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The angle sum of a triangle: From Euclid to Einstein

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Abstract

Einstein's general relativity theory changed our understanding of gravity. First defined by Newton as an attractive force between matter, general relativity connects it to the geometry of the space-time and particularly to its curvature. This thesis focuses on the historical role of the triangle's angle sum in defining and deriving curvature and developing non-Euclidean geometry. This was needed for Einstein to be able to formulate the relation between the curvature and the energy and momentum. The curvature of a surface is shown to be an intrinsic property of the space, calculable by the angle sum of a triangle. From the results of an experiment testing general relativity in 1919, the angle sum of a triangle is shown to exceed 180 ° verifying that space is curved. The thesis also deals with Einstein's special relativity theory, proving that time and distances are relative to the frame of reference.

Contents

1 Introduction

If you ask a high school student what the angle sum of a triangle is, the answer will most likely be that it is always 180°. This is true in Euclidean geometry but the students will probably not know that there are different non-Euclidean geometries where the angle sum can be greater or smaller. Up until the 19th century this was not known to most mathematicians either. The development of non-Euclidean geometry, that originated from trying to prove the parallel postulate, has not only led to new understandings of geometry but also insights about space and physics. Without it Einstein couldn't have formulated the relation between the energy and the curvature of space and time in his general relativity theory.

This thesis will deal with the triangle's angle sum through history as it laid the ground to differential geometry that later on became the tool used in general relativity, which was one of the starting points of modern physics.

First, in section 2 we will look at Euclidean geometry; how it was structured by Euclid and how the angle sum of a triangle played a part in it. The main source of information for this part will be Wolfe (1945), which will also be relevant when we look at how non-Euclidean geometry started taking form in section 3 . Section 2 will also deal with the history of spherical triangles and we show how their angle sum differs from that of a triangle in the plane, using Rosenfeld (1988) as the primary source.

In section 3, we will look at how C.F. Gauss , using triangles, defined curvature in $\mathbb{R}^3$ and proved that it is intrinsic. We will also mention how B. Riemann extended Gauss' work in an n-dimensional space. For this we will use mostly Spivak (1999) as an inspiration.

In section 4 and 5 we will, mostly following Ellis and Ruth (2000), look at Einstein's theory of relativity by first looking at the special relativity and how the invariance of the speed of light led to time and distances being dependent on the frame of reference of the observer. We will continue in section 5 by briefly looking at the principles behind general relativity and how it connects with what Gauss and Riemann discovered. We will end by showing the results from Eddington's experiment in 1919 that proved that a triangle in general relativity can have an angle sum greater than 180°.

> *I entreat you, leave the doctrine of parallel lines alone; you should fear it like a sensual passion; it will deprive you of health, leisure and peace; it will destroy all joy in your life.*
>
> (Farkas Bolyai to his son Janos Bolyai)

2 Euclidean geometry

In about 300 B.C., Euclid wrote *Elementa*, a compilation of his work as well as the work of the Greek and Egyptian mathematicians before him (Waerden, 1975). Even though most of the discovering have been made before him by mathematicians like Thales or Pythagoras, Euclid contributed by organizing all the knowledge into a logical structure (Waerden, 1975). It was Euclid's understanding that not everything can be proven, that you have to draw the line somewhere. So he used postulates he assumed to be true based on experience and intuition as a starting point from which his other propositions would then logically follow. This was to be the standard in how mathematical reasoning would be done for over two thousand years until the late 1900th century when a more rigorous axiomatic method started taking its place (Mueller, 1969).

Euclid based his geometrical proofs on ten assumptions divided into two groups called "common notations" and "postulates". Though it is not clear what he thought the difference was, it is believed that the common notations were applicable to all sciences and the postulates were more specific to geometry.

Euclid's five common notations were (Fitzpatrick, 2008):

1. Things that are equal to the same thing are also equal to one.
2. If equals are added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to another.
5. The whole is greater than the part.

The first notation defines transitive property in relations between elements and can be expressed as: If $a = b$ and $b = c$ then it follows that $a = c$.

The second and third are known as addition and subtraction of equals and denotes as : If $a = b$ then it follows that $a+c = b+c$ and that $a-c = b-c$

The forth notation, that says that for example any e.g. number or set, a, is equal to itself, seems to be redundant but it is used in proofs when comparing angles or line segments with themselves.

The fifth notation is often attributed to Aristoteles even though he phrased it differently than what Euclid wrote in Elementa. Euclid's way of phrasing it meant that the whole can be divided into parts and that any of those parts is lesser than the whole. This can be written as:

If $a = |b| + |c|$ than $a > b$ and $a > c$

Euclid's postulates were (Fitzpatrick, 2008):

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

In Euclid's first postulate it is presumed that he says that there can only be one unique straight line connecting two points. This seems intuitively clear the shortest distance between the two points. Looking at a sphere we could easily see that this isn't always true. We could e.g. draw two lines of equal lengths between two antipodal points that collate the shortest distance. The postulate infer that you can't enclose a space using two unique straight lines since they can't coincide more than once.

The second postulate explains that a finite line can only be continued in its both extremities in only one way, which leads to the conclusion that two separate lines can't share a segment.

In the third and fourth postulate Euclid defines circles with a distance from its center to every point on its circumference and provides a measurement for angles.

The fifth postulate, also known as the parallel postulate, is equivalent to the claim that the sum of the internal angles in any triangle is 180°. The parallel postulate was early on criticized for being too complex and not self evident to be a postulate, but was regarded more as a proposition. The order of the postulates and Euclid postponing the usage of the postulate indicates that he himself was aware of this but could neither prove it nor proceed without it (Lewis, 1920).

Many tried to prove the parallel postulate using Euclid's four other postulates and for a long time it was believed that Proclus (410-485 A.D.) already had done so in the 5th century. The problem was that Proclus and those after him had to make new assumptions to make it work, such as Proclus assuming parallel lines have a constant distance between them when parallel lines are only defined as two straight lines that do not intersect at any point.

In trying to simplify the statement, the parallel postulate has often been substituted by equivalent postulates, of which one of the most common is *Playfair's axiom* that says

In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

Playfair's axiom, unlike the Parallel postulate, uses parallel lines and from it the parallel postulate, as Euclid defined it, can be deduced.

2.1 Triangles in spherical geometry

The earliest systematic work on spherical geometry has been written by Theodosius (169-100 B.C) and is called *Sphaerica*. In it, he defines many concepts of spherical geometry like the center and the diameter. To create circles he uses planes intersecting with the surface of the sphere. The planes that also go through the center of the sphere creates so-called great circles, the largest circles on the sphere. Thedosius created angles on the surface of the sphere by the inclination of two planes that intersect the sphere. He didn't consider spherical triangles though, that was first introduced and defined by Menelaus (70 -140) in his book *On the sphere*. To create a triangle on the sphere Menelaus used three great circles that would bound a triangle. Menelaus' work was much more concentrated on angles and sides of spherical triangles. Many of those propositions were analogous to that of Euclid in plane geometry and are interchangeable. The one that wasn't analogous was a proposition on the relation between the exterior angle of a spherical triangle and the interior angles opposite to it. The proposition also infer the sum of the interior angles of a spherical triangle and was worded as follows:

Theorem 2.1. *An exterior angle of any three-sided figure is smaller than the sum of both interior angles opposite to it.*

To prove this Menelaus used another proposition which is formulated below:

Lemma 2.2. *Let there be a triangle $\triangle ABC$ on the sphere with the vertices A,B and C and exterior angle $\angle BCD$ created by extending the line AC to an arbitrary point D. We have then the following equivalences:*

$\angle BCD$ is equal to $\angle A$ if and only if $AB + BC$ is equal to a semicircle
$\angle BCD$ is greater than $\angle A$ if and only if $AB + BC$ is less than a semicircle
$\angle BCD$ is lesser than $\angle A$ if and only if $AB + BC$ is greater than a semicircle

Proof. We create a spherical triangle $\triangle ABC$ on the sphere with the vertices A,B and C. We extend the line AC to an arbitrary point D creating the exterior angle $\angle BCD$.

To prove this proposition we need to show that $\angle BCD < \angle ABC + \angle BAC$.

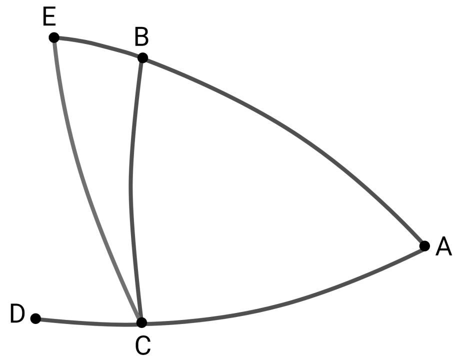

Figure 1: A triangle on a sphere with the vertices A,B and C where the segment AC has been extended to point D and AB has extended to E.

We assume that $\angle BCD$ is greater than $\angle BAC$ otherwise the proof would be trivial. This allows us to construct a new angle $\angle DCE$ equal to $\angle BAC$ but less than $\angle BCD$ by drawing a line from C to E where E is on the extension of the segment AB creating the figure shown in figure 1. Since $\angle DCE$ is an exterior angle to $\triangle AEC$ and is equal to $\angle BAC$ the segments $AE + EC$ must be equal to a semicircle according to lemma 2.2 . Therefore the segments $BE + EC$ must be smaller than a semicircle and thus the exterior angle $\angle ABC$ to the triangle $\triangle BCE$ must be greater than the interior angle $\angle BCE$ according to 2.2. If we add the angles $\angle BAC$ to both sides we get

$$\angle BCE + \angle BAC < \angle ABC + \angle BAC$$

If we then use that we created $\angle ECD$ so that $\angle ABC = \angle ECD$ and $\angle BCE + \angle DCE = \angle BCD$ we get

$$\angle BCE + \angle BAC = \angle BCE + \angle DCE = \angle BCD < \angle ABC + \angle BAC \; Q.E.D$$

□

If we were to add the remaining interior angle $\angle ACB$ to the inequality we would get that the angle sum of the triangle would be greater than that of a straight line. This would be contradictory to Euclid's parallel postulate but many thought of triangles on the sphere as a separate entity from that on the plane. The spherical triangle was instead often compared to that of a triangle in plane geometry. Comparing the two's angle sum, the spherical triangle would be greater than that in the plane. The difference, called angular excess, would be $\alpha + \beta + \gamma - \pi$, if the angles in the spherical triangle are α, β and γ.

