$

where, t_f is the final state of a continuous time interval, Ψ is the endpoint cost, $\mathscr{L}$ is the running cost, and

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \quad u \in \mathcal{U}_{ad}. \tag{7.2.2}$$

From here, instead of solving (7.2.1) directly, let $t_f > 0$ be fixed and consider

$$\begin{cases} \dot{x}(s) = f(x(s), u(s)), & 0 \leq t \leq s \leq t_f \\ x(t) = x, & \forall x \in \mathbb{R}^n \end{cases} \tag{7.2.3}$$

with

$$\mathcal{J}_{x,t}(u(\cdot)) = \int_t^{t_f} \mathscr{L}(x(s), u(s))ds + \Psi(x(t_f)). \tag{7.2.4}$$

From here the definition of the *Hamilton-Jacobi-Bellman equation* can be formulated.

Definition 7.1. For all $x \in \mathbb{R}^n$ and $0 \leq t \leq t_f$, the *value function*, $V(x,t)$, is given by

$$V(x,t) = \min_{u(\cdot)\in\mathcal{U}_{ad}} \mathcal{J}_{x,t}(u(\cdot)) \tag{7.2.5}$$

for all trajectories given by (7.2.2) whereas

$$V(x,t_f) = \Psi(x,t_f). \tag{7.2.6}$$

7.3 Derivation

(i) To begin with, consider

$$\forall\ x \in \mathbb{R}^n, \quad 0 \leq t \leq t_f, \quad \Delta t > 0 \tag{7.3.1}$$

where Δt, is arbitrarily small. Then take an arbitrary *control*, $\alpha \in \mathcal{U}_{ad}$, i.e. $u(\cdot) = \alpha$. Apply, $u(\cdot) = \alpha$ on $t \leq s \leq t+\Delta t < t_f$. Then, with, $\dot{x}(s) = f(x(s),\alpha)$ and $x(t) = x$, the *cost* for the time interval $[t, t+\Delta t]$ can be represented by

$$\int_t^{t+\Delta t} \mathscr{L}(x(s),\alpha)ds. \tag{7.3.2}$$

Furthermore, the *cost* for, $s \in [t+\Delta t, t_f]$, can be considered as

$$V(x(t+\Delta t), t+\Delta t), \tag{7.3.3}$$

as per definition of the *value function*. With (7.3.2) and (7.3.3) the total cost can now be described as

$$\int_t^{t+\Delta t} \mathscr{L}(x(s),\alpha)ds + V(x(t+\Delta t), t+\Delta t). \tag{7.3.4}$$

which makes the least possible cost, starting from (x,t), $V(x,t)$. Thus,

$$V(x,t) \leq \int_t^{t+\Delta t} \mathscr{L}(x(s),\alpha)ds + V(x(t+\Delta t), t+\Delta t). \tag{7.3.5}$$

(ii) Now, to get the differential form of (7.3.5), the inequality is rearranged as

$$\frac{V(x(t+\Delta t), t+\Delta t) - V(x,t)}{\Delta t} + \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathscr{L}(x(s),\alpha)ds \leq 0. \tag{7.3.6}$$

By letting, $\Delta t \to 0$, (7.3.6), can be written as

$$V_t(x,t) + V_x(x,t)\cdot \dot{x}(t) + \mathscr{L}(x(t),\alpha) \leq 0. \tag{7.3.7}$$

It is clear that, x solves the ODE $\dot{x}(s) = f(x(s),\alpha)$ for $(t \leq s \leq t+\Delta t)$ with $x(t) = x$. Applying this in (7.3.7) results in

$$V_t(x,t) + V_x(x,t)\cdot f(x,\alpha) + \mathscr{L}(x(t),\alpha) \leq 0. \tag{7.3.8}$$

Then, for all *controls*, $\alpha \in \mathcal{U}_{ad}$:

$$\min_{\alpha\in\mathcal{U}_{ad}} \{V_t(x,t) + V_x(x,t)\cdot f(x,\alpha) + g(x,\alpha)\} \leq 0. \tag{7.3.9}$$

(iii) Now, to show that the inequality in fact is an equality, assume that, $u^*(\cdot)$ and $x^*(\cdot)$ are optimal for (7.3.9). Then, by using the *optimal control*, $u^*(\cdot)$, the *optimal cost*, for $t \leq s \leq t+\Delta t$, can be considered as

$$\int_t^{t+\Delta t} \mathscr{L}(x^*(s),u^*(s))ds \tag{7.3.10}$$

where the rest of the cost is $V(x^*(t+\Delta t), t+\Delta t)$, which make the total cost

$$\int_t^{t+\Delta t} \mathscr{L}(x^*(s),u^*(s))ds + V(x^*(t+\Delta t), t+\Delta t) = V(x^*,t). \tag{7.3.11}$$

(7.3.11) can then be rewritten as

$$\frac{V(x^*(t+\Delta t), t+\Delta t) - V(x,t)}{\Delta t} + \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathscr{L}(x^*(s), u^*(s))ds = 0. \qquad (7.3.12)$$

Now, yet again, let $\Delta t \to 0$, where $u^*(t) = \alpha \in \mathcal{U}_{ad}$, which leads to

$$\begin{aligned} V_t(x,t) + V_x(x,t) \cdot \dot{x}^*(t) + \mathscr{L}(x^*, \alpha^*) = 0 \to \\ V_t(x^*,t) + V_x(x^*,t) \cdot f(x^*,t) + \mathscr{L}(x^*, \alpha^*) = 0 \end{aligned} \qquad (7.3.13)$$

for any, $\alpha^* \in \mathcal{U}_{ad}$. From the above derivation, the following necessary condition for optimality is obtained.

