SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

On the convergence and divergence of Fourier series

av

Nell Jacobsson

2021 - No K24

On the convergence and divergence of Fourier series

Nell Jacobsson

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Alan Sola

2021

Abstract

The purpose of this thesis is to study convergence and divergence phenomena for Fourier series. In particular, we seek sufficient criteria for functions to guarantee that their Fourier series converge. We present several positive results as we discover conditions that ensure different types of convergence. On the other hand, we construct a continuous function whose Fourier series diverges at one point, establishing that continuity in a function does not in fact guarantee the convergence of its Fourier series.

Acknowledgements

First and foremost I would like to thank my supervisor Alan Sola, for all the great support and guidance he has provided throughout this project. I would also like to thank my friend and fellow student Markus Gustafsson, for numerous discussions about mathematical concepts and the writing process in general.

Contents

1 Overview

The purpose of this thesis is to study the convergence behavior of Fourier series. We seek to find sufficient criteria for functions to guarantee the convergence of their Fourier series. We assume that the reader is familiar with basic principles of analysis and concepts such as convergence, completeness and compactness.

Firstly, we introduce some basic definitions and examples and study some fundamental results concerning the properties of Fourier series. We then move on to seeking explicit criteria which ensure the convergence of Fourier series. We study several different types of convergence, and we will find that the necessary conditions on a given function f differ considerably depending on whether we require mean-square, pointwise or uniform convergence of its Fourier series.

Lastly, we will illuminate the intricate nature of this topic by constructing a continuous function whose Fourier series diverges at one point. We will see that the question of convergence is a very delicate one, and gain some insight into why it has occupied mathematicians for so long.

2 Introduction

In this particular section, we will define the notions of Fourier coefficients and Fourier series, and study some simple examples. We will then move on to investigating the uniqueness of Fourier series, and ask ourselves what can be said about two functions if they have the same Fourier coefficients. Moreover, we will study certain families of functions, *kernels*, which turn out to be very useful in the context of Fourier series.

Finally, we will take a look at what can be said about the convergence of Fourier series when we apply a different summation method than the one we are used to, so called *Cesàro summability*. We will see that in this case, we can prove a rather strong result; namely that if a function f is continuous, its Fourier series will be uniformly Cesàro summable to f.

2.1 Basic definitions and examples

Definition 2.1. *Let $f : [a, b] \to \mathbb{C}$ be a Riemann integrable function. We define the n^{th}* ***Fourier coefficient*** *of f by*

$$\hat{f}(n) = \frac{1}{b-a} \int_a^b f(x) e^{-2\pi i n x/(b-a)}\, dx,$$

for $n \in \mathbb{Z}$. The ***Fourier series*** *associated with the function f is defined as*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x/(b-a)},$$

with partial sums $S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x/(b-a)}$.

We use the following notation

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x/(b-a)}$$

for a function f and its corresponding Fourier series. Note that we have not yet said anything about the convergence of Fourier series - this issue will be studied in detail in the next section.

Remark 2.1. For two functions f and g, the Fourier coefficients of their sum are $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$, and for $\alpha \in \mathbb{C}$, we have $\widehat{(\alpha f)}(n) = \alpha \hat{f}(n)$. This follows immediately from the definition. □

Let us calculate the Fourier series for some simple functions.

Example 2.1. Let $f(x) = e^x$ for $x \in [0, 1]$. The Fourier coefficients of this function are given by

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^x e^{-2\pi i n x}\, dx \\ &= \int_0^1 e^{x(1-2\pi i n)}\, dx \\ &= \left[\frac{e^{x(1-2\pi i n)}}{1-2\pi i n} \right]_0^1 \\ &= \frac{e^1 e^{-2\pi i n} - 1}{1-2\pi i n}. \end{aligned}$$

Recall Euler's identity, $e^{i\pi} = -1$. With this in mind, we finally arrive at

$$\hat{f}(n) = \frac{e-1}{1-2\pi i n}.$$

The Fourier series of f is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{e-1}{1-2\pi i n} e^{2\pi i n x}.$$

□

Example 2.2. Let $f(x) = x$, for $x \in [-\pi, \pi]$. For $n = 0$,

$$\begin{aligned} \hat{f}(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^0\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx \\ &= 0. \end{aligned}$$

For $n \neq 0$, we use integration by parts;

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx \\ &= \frac{1}{2\pi}\left(\left[-\frac{x}{in}e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx\right) \\ &= \frac{1}{2\pi}\left[e^{-inx}\left(-\frac{x}{in}+\frac{1}{n^2}\right)\right]_{-\pi}^{\pi}.\end{aligned}$$

Using Euler's identity, we get

$$\begin{aligned}&\frac{1}{2\pi}\left[e^{-inx}\left(-\frac{x}{in}+\frac{1}{n^2}\right)\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}(-1)^n\left(-\frac{\pi}{in}+\frac{1}{n^2}-\left(\frac{\pi}{in}+\frac{1}{n^2}\right)\right) \\ &= \frac{(-1)^n}{2\pi}\cdot\frac{(-2\pi)}{in} \\ &= \frac{(-1)^{n+1}}{in}.\end{aligned}$$

Hence

$$f(x) \sim \sum_{n\neq 0}\frac{(-1)^{n+1}}{in}e^{inx}.$$

Recall Euler's formulas;

$$\begin{aligned}\cos(nx) &= \frac{e^{inx}+e^{-inx}}{2}, \\ \sin(nx) &= \frac{e^{inx}-e^{-inx}}{2i}.\end{aligned}$$

Since $(-1)^n = (-1)^{-n}$, we can combine the n^{th} and $(-n)^{th}$ terms to

$$(-1)^{n+1}\left(\frac{e^{inx}}{in}+\frac{e^{-inx}}{-in}\right) = 2\sin(nx)\cdot\frac{(-1)^{n+1}}{n}.$$

We conclude that

$$f(x) \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$$

□

When f is 2π-periodic, the Fourier coefficients are reduced to

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,. \end{aligned}$$

From here on, our primary focus will lie with Riemann integrable, 2π-periodic functions.

Example 2.3. Let f be an (Riemann) integrable, 2π-periodic function with Fourier coefficients $c_n = \hat{f}(n)$ and Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$. Recall that Euler's formulas can be expressed as

$$\begin{aligned} e^{inx} &= \cos(nx) + i\sin(nx), \\ e^{-inx} &= \cos(nx) - i\sin(nx). \end{aligned}$$

Thus,

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx - \frac{i}{2\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx,$$

and with the following notation

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx, \end{aligned}$$

the partial sums $S_N(f)(x)$ can be expressed as

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \frac{a_n - ib_n}{2} e^{inx} \\ &= \frac{a_0}{2} + \sum_{n=-N}^{-1} \frac{a_n - ib_n}{2}(\cos(nx) + i\sin(nx)) \\ &\quad + \sum_{n=1}^{N} \frac{a_n - ib_n}{2}(\cos(nx) + i\sin(nx)). \end{aligned}$$

Note that $a_{-n} = a_n$ and $b_{-n} = -b_n$. For this reason, we can rewrite the sums above as

$$\begin{aligned} &\frac{a_0}{2} + \sum_{n=1}^{N} \frac{a_n + ib_n}{2}(\cos(nx) - i\sin(nx)) \\ &+ \sum_{n=1}^{N} \frac{a_n - ib_n}{2}(\cos(nx) + i\sin(nx)). \end{aligned}$$

Expansion of the summands finally yields this alternate way of expressing the partial sums:

$$S_N(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)).$$

□

2.2 Uniqueness

Under the assumption that the Fourier series of a function f actually converges to f, the function would be uniquely determined by its Fourier coefficients. In that case, two functions f and g would be equal if all their Fourier coefficients were identical. By defining $h = f - g$, we can rephrase this as the following: if $\hat{h}(n) = 0$ for all $n \in \mathbb{Z}$, then h is the zero function. Unfortunately, this does not hold generally, but we have the following theorem.

Theorem 2.1. *Let f be a Riemann integrable, 2π-periodic function with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever f is continuous at x_0.*

Before we prove this result, we must give one definition:

Definition 2.2. *A function of the form*

$$\begin{aligned} P(x) &= a_0 + \sum_{n=1}^{N} a_n \cos(nx) + i \sum_{n=1}^{N} b_n \sin(nx) \\ &= \sum_{n=-N}^{N} c_n e^{inx}, \end{aligned}$$

where $a_n, b_n \in \mathbb{C}$, $x \in \mathbb{R}$ and $N \geq 0$ is an integer, is called a ***trigonometric polynomial*** *of degree N.*

The proof below closely follows the one given in section 2 in chapter 2 of [5].

Proof. (*Of Theorem 2.1.*) We begin with the case where f is real-valued. Without loss of generality, suppose $x_0 = 0$ and $f(0) > 0$. (Otherwise, study $f(x - x_0)$ or $-f(x)$ respectively.)

The plan is to construct a family of trigonometric polynomials $\{p_k\}$ which "peak" at 0 and satisfy

$$\int_{-\pi}^{\pi} f(x) p_k(x)\, dx \to \infty$$

for $k \to \infty$. However, the Fourier coefficients of f are all equal to zero (by assumption), which implies that these integrals should equal zero too. In this way, we will arrive at a contradiction.

The continuity of f enables us to choose a $\delta \in (0, \frac{\pi}{2}]$ such that $f(x) > \frac{f(0)}{2}$ whenever $|x| < \delta$. Define

$$p(x) = \varepsilon + \cos(x)$$

and

$$p_k(x) = (p(x))^k,$$

with $\varepsilon > 0$ chosen such that $|p(x)| < 1 - \frac{\varepsilon}{2}$ whenever $\delta \leq |x| \leq \pi$. Moreover, choose an η such that $0 < \eta < \delta$, and $p(x) \geq 1 + \frac{\varepsilon}{2}$ whenever $|x| < \eta$.

Since f is Riemann integrable, it is bounded, and we can find some $B \in \mathbb{R}$ such that $|f(x)| \leq B$ always holds. We can now estimate

$$\begin{aligned}
\left| \int_{\delta \leq |x| \leq \pi} f(x) p_k(x)\, dx \right| &\leq \int_{\delta \leq |x| \leq \pi} |f(x) p_k(x)|\, dx \\
&\leq \int_{\delta \leq |x| \leq \pi} B \Big(1 - \frac{\varepsilon}{2}\Big)^k dx \\
&\leq 2\pi B \Big(1 - \frac{\varepsilon}{2}\Big)^k.
\end{aligned}$$

We conclude that $\int_{\delta \leq |x| \leq \pi} f(x) p_k(x)\, dx \to 0$ as $k \to \infty$.

Moreover,

$$\int_{\eta \leq |x| < \delta} f(x) p_k(x)\, dx \geq 0,$$

since $f(x) > \frac{f(0)}{2} > 0$ and $p(x) \geq 0$ on this interval.

Lastly,

$$\int_{|x| < \eta} f(x) p_k(x)\, dx \geq 2\pi \frac{f(0)}{2} \left(1 + \frac{\varepsilon}{2}\right)^k,$$

which tends to infinity as $k \to \infty$.