Much later in the 17th century Albert Girard (1595-1632) showed that this angular excess is equivalent to the area of the triangle . His proof was not rigorous so we will use a version of Euler's proof made about 150 years later. He used Menelaus' definition of triangles as the surface bound by three great circles. These great circles will intersect with each other in three points creating a spherical triangle.

Theorem 2.3. *The area, A, of a triangle on a sphere with the radius R is*

$$A = R^2(\alpha + \beta + \gamma - \pi)$$

if α,β and γ is its angles.

To prove this we need to use the notion of a spherical digon. A digon is defined by two great circles that intersect twice on a sphere in two antipodal points that are the digon's vertices. We will use that the area is proportional to the angle between the two great circles.

Lemma 2.4. *The area, A, of a digon on a sphere with the radius R equals*

$$A = 2\theta R^2$$

where θ is the angle between the two sides of the digon.

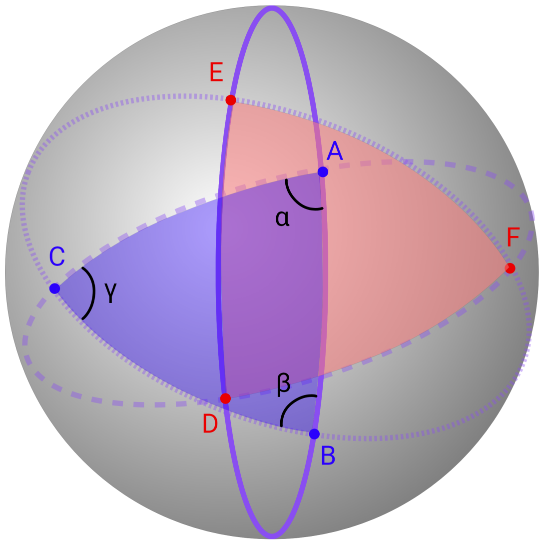

Figure 2: A sphere with two triangles bounded by three great circles with the vertices A, B and C and their antipodal points D,E and F respectively. The triangles have the angles α, β and γ

Proof. We create a triangle ABC using three great circles intersecting at A, B and C, which are the triangle's vertices. The great circle will also intersect at the vertices' antipodal points D, E and F respectively, creating the triangle DEF with an equal area to ABC. The great circles will also create three pairs of digons, AD, BE and CF that have the angles α,β and γ

respectively. These six digons will cover up the entire surface of the sphere, but will overlap over both of the triangle seen in figure 2. So we can equate the area of the digons if we remove four times the area of the triangles. Thus we have

$$2 \cdot Area(AD) + 2 \cdot Area(BE) + 2 \cdot Area(CF) - 4 \cdot Area(\triangle ABC) = 4\pi R^2.$$

Using lemma (2.4) we have

$$2 \cdot 2\alpha R^2 + 2 \cdot 2\beta R^2 + 2 \cdot 2\gamma R^2 - 4 \cdot Area(\triangle ABC) = 4\pi R^2.$$

We simplify and solve for the area of the triangle and get

$$Area(\triangle ABC) = \alpha R^2 + \beta R^2 + \gamma R^2 - \pi R^2 = R^2(\alpha + \beta + \gamma - \pi). \quad (2.1)$$

□

This leads to the conclusion that the angle sum must be greater than π if the area is to have a non-zero positive value. One also concludes that the greater the triangle on a sphere is the greater is its angular excess and therefore also the greater is the angle sum.

3 Non-Euclidean geometry

When there still was no sound proof of Euclid's parallel postulate in the beginning of the 18th century Girolamo Saccheri (1667-1733) tried a new approach nobody had thought of before. He used the Reductio ad absurdum method, trying to prove the parallel postulate by assuming its opposite and showing that it would lead to contradiction. He used a quadrilateral with right angles at the base and that the segments from the base to the summit were of equal length. Saccheri then stated three hypotheses about the length of the summit being equal to, greater or lesser than the base. Using the parallel postulate, the summit's length must be equal to the base and the angles at the summit must be that of right angles. For a quadrilateral Saccheri proved that if the line joining the midpoints of the base and the summit is perpendicular to the both the base and the summit then the angles at the summit are equal and acute if the summit is greater than the base or obtuse if the summit is less. These are named as *Saccheri quadrilaterals*.

Showing that the angles of the summit being acute or obtuse would lead to contradictions would then prove that the parallel postulate to be true by elimination of alternatives. Sacchieri named these as the hypothesis of the acute and the hypothesis of the obtuse angles.

To find the contradictions we look at a quadrilateral where both the base angles and one of the summit angles are right under the hypothesis of the acute angle and the hypothesis of the obtuse angle separately and then deduce the fourth angle.

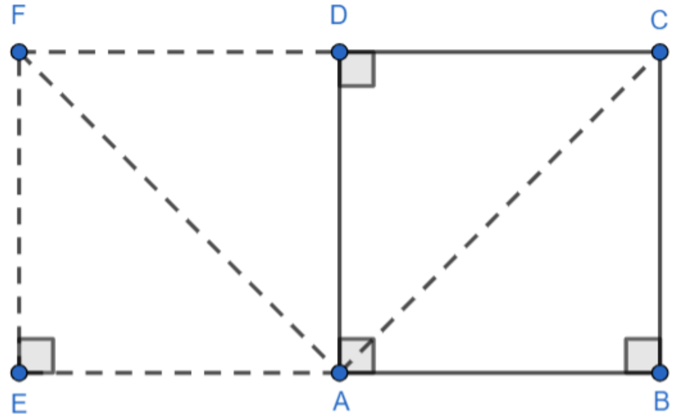

Figure 3: A quadrilateral ABCD with extended segment from AB to E. The line EF is perpendicular to AE.

We create a quadrilateral ABCD where the angles at A, B and D are right. We extend the segment BA to E so that AE is equal to BA. From E we draw the line EF perpendicular to AE so that EF is equal to BC. From F we draw lines to D and to A. To join A and C we also draw the line AC. This would give us the figure shown in figure 3 .

By construction, the triangles AEF and ABC have two sides and their included angle congruent to the corresponding sides and angle (SAS) and therefore the triangles are also congruent. This implies that the angles EAF and BAC are equal.

The angle EAD is a supplementary angle with the right angle BAD and is therefore also a right angle. Thus the angles DAF and DAC must be equal. Then we have in triangles ADF and ACD a shared common side in AD and congruent sides in AF and AC with an equal included angle with DAF and DAC and thus the triangles are also congruent. It follows that the angle ADF is a right angle and therefore F, D and C are collinear since ADF and ADC are supplementary angles to the line FC. This makes $BCFE$ a Saccheri quadrilateral and thus the angles at C and F must be acute under the acute hypothesis and obtuse under the obtuse hypothesis respectively. We use this lemma to deduce the angle sum of a triangle.

We first look at the situation of a right triangle ABC with the right angle at C. From the midpoint M of A and B we draw the line MP perpendicular to BC with P located on BC. We then draw the line AD so that the angle BAD is equal to ABC and so AQ is equal to PB, with Q located on AD.

Joining Q and M we get the picture shown in figure 4. In the triangle BMP sides BM and BP will be equal to AM and AQ with an equal included angle. Therefore the triangles BMP and AQM are congruent and

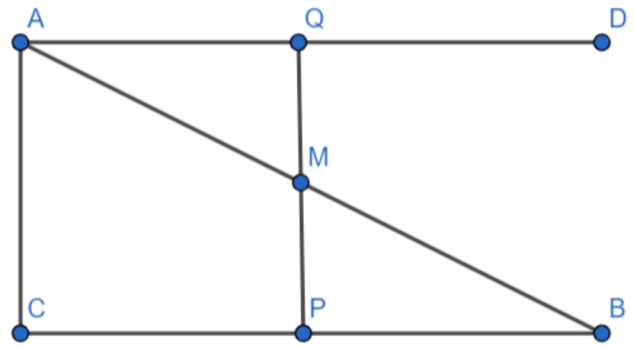

Figure 4: A triangle ABC with a right angle at C. M is the midpoint of AB. AD is drawn so the angle BAD is equal to ABC and the point Q lies on AD so AQ is equal to PB.

that implies that the angle AQM is right. Since the angle AQM is equal to BMP and BMP is a supplementary angle with AMP then AQM must also be a supplementary angle with AMP and therefore the points Q, M and P must be collinear.

This makes $AQPC$ to be a quadrilateral with three right angles and an acute or obtuse angle at A under the acute and obtuse hypothesis respectively. Since the angle ABC is equal to BAD and the angle at A is the sum of BAC and BAD we get that the sum of ABC and BAC is equal to the angle at A. Therefore the sum of the angles of the triangle ABC is the sum of the angles at A and C. Under the acute hypothesis, where the angle at A is less than 90°, the angle sum would be less than two right angles and under the obtuse hypothesis it would be greater than two right angles .

Saccheri's assumption that the angels were obtuse led him to the contradiction that the extensions of the parallel base and summit intersect. Assuming Euclid's second postulate also to be true, that you can extend a finite line indefinitely. Trying to refute the acute hypothesis never gave him a contradiction he wanted but led him to many strange conclusions like the two parallel straight lines intersecting, or having a common perpendicular on each side of which they diverge, or diverge in one direction and converge in the other. Even though not able to find a contradiction, he stated the acute hypothesis to be false claiming it to be *"...repugnant to the nature of straight lines"* (Rosenfeld, 1988). In his attempt to disprove it he was forced to prove several propositions along the way, thus unknowingly providing some of the classical theorems in what will be later known as hyperbolic geometry.

Half a century later a German mathematician Johann Hemrich Lambert (1718-1777) tried to prove the parallel postulate similarly to Saccheri's method. Lambert, like Saccheri, refutes the obtuse angle hypothesis and even though he acknowledges the flaws of his arguments he also refuted the acute angle. In his work Lambert proved not only that under the obtuse and acute hypothesis the angle sum of a triangle is greater respectively lesser than 180°, but also how this angular defect relates to the triangle's area under the acute hypothesis.

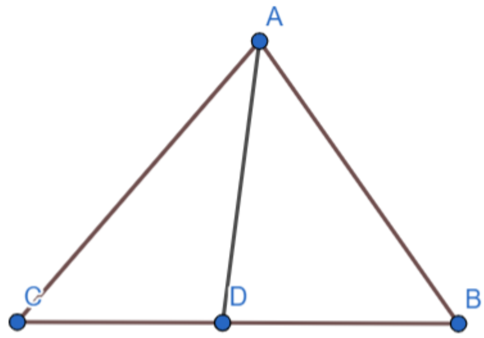

Figure 5: A triangle ABC with a line AD drawn from A to any point D on the line CB

If we look at a triangle ABC and draw a line from A to D on the line BC we get the triangle in figure 5. The sum of the angular defect for the triangle ABD and ACD is

$$180° - \angle ABD - \angle ADB - \angle BAD + 180° - \angle ACD - \angle ADC - \angle CAD$$

With ADB and ADC being supplementary angles equal to 180° and BAD and CAD being adjacent angles equal to BAC we get

$$180° - \angle ABD - \angle BAC - \angle ACD.$$

Since $\angle ABD = \angle ABC$ and $\angle ACD = \angle ACB$ the sum of the angular defect is

$$180° - \angle ABC - \angle BAC - \angle ACB$$

,which is the angular defect of the triangle ABC. This together with Euclid's fifth common notation lead Lambert to the conclusion that not only that the greater area a triangle has the greater the angular defect and therefore the lesser the angle sum but also that the triangles area A must be proportional to the angular defect : $A = C(180 - (\alpha + \beta + \gamma))$, where C is an unknown constant and α, β, γ are the triangle's angles. Lambert also proved that for the obtuse hypothesis, the triangle's area would also be proportional to the angular defect but that the angle sum would always be greater than 180°. He observed the resemblance with the spherical triangle, stating that the obtuse hypothesis would work on a sphere instead of a plane. From this he also stated that the acute angle hypothesis must hold on some imaginary sphere with an imaginary radius. If we put the radius as $c \cdot i$ in the formula for the area of a spherical triangle 2.1 we would get

$$Area = c^2 i^2 (\alpha + \beta + \gamma - \pi) = c^2(\pi - (\alpha + \beta + \gamma)),$$

which resembles the result for the acute hypothesis.

Another consequence Lambert found was that under the acute hypothesis there is an absolute measure of length for every line, of area for every surface and of volume for every physical space. This differed from the consensus at that time that the measure depended on how it was represented.

3.1 Gaussian geometry

In the beginning of the 19th century many mathematicians were trying to solve the problem with the parallel postulate by e.g. using the Reductio ad absurdum method, trying to prove that the opposite would be absurd (Wolfe, 1945). One who spent a lot of his work on Euclidian geometry was Carl Friedrich Gauss(1777-1855). He had, from the assumption that a triangle could be greater than 180°, formed a new geometry which he called "non-Euclidean geometry", a geometry not built on Euclid's parallel postulate(ibid). Like Girard, he found that on a surface matching that of a sphere the sum of the angles of a triangle could be greater than 180° and that the difference would be proportional to its area.

When studying curved surfaces Gauss found it unnecessary to use Cartesian coordinates to indicate a point in the Euclidean space (Fre, 2018). Instead he used two different systems of curves where the curves within the system all are considered as parallel to each other. Then for all points of the surface there would be one curve from both systems which the point would lie on. Since there can only be one specific curve from each system which intersect at this point, the curves would then indicate the point's position. Distances on the surface couldn't be measured using euclidean geometry but instead Gauss thought of distances as the sum of infinitesimal segments along the surface between two points. A segment that starts in the point (u, v) where the curves u and v intersect would then end at $(u + du, v + dv)$ where du and dv are infinitesimals. Since the change is so small, both points and the segment could be approximately lie in the tangent plane of point (u, v). In this plane Euclidean geometry could be applied and if the change of coordinates in this planes axis is dx, dy and dz the distance in quadratic form, ds^2, could be calculated using Pythagoras theorem: $ds^2 = dx^2 + dy^2 + dz^2$. Studying the geometry of the surface, Gauss used the unit sphere in defining the curvature. By mapping the unit normals from the neighbourhood of a point P on the surface to a fixed point, the unit vectors would point to the surface of the unit sphere. The curvature of the surface of P is then determined by the area on the unit sphere divided by the area on the surface. The more the surface would curve at P the more the unit normal would diverge and the greater the area would become on the unit sphere. When defining the curvature Gauss arrived at his result using a cumbersome calculus method, we will instead use a more modern approach with vector calculus but getting the same result as Gauss.

Definition 3.1. *Let M be a 2-dimensional submanifold of $\mathbb{R}^3$ with the inclusion map $i\colon M \hookrightarrow \mathbb{R}^3$ At each point $p\epsilon M$ there is a unique unit vector ν that is perpendicular to M_p Then we have a normal map $\nu\colon M \mapsto S^2 \subset \mathbb{R}^3$ where S^2 is the surface of the unit sphere. If $U\epsilon M$ is the neighbourhood around the point p then he curvature K at point p is then given by*

$$K(p) = \lim_{U \to p} \frac{Area\,of\,\nu(U)}{Area\,of\,U}.$$

If we represent the points on M using two independent system of curves (s,t) on M with the coordinate system $X\colon M \mapsto \mathbb{R}^2$ the inverse function f will be $f = X^{-1}\colon \mathbb{R}^2 \mapsto M \subset \mathbb{R}^3$. If $p = f(s,t)$ then the neighbourhood of p is made up by the points

$$f(s,t),\ f(s+\Delta s,t),\ f(s,t+\Delta t) \text{ and } f(s+\Delta s,t+\Delta t).$$

This surface can be approximated by the parallelogram in the tangent plane $T_{M(p)}$ of M at p, which is spanned by $\frac{\delta f}{\delta s}\Delta s$ and $\frac{\delta f}{\delta t}\Delta t$.

The area of the neighbourhood of p is then given by

$$|\frac{\delta f}{\delta s}\Delta s \times \frac{\delta f}{\delta t}\Delta t| = |\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}|\Delta s \Delta t.$$

This tangent plane must be parallel to the tangent plane $T_{S^2(\nu(p))}$ of the unit sphere at the point $\nu(p)$ because $\nu(p)$ is perpendicular to $T_{M(p)}$ by definition. The tangent plane of the unit sphere is given by the change of the normal vector.

Definition 3.2. *If the normal map is defined by $\nu\colon M \mapsto S^2 \subset \mathbb{R}^3$ then $\delta(\nu) : T_{M(p)} \mapsto T_{S^2(\nu(p))}$.*

Thus the curvature $K(p)$ can be calculated using the areas on the tangent planes spanned by the tangent vectors where the tangent vectors on $T_{S^2(\nu(p))}$ are given by $\frac{\delta \nu}{\delta s}$ and $\frac{\delta \nu}{\delta t}$. If M is the graph of $g : \mathbb{R}^2 \mapsto \mathbb{R}$ so that

$$M = (x, y, (g(x,y)) : (x,y)\epsilon\mathbb{R}^2,$$

then M is the image of$f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by

$$f(s,t) = (s,t,g(s,t))$$

We choose the vectors u and w as

$$\begin{cases} u = \frac{\delta f}{\delta s} = (1, 0, \frac{\delta g}{\delta s}) \\ w = \frac{\delta f}{\delta t} = (0, 1, \frac{\delta g}{\delta t}). \end{cases} \tag{3.1}$$

The area on $T_{M(p)}$ is then given by

$$Area\, of\, U = |u \times w| = |(-\frac{\delta g}{\delta s}, -\frac{\delta g}{\delta t}, 1| = \sqrt{(\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1}.$$

The normal vector is given by

$$\nu(f(s,t)) = \frac{u \times w}{|u \times w|} = \frac{(-\frac{\delta g}{\delta s}, -\frac{\delta g}{\delta t}, 1)}{\sqrt{(\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1}}. \tag{3.2}$$

Therefore the area on the sphere is then given by

$$Area\, of\, \nu(U) = |\frac{\delta \nu}{\delta s} \times \frac{\delta \nu}{\delta t}| = \frac{\frac{\delta^2 g}{\delta s^2}\frac{\delta^2 g}{\delta t^2} - (\frac{\delta^2 g}{\delta s \delta t})^2}{((\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1)^{3/2}}.$$

(for a more detailed derivation see Appendix A).

This leads us to the conclusion that the curvature equals

$$K(p) = \lim_{U \to p} \frac{Area\, of\, \nu(U)}{Area\, of\, U} = \frac{|\frac{\delta \nu}{\delta s} \times \frac{\delta \nu}{\delta t}|}{|u \times w|} = \frac{\frac{\delta^2 g}{\delta s^2}\frac{\delta^2 g}{\delta t^2} - (\frac{\delta^2 g}{\delta s \delta t})^2}{((\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1)^2}. \tag{3.3}$$

If we take the squared first partial derivatives of f and define them as E,F and G we would get

$$\begin{cases} E = \frac{\delta f}{\delta s}\frac{\delta f}{\delta s} = 1 + \frac{\delta g}{\delta s}^2 \\ G = \frac{\delta f}{\delta t}\frac{\delta f}{\delta t} = 1 + \frac{\delta g}{\delta t}^2 \\ F = \frac{\delta f}{\delta s}\frac{\delta f}{\delta t} = \frac{\delta g}{\delta s}\frac{\delta g}{\delta t}. \end{cases} \tag{3.4}$$

Then we obtain (like Gauss did) the area of U to be given by

$$Area\, of\, U = |u \times w| = \sqrt{EG - F^2}. \tag{3.5}$$

Furthermore if we denote the second partial derivatives times the normal vector as L, M, and N we would have

$$\begin{cases} L = \frac{\delta^2 f}{\delta s^2} \cdot \frac{\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}}{|\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}|} = \frac{\frac{\delta^2 g}{\delta s^2}}{\sqrt{(\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1}} \\ N = \frac{\delta^2 f}{\delta t^2} \cdot \frac{\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}}{|\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}|} = \frac{\frac{\delta^2 g}{\delta t^2}}{\sqrt{(\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1}} \\ M = \frac{\delta^2 f}{\delta s \delta t} \cdot \frac{\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}}{|\frac{\delta f}{\delta s} \times \frac{\delta f}{\delta t}|} = \frac{\frac{\delta^2 g}{\delta s \delta t}}{\sqrt{(\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1}} \end{cases} \tag{3.6}$$

which we could use to rewrite the area of $\nu(U)$ to be

$$Area\,of\;\nu(U) = |\frac{\delta\nu}{\delta s} \times \frac{\delta\nu}{\delta t}| = \frac{LN - M^2}{\sqrt{EG - F^2}} \tag{3.7}$$

and we would finally have the curvature as follows

$$K = \frac{LN - M^2}{EG - F^2}. \tag{3.8}$$

This is how Gauss expressed it except he used D, D' and D'' instead of L, M and N which is a more modern notation. Now we will prove that the surface curvature isn't dependent on how it is embedded in space but only of the properties within the surface. To do this we will first define a geodesic and the curvature of a curve.

3.1.1 Curves

In defining the curvature of a curve we will measure its deviation from a straight line, which would be represented by the tangent vector given by the partial derivatives of the curve. Calculating how much the tangent vector changes along the normal vector gives us the curvature of the curve.

Definition 3.3. *Let α be a regular curve on a surface $S \epsilon \mathbb{R}^3$ and s be a segment on S. Let also the tangent vector t be given by $\frac{d\alpha}{ds}$ and the curvature κ of α given by*

$$\kappa = |\frac{dt}{ds}|.$$

If we consider an orthonormal basis $(t, t \times N, N)$,where N is the normal vector to the surface S, then $\frac{dt}{ds}$ can be written as a a combination of N that is orthogonal to the surface and $t \times N$ that is tangential to the surface since t is orthogonal to $\frac{dt}{ds}$. Namely

$$\frac{dt}{ds} = \kappa_g t \times N + \kappa_N N. \tag{3.9}$$

The coefficient κ_g is called the *geodesic curvature*, it represents the amount the curve deviates from a geodesic on S, where a geodesic is a curve between two points on a surface that realizes the shortest path. In the Euclidean space the geodesic is a straight line.

Definition 3.4. *Let α be a regular curve on a surface $S \epsilon \mathbb{R}^3$ and s be a segment on S. Let the tangent vector t be given by $\frac{d\alpha}{ds}$ and N be the normal vector to S then α is a geodesic if at every point on α:*

$$t \times N = 0$$

This means the geodesic follows the surface and the curvature of the curve is dependent only on the surface.

The coefficient κ_N is called *the normal curvature* and is related to how the surface curves in $\mathbb{R}^3$.

If we take the scalar product of equation (3.9) with $t \times N$ we get that the geodesic curvature equals

$$\kappa_g = \frac{dt}{ds} \cdot t \times N. \tag{3.10}$$

3.1.2 Curvature being intrinsic

Theorem 3.1 (The Gauss-Bonnet theorem for a triangle(Woodward and Bolton, 2018))**.** *Let T be a triangle on a surface $S\epsilon\mathbb{R}^n$ and let A, B and C be the interior angles of T then the integral of the geodesic curvature k_g along the boundary of T and the total curvature of the triangle satisfies the relation*

$$\int_{\delta T} k_g\, ds + \iint_T K dA = A + B + C - \pi$$

Proof. We will first define an orthonormal basis $(e^1, e^2,)$ in the tangent plane at every point of δT where e^1 and e^2 are functions of the parameters (u, v). If N is the normal vector then (e^1, e^2, N) forms an orthonormal basis of $\mathbb{R}^3$. Let θ be the angle between the tangent vector $\dot{\alpha}$ and the unit vector e^1. This gives us

$$\dot{\alpha} = \cos\theta e^1 + \sin\theta e^2$$

and the second derivative then is

$$\ddot{\alpha} = \cos\theta \dot{e}^1 + \sin\theta \dot{e}^2 + \theta'(-\sin\theta e^1 + \cos\theta e^2). \tag{3.11}$$

With $N = e^1 \times e^2$ we also have

$$N \times \dot{\alpha} = -\sin\theta e^1 + \cos\theta e^2. \tag{3.12}$$

Equations 3.11 and 3.12 in the definition for geodesic curvature from 3.10

$$k_g = (N \times \dot{\alpha}) \cdot \ddot{\alpha} =$$
$$= (-\sin\theta e^1 + \cos\theta e^2)(\cos\theta \dot{e}^1 + \sin\theta \dot{e}^2 + \dot{\theta}(-\sin\theta e^1 + \cos\theta e^2)) =$$
$$= \dot{\theta} + \cos^2\theta(\dot{e}^1 \cdot e^2) - \sin^2\theta(\dot{e}^2 \cdot e^1) + \sin\theta\cos\theta(\dot{e}^2 \cdot e^2 - \dot{e}^1 \cdot e^1).$$

Since e^1 and e^2 are orthogonal unit vectors we have

$$\dot{e}^1 \cdot e^1 = \dot{e}^2 \cdot e^2 = 0, \;\; \dot{e}^1 \cdot e^2 = -e^1 \cdot \dot{e}^2.$$

Thus

$$k_g = \dot{\theta} - e^1 \cdot \dot{e}^2. \tag{3.13}$$

The integral along the boundary δT then gives us

$$\int_{\delta T} k_g ds = \int_{\delta T} \theta' - e^1 \cdot \dot{e}^2 ds = \int_{\delta T} \dot{\theta} ds - \int_{\delta T} (e^1 \cdot \frac{e^2}{du}) du + (e^1 \cdot \frac{e^2}{dv}) dv. \quad (3.14)$$

With Green's theorem the second integral can be rewritten as

$$\begin{aligned} &\iint_T (\frac{d}{du}(e^1 \cdot e^2_v) - \frac{d}{dv}(e^1 \cdot e^2_u)) dudv = \\ &= \iint_T (e^1_u \cdot e^2_v - e^1_v \cdot e^2_u) dudv. \end{aligned} \quad (3.15)$$

Since e^1, e^2 and N constitute an orthonormal basis we can use them to express the partial derivatives of e^1 and e^2. Because the partial derivatives of e^1 and e^2 are perpendicular to e^1 respectively e^2 they will only be dependent on e^2 and N respectively e^1 and N. Therefore

$$\begin{cases} e^1_u = a_1 e^2 + \lambda_1 N \\ e^1_v = b_1 e^2 + \mu_1 N \\ e^2_u = a_2 e^1 + \lambda_2 N \\ e^2_v = b_2 e^1 + \mu_2 N \end{cases} \quad (3.16)$$

where $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2, \mu_1$, and μ_2 are scalars that could depend on u and v. Using that e^1 and e^2 are perpendicular, we have $e^1 \cdot e^2 = 0$. The derivative of this equation with respect to u gives

$$e^1_u \cdot e^2 + e^2 \cdot e^2_u = 0.$$

With (3.16) it becomes

$$(a_1 e^2 + \lambda_1 N) \cdot e^2 + (a_2 e^1 + \lambda_2 N) \cdot e^1 = a_1 + \lambda_1 N \cdot e^2 + a_2 + \lambda_2 N \cdot e^1 = 0$$

From this relation it is clear that $a_1 = -a_2$ and differentiating with respect to v we would analogously get $b_1 = -b_2$ and (3.16) becomes

$$\begin{cases} e^1_u = a_1 e^2 + \lambda_1 N \\ e^1_v = b_1 e^2 + \mu_1 N \\ e^2_u = -a_1 e^1 + \lambda_2 N \\ e^2_v = -b_1 e^1 + \mu_2 N. \end{cases} \quad (3.17)$$

It follows that :

$$e^1_u \cdot e^2_v - e^1_v \cdot e^2_u = \lambda_1 \mu_2 - \lambda_2 \mu_1. \quad (3.18)$$

If we now instead take the scalar product between the vectors e^1 and e^2 with N and differentiate it with respect to u and w we would get

$$\begin{cases} e^1 \cdot N_u = -e^1_u \cdot N \\ e^1 \cdot N_v = -e^1_v \cdot N \\ e^2 \cdot N_u = -e^2_u \cdot N \\ e^2 \cdot N_v = -e^2_v \cdot N. \end{cases} \quad (3.19)$$

Determining N_u and N_v we use the relation between the normal vector and the curvature given by (3.3). Namely

$$K = \frac{|N_u \times N_v|}{|u \times v|}$$

and we rewrite it to

$$|N_u \times N_v| = K \cdot (u \times v). \tag{3.20}$$

Using (3.5) in the formula for the normal vector from (3.2) we could use the equation $u \times v = N\sqrt{EG - F^2}$. This is substituted in (3.20) together with the formula for K from (3.8) and we get that

$$N_u \times N_v = \frac{LN - M^2}{EG - F^2} N \Leftrightarrow (N_u \times N_v)N = \frac{LN - M^2}{EG - F^2}.$$

Since $N = e^1 \times e^2$ the equation results in

$$\frac{LN - M^2}{EG - F^2} = (N_u \times N_v)(e^1 \times e^2) = (N_u \cdot e^1)(N_v \cdot e^2) - (N_u \cdot e^2)(N_v \cdot e^1).$$

Using (3.19) we could rewrite it as

$$\frac{LN - M^2}{EG - F^2} = (e_u^1 \cdot N)(e_v^2 \cdot N) - (e_u^2 \cdot N)(e_v^1 \cdot N).$$

Using (3.17) we have

$$\frac{LN - M^2}{EG - F^2} = \lambda_1 \mu_2 - \lambda_2 \mu_1$$

and finally by equating it with (3.18) we get

$$e_u^1 \cdot e_v^2 - e_v^1 \cdot e_u^2 = \frac{LN - M^2}{EG - F^2} = K.$$

Thus is (3.15) equal the integral of the curvature K. The integral of the geodesic curvature in (3.14) then becomes:

$$\int_{\delta T} k_g ds + \iint_T K dA = \int_{\delta T} \dot{\theta} ds. \tag{3.21}$$

To conclude the proof we will use .a standard argument from the *Hopf Umlaufsatz* called *Theorem of Turning Tangents*. It says that the total rotational angle of the tangent vector given by $\int \dot{\theta} ds$ along a simple and closed curve is 2π . If there are singularities and the curve is piecewise regular by $(\alpha_1, ..., \alpha_n)$ the total rotation angle is $2\pi - \sum_{i=1}^{n} \phi$ where ϕ is the angle between α_i and α_{i+1}. For a triangle with vertices (A, B, C) we then have (Woodward and Bolton, 2018)

$$\int_{\delta T} \theta' ds = 2\pi - \sum_{i=1}^{n} \phi = 2\pi - ((\pi - A) + (\pi - B) + (\pi - C)) = A + B + C - \pi$$

This in Eq. 10 and it gives us

$$\int_{\delta T} k_g ds + \iint_T K dA = A + B + C - \pi$$

and our proof is completed. □

Since a triangle is constructed by geodesics the geodesic curvature by definition is 0 we can conclude

$$\iint_T K dA = A + B + C - \pi$$

This shows that the curvature of a surface is independent of the surface's embedding in space, but is an intrinsic invariant and can be determined by only measuring distances, angles and areas on the surface. Gauss did a more general proof of this fact in 1827 and called it the remarkable theorem *Theorema Egregium* since it means if the Gaussian curvature is the same then the distances,angles and areas are unchanged.

3.2 Riemannian geometry

Until the 19th century the concept of physical space and Euclidean geometry were indistinguishable. Euclidean geometry was the only accepted geometry at that time and was thought to be applicable to all spaces (Farwell and Knee, 1990). Non-Euclidean geometry had been independently invented by Nikolai Lobachevsky and Janos Bolyai in the 1830s, but it was assumed by many that a contradiction would be soon found.

Bernhard Riemann (1826-1866), who studied under Gauss, gave a famous lecture called *About the hypothesis which geometry is based upon* for his lectureship in 1854 (Mlodinow, 2000). Riemann wanted to move away from applying Euclidean geometry to any space a-priori, but instead approached it by letting experience determine which geometry to apply (Farwell and Knee, 1990). In his lecture he gave an early definition of a manifold which he called an "n fold extended quantity" that was locally like a n-dimensional Euclidean space. Like Gauss, Riemann thought of distances as integrals of infinitesimal segments along a curve where the segments are given by the tangent vectors of the curve. If the curve on the manifold is given by $\gamma = (x^1, x^2, ..., x^n)$ then the infinitesimal distance ds can be determined by using the quadratic form $ds^2 = \dot{\gamma} \cdot \dot{\gamma} = \sum_{i=1} \sum_{j=1}^{n} g_{ij} dx^i dx^j$, where g_{ij} is a number and $g = \sum_{i=1} g_{ij}$ is called the *Riemann metric*. Riemann assumed that this metric could be determined by the curvature.

He introduced the *Riemann curvature tensor* R that assigns a tensor to every point on a manifold that represents the curvature at that point. A modern definition of the Riemann curvature tensor uses the affine connections and is expressed with the *Levi-Civita connection*∇:

$$R(\mathbf{u}, \mathbf{v})\mathbf{w} = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}, \tag{3.22}$$

where u and v are vector fields on the manifold and w is a vector. The $\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w}$ part transports w an infinitesimal distance in v's direction then in u's direction . Likewise $\nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w}$ transports w but in reversed order. It can therefore be interpreted as the discrepancy when transporting a vector along two vector fields in different orders. We can say that a vector is transported along a curve if the angle between the transported vector and the tangent to the curve remains constant throughout the entire transport. On a flat manifold it would give the same vector independent of order but on a curved manifold the resulting vector would differ. The last part $\nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}$ is a necessary correction if the transport of the vector doesn't end up at the same point.

4 Special relativity

4.1 The invariance of the speed of light

After a double slit experiment in 1801, that showed the interference of light, the wave theory of light was established followed by the aether theory (Mlodinow, 2000). Light moving like a wave through space without a medium seemed implausible at the time so the idea of a medium, called the aether, that was thought to be everywhere but undetectable and only affecting light, became leading . The idea was that light moved with different speeds depending on the inertial system where the aether's frame would be absolute and the only one where the speed would be constant, noted as c (Resnick, 1968). In the 1880's Michelson and Morley tried to detect the Earth's motion through this aether by using addition of velocities. According to Galilean relativity used at the time, the speed of light in Earth's frame, c', would be the sum of the speed of light in the ethers frame, c, and the speed of the Earth through the ether ,v (Resnick, 1968). The experiment was carried out by transmitting a beam of light with a single frequency against the Earth's movement through the assumed ether. The beam is then divided by a mirror, M, into two beams with one beam continuing along the same path and the other one reflected 90 °. Both beams travel then the length L to two mirrors M_1 respectively M_2 to be reflected back to mirror M where they reflect one last time to a telescope T where they converge. The difference in time the beams travel will emerge from travelling back and forth to mirror M. The beam that the first time passes through M will first move in the same direction as the Earth and then against in on the way back. Accordingly to Galilean relativity it will take the round trip in the time

$$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = L\frac{(c+v)+(c-v)}{c^2-v^2} = L\frac{2c}{c^2(1-\frac{v^2}{c^2}} = \frac{2L}{c}\frac{1}{1-\frac{v^2}{c^2}}.$$

Mirror M_2 is at distance L from M but since the Earth has moved $vt_2/2$ from when the beam travels from M to M_2 the distance the beam travels is given by Pythagoras theorem as:

$$2(c\frac{t_1}{2})^2 = L^2 + (v\frac{t_1}{2})^2 \leftrightarrow 4L^2 = c^2t_2^2 - v^2t_2^2.$$

We solve for t_2 and get

$$t_2 = 2\frac{L}{\sqrt{c^2-v^2}} = 2L\frac{1}{\sqrt{c^2(1-\frac{v^2}{c^2})}} = \frac{2L}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

The difference in time between the two beams are therefore:

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\frac{1}{1-\frac{v^2}{c^2}} - \frac{2L}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2L}{c}((1-\frac{v^2}{c^2})^{-1} - (1-\frac{v^2}{c^2})^{-1/2}).$$

Since $\frac{v^2}{c^2} << 1$ we can use the binomial approximation that states $(1+x)^a \approx 1 + ax$ we then get

$$\Delta t = \frac{2L}{c}((1 + \frac{v^2}{c^2}) - (1 + \frac{v^2}{2c^2})) = \frac{L}{c}\frac{v^2}{c^2}.$$

To find the path difference, $\Delta\lambda$, we multiply with c and get

$$\Delta\lambda = c\Delta t = L\frac{v^2}{c^2}$$

If the path difference is an integer of the wavelength, λ, there will be constructive interference, white fringes will appear on the screen of the telescope. If the whole apparatus would then rotate 90° the path difference would then analogously be $\Delta\lambda' = -L\frac{v^2}{c^2}$ and the fringes should shift. To find the fringe shift, ΔN, we subtract the path differences and divide by the wavelength:

$$\Delta N = \frac{\Delta\lambda - \Delta\lambda'}{\lambda} = \frac{2Lv^2}{\lambda c^2}.$$

Michelson and Morely predicted a shift of $\Delta N = 0,4$ but the experiment showed almost no shift at all (Resnick, 1968)). The experiment was repeated during all time of the day and all seasons of the year to counter the effect of Earth's spin and velocity relative to the aether, but with no change in outcome (Resnick, 1968).

One explanation for the negative results would be if the speed of light was the same in all frames of reference and there is no absolute frame.

4.2 Lorentz transformation

Lorentz, a supporter of the aether theory, wanted to explain the null result and went to prove ad hoc how the ether contracted bodies with a factor of $\sqrt{1 - \frac{v^2}{c^2}}$ (Resnick, 1968).

Einstein did the same thing, probably unaware of Lorentz contribution, but based on two axioms: the speed of light is an invariant, the same in all inertial systems, and the laws of physics apply to all systems.

If we define two inertial systems, S and S', such as:

1. x and x' are positive axis are in the same direction
2. S' moves with the constant speed, v, from S along the positive x-axis
3. The y and y' axis and the z and z' axis are parallel
4. The origo in S and S' coincide at $t = t' = 0$.

Due to the fact that accelerations only can occur from forces and not from transformations and that a point in one system can't have two corresponding

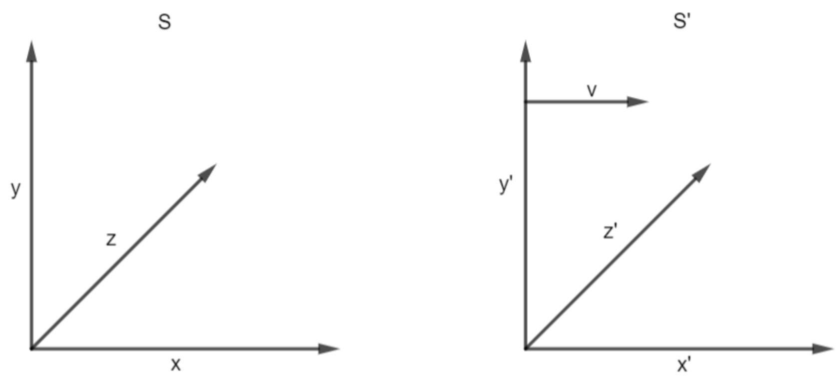

Figure 6: Two inertial systems S and S' that move with the speed v along the x-axis direction

points in another system the relation between x and x' must be linear. We then get

$$x' = \gamma x + bt \tag{4.1}$$

where γ and b are unknown constants. During the time ,t , since S and S' coincided the distance S' has moved in S is equal to vt . Therefore $x' = 0$ in S' has the coordinate $x = vt$ in S. This would lead to the equation (4.1) becoming

$$0 = \gamma vt + bt \leftrightarrow b = -\gamma v.$$

We substitute b in (4.1) and then we acquire that x' becomes

$$x' = \gamma x - \gamma vt = \gamma(x - vt). \tag{4.2}$$

Analogously we get the reverse transformation, transforming from S' to S, using that the distance now is instead $-vt$. This would instead result in the relation

$$x = \gamma(x' + vt). \tag{4.3}$$

If we now use a light that moves in the x- and x'-axis direction from S and S' origo at $t = t' = 0$. The distance the light traveled would be $x = ct$ in S and $x' = ct'$ in S'. Using this we get that (4.3) becomes

$$ct' = \gamma(ct - vt) = \gamma t(c - v).$$

If we then solve for $\frac{t}{t'}$, we get

$$\frac{t}{t'} = \frac{c}{\gamma(c - v)}. \tag{4.4}$$

In the same way (4.3) becomes

$$ct = \gamma(ct' + vt') = \gamma t'(c - v).$$

Solving for $\frac{t}{t'}$, we get

$$\frac{t}{t'} = \frac{\gamma(c+v)}{c}. \tag{4.5}$$

Equating (4.4) and (4.5) we get that

$$\frac{\gamma(c+v)}{c} = \frac{c}{\gamma(c-v)},$$

solve it for γ^2, we get

$$\gamma^2 = \frac{c^2}{(c-v)(c+v)} = \frac{c^2}{c^2 - v^2} = \frac{c^2}{c^2(1 - \frac{v^2}{c^2})} = \frac{1}{1 - \frac{v^2}{c^2}},$$

giving

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.6}$$

The number γ is know as the Lorentz factor. The transformation equation (4.1) and the reversed transformation equation (4.3) is defined as

$$x' = \gamma(x - vt) = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4.7}$$

and

$$x = \gamma(x' + vt') = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.8}$$

To find the transformation equation between t' and t we substitute x' from (4.7) and put it into (4.8) and get

$$x = \frac{x - vt}{1 - \frac{v^2}{c^2}} + \frac{vt'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.9}$$

Multiplying by $1 - \frac{v^2}{c^2}$, we get

$$x(1 - \frac{v^2}{c^2}) = x - vt + vt'\sqrt{1 - \frac{v^2}{c^2}}. \tag{4.10}$$

Solving for t', one obtains

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(t - \frac{vx}{c^2}). \tag{4.11}$$

The inverse function would analogously be

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(t' + \frac{vx'}{c^2}). \tag{4.12}$$

Lorentz called t the local time and t' the universal time thinking that the universal time was the correct time and local time was just a mathematical construct needed to make his hypothesis consistent with experimental facts (Mlodinow, 2000). It was Einstein who showed that time is relative and not the same for all observers.

4.3 Minkowski and space-time

In 1908 Minkowski gave a lecture called *Space and time* where he introduced using the four variables (ct, x, y, z) of space and time in coordinate form where a point in space would be equivalent to an event in space-time. According to Minkowski, space and time can not be seen geometrically by themselves but only as a union. According to special relativity, an event would be different in different inertial frames. To be able to measure intervals in space-time the invariant speed of light, c, is used as a conversion factor for the time dimension. Interval in space-time between two events is defined as

$$ds = \sqrt{c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)} \tag{4.13}$$

To compare the interval in different inertial frames we use equations (4.8) and (4.12) and get

$$ds = \sqrt{c^2(\gamma(\Delta t' + \frac{v\Delta x'}{c^2})^2 - ((\gamma(\Delta x' + vt'))^2 + \Delta y'^2 + \Delta z'^2)}.$$

Expanding it gives us

$$ds = \sqrt{c^2\gamma^2(\Delta t'^2 + \frac{2v\Delta x'\Delta t'}{c^2} + \frac{v^2\Delta x'^2}{c^4}) - \gamma^2(\Delta x' + 2v\Delta t'\Delta x' + v^2\Delta t'^2) - \Delta y'^2 - \Delta z'^2}.$$

By simplifying and factorising $\Delta t'^2$ and $\Delta x'^2$ we get

$$ds = \sqrt{\gamma^2(c^2 - v^2)\Delta t'^2 - \gamma^2(1 - \frac{v^2}{c^2})\Delta x'^2 - \Delta y'^2 - \Delta z'^2}.$$

Using $c^2 - v^2 = c^2(1 - \frac{v^2}{c^2})$ and $\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$ we would finally have

$$ds = \sqrt{c\Delta t'^2 - \Delta x' - \Delta y'^2 - \Delta z'^2} = ds'. \tag{4.14}$$

This shows that under transformations the interval is an invariant and will not change.

Since according to special relativity matter moves with a speed lower than the speed of light c therefore we get $\frac{\Delta x^2+\Delta y^2+\Delta z^2}{\Delta t^2} < c^2$, which gives us an interval greater than zero. This interval is dominated by the time separation and is therefore is called *timelike*. If instead the interval $ds = 0$ the time separation and distance are equal, the two events could only be connected by light and the interval is then called *lightlike*. The last possibility is that $ds < 0$. Such an interval is called *spacelike* and means the two events cannot be connected at all since nothing can move faster than the speed of light.

4.4 General Lorentz transformation

Let us instead consider at a general transformation between two inertial systems that moves with respect to each other with a constant speed in an arbitrary direction.

We look at an event with the space-time coordinates (ct, x_1, x_2, x_3) in S inertial system where the distance vector is $\vec{r} = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}$. Another inertial system S' moves with the velocity vector $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + v_3\vec{e_3}$ relative S. In S' the same event has the coordinates (ct', x'_1, x'_2, x'_3) with the distance vector $\vec{r'} = x'_1\vec{e_1} + x'_2\vec{e_2} + x'_3\vec{e_3}$.

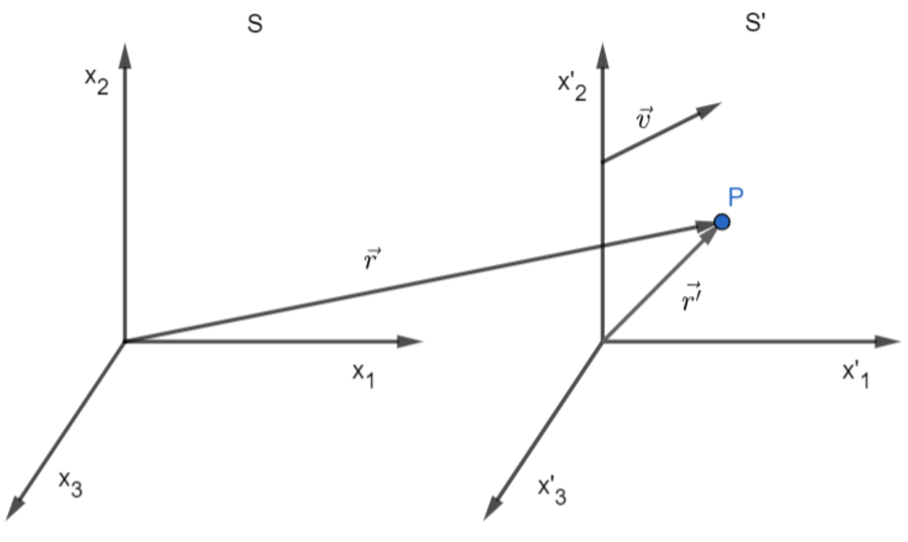

Figure 7: Two inertial systems S and S' in $\mathbb{R}^3$ with the relative speed v between the two systems in an arbitrary direction. The position vectors $\vec{r}$ and $\vec{r'}$ points to the same point from the two systems.

Transforming time between the systems according to (4.11):

$$t' = \gamma(t - \frac{\vec{r}\cdot\vec{v}}{c^2}) = \gamma t - \gamma\frac{x_1 v_1}{c^2} - \gamma\frac{x_2 v_2}{c^2} - \gamma\frac{x_3 v_3}{c^2}.$$

Extending with c on both sides and introducing $\vec{\beta} = \frac{\vec{v}}{c} = (\frac{v_1}{c}, \frac{v_2}{c}, \frac{v_3}{c})$ we would get

$$ct' = \gamma ct - \gamma x_1\beta_1 - \gamma x_2\beta_2 - \gamma x_3\beta_3. \tag{4.15}$$

Now divide $\vec{r}$ into two vectors $\vec{r_\parallel}$ and $\vec{r_\perp}$ that are respectively parallel and perpendicular to $\vec{v}$. When transforming between $\vec{r'}$ and $\vec{r}$ it is only the part that is in the same direction as the velocity vector that is affected. According to (4.7) the relation is

$$\vec{r'} = \vec{r_\perp} + \gamma(\vec{r_\parallel} - \vec{v}t).$$

We can use the relation $\vec{r} = \vec{r_\parallel} + \vec{r_\perp}$ and solve for $\vec{r_\perp}$ and substitute it in the transformation

$$\vec{r'} = \vec{r} + \vec{r_\parallel} + \gamma(\vec{r_\parallel} - \vec{v}t) = -\gamma\vec{v}t + \vec{r} + (\gamma - 1)\vec{r_\parallel}. \tag{4.16}$$

Seeing that $\vec{r_\parallel}$ has the same direction as the velocity, $\vec{v}$, between the systems, the dot product between $\vec{r}$ and $\vec{v}$ is the same as the scalar product between $\vec{r_\parallel}$ and $\vec{v}$ which in turn is the product of their magnitudes. Solving for $|\vec{r_\parallel}|$, we get

$$|\vec{r_\parallel}| = \frac{\vec{r_\parallel}\cdot\vec{v}}{|\vec{v}|} = \frac{\vec{r}\cdot\vec{v}}{|\vec{v}|}.$$

If we combine this with writing $\vec{r_\parallel}$ as its magnitude times $\vec{v}$'s direction we get

$$\vec{r_\parallel} = |\vec{r_\parallel}|\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{r}\cdot\vec{v}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = \vec{r}\cdot\vec{v}\frac{\vec{v}}{|\vec{v}|^2}.$$

Using this in (4.16) we get the transformation of the position vectors

$$\vec{r'} = -\gamma\vec{v}t + \vec{r} + (\gamma - 1)\vec{r}\cdot\vec{v}\frac{\vec{v}}{|\vec{v}|^2}.$$

Transformation of the components are

$$x_i' = \gamma v_i t + x_i + (\gamma - 1)\frac{v_i}{|v|^2}(x_1 v_1 + x_2 v_2 + x_3 v_3)$$

for $i = 1, 2, 3$.
Using $\beta_1 = v_1/c,\ \beta_2 = v_2/c,\ \beta_3 = v_3/c$ and (4.15) we can write the coordinate transformation as

$$x_i' = \gamma\beta_i ct + x_i + (\gamma - 1)\frac{\beta_i}{|\beta|^2}(x_1\beta_1 + x_2\beta_2 + x_3\beta_3)$$

$$ct' = \gamma ct - \gamma x_1\beta_1 - \gamma x_2\beta_2 - \gamma x_3\beta_3$$

$$\begin{bmatrix} ct' \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta_1 & \gamma\beta_2 & \gamma\beta_3 \\ \gamma\beta_1 & 1+(\gamma-1)\frac{\beta_1^2}{|\beta|^2} & (\gamma-1)\frac{\beta_1\beta_2}{|\beta|^2} & (\gamma-1)\frac{\beta_1\beta_3}{|\beta|^2} \\ \gamma\beta_2 & (\gamma-1)\frac{\beta_1\beta_2}{|\beta|^2} & 1+(\gamma-1)\frac{\beta_2^2}{|\beta|^2} & (\gamma-1)\frac{\beta_2\beta_3}{|\beta|^2} \\ \gamma\beta_3 & (\gamma-1)\frac{\beta_1\beta_3}{|\beta|^2} & (\gamma-1)\frac{\beta_2\beta_3}{|\beta|^2} & 1+(\gamma-1)\frac{\beta_3^2}{|\beta|^2} \end{bmatrix} \begin{bmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the Lorentz factor in the vector form is $\gamma = \frac{1}{\sqrt{1-\frac{\vec{v}\cdot\vec{v}}{c^2}}} = \frac{1}{\sqrt{1-\frac{|\vec{v}|^2}{c^2}}}$.

4.5 Time dilation and length contraction

If we look at how much time has elapsed between two points in time, t_A and t_B in S from S' point of view the time that has taken place would be according to Lorentz time transformation (4.11) be given by

$$\Delta t' = t'_B - t'_A = \gamma(t_B - \frac{vx_B}{c^2}) - \gamma(t_A - \frac{vx_A}{c^2}) =$$

$$= \gamma(t_B - t_A + \frac{v}{c^2}(x_A - x_B)) = \gamma(t_B - t_A - \frac{v}{c^2}(t_B - t_A)(\frac{x_B - x_A}{t_B - t_A}))$$

$$= \gamma(t_B - t_A)(1 - \frac{v}{c^2}\frac{x_A - x_B}{t_B - t_A}).$$

Using that $t_B - t_A = \Delta t$ and that $\frac{x_B - x_A}{t_B - t_A} = v$ we get

$$\Delta t' = \gamma\Delta t(1 - \frac{v^2}{c^2}).$$

Since $1 - \frac{v^2}{c^2} = \gamma^{-2}$ we finally have

$$\Delta t' = \gamma^{-1}\Delta t \tag{4.17}$$

From the local frame of reference of S the time in S' is perceived as going slower and vice versa from the S' frame of reference.

Looking at how distance might be distorted, we use Lorentz transformation (4.8) on a length L in S that goes from coordinate x_a to x_b and transform it to S' coordinates. Obtaining

$$L = x_b - x_a = \gamma(x'_b - vt'_b) - \gamma(x'_a - vt'_a) = \gamma(x'_b - x'_a) + \gamma(vt'_b - vt'_a).$$

We assume when the length is measured the coordinates are determined simultaneously and therefore $t'_a = t'_b$. The length L' in S' is $x'_b - x'_a$ making the equation

$$L = \gamma L'.$$

Lengths in S' that are in the same direction as S' velocity relative S will seem to be contracted from S perspective, but any length that isn't in the same direction will be unaffected.

5 Beginning of general relativity

Einstein wasn't content with the special relativity, only working on inertial systems that have an uniform motion, but wanted to create a model that worked for any frame of reference. To understand the frame of reference with acceleration Einstein used thought experiments and set up a situation with an elevator that was free falling toward the Earth. He concluded that any object in the elevator would free fall with the same motion as the elevator and would be stationary to the elevator's frame of reference (unless a force acted upon it). The inside of the accelerated elevator could thus be seen as an inertial system.

In another thought experiment the elevator is accelerated in space with the constant value of Earth's gravitational acceleration. Since anyone in the elevator would be pressed toward the floor with the same amount of force as on Earth and thus be unable to distinguish between the two, they are therefore equivalent. Einstein concluded that (Mlodinow, 2000):

> It is impossible to distinguish, except in comparison to other bodies, whether a body is undergoing uniform acceleration or is at rest in a uniform gravitational field.

This is called *the principle of equivalence* and is the third axiom in relativity. By using the work of Gauss and Riemann, Einstein could later on also include non-uniform gravitational fields by seeing them as infinitesimal patches of uniform gravitational fields.

If we see space-time as infinitesimal patches, the distance, L between two events using (4.13) can be seen as the sum of these patches using the Riemann metric.

$$L = \int \sqrt{ds^2} = \int \sum \sqrt{g_{ij} dx^i dx^j}$$

Since we have shown in (4.14) that ds is intrinsic, L must also be intrinsic. The shortest distance and the curve that is a geodesic is then the path that minimizes L. This is realized for light rays since their interval is $ds = 0$ at each point. In other words, lights are geodesics in curved space-time.

In a third thought experiment a light beam entered the elevator through one side and hit the opposite wall. When the elevator is at rest the light travels the shortest distance in a straight line. If the elevator instead would be accelerating upwards the light would go in a curved path hitting the opposite wall a small distance below where it would have if the elevator hadn't been accelerating. The path that the light takes is therefore curved and since the accelerated frame of reference is equivalent to a uniform gravitational field it means that the mass creating the gravitational field curves the space-time.

From this Einstein concluded that unless a force acts upon it matter follows the shortest path in space-time but the space-time could be curved and then the shortest path is a geodesic. The source of this curvature would be matter itself, so gravity wouldn't be a force but simply the result of the curvature of space-time.

Einstein formulated an equation for how the curvature relates to matter distribution called *Einstein field equation* and can be written as :

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $G_{\mu\nu}$, the *Einstein tensor*, is dependent on the Riemann metric and its first and second derivatives. G is *Newton's constant of gravitation* and $T_{\mu\nu}$ is the *mass-energy tensor* that describes the density and flux of energy and momentum in space-time.

5.1 Testing general relativity

In 1919 Eddington tested Einstein's general relativity theory by measuring if the sun would bend light from other stars. During an eclipse that year the sun would be in front of a prominent group of stars known as the Hyades. The experiment from the Island of Principe showed that the light's path from the stars was bent resulting in the stars seeming to be displaced when the sun was in front of them. From the displacements Eddington concluded that the deflection by the sun to be 1,61 arcseconds equal to $4,47 \cdot 10^{-4}$ ° confirming general relativity within a 10 error margin (Dyson et al., 1920). We define ϵ_1 as the angle between the light direction toward Earth from the star when the sun is between the star and Earth and not, we define ϵ_2 likewise but as the received light on Earth shown in figure 8.

If we draw asymptotes to the lights path from the star S and the Earth O the deflection of the light δ will be the angle between them. If we then draw a parallel line to SO where the asymptotes intersect we will split δ in two angles θ_1 and θ_2. θ_2 and ϵ_2 are corresponding angles and therefore equal and θ_1 and ϵ_1 are alternate angles and are also equal. Then the deflection of the light is $\delta = \epsilon_1 + \epsilon_2$.

If we then have two stars that with the Earth make up a triangle with the angles α, β and γ that adds up to 180°

The sum of the angles of this triangle is:

$$(\alpha+\epsilon_1)+(\beta+\epsilon_2)+(\alpha+\epsilon_2+\epsilon_4) = \alpha+\beta+\gamma+(\epsilon_1+\epsilon_2)+(\epsilon_3+\epsilon_4) = \alpha+\beta+\gamma+2\delta$$

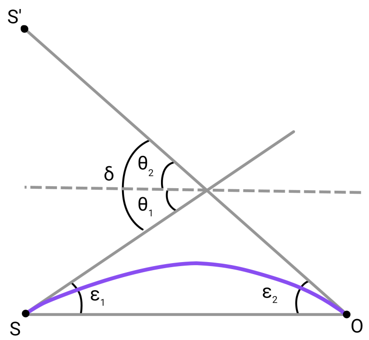

Figure 8: The deflection of light by a mass between the source, S and the observer, O. S' is the perceived source by the observer

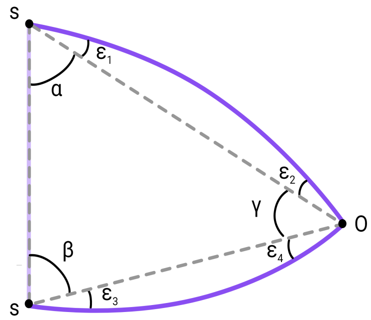

Figure 9: The triangle made by deflected light from two stars S_1 and S_2. α, β and γ are the angles without the deflection.

With δ being $4,47 \cdot 10^{-4}\,°$ we have that the angle sum is equal

$$180 + 2 \cdot 4,47 \cdot 10^{-4} = 180,00894°$$

Therefore we have a triangle greater than two right angles and space must be curved.

6 Conclusions

In this thesis we have shown that one of Euclid's axioms, the parallel postulate, doesn't hold on a sphere since the angle sum of triangles is greater than 180°. We concluded that the difference between the angle sum of the triangle and 180°, known as angular defect, is proportional to the area of the triangle, not only on the sphere but also if we start with the hypothesis that the angle sum is greater or lesser than 180°.

We have also seen this relation between the area of the triangle and it's angle sum in the curvature of a surface. Using the area of the triangle in the definition of curvature, we deduced that the curvature only depends on the angles of the triangle and are therefore intrinsic. This curvature tells us the geometry of space and is affected by the mass-energy density as shown by Einstein in his general relativity.

With the experiment made by Eddington that tested general relativity, we have produced a light triangle that shows that the parallel postulate isn't true but that space is partly curved giving the angle sum a greater value than 180°.

We have also shown how Einstein, using the invariance of the light axiom in special relativity came to the conclusion that space and time are relative to the inertial system of the observer.

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A Gauss curvature

Let M be a 2-dimensional submanifold of $\mathbb{R}^3$ with the inclusion map $i\colon M \mapsto \mathbb{R}^3$ At each point of $p\epsilon M$ there is a unique unit vector ν that is perpendicular to M_p Then we have a normal map $\nu\colon M \mapsto S^2 \subset \mathbb{R}^3$ where S^2 is the surface of the unit sphere.

If we express the points on M using two independent system of curves (s,t) on M with the coordinate system $X\colon M \mapsto \mathbb{R}^2$ the inverse function function f will be $f = X^{-1}\colon \mathbb{R}^2 \mapsto M \subset \mathbb{R}^3$.

If $p = f(s,t)$ and if M is the graph of $g : \mathbb{R}^2 \mapsto \mathbb{R}$ so that

$$M = (x, y, (g(x, y)) : x, y\epsilon\mathbb{R}^2,$$

then M is the image of$f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by:

$$f(s,t) = (s,t,g(s,t))$$

We choose the vectors u and w:

$$u = \frac{\delta f}{\delta s} = (1, 0, \frac{\delta g}{\delta s})$$

$$w = \frac{\delta f}{\delta t} = (0, 1, \frac{\delta g}{\delta t})$$

the cross product of u and w and its scalar is:

$$u \times w = (-\frac{\delta g}{\delta s}, -\frac{\delta g}{\delta t}, 0)$$

$$|u \times w| = \sqrt{\frac{\delta g}{\delta s}^2 + \frac{\delta g}{\delta t}^2 + 1}$$

To make it easier to follow we will make these notations:

$$a = \frac{\delta g}{\delta s}, b = \frac{\delta g}{\delta t}$$

$$\alpha = \frac{\delta a}{\delta s} = \frac{\delta^2 g}{\delta s^2}, \beta = \frac{\delta b}{\delta t} = \frac{\delta^2 g}{\delta t^2}$$

and

$$\gamma = \frac{\delta a}{\delta t} = \frac{\delta b}{\delta s} = \frac{\delta^2 g}{\delta s \delta t}$$

The tangent vector to the unit sphere will then be:

$$\frac{\delta \nu}{\delta t} = \frac{\delta}{\delta t}\frac{u \times w}{|u \times w|} = \frac{(\frac{\delta u}{\delta t} \times w + u \times \frac{\delta w}{\delta t})\dot{|}u \times w| - u \times w\frac{\delta}{\delta t}|u \times w|}{|u \times w|^2}$$

$$\frac{\delta u}{\delta t} \times w = (0, 0, \gamma) \times (0, 1, b) = (-\gamma, 0, 0)$$

$$u \times \frac{\delta w}{\delta t} = (1, 0, a) \times (0, 0, \beta) = (0, -\beta, 0)$$

$$\frac{\delta u}{\delta t} \times w + u \times \frac{\delta w}{\delta t} = (-\gamma, -\beta, 0)$$

$$\frac{\delta |u \times w|}{\delta t} = \frac{\delta}{\delta t}\sqrt{a^2 + b^2 + 1} = \frac{a\gamma + b\beta}{\sqrt{a^2 + b^2 + 1}}$$

then the tangent vector can be written as:

$$\frac{\delta \nu}{\delta t} = \frac{(-\gamma, -\beta, 0)\sqrt{a^2 + b^2 + 1} - (-a, -b, 1)\frac{b\beta + a\gamma}{\sqrt{a^2+b^2+1}}}{a^2 + b^2 + 1} =$$

$$= \frac{(ab\beta - b^2(\gamma + 1), ab\gamma - \beta(a^2 + 1), -b\beta - a\gamma)}{(a^2 + b^2 + 1)^{3/2}}$$

The tangent vector $\frac{\delta \nu}{\delta s}$ on the unit sphere can likewise be written as:

$$\frac{\delta \nu}{\delta s} = \frac{\delta}{\delta s}\frac{u \times w}{|u \times w|} = \frac{(\frac{\delta u}{\delta s} \times w + u \times \frac{\delta w}{\delta s})\dot{|}u \times w| - u \times w\frac{\delta}{\delta s}|u \times w|}{|u \times w|^2}$$

$$\frac{\delta u}{\delta s} \times w = (0, 0, \alpha) \times (0, 1, b) = (0, -\alpha, 0)$$

$$u \times \frac{\delta w}{\delta t} = (1, 0, a) \times (0, 0, \gamma) = (-\gamma, 0, 0)$$

$$\frac{\delta u}{\delta s} \times w + u \times \frac{\delta w}{\delta s} = (-\gamma, -\alpha, 0)$$

$$\frac{\delta |u \times w|}{\delta s} = \frac{\delta}{\delta s}\sqrt{a^2 + b^2 + 1} = \frac{a\alpha + b\gamma}{\sqrt{a^2 + b^2 + 1}}$$

$$\frac{\delta \nu}{\delta s} = \frac{(-\gamma, -\alpha, 0)\sqrt{a^2 + b^2 + 1} - (-a, -b, 1)\frac{a\alpha + b\gamma}{\sqrt{a^2+b^2+1}}}{a^2 + b^2 + 1} =$$

$$= \frac{(ab\gamma - \alpha(b^2 + 1), ab\alpha - \gamma(a^2 + 1), -a\alpha - b\gamma)}{(a^2 + b^2 + 1)^{3/2}}$$

The area of the parallelogram made by the tangent vectors on the unit sphere is:

$$\frac{\delta \nu}{\delta s} \times \frac{\delta \nu}{\delta t} = (\frac{(ab\alpha - \gamma(a^2 + 1))(-b\beta - a\gamma) - (ab\gamma - \beta(a^2 + 1))(-a\alpha - b\gamma))}{(a^2 + b^2 + 1)^3},$$

$$\frac{(-a\alpha - b\gamma)(ab\beta - b^2(\gamma + 1)) - (ab\gamma - \alpha(b^2 + 1))(-b\beta - a\gamma)}{(a^2 + b^2 + 1)^3},$$

$$\frac{(ab\gamma - \alpha(b^2 + 1))(ab\gamma - \beta(a^2 + 1)) - (ab\alpha - \gamma(a^2 + 1))(ab\beta - b^2(\gamma + 1))}{(a^2 + b^2 + 1)^3}) =$$

$$= \frac{(a\gamma^2 - a\alpha\beta)(a^2+b^2+1), (b\gamma^2 - b\alpha\beta)(a^2+b^2+1), (\alpha\beta - \gamma^2)(a^2+b^2+1)}{(a^2+b^2+1)^3} =$$

$$= \frac{(a\gamma^2 - a\alpha\beta, b\gamma^2 - b\alpha\beta, \alpha\beta - \gamma^2)}{(a^2+b^2+1)^2}$$

$$|\frac{\delta\nu}{\delta s} \times \frac{\delta\nu}{\delta t}| = \frac{\sqrt{(a\gamma^2 - a\alpha\beta)^2 + (b\gamma^2 - b\alpha\beta)^2 + (\alpha\beta - \gamma^2)^2}}{(a^2+b^2+1)^2} =$$

$$= \frac{\sqrt{\gamma^4(a^2+b^2+1) - 2\alpha\beta\gamma^2(a^2+b^2+1) + \alpha^2\beta^2(a^2+b^2+1)}}{(a^2+b^2+1)^2} =$$

$$= \frac{\sqrt{(\alpha\beta - \gamma^2)^2}}{(a^2+b^2+1)^{3/2}} = \frac{\alpha\beta - \gamma^2}{(a^2+b^2+1)^{3/2}}$$

replacing our notations we then conclude:

$$\therefore |\frac{\delta\nu}{\delta s} \times \frac{\delta\nu}{\delta t}| = \frac{\frac{\delta^2 g}{\delta s^2}\frac{\delta^2 g}{\delta t^2} - (\frac{\delta^2 g}{\delta s \delta t})^2}{((\frac{\delta g}{\delta s})^2 + (\frac{\delta g}{\delta t})^2 + 1)^{3/2}}$$