Theorem 7.2. *Assume that, $V(x,t)$ is a C^1-function of the variables, (x,t). Then, V, solves* the Hamilton-Jacobi-Bellman equation *as*

$$V_t(x,t) + \min_{\alpha \in \mathcal{U}_{ad}} \{V_x(x,t) \cdot f(x,\alpha) + \mathscr{L}(x,\alpha)\} = 0 \qquad (7.3.14)$$

with the boundary condition

$$V(x,t_f) = \Psi(x(t_f)), x \in \mathbb{R}^n. \qquad (7.3.15)$$

Comment:

Let,

$$\mathcal{H}(x,\rho) = \min_{\alpha \in \mathcal{U}_{ad}} (x,\rho,\alpha) \qquad (7.3.16)$$

with

$$\mathcal{H}(x,\rho,\alpha) = \rho^T \cdot f(x,\alpha) + \mathscr{L}(x,\alpha), \quad (x,\rho \in \mathbb{R}^n) \qquad (7.3.17)$$

as the *Hamiltonian.* Furthermore, it is possible to design the *optimal cost* by first solving *the Hamilton-Jacobi-Bellman equation* (i.e. determining V), and thereafter determining a *feedback control*, $u^*(\cdot)$.

To find the feedback control, define for all, $x \in \mathbb{R}^n$, and, $0 \leq t \leq t_f$,

$$u(x,t) = \alpha \in \mathcal{U}_{ad}, \tag{7.3.18}$$

such that *the Hamilton-Jacobi-Bellman equation* is minimized. In other word, $u(x,t)$, is chosen such that,

$$V_t(x,t) + V_x(x,t) \cdot f(x, u(x,t)) + \mathscr{L}(x, u(x,t)) = 0. \tag{7.3.19}$$

Then, the ODE,

$$\begin{cases} \dot{x^*}(s) = f(x^*(s), u(s), s), & 0 \leq t \leq s \leq t_f \\ x(t) = x, & \forall x \in \mathbb{R}^n \end{cases} \tag{7.3.20}$$

which lastly, leads to the formulation of the *feedback control*

$$u^*(s) := u(x^*(s), s) \tag{7.3.21}$$

Moreover, it is also natural to ponder whether the *Hamilton-Jacobi-Bellman equation* is a sufficient condition for optimality. This is indeed the case and will be shown via the so called *verification theorem.*

Theorem 7.3 (Verification Theorem). *Recall (7.2.4) and consider its optimal form as*

$$\mathcal{J}_{x,t}(u^*(\cdot)) = \int_t^{t_f} \mathscr{L}(x^*(s), u^*(s))ds + \Psi(x^*(t_f))). \tag{7.3.22}$$

This enables the following rephrasing:

$$
\begin{aligned}
\mathcal{J}_{x,t}(u^*(\cdot)) &= \int_t^{t_f} (-V_t(x^*(s),s) - V_x(x^*(s),s))ds + \Psi(x^*(t_f))) \\
&= - \int_t^{t_f} \frac{d}{ds} V(x^*(s),s)ds + \Psi(x^*(t_f))) \\
&= - V(x^*(t_f),t_f) + V(x^*(t),t) + \Psi(x^*(t_f))) \\
&= V(x^*(t),t) = \min_{u(\cdot)\in\mathcal{U}_{ad}} \mathcal{J}_{x,t}(u^*(\cdot)),
\end{aligned} \tag{7.3.23}
$$

which concludes the verification theorem*.*

7.4 The Hamilton-Jacobi-Bellman Equation for Stochastic Optimal Control Problems

The *Hamilton-Jacobi-Bellman equation* must, moreover, in order to suit the stochastic character of control problems, be altered. Now instead, consider *Markov controls*

$$u = u(t, X_t(\omega)). \tag{7.4.1}$$

The system equation, as an Itô process (Øksendal, 2013) becomes

$$dY_t = b(Y_t, u(Y_t))dt + \sigma(Y_{,t}, u(Y_t))dB_t \tag{7.4.2}$$

and for $v \in U$, and $\phi \in C_0^2$, define

$$(\mathscr{L}^v\phi)(y) = \frac{\partial \phi}{\partial s}(y) + \sum_{i=1}^n b_i(y,v)\frac{\partial \phi}{\partial x_i} + \sum_{i,j=1}^n a_{i,j}(y,v)\frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{7.4.3}$$

Here, $a_{i,j} = \frac{1}{2}(\sigma\sigma^T)_{i,j}$, $y = (s,x)$ and, $x = x_1, ..., x_n$. Then for each, u, $Y_t = Y_t^u$, is an Itô diffusion with generator $\mathcal{A}$ given by

$$(\mathcal{A}\phi)(y) = (\mathscr{L}^{u(y)}\phi)(y), \phi \in C_0^2. \tag{7.4.4}$$

Theorem 7.4. *For, $v \in U$, define $f^v(y) = f(y,v)$ and suppose that $\Phi \in C^2(G) \cap C(\overline{G})$, where G is a fixed domain in $\mathbb{R} \times \mathbb{R}^n$. In that case*

$$E^y\left[|\Phi(Y_\alpha)| + \int_0^\alpha |\mathscr{L}^y\Phi(Y_t)|dt\right] < \infty \tag{7.4.5}$$

is satisfied for all bounded stopping times $\alpha \le \tau_G$, all $y \in G$ and all $v \in U$. Furthermore, suppose that u^*, the optimal Markov control, exist and that ∂G is regular for $Y_t^{u^*}$. In that case the value function, Φ, *satisfies*

$$\sup_{v\in U}\{f^v(y) + \mathscr{L}^v\Phi(y)\} = 0, \forall\ y \in \partial G \tag{7.4.6}$$

and

$$\Phi(y) = g(y), \forall y \in \partial G. \tag{7.4.7}$$

Thus, the supremum in (7.4.6) is found if $v = u^*(y)$, which means

$$f(y, u^*(y) + (\mathscr{L}^{u^*(y)}\Phi)(y) = 0, \forall\ y \in G \tag{7.4.8}$$

Proof. As $u^* = u^*(y)$, the following equality is given

$$\Phi(y) = J^{u^*}(y) = E^y\left[\int_0^{\tau_G} f(Y_s, u^*(Y_s))ds + g(Y_{\tau_G})\cdot\chi_{\{\tau_G<\infty\}}\right]. \tag{7.4.9}$$

If $y \in \partial G$ then $\tau_G = 0$ and (7.4.7) will follow. For the next part of the proof the solution for the stochastic Dirichlet-Poisson problem is used (Øksendal, 2013).

Assuming that

$$w(x) = E^x[\phi(X_{\tau_D})] + E^x\left[\int_0^{\tau_D} g(X_t)dt\right] \tag{7.4.10}$$

holds and, $\phi \in C(\partial D)$*(bounded)* and $g \in C(D)$ satisfy

$$E^x\left[\int_0^{\tau_D} |g(X_s)|ds\right] < \infty, \forall\ x \in D. \tag{7.4.11}$$

If

$$w(x) = E^x[\phi(X_{\tau_D})] + E^x\left[\int_0^{\tau_D} g(X_s)ds\right], x \in D \tag{7.4.12}$$

then

$$\mathcal{A}w = -g, x \in D \tag{7.4.13}$$

and

$$\lim_{t \uparrow \tau_D} w(X_t) = \phi(X_{\tau_D}), \forall\ x \in D. \tag{7.4.14}$$

which in the case of this theorem proves

$$(\mathscr{L}^{u^*(y)}\Phi)(y) = -f(y, u^*(y)), \forall\ y \in G\ \textit{(7.4.8)}. \tag{7.4.15}$$

Lastly, is the proof of (7.4.7). Have $y = (s, x) \in G$, a Markov control u and $\alpha \leq \tau_G$ being a bounded stopping time.

Since

$$J^u(y) = E^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}) \cdot \chi_{\{\tau_G < \infty\}}\right] \tag{7.4.16}$$

the strong Markovian property

$$E^x[\theta_\tau \eta | \mathcal{F}_\tau^{(m)}] = E^{X_\tau}[\eta], \tag{7.4.17}$$

where θ_τ is the *shift operator* ($\theta_\tau : \mathcal{H} \to \mathcal{H}$, $\mathcal{H}$ being the set of all real $\mathcal{M}_\infty$-measurable functions.), combined with

$$\theta_\alpha \eta \cdot \chi_{\{\alpha < \infty\}} = g(X_{\tau_H^\alpha})\chi_{\{\tau_H^\alpha < \infty\}}, \tag{7.4.18}$$

where $\eta = g(X_{\tau_H})\chi_{\{\tau_H < \infty\}}$ (g being a bounded continuous function on $\mathbb{R}^n$, $H \subset \mathbb{R}^n$ and τ_H is the first exit time from H for an Itô diffusion, X_t) and $\tau_H^\alpha = \inf\{t > \alpha; X_t \notin H\}$, together with

$$\theta_\tau \eta = \int_\tau^{\tau_D} g(X_s)ds, \tag{7.4.19}$$

gives:

$$\begin{aligned}
E^y[J^u(Y_\alpha)] = E^y &\left[E^{Y_\alpha}\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}\chi_{\{\tau_G<\infty\}}\right]\right] \\
=&E^y\left[E^y\left[\theta_\alpha\left(\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}\chi_{\{\tau_G<\infty\}}\right)|\mathcal{F}_\alpha\right]\right] \\
=&E^y\left[E^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}\chi_{\{\tau_G<\infty\}}|\mathcal{F}_\alpha\right]\right] \\
=&E^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}\chi_{\{\tau_G<\infty\}} - \int_0^\alpha f^u(Y_r)dr\right] \\
=&J^u(y) - E^y\left[\int_0^\alpha f^u(Y_r)dr\right].
\end{aligned} \tag{7.4.20}$$

Meaning that

$$J^u(y) = E^y\left[\int_0^\alpha f^u(Y_r)dr\right] + E^y[J^u(Y_\alpha)]. \tag{7.4.21}$$

Furthermore, let $W \subset G$ be of the form $W = \{(r,z) \in G; r < t_1\}$ where $s < t_1$. Set $\alpha = \inf\{t \geq 0; Y_t \notin W\}$. The optimal control $u^*(y) = u^*(r,z)$ and let

$$u(r,z) = \begin{cases} v & \textit{if } (r,z) \in W \\ u^*(r,z) & \textit{if } (r,z) \in G \backslash W \end{cases} \tag{7.4.22}$$

Then

$$\Phi(Y_\alpha) = J^{u^*}(Y_\alpha) = J^u(Y_\alpha) \tag{7.4.23}$$

and thus, by combing (7.4.13) and (7.4.15) the following is obtained

$$\Phi(y) \geq J^u(y) = E^y\left[\int_0^\alpha f^u(Y_r)dr\right] + E^y[\Phi(Y_\alpha)]. \tag{7.4.24}$$

As $\Phi \in C^2(G)$, Dynkin's formula,

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right] \tag{7.4.25}$$

provides the following expression

$$E^y[\Phi(Y_\alpha)] = \Phi(y) + E^y\left[\int_0^\alpha (\mathscr{L}^v\Phi)(Y_r)dr\right], \tag{7.4.26}$$

which by substitution with (7.4.16) gives

$$\Phi(y) \geq E^y\left[\int_0^\alpha f^v(Y_r)dr\right] + \Phi(y) + E^y\left[\int_0^\alpha (\mathscr{L}^v\Phi)(Y_r)dr\right] \tag{7.4.27}$$

$$\Rightarrow \frac{E^y\left[\int_0^\alpha (f^v(Y_r) + (\mathscr{L}^v\Phi)(Y_r))dr\right]}{E^y[\alpha]} \leq 0, \quad \forall W \tag{7.4.28}$$

Letting, $t_1 \downarrow s$, and because $f^v(\cdot)$ and $\mathscr{L}^v\Phi(\cdot)$ are continuous at y, gives $f^v(y) + (\mathscr{L}^v\Phi(y)) \leq 0$. This, together with (7.4.8) provides (7.4.6) which completes the proof (Øksendal, 2013). $\square$

Theorem 7.5 (Verification Theorem)**.** *Let ϕ be a function in $C^2(G) \cap C(\overline{G})$, such that, $\forall\ v \in U$,*

$$f^v(y) + (\mathscr{L}^v\phi)(y) \leq 0, \quad y \in G \tag{7.4.29}$$

with the boundary condition

$$\lim_{t\to\tau G} \phi(Y_t) = g(Y_{\tau G}) \cdot \chi_{\{\tau G<\infty\}}, \qquad a.s.\ Q^y \tag{7.4.30}$$

and such that, $\{\phi^-(Y_\tau); \tau,$ stopping time, $\tau \leq \tau_G\}$ is uniformly Q^y-integrable for all Markov controls u and all, $y \in G$.

Furthermore, if for each $y \in G$, we have found $u_0(y)$, such that

$$f^{u_0(y)}(y) + (\mathscr{L}^{u_0(y)}\phi)(y) = 0 \tag{7.4.31}$$

and, $\{\phi(Y_\tau^{u_0}); \tau,$ stopping time, $\tau \leq \tau_G\}$ is uniformly Q^y-integrable for all, $y \in G$.

Then, $u_0 = u_0(y)$ *is a Markov control such that*

$$\phi(y) = J^{u_0}(y) \tag{7.4.32}$$

and hence, if u_0 *is admissible, then* u_0 *must be an* optimal control *and* $\phi(y) = \Phi(y)$.

7.5 The Difference Between the Hamilton-Jacobi-Bellman Equation and Pontryagin's Minimum Principle

As mentioned, both methods are used to solve *optimal control* problems but they do so in different ways. The *Hamilton-Jacobi-Bellman equation* solves for the *value function* by minimizing the cost associated with the problem. To find the *optimal control* all possible trajectories are considered which exhausts all possible solutions. Hence, the method itself creates both the sufficient and necessary condition for the found control to be optimal.

On the other hand, *Pontryagin's minimum principle* tests specific control candidates, which then must be tested for optimality. Consequently, this method does not necessarily find the optimal solution and might thus not be sufficient. In a way, *Pontryagin's minimum principle* is therefore much simpler as it solves an easier problem whereas the *Hamilton-Jacobi-Bellman equation* is more thorough and comprehensive, even though it sometimes can be surfeited.

8

Solving the Brachistochrone Problem

8.1 Using Calculus - Bernoulli's Solution

The solution that *Johann Bernoulli* posed was quite ingenious in itself as he successfully solved the problem indirectly using knowledge not directly tied to the problem at hand (Bernoulli, 1696). Bernoulli used the laws of light, more precisely its' ability to find the quickest route, to show how light would act in a similar situation as that of the of the Brachistochrone problem.

Consider a light particle traveling from point A, to point B, through two different mediums with the corresponding velocities v_1 and v_2. Then the time, t, it takes for the light to travel from point A to point B can, with the notations from *Figure 1*, be described as

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c - x)^2}}{v_2}, \tag{8.1.1}$$

where, x ($x \in [o, c]$), is a variable and a, b, c, v_1, v_2 are fixed.

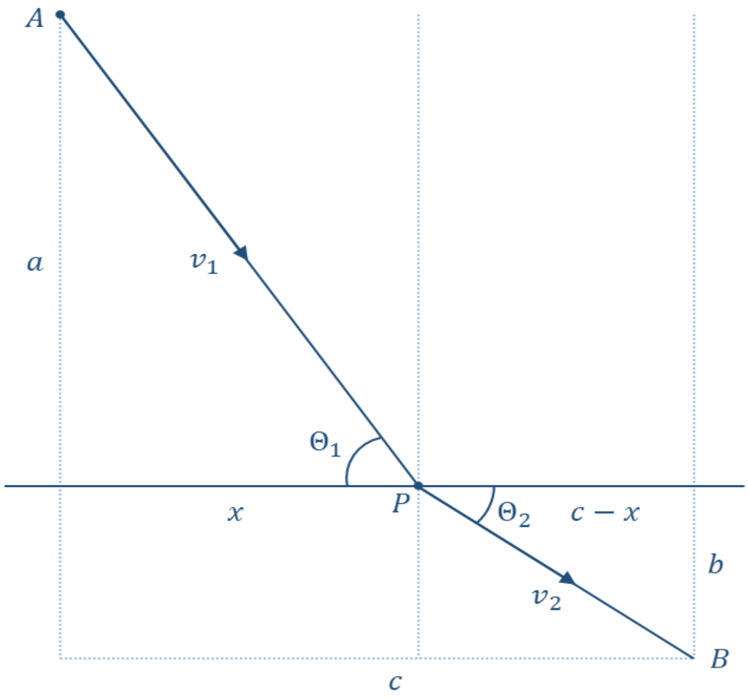

Figure 2: *A light particle traveling from point, A, to point, B, through two different mediums with the corresponding velocities v_1 and v_2.*

To minimize the function (8.1.1), the derivative $\frac{dt}{dx}$ is set to 0 which gives:

$$\begin{aligned}\frac{dt}{dx} =& \frac{d}{dx}\left(\frac{\sqrt{a^2+x^2}}{v_1}+\frac{\sqrt{b^2+(c-x)^2}}{v_2}\right)\\ =& \frac{\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}}(2x)}{v_1}-\frac{\frac{1}{2}(b^2+(c-x)^2)^{-\frac{1}{2}}(2)(c-x)}{v_2}\\ =& \frac{x}{v_1\sqrt{a^2+x^2}}-\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}=0\Rightarrow\end{aligned} \tag{8.1.2}$$

$$\frac{x}{v_1\sqrt{a^2+x^2}}=\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}. \tag{8.1.3}$$

Furthermore, from *Figure 1* it also is clear that

$$\sin(\theta_1)=\frac{x}{\sqrt{a^2+x^2}} \tag{8.1.4}$$

and

$$\sin(\theta_2) = \frac{c-x}{\sqrt{b^2+(c-x)^2}}. \tag{8.1.5}$$

By substituting (8.1.4) and (8.1.5) into (8.1.6) the following is given

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}. \tag{8.1.6}$$

This, (8.1.6), is what is formally known as *Snell's law of refraction* and displays how light travels along the path requiring the least amount of time.

By letting light travel through denser and denser mediums the angle, θ_i, decreases. See *Figure 2* to see how light would behave if $v_1 \geq v_2 \geq v_3 \geq v_4$.

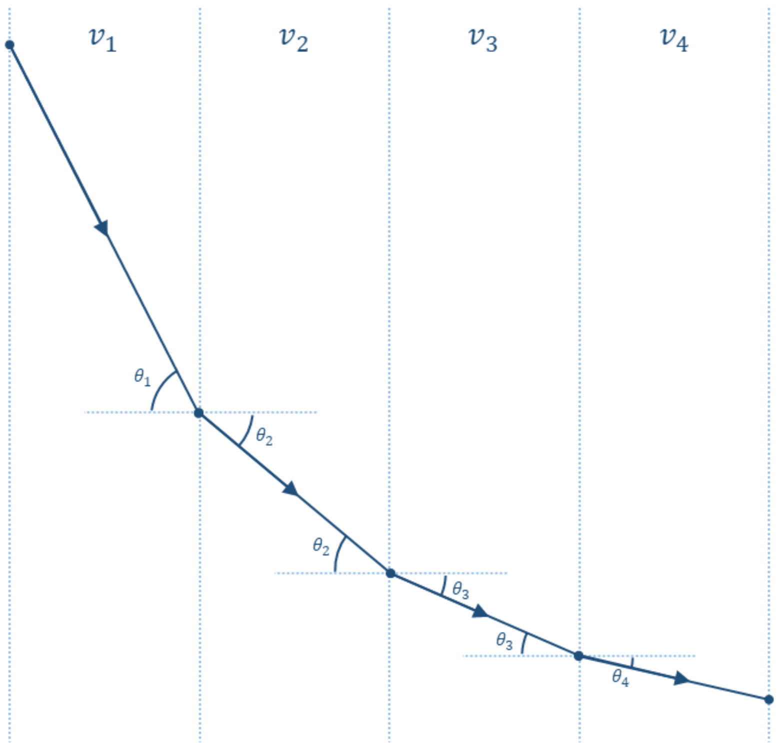

Figure 3: *A light particle traveling from point, A, to point, B, through four different mediums with the corresponding velocities v_1-v_4.*

Applying *Snell's law* to the situation displayed in *Figure 2* yields

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2} = \frac{\sin(\theta_3)}{v_3} = \frac{\sin(\theta_4)}{v_4} \tag{8.1.7}$$

The sections can be divided into smaller and smaller pieces up until the point where the path would approach a smooth curve where the velocity, v, decreases continuously. Then

$$\frac{\sin(\theta)}{v} = C, \tag{8.1.8}$$

where, C, is a constant.

Up until this point, the situation regarding light through different mediums has been considered. However, the same methodology and rationale can be used to find the optimal path for an object with respect to gravity, i.e. the case of the Brachistochrone problem.

Assume that an object travels from point A to point B, and to simplify the problem assume a friction-free environment. An object with regards to energy can either have kinetic or potential energy. As the object starts in rest, the relationship between the potential energy and the kinetic energy, together with the notation from *Figure 3* can be written as

$$mgy = \frac{mv^2}{2} \rightarrow gy = \frac{v^2}{2}, \tag{8.1.9}$$

where m is the mass of the object, g is the gravitational constant, v, is the velocity, and y is the horizontal distance between the object and its starting point, A. According to *Figure 4*, a relationship between α, β and y can be observed which also mathematically can be interpreted as

$$\sin(\alpha) = \cos(\beta) = \frac{1}{\sqrt{(1+\tan^2(\beta))}} = \frac{1}{\sqrt{(1+(\frac{dy}{dx})^2)}}. \tag{8.1.10}$$

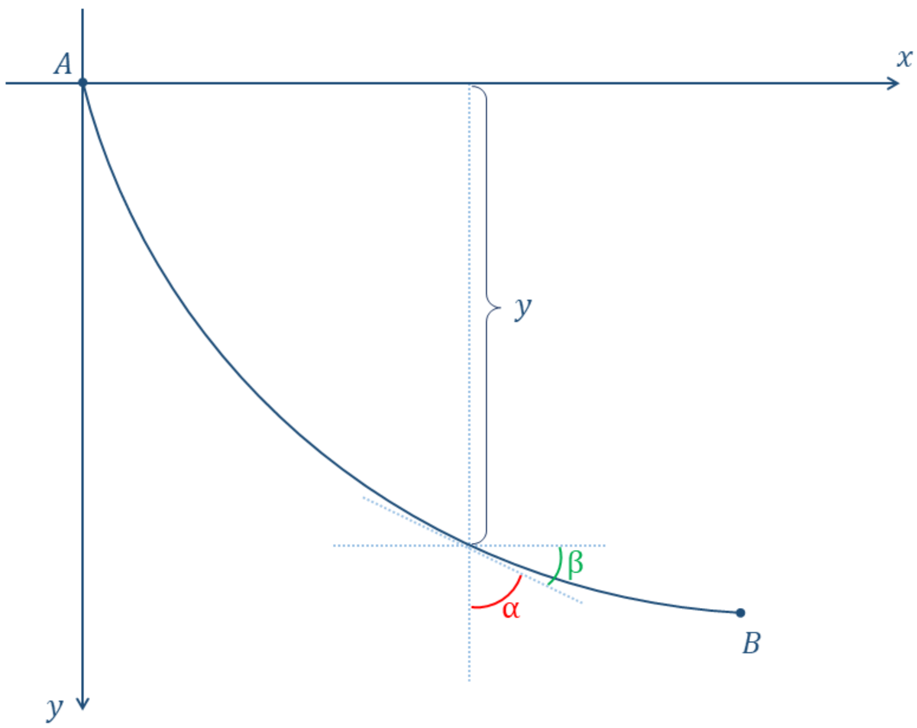

Figure 4: The relationship between α, β and y with respect to the x-axis which represent time.

Furthermore, from (8.1.9), $v = \sqrt{2gy}$, can be derived and substituting this, together with (8.1.10) into (8.1.8) gives

$$\begin{aligned} \frac{1}{\sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}} &= c \Rightarrow \\ \sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \frac{1}{c} \Rightarrow \\ \sqrt{y}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \frac{1}{c\sqrt{2g}} \Rightarrow \\ y\left[1+\left(\frac{dy}{dx}\right)^2\right] &= \frac{1}{2gc^2} \end{aligned} \tag{8.1.11}$$

Note that the last notation on the right hand side in (8.1.11) only consists of constants, making it possible to state the equality as

$$y\left[1+\left(\frac{dy}{dx}\right)^2\right] = C, \tag{8.1.12}$$

which is the differential equation that solves the brachistochrone problem.

This differential equation can be solved by rewriting the equation as:

$$\begin{aligned} y\left[1+\left(\frac{dy}{dx}\right)^2\right] &= C \\ \left(\frac{dy}{dx}\right)^2 &= \frac{C-y}{y} \\ \frac{dy}{dx} &= (\pm)\left(\frac{C-y}{y}\right)^{\frac{1}{2}} \rightarrow \\ dx &= \left(\frac{y}{C-y}\right)^{\frac{1}{2}} dy \end{aligned} \tag{8.1.13}$$

Then by letting

$$\left(\frac{y}{C-y}\right)^{\frac{1}{2}} = \tan(\psi) \tag{8.1.14}$$

it is possible to continue the solution accordingly:

$$\begin{aligned}
\frac{y}{C-y} &= \tan^2(\psi) \Rightarrow \\
y &= C\tan^2(\psi) - y\tan^2(\psi) \Rightarrow \\
y(1+\tan^2(\psi)) &= C\tan^2(\psi) \Rightarrow \\
y &= C\left(\frac{\tan^2(\psi)}{1+\tan^2(\psi)}\right) \\
&= C\sin^2(\psi) \\
&= C\left(\frac{1-\cos(2\psi)}{2}\right) \\
&= \frac{C}{2}(1-\cos(2\psi)).
\end{aligned} \tag{8.1.15}$$

Taking the derivative, $\frac{dy}{d\psi}$, gives

$$dy = 2C\sin(\psi)\cos(\psi)d\psi \tag{8.1.16}$$

and by substituting (8.1.14) and (8.1.16) into (8.1.13) the following is given:

$$dx = 2C\sin^2(\psi)d\psi. \tag{8.1.17}$$

Then, integrating both sides gives:

$$\begin{aligned}
\int dx = x &= \int 2C\sin^2(\psi)d\psi \\
&= 2C\int \frac{1-\cos(2\psi)}{2}d\psi \\
&= C\left(\int 1d\psi - \int \cos(2\psi)d\psi\right) \\
&= C\psi - \frac{C}{2}\sin(2\psi) + c_1 \\
&= \frac{C(2\psi - \sin(2\psi))}{2} + c_1.
\end{aligned} \tag{8.1.18}$$

As the initial state, when the object lays still at point A, has $\psi = 0$, the constant,

c_1, must also 0. Thus

$$x = \frac{C(2\psi - \sin(2\psi))}{2}. \tag{8.1.19}$$

Lastly, (7.1.15) and (7.1.19) can be rewritten in a more "clean" form by letting, $r = \frac{C}{2}$ and $\phi = 2\psi$, which leaves:

$$x = r(\phi - \sin(\phi)) \tag{8.1.20}$$

and

$$y = r(1 - \cos(\phi)). \tag{8.1.21}$$

Here, x and y, are the parametric equations of a cycloid, where, r, is the radius of a circle, which "roles" down the x-axis. For some value, r, the curve passes through the end-point, B, which then provides the optimal route for the object, the brachistochrone curve (see *Figure 4*).

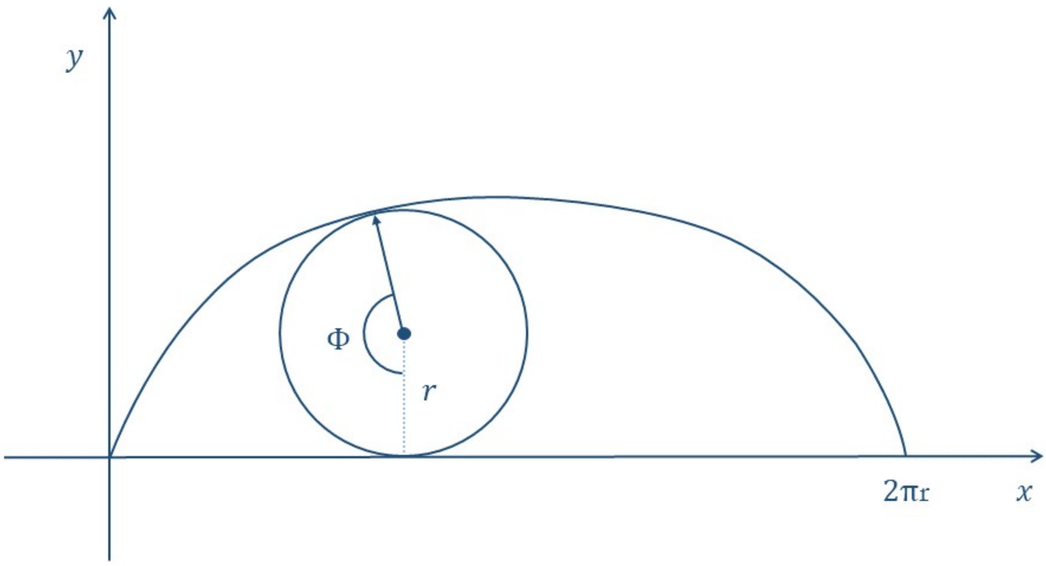

Figure 5: *The relationship between a circle and a cycloid (The brachistochrone curve).*

8.2 Using Calculus of Variation

Bernoulli's indirect solution to the brachistochrone problem is far from the only solution and arguably also not the best solution, as will be discussed later on. Another way is to use the previously discussed *Euler-Lagrange equation*, which is a central aspect in *calculus of variation*. The *Euler-Lagrange equation* previously stated in (4.2.8), can also be written as

$$\frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}}\right) = \frac{\partial \mathscr{L}}{\partial y}. \tag{8.2.1}$$

When using the calculus of variation approach, the velocity, previously denoted just as v, will instead be considered as vectors in the x- and y-direction, displaying the change in each direction. Here, it is necessary to postulate that it is sufficient to examine graphs of functions as, $x \to y(x)$. Furthermore, the mass of the object can be chosen to be 1. Then, the kinetic energy can be described as

$$\frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2} \tag{8.2.2}$$

and the potential energy can be described as

$$-gy(t). \tag{8.2.3}$$

To further simplify the conditions, the case of when the total energy equals 0 can be considered as well as a renotation of the gravitation force, leaving the gravitational-effect being 0.5 of an arbitrary unit. Then, the relationship between the kinetic and potential energy can be written as

$$dx^2 + dy^2 = y\,dt^2, \tag{8.2.4}$$

which further gives

$$dt = \frac{\sqrt{dx^2+dy^2}}{\sqrt{y}} = \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{y}}dx = \mathscr{L}(x,y,\dot{y})dx, \tag{8.2.5}$$

where

$$\mathscr{L}(x,y,u) = \frac{\sqrt{1+u^2}}{\sqrt{y}}. \tag{8.2.6}$$

Thus, the *Euler-Lagrange equation* in this case state that the solution is found when

$$\frac{d}{dx}\frac{\left(\frac{dy}{dx}\right)}{\sqrt{y\left((1+\left(\frac{dy}{dx}\right)^2\right)}} = -\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y^3}} \tag{8.2.7}$$

as

$$\frac{\partial\mathscr{L}(x,y,\dot{y})}{\partial y} = -\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y^3}} \tag{8.2.8}$$

and

$$\frac{\partial\mathscr{L}(x,y,\dot{y})}{\partial \dot{y}} = \frac{d}{dx}\frac{\left(\frac{dy}{dx}\right)}{\sqrt{y\left((1+\left(\frac{dy}{dx}\right)^2\right)}}. \tag{8.2.9}$$

From here (8.2.9) can be rewritten by taking the total derivative resulting in

$$\begin{aligned}\frac{d}{dx}\frac{\left(\frac{dy}{dx}\right)}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}} &= \frac{\frac{d^2y}{dx^2}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\\ -\frac{1}{2}\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{y^3\left(1+\left(\frac{dy}{dx}\right)^2\right)}} &- \frac{\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)^3}}.\end{aligned} \tag{8.2.10}$$

Now (8.2.7) can be written as

$$\frac{\frac{d^2y}{dx^2}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}} - \frac{1}{2}\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{y^3\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$$
$$-\frac{\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)^3}} = -\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y^3}} \tag{8.2.11}$$

and by multiplying the equation with $\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}$, the following is given

$$-\frac{1}{2}\frac{1+\left(\frac{dy}{dx}\right)^2}{y} = \frac{d^2y}{dx^2} - \frac{1}{2}\frac{\left(\frac{dy}{dx}\right)^2}{y} - \frac{\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}. \tag{8.2.12}$$

This can further be simplified to

$$1+2y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \tag{8.2.13}$$

and by multiplying (8.2.13) with $\frac{dy}{dx}$ the following is given

$$\frac{dy}{dx}+2y\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3 = 0. \tag{8.2.14}$$

The left hand side of (8.2.14) is in fact the derivative of the function

$$y+y\left(\frac{dy}{dx}\right)^2+C. \tag{8.2.15}$$

which is the exact same equation as (8.1.12), which of thence the solution can proceed.

8.3 Using Optimal Control

To begin with, it is wise to formulate the brachistochrone problem such that the motion of object takes place in the x and y plane with the dynamic behaviour given by

$$\dot{x} = u\sqrt{y}, \quad \dot{y} = v\sqrt{y}. \tag{8.3.1}$$

In this case *the control* is a 2-dimensional vector field taking values in the set

$$U = \Big\{(u, v) : u^2 + v^2 = 1\Big\}. \tag{8.3.2}$$

Consider the *Hamiltonian* $\mathcal{H}$,

$$\mathcal{H}(x, y, u, v, p, q, p_0, t) = (pu + qv)\sqrt{y} - p_0, \tag{8.3.3}$$

where p, q and p_0 denotes the momentum variables conjugate to x and y, and the abnormal multiplier (Sussmann, 2002). In that case, *Pontryagin's maximum principle* states that if a curve

$$t \in [0, T] \to \xi(t) = (x(t), y(t))$$

is *optimal* then, there exists continuous functions for p and q, ($p_0 \geq 0$), where the Hamiltonian maximization conditions

$$u(t) = \frac{p(t)}{\parallel \overrightarrow{p}(t) \parallel} \text{ and } v(t) = \frac{q(t)}{\parallel \overrightarrow{p}(t) \parallel} \tag{8.3.4}$$

as well as the adjoint system of differential equations:

$$\dot{p}(t) = 0, \quad \dot{q} = -\frac{p(t)u(t) + q(t)v(t)}{2\sqrt{y(t)}} = -\frac{\parallel \overrightarrow{p} \parallel}{2\sqrt{y(t)}} \tag{8.3.5}$$

are satisfied for all t, where $u = \dot{x}$, $v = \dot{y}$, $\overrightarrow{p}$ denotes the momentum vector and, $\parallel \overrightarrow{p} \parallel$ is its Euclidean norm ($\parallel \overrightarrow{p} \parallel \neq 0$).

Pontryagin's minimum principle also states that $\mathcal{H} = 0$, meaning that, $\parallel \overrightarrow{p} \parallel$ would imply that, $p_0 = 0$. If the constant p disappears, then $\dot{x} \equiv 0$, resulting in a vertical line. Otherwise, $\dot{x}$ remains non-zero, enabling a parameterization of x as a solution. Furthermore,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{v}{u} = \frac{q}{p} \tag{8.3.6}$$

enables,

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{\| \overrightarrow{p} \|^2}{p^2} \tag{8.3.7}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{p}\frac{dq}{dx} = \frac{\dot{q}}{p\dot{x}}. \tag{8.3.8}$$

From here, (8.3.1) together with (8.3.4) gives

$$\dot{x} = \frac{p\sqrt{y}}{\| \overrightarrow{p} \|} \tag{8.3.9}$$

and, (8.3.5) together with (8.3.8) yield

$$\frac{d^2y}{dx^2} = -\frac{\| \overrightarrow{p} \|^2}{2yp^2}. \tag{8.3.10}$$

A trivial rearrangement of (8.3.10) then gives

$$2y\frac{d^2y}{dx^2} = -\frac{\| \overrightarrow{p} \|^2}{p^2} = -\left(1 + \left(\frac{dy}{dx}\right)^2\right) \tag{8.3.11}$$

which then lastly can be rewritten as

$$1 + 2y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \tag{8.3.12}$$

being the exact same formulation as (8.2.13), concluding this solution as the rest can be followed from (8.2.13).

8.4 Comparing the Different Methods

Even though the three different methods by and large come to the same conclusion, they do so with different assumptions and accuracy. One issue that is resolved, using *calculus of variation* and *optimal control theory* in contrast to *Bernoulli's* solution, is the existence of a spurious solution. This can be seen in (8.1.13) where the solution should also be able to have a negative slope. However, this is not the case and as such the negative case is not considered. Furthermore, $\frac{dy}{dx}$ should be continuous which will not be the case if it can hold both positive and negative numbers. Consequently, this method is somewhat flawed mathematically, as the negative root should in fact exist.

Instead, the method of using *calculus of variation* provides a solution that circumvent these issues as (8.2.13) is the same solution as (8.1.13) without the spurious solution. Thus, the issue with the negative case and the non-continuity is solved.

However, using *optimal control theory* can provide an even better solution as with *optimal control theory*, it is no longer necessary to postulate that the solution can be represented by graphs such that, $x \to y(x)$. Thus, the *optimal control theory* method can be considered as superior as it provides the most accurate and non-postulated solution.

9

Solving Merton's Portfolio Problem Using Optimal Control

Consider an investor with the lifetime from time 0 to time T i.e. $t \in [0, T]$, with the corresponding wealth R_t at each time, t, and a known initial wealth, R_0. The investor can, as stated previously, choose to invest in two different assets, a risky asset, S, and a risk-free asset, S_{rf} with the purpose to maximize its wealth's, R, utility until the time, T. The parameters, μ and σ will be used to denote the drift and the volatility of the risky asset respectively, and the risk-free asset S_{rf} will at time t be denoted as $S_{(rf,t)}$. From here the dynamics of the price processes are

$$dS_{(rf,t)} = rS_{(rf,t)}dt \tag{9.0.1}$$

and

$$dS_t = S_t[\mu dt + \sigma dW_t] \tag{9.0.2}$$

where r is the continuously compounding interest rate of S_{rf} and W_t is the one-dimensional Brownian motion often called the Wiener-process named after *Norbert Wiener* (Schilling, 2012). Furthermore, the proportion, α of the wealth R will be invested in the risky asset which means that the proportion invested in the risk-free asset can be denoted as, $1 - \alpha$. Hence, the wealth process can be described as

$$
\begin{aligned}
dR_t &= \alpha_t R_t \frac{dS_t}{S_t} + (1-\alpha_t) R_t \frac{dS_{rf,t}}{S_{rf,t}} \\
&= R_t[(r + (\mu - r)\alpha_t)dt + \sigma \alpha_t dW_t].
\end{aligned} \tag{9.0.3}
$$

The process α is admissible if

$$
E\left[\int_0^T |\alpha_t|^2 dt\right] < \infty, \tag{9.0.4}
$$

where all admissable portfolios are denoted as $\mathcal{U}_{ad}$. Moreover, let γ denote an arbitrary parameter where, $\gamma \in [0,1]$ and let the *power utility function*, $U(x)$ be given by

$$
U(x) := x^\gamma \ \textit{for} \ x \geq 0. \tag{9.0.5}
$$

Here, the parameter γ, is the *relative risk premium coefficient.* From this point, with the information given above, it is possible to frame the problem more clearly. The investor has the objective to maximize the utility of the terminal wealth. Thus, the problem, or more rather, the *value function*, $\Phi(t)$, can be stated as

$$
\Phi(t,x) := \sup_{\alpha_t \in \mathcal{U}_{ad}} \left\{ E_{t,x}\left[U(R_T)\right]\right\}. \tag{9.0.6}
$$

where the associated *Hamilton-Jacobi-Bellman equation* is

$$
\begin{gathered}
\frac{\partial \Phi}{\partial t}(t,x) + \sup_{\alpha_t \in \mathcal{U}_{ad}} \{\mathscr{L}^v \Phi(t,x)\} = \\
\frac{\partial \Phi}{\partial t}(t,x) + \sup_{\alpha_t \in \mathcal{U}_{ad}} \left\{ (r + (\mu - r)\alpha_t)x \frac{\partial \Phi}{\partial x}(t,x) + \frac{1}{2}\alpha^2 \sigma^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}(t,x) \right\} = 0,
\end{gathered} \tag{9.0.7}
$$

for, $(t,x) \in [0,T]$, and $\Phi(T,x) = U(x)$.

Moreover, one contender for the *optimal control* is obtained from the first-order condition for the *Hamilton-Jacobi-Bellman equation* above (Pham, 2007) which yields:

$$\hat{\alpha}(t,x) = u_t^* = -\frac{\mu - r}{\sigma^2}\frac{\frac{\partial \Phi}{\partial x}}{x\frac{\partial^2 \Phi}{\partial x^2}}(t,x) = \frac{\mu - r}{\sigma^2(1-\gamma)}. \tag{9.0.8}$$

Then, consider the *power utility function* (9.0.5) and suppose that the *value function*, $\Phi(t,x)$, has the form

$$\Phi(t,x) = \varphi(t)x^\gamma \tag{9.0.9}$$

for some deterministic function, φ. Now, substituting this into (9.0.7) leads to the ordinary differential equation (ODE):

$$\begin{aligned} \frac{d\varphi}{dt} + \lambda\varphi(t) &= 0 \\ \varphi(T) &= 1 \end{aligned} \tag{9.0.10}$$

where

$$\lambda = \sup_{\alpha_t \in \mathcal{U}_{ad}} \left\{ (r + (\mu - r)\alpha_t)\gamma - \frac{1}{2}\alpha^2\sigma^2\gamma(1-\gamma) \right\} = \frac{(\mu - r)^2}{2\sigma^2}\frac{\gamma}{1-\gamma} + r\gamma. \tag{9.0.11}$$

The solution to this ODE is $\varphi(t) = e^{\lambda(T-t)}$ meaning that the *value function*, $\Phi(t,x) = e^{\lambda(T-t)}x^\gamma$, satisfies the *Hamilton-Jacobi-Bellman equation* (9.0.7). Moreover, by using the constant *optimal control* from (9.0.8), it is possible to conclude that the *value function* to Merton's portfolio (with CRRA utility) is given by

$$\Phi(t,x) = e^{\lambda(T-t)x^\gamma} \tag{9.0.12}$$

and the *optimal control* is given by

$$u_t^* = \frac{\mu - r}{\sigma^2(1-\gamma)}. \tag{9.0.13}$$

10

Conclusion

Control theory is, as shown, a rather effective and accurate tool, which today after many additions by a multitude of prolific mathematicians, can solve a multitude of problems. Not only problems of deterministic nature but also stochastic problems just like *Merton's portfolio problem*. *Optimal control theory* is, however, quite a new area of applied mathematics as it was first introduced in its current form in 1950. There is much more research to be done, where new findings can be pivotal for the society we live in. *Control theory*, after all, put humankind on the moon and may very well solve many other important issues in the future.

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