We conclude that

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) p_k(x)\, dx \\
&= \int_{\delta \leq |x| \leq \pi} f(x) p_k(x)\, dx + \int_{\eta \leq |x| < \delta} f(x) p_k(x)\, dx \\
&+ \int_{|x| < \eta} f(x) p_k(x)\, dx \longrightarrow \infty
\end{aligned}$$

as $k \to \infty$, giving us the desired contradiction and proving the theorem for real-valued functions f.

When f is complex-valued, it can be written as $f(x) = u(x) + iv(x)$ for real-valued functions u, v where

$$u(x) = \frac{f(x) + \overline{f(x)}}{2},$$
$$v(x) = \frac{f(x) - \overline{f(x)}}{2i}.$$

By assumption, $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. It is easily checked that the n^{th} Fourier coefficient of $\overline{f(x)}$ is given by $\overline{\hat{f}(-n)}$, which must then be zero as well. Therefore,

$$\hat{u}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}(f(x) + \overline{f(x)})e^{-inx}\,dx = \frac{1}{4\pi}(\hat{f}(n) + \overline{\hat{f}(-n)}) = 0,$$
$$\hat{v}(n) = \frac{1}{4i\pi}\int_{-\pi}^{\pi}(f(x) - \overline{f(x)})e^{-inx}\,dx = \frac{1}{4i\pi}(\hat{f}(n) - \overline{\hat{f}(-n)}) = 0.$$

By the argument for real-valued functions, both u and v must be zero at points of continuity, and so f must be too. □

Remark 2.2. It is worth pointing out that the issue of uniqueness is rather more complex than implied here. The theory of Fourier analysis can be extended to cover *Lebesgue integrable* functions, in which case two functions may differ and be discontinuous at some points (to be exact, on sets of so called *measure zero*) but still have identical Fourier coefficients. In this context, we work with equivalence classes of functions that are equal *almost everywhere*, meaning that they are equal for all $x \notin E$ where E is some set of measure zero. □

2.3 Kernels and convolutions

We will now define a family of functions which will be central in the context of Fourier series.

Definition 2.3. *Define the trigonometric polynomial*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}, \quad x \in [-\pi, \pi].$$

This is called the N^{th} ***Dirichlet kernel****.*

Remark 2.3. There is a closed form formula for $D_N(x)$. For $x \neq 0$, we use the formula for geometric sums and obtain

$$\begin{aligned}
D_N(x) &= \sum_{n=-N}^{-1} e^{inx} + \sum_{n=0}^{N} e^{inx} \\
&= \frac{(e^{ix})^{-N} - 1}{1 - e^{ix}} + \frac{1 - (e^{ix})^{N+1}}{1 - e^{ix}} \\
&= \frac{(e^{ix})^{-N} - (e^{ix})^{N+1}}{1 - e^{ix}} \\
&= \frac{(e^{ix})^{-N-\frac{1}{2}} - (e^{ix})^{N+\frac{1}{2}}}{(e^{ix})^{-\frac{1}{2}} - (e^{ix})^{\frac{1}{2}}} \\
&= \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}.
\end{aligned}$$

For $x = 0$, we have $D_N(0) = \sum_{n=-N}^{N} 1 = 2N + 1$, which coincides with the limit of $\frac{\sin(x(N+\frac{1}{2}))}{\sin(\frac{x}{2})}$ as $x \to 0$. □

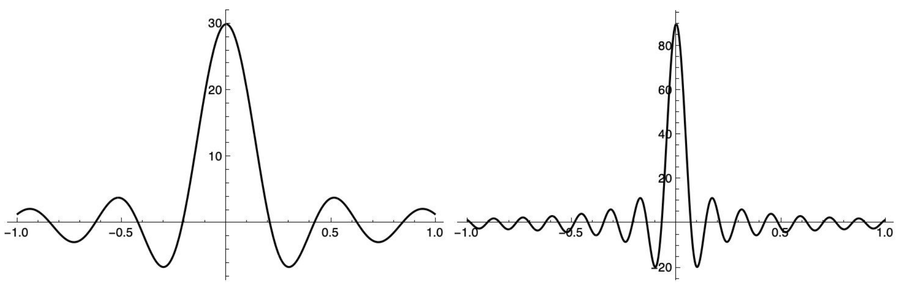

Figure 1: The Dirichlet kernel for $N = 10$ and $N = 30$.

To see the relevance of the Dirichlet kernel in Fourier analysis, we must first introduce another concept.

Definition 2.4. *Let f and g be two Riemann integrable, 2π-periodic functions on $\mathbb{R}$. We define their* ***convolution*** *$f * g$ by*

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x-y)\, dy\,.$$

The connection between the Fourier series of a Riemann integrable, 2π-

periodic function f and the Dirichlet kernel is given by the following relation;

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny}\, dy \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy \\ &= (f * D_N)(x). \end{aligned}$$

Thus we can gain insight in to the behavior of $S_N(f)$ by studying $f * D_N$. First, let us establish some basic properties to better understand convolutions.

Proposition 2.2. *Let f, g and h be Riemann integrable, 2π-periodic functions. Then*

(i) $f * (g + h) = (f * g) + (f * h)$.

(ii) $(cf) * g = c(f * g) = f * (cg), \quad c \in \mathbb{C}$.

(iii) $f * g = g * h$.

(iv) $(f * g) * h = f * (g * h)$.

(v) $f * g$ *is continuous.*

(vi) $(\widehat{f * g})(n) = \hat{f}(n)\hat{g}(n)$.

The proof is quite straight-forward when we assume f and g are continuous; extending it to the case where the functions are merely integrable requires a bit more work. We omit the details here, and refer the curious reader to section 3 in chapter 2 of [5].

Let us now introduce another concept relating to kernels:

Definition 2.5. *We call $\{K_n(x)\}_{n=1}^{\infty}$ a family of* ***good kernels*** *if*

(a) For all $n \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1.$$

(b) There exists a constant $M > 0$ such that

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$$

for all $n \geq 1$.

(c) For every $\delta \in (0, \pi]$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0 \text{ as } n \to \infty.$$

Good kernels happen to be very useful in the context of convolutions, as is shown by the following result.

Theorem 2.3. *Let $\{K_n(x)\}_{n=1}^{\infty}$ be a family of good kernels, and f a Riemann integrable, 2π-periodic function. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, the limit is uniform.

In other words, by calculating the convolution of a given a function and a good kernel and then taking the limit of the resulting integral, we can get our function back. The proof we give of this theorem can be found in section 4 in chapter 2 of [5].

Proof. The function f being continuous at x means that given any $\varepsilon > 0$, we can choose a $\delta > 0$ such that

$$|f(x - y) - f(x)| < \varepsilon$$

whenever $|y| < \delta$. By property (a) of good kernels,

$$\begin{aligned}(f * K_n)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) f(x-y)\, dy - f(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)(f(x-y) - f(x))\, dy\,.\end{aligned}$$

Note that since f is Riemann integrable, it is bounded by some $B > 0$ for all x. We estimate the absolute value of $(f * K_n)(x) - f(x)$;

$$\begin{aligned}|(f * K_n)(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)(f(x-y) - f(x))\, dy\right| \\ &\leq \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)||(f(x-y) - f(x))|\, dy \\ &+ \frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)||(f(x-y) - f(x))|\, dy \\ &\leq \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|\, dy + \frac{2B}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)|\, dy \\ &\leq \frac{\varepsilon M}{2\pi} + \frac{2B}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)|\, dy,\end{aligned}$$

where we have used property (b) of good kernels in the last step. For all n large enough,

$$\frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi}|K_n(y)|\,dy<\varepsilon,$$

by property (c). Hence we conclude that

$$|(f*K_n)(x)-f(x)|\leq C\varepsilon$$

for some constant $C>0$ and all large n. Since ε is arbitrarily small, this proves that

$$\lim_{n\to\infty}(f*K_n)(x)=f(x)$$

whenever f is continuous at x.

Now suppose f is continuous everywhere. Then f is continuous on any closed bounded interval, and thus uniformly continuous on any closed bounded interval (since these are compact sets in $\mathbb{R}$). In particular, f is uniformly continuous on any closed bounded interval of length 2π. Since f is 2π-periodic, this implies that f is uniformly continuous on all of $\mathbb{R}$; then $\delta>0$ can be chosen independently of x. In this case, $f*K_n$ converges to f uniformly. □

The question begs to be raised; is $D_N(x)$ a good kernel? If so, we could use the relation $S_N(f)(x)=(f*D_N)(x)$ and happily declare that the Fourier series of a function f converges to f at all points of continuity.

Sadly, this is not the case. (We will actually construct a continuous function whose Fourier series diverges at a point in a later section.)

Proposition 2.4. *The Dirichlet kernel satisfies*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(x)|\,dx\geq\frac{4}{\pi^2}\ln(2(N+1)).$$

To prove this, we follow the technique used in chapter 18 of [4].

Proof. We apply the closed form formula,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_N(x)|\,dx&=\int_{-\pi}^{\pi}\left|\frac{\sin(x(N+\frac{1}{2}))}{\sin(\frac{x}{2})}\right|dx\\&=2\int_{0}^{\pi}\left|\frac{\sin(x(N+\frac{1}{2}))}{\sin(\frac{x}{2})}\right|dx\\&\geq4\int_{0}^{\pi}\left|\frac{\sin(x(N+\frac{1}{2}))}{x}\right|dx,\end{aligned}$$

where the second equality follows from the fact that the integrand is an even function. In the last step, we have used that $\left|\frac{x}{2}\right| \geq \left|\sin\left(\frac{x}{2}\right)\right|$ in the interval $[0, \pi]$. Moreover, by dividing up the integral over sub-intervals of $[0, \pi]$, we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(x)|\, dx &\geq 4 \int_0^{\pi} \left| \frac{\sin((N+\frac{1}{2})x)}{x} \right| dx \\
&= 4 \sum_{k=0}^{2N} \int_{k\pi/(2N+1)}^{(k+1)\pi/(2N+1)} \frac{|\sin((N+\frac{1}{2})x)|}{x}\, dx \\
&\geq 4 \sum_{k=0}^{2N} \int_{k\pi/(2N+1)}^{(k+1)\pi/(2N+1)} \frac{|\sin((N+\frac{1}{2})x)|}{(k+1)\pi/(2N+1)}\, dx \\
&= 4 \sum_{k=0}^{2N} \frac{2N+1}{(k+1)\pi} \int_{k\pi/(2N+1)}^{(k+1)\pi/(2N+1)} \left| \sin\left(\left(N+\frac{1}{2}\right)x\right) \right| dx\,. \qquad (1)
\end{aligned}$$

Note that, for all $k = 0, \ldots, 2N$,

$$\begin{aligned}
\int_{k\pi/(2N+1)}^{(k+1)\pi/(2N+1)} \left| \sin\left(\left(N+\frac{1}{2}\right)x\right) \right| dx &= \int_0^{\pi/(2N+1)} \sin\left(\left(N+\frac{1}{2}\right)x\right) dx \\
&= \frac{1}{N+\frac{1}{2}} \Big(-\cos\left(\frac{\pi}{2}\right) + \cos(0) \Big) \\
&= \frac{1}{N+\frac{1}{2}} \\
&= \frac{2}{2N+1}.
\end{aligned}$$

Hence (1) yields

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq \frac{8}{\pi} \sum_{k=0}^{2N} \frac{1}{k+1}.$$

Recall that

$$\sum_{k=0}^{2(N-1)} \frac{1}{k+1} \geq \sum_{k=1}^{2N} \int_k^{k+1} \frac{1}{x}\, dx = \int_1^{2(N+1)} \frac{1}{x}\, dx = \ln(2(N+1)),$$

and so

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq \frac{8}{\pi} \sum_{k=0}^{2N} \frac{1}{k+1} \geq \frac{8}{\pi} \ln(2(N+1)).$$

We conclude that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(x)|\, dx \geq \frac{4}{\pi^2} \ln(2(N+1)).$$

□

It is clear from the proposition above that the Dirichlet kernel violates property (b) of good kernels.

Remark 2.4. Note, however, that $D_N(x)$ does satisfy property (a) of good kernels:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-N}^{n} e^{inx}\right)dx \\ &= \frac{1}{2\pi}\sum_{n=-N}^{n}\left(\int_{-\pi}^{\pi} e^{inx}\,dx\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{0}\,dx \\ &= 1.\end{aligned}$$

□

2.4 A new way to sum

Although the Dirichlet kernel did not behave as well as we had hoped, we can still use convolutions to establish some interesting facts about Fourier series. While convergence of Fourier series in the usual sense is a tricky topic, it turns out that we can prove some related results using other kinds of summation. Let us define one such type:

Definition 2.6. *Given a series $\sum_{k=0}^{\infty} c_k$, $c_k \in \mathbb{C}$, and partial sums $S_n = \sum_{k=0}^{n} c_k$, we call*

$$\sigma_N = \frac{S_0 + \cdots + S_{N-1}}{N}$$

*its N^{th} **Cesàro sum**. We say the series is **Cesàro summable** to σ if*

$$\lim_{N\to\infty} \sigma_N = \sigma.$$

Example 2.4. The series $\sum_{k=0}^{\infty}(-1)^k$ does not converge; note that its partial sums follow the pattern

$$S_n = \begin{cases} 1, & \text{for even } n, \\ 0, & \text{for odd } n. \end{cases}$$

Then

$$\sigma_N = \frac{S_0 + \cdots + S_{N-1}}{N} = \frac{\lceil \frac{N+1}{2} \rceil}{N},$$

where $\lceil \frac{N+1}{2} \rceil$ represents the unique integer such that $\lceil \frac{N+1}{2} \rceil - 1 < \frac{N+1}{2} \leq \lceil \frac{N+1}{2} \rceil$.

We observe that σ_N converges to $\frac{1}{2}$ as $N \to \infty$; hence the series $\sum_{k=0}^{\infty}(-1)^k$ is Cesàro summable to $\frac{1}{2}$. Seeing that the partial sums alternate between 0 and 1, this lines up with basic intuition that the "limit" of the series should lie midway between these values. □

The example above shows us that there are divergent series that are Cesàro summable. What about the converse?

Lemma 2.5. *Let* $\sum_{k=0}^{\infty} c_k$ *be a series with* $c_k \in \mathbb{C}$. *If the partial sums* S_n *converge to a limit* S *as* $n \to \infty$, *then the Cesàro sums satisfy*

$$\lim_{N\to\infty} \sigma_N = S.$$

Our proof of this lemma will closely follow the one given in chapter 1 of [4].

Proof. Let $\varepsilon > 0$ be given. Since $S_n \to S$ as $n \to \infty$, we can find an integer $m > 0$ such that $n \geq m$ imples $|S_n - S| \leq \frac{\varepsilon}{2}$.

Now define

$$A = \sum_{k=0}^{m} |S_k - S|,$$

and choose $m' > m$ such that

$$m' \geq \frac{2A}{\varepsilon} \Leftrightarrow A \leq \frac{m'\varepsilon}{2}.$$

Then, whenever $N \geq m'$, we have

$$\begin{aligned}
|\sigma_N - S| &= \left| \left(\frac{1}{N} \sum_{k=0}^{N-1} S_k \right) - \frac{NS}{N} \right| \\
&= \left| \frac{1}{N} \sum_{k=0}^{N-1} (S_k - S) \right| \\
&\leq \frac{1}{N} \sum_{k=0}^{N-1} |S_k - S| \\
&= \frac{1}{N} \left(\sum_{k=0}^{m} |S_k - S| + \sum_{k=m+1}^{N-1} |S_k - S| \right) \\
&\leq \frac{1}{N} \left(A + (N - 1 - m)\frac{\varepsilon}{2} \right) \\
&\leq \frac{1}{N} \left(\frac{\varepsilon N}{2} + \frac{\varepsilon N}{2} \right) \\
&= \varepsilon.
\end{aligned}$$

This proves that $\sigma_N \to S$ as $N \to \infty$. □

Let us take a look at the Cesàro sums of the Fourier series of an (Riemann) integrable, 2π-periodic function f. We get

$$\begin{aligned}\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\ &= \frac{(f * D_0)(x) + \cdots + (f * D_{N-1})(x)}{N} \\ &= f * \left(\frac{D_0(x) + \cdots + D_{N-1}(x)}{N} \right),\end{aligned}$$

by the properties of convolutions given in Proposition 2.2. The term $\frac{D_0(x)+\cdots+D_{N-1}(x)}{N}$ actually has its own name:

Definition 2.7. *We define the* N^{th} ***Fejér kernel*** $F_N(x)$ *by*

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

Using the definition above, we see that given a (Riemann integrable, 2π-periodic) function f, the N^{th} Cesàro sum of its Fourier series is

$$\sigma_N(f)(x) = (f * F_N)(x).$$

Let us now establish some facts about the Fejér kernel.

Lemma 2.6. *(i) The Fejér kernel has a closed form formula given by*

$$F_N(x) = \frac{1}{N} \cdot \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}.$$

(ii) The Fejér kernel is a good kernel.

Proof. (i) Recall from Remark 2.3 that the N^{th} Dirichlet kernel can be written as

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{(e^{ix})^{-N} - (e^{ix})^{N+1}}{1 - e^{ix}},$$

for $x \neq 0$, so

$$\begin{aligned}NF_N(x) &= \sum_{k=0}^{N-1} D_k(x) \\ &= \sum_{k=0}^{N-1} \frac{(e^{ix})^{-k} - (e^{ix})^{k+1}}{1 - e^{ix}} \\ &= \frac{1}{1 - e^{ix}} \left(\sum_{k=0}^{N-1} (e^{ix})^{-k} - \sum_{k=0}^{N-1} (e^{ix})^{k+1} \right)\end{aligned}$$

whenever $x \neq 0$.

By applying the usual formula for geometric sums, we get

$$\begin{aligned}
NF_N(x) &= \frac{1}{1-e^{ix}}\left(\sum_{k=0}^{N-1}(e^{ix})^{-k} - \sum_{k=0}^{N-1}(e^{ix})^{k+1}\right) \\
&= \frac{e^{ix}}{1-e^{ix}}\left(\frac{(e^{ix})^{-N}-1}{1-e^{ix}} - \frac{1-(e^{ix})^{N}}{1-e^{ix}}\right) \\
&= e^{ix}\cdot\frac{(e^{ix})^{-N}-2+(e^{ix})^{N}}{(1-e^{ix})^2} \\
&= \frac{(e^{ix})^{-N}-2+(e^{ix})^{N}}{(e^{-ix/2})^2(1-e^{ix})^2} \\
&= \frac{((e^{ix})^{-N/2}-(e^{ix})^{N/2})^2}{((e^{ix})^{-1/2}-(e^{ix})^{1/2})^2} \\
&= \frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})}.
\end{aligned}$$

We conclude that

$$F_N(x) = \frac{1}{N}\cdot\frac{\sin^2(\frac{Nx}{2})}{\sin^2(\frac{x}{2})},$$

for $x \neq 0$. As was the case with the Dirichlet kernel, the limit of the closed form $\frac{\sin^2(\frac{Nx}{2})}{N\sin^2(\frac{x}{2})}$ as $x \to 0$ coincides with the value of $F_N(0)$.

(ii) (a) For each $N \geq 1$,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_{k=0}^{N-1}D_k(x)}{N}\,dx \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}D_k(x)\,dx\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}1 \\
&= 1,
\end{aligned}$$

following from the fact that the Dirichlet kernel has property (a) of good kernels.

(b) $F_N(x)$ is positive, and thus

$$\begin{aligned}
\int_{-\pi}^{\pi}|F_N(x)|\,dx &= \int_{-\pi}^{\pi}F_N(x)\,dx \\
&= 2\pi,
\end{aligned}$$

by (a).

(c) Given $\varepsilon \in (0, 1]$, there exists a $\delta > 0$ such that $\delta \le |x| \le \pi$ implies $\sin^2(\frac{x}{2}) \ge \varepsilon$. Then

$$F_N(x) = \frac{\sin^2(\frac{Nx}{2})}{N \sin^2(\frac{x}{2})} \le \frac{1}{N\varepsilon}.$$

Therefore

$$\int_{\delta \le |x| \le \pi} |F_N(x)|\, dx = \int_{\delta \le |x| \le \pi} F_N(x)\, dx \to 0$$

as $N \to \infty$.

□

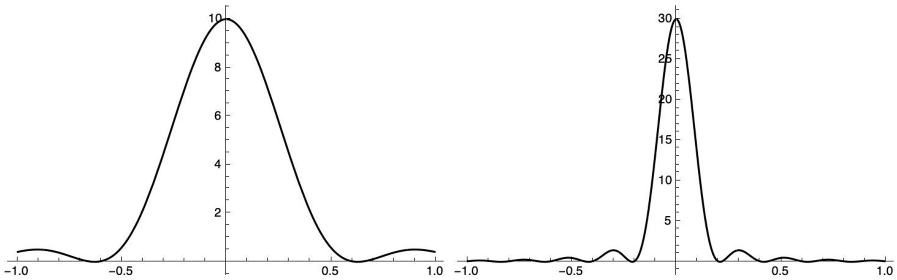

Figure 2: The Fejér kernel for $N = 10$ and $N = 30$.

The lemma above leads us to an interesting result concerning Fourier series, due to Fejér.

Theorem 2.7. ***(Fejér)*** *Let f be a Riemann integrable, 2π-periodic function. Then the Fourier series of f is Cesàro summable to f at every point of continuity of f. Moreover, if f is continuous everywhere, its Fourier series is uniformly Cesàro summable to f.*

Proof. Apply Theorem 2.3 to $\sigma_N(f) = (f * F_N)(x)$. □

The following corollary can be seen as the trigonometric equivalent of Weierstrass polynomial approximation theorem, and will be used in the next section.

Corollary 2.7.1. *Functions that are continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ can be uniformly approximated by trigonometric polynomials; i.e., given $\varepsilon > 0$ there exists a trigonometric polynomial P such that*

$$|f(x) - P(x)| < \varepsilon$$

for $x \in [-\pi, \pi]$.

Proof. Since the partial sums $S_N(f)$ are trigonometric polynomials, so are the Cesàro sums, which uniformly sum up to f as $N \to \infty$ by Theorem 2.7. □

It turns out that the uniqueness property of Fourier series (Theorem 2.1) can be seen as a direct corollary of the theorem above. The proof is simple; if $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $\sigma_N(f) = 0$, and by Theorem 2.7, $f = 0$ at its points of continuity.

There are other summation methods of relevance in the context of Fourier series. We can define the notion of a series being *Abel summable* to a limit, and prove a similar result to Theorem 2.7 using Abel summability instead of Cesàro summability. The interested reader is referred to section 5.3 in chapter 2 of [5] for further exploration of this topic.

3 Criteria for convergence

In this section, we will systematically present and prove some notable results concerning the convergence of Fourier series. Given a function f, our goal is to find criteria which will ensure different types of convergence of its corresponding Fourier series.

Firstly, we will study so called *mean-square convergence* by using concepts the reader might recognize from linear algebra. From this, we will be able to derive a result which can be ingeniously applied to finding the limits of infinite series.

We will then move on to the familiar notion of uniform convergence, and see how it ties in with the absolute convergence of Fourier series. We will find that f being continuously differentiable is enough to ensure uniform convergence of its Fourier series, and then discover that uniform convergence can be guaranteed even when we place weaker conditions - certain kinds of so called *Hölder conditions* - on f.

Finally, we will investigate what needs to be required of f to guarantee pointwise convergence of its associated Fourier series. We will see that in this case, it is sufficient for f to simply be differentiable.

3.1 Mean-square convergence

Let us first state the theorem which will be the main focus of this particular section.

Theorem 3.1. ***(Mean-square convergence)*** *Let f be a 2π-periodic, Riemann integrable function. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 \, dx \to 0$$

as $N \to \infty$.

To prove this, we will use the concept of orthogonality. Our setting must thus be a vector space with an inner product. If the reader wishes to refresh their memory on the topic of vector spaces and inner products, we refer them to section 1.1 in chapter 3 of [5]. In this text, we will use the properties of such objects freely.

We mainly restrict ourselves to complex-valued, 2π-periodic, Riemann integrable functions, and define the inner product of two such functions f and g as

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx\,.$$

This yields norm $\|f\|$ such that

$$\|f\|^2 = (f,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx\,.$$

Hence we can rephrase the statement of Theorem 3.1 as $\|f - S_N(f)\| \to 0$ as $N \to \infty$.

To simplify notation, let $e_n(x) = e^{inx}$ for every $n \in \mathbb{Z}$. Note that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal family with respect to the inner product we just defined;

$$(e_n, e_m) = \begin{cases} 1 \text{ for } n = m, \\ 0 \text{ for } n \neq m. \end{cases}$$

Another important observation is

$$(f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \hat{f}(n).$$

We see that

$$S_N(f) = \sum_{|n|\leq N} \hat{f}(n)e_n = \sum_{|n|\leq N} a_n e_n,$$

if we let $a_n = \hat{f}(n)$ denote the Fourier coefficients of f. Moreover, note that

$$\Big(f - \sum_{|n|\leq N} a_n e_n\Big) \perp \sum_{|n|\leq N} b_n e_n \tag{2}$$

for any $b_n \in \mathbb{C}$, since

$$\begin{aligned}\Big(\Big(f - \sum_{|n|\leq N} a_n e_n\Big), \sum_{|n|\leq N} b_n e_n\Big) &= \Big(f, \sum_{|n|\leq N} b_n e_n\Big) - \Big(\sum_{|n|\leq N} a_n e_n, \sum_{|n|\leq N} b_n e_n\Big) \\ &= \sum_{|n|\leq N} a_n \overline{b_n} - \sum_{|n|\leq N} a_n \overline{b_n} \\ &= 0\end{aligned}$$

by the orthonormality of $\{e_n\}_{n\in\mathbb{Z}}$ and the relation $(f, e_n) = a_n$.

Since b_n are arbitrary complex numbers, we can set $b_n = a_n$ and conclude that

$$\Big(f - \sum_{|n|\leq N} a_n e_n\Big) \perp \sum_{|n|\leq N} a_n e_n.$$

By applying the Pythagorean theorem (see [5]) to

$$f = f - \sum_{|n|\leq N} a_n e_n + \sum_{|n|\leq N} a_n e_n,$$

we get

$$\begin{aligned}\|f\|^2 &= \Big\|f - \sum_{|n|\leq N} a_n e_n\Big\|^2 + \Big\|\sum_{|n|\leq N} a_n e_n\Big\|^2 \\ &= \Big\|f - \sum_{|n|\leq N} a_n e_n\Big\|^2 + \sum_{|n|\leq N} |a_n|^2, \qquad (3)\end{aligned}$$

where we have again used the orthonormality of $\{e_n\}_{n\in\mathbb{Z}}$ in the last step. This leads us to the following lemma:

Lemma 3.2. ***(Best approximation)*** *If f is a Riemann integrable, 2π-periodic function with Fourier coefficients $a_n = \hat{f}(n)$, then*

$$\|f - S_N(f)\| \leq \Big\|f - \sum_{|n|\leq N} c_n e_n\Big\|$$

for any $c_n \in \mathbb{C}$. Moreover, the equality holds precisely when $c_n = a_n$ for all $|n| \leq N$.

Proof. Let $b_n = a_n - c_n$. Then

$$f - \sum_{|n|\leq N} c_n e_n = f - S_N(f) + \sum_{|n|\leq N} b_n e_n.$$

Since $(f - S_N(f)) \perp \sum_{|n|\leq N} b_n e_n$ by (2), we can apply the Pythagorean theorem:

$$\Big\|f - \sum_{|n|\leq N} c_n e_n\Big\|^2 = \Big\|f - S_N(f)\Big\|^2 + \Big\|\sum_{|n|\leq N} b_n e_n\Big\|^2.$$

Note that $\|\cdot\|^2$ is always non-negative. Thus

$$\|f - S_N(f)\| \leq \Big\|f - \sum_{|n|\leq N} c_n e_n\Big\|$$

with equality only when $c_n = a_n$ (i.e. $b_n = 0$) for all $|n| \leq N$. □

We can interpret this lemma as saying that the trigonometric polynomial of degree $M \leq N$ that is "closest" to f in our norm is precisely $S_N(f)$. For this reason, the lemma is often called the best approximation lemma.

Before we move on to proving Theorem 3.1, we must first establish the following result:

Lemma 3.3. *Suppose f is a 2π-periodic, Riemann integrable function bounded by $B > 0$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous, 2π-periodic functions such that*

$$\sup_{x \in [-\pi, \pi]} |f_k(x)| \leq B$$

for each k, and

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)| \, dx \to 0$$

as k tends to infinity.

Our proof uses the same argument as the one given in section 1 in the appendix of [5].

Proof. We restrict ourselves to the case where f is real-valued. (When considering complex-valued functions $f = u + iv$, apply the same kind of argument to the real-valued u and v.)

Let $\varepsilon > 0$ be given, and choose a partition, say P,

$$-\pi = x_0 < x_1 < \cdots < x_N = \pi$$

of $[-\pi, \pi]$ such that the upper and lower Riemann sums of f satisfy

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) < \frac{\varepsilon}{2}.$$

(This is possible since f is assumed to be Riemann integrable.)

Consider the step function defined by

$$f^*(x) = \sup_{x_{j-1} \leq y \leq x_j} f(y) \quad \text{if } x \in [x_{j-1}, x_j) \text{ for } 1 \leq j \leq N.$$

Note that this yields $|f^*| \leq B$, and

$$\begin{aligned} \int_{-\pi}^{\pi} |f^*(x) - f(x)| \, dx &< \mathcal{U}(P, f) - \mathcal{L}(P, f) \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

This function f^* approximates f, but it is not continuous. However, we can use f^* to construct a new continuous function $\tilde{f}$. For small $\delta > 0$, let $\tilde{f}(x) = f^*(x)$ when the distance from x to any of the points x_j of the partition is greater or equal to δ. When x is in the δ-neighborhood of a point x_j, $j = 1, \ldots, N-1$, let $\tilde{f}(x)$ be the linear function for which $\tilde{f}(x_j \pm \delta) = f^*(x_j \pm \delta)$. Near $x_0 = -\pi$, let $\tilde{f}$ be linear with $\tilde{f}(-\pi) = 0$ and $\tilde{f}(-\pi + \delta) = f^*(-\pi + \delta)$. Near $x_N = \pi$, let $\tilde{f}$ be linear with $\tilde{f}(\pi - \delta) = f^*(\pi - \delta)$ and $\tilde{f}(\pi) = 0$.

Visually, we have modified the step function f^* by shortening the ends of each step by δ and connecting these updated ends with linear segments, as shown in Figure 3. Since we have constructed $\tilde{f}$ such that $\tilde{f}(-\pi) = \tilde{f}(\pi)$, we may also extend it to be 2π-periodic on $\mathbb{R}$. By our construction, this extension will still be bounded by B.

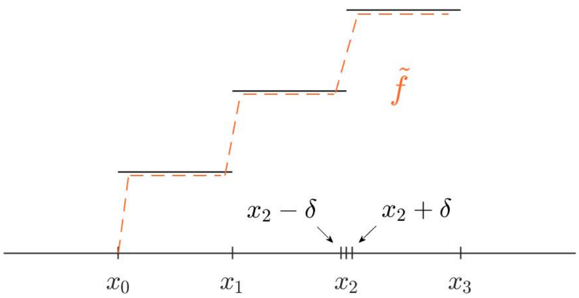

Figure 3: Construction of $\tilde{f}$. The orange graph is shifted slightly below the step function for the sake of clarification.

Note that $\tilde{f}$ differs from f^* only in the N intervals of length 2δ around points $x_0, \ldots, x_N$. We observe that

$$\int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|\, dx \leq 2BN \cdot 2\delta = 4BN\delta.$$

Choosing δ sufficiently small yields

$$\int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|\, dx < \frac{\varepsilon}{2}.$$

Using our previously established result along with the triangle inequality finally yields

$$\begin{aligned}\int_{-\pi}^{\pi} |f(x) - \tilde{f}(x)|\, dx &\leq \int_{-\pi}^{\pi} |f(x) - f^*(x)|\, dx \\ &\quad + \int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|\, dx \\ &< \varepsilon.\end{aligned}$$

By denoting $\tilde{f}$ by f_k for $\varepsilon = \frac{1}{k}$, we obtain our desired sequence of continuous functions $\{f_k\}$. □

The main idea for the proof above is fairly simple, especially when we visualize our constructed function $\tilde{f}$; we created a step function using the supremum of our original function on different intervals, and then proceeded to connect the steps with linear segments. However, as we saw, formalizing this idea required some attention to detail. The good news is that we now have all the machinery we need to prove Theorem 3.1.

Proof. (*Of Mean-square convergence.*) Firstly, recall that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2\, dx = \|f(x) - S_N(f)(x)\|^2.$$

Hence we aim to prove that $\|f - S_N(f)\| \to 0$ as $N \to \infty$.

We begin with the case where f is continuous. Let $\varepsilon > 0$ be given. By Corollary 2.7.1, there exists some trigonometric polynomial P - say of degree M - such that

$$|f(x) - P(x)| < \varepsilon$$

for all x. In particular,

$$\begin{aligned}\|f(x) - P(x)\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - P(x)|^2\, dx \\ &< \frac{1}{2\pi}\int_{-\pi}^{\pi} \varepsilon^2\, dx \\ &< \frac{(\pi - (-\pi))}{2\pi}\varepsilon^2 \\ &= \varepsilon^2,\end{aligned}$$

and so

$$\|f(x) - P(x)\| < \varepsilon.$$

By the best approximation lemma, whenever $N \geq M$, we have

$$\|f - S_N(f)\| < \varepsilon.$$

Since ε can be chosen arbitrarily small, this proves that $\|f - S_N(f)\| \to 0$ as N tends to infinity for continuous functions f.

Now suppose f is merely (Riemann) integrable. Using Lemma 3.3, we choose a continuous, 2π-periodic function g satisfying

$$\sup_{x\in[-\pi,\pi]} |g(x)| \leq \sup_{x\in[-\pi,\pi]} |f(x)| = B$$

and

$$\int_{-\pi}^{\pi} |f(x) - g(x)|\, dx < \varepsilon^2.$$

In this case,

$$\begin{aligned}
\|f - g\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g(x)|^2\, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g(x)||f(x) - g(x)|\, dx \\
&\leq \frac{2B}{2\pi}\int_{-\pi}^{\pi} |f(x) - g(x)|\, dx \\
&< C\varepsilon^2
\end{aligned}$$

for a constant $C > 0$. Since g is continuous, we can apply Corollary 2.7.1 and approximate g with some trigonometric polynomial P of degree $M \leq N$ such that $\|g - P\| < \varepsilon$. By the triangle inequality,

$$\begin{aligned}
\|f - P\| &\leq \|f - g\| + \|g - P\| \\
&< \varepsilon + C'\varepsilon \\
&< C''\varepsilon
\end{aligned}$$

for constants $C', C'' > 0$. Applying the best approximation lemma yields

$$\|f - S_N(f)\| < C''\varepsilon$$

for all $N \geq M$. This concludes our proof. □

We can use mean-square convergence to establish the following result, usually called Parseval's identity.

Theorem 3.4. ***(Parseval's identity)*** *Let f be a Riemann integrable, 2π-periodic function. With notation $a_n = \hat{f}(n)$, we have*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx\,.$$

Proof. Recall from (3) that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2.$$

By Theorem 3.1, we have

$$\sum_{|n|\leq N} |a_n|^2 \to \|f\|^2$$

as $N \to \infty$. □

Example 3.1. Let us take a small detour and see a clever application of Parseval's identity, found in section 11 in chapter 7 of [1]. Recall that while we have mainly restricted ourselves to 2π-periodic functions, all theory presented here can be extended to any Riemann integrable function $f : [a, b] \to \mathbb{C}$ (see Definition 2.1). Parseval's identity still holds if we decide to work with the vector space of such functions instead, and define the inner product of $f, g : [a, b] \to \mathbb{C}$ by

$$(f, g) = \frac{1}{b-a} \int_a^b f(x)\overline{g(x)}\, dx\,.$$

Having established this, consider $f(x) = x$ for $x \in [-1, 1]$. The Fourier coefficients of f are given by

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi n x}\, dx \\ &= \frac{1}{2} \int_{-1}^{1} x e^{-i\pi n x}\, dx\,. \end{aligned}$$

For $n = 0$, we have

$$\hat{f}(0) = \frac{1}{2} \int_{-1}^{1} x\, dx = 0.$$

For $n \neq 0$, we can use integration by parts to obtain

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2} \left[\frac{e^{-i\pi n x}(1 + i\pi n x)}{\pi^2 n^2} \right]_{-1}^{1} \\ &= \frac{1}{2\pi^2 n^2} (e^{-i\pi n}(1 + i\pi n) - e^{i\pi n}(1 - i\pi n)), \end{aligned}$$

and using Euler's identity $e^{i\pi} = -1$ yields

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi^2 n^2} ((-1)^n + i\pi n(-1)^n - (-1)^n + i\pi n(-1)^n) \\ &= \frac{1}{2\pi^2 n^2} (2i\pi n(-1)^n) \\ &= \frac{i}{\pi n} (-1)^n. \end{aligned}$$

The Fourier series of f is given by

$$f(x) \sim \frac{i}{\pi} \sum_{n \neq 0} \frac{(-1)^n}{n} e^{i\pi n x}.$$

Note that

$$\|f\|^2 = \frac{1}{2} \int_{-1}^{1} x^2 \, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3}.$$

By Parseval's identity,

$$\begin{aligned} \frac{1}{3} &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \\ &= \sum_{n \neq 0} \frac{1}{\pi^2 n^2} \\ &= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

We finally arrive at a familiar result;

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Finding the limit of this series has historically been called the Basel problem. It was first solved by Leonhard Euler in the early 1700s. □

We conclude this section with the following lemma:

Lemma 3.5. ***(Riemann-Lebesgue)*** *Suppose f is Riemann integrable and 2π-periodic. Then $\hat{f}(n) \to 0$ as $|n| \to \infty$.*

Proof. By Parseval's identity, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ converges. Thus $|\hat{f}(n)|^2 \to 0$ as $|n| \to \infty$, and so $\hat{f}(n) \to 0$ as $|n|$ tends to infinity. □

Let us not forget that while this looks like a simple enough result, it did take quite a bit of work to get here. Recall that convergence of a series $\sum a_n$ implies that $a_n \to 0$ as $n \to \infty$, although the converse implication does not hold. If it did, we could have concluded that $\sum_{n=-\infty}^{\infty} \hat{f}(n)$ converged as well, but this in turn would not guarantee a convergent Fourier series. We have no choice but to continue our search for necessary criteria to ensure the convergence of Fourier series.

3.2 Uniform convergence

We now prove our first theorem concerning the uniform convergence of Fourier series, using the uniqueness property. We apply the technique used in section 2 in chapter 2 of [5].

Theorem 3.6. *Suppose f is a continuous, 2π-periodic function. If*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty,$$

then $S_N(f)$ converges uniformly to f as $N \to \infty$.

Proof. Define the function

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

Note that

$$\begin{aligned} |g(x) - S_N(f)(x)| &= \left| \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} - \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \right| \\ &= \left| \sum_{|n|>N} \hat{f}(n) e^{inx} \right| \\ &\leq \sum_{|n|>N} |\hat{f}(n)|. \end{aligned}$$

The last sum can be made arbitrarily small, by our assumption of absolute convergence. Therefore, the partial sums $S_N(f)$ converge uniformly to g. Since $\{S_N(f)\}$ is a sequence of continuous, 2π-periodic functions converging uniformly, their limit g must be continuous and 2π-periodic as well.

Let us take a look at the Fourier coefficients of g:

$$\begin{aligned} \hat{g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} \hat{f}(m) e^{imx} \right) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \hat{f}(m) \int_{-\pi}^{\pi} e^{ix(m-n)}\, dx, \end{aligned}$$

where the sum and integral can be interchanged thanks to the uniform convergence of the series. Note that

$$\int_{-\pi}^{\pi} e^{ix(m-n)}\, dx = 0$$

for $m \neq n$. Therefore,

$$\begin{aligned}\hat{g}(n) &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\hat{f}(m)\int_{-\pi}^{\pi} e^{ix(m-n)}\,dx\\ &= \frac{1}{2\pi}\hat{f}(n)\int_{-\pi}^{\pi} e^{0}\,dx\\ &= \frac{1}{2\pi}\hat{f}(n)(\pi-(-\pi))\\ &= \hat{f}(n).\end{aligned}$$

By applying Theorem 2.1 to $f - g$, we finally conclude that $f = g$. □

With this theorem, we have found conditions that will ensure the convergence of a Fourier series to its corresponding function - namely, continuity of the function and absolute convergence of the Fourier series. The question is now, when is the Fourier series associated with a function f absolutely convergent? What must be required of f for $\sum_{n=-\infty}^{\infty}|\hat{f}(n)| < \infty$ to hold? The purpose of this section is to find some conditions which will guarantee absolute convergence of a given function's Fourier series.

Before we move on to such matters, we must establish the following lemma:

Lemma 3.7. *Suppose f is a 2π-periodic, differentiable function with a continuous derivative f'. Then*

$$\widehat{f'}(n) = in\hat{f}(n).$$

Proof. For $n = 0$, we have $in\hat{f}(n) = 0$, and

$$\begin{aligned}\widehat{f'}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{0}\,dx\\ &= \frac{1}{2\pi}\big[f(x)\big]_{-\pi}^{\pi}\\ &= 0,\end{aligned}$$

since f is 2π-periodic. It is clear that the formula holds for $n = 0$.

For $n \neq 0$, we use integration by parts;

$$\begin{aligned}2\pi\hat{f}(n) &= \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\\ &= \left[-f(x)\frac{e^{-inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx\,.\end{aligned}$$

The first term is zero, since f is 2π-periodic. We conclude that

$$\hat{f}(n) = \frac{1}{in\cdot 2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx = \frac{1}{in}\widehat{f'}(n).$$

□

With the help of this lemma, we can prove the following result.

Theorem 3.8. *Let f be a 2π-periodic, differentiable function with a continuous derivative. Then the Fourier series of f converges absolutely and uniformly to f.*

Proof. Applying the formula $\widehat{f'}(n) = in\hat{f}(n)$ as well as Parseval's identity, we obtain

$$\sum_{n=-\infty}^{\infty} n^2|\hat{f}(n)|^2 = \sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2 = \|f'\|^2. \tag{4}$$

Since f' is continuous, we know $\|f'\|^2$ to be finite.

Note that we can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\hat{f}(n)| &= |\hat{f}(0)| + \sum_{n\neq 0} |\hat{f}(n)| \\ &= |\hat{f}(0)| + \sum_{n\neq 0} n|\hat{f}(n)| \cdot \frac{1}{n}.\end{aligned}$$

To prove the convergence of the last sum, we apply the Cauchy-Schwarz inequality;

$$\left(\sum_{n\neq 0} n \cdot |\hat{f}(n)| \cdot \frac{1}{n}\right)^2 \leq \sum_{n\neq 0} \frac{1}{n^2} \sum_{n\neq 0} n^2|\hat{f}(n)|^2.$$

Since $\sum_{n\neq 0} \frac{1}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$, we get

$$\begin{aligned}\left(\sum_{n\neq 0} n \cdot |\hat{f}(n)| \cdot \frac{1}{n}\right)^2 &\leq \sum_{n\neq 0} \frac{1}{n^2} \sum_{n\neq 0} n^2|\hat{f}(n)|^2 \\ &= \frac{\pi^2}{3}\|f'\|^2\end{aligned}$$

by (4). The quantity on the right-hand side is finite, so

$$\sum_{n\neq 0} |\hat{f}(n)| \leq \frac{\pi}{\sqrt{3}}\|f'\|.$$

We finally arrive at

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{n\neq 0} |\hat{f}(n)| < \infty.$$

By Theorem 3.6, the Fourier series converges uniformly to f. □

The theorem above states that it is enough for f to be continuously differentiable to guarantee that its Fourier series is absolutely convergent (and thus converges uniformly to f). We will conclude this section with an even stronger result, but we must first define a new concept.

Definition 3.1. *We say that a function f satisfies a* ***Hölder condition of order α*** *if there exists some $0 < \alpha \leq 1$ and some constant $C > 0$ such that*

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

holds for all x and all y.

Note that a function f satisfying a Hölder condition implies that f is uniformly continuous; given $\varepsilon > 0$, choose a δ such that $\delta < (\frac{\varepsilon}{C})^{1/\alpha}$. It follows that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Moreover, if f is continuously differentiable on an interval $[a, b]$, it satisfies a Hölder condition for $\alpha = 1$ (a so called *Lipschitz condition*) on that interval. By the Mean Value Theorem, $f(x) - f(y) = f'(\xi)(x - y)$ for some $\xi \in (y, x) \subset [a, b]$. Since f' is assumed to be continuous, it must be bounded on $[a, b]$ (recall that closed and bounded intervals are compact in $\mathbb{R}$). Hence there exists some constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ holds.

The following example shows that the converse does not hold; f satisfying a Lipschitz condition does not ensure differentiability.

Example 3.2. The function $f(x) = |x|$ satisifes a Lipschitz condition on all of $\mathbb{R}$; by the reverse triangle inequality, we have that

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|$$

for all $x, y \in \mathbb{R}$. □

Now that we have introduced Hölder conditions, we can state the theorem we intend to prove:

Theorem 3.9. ***(Bernstein)*** *Suppose f is a continuous, 2π-periodic function that satisfies a Hölder condition of order $\alpha > 1/2$, i.e. there exists some $C > 0$ and some $\alpha > 1/2$ such that*

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

holds for all x, y. Then the Fourier series of f converges absolutely and uniformly to f.

To prove this, we first need to establish a couple of lemmas. (We follow the proof outlined in the exercises in chapter 3 of [5].)

Lemma 3.10. *Let f be a Riemann integrable, 2π-periodic function for which there exists some $C > 0$ and some $\alpha > 1/2$ such that*

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

holds for all x, y*. Define*

$$g_h(x) = f(x+h) - f(x-h)$$

for $h > 0$*. Then*

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx = \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2$,

(b) $\sum_{n=-\infty}^{\infty} |\sin(nh)|^2|\hat{f}(n)|^2 \le \frac{C^2h^{2\alpha}2^{2\alpha}}{4}$.

Proof. (a) Let us take a look at the Fourier coefficients of g_h:

$$\begin{aligned}\widehat{g_h}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x+h) - f(x-h))e^{-inx}\,dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x+h)e^{-inx}\,dx - \int_{-\pi}^{\pi} f(x-h)e^{-inx}\,dx\right).\end{aligned}$$

We perform a change of variables $y = x + h$ and $z = x - h$. Note that the bounds of integration do not need to be changed, since f is 2π-periodic and we are still integrating over intervals of length 2π.

$$\begin{aligned}\widehat{g_h}(n) &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)e^{-iny}e^{inh}\,dy - \int_{-\pi}^{\pi} f(z)e^{-inz}e^{-inh}\,dz\right) \\ &= \hat{f}(n)(e^{inh} - e^{-inh}) \\ &= 2i\sin(nh)\hat{f}(n),\end{aligned}$$

where we have used the fact that $\sin(nh) = \frac{e^{inh}-e^{-inh}}{2i}$. We can now apply Parseval's identity;

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx &= \sum_{n=-\infty}^{\infty} |2i\sin(nh)\hat{f}(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2.\end{aligned}$$

This concludes our proof of the first part of the lemma.

(b) Since we supposed $|f(x) - f(y)| \le C|x-y|^\alpha$, we have

$$|g_h(x)| = |f(x+h) - f(x-h)| \le C(2h)^\alpha.$$

Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_h(x)|^2\,dx &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |C(2h)^\alpha|^2\,dx \\ &= C^2 2^{2\alpha}h^{2\alpha}.\end{aligned}$$

By (a),

$$\sum_{n=-\infty}^{\infty} |\sin(nh)|^2 |\hat{f}(n)|^2 \leq \frac{C^2 2^{2\alpha} h^{2\alpha}}{4}.$$

This concludes our proof of the lemma.

□

We need one more lemma before we arrive at the much anticipated proof of Bernstein's theorem:

Lemma 3.11. *Suppose f is a Riemann integrable, 2π-periodic function for which there exists some $C > 0$ and some $\alpha > 1/2$ such that*

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

holds for all x, y. Let p be a positive integer. Then

$$\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \leq \frac{C^2 \pi^{2\alpha}}{2^{2\alpha p + 1}}.$$

Proof. To prove this, we will apply part (b) of Lemma 3.10 with $h = \frac{\pi}{2^{p+1}}$. Then

$$\sum_{2^{p-1} < |n| \leq 2^p} \left| \sin\left(\frac{\pi n}{2^{p+1}}\right) \right|^2 |\hat{f}(n)|^2 \leq \frac{C^2 \pi^{2\alpha}}{2^{2\alpha p + 2}}.$$

Note that we are only summing over $2^{p-1} < |n| \leq 2^p$, where

$$\frac{\pi 2^{p-1}}{2^{p+1}} < \left| \frac{\pi n}{2^{p+1}} \right| \leq \frac{\pi 2^p}{2^{p+1}} \Leftrightarrow \frac{\pi}{4} < \left| \frac{\pi n}{2^{p+1}} \right| \leq \frac{\pi}{2}.$$

This implies

$$\frac{1}{2} \leq \left| \sin\left(\frac{\pi n}{2^{p+1}}\right) \right|^2 \leq 1.$$

Hence

$$\begin{aligned} \frac{1}{2} \sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 &\leq \sum_{2^{p-1} < |n| \leq 2^p} \left| \sin\left(\frac{\pi n}{2^{p+1}}\right) \right|^2 |\hat{f}(n)|^2 \\ &\leq \frac{C^2 \pi^{2\alpha}}{2^{2\alpha p + 2}}. \end{aligned}$$

We conclude that

$$\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \leq \frac{C^2 \pi^{2\alpha}}{2^{2\alpha p + 1}}.$$

□

We now have all the machinery needed to prove Bernstein's theorem.

Proof. (Of Bernstein.) We apply the Cauchy-Schwarz inequality and obtain

$$\left(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|\right)^2 \leq \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \sum_{2^{p-1}<|n|\leq 2^p} 1.$$

By Lemma 3.11, we have

$$\begin{aligned}\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \sum_{2^{p-1}<|n|\leq 2^p} 1 &\leq \frac{C^2\pi^{2\alpha}}{2^{2\alpha p+1}} \cdot 2^p \\ &= \frac{C^2\pi^{2\alpha}}{2^{p(2\alpha-1)+1}}.\end{aligned}$$

Combined, this yields

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \leq \frac{C\pi^\alpha}{\sqrt{2^{p(2\alpha-1)+1}}}.$$

Note that we can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\hat{f}(n)| &= |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + \sum_{p=1}^{\infty} \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \\ &\leq |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + C\pi^\alpha \sum_{p=1}^{\infty} \frac{1}{\sqrt{2^{p(2\alpha-1)+1}}}. \qquad (5)\end{aligned}$$

We simplify the sum to

$$\begin{aligned}C\pi^\alpha \sum_{p=1}^{\infty} \frac{1}{\sqrt{2^{p(2\alpha-1)+1}}} &= \frac{C\pi^\alpha}{\sqrt{2}} \sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha-1/2)}} \\ &= \frac{C\pi^\alpha}{\sqrt{2}} \sum_{p=1}^{\infty} \left(\frac{1}{2^{\alpha-1/2}}\right)^p.\end{aligned}$$

Moreover, by the formula for geometric sums, we have

$$\sum_{p=1}^{\infty} \left(\frac{1}{2^{\alpha-1/2}}\right)^p = -1 + \sum_{p=0}^{\infty} \left(\frac{1}{2^{\alpha-1/2}}\right)^p = -1 + \frac{1}{1 - \left(\frac{1}{2^{\alpha-1/2}}\right)}$$

whenever $\left|\frac{1}{2^{\alpha-1/2}}\right| < 1$, i.e. whenever $\alpha > \frac{1}{2}$. Applying this to (5), we arrive at

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + \frac{C\pi^\alpha}{\sqrt{2}} \left(-1 + \frac{1}{1 - \left(\frac{1}{2^{\alpha-1/2}}\right)}\right)$$

for $\alpha > \frac{1}{2}$. In this case, the Fourier series of f converges uniformly to f by Theorem 3.6. □

Remark 3.1. Our purpose in this section has been to give a brief overview of some conditions which ensure absolute convergence of the Fourier series associated with a given function. The space of continuous functions on the unit circle (i.e. 2π-periodic) with absolutely convergent Fourier series is called the *Wiener algebra*. It is denoted by $A(\mathbb{T})$ (where $\mathbb{T}$ denotes the unit circle). By letting $\mathcal{C}^1(\mathbb{T})$ denote the space of continuously differentiable functions on the unit circle, Theorem 3.8 can be rephrased as $\mathcal{C}^1(\mathbb{T}) \subset A(\mathbb{T})$.

The Wiener algebra has many interesting properties; it is a so called *Banach algebra*, an algebra that forms a complete normed vector space (a *Banach space*) while its norm also satisfies $\|xy\| \leq \|x\|\|y\|$ for all its elements x, y. For more information about $A(\mathbb{T})$ and its structure, we refer the reader to section 6 in chapter 1, as well as chapter 8, of [3]. □

3.3 Pointwise convergence

It turns out that when we settle for pointwise convergence instead of uniform convergence, we can put even fewer restrictions on a function f to ensure convergence of its associated Fourier series. We will prove the following:

Theorem 3.12. *Let f be a Riemann integrable, 2π-periodic function which is differentiable at a point x_0. Then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.*

Here, as opposed to in Theorem 3.8, we do not need f to be continuously differentiable - differentiability in itself is enough. However, we can only guarantee pointwise convergence of the partial sums, not uniform convergence.

To prove this result, we will apply the same technique as the one used in section 2.1 in chapter 3 of [5]. Firstly, we must establish a result concerning Riemann integrability in general.

Proposition 3.13. *Suppose f is a bounded function on the (closed, bounded) interval $[a, b]$, and let $c \in (a, b)$. If f is Riemann integrable on $[a, c-\delta]$ and $[c+\delta, b]$ for all small $\delta > 0$, then f is integrable on $[a, b]$.*

Proof. Suppose $|f| \leq M$, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ small enough to satisfy

$$4\delta M \leq \frac{\varepsilon}{3}.$$

Since f is integrable on $[a, c-\delta]$ and $[c+\delta, b]$, we can find partitions P_1, P_2 of each of these intervals such that

$$\mathcal{U}(P_i, f) - \mathcal{L}(P_i, f) < \frac{\varepsilon}{3}$$

for $i = 1, 2$. We define the common refinement $P = P_1 \cup P_2$, a partition of the entire interval $[a, b]$. Now observe that

$$\begin{aligned}\mathcal{U}(P, f) - \mathcal{L}(P, f) = &\,(\mathcal{U}(P_1, f) - \mathcal{L}(P_1, f)) \\ &+ \Big(\sup_{c-\delta \leq x \leq c+\delta} f(x) - \inf_{c-\delta \leq x \leq c+\delta} f(x)\Big)((c+\delta) - (c-\delta)) \\ &+ (\mathcal{U}(P_2, f) - \mathcal{L}(P_2, f)).\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}\mathcal{U}(P,f)-\mathcal{L}(P,f) &\leq (\mathcal{U}(P_1,f)-\mathcal{L}(P_1,f)) \\ &+ 2\delta\left|\sup_{c-\delta\leq x\leq c+\delta} f(x) - \inf_{c-\delta\leq x\leq c+\delta} f(x)\right| \\ &+ (\mathcal{U}(P_2,f)-\mathcal{L}(P_2,f)) \\ &\leq \frac{\varepsilon}{3} + 2\delta\cdot 2M + \frac{\varepsilon}{3} \\ &\leq \varepsilon.\end{aligned}$$

This proves that f is integrable over $[a,b]$. □

We now have the machinery we need to prove Theorem 3.12.

Proof. (Of Theorem 3.12.) We begin by defining

$$F(t) = \begin{cases} \frac{f(x_0-t)-f(x_0)}{t}, & \text{for } t \neq 0 \text{ and } |t| < \pi, \\ -f'(x_0), & \text{for } t = 0. \end{cases}$$

Note that F is differentiable at $t=0$, and thus bounded around this point. Moreover, for every small $\delta > 0$, F is integrable on $[-\pi,-\delta]\cup[\delta,\pi]$, since f is integrable there and $|t| > \delta$ on these intervals. By Proposition 3.13, F must be integrable on $[-\pi,\pi]$.

Recall now that

$$S_N(f)(x) = (f * D_N)(x),$$

where $D_N(x)$ is the N^{th} Dirichlet kernel. Moreover, we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1.$$

Hence

$$\begin{aligned}S_N(f)(x_0) - f(x_0) &= (f * D_N)(x_0) - f(x_0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-t)D_N(t)\,dt - f(x_0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0-t)-f(x_0))D_N(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)tD_N(t)\,dt\,.\end{aligned}$$

By the closed form formula of D_N,

$$tD_N(t) = \frac{t}{\sin(\frac{t}{2})}\sin\left(\left(N+\frac{1}{2}\right)t\right),$$

where $\frac{t}{\sin(\frac{t}{2})}$ is continuous on $[-\pi, \pi]$ and

$$\sin\left(\left(N+\frac{1}{2}\right)t\right) = \sin(Nt)\cos\left(\frac{t}{2}\right) + \cos(Nt)\sin\left(\frac{t}{2}\right).$$

Applying this yields

$$\begin{aligned}
S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)tD_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\frac{t}{\sin(\frac{t}{2})}\Big(\sin(Nt)\cos\left(\frac{t}{2}\right) + \cos(Nt)\sin\left(\frac{t}{2}\right)\Big)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)t\sin(Nt)\frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\,dt \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)t\cos(Nt)\,dt\,.
\end{aligned}$$

Applying the terminology of Example 2.3, we observe that the first integral above is the Fourier coefficient b_N of $\frac{1}{2}\big(F(t)\frac{t}{\sin(t/2)}\cos(t/2)\big)$, and the second integral is the Fourier coefficient a_N of $\frac{1}{2}F(t)t$. These functions are Riemann integrable, and so by the Riemann-Lebesgue lemma, their Fourier coefficients tend to 0 as $N \to \infty$. We conclude that

$$S_N(f)(x_0) - f(x_0) \to 0$$

as N tends to infinity. □

At this point, it is easy to get lulled into a false sense of security, and imagine that the differentiability condition in Theorem 3.12 can be replaced with a plain and simple condition of continuity. Many distinguished mathematicians thought so for a long time. We are, however, rapidly approaching one of the highlights of this thesis; the construction of a continuous function whose Fourier series diverges at one point. Before taking on this noble task, we establish one last result.

Theorem 3.14. ***(Riemann localization principle)*** *Suppose f and g are two Riemann integrable, 2π-periodic functions, and for some x_0 there exists an open interval I containing x_0 such that*

$$f(x) = g(x) \quad \text{for all } x \in I.$$

Then $S_N(f)(x_0) - S_N(g)(x_0) \to 0$ as $N \to \infty$.

Proof. Note that for all $x \in I$, we have that $f(x) - g(x) = 0$. This implies that $f - g$ is differentiable at x_0. By Theorem 3.12, $S_N(f)(x_0) - S_N(g)(x_0) \to 0$ as $N \to \infty$. □

From this theorem, we conclude - with some surprise - that the convergence of $S_N(f)(x_0)$ depends solely on f's behavior near x_0, despite the fact that we need to integrate f over an interval of length 2π to obtain its Fourier coefficients.

4 A special function

At last we arrive at the much anticipated construction of a continuous function f whose Fourier series diverges at one point (which we, without loss of generality, choose to be the origin). The first construction of such a function was produced by the German mathematician Paul Du Bois-Reymond in 1876. At this time, many prominent mathematicians - including Riemann, Dirichlet and Weierstrass - believed that the Fourier series of every continuous function must converge (as told in chapter 18 of [4]). In this context, it is easy to see why Du Bois-Reymond's counter example came as such a surprise.

Since the first construction of a continuous function with a diverging Fourier series, several other examples have been produced. We will follow the technique used in chapter 18 of [4]. For alternative constructions, the reader is referred to section 10.3 in chapter 10 of [2] and section 2.2 in chapter 3 of [5].

Theorem 4.1. ***(Du Bois-Reymond)*** *There exists a 2π-periodic, continuous function $f : \mathbb{R} \to \mathbb{C}$ such that $\limsup_{N\to\infty} |S_N(f)(0)| = \infty$.*

Remark 4.1. If one would rather that the Fourier series of f diverges at an arbitrary point x_0, one may study $f(x - x_0)$. □

The question is, where to begin? To make things easier for ourselves, we will break our construction into steps by setting some sub-goals. Firstly, we aim to find a reasonably well-behaved, but not necessarily continuous, 2π-periodic function h for which $\sup_x |h(x)|$ is small but $\sup_N |S_N(h)(0)|$ is very large. Our next goal is to use our function h to find a continuous, 2π-periodic function g for which $\sup_x |g(x)|$ is small but $\sup_N |S_N(g)(0)|$ is very large. Finally, we will try to modify g and construct our desired function f of Theorem 4.1.

Recall from our discussion about kernels and convolutions that the Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ satisfies

$$S_N(h)(x) = (h * D_N)(x),$$

and so

$$S_N(h)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)D_N(-x)\,dx\,.$$

By Proposition 2.4,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx \geq \frac{4}{\pi^2}\ln(2(N+1)).$$

These results will be useful to us as we prove the following;

Lemma 4.2. *Let $h_N(x) = \operatorname{sgn}(D_N(-x))$. Then*

(a) h_N is constant on intervals $\left(\frac{k\pi}{2N+1}, \frac{(k+1)\pi}{2N+1}\right)$ for $k \in \mathbb{N}$,

(b) $|h_N(x)| \leq 1$ for all x,

(c) $S_N(h_N)(0) \geq \frac{4}{\pi^2}\ln(2(N+1))$.

Proof. By definition,

$$h_N(x) = \begin{cases} 1 & \text{for } D_N(-x) > 0, \\ 0 & \text{for } D_N(-x) = 0, \\ -1 & \text{for } D_N(-x) < 0. \end{cases}$$

This, along with the closed form formula $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$ $(x \neq 0)$, proves properties (a) and (b) (meaning h_N is reasonably well-behaved). Moreover,

$$\begin{aligned} S_N(h_N)(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h_N(x)D_N(-x)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(-x)|\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx \\ &\geq \frac{4}{\pi^2}\ln(2(N+1)). \end{aligned}$$

This concludes our proof. □

With that, we have found our desired function $h = h_N$. Before we move on, let us establish some results that will be needed in our next step of the construction.

Lemma 4.3. *If $g, h : \mathbb{R} \to \mathbb{C}$ are 2π-periodic, Riemann integrable functions, then*

(a) $|\hat{g}(n) - \hat{h}(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - h(x)|\,dx$ *for all* n,

(b) $|S_N(g)(0) - S_N(h)(0)| \leq \frac{2N+1}{2\pi}\int_{-\pi}^{\pi} |g(x) - h(x)|\,dx$ *for all* $n \geq 0$.

Proof. (a) Simply observe that

$$\begin{aligned} |\hat{g}(n) - \hat{h}(n)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (g(x) - h(x))e^{-inx}\,dx\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |(g(x) - h(x))e^{-inx}|\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - h(x)|\,dx\,. \end{aligned}$$

(b) We use (a) to obtain

$$\begin{aligned}|S_N(g)(x) - S_N(h)(x)| &= \left|\sum_{n=-N}^{N} (\hat{g}(n) - \hat{h}(n))e^{inx}\right| \\ &\leq \sum_{n=-N}^{N} |(\hat{g}(n) - \hat{h}(n))e^{inx}| \\ &= \sum_{n=-N}^{N} |\hat{g}(n) - \hat{h}(n)| \\ &\leq \sum_{n=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - h(x)|\, dx\right) \\ &= \frac{2N+1}{2\pi}\int_{-\pi}^{\pi} |g(x) - h(x)|\, dx\,.\end{aligned}$$

□

We now find our desired function g;

Lemma 4.4. *For each $N \geq 0$ there exists a continuous, 2π-periodic function $g_N : \mathbb{R} \to \mathbb{R}$ such that*

(a) $|g_N(x)| \leq 1$ for all x,

(b) $|S_N(g_N)(0)| \geq \frac{4}{\pi^2}\ln(N+1)$.

Proof. We can construct a continuous function g_N such that $|g_N(x)| \leq 1$ for all x and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_N(x) - h_N(x)|\, dx \leq \frac{1}{4(2N+1)}.$$

As $h_N(x)$ is a step function, this can be done using the same kind of technique as in the proof of Lemma 3.3 (see Figure 3).

Our previous results, Lemmas 4.2 and 4.3, finally yield

$$\begin{aligned}|S_N(g_N)(0)| &\geq |S_N(h_N)(0)| - |S_N(g_N)(0) - S_N(h_N)(0)| \\ &\geq \frac{4}{\pi^2}\ln(2(N+1)) - \frac{1}{4} \\ &= \frac{4}{\pi^2}\ln(N+1) + \frac{4}{\pi^2}\ln(2) - \frac{1}{4} \\ &\geq \frac{4}{\pi^2}\ln(N+1).\end{aligned}$$

□

For convenience's sake, we replace g_N with a trigonometric polynomial G_N, which is still continuous and well-behaved with $S_N(G_N)(0)$ large.

Lemma 4.5. *For each $N \geq 0$ there exists a trigonometric polynomial G_N such that*

(a) $|G_N(x)| \leq 2$ *for all* x,

(b) $|S_N(G_N)(0)| \geq \frac{4}{\pi^2} \ln(N+1) - 1$.

Proof. Recall Corollary 2.7.1, by which we can find a trigonometric polynomial G_N with

$$|G_N(x) - g_N(x)| \leq \frac{1}{2N+1}$$

for all x. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G_N(x) - g_N(x)| \, dx \leq \frac{1}{2N+1}. \tag{6}$$

By Lemma 4.4,

$$\begin{aligned} |G_N(x)| &\leq |G_N(x) - g_N(x)| + |g_N(x)| \\ &\leq \frac{1}{2N+1} + 1 \\ &\leq 2 \end{aligned}$$

for all x. Using Lemma 4.3 and (6), we obtain

$$\begin{aligned} |S_N(G_N)(0)| &\geq |S_N(g_N)(0)| - |S_N(G_N)(0) - S_N(g_N)(0)| \\ &\geq \frac{4}{\pi^2} \ln(N+1) - 1. \end{aligned}$$

□

At last, we are ready to prove Theorem 4.1.

Proof. (Of Du Bois-Reymond.) By Lemma 4.5, we can find a sequence of trigonometric polynomials H_k, $k = 1, 2, \ldots$, and positive integers n_k such that

(a) $|H_k(x)| \leq 1$ for all x,

(b) $|S_{n_k}(H_k)(0)| \geq 2^{2k}$.

Moreover, let q_k be a sequence of integers such that $q_k > n_k$ and $q_k \geq q_{k-1}$ for each $k = 1, 2, \ldots$ Then we can write

$$H_k(x) = \sum_{r=-q_k}^{q_k} \alpha_{k,r} e^{irx},$$

for some complex numbers $\alpha_{k,r}$. Additionally, we define $p_k = \sum_{j=1}^{k} (2q_j + 1)$. (The reasons behind these definitions will become clear further along the proof.)

Now we can define, for positive integers m, the functions

$$f_m(x) = \sum_{k=1}^{m} 2^{-k} e^{ip_k x} H_k(x).$$

Note that each f_m is the sum of continuous, 2π-periodic functions, and thus continuous and 2π-periodic itself.

We observe that, for $m \geq m' + 1$,

$$\begin{aligned}
|f_m(x) - f_{m'}(x)| &= \left| \sum_{k=m'+1}^{m} 2^{-k} e^{ip_k x} H_k(x) \right| \\
&\leq \sum_{k=m'+1}^{m} |2^{-k} e^{ip_k x} H_k(x)| \\
&\leq \sum_{k=m'+1}^{m} 2^{-k} \\
&= 2^{-m'}(2^{-1} + 2^{-2} + \cdots + 2^{-(m-m')}) \\
&\leq 2^{-m'} \to 0
\end{aligned}$$

as $m' \to \infty$. Then the sequence of functions f_m is a Cauchy sequence, and since we are operating in a complete metric space, we may conclude that $f_m \to f$ for some f. By the inequality above, the convergence is uniform. Since f is the uniform limit of continuous, 2π-periodic functions, f must be continuous and 2π-periodic as well.

Moreover,

$$\widehat{f_m}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f_m(x)\, dx \to \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx = \hat{f}(n)$$

as $m \to \infty$. For this reason, we must study the Fourier coefficients of $f_m(x)$;

$$\begin{aligned}
\widehat{f_m}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f_m(x)\, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \left(\sum_{k=1}^{m} 2^{-k} e^{ip_k x} H_k(x) \right) dx\,.
\end{aligned}$$

By the uniform convergence of f_m, we can interchange integration and summation:

$$\begin{aligned}
\widehat{f_m}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \left(\sum_{k=1}^{m} 2^{-k} e^{ip_k x} H_k(x) \right) dx \\
&= \sum_{k=1}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} H_k(x)\, dx\,.
\end{aligned}$$

Let ℓ be a positive integer such that $m \geq \ell$. Then

$$\begin{aligned}
\widehat{f}_m(n) &= \sum_{k=1}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} H_k(x)\, dx \\
&= \frac{2^{-\ell}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_\ell)} H_\ell(x)\, dx + \sum_{\substack{k=1 \\ k\neq\ell}}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} H_k(x)\, dx \\
&= 2^{-\ell} \widehat{H}_\ell(n-p_\ell) + \sum_{\substack{k=1 \\ k\neq\ell}}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} H_k(x)\, dx\,. \qquad (7)
\end{aligned}$$

We now study the sum in (7). Note that, by using the definition of $H_k(x)$, we obtain

$$\begin{aligned}
&\sum_{\substack{k=1 \\ k\neq\ell}}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} H_k(x)\, dx \\
&= \sum_{\substack{k=1 \\ k\neq\ell}}^{m} \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k)} \left(\sum_{r=-q_k}^{q_k} \alpha_{k,r} e^{irx} \right) dx \\
&= \sum_{\substack{k=1 \\ k\neq\ell}}^{m} 2^{-k} \sum_{r=-q_k}^{q_k} \frac{\alpha_{k,r}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k-r)}\, dx\,. \qquad (8)
\end{aligned}$$

Recall that the last integral satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k-r)}\, dx = \begin{cases} 1 & \text{if } n-p_k-r=0, \\ 0 & \text{if } n-p_k-r\neq 0. \end{cases} \qquad (9)$$

Now let u be an integer such that $|u| \leq q_\ell$, and let $n = p_\ell + u$. For $\ell < k$, we have

$$\begin{aligned}
n-p_k-r &= p_\ell + u - p_k - r \\
&= u - r + \sum_{j=1}^{\ell} (2q_j+1) - \sum_{j=1}^{k} (2q_j+1) \\
&= u - r - \sum_{j=\ell+1}^{k} (2q_j+1).
\end{aligned}$$

Since $|r| \le q_k$ and $|u| \le q_\ell$,

$$\begin{aligned} n - p_k - r &= u - r - \sum_{j=\ell+1}^{k} (2q_j + 1) \\ &\le q_\ell + q_k - \sum_{j=\ell+1}^{k} (2q_j + 1) \\ &< 0, \end{aligned}$$

where we have used that all q_j are positive integers and that $q_j \ge q_{j-1}$ for all j by definition. For $\ell > k$,

$$\begin{aligned} n - p_k - r &= u - r + \sum_{j=k+1}^{\ell} (2q_j + 1) \\ &\ge -q_\ell - q_k + \sum_{j=k+1}^{\ell} (2q_j + 1) \\ &> 0. \end{aligned}$$

We have established that $n - p_k - r \ne 0$ for $n = p_\ell + u$, and so

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k-r)} \, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix(p_\ell+u-p_k-r)} \, dx \\ &= 0. \end{aligned}$$

Hence the sum in (8) equals zero for $n = p_\ell + u$. By (7),

$$\begin{aligned} \widehat{f}_m(p_\ell + u) &= 2^{-\ell} \widehat{H}_\ell(p_\ell + u - p_\ell) + 0 \\ &= 2^{-\ell} \widehat{H}_\ell(u). \end{aligned}$$

Since we have established that $\widehat{f}_m(n) \to \hat{f}(n)$ as $m \to \infty$ for fixed n, we conclude that

$$\hat{f}(p_\ell + u) = 2^{-\ell} \widehat{H}_\ell(u) \quad \text{for all } |u| \le q_\ell. \tag{10}$$

Similarly to above, we note that for $n < 0$,

$$\begin{aligned} n - p_k - r &= n - \sum_{j=1}^{k} (2q_j + 1) - r \\ &\le n + q_k - \sum_{j=1}^{k} (2q_j + 1) \\ &< 0, \end{aligned}$$

by the way we defined q_k and the fact that $|r| \leq q_k$. Using (9), we see that

$$\begin{aligned} \widehat{f_m}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f_m(x)\, dx \\ &= \sum_{k=1}^{m} 2^{-k} \sum_{r=-q_k}^{q_k} \frac{\alpha_{k,r}}{2\pi} \int_{-\pi}^{\pi} e^{-ix(n-p_k-r)}\, dx \\ &= 0 \end{aligned}$$

for $n < 0$. Thus

$$\hat{f}(n) = 0 \quad \text{for all } n < 0.$$

Let us now take a look at the partial sums

$$S_{p_k+n_k}(f)(0) = \sum_{u=-(p_k+n_k)}^{p_k+n_k} \hat{f}(u) = \sum_{u=0}^{p_k+n_k} \hat{f}(u),$$

$$S_{p_k-n_k-1}(f)(0) = \sum_{u=-(p_k-n_k-1)}^{p_k-n_k-1} \hat{f}(u) = \sum_{u=0}^{p_k-n_k-1} \hat{f}(u).$$

We observe that

$$|S_{p_k+n_k}(f)(0) - S_{p_k-n_k-1}(f)(0)| = \left| \sum_{u=p_k-n_k}^{p_k+n_k} \hat{f}(u) \right| = \left| \sum_{u=-n_k}^{n_k} \hat{f}(p_k+u) \right|.$$

Note now that since we are summing over all integers u such that $|u| \leq n_k < q_k$, we can use (10) and conclude that

$$\begin{aligned} |S_{p_k+n_k}(f)(0) - S_{p_k-n_k-1}(f)(0)| &= \left| \sum_{u=-n_k}^{n_k} \hat{f}(p_k+u) \right| \\ &= \left| \sum_{u=-n_k}^{n_k} 2^{-k} \widehat{H}_k(u) \right| \\ &= 2^{-k} |S_{n_k}(H_k)(0)| \\ &\geq 2^{-k} \cdot 2^{2k} = 2^k \to \infty \end{aligned}$$

as $k \to \infty$. Hence

$$\lim_{k\to\infty} \max(|S_{p_k+n_k}(f)(0)|, |S_{p_k-n_k-1}(f)(0)|) = \infty.$$

From this, we may finally conclude that

$$\limsup_{N\to\infty} |S_N(f)(0)| = \infty.$$

□

Remark 4.2. After Du Bois-Reymond presented his construction of a continuous function whose Fourier series diverges at one point, public opinion in the mathematical community began to change - it was now suspected that the Fourier series of a continuous function could possibly diverge at *every* point. In 1926, Andrey Kolmogorov presented a Lebesgue integrable function whose Fourier series diverges everywhere, strengthening the belief that there might exist a continuous function with the same property. However, in 1964, Swedish mathematician Lennart Carleson proved that the Fourier series of any continuous function f converges to f almost everywhere, i.e. for all $x \notin E$ where E is some set of measure zero. (All this history and more can be found in chapter 19 of [4].)

We restrict ourselves to simply mentioning these results here, as they are quite advanced and require in-depth knowledge of measure theory and Lebesgue integrability. The interested reader is referred to section 3 in chapter 2 of [3] as well as section 10.4 in chapter 10 of [2]. □

References

[1] Mary L. Boas. *Mathematical methods in the physical sciences*. Wiley, Hoboken, 3 edition, 2005.

[2] Robert E. Edwards. *Fourier Series: A Modern Introduction Volume 1*. Springer-Verlag, New York, 2 edition, 1979.

[3] Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge University Press, Cambridge, 3 edition, 2004.

[4] Thomas W. Körner. *Fourier Analysis*. Cambridge University Press, Cambridge, 1988.

[5] Elias M. Stein and Rami Shakarchi. *Fourier analysis: An introduction*. Princeton University Press, Princeton, 2003.