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On holomorphic functions; analytic continuation, boundary values on the disk of convergence and the Hardy spaces.

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On holomorphic functions; analytic continuation, boundary values on the disk of convergence and the Hardy spaces.

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Abstract

In this thesis we are looking at holomorphic functions, their connection to power series and behaviour on the boundary of the disk of convergence. We investigate analytic continuation with several examples, displaying how one can use the Poisson kernel to analyse values on, or restore a function from, the boundary of the disk of convergence. We are also going to explore some theory of the Hardy spaces on the unit disk.

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Beside my advisor, I would like to thank Nell Paiz Jacobsson for being the best study companion for the past three years and of which have become a very close friend.

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Introduction

We will look at properties of holomorphic functions within their domain of being holomorphic. As we are going to see, if a function f is holomorphic, then this implies that f is infinitely differentiable. We will then alternatively define holomorphic functions as converging power series about each point within its domain of holomorphicity. Questions arises concerning behaviours on the boundary of the disk of convergence. Does it converge? Does it diverge? Are there some points on the boundary where it converges and other where it does not. Here, Niels Henrik Abel is going to provide us with some tools for where we can further investigate the behaviour on the boundary of the disk of convergence. So far we've only been working inside and up to the circle of convergence. However, is it possible to go further? The chapter about analytic continuation will show us that we sometimes can extend the domain for where a function is holomorphic. This section is pretty technical with some interesting examples of what can happen when a function is analytically continued around a singularity. On the other hand there are conditions for where it doesn't matter along which curve we go. We are still going to arrive at the same function. This is known as the Monodromy theorem. After we've explored analytic continuation we go back to the unit disk. Introducing approximate identities, in particular the Poisson kernel, and see how we can use it to restore a function from values on the unit circle. E.g. if f is in the class of continuous functions defined on the unit circle, is it possible for us to restore a function inside the unit circle that is equal to f on the unit circle? This is the Dirichlet problem. With help from the Poisson kernel we will also show that if a function f is holomorphic in the unit disk then the radial limits exists for almost all points on the unit circle. The last section explore some theory about the Hardy spaces on the unit disk.

2

Basics

Throughout the text we are going to denote the unit disk as

$$\mathbb{D} = \{z \in \mathbb{C} : |z < 1|\}$$

and the boundary of the unit disk with

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : t \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

More general domains will most commonly be denoted as B, D or G and as for more general circles or curves we will mostly use C, γ or Γ.

2.1 Power series

Definition 2.1 ([1], page 38)**.** A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n, z_0 \in \mathbb{C}$ is called a power series centered at z_0. For each power series there exists a number $R \in [0, \infty]$, called the radius of convergence, such that

(i) the series converges uniformly on every closed disk $|z - z_0| \leq R' < R$,

(ii) the series diverges if $|z - z_0| > R$.

If $R = \infty$ the series is said to have an infinite radius of convergence and thus converges for all points in the whole complex plane. If $R = 0$ the series diverges for any $z \neq z_0$.

We consider some simple examples.

Example 2.1. We begin with

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We have $a_n = \frac{1}{n!}$ and thus by the ratio test we get

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

That implies that $R = \infty$ and the series converges for all z in $\mathbb{C}$.

Example 2.2. Let us continue and consider the power series

$$\sum_{n=0}^{\infty} n! z^n.$$

So we have $a_n = n!$ and again using the ratio test we get

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} n+1 = \infty.$$

That is $R = 0$ which implies that the series is divergent for every z except at one point, $z = 0$.

Example 2.3. Finally we look at

$$\sum_{n=0}^{\infty} \frac{1}{n} z^n.$$

This time we use Cauchy-Hadamard's formula to compute the circle of convergence and we get

$$\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|n|}} = 1.$$

And thus the radius of convergence is 1 and it follows that the series converges uniformly for all $|z| < 1$.

2.2 Contour Integrals

Consider a function defined over a directed smooth curve γ with initial point α and terminal point β (possibly coinciding with α). For any positive integer n, we define a partition $\mathcal{P}_n$ of γ to be a finite number of points $z_0, z_1, ..., z_n$ on γ ordered in accordance to the direction and where $\alpha = z_0$ and $\beta = z_n$. If we compute the arc length along γ between every consecutive pair of points, the largest of these lengths provides a measure of the fineness of the partition; this maximum length is called the mesh of the partition and is denoted $\mu(\mathcal{P}_n)$. Now let $c_1, ..., c_n$ be any points on γ such that c_1 lies between z_0 and z_1, c_2 lies between z_1 and z_2, etc. Then the sum $S(\mathcal{P}_n)$ is defined by

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}),$$

is called the Riemann sum of the partition $\mathcal{P}_n$.

Definition 2.2 ([8], page 162)**.** Let f be a complex-valued function defined on the directed smooth curve γ. We say that f is integrable along γ if there exists a complex number L that is the limit of every sequence of Riemann sums

$$S(\mathcal{P}_1), S(\mathcal{P}_2), ..., S(\mathcal{P}_n), ...$$

corresponding to any sequence of partitions of γ satisfying $\lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$, i.e.

$$\lim_{n\to\infty} S(\mathcal{P}_n) = L \quad \text{whenever} \quad \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0.$$

The constant L is called the integration of f along γ, and we write

$$L = \lim_{N\to\infty} \sum_{n=1}^{N} f(c_n)\Delta z_n = \int_\gamma f(z)dz.$$

Theorem 2.1 ([8], page 165)**.** *Let f be a continuous function on the directed smooth curve γ. Then if $z = z(t)$, $a \le t \le b$, is any parametrization of γ consistent with its direction, we have*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Definition 2.3 ([8], page 167)**.** Suppose that Γ is a contour consisting of the directed smooth curves $(\gamma_1, ..., \gamma_n)$, and let f be a continuous function on Γ. Then the contour integral of f along Γ is denoted by

$$\int_\Gamma f(z)dz,$$

and is defined by the equation

$$\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

We will now look at an example.

Example 2.4. Compute

$$\int_\Gamma \frac{1}{z - z_0}dz,$$

where Γ is the circle $|z - z_0| = r$ going twice in the counterclockwise direction starting from the point $z_0 + r$.

Let C_r denote the circle $|z - z_0| = r$ going around the circle once in the counterclockwise direction. Then we have two directed smooth curves and by definition 2.3 we write $\Gamma = (C_r, C_r)$. Then

$$\int_\Gamma \frac{1}{z - z_0}dz = \int_{C_r} \frac{1}{z - z_0}dz + \int_{C_r} \frac{1}{z - z_0}dz$$

From theorem 2.1 let $z = z_0 + re^{it}$ with $dz = rie^{it}dt$, $0 \leq t \leq 2\pi$ then

$$\begin{aligned}\int_{C_r} \frac{1}{z - z_0} dz + \int_{C_r} \frac{1}{z - z_0} dz &= \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt + \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt \\ &= i \int_0^{2\pi} dt + i \int_0^{2\pi} dt \\ &= 2\pi i + 2\pi i \\ &= 4\pi i.\end{aligned}$$

2.3 Holomorphic functions

Definition 2.4 ([8], page 70)**.** A complex-valued function $f(z)$, defined on an open set D, is said to be holomorphic in D if it has a derivative at each point of D.

In essence, the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1}$$

must exist for all $z \in D$.

As a first consequence f must necessarily be continuous. And indeed, from (1) we have

$$\lim_{h \to 0} f(z+h) - f(z) = \lim_{h \to 0} \frac{h\left(f(z+h) - f(z)\right)}{h} = 0 \cdot f'(z) = 0.$$

If $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$ then it follows that $u(x, y)$ and $v(x, y)$ are both continuous. We also have that the limit of (1) must be equal regardless of the way in which $h \to 0$. Let us start by first consider $h = k$, then the imaginary part is kept fixed and the derivative becomes a partial derivative with respect to x,

$$\frac{\partial f}{\partial z} = \lim_{k \to 0} \frac{f(z+k) - f(z)}{k} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

If we instead choose to substitute h with purely imaginary values $h = ik$ then we keep the real part fixed and the derivative becomes a partial derivative with respect to y and we obtain,

$$\frac{\partial f}{\partial z} = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

For the limit (1) to exist, the condition

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0,$$

must be met. That is,

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0.$$

So in particular, the real and imaginary part of a complex-valued function $f(z) = u(x, y) + iv(x, y)$ must satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{2}$$

for every point in D to be holomorphic there. These equations in (2) are known as the Cauchy-Riemann equations.

We emphasize that holomorphicity is a property defined over open sets while differentiability could possibly hold at one point. So when we say, "$f(z)$ is holomorphic at the point z_0", we mean that $f(z)$ is holomorphic in some neighborhood of z_0. A point where f is not holomorphic but is the limit of points for which f is holomorphic is called singular point. An example would be a rational function f for which is holomorphic at every point except when the denominator is zero, the zeroes of the denominator are then the singular points or singularities. If a function is holomorphic on the whole complex plane we say that the function is entire. We are also going to see that we can define a holomorphic function using power series. This relation will be treated after we have derived the Cauchy Integral formula and showed that a holomorphic function is infinitely differentiable.

Before we get to Cauchy's Integral formula we have Cauchy's Integral theorem ([8],page $191 - 194$), which says that if a function f is holomorphic in a simply connected domain D and Γ is any closed contour, in D then

$$\int_\Gamma f(z)dz = 0.$$

The result can be obtained from Greens theorem that says that a contour integral can be computed as a double integral over the region contained inside the contour. When you then apply Green's theorem to a holomorphic function the Cauchy-Riemann equation shows up and cancel out both the real and imaginary part. For a full proof we refer to Saff and Snider ([8], page $193 - 194$). From this result we will now derive Cauchy's Integral formula.

Theorem 2.2 ([8], page 204)**.** *Let Γ be a simple closed positively oriented contour. If f is holomorphic in some simply connected domain $D \subset \mathbb{C}$ containing Γ and z_0 is any point inside Γ, then*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz. \tag{3}$$

Proof. Define $g(z) = \frac{f(z)}{z-z_0}$. Since f is holomorphic on and inside Γ so is g except for the point $z = z_0$, that means that Γ can be continuously deformed without passing through the point z_0 into the positively oriented unit circle γ around z_0. Thus, from ([8], page 182) and by Deformation Invariance theorem ([8], page 183)

$$\int_\Gamma g(z)dz = \int_\gamma g(z)dz. \tag{4}$$

Now by letting $z = z_0 + re^{i\theta}$ with $dz = rie^{i\theta}d\theta$, $0 \leq \theta \leq 2\pi$, we can write the right side of equation (4) as

$$\int_\gamma g(z)dz = \int_\gamma \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}}rie^{i\theta}d\theta = i\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)d\theta.$$

As f is holomorphic for every point in a simply connected domain in D we can, without changing the value of the integral, let r tend to 0:

$$\lim_{r\to 0} i\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)d\theta = i\int_0^{2\pi} f\left(z_0\right)d\theta = 2\pi i f(z_0)$$

It now follows that

$$\int_\Gamma g(z)dz = 2\pi i f(z_0) \iff f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz.$$

□

The conclusion of Cauchy's Integral formula is that if f is holomorphic in a simply connected region containing a simple closed positively oriented contour then the behaviour of f inside the contour is completely determined by its behaviour on that contour.

We continue by showing that if f is holomorphic then f is infinitely differentiable. The property of a holomorphic function f to be infinitely differentiable is going to be a necessity in the construction of an infinite Taylor expansion of f defined above as a power series.

Theorem 2.3 ([8], page 211)**.** *If f is holomorphic inside and on a simple closed positively oriented contour Γ and if z is any point inside Γ, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta \quad n = 1, 2, 3, ... \tag{5}$$

Proof. For z inside Γ we know from Cauchy's Integral formula that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

We differentiate both sides with respect to z:

$$\frac{\partial f(z)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta\right)$$

Since we integrate with respect to ζ and the interval of the integration is between 0 and 2π, i.e. two constants, and we want to differentiate with respect to z, then by Leibniz[1] integral rule we can interchange the order of differentiation and integration:

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right) d\zeta \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}d\zeta \end{aligned}$$

A similar computation yields the second derivative:

$$\begin{aligned} f''(z) &= \frac{1\cdot 2}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^3}d\zeta \\ f''(z) &= \frac{2!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^3}d\zeta \end{aligned}$$

And for the third derivative:

$$\begin{aligned} f'''(z) &= \frac{1\cdot 2\cdot 3}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^4}d\zeta \\ f'''(z) &= \frac{3!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^4}d\zeta \end{aligned}$$

Now suppose that it holds for $n = k$, that is

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta.$$

When differentiating $f^{(k)}(z)$ with respect to z we can again interchange the order of differentiation and integration by the same reason as before, which yields

$$\begin{aligned} f^{(k+1)}(z) &= \frac{k!}{2\pi i}\int_\Gamma \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{(\zeta - z)^{k+1}}\right) d\zeta \\ &= \frac{k!}{2\pi i}\int_\Gamma \frac{(k+1)f(\zeta)}{(\zeta - z)^{k+2}}d\zeta \\ &= \frac{(k+1)!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}}d\zeta. \end{aligned}$$

And thus we conclude that if f is holomorphic inside and on some simply closed positively oriented contour then f is infinitely differentiable at each point inside that contour. □

[1]Leibniz integral rule can be found in the appendix.

It is a remarkable result since just knowing a function is holomorphic in some domain D, i.e. satisfies the Cauchy-Riemann equations in D, we get that it is also infinitely differentiable at every point in D.

