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An Introduction to Abstract Harmonic Analysis

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Abstract

This thesis will provide an introduction to Fourier analysis on locally compact abelian Hausdorff groups. The abstract Fourier transform unifies the classical Fourier transforms and will enable us not only to see the classical transforms as realizations in different groups but also lets us prove theorems regarding all of them in one stroke. The theory also gives us the opportunity to apply Fourier analysis in more exotic settings. Examples of Fourier Transforms are among many others the Fourier transform on the real line, Fourier series and the discrete Fourier transform. In the end two applications in number theory will be given.

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Contents

1 Introduction

The area of Fourier analysis studies how signals can be decomposed with its frequency components and is used throughout science, mathematics and engineering. The Fourier transform of an integrable function f is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx.$$

Here each $\xi \in \mathbb{R}$ can be seen as a frequency corresponding to the frequency component $e^{i\xi x}$ and $\mathbb{R}$ is called the frequency space. The Fourier transform evaluated at ξ is a complex number which shows how f depends on the frequency component $e^{i\xi x}$. The magnitude of the complex number $\hat{f}(\xi)$ is the amplitude of the frequency component $e^{i\xi x}$ and the angle of $\hat{f}(\xi)$ is the phase of $e^{i\xi x}$. You can also (with some assumptions) recover f by integrating each frequency component with its amplitude and adjust for its phase over the frequency space:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi.$$

This is the content of Fourier's inversion theorem. When studying periodic function over $[0, 2\pi]$ the frequency components are instead the functions e^{inx} for $n \in \mathbb{Z}$ hence $\mathbb{Z}$ is the frequency space. For a given periodic function f the amplitude and phase of the component e^{inx} are now given by its Fourier coefficient

$$\hat{f}(n) = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{-inx}dx$$

and f can (with some assumptions) be recovered with the formula

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}.$$

The so called Fourier series. Similarly, if f is defined on $\mathbb{Z}/N\mathbb{Z}$ then the frequency components are given by the functions $\gamma(n) = e^{i2\pi kn/N}$ for $[k] \in \mathbb{Z}/N\mathbb{Z}$ and the amplitude and phase is calculated with the discrete Fourier transform

$$\hat{f}(k) = \sum_{n=0}^{N-1} f(n)e^{-i2\pi kn/N}.$$

The function can recovered by

$$f(n) = \frac{1}{2\pi}\sum_{k=0}^{N-1} \hat{f}(k)e^{i2\pi kn/N}.$$

These entirely different situations are remarkably similar. In each of the three situations the frequency components are complex exponentials and each Fourier transform is calculated by integrating or summing the function with the inverse of the frequency component over its domain of definition. The function is then recovered by integrating or summing the Fourier transform with each frequency component over the frequency space.

It turns out that the unifying factor in these different situations is that the domain of definition for the given functions is a locally compact abelian Hausdorff group. In our case the real numbers, the unit circle and $\mathbb{Z}/n\mathbb{Z}$ respectively. The frequency components in each case correspond to continuous homomorphisms into the unit circle, the so called dual group. Generalizations of the idea of decomposing functions with their frequency components is called abstract harmonic analysis.

In this thesis an introduction to Fourier analysis on locally compact abelian Hausdorff groups will be given. The first 6 chapters will give an introduction to the general theory and the presentation is inspired by chapter 1 in Rudin's Fourier Analysis on Groups [3]. After this is done we show that the abstract Fourier transform is truly a generalization by showing that it coincides with the usual transforms when applying it to various classical groups. In the last two chapters two applications in number theory will be given, one about the Collatz conjecture and one about the partition function.

The two main components in the general theory are the Haar-measure and the dual group. The Haar-measure will enable us to apply integration theory on our groups and the dual group can be seen as the frequency space corresponding to our group. Once they have been established the abstract Fourier transform will be defined and theorems such as the inversion theorem and Plancherel's theorem will be transferred to this abstract setting. We will also prove the remarkable Pontryagin's duality theorem which roughly states that the frequency space of the frequency space, that is the dual group of the dual group is isomorphic the group itself.

The groups in question that this thesis studies are the locally compact abelian Hausdorff groups. In order to not get into too much technicalities we will also assume that the Haar-measure of our group is σ-finite. This is not a big restriction on scope of the theory.

As preliminaries a firm grasp of the first 9 chapters in [1] is recommended where chapter 1, 2, 3 and 6 are especially important. Most of the topology needed can also be found in [1] but for a more comprehensive treatment relevant parts of chapter 2, 3 and 4 in [6] can be consulted. The theory of Banach algebras will be relevant in some proofs but has been tried to be kept at a minimum in order to not expand the scope of the thesis too much. A nice but not completely sufficient introduction is chapter 18 in [1] and a more complete treatment is given in chapter 10 and 11 in [2].

2 Topological Groups and Haar Measures

In this section we will define and prove basic properties regarding topological groups and define the Haar-measure. If one needs a reminder of the various topological definitions that I take for granted section 1.2 and 2.3 in [1] can be consulted for a minimal treatment or relevant parts of Chapter 2, 3 and 4 in [6] can be consulted for a more complete treatment on the topic.