From this result we establish the relation between holomorphic functions and power series as follows;

Claim. If f is a holomorphic function in a simply connected domain $D \subset \mathbb{C}$ and C is an open disk contained in D with radius $r \in \mathbb{R}$ centered at z_0. Then for any $z \in C$, $f(z)$ can be represented as a convergent power series with positive radius of convergence.

Let $|\zeta - z_0| = r$, then by the assumption $z \in C$, we get the inequality

$$|z - z_0| < |\zeta - z_0|. \tag{6}$$

We write

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 + z_0 - z} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}. \end{aligned} \tag{7}$$

By the assumption on z we observe that the second factor on the last line in (7) can be written as a geometric sum, this implies that

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{8}$$

Continuing by substituting (8) into equation (3)

$$f(z_0) = \frac{1}{2\pi i} \int_C f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta,$$

and notice that since the series is uniformly convergent we can interchange the summation and integration and thus integrate term by term,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta + \cdots + \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta + \cdots \quad . \tag{9}$$

From Cauchy's Integral formula we use the equality

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

and write f as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + ... + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + ... = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

We can conclude that a holomorphic function f has a convergent power series representation for all z inside C. And thus we've arrived at the following theorem.

Theorem 2.4. *If f is holomorphic in a simply connected domain $D \subset \mathbb{C}$, then f has a convergent power series representation at every point of D.*

Remark. From Cauchy's Integral formula we get that the restriction $|z - z_0| < |\zeta - z_0|$ is a necessity for us to derive theorem 2.4. This however, also implies that the largest disk for where a power series converges representing a holomorphic function, is the disk of radius equal to the distance from the center (z_0 in our case) to the closest singularity of f.

Remark. We also note that the power series of all derivatives of f can be obtained by termwise differentiation of the power series of f and moreover, all its derivative expansions converges in the same disk as the power series of f.

Remark. On the other hand there might be other points $z \neq z_0$ inside the circle of convergence for where we can construct a new circle of convergence that goes beyond the circle of convergence around z_0 and thus extend the domain for where the function is holomorphic. This is called analytic continuation, which we will come back to later.

We are now going to look at two examples. The first one is a simple demonstration of how one can determine the radius of convergence of a function just by observing the distance to the closest singularity.

Example 2.5. Let $f(z) = \frac{1}{1+z^2}$. Straight away we see that f is undefined at $\pm i$ and the distance from 0 to $\pm i$ is 1. Without using any formula to determine the radius of convergence of the power series representing f, we can by just looking at the function conclude that the radius of convergence is 1. Moreover, this implies that f can be represented as a convergent power series for $|z| < 1$. We can write the function as

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - z^6 + z^8 - ... = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

Thus the function $f(z) = \frac{1}{1+z^2}$ has the power series representation $\sum_{n=0}^{\infty}(-1)^n z^{2n}$ for $|z| < 1$.

In our second example we want to display that differentiating f by the standard method yields the same result as termwise differentiating the power series of f.

Example 2.6. Consider the well known geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... + z^n + ... \tag{10}$$

whose partial sum $S_N(z) = \sum_{n=0}^{N} z^n$ can be written as

$$1 + z + z^2 + ... + z^N = \frac{1 - z^{N+1}}{1 - z}$$

which for $|z| < 1$ converges to the function $f(z) = \frac{1}{1-z}$ as $n \to \infty$. It is clear that the radius of convergence for (10) is 1, however we can do the same observation as in the previous example which confirms the radius of convergence to be 1. By termwise differentiation of the partial sum of the geometric series we get

$$1 + 2z + 3z^2 + 4z^3 + ... + Nz^{N-1} = \sum_{n=1}^{N} nz^{n-1}.$$

If we now apply the ratio test to the termwise differentiated partial sum we see that they have the same radius of convergence,

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

And by differentiating the function $f(z) = \frac{1}{1-z}$ it comes as no surprise that f' as well has the same radius of convergence as f,

$$f'(z) = \frac{1}{(1-z)^2},$$

where $\sum_{n=1}^{\infty} nz^{n-1}$ is the power series respresentation of $f'(z) = \frac{1}{(1-z)^2}$.

By different methods like the Cauchy-Hadamard's formula and the ratio or the root test, we've been able to investigate for where a power series is uniformly convergent. However, there might be points on the circle of convergence for where the power series converges as well. Abel's limit theorem will give us a tool to investigate pointwise convergence on the boundary.

2.4 Abel's limit theorem

Abel's limit theorem refers to where a power series converges on its circle of convergence.

Theorem 2.5 (Abel's limit theorem ([1], page 41). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series, let z_1 be a point on the boundary of the disk of convergence and assume that the series $\sum_{n=0}^{\infty} a_n z_1^n = s$ is convergent. Then $f(z)$ converges to s if z approaches z_1 in such a way that $|z_1 - z|/(|z_1| - |z|)$ remains bounded.*

Proof. Without loss of generality we assume that $R = 1$ and that the convergence takes place at $z_1 = 1$.
Define

$$s = \sum_{n=0}^{\infty} a_n$$

and the k:th partial sum of s as

$$s_k = \sum_{n=0}^{k} a_n.$$

Then the k:th partial sum of the power series s, $s_k(z) = \sum_{n=0}^{k} a_n z^n$, is

$$s_k(z) = a_0 + a_1 z + a_2 z^2 + ... + a_k z^k. \tag{11}$$

We can rewrite (11) as

$$\begin{aligned} s_k(z) &= s_0 + (s_1 - s_0)z + ... + (s_k - s_{k-1})z^k \\ &= s_0(1 - z) + s_1(z - z^2) + ... + s_{k-1}(z^{k-1} - z^k) + s_k z^k \\ &= (1 - z)(s_0 + s_1 z + ... + s_{k-1} z^{k-1}) + s_k z^k \end{aligned}$$

Assume $|z| < 1$ then $z^k \to 0$ as $k \to \infty$, hence $s_k z^k \to 0$ and we arrive at

$$f(z) = (1 - z) \sum_{n=0}^{\infty} s_n z^n.$$

Now, the formula of the geometric series implies that

$$1 = (1 - z) \sum_{n=0}^{\infty} z^n,$$

and by multiplying both sides with s yields

$$s = (1 - z) \sum_{n=0}^{\infty} s z^n.$$

Assume $|1-z| \leq K(1-|z|)$. Since $s_n \to s$ we have that given an $\epsilon > 0$ we find an $N \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{2K}$ for $n \geq N$. We have

$$
\begin{aligned}
|f(z)-s| &= \left|(1-z)\sum_{n=0}^{\infty} s_n z^n - (1-z)\sum_{n=0}^{\infty} s z^n\right| \\
&= \left|(1-z)\sum_{n=0}^{\infty}(s_n - s)z^n\right| \\
&= \left|(1-z)\sum_{n=0}^{N}(s_n - s)z^n + (1-z)\sum_{n=N+1}^{\infty}(s_n - s)z^n\right| \\
&\leq \left|(1-z)\sum_{n=0}^{N}(s_n - s)z^n\right| + \left|(1-z)\sum_{n=N+1}^{\infty}(s_n - s)z^n\right|
\end{aligned}
\tag{12}
$$

Since $s_n \to s$ and $|z| < 1$ we get the inequality for the second term on the last line in (12) as

$$
\begin{aligned}
\left|(1-z)\sum_{n=N+1}^{\infty}(s_n - s)z^n\right| &\leq |1-z|\sum_{n=N+1}^{\infty}|s_n - s||z|^n \\
&\leq |1-z|\frac{\epsilon}{2K}\sum_{n=N+1}^{\infty}|z|^n \\
&= \frac{|1-z||z|^{N+1}}{1-|z|}\frac{\epsilon}{2K} < K\frac{\epsilon}{2K} = \frac{\epsilon}{2}.
\end{aligned}
$$

And for the first term on the last line in (12) we can make $(1-z)$ arbitrary small by choosing z sufficiently close to 1. Namely, there exists an $M \in \mathbb{R}$ such that $\left|\sum_{n=0}^{N}(s_n - s)z^n\right| \leq M$ for all $|z| < 1$. So for any given $\epsilon > 0$ we can choose z sufficiently close to 1 such that

$$
|1-z|M < \frac{\epsilon}{2}.
$$

Thus we have

$$
\begin{aligned}
|f(z)-s| &< |1-z|M + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

and it follows that $\lim_{z\to 1^-} f(z) = s$. □

Remark. The condition of $\frac{|1-z|}{1-|z|}$ to be bounded when z approaches 1 means that z needs to approach the vertex inside an angle of $< \pi$. One say, z must tend to the vertex non-tangentially or - be inside a Stolz angle of the vertex. According to the figure below, if z is approaching the vertex within the grey area then z is said to be approaching the vertex within a Stolz angle.

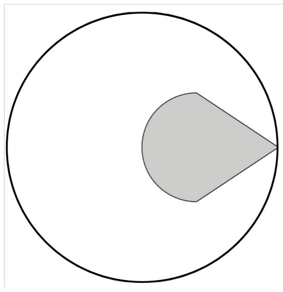

Figure 1: Stolz angle.

Lets look at two examples.

Example 2.7. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n.$$

for $|z| < 1$. We note that by differentiating we get the geometric series

$$\frac{d}{dz}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n\right] = \sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1}.$$

Let $m = n - 1$ then

$$\begin{aligned}\sum_{m=0}^{\infty} (-1)^m z^m &= \sum_{m=0}^{\infty} (-z)^m \\ &= \frac{1}{1+z}.\end{aligned}$$

By integrating both sides along some curve γ, from 0 to z, such that both γ and z are contained in $\mathbb{D}$:

$$\int_0^z \sum_{m=0}^{\infty} (-1)^m t^m dt = \int_0^z \frac{1}{1+t} dt$$

Computing both sides yields

$$\sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{m+1} = \log(1+z) + \log(1) = \mathrm{Log}(1+z),$$

by the definition of the multivalued logarithm function.
Now by changing back the index we arrive at

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} = \mathrm{Log}(1+z), \quad |z| < 1.$$

By Abel's limit theorem; if we can show that

$$\sum_{n=0}^{\infty} a_n z_1^n$$

converges for a point z_1 on the circle of convergence then

$$f(z) \to s \quad \text{as} \quad z \to z_1.$$

So for $z_1 = 1$ and by the alternating series test, the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges. Thus by Abel's limit theorem it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \operatorname{Log}(2).$$

Instead of $z_1 = 1$, consider $z_1 = -1$. Then we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n z_1^n &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} \\ &= -\sum_{n=1}^{\infty} \frac{1}{n},\end{aligned}$$

which we recognize as the harmonic series which we know diverges.
We conclude that the series representing $\operatorname{Log}(1+z)$ converges as z tends to 1 which tells us that we can extend the function to be well defined at $z = 1$ as well, on the other hand as z tends to -1 the series diverges and the function in our case ($\operatorname{Log}(0)$) is undefined. However, the function in a general case does not need to be undefined. Since look at the example of

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which is no where convergent on the circle of convergence. However, f is a well-defined function at $z = -1$.

However, as we going to see, there are conditions that guarantee pointwise convergence for all points on the boundary of convergence except for possibly $z = 1$.

Theorem 2.6 (Abel's criterion). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence equal to 1. If there is an $N \in \mathbb{N}$ such that $\{a_n\}_{n\in\mathbb{N}}$ is monotonically decreasing for $n \geq N$ and $\lim_{n\to\infty} a_n = 0$ then the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges pointwise on the boundary except for possibly $z = 1$.*

Proof. Let

$$f_N(z) = \sum_{n=0}^{N} a_n z^n \tag{13}$$

denote the N:th partial sum of $f(z) = \sum_{n=0}^{\infty} a_n z^n$. By multiplying by $(1-z)$ on both sides in (13) we obtain

$$\begin{aligned}(1-z)f_N(z) &= \sum_{n=0}^{N} a_n(1-z)z^n \\ &= \sum_{n=0}^{N} a_n z^n - \sum_{n=0}^{N} a_n z^{n+1} \\ &= a_0 + \sum_{n=1}^{N} a_n z^n - \sum_{n=1}^{N+1} a_{n-1} z^n \\ &= a_0 - a_N z^{N+1} + \sum_{n=1}^{N} a_n z^n - \sum_{n=1}^{N} a_{n-1} z^n,\end{aligned}$$

where a_0 is a constant and $a_N z^{N+1} \to 0$ for $|z| < 1$ as $N \to \infty$. We are left to show that

$$\sum_{n=1}^{N} (a_n - a_{n-1}) z^n$$

converges. We have

$$\left| \sum_{n=1}^{N} (a_n - a_{n-1}) z^n \right| \leq \sum_{n=1}^{N} |a_n - a_{n-1}||z|^n,$$

and since $a_{n-1} \geq a_n$ we have

$$\sum_{n=1}^{N} |a_n - a_{n-1}||z|^n \leq \sum_{n=1}^{N} (a_{n-1} - a_n),$$

which is a convergent telescoping sum. We conclude that the sum $\sum_{n=0}^{\infty} a_n z^n$ is converging pointwise on the boundary except for possibly $z = 1$. $\square$

Example 2.8. Consider the following two power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{z^n}{n}.$$

Both power series have monotonically decreasing sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 0$. Thus both power series converging pointwise on the boundary except for possibly $z = 1$ by Abel's criterion. For the first series with $z = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{14}$$

which we recognize as a p-series. From basic knowledge about p-series we know that they converges for $p > 1$, and thus it follows that (14) is pointwise convergent for every point on the boundary. For $z = 1$ in the second power series we get the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which we know is divergent. The power series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ is thus pointwise convergent for every point on the boundary except for $z = 1$ by Abel's criterion.

So far we've been looking at holomorphic functions and their relation to power series within the circle of convergence. Furthermore, from the theorems provided by Abel we've been able to investigate points on the circle of convergence. We saw that there is no general fact about the convergence on the boundary. Namely, there are power series for which diverges everywhere on the boundary as well as there are power series that converges pointwise everywhere on its circle of convergence. Onwards in this section we are going to look at properties of holomorphic functions within its radius of convergence.

2.5 Mean Value property

Let $f(x)$ be a real-valued continuous function defined on an interval $[a, b]$ with $F'(x) = f(x)$ throughout $[a, b]$. Then the average value of $f(x)$ over $[a, b]$ is given by

$$f_{a.v.} = \frac{1}{b-a} \int_a^b f(x)dx = \frac{F(b) - F(a)}{b-a}.$$

Furthermore, by the mean value theorem, we have that $f_{a.v.} = F'(\zeta)$ for some $\zeta \in [a, b]$. Our next theorem will show us that if f is holomorphic inside and on a circle, say C, contained in a domain for which f is holomorphic, then the average value of f on the circle C is equal to f's value in the center of the circle. It should not come as a surprise, since from Cauchy's Integral formula we know that the behaviour of f inside C is completely determined by its behaviour on C. From Cauchy's Integral formula we derive the following theorem.

Theorem 2.7 (Gauss's Mean Value Theorem ([5], page 275))**.** *Suppose $f(z)$ is holomorphic in the closed disk $|z-z_0| \leq r$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Proof. By Cauchy's Integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

With the parameterization $\zeta = z_0 + re^{i\theta}$ with $d\zeta = rie^{i\theta}d\theta$, $0 \leq \theta \leq 2\pi$. Then

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

We therefore conclude that holomorphic functions possess the mean value property. □

The obvious example is when f equals the constant function K. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} K d\theta = \frac{1}{2\pi}(2\pi K - 0 \cdot K) = K.$$

2.6 Maximum modulus theorem

The Maximum Modulus theorem shows us that for a non-constant function, there must be points on the circle $|z - z_0| = r$ for which

$$|f(z)| > |f(z_0)|.$$

We will now prove it in two parts.

Theorem 2.8 (Maximum modulus theorem part I ([5], page 275))**.** *If $f(z)$ is holomorphic in an open domain $D \subset \mathbb{C}$, then $|f(z)|$ cannot attain a maximum in D unless $f(z)$ is constant.*

Proof. Suppose $|f(z)|$ attains a maximum in a point $z_0 \in D$. Then we choose a disk $|z - z_0| \leq r$ contained in D and by Gauss's mean value theorem we have the equality

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

If we take the absolute value of both sides we get the inequality,

$$\begin{aligned} |f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|d\theta. \end{aligned} \tag{15}$$

But by assumption we have

$$|f(z_0 + re^{i\theta})| \leq |f(z_0)|,$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| d\theta = |f(z_0)|. \tag{16}$$

From (15) and (16) we get

$$\frac{1}{2\pi}\int_0^{2\pi} (|f(z_0)| - |f(z_0 + re^{i\theta})|) d\theta = 0.$$

Since $|f(z_0+re^{i\theta})| \leq |f(z_0)|$ it follows that the integrand is non-negative and therefore

$$|f(z_0)| = |f(z_0 + re^{i\theta})|$$

for all $\theta \in [0, 2\pi]$. Hence, $f(z) = f(z_0)$ for every z on $|z - z_0| = r$ and by the Identiy theorem, which is stated and proven further down in this section (page 28 theorem 2.14), $|f(z)| = |f(z_0)|$ for all $z \in |z - z_0| \leq r$. In fact, by the Identity theorem $f(z)$ is constant in the whole domain D and therefore $f(z)$ cannot attain a maximum at a point $z \in D$ unless $f(z)$ is constant. □

Theorem 2.9 (Maximum modulus theorem part II ([5], page 276))**.** *If $f(z)$ is holomorphic in a bounded domain D and continuous on its closure $\overline{D}$, then $|f(z)|$ attains maximum on the boundary ∂D. Furthermore, $|f(z)|$ does not attain a maximum at an interior point unless $f(z)$ is constant.*

Proof. We observe that $\overline{D}$ is a compact set because D is bounded. Since it is compact and $|f(z)|$ is a continuous real function on $\overline{D}$, $|f(z)|$ attains a maximum somewhere on $\overline{D}$. But by the first part of the Maximum modulus theorem, theorem 2.8, the maximum cannot be attained at an interior point, hence the maximum must occur on the boundary ∂D. □

Two remarks:

Remark. We display by an example that the domain D does not need to be simply connected.

Example 2.9. Let f be a holomorphic function in the open annulus

$$\frac{1}{r} < |z| < R,$$

and continuous on the closed annulus

$$\frac{1}{r} \leq |z| \leq R.$$

Then by theorem 2.9, f must attain its maximum on the boundary. If $f(z) = z$ then the maximum is attained at the outer boundary and if $f(z) = \frac{1}{z}$ then maximum is attained on the inner boundary.

Remark. Note also the necessity for D to be bounded. For example; if

$$f(z) = e^z \quad D = \{z \in \mathbb{C} : Re(z) > 0\},$$

then ∂D is the imaginary axis and $|f(iy)| = |e^{iy}| = 1$. But $|f(x)| = |e^x| \to \infty$ as $x \to \infty$ and thus the condition for D to be bounded is essential for 2.9.

2.7 Liouville's theorem

Interesting facts occur when we consider upper bounds of the moduli of holomorphic functions. As we apply it to the Cauchy integral formula we get what is called the Cauchy estimates for the derivatives of a holomorphic function.

Theorem 2.10 (Cauchy estimates ([8], page 215))**.** *Let f be holomorphic inside and on a circle C_r of radius $r > 0$ centered about z_0. If $|f(z)| \leq M$ for all z on C_r, then the derivatives of f at z_0 satisfy*

$$\left|f^{(n)}(z_0)\right| \leq \frac{Mn!}{r^n} \quad n = 1, 2, 3, \ldots.$$

Proof. Let C_r be positively oriented. Then we have by the Cauchy Integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

By assumption $|f(z)| \leq M$ for any ζ on C_r, and the length of $C_r = 2\pi r$. Let C_r be parametrized as $\zeta = z_0 + re^{i\theta}$ with $d\zeta = rie^{i\theta} d\theta$, $0 \leq \theta \leq 2\pi$ then

$$\begin{aligned}
|f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} d\theta \right| \\
&\leq \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} \right| d\theta \\
&= \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + re^{i\theta})|}{r^n |e^{in\theta}|} d\theta \\
&= \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + re^{i\theta})|}{r^n} d\theta
\end{aligned}$$

Since $|f(z)| \leq M$, the theorem now follows,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^n} d\theta = \frac{Mn!}{r^n}.$$

□

Consider the case where f is holomorphic and bounded by some number M over the whole complex plane. Then f is bounded for any point $z_0 \in \mathbb{C}$ and any $r > 0$. If we choose n to be equal to 1 and let $r \to \infty$ we see that f' vanishes everywhere and thus f must be constant. This result is known as Liouville's theorem.

Theorem 2.11 (Liouville's theorem ([8], page 215))**.** *The only bounded entire functions are the constant functions.*

2.8 Uniqueness/Identity

In this last section of basics we are going to look at the uniqueness/identity theorems for a holomorphic function within its domain of holomorphicity. We are actually laying the foundation for analytic continuation where the upcoming properties will play a vital role.

The next theorem is about how the behaviour of a holomorphic function at a sequence of points influences its behaviour elsewhere.

Theorem 2.12 ([5], page 269). *Suppose $f(z)$ is holomorphic in the open disk $D = \{z \in \mathbb{C} : |z - z_0| < R\}$, and that there is a sequence $\{z_n\}$ of distinct points with accumulation point z_0. If $f(z_n) = 0$ for each $n \in \mathbb{N}$, then $f(z) \equiv 0$ for all $z \in D$.*

Proof. Since f is holomorphic inside the disk $|z - z_0| < R$ it can be represented as a power series,

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad |z - z_0| < R.$$

For a_0 we have

$$a_0 = f(z_0) = f(\lim_{n\to\infty} z_n).$$

Since f is holomorphic, and hence continuous on D, by continuity

$$f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = 0. \tag{17}$$

So from (17) we know that the series does not have a constant term. Therefore we can write

$$f(z) = (z - z_0)\left[a_1 + \sum_{n=2}^{\infty} a_n (z - z_0)^{n-1}\right].$$

We divide both sides with $(z - z_0)$ which yields

$$\frac{f(z)}{z - z_0} = a_1 + \sum_{n=2}^{\infty} a_n (z - z_0)^{n-1}.$$

We get the value for a_1 at $z = z_0$ by applying the same kind of argument as in the first case,

$$a_1 = \lim_{\substack{z\to z_0 \\ z\neq z_0}} \frac{f(z)}{z - z_0} = \lim_{n\to\infty} \frac{f(z_n)}{z_n - z_0} = 0.$$

Now assume $a_j = 0$ for $j \leq k$. Then we have

$$f(z) = (z - z_0)^{k+1} \left(a_{k+1} + \sum_{n=k+2}^{\infty} a_n (z - z_0)^{n-(k+1)} \right).$$

Like in the previous case dividing both sides with $(z - z_0)^{k+1}$ yields

$$\frac{f(z_n)}{(z - z_0)^{k+1}} = \left(a_{k+1} + \sum_{n=k+2}^{\infty} a_n (z - z_0)^{n-(k+1)} \right).$$

And again from continuity it follows that

$$a_{k+1} = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z)}{(z - z_0)^{k+1}} = \lim_{n \to \infty} \frac{f(z_n)}{(z_n - z_0)^{k+1}} = 0.$$

Thus f has a power series representation that vanishes in the open disk $|z - z_0| < R$ and therefore we conclude that $f(z) \equiv 0$ for every point on $|z - z_0| < R$. $\square$

Corollary 2.1 ([5], page 270)**.** *Suppose $f(z)$ is holomorphic at a point $z = z_0$. Then either $f(z) \equiv 0$ in a neighborhood of z_0, or there exists a real number R' such that $f(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq R'$.*

Proof. Assume there is no such R'. That means in each punctured disk $0 < |z - z_0| < \frac{1}{n}$ there exists a point z_n such that $f(z_n) = 0$. Since $z_n \to z_0$, theorem 2.12 implies that $f(z) \equiv 0$ in some neighborhood of z_0. $\square$

In other words, if there is a z in every neighborhood of z_0 where $f(z) = 0$ then $f(z) \equiv 0$. The uniqueness theorem will now follow from theorem 2.12 and corollary 2.1.

Theorem 2.13 (Uniqueness theorem ([5], page 270))**.** *Suppose f is holomorphic in a domain D, and that $\{z_n\}$ is a sequence of distinct points with accumulation point $z_0 \in D$. If $f(z_n) = 0$ for each n, then $f(z) \equiv 0$ throughout D.*

Proof. Since z_0 is an accumulation point of $\{z_n\}$ there is at least one point $z \in \{z_n\}$ in every neighbourhood of z_0 where $f(z) = 0$. The theorem now follows from corollary 2.1 and is just a generalisation of theorem 2.12. $\square$

Theorem 2.14 (Identity theorem ([5], page 271))**.** *Suppose $\{z_n\}$ is a sequence of points with accumulation point z_0 in a domain D. If f and g are holomorphic in D, with $f(z_n) = g(z_n)$ for each n, then $f \equiv g$ throughout D.*

Proof. Let $h(z) = f(z) - g(z)$ then h is holomorphic in D and $h(z_k) = f(z_k) - g(z_k) = 0$ for all points $\{z_k\}$. Then by the uniqueness theorem $h \equiv 0$ throughout D which implies that $f \equiv g$ throughout D. $\square$

Remark. The Identity theorem is often referred to as the principle of analytic continuation.

3

Analytic continuation

We have so far been examining holomorphic functions and their power series representation inside the circle of convergence as well as done some investigation of the behaviour on the boundary of the disk of convergence. We will now see if we can go beyond that boundary. That is, let f be a holomorphic function in a domain $D_0 \subset \mathbb{C}$. Is there a way for us to extend f to a larger domain $D_0 \cup D_1$ where $D_0 \cap D_1 \neq \emptyset$? More concretely, if f_0 is holomorphic in a domain D_0, is there a holomorphic function f_1 in a different domain D_1 that agrees with f_0 in $D_0 \cap D_1$? Analytic continuation deals with the problem of properly redefining a holomorphic function so its domain of holomorphicity gets bigger.

3.1 Analytic continuation by Weierstrass [2]

Let

$$P_k = \sum_{n=0}^{\infty} a_{n,k}(z - z_k)^n$$

be a power series centered at $z_k \in \mathbb{C}$ with positive radius of convergence denoted $R(P_k) > 0$. The series is determined by its center z_k, and the sequence $(a_{n,k})_{n,k\in\mathbb{N}}$ of complex coefficients. As we have seen, a power series P_k with $R(P_k) > 0$ defines a holomorphic function f_k in the open disk

$$D_k = \{z \in \mathbb{C} : |z - z_k| < R(P_k)\}.$$

Now let f_0 be the holomorphic function in D_0 for which P_0 with $R(P_0) > 0$ is the power series representation centered at z_0. For any point $z_1 \neq z_0 \in D_0$ there is a power series expansion P_1, centered at z_1, for which P_1 converges uniformly in the disk D_1 with positive radius of convergence

$$R(P_1) \geq R(P_0) - |z_1 - z_0|.$$

The new power series, P_1, defines a holomorphic function f_1 in D_1 where

$$f_0(z) = f_1(z) \quad \forall z \in D_0 \cap D_1.$$

[2]We follow closely Lars Ahlfors reasoning about Analytic continuation by Weierstrass which can be found in [1], page 283-284.

We say that f_1 is obtained from f_0 by direct analytic continuation. Together f_0 and f_1 define a holomorphic function in $D_0 \cup D_1$. If $D_1 \not\subseteq D_0$ then the new function f_1 is an extension of f_0 into D_1, which is the purpose of the construction. In the general case we have to consider a succession of power series, $P_0, P_1, ..., P_n$,

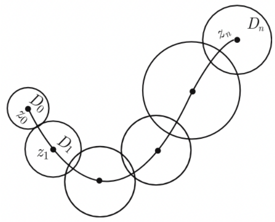

Figure 2: Analytic continuation.

each of which is a direct analytic continuation of the previous one,

$$f_{k-1} = f_k \quad \forall z \in D_{k-1} \cap D_k,$$

then we say that P_n is an analytic continuation of P_0. The set of all power series P_k that can be obtained from analytic continuation from P_0 is called an analytic function in the sense of Weierstrass.

Remark. It is important to understand that it does not always follow that $f_0, ..., f_n$ defines a single-valued function in $D_0 \cup ... \cup D_n$, because if $D_k \cap D_h \neq \emptyset$ with $h \neq k-1$ or $h \neq k+1$, there is no guarantee that

$$f_k = f_h \quad \forall z \in D_k \cap D_h.$$

We are going to see examples of this later on but it is precisely this possibility that leads beyond the notion of a function to have just one value at each point of its domain.

Instead of thinking about analytic continuation in terms of power series and their circle of convergence we introduce a more direct approach.

3.2 Analytic continuation by function elements

Definition 3.1. ([5],page 447) A function f, together with a domain D in which f is holomorphic is said to be a function element (f, D). Two function elements (f_0, D_0) and (f_1, D_1) are called a direct analytic continuation of each other if and only if $D_0 \cap D_1 \neq \emptyset$ and $f_0 = f_1$ on $D_0 \cap D_1$.

Whenever there is an analytic continuation of (f_i, D_i) into D_j it must be uniquely determined since two analytic continuations have to agree on $D_i \cap D_j$ and therefore by the Uniqueness and Identity theorem 2.13 and 2.14 would consequently agree throughout D_j. That means that there is at most one way to analytic continue a function f from one domain to another. Suppose $\{(f_0, D_0), ..., (f_n, D_n)\}$ is a finite set of function elements such that (f_k, D_k) and (f_{k+1}, D_{k+1}) are direct analytic continuations of each other for $k = 0, 1, ..., n-1$. Then the set of function elements are said to be analytic continuations of one another. Such a set of function elements is called a chain.

Example 3.1. As a trivial example of analytic continuation let $f(z) = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Thus we have the function element $(f, \mathbb{D})$ with

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

A direct analytic continuation of $(f, \mathbb{D})$ to (g, D) where $D = \mathbb{C} \setminus \{1\}$ is

$$(g, D) = \left(\frac{1}{1-z}, \mathbb{C} \setminus \{1\} \right).$$

Then we have $f = g$ on $\mathbb{D}$ and g is the direct analytic continuation of $(f, \mathbb{D})$ to the punctured complex plane $\mathbb{C} \setminus \{1\}$. Note that this is the maximal continuation of $(f, \mathbb{D})$ and f can not be analytically continued to $z = 1$ since it is not a removable singularity.

Let's continue by another example but this time looking at an analytic continuation along a smooth curve γ. Let $\gamma : [0, 1] \to \mathbb{C}$ be a smooth curve and assume there exists a chain $\{(f_i, D_i)\}_{1 \le i \le n}$, of function elements such that

$$\gamma([0, 1]) \subset \bigcup_{i=1}^{n} D_i, \quad z_0 = \gamma(0) \in D_0, \quad z_n = \gamma(1) \in D_n,$$

then we say that the function element (f_n, D_n) is an analytic continuation of (f_0, D_0) along the curve γ. We define (f, D) to be the function analytically continued along the curve γ if there is a chain $\{(f_i, D_i)\}_{1 \le i \le n}$ such that each point on the curve is contained in the domain of some function element of the chain.
Given a chain $\{(f_i, D_i)\}_{0 \le i \le n}$, is it possible for a function $f(z)$ to be defined such that $f(z) = f_i(z)$ for $z \in D_i$? Indeed, for $n = 2$ and $D_1 \cap D_2 \neq \emptyset$ we can write the function

$$f(z) = \begin{cases} f_0(z), & \text{if } z \in D_0 \\ f_1(z), & \text{if } z \in D_1 \end{cases}$$

which is holomorphic in $D_1 \cup D_2$. However, as was mentioned in the previous remark, the general case fails as we are going to see now in the upcoming examples.

Example 3.2. Consider the chain $(f_1, D_1), ..., (f_4, D_4)$ of function elements that is analytically continued along the unit circle in the counterclockwise direction starting at $z = 1$ and such that no domain contains the origin and $D_1 \cap D_4 \neq \emptyset$.

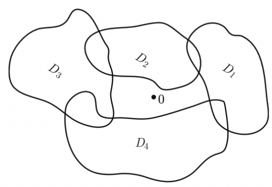

Figure 3: Continuation around the origin.

For a fixed branch of the multi-valued logarithm function we have $f_1(z) = \log(z)$ in D_1. Since (f_2, D_2) is an direct analytic continuation of (f_1, D_1) it is uniquely determined by f_1 in D_1. That is $f_1 = f_2$ in $D_1 \cap D_2$. Further we have that (f_2, D_2) determines (f_3, D_3) uniquely which in turn determines (f_4, D_4) uniquely. When we again arrive at the starting point we have that even though the domains D_1 and D_4 overlap, $f_1 \neq f_4$ in $D_1 \cap D_4$ since there are not an direct analytic continuation between them. Actually, $f_4 = f_1 + 2\pi i$ for all points in $D_1 \cap D_4$, which is the difference of the argument of the multi-valued logarithm function going around the origin one time. Note that if we instead gone in the clockwise direction we would have end up with $f_4 = f_1 - 2\pi i$ for all points in $D_1 \cap D_4$.

Lets consider another multi-valued example, namely $f(z) = \sqrt{z}$.

Example 3.3 ([9], page 120). Let $f_0(z) = \sqrt{z}$ be defined on the disk $D_0 = \{z \in \mathbb{C} : |z-1| < 1\}$. Also, let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed contour given by $\gamma(t) = e^{it}$ starting from $z_0 = \gamma(0) = 1$. Then $f(z)$ has an analytic continuation along γ.

First, the binomial series formula lets us write

$$f_0(z) = \sqrt{z} = (1 + (z-1))^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{1/2}{n} (z-1)^n, \tag{18}$$

which is the power series expansion of the principal branch of $\sqrt{z}$. We apply the ratio test to the sum in (18) which yields

$$\lim_{n\to\infty} \left| \frac{\binom{1/2}{n+1}}{\binom{1/2}{n}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(1/2)(-1/2)...(1/2-n+1)(1/2-(n+1)+1)}{(n+1)!}}{\frac{(1/2)(-1/2)...(1/2-n+1)}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1/2 - n}{n+1} \right| = 1.$$

Now let us choose a point z_1 on γ that lies within the circle of convergence of f_0 and see if we can extend the domain for which f is holomorphic. Since we know that every point inside the circle of convergence of a power series has again a power series expansion with radius of at least $R(z_0) - |z - z_0|$. Again by the binomial series formula we can write

$$\begin{aligned} f_1(z) = \sqrt{z} &= \sqrt{z_1}\sqrt{\frac{z}{z_1}} \\ &= \sqrt{z_1}\sqrt{1+\frac{z}{z_1}-1} \\ &= \sqrt{z_1}\sum_{n=0}^{\infty}\binom{1/2}{n}(\frac{z}{z_1}-1)^n \\ &= \sqrt{z_1}\sum_{n=0}^{\infty} z_1^{-n}\binom{1/2}{n}(z-z_1)^n. \end{aligned}$$

Now let $z_1 = e^{\frac{2\pi i}{9}}$, then the power series expansion about z_1 is

$$f_1(z) = e^{\frac{i\pi}{9}}\sum_{n=0}^{\infty} e^{\frac{-2\pi i n}{9}}\binom{1/2}{n}(z-e^{\frac{2\pi i}{9}})^n.$$

Computing the radius of convergence for $f_1(z_1)$ by the ratio test yields

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{e^{\frac{-2\pi i(n+1)}{9}}\binom{1/2}{n+1}}{e^{\frac{-2\pi i n}{9}}\binom{1/2}{n}}\right| &= \lim_{n\to\infty}\left|\frac{e^{\frac{-2\pi i(n+1)}{9}}}{e^{\frac{-2\pi i n}{9}}}\right|\left|\frac{\binom{1/2}{n+1}}{\binom{1/2}{n}}\right| \\ &= \lim_{n\to\infty}\left|e^{\frac{-2\pi i}{9}}\right|\left|\frac{\binom{1/2}{n+1}}{\binom{1/2}{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\binom{1/2}{n+1}}{\binom{1/2}{n}}\right| = 1. \end{aligned}$$

That means that we have extended f_0 to a larger domain with $f_0 = f_1$ in $D_0 \cap D_1$. Now for $k = 0, 1, ..., 9$ we define $z_k = e^{\frac{2\pi i k}{9}}$. From

$$f_k(z) = (z_k)^{\frac{1}{2}}\sum_{n=0}^{\infty} z_k^{-n}\binom{1/2}{n}(z-z_k)^n,$$

we get that

$$D_k = \{z \in \mathbb{C} : |z - z_k| < 1\},$$

and thus all power series have radius of convergence equal to 1. They also form a chain of function elements $\{(f_i, D_i)\}_{0\leq i\leq 9}$ as seen in figure 4 where $D_0 \cap D_9 = D_0$,

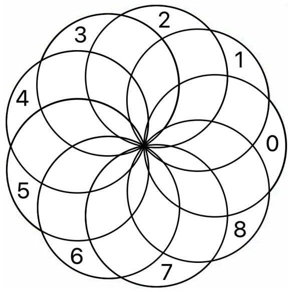

Figure 4: A chain of function elements each centered at a point on the unit circle with radius of convergence equal to 1.

each of which is a direct analytic continuation of the previous one. However (f_0, D_0) is not a direct analytic continuation of neither $(f_6, D_6)(f_7, D_7)(f_8, D_8)$, in particular

$$f_0(z) \neq f_9(z) \quad \forall z \in D_0 \cap D_9.$$

In fact

$$\begin{aligned} f_9(z) &= (e^{\frac{2\pi i 9}{9}})^{\frac{1}{2}} \sum_{n=0}^{\infty} (e^{\frac{2\pi i 9}{9}})^{-n} \binom{1/2}{n} (z - e^{\frac{2\pi i 9}{9}})^n \\ &= e^{i\pi} \sum_{n=0}^{\infty} e^{-2\pi i n} \binom{1/2}{n} (z - e^{2\pi i})^n \\ &= -\sum_{n=0}^{\infty} \binom{1/2}{n} (z-1)^n = -\sqrt{z}, \end{aligned}$$

we've reached the other branch of the square root. Therefore we have

$$f_0(z) = -f_9(z) \quad \forall z \in D_0 \cap D_9.$$

With these two examples on our mind one might ask if there are conditions for where we can perform analytic continuation along different paths with common end points and be sure that at the terminal point we have the same function. And indeed, the condition is that two paths, in a region D, need to be continuously deformable into each other, one say; two paths need to be homotopic in D. A more precise definition of two curves being homotopic will follow further down in the following section.

3.3 Monodromy

Definition 3.2 ([3], page 214)**.** For a given function element (f, D) define the germ of f at z_0 to be the collection of all function elements (g, B) such that $z_0 \in D$ and $f(z) = g(z)$ for all z in a neighborhood of z_0. We denote the germ by $[f]_{z_0}$.

Note that $(g, B) \in [f]_{z_0}$ if and only if $(f, D) \in [g]_{z_0}$.

Definition 3.3 ([3], page 214)**.** Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and suppose that for each $t \in [0, 1]$ there is a function element (f_t, D_t) such that

(i) $\gamma(t) \in D_t$,

(ii) for each $t \in [0, 1]$ there is a $\delta > 0$ such that $|s - t| < \delta$, $s \in [0, 1]$, implies that $\gamma(s) \in D_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}.$$

Then $\{(f_t, D_t) : 0 \le t \le 1\}$ is an analytic continuation of (f_0, D_0) along the path γ; or, $\{(f_t, D_t) : 0 \le t \le 1\}$ is obtained from (f_0, D_0) by analytic continuation along γ.

Whether or not there is an analytic continuation along a curve and a given function element is not always an easy question. However for a fixed curve the following proposition is going to show us that two different analytic continuations along the same curve results in the same function elements.

Proposition 3.1 ([3], page 215)**.** *Let $\gamma : [0, 1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$.*

Proof. The proposition follows if we can show that the set

$$T = \{t \in [0, 1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$$

is both open and closed in $[0, 1]$. Since $0 \in T$ so T is nonempty and it will follow that $T = [0, 1]$ so that in particular $1 \in T$.
To show that T is open we fix $t \in T$. By the definition of analytic continuation there is a $\delta > 0$ such that for $|s - t| < \delta$, then $\gamma(s) \in B_t \cap D_t$ and

$$\begin{cases} [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, \\ [g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}. \end{cases} \tag{19}$$

But since $t \in T$,

$$f_t(z) = g_t(z) \quad \forall z \in B_t \cap D_t,$$

hence

$$[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} \quad \forall \gamma(s) \in B_t \cap D_t.$$

It follows from (19) that

$$[f_s]_{\gamma(s)} = [g_s]_{\gamma(s)} \quad \text{whenever} \quad |s - t| < \delta.$$

That is, $(t-\delta, t+\delta) \cap [0,1] \subset T$ and T is open.
To show that T is closed let t be an accumulation point of T. Then there exists a $\delta > 0$ such that $\gamma(s) \in B_t \cap D_t$ and (19) is satisfied whenever $|s-t| < \delta$. Since t is an accumulation point of T, there is a point s in T with $|s-t| < \delta$, so that $\gamma(s)$ is contained in

$$G = B_t \cap D_t \cap B_s \cap D_s,$$

and therefore G is a nonempty set. Then

$$f_s(z) = g_s(z) \quad \forall z \in G$$

by the definition of T. But that means that

$$f_t(z) = g_t(z) \quad \forall z \in G,$$

and because G has an accumulation point in $B_t \cap D_t$ we obtain

$$[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}.$$

That is t in T and thus T is closed. □

Lemma 3.1 ([3], page 218)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. For $0 \leq t \leq 1$ let $R(t) > 0$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(t) \equiv \infty$ or $R : [0,1] \to (0, \infty)$ is continuous.*

Proof. If $R(t) = \infty$ for some $t \in [0,1]$ we are done since then f_t can be extended to an entire function. Because by the definition of analytic continuation we can conclude that

$$f_s(z) = f_t(z) \quad \text{for all } z \in D_s$$

which implies that

$$R(s) = \infty \text{ for each } s \in [0,1] \text{ and } R(s) \equiv \infty.$$

Let $R(t) < \infty$ for all $t \in [0,1]$. Then fix t and define $z_t = \gamma(t)$ and let

$$f_t = \sum_{n=0}^{\infty} a_{n,t}(z - z_t)^n$$

be the power series expansion of f_t about z_t. Now let $\delta_1 > 0$ be such that $|s-t| < \delta_1$ implies that

$$\gamma(s) \in D_t \cap B(z_t; R(t)) \quad \text{and} \quad [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}.$$

Fix s with $|s-t|<\delta_1$ and let $\sigma=\gamma(s)$. Now f_t can be extended to a holomorphic function on $B(z_t;R(t))$. By the Identity theorem, f_s agrees with f_t on a neighborhood of σ, thus f_s can be extended to a holomorphic function on $B(z_t,R(t))\cup D_s$. If f_s has power series expansion

$$f_s=\sum_{n=0}^{\infty}\sigma_n(z-\sigma)^n \quad \text{about} \quad z=\sigma,$$

then the radius of convergence $R(s)$ must be at least as big as the distance from σ to the circle $\Gamma=\{|z-z_t|=R(t)\}$. That is,

$$R(s)\geq d(\sigma,\Gamma)\geq R(t)-|z_t-\sigma|.$$

Which implies that

$$R(t)-R(s)\leq|z_t-\sigma|$$

and thus

$$R(t)-R(s)\leq|\gamma(t)-\gamma(s)|.$$

A similar argument yields that

$$R(s)-R(t)\leq|\gamma(t)-\gamma(s)|$$

and it follows that

$$\max\{R(t)-R(s),-(R(t)-R(s))\}\leq|\gamma(t)-\gamma(s)| \tag{20}$$

for $|s-t|<\delta_1$. Since γ is continuous and defined in a compact domain, then γ is uniformly continuous and thus we have that: for a given $\epsilon>0$ there exists a $\delta_1>0$ such that

$$\gamma(s)\in D_t\cap B(z_t,R(t)),$$

f_s is holomorphic in $B(z_t,R(t))$ and

$$|s-t|<\delta_1 \quad \text{implies} \quad |\gamma(t)-\gamma(s)|<\epsilon.$$

From (20) it follows that if $|s-t|<\delta_1$ then

$$|R(t)-R(s)|<\epsilon$$

and thus R is uniformly continuous in a neighborhood $|s-t|<\delta_1$ of t. $\square$

By using the result of lemma 3.1, the next lemma is going to show us that if two paths are lying sufficiently close, then the analytic continuation along any of the two paths are going to result in the same function elements.

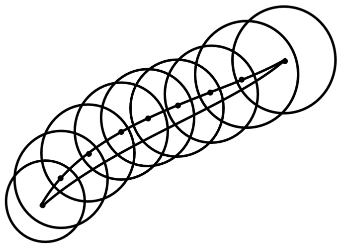

Figure 5: Two paths lying sufficiently close to each other such that analytic continuation along the paths results in the same function elements.

Lemma 3.2 ([3], page 218)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Then there is an $\epsilon > 0$ such that if $\sigma : [0,1] \to \mathbb{C}$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \epsilon$ for all $t \in [0,1]$, and if $\{(g_t, B_t) : 0 \leq t \leq 1\}$ is any analytic continuation along σ with $[g_0]_a = [f_0]_a$, then $[g_1]_b = [f_1]_b$.*

Proof. Let the power series expansion of f_t about $z_t = \gamma(t)$, $0 \leq t \leq 1$ be

$$f_t(z) = \sum_{n=0}^{\infty} a_{n,t}(z - z_t)^n$$

let $R(t) > 0$ be the radius of convergence. By preceding lemma $R(t)$ is a continuous function and since $R(t) > 0$ for all t. Let

$$0 < \epsilon < \frac{1}{2} \min_{0 \leq t \leq 1} R(t) \tag{21}$$

and suppose σ and $\{(g_t, B_t)\}$ are as in the statement of the lemma. Furthermore suppose that D_t is the disk given by

$$D_t = \{z \in \mathbb{C} : |z - z_t| < R(t)\}.$$

Since $|\gamma(t) - \sigma(t)| < \epsilon < R(t)$, then $\sigma(t) \in B_t \cap D_t$ for all t in $[0,1]$. To show that $g_t(z) = f_t(z)$ for all $z \in B_t \cap D_t$ we show that this is precisely the case for $t = 1$. To do so we define the set

$$T = \{t \in [0,1] : f_t(z) = g_t(z) \quad \forall z \in B_t \cap D_t\},$$

and show that $1 \in T$. We show that $1 \in T$ by showing that T is nonempty, opened and closed subset of $[0, 1]$. From the hypothesis of the lemma we have

$$[f_0]_a = [g_0]_a \quad \text{and thus} \quad f_0(z) = g_0(z) \quad \forall z \in D_t \cap B_t,$$

so $0 \in T$ and thus T is nonempty. To show that T is open, fix t in T and choose $\delta > 0$ such that

$$\begin{cases} |\gamma(s) - \gamma(t)| < \epsilon, & [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, \\ |\sigma(s) - \sigma(t)| < \epsilon, & [g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}, \text{ and} \\ \sigma(s) \in B_t \end{cases} \tag{22}$$

whenever $|s-t| < \delta$. We will now show that $G = B_s \cap B_t \cap D_s \cap D_t \neq \emptyset$ for $|s-t| < \delta$; in fact, $\sigma(s)$ is in the intersection. If $|s-t| < \delta$ then

$$|\sigma(s) - \gamma(s)| < \epsilon < R(s)$$

so that $\sigma(s) \in D_s$. Also

$$\begin{aligned} |\sigma(s) - \gamma(t)| &= |\sigma(s) - \gamma(s) + \gamma(s) - \gamma(t)| \\ &\leq |\sigma(s) - \gamma(s)| + |\gamma(s) - \gamma(t)| \\ &< 2\epsilon \\ &< R(t), \end{aligned}$$

by (21), so $\sigma(s)$ in D_t. Since we already have that $\sigma(s) \in B_s \cap B_t$ by (22),

$$\sigma(s) \in G.$$

Because $t \in T$ it follows that $f_t(z) = g_t(z)$ for all $z \in G$. Also from (22) we have that

$$f_s(z) = f_t(z) \quad \text{and} \quad g_s(z) = g_t(s) \quad \forall z \in G,$$

which implies $f_s(z) = g_s(z)$ for all $z \in G$. But since G has an accumulation point in $B_s \cap D_s$, it must follow that $s \in T$. That is, $(t-\delta, t+\delta) \cap [0, 1] \subset T$ and thus T is open.
To show that T is closed, let t be an accumulation point in T. Then there is a $\delta > 0$ such that (22) holds whenever $|s-t| < \delta$. Since t is an accumulation point, there exists a point s in T such that $|s-t| < \delta$; so that $\gamma(s)$ is contained in G and therefore; G is a nonempty set. By definition of T we have that

$$f_s(z) = g_s(z) \quad \forall z \in G.$$

From (19) we have that

$$f_s(z) = f_t(z) \quad \text{and} \quad g_s(z) = g_t(z) \quad \forall z \in G,$$

which implies

$$f_t(z) = g_t(z) \quad \forall z \in G.$$

And since G has an accumulation point in $B_t \cap D_t$ we obtain

$$[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}.$$

That is $t \in T$ and thus T is closed since it contains all its accumulation points. □

Definition 3.4 ([3], page 219)**.** Let (f, D) be a function element and let G be a region which contains D; then (f, D) admits unrestricted analytic continuation in G if for any path γ in G with initial point in D there is an analytic continuation of (f, D) along γ.

Definition 3.5. Let $G \subset \mathbb{C}$ and let $\gamma_0 : [0,1] \to G$ and $\gamma_1 : [0,1] \to G$ be two curves. Assume $\gamma_0(0) = \gamma_1(0) = \alpha$ and $\gamma_0(1) = \gamma_1(1) = \beta$. We say that γ_0 and γ_1 are homotopic in G (with fixed endpoints) if there is a continuous function

$$\Gamma : [0,1] \times [0,1] \to G$$

such that

(i) $\Gamma(t, 0) = \gamma_0(t)$ for all $t \in [0,1]$,

(ii) $\Gamma(t, 1) = \gamma_1(t)$ for all $t \in [0,1]$,

(iii) $\Gamma(0, s) = \alpha$ for all $s \in [0,1]$,

(iv) $\Gamma(1, s) = \beta$ for all $s \in [0,1]$.

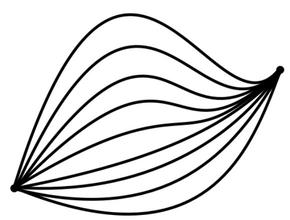

Figure 6: Homotopic paths.

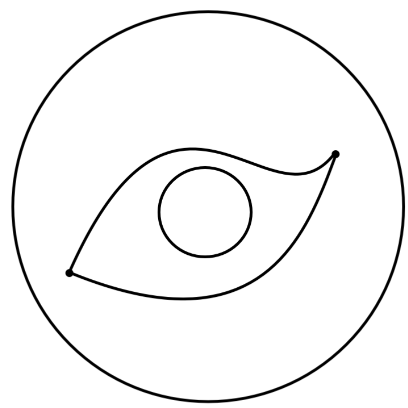

Figure 7: Non-homotopic paths.

Theorem 3.1 (Monodromy ([3], 219)). *Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let α in D and β in G and let γ_0 and γ_1 be paths in G from α to β. let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t, D_t) : 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed-end-points homotopic in G then*

$$[f_1]_\beta = [g_1]_\beta.$$

Proof. Since γ_0 and γ_1 are fixed-end-points homotopic in G there is a continuous function $\Gamma : [0, 1] \times [0, 1] \to G$ such that

$$\Gamma(t, 0) = \gamma_0(t), \quad \Gamma(t, 1) = \gamma_1(t),$$
$$\Gamma(0, u) = \alpha, \quad \Gamma(1, u) = \beta,$$

for all t and u in $[0, 1]$. Fix $0 \leq u \leq 1$ and consider the path γ_u, defined by

$$\gamma_u(t) = \Gamma(t, u),$$

from α to β. By hypothesis on G there is an analytic continuation

$$\{(h_{t,u}, D_{t,u}) : 0 \leq t \leq 1\}$$

of (f, D) along γ_u. It now follows from proposition 3.1 that $[g_1]_\beta = [h_{1,1}]_\beta$ and $[f_1]_\beta = [h_{1,0}]_\beta$. Hence, it is sufficient to show that $[h_{1,1}]_\beta = [h_{1,0}]_\beta$. To do so we introduce the set

$$U = \{u \in [0, 1] : [h_{1,u}]_\beta = [h_{1,0}]_\beta\},$$

and showing that U is a nonempty open and closed subset of $[0, 1]$. Since $0 \in U$, $U \neq \emptyset$. To show that U is both open and closed we have the following claim.

Claim. For $0 \leq u \leq 1$ there is a $\delta > 0$ such that if $|u - v| < \delta$ then

$$[h_{1,u}]_\beta = [h_{1.v}]_\beta.$$

Proof. Fix $0 \leq u \leq 1$ and apply lemma 3.2 to find a $\epsilon > 0$ such that if σ is a path from α to β with

$$|\gamma_u(t) - \sigma(t)| < \epsilon \quad \forall t \in [0, 1],$$

and if $\{(k_t, E_t)\}$ is any continuation of (f, D) along σ, then

$$[h_{1,u}]_\beta = [k_1]_\beta.$$

Now since Γ is a uniformly continuous function, there is a $\delta > 0$ such that if $|u-v| < \delta$ then

$$|\gamma_u(t) - \gamma_v(t)| = |\Gamma(t, u) - \Gamma(t, v)| < \epsilon, \quad \forall t \in [0, 1].$$

The claim now follows by applying that $[h_{1,u}]_\beta = [k_1]_\beta$. □

So suppose $u \in U$ and let $\delta > 0$ be the number given by the previous claim. By the definition of U,

$$(u - \delta, u + \delta) \subset U.$$

So U is open.

If $u \in \overline{U}$, then from the claim there is a $\delta > 0$ and a $v \in U$ such that if $|u - v| < \delta$ then

$$[h_{1,u}]_\beta = [h_{1,v}]_\beta.$$

And since $v \in U$ it follows that

$$[h_{1,v}]_\beta = [h_{1,0}]_\beta.$$

Therefore $[h_{1,u}]_\beta = [h_{1,0}]_\beta$ so that $u \in U$ and hence U is closed. $\square$

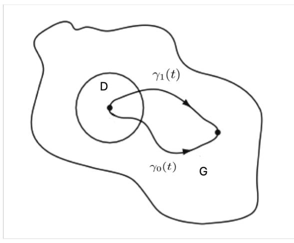

Figure 8: Two paths starting in D ending in G.

The most important consequence of the Monodromy theorem is that given a function element (f, D) which admits unrestricted continuation in a simply connected region G. If $z_0 \in D$, $z \in G$ and γ is a path from z_0 to z and $\{(f_t^\gamma, D_t : 0 \leq t \leq 1)\}$ is an analytic continuation of (f, D) along γ. Then let $F(z, \gamma) = f_1^\gamma(z)$ and since G is simply connected it follows that

$$F(z, \gamma) = F(z, \sigma)$$

for any two paths γ and σ in G from z_0 to z.

Another obvious question would be if there are conditions under which we can't analytically continue a function outside its domain. And indeed, a power series is under certain conditions due to Ostrowski-Hadamard's gap theorem and Fabry's gap theorem, "badly behaved" in the sense that it cannot be extended to be a holomorphic function anywhere on the boundary of its disk of convergence. These theorems are left out but we have the following definition of a natural boundary and an example of a power series that cannot be extended outside its domain.

Definition 3.6 ([6], page 320)**.** If f is a holomorphic function in a region $G \subset \mathbb{C}$ then ∂G is said to be the natural boundary of f if f has no holomorphic extension to any domain which properly contains G.

Consider the following example with $G = \mathbb{D}$ and $\partial G = \mathbb{T}$.

Example 3.4. Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

We want to show that this function cannot be analytically continued beyond its disk of convergence. It is sufficient to show that the singularities of f are dense on $\mathbb{T}$. Let t and N be positive integers and let $z = re^{\frac{2\pi in}{2^N}}$ for some $r < 1$, then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} z^{2^n} \\ &= \sum_{n=0}^{\infty} \left(re^{\frac{2\pi it}{2^N}}\right)^{2^n} \\ &= \sum_{n=0}^{N-1} \left(re^{\frac{2\pi in}{2^N}}\right)^{2^n} + \sum_{n=N}^{\infty} \left(re^{\frac{2\pi it}{2^N}}\right)^{2^n} \end{aligned}$$

For $n \geq N$, $\left(e^{\frac{2\pi it}{2^N}}\right)^{2^n} = 1$ thus

$$\sum_{n=0}^{N-1} \left(re^{\frac{2\pi in}{2^N}}\right)^{2^n} + \sum_{n=N}^{\infty} \left(re^{\frac{2\pi it}{2^N}}\right)^{2^n} = \sum_{n=0}^{N-1} \left(re^{\frac{2\pi in}{2^N}}\right)^{2^n} + \sum_{n=N}^{\infty} r^{2^n}.$$

But as r tends to 1^- then the sum $\sum_{n=N}^{\infty} r^{2^n}$ is unbounded. So for each choice of positive integers t, N, $z = re^{\frac{2\pi it}{2^N}}$ is a singularity of f and since this gives a dense set of singular points on the unit circle and hence a natural boundary to f.

4

Harmonic vs Holomorphic functions

In this section we will investigate the relation between holomorphic functions and harmonic functions. As we are going to see, both the real and imaginary part of a holomorphic function are harmonic functions. As already been implied, this section will be about facts of harmonic functions that are closely related with Cauchy's theorem. Lets start with the definition of a harmonic function.

Definition 4.1 ([1], page 162). A real valued function $u(z)$ or $u(x,y)$ where $z = x+iy$, defined on a domain D, is said to be harmonic in D if it is continuous together with its partial derivatives of the first two orders and satisfies the Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Further, if f is a complex-valued function $f : D \to \mathbb{C}$ then f is harmonic if

$$Re(f) : D \to \mathbb{R},$$
$$Im(f) : D \to \mathbb{R}$$

are both harmonic.

We continue by showing that real valued harmonic functions are the real part of holomorphic functions.

Theorem 4.1 ([4], page 2). *If $u(x,y)$ is harmonic on a simply connected region $D \subset \mathbb{C}$, then $u(x,y)$ is the real part of an holomorphic function $f(z) = u(x,y) + iv(x,y)$.*

Proof. Let $g(x,y) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ and put $p = \frac{\partial u}{\partial x}$ and $q = -\frac{\partial u}{\partial y}$. Then we have $g(x,y) = p + iq$ and

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \Leftrightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y} \Leftrightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x},$$

thus g satisfies the Cauchy-Riemann equations which implies that g is holomorphic and can be denoted $g(z)$. Now for two points, z_0 and z in D, connected via a simple curve γ contained in D, we define f by $f(z) = \int_\gamma g(\zeta)d\zeta$. Since f is holomorphic in D, and D being a simply connected region, we can choose to go from $z_0 = (x_0, y_0)$ to $z = (x,y)$ by integrating over the two straight lines; (x_0, y_0) to (x, y_0) and (x, y_0) to (x,y). That is

$$\begin{aligned} Re\,[f(z)] &= Re\left[\int_{x_0}^{x} \left(\frac{\partial u}{\partial x}(t, y_0) - i\frac{\partial u}{\partial y}(t, y_0)\right) dt + \int_{y_0}^{y} \left(\frac{\partial u}{\partial x}(x,t) - i\frac{\partial u}{\partial y}(x,t)\right) idt\right] \\ &= \int_{x_0}^{x} \frac{\partial u}{\partial x}(t, y_0)dt + \int_{y_0}^{y} \frac{\partial u}{\partial y}(x,t)dt \\ &= u(x, y_0) - u(x_0, y_0) + u(x,y) - u(x, y_0) \\ &= u(x,y) - u(x_0, y_0) \end{aligned}$$

where $u(x_0, y_0)$ is a constant. And thus the theorem follows from $\mathrm{Re} f(z) = u(x,y) - u(x_0, y_0)$. □

In fact, as the following theorem will show; both the real and imaginary part of a holomorphic function are harmonic functions.

Theorem 4.2 ([4], page 2)**.** *Let $z = x + iy$ and write $f(z) = u(x, y) + iv(x, y)$. If f is holomorphic in a region $D \subset \mathbb{C}$ then both u and v are harmonic functions on D.*

Proof. By the Cauchy-Riemann equations we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{23}$$

From the Cauchy-Riemann equations in (23) it follows that

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial y \partial x} = 0 \tag{24}$$

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{25}$$

and now combining (24) and (25) yields

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0, \tag{26}$$

which implies that u is harmonic on D. Similarly we can do the same for v which implies that both the real and imaginary part of a holomorphic function are harmonic on the region where its defined. □

4.1 Mean Value property

We now derive the mean value property for a harmonic function u, from u being the real part of a holomorphic function.

Theorem 4.3 (Mean value property)**.** *If u is a harmonic function then u satisfies the mean value property. That is, suppose u is harmonic on and inside a circle of radius $r > 0$ centered at $z_0 = x_0 + iy_0$. Then*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

Proof. Let $f = u + iv$ be an holomorphic function with u as its real part. The mean value property for f says

$$\begin{aligned} u(x_0, y_0) + iv(x_0, y_0) = f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta}) \right] d\theta \end{aligned}$$

and the theorem now follows by looking at the real parts of the equation. □

Moreover, the following holds;

Theorem 4.4. *If a continuous function $u : D \to \mathbb{C}$ possesses the mean value property, then u is harmonic.*

The proof is left out.

5

Poisson kernel

5.1 Approximate Identity

We are now going to introduce the concept of approximate identity. We will use the notation

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : -\pi \leq \theta \leq \pi\}$$

for the unit circle and

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

for the open unit disk.

Definition 5.1 ([7], page 13)**.** Let $\{\varphi_r\}_{r<1}$ be a family of integrable functions on $\mathbb{T}$. We say that they form an approximate identity as $r \to 1^-$ if

(i) $\int_{\mathbb{T}} |\varphi_r(t)|dt \leq C$ for some constant C and $0 \leq r < 1$

(ii) $\int_{\mathbb{T}} \varphi_r(t)dt = 1$ for $0 \leq r < 1$

(iii) for every $\delta > 0$

$$\lim_{r \to 1^-} \int_{\delta \leq |t| \leq \pi} |\varphi_r(t)|dt = 0.$$

Before we justify the concept of an approximate identity, we introduce the notion of convolution.

Definition 5.2. The convolution of two 2π-periodic functions f and g is a third function denoted $f * g$ defined as

$$h(x) = (f * g)(x) = \int_{-\pi}^{\pi} f(t)g(x-t)dt = \int_{-\pi}^{\pi} f(x-t)g(t)dt.$$

where equality holds due to periodicity.

Proposition 5.1 ([7], page 15)**.** *Let $\{\varphi_r\}$, $0 \leq r < 1$ be an approximate identity on $\mathbb{T}$. If $f \in C(\mathbb{T})$, then*

$$\lim_{r \to 1^-} f * \varphi_r = f$$

uniformly on $\mathbb{T}$.

Proof. Suppose f is continuous and let t be fixed. Then for any $\epsilon > 0$ there exists a $\delta > 0$, not depending on t since f is continuous on a compact set and thus bounded, such that

$$|f(t-u) - f(t)| < \epsilon, \quad |u| < \delta. \tag{27}$$

Note that by the second condition we can write

$$f(t) = f(t) \int_{\mathbb{T}} \varphi_r(u)du = \int_{\mathbb{T}} f(t)\varphi_r(u)du,$$

and by the third condition we have for r_0 sufficiently close to 1, $r_0 < 1$, such that if $r_0 < r < 1$ then

$$\int_{\delta \leq |u| \leq \pi} |\varphi_r(u)|du < \epsilon. \tag{28}$$

So for $r_0 < r < 1$ we have

$$\begin{aligned} |(f * \varphi_r)(t) - f(t)| &= \left| \int_{\mathbb{T}} f(t-u)\varphi_r(u)du - f(t) \right| \\ &= \left| \int_{\mathbb{T}} f(t-u)\varphi_r(u)du - f(t) \int_{\mathbb{T}} \varphi_r(u)du \right| \\ &= \left| \int_{\mathbb{T}} \left(f(t-u) - f(t)\right) \varphi_r(u)du \right| \\ &\leq \int_{\mathbb{T}} |f(t-u) - f(t)| \, |\varphi_r(u)| \, du. \end{aligned} \tag{29}$$

We are now splitting the integral into two as

$$\int_{|u|<\delta} |f(t-u) - f(t)| \, |\varphi_r(u)| \, du + \int_{\delta \leq |u| \leq \pi} |f(t-u) - f(t)| \, |\varphi_r(u)| \, du. \tag{30}$$

For the first integral in (30) we have by (27) that

$$\int_{|u|<\delta} |f(t-u) - f(t)|\,|\varphi_r(u)|\,du < \epsilon||\varphi_r||_1.$$

For the second integral in (30) we have from (28) and by applying Hölders[3] inequality twice

$$\int_{\delta\le u\le\pi} |f(t-u) - f(t)|\,|\varphi_r(u)|du \le 2\sup_{t\in\mathbb{T}}|f(t)| \times \int_{\delta\le u\le\pi} |\varphi_r(u)|du < 2||f||_\infty\epsilon.$$

And thus it follows that

$$|f*\varphi_r(t) - f(t)| < \epsilon\,(C + 2||f||_\infty)$$

which proves that $f*\varphi_r \to f$ as $r\to 1^-$ for $f\in C(\mathbb{T})$. □

We are now continuing by looking at an example of an approximate identity.

5.2 Poisson kernel

Definition 5.3. The function

$$P_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$$

for $0\le r<1$ and $-\infty<\theta<\infty$ is called the Poisson kernel.

We are sometimes going to denote the Poisson kernel as $P(r,\theta)$ since it is a function depending on both r and θ.

Lemma 5.1. *The Poisson kernel equals*

$$P_r(e^{i\theta}) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

for $r<1$ and $P_r(e^{i\theta})$ is harmonic in $\mathbb{D}$.

Proof. From the definition we have

$$P_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$$

which we can write as

$$\begin{aligned}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} &= \sum_{n=-\infty}^{-1} r^{-n}e^{in\theta} + \sum_{n=0}^{\infty} r^{n}e^{in\theta}\\ &= \sum_{n=0}^{\infty}\left(re^{-i\theta}\right)^n - 1 + \sum_{n=0}^{\infty}\left(re^{i\theta}\right)^n.\end{aligned}$$

[3] Hölder inequality can be found in the appendix.

Since each of the two sums are convergent geometric series we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(re^{-i\theta}\right)^n+\sum_{n=0}^{\infty}\left(re^{i\theta}\right)^n-1&=\frac{1}{1-re^{-i\theta}}+\frac{1}{1-re^{i\theta}}-1\\&=\frac{1-re^{i\theta}+1-re^{-i\theta}-(1-re^{i\theta})(1-re^{-i\theta})}{(1-re^{-i\theta})(1-re^{i\theta})}\\&=\frac{1-re^{i\theta}+1-re^{-i\theta}-1+re^{i\theta}+re^{-i\theta}-r^2}{(1-re^{-i\theta}-re^{i\theta}+r^2)}\\&=\frac{1-r^2}{1-2r\cos\theta+r^2}\end{aligned}$$

and thus

$$P_r(e^{i\theta})=\frac{1-r^2}{1-2r\cos\theta+r^2},$$

for $r<1$ in $\mathbb{D}$.

The harmonicity of the Poisson kernel is shown with the help of the following lemma.

Lemma 5.2. *For $z=re^{i\theta}$, $r\in[0,1)$ and $-\infty<\theta<\infty$*

$$P(r,\theta)=Re\left(\frac{1+z}{1-z}\right).$$

Proof. We have

$$\begin{aligned}Re\left(\frac{1+z}{1-z}\right)&=Re\left(\frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})}\right)\\&=Re\left(\frac{1-\bar{z}+z-|z|^2}{1-\bar{z}-z+|z|^2}\right)\end{aligned}$$

By substituting $z=re^{i\theta}$ and $\bar{z}=re^{-i\theta}$ we get

$$\begin{aligned}Re\left(\frac{1-re^{-i\theta}+re^{i\theta}-|re^{i\theta}|^2}{1-re^{-i\theta}-re^{i\theta}+|re^{i\theta}|^2}\right)&=Re\left(\frac{1+2i\sin(\theta)-r^2|e^{i\theta}|^2}{1-2r\cos(\theta)+r^2|e^{i\theta}|^2}\right)\\&=\frac{1-r^2}{1-2r\cos(\theta)+r^2}\end{aligned}$$

□

From lemma 5.2 and theorem 4.1 it now follows that the Poisson kernel is harmonic. □

Properties of the Poisson kernel are given in the following proposition.

Proposition 5.2. *For $0 \leq r < 1$ and $-\infty < \theta < \infty$ we have*

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1$

(ii) $P(r,\theta) > 0$ *for all* θ

(iii) $P(r,-\theta) = P(r,\theta)$

(iv) $P(r,\theta) < P(r,\delta)$ *if* $0 < \delta < |\theta| \leq \pi$

and for each $\delta > 0$ *we have that*

$$\lim_{r\to 1^-} P(r,\delta) = 0$$

converges uniformly for $\delta \leq |\theta| \leq \pi$.

Proof. By uniform convergence of $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$ it is justified to interchange the summation and the integration

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|}e^{in\theta}d\theta$$

so that we can integrate termwise. We note that for $n \neq 0$ the integral will be 0 and for $n = 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} r^{|0|}e^{0}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1d\theta = \frac{1}{2\pi}\left[\pi - (-\pi)\right] = 1,$$

and thus (i) is proven.
The second part (ii) follows from

$$1 - 2r\cos\theta + r^2 = |1-z|^2$$

for $z = re^{i\theta}$ and $r < 1$.
The third claim is immediate from $\cos(-\theta) = \cos(\theta)$.
The fourth claim follows from the fact that the derivative of $P(r,\theta)$ with respect to θ is less than or equal to 0,

$$P'_\theta(r,\theta) = \frac{-(1-r^2)(2r\sin\theta)}{(1-2r\cos\theta+r^2)^2} \leq 0 \tag{31}$$

for $\delta \leq \theta \leq \pi$. And finally the limit relation holds pointwise for $\theta \neq 0$ and by (iv) hence uniformly in the region $\delta \leq |\theta| \leq \pi$. $\square$

5.3 The Dirichlet problem

The Dirichlet problem consists of finding a harmonic function in a bounded domain given boundary values. For the unit disc we have the following theorem.

Theorem 5.1. *Let $\mathbb{D}$ be the open unit disc and assume that $f \in C(\mathbb{T})$. Then there exists a continuous function u on the closure of $\mathbb{D}$ such that $u = f$ on $\mathbb{T}$ and u is harmonic in $\mathbb{D}$. Moreover, u is unique and is defined by*

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t - \theta) f(e^{it}) dt$$

for $0 \leq r < 1$ and $\theta \in [-\pi, \pi]$.

Proof. To show that u is harmonic in the interior of $\mathbb{D}$ we note that the integration is over t and the limits of the integration are constants and since we want to apply the Laplacian with respect to r and θ it allows us by Leibniz integral rule to bring the Laplacian inside the integral sign. That means, applying the Laplacian with respect to r and θ to the integral is the same as applying the Laplacian to $P(r, t - \theta) f(e^{it})$ which in turn is the same as the Laplacian applied on $P(r, t - \theta)$ times $f(e^{it})$. So let t be fixed and from lemma 5.2 it follows that

$$\begin{aligned} \Delta \left(\frac{1}{2\pi} \int_{\mathbb{T}} P(r, t - \theta) f(e^{it}) dt \right) &= \frac{1}{2\pi} \int_{\mathbb{T}} \Delta \left(P(r, t - \theta) \right) f(e^{it}) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} 0 \cdot f(e^{it}) dt \\ &= 0 \end{aligned}$$

And for continuity to the unit circle we have the upcoming lemma.

Lemma 5.3. *Let $t_0 \in [-\pi, \pi]$ and $\epsilon > 0$ be given. Then for f and u as above there exist $\rho \in (0, 1)$ and an arc γ of the unit circle with center e^{it_0} such that for $\rho < r < 1$ and any point $e^{i\theta}$ on γ the inequality*

$$|u(r, \theta) - f(e^{it_0})| < \epsilon \tag{32}$$

holds.

Proof. We need to show that given any $\epsilon > 0$ we can find a $\delta > 0$ such that whenever $|re^{i\theta} - e^{it_0}| < \delta$ then inequality (32) holds.
First off we have that

$$u(r, \theta) - f(e^{it_0}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - t) f(e^{it}) dt - f(e^{it_0})$$

From proposition 1 it follows that we can write

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(e^{it})dt - f(e^{it_0})\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(e^{it})dt - f(e^{it_0})\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)dt\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)(f(e^{it}) - f(e^{it_0}))dt.
\end{aligned}$$

From continuity of f there exists an α such that

$$|f(e^{it}) - f(e^{it_0})| < \frac{\epsilon}{2} \quad \text{whenever} \quad |t-t_0| < \alpha. \tag{33}$$

Now if we break up the integral into two pieces

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|t-t_0|<\alpha} P(r,\theta-t)(f(e^{it}) - f(e^{it_0}))dt +\\
&+ \frac{1}{2\pi}\int_{\alpha\le|t-t_0|\le\pi} P(r,\theta-t)(f(e^{it}) - f(e^{it_0}))dt = I_1 + I_2,
\end{aligned}$$

we get two integrals which we can estimate.

For I_1 we have by (33)

$$\begin{aligned}
|I_1| &= \left|\frac{1}{2\pi}\int_{|t-t_0|<\alpha} P(r,\theta-t)(f(e^{it}) - f(e^{it_0}))dt\right|\\
&\le \frac{1}{2\pi}\int_{|t-t_0|<\alpha} |P(r,\theta-t)|\left|f(e^{it}) - f(e^{it_0})\right| dt\\
&\le \frac{\epsilon}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)dt\right)\\
&= \frac{\epsilon}{2}.
\end{aligned}$$

For I_2 we note that $|f(e^{it}) - f(e^{it_0})|$ is bounded on the unit circle, $|f(e^{it}) - f(e^{it_0})| \le 2M$. Now assume $|\theta - t_0| < \frac{\alpha}{2}$ and if $t \in [-\pi, t_0-\alpha] \cup [t_0+\alpha, \pi]$ then we have $|\theta - t| > \frac{\alpha}{2}$ and so the denominator of $P(r,\theta)$ is bounded away from zero say by a constant C then we have

$$\begin{aligned}
|I_2| &= \left| \frac{1}{2\pi} \int_{\alpha \le |t-t_0| \le \pi} P(r, \theta - t)(f(e^{it}) - f(e^{it_0}))dt \right| \\
&\le \frac{1}{2\pi} \int_{\alpha \le |t-t_0| \le \pi} P(r, \theta - t) \left|(f(e^{it}) - f(e^{it_0})\right| dt \\
&\le \frac{1}{2\pi} \int_{\alpha \le |t-t_0| \le \pi} P(r, \theta - t) 2M dt \\
&\le \frac{1}{2\pi} \int_{\alpha \le |t-t_0| \le \pi} \frac{(1-r^2)}{C} 2M dt \\
&\le \frac{(1-r)(1+r)}{2\pi C} 2M 2\pi \\
&\le \frac{2M}{C}\delta \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

provided $|1-r| < \delta$ and $\delta = \frac{C\epsilon}{4M}$.
The lemma now follows by

$$\begin{aligned}
\left|u(re^{i\theta}) - f(e^{it_0})\right| &= |I_1 + I_2| \\
&\le |I_1| + |I_2| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

The continuity u to the unit circle now follows from lemma 5.3. □

So given a continuous function on the boundary of the unit disk then by the Poisson kernel we can restore a harmonic function inside the unit disk and which is continuous and equal to the continuous function on the boundary.

5.4 Fatou's theorem

Theorem 5.2 (Fatou's theorem)**.** *Let F be holomorphic and bounded on $\mathbb{D}$. Then for almost all $e^{i\theta}$ on $\mathbb{T}$ the radial limit $F(e^{i\theta})$ exists, i.e. $\lim_{r\to 1^-} F(re^{i\theta})$. A bounded holomorphic function on $\mathbb{D}$ has radial limits almost everywhere.*

Remark. Almost everywhere means that the set of which the limit does not exist must be a set of measure zero. I.e., a property holds almost everywhere on a set D if there is a subset $S \subset D$ whose measure, $m(S) = 0$, and such that it holds on all $D \setminus S$.

Proof. ([2], page 3) If $F : \mathbb{D} \to \mathbb{C}$ is a holomorphic function, let it have the power series representation

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}.$$

By the Cauchy integral formula for $n \geq 0$ and any $0 \leq r < 1$ with $\gamma_r(\theta) = re^{i\theta}$, we have

$$a_n = \frac{1}{2\pi i} \int_{r\mathbb{T}} \frac{F(\zeta)}{\zeta^{n+1}} d\zeta.$$

Let $\zeta = re^{i\theta}$, $d\zeta = rie^{i\theta} d\theta$ over $0 \leq \theta < 2\pi$ then

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{r\mathbb{T}} \frac{F(\zeta)}{\zeta^{n+1}} d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{F(re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} d\theta \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned} \tag{34}$$

Then, for $n \geq 0$ we have

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta, \tag{35}$$

and since $\frac{F(\zeta)}{\zeta^{n+1}}$ is a holomorphic function on $\mathbb{D}$ for $n < 0$, by Cauchy's Integral theorem and (35) it yields

$$\frac{1}{2\pi r^n} \int_0^{2\pi} F(re^{i\theta}) e^{-in\theta} d\theta = \frac{1}{2\pi i} \int_{r\mathbb{T}} \frac{F(\zeta)}{\zeta^{n+1}} d\zeta = 0. \tag{36}$$

Now let $F : \mathbb{D} \to \mathbb{C}$ be holomorphic and suppose there is a constant M such that $|F(z)| \leq M$ for all $z \in \mathbb{D}$. For $0 < r < 1$ define $f_r : \mathbb{T} \to \mathbb{C}$ by $f_r(\theta) = F(re^{i\theta})$. From (35) and (36) we have

$$\hat{f}_r(n) = \begin{cases} a_n r^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

For $0 < r < 1$ we note that $||f_r||_{L^2}^2 \leq M^2$ and thus by Parseval's[4] identity,

$$\sum_{n=-\infty}^{\infty} |\hat{f}_r(n)|^2 \leq M^2.$$

[4]For definition see appendix.

On the other hand

$$\sum_{n=-\infty}^{\infty} |\hat{f}_r(n)|^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

It follows that

$$\sum_{n=0}^{\infty} |a_n|^2 \leq M^2, \tag{37}$$

since $r \in (0,1)$. So now define $f \in L^2(\mathbb{T})$ by the Fourier coefficients

$$\hat{f}(n) = \begin{cases} a_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

This defines an element of $L^2(\mathbb{T})$ if and only if

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty,$$

and indeed since

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 \leq M^2.$$

As $f \in L^2(\mathbb{T})$ it is in $L^1(\mathbb{T})$ and thus for almost all $\theta \in \mathbb{T}$

$$\lim_{r \to 1^-} \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} \hat{f}(n) = f(\theta),$$

which if we do the substitution $\hat{f}(n) = a_n$ yields

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^{|n|} e^{int} = f(\theta).$$

And finally we can conclude that

$$\lim_{r \to 1^-} F(re^{i\theta}) = f(\theta),$$

and thus we've proved that if f is a bounded holomorphic function on the unit disk then f has radial limits at almost every angle. $\square$

6

Hardy Space

We are going to present some of the main properties of Hardy spaces on the unit disk $\mathbb{D}$. The natural identification between $\mathbb{T}$ and $\mathbb{R}/2\pi\mathbb{Z}$ is going to be assumed and hence functions defined on $\mathbb{T}$ will be identified with functions on $\mathbb{R}/2\pi\mathbb{Z}$, i.e. with functions on the real line, periodic with period 2π. Integrals on $\mathbb{T}$ will be understood with respect to the normalized Lebesgue measure $\frac{1}{2\pi}dt$. We will also use the notation of D_r defined as the disk with radius r centered at the origin.

Let f be a holomorphic function on $\mathbb{D}$. Given $r \in [0,1)$ and $p \geq 1$ we define

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p dt\right)^{\frac{1}{p}},$$

and for $p = \infty$ we define

$$M_\infty(f,r) = \sup_{e^{it}\in\mathbb{T}} |f(re^{it})|.$$

If we set $f_r(e^{it}) = f(re^{it})$ for $0 \leq r < 1$, we can then say that

$$M_p(f,r) = ||f_r||_p.$$

Definition 6.1 ([7], page 4)**.** Let $1 \leq p \leq \infty$. We denote by $H^p(\mathbb{D})$ the vector space of holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{0\leq r<1} M_p(f,r) = ||f||_{H^p} < \infty. \tag{38}$$

Let us confirm that $\sup_{0\leq r<1} M_p(f,r) = ||f||_{H^p}$ really defines a norm.

First of all we note that by definition $M_p(f,r) \geq 0$. We then observe that any $f \in H^p(\mathbb{D})$ is continuous on $\mathbb{D}$, so that $||f_r||_p = 0$ implies that $f_r = 0$, if $r < 1$ which implies that $f \equiv 0$ in $\mathbb{D}$.

We continue by showing $\sup_{0\le r<1} M_p(af,r) = |a| \sup_{0\le r<1} M_p(f,r)$, we have

$$\begin{aligned}
\sup_{0\le r<1} M_p(af,r) &= \sup_{0\le r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |af(re^{it})|^p dt \right)^{\frac{1}{p}} \\
&= |a| \sup_{0\le r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}} \\
&= |a| \sup_{0\le r<1} M_p(f,r).
\end{aligned}$$

Lastly we need to show that

$$\sup_{0\le r<1} M_p(f+g,r) \le \sup_{0\le r<1} M_p(f,r) + \sup_{0\le r<1} M_p(g,r).$$

From Minkowski's inequality, $||f+g||_p \le ||f||_p + ||g||_p$, we have

$$\begin{aligned}
\sup_{0\le r<1} M_p(f+g,r) &= \sup_{0\le r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it}) + g(re^{it})|^p dt \right)^{\frac{1}{p}} \\
&\le \sup_{0\le r<1} \left[\left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}} + \left(\frac{1}{2\pi} \int_0^{2\pi} |g(re^{it})|^p dt \right)^{\frac{1}{p}} \right] \\
&\le \sup_{0\le r<1} M_p(f,r) + \sup_{0\le r<1} M_p(g,r),
\end{aligned}$$

thus we've confirmed that (38) defines a norm.

We want to show that $H^p(\mathbb{D})$ is complete, i.e. $H^p(\mathbb{D})$ is a Banach space.

Definition 6.2 (Banach space [5])**.** A Banach space is a complete normed vector space.

And to help us with that we have the following lemmas where the first one is stated with out proof.

Lemma 6.1 ([7], page 5)**.** *For $s > 1$ fixed, consider the integral*

$$I_s(r) = \int_{-\pi}^{\pi} \frac{1}{|1 - re^{it}|^s} dt,$$

as a function of $r \in [0,1)$. Then for $r \to 1$,

$$\frac{I_s(r)}{(1-r)^{-(s-1)}}$$

is bounded from above and below by a positive constant on a neighborhood of 1.

[5] For a more detailed description see appendix.

Lemma 6.2 ([7], page 6). *Let $f \in H^p(\mathbb{D})$. Then, for every $z \in \mathbb{D}$ and $p > 1$,*

$$|f(z)| \leq C \frac{||f||_{H^p}}{(1-|z|)^{\frac{1}{p}}}.$$

Proof. Assume p is finite and take r such that $|z| < r < 1$. Let γ_r be the circle of radius r, centered at the origin with counterclockwise orientation. Then by Cauchy's Integral formula we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Setting $\zeta = re^{it}$ with $d\zeta = rie^{it}dt$, $-\pi \leq t \leq \pi$ so that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(re^{it})}{re^{it} - z} rie^{it} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(re^{it})}{1 - \frac{z}{r}e^{-it}} dt \end{aligned}$$

If $z = |z|e^{i\theta}$, we have

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(re^{it})}{1 - \frac{|z|}{r}e^{i(\theta - t)}} dt.$$

Let

$$f = f(re^{it}) \text{ and } g = \frac{1}{1 - \frac{|z|}{r}e^{i(\theta-t)}}$$

then from Hölder's inequality, $||fg||_1 \leq ||f||_p ||g||_q$, we get

$$|f(z)| \leq M_p(f, r) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1 - \frac{|z|}{r}e^{i(\theta-t)}|^q} dt \right)^{\frac{1}{q}}.$$

We do the substitution $u = \theta - t$ with $du = -dt$, and using the periodicity of the integrand which yields

$$\int_{-\pi}^{\pi} \frac{1}{|1 - \frac{|z|}{r}e^{i(\theta-t)}|^q} dt = \int_{-\pi}^{\pi} \frac{1}{|1 - \frac{|z|}{r}e^{i(u)}|^q} du = I_q\left(\frac{|z|}{r}\right).$$

Since $q > 1$ we have by lemma 6.1

$$\begin{aligned}
|f(z)| &\leq M_p(f,r)\left(\frac{1}{2\pi}I_q\left(\frac{|z|}{r}\right)\right)^{\frac{1}{q}} \\
&= ||f_r||_p C\left(\left(1-\frac{|z|}{r}\right)^{-(q-1)}\right)^{\frac{1}{q}} \\
&= ||f_r||_p C\left(1-\frac{|z|}{r}\right)^{-\frac{(q-1)}{q}} \\
&= ||f_r||_p C\left(1-\frac{|z|}{r}\right)^{-\frac{1}{p}}
\end{aligned}$$

The proof now follows if we let $r \to 1$. □

The next corollary is about uniform convergence on compact subsets of $\mathbb{D}$.

Corollary 6.1 ([7], page 6)**.** *Convergence in $H^p(\mathbb{D})$ implies uniform convergence on compact subsets of $\mathbb{D}$.*

Proof. Let $K \subset \mathbb{D}$ be compact. Then there is an $r < 1$ such that $K \subset \overline{D_r}$ of radius r centered at the origin. From lemma 6.2 we concluded that if

$$f_n \to f \quad \text{in } H^p(\mathbb{D}),$$

then

$$||f_n - f||_{\infty,K} = \sup_{z\in K}|f_n(z) - f(z)| \leq C\frac{||f_n - f||_{H^p}}{(1-r)^{\frac{1}{p}}}.$$

If we let $n \to \infty$, then

$$\lim_{n\to\infty} C\frac{||f_n - f||_{H^p}}{(1-r)^{\frac{1}{p}}} = 0.$$

And thus f_n converges uniformly to f on compact subsets of $\mathbb{D}$. □

Theorem 6.1 ([7], page 7)**.** *$H^p(\mathbb{D})$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $H^p(\mathbb{D})$. For every $r < 1$ and by lemma 6.2, $\{f_n\}$ is a Cauchy sequence in $C(\overline{D_r})$ with uniform norm. By the completeness of $C(\overline{D_r})$, there is a $g_r \in C(\overline{D_r})$ such that $f_n \to g_r$ uniformly on $\overline{D_r}$. For $r_1 < r_2 < 1$ corresponding to g_{r_1} and g_{r_2} respectively, then g_{r_1} and g_{r_2} coincide on $\overline{D_{r_1}}$ since all various g_r, with $r < 1$, are all restrictions of a unique function g continuous on $\mathbb{D}$. To prove that g is holomorphic in $\mathbb{D}$ we use Morera's theorem which states that; a continuous complex-valued function defined on an open set G in the complex plane that satisfies

$$\oint_\gamma g(z)dz = 0$$

for every closed piecewise C^1 curve γ in G must be holomorphic on G.
Now let γ be such a curve. Since γ is contained in $\overline{D_r}$ for some $r \in (0,1)$, then $f_n \to g$ uniformly on γ. Therefore

$$\oint_\gamma g(z)dz = \lim_{n\to\infty} \oint_\gamma f_n(z)dz = 0,$$

since f_n is holomorphic for every n.

Finally we show that $f_n \to g$ in $H^p(\mathbb{D})$. Given any $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$||f_n - f_m||_{H^p} < \epsilon \quad \text{for } n, m > N.$$

Let $r < 1$ and since $f_m \to g$ uniformly on the circle $|z| \leq r$, if $n > N$ we have

$$M_p(f_n - g, r) = \lim_{m\to\infty} M_p(f_n - f_m, r) \leq \lim_{m\to\infty} ||f_n - f_m||_p < \epsilon.$$

And since this holds for every $r < 1$, $||f_n - g||_{H^p} < \epsilon$, and thus $H^p(\mathbb{D})$ is a Banach space. $\square$

7

Appendix

In this appendix we will mostly provide results without proofs and definitions that we used throughout the text.

Theorem 7.1 (Leibniz integral rule)**.** *Let the function $f(x, y)$ and its partial derivatives with respect to x and y be coninuous in some domain in the complex plane such that*

$$a(x) \leq y \leq b(x) \quad \textit{and} \quad x_1 \leq x \leq x_2.$$

Also suppose that $a(x)$ and $b(x)$ and their respectively derivatives are continuous for $x_1 \leq x \leq x_2$. Then for $x_1 \leq x \leq x_2$ we have

$$\frac{\partial}{\partial x}\left(\int_{a(x)}^{b(x)} f(x, y)dy\right) =$$
$$= f(x, b(x))\frac{\partial}{\partial x}(b(x)) - f(x, a(x))\frac{\partial}{\partial x}(a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, y)dy$$

Definition 7.1. For Lebesgue integrable functions f on $\mathbb{T}$ we define

$$||f||_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$||f||_\infty = \operatorname{ess\,sup}|f|,$$

where $\operatorname{ess\,sup} f = \inf\{k : k \geq f a.e.\}$.

Definition 7.2 (L^p-space)**.** We define the $L^P(\mathbb{T})$-space as the space containing all functions f such that

$$||f||_p \leq \infty,$$

which is valid for all $1 \leq p \leq \infty$.

Definition 7.3. A Banach space is a vector space V with a vector space addition and scalar multiplication equipped with a norm, $(V, +, \cdot, ||\cdot||)$. A vector space norm is a map

$$||\cdot|| : V \to \mathbb{R}.$$

And even though we are looking at a complex vector space the map that defines a norm takes us into the real numbers. To be a map that defines a norm the following must be satisfied for the f and g in V.

(i) $||\lambda f|| = |\lambda| \cdot ||f||$ where $\lambda \in \mathbb{C}$,

(ii) $||f +_V g|| \leq ||f|| +_{\mathbb{R}} ||g||$, and

(iii) $||f|| \geq 0$ if and only if $f = 0$.

So a Banach space is a vector space with a norm which is complete with respect to the norm and by complete means that every Cauchy sequence in the space converges to another element within the space. So if we know we are dealing with a Banach space we know that a sequence in this space converges to an element in the space.

Theorem 7.2 (Hölder inequality). *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$ then*

$$||fg||_1 \leq ||f||_p ||g||_q.$$

Theorem 7.3 (Parseval's theorem). *Let $f, g \in L^2(\mathbb{T})$ then*

$$\frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2$$

where a_n and b_n is the coefficients of f and g respectivly, given by

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int}dt.$$

Definition 7.4 (Minkowski's Inequality). The Minkowski inequality is

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Proof. We have

$$
\begin{aligned}
||f+g||_p^p &= \int_{\mathbb{T}} |f+g|^p dt \\
&= \int_{\mathbb{T}} |f+g||f+g|^{p-1} dt \\
&\leq \int_{\mathbb{T}} |f||f+g|^{p-1} dt + \int_{\mathbb{T}} |g||f+g|^{p-1} dt \\
&\leq \left[\left(\int_{\mathbb{T}} |f|^p dt \right)^{\frac{1}{p}} + \left(\int_{\mathbb{T}} |g|^p dt \right)^{\frac{1}{p}} \right] \left(\int_{\mathbb{T}} |f+g|^{(p-1)\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\
&= (||f||_p + ||g||_p) \frac{||f+g||_p^p}{||f+g||_p},
\end{aligned}
$$

and by multiplying both sides with $\frac{||f+g||_p}{||f+g||_p^p}$ yields the result we were looking for. $\square$

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