2.1 Topological Groups and their basic properties

We begin by recalling the definition of a group

Definition 1. *Let G be a set and $\cdot : G \times G \to G$ be a binary operator, called multiplication. We say that $(G, \cdot)$ is a group if the following properties hold*
i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for any $x, y, z \in G$
ii) There exists an element $1 \in G$ such that $x \cdot 1 = 1 \cdot x = 1$ for any $x \in G$
iii) For each $x \in G$ there exists an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$

Remark. *Axiom i) is called associativity, the element 1 is called the identity, the element x^{-1} is called the inverse of x for each $x \in G$ and the function $i(x) = x^{-1}$ is called inversion. It is common practice to write xy instead of $x \cdot y$ and call G a group when the operation is known from the context. Furthermore, we say that G is abelian if $xy = yx$ for any $x, y \in G$. When G is an abelian group, additive notation is often used where the binary operator is denoted by $+$, inversion is denoted by $i(x) = -x$ and the identity element is called 0.*

Definition 2. *We say that a group, G equipped with a topology τ is a topological group if multiplication and inversion are continuous functions with respect to this topology. In other words: G is a topological group if the functions $i : G \to G$ and $m : G \times G \to G$ given by $i(x) = x^{-1}$ and $m(x, y) = xy$ are continuous. Here it is assumed that $G \times G$ has been given the product topology (Sets of the form $U \times V$ where U and V are open in G is a basis for τ).*

Proposition 1. *For any fixed $x \in G$ in the abelian group G the map $T_x : G \to G$ given by $T_x(y) = x + y$ and the map $i(y) = -y$ are both homeomorphisms (a bijective continuous map with continuous inverse).*

Proof. Choose $x \in G$. It is clear that T_x is surjective and injective. Since the inverse of T_x is given by T_{-x} we only need to prove continuity of T_x. The map T_x is equal to the composition $m \circ \iota_x$ where $\iota_x : G \to G \times G$ is given by $\iota_x(y) = (x, y)$. The map m is continuous since G is topological group and ι_x is continuous by the definition of the product topology hence T_x is continuous. For the inversion map i it is easy to check that it is a bijection with inverse i since $i(i(y)) = y$. Since i is continuous it follows that the inverse is continuous so i is homeomorphism. $\square$

Remark. *Since T_x is a homeomorphism the set $x + E := T_x(E)$ is compact, open or closed whenever E is compact, open or closed respectively for any $x \in G$.*

Another useful property of topological groups is the following

Proposition 2. *For any open set V of 0 in the abelian group G there exists an open set W containing 0 such that $W = -W$ (where $-W$ is the set of inverses of elements of W) and*

$$W + W \subseteq V.$$

Proof. Let V be an open set containing 0. By continuity of multiplication the set $m^{-1}(V)$ is open in $G \times G$ and contains $(0, 0)$. By definition of the product topology there exists a basis element $B = B_1 \times B_2$ containing $(0, 0)$ where B_1 and B_2 are open in G. If we let $W = (B_1 \cap -B_1) \cap (B_2 \cap -B_2)$ then it follows that W is open since inversion is a homeomorphism (and therefore takes open sets to open sets) and W contains 0. If x is an element of W then x is an element of $B_1 \cap -B_1$ hence $-x \in W$ so $W = -W$. Lastly we recall that $B_1 \times B_2 \subseteq m^{-1}(V)$ hence

$$W + W \subseteq B_1 + B_2 \subseteq V.$$

□

Remark. *If G is any topological group and H is a subgroup of G then H equipped with the subspace topology is also a topological group since the restrictions $m|_{H \times H} : H \times H \to H$ and $i|_H : H \to H$ are continuous functions in the subspace topology.*

Proposition 3. *If G is an abelian group and H is a subgroup of G that is open in G then H is closed in G.*

Proof. Define $x + H = T_x(H)$ for any $x \in G$. Since H is open and T_x is a homeomorphism $x + H$ is open for any $x \in G$. If $T_x(H) \cap H \neq \varnothing$ then for some $h \in H$ there exists $h_0 \in H$ such that $h = h_0 + x$ hence $x = h - h_0 \in H$. If we define

$$V := \bigcup_{x \in G - H} (x + H).$$

Then it follows that V is open, $V \cup H = G$ and $V \cap H = \varnothing$. It follows that $H = G - V$ is closed. □

Proposition 4. *If H is a topological subgroup of a topological group G then $\overline{H}$ is also a topological group.*

Proof. Since G is a topological group it is sufficient to prove that the group operations restricted to $\overline{H}$ map into $\overline{H}$ which happens if and only if the map $f : \overline{H} \times \overline{H} \to G$ given by $f(x, y) = xy^{-1}$ has its image contained in $\overline{H}$. Since $f = m \circ g$ where $g : \overline{H} \times \overline{H} \to G \times G$ is given by $g(x, y) = (x, i(y))$ it follows that f is a continuous map. Note that $H \times H \subseteq f^{-1}(\overline{H})$ since H is a group. By continuity of f we know that $f^{-1}(\overline{H})$ is closed and it follows that

$$\overline{H \times H} \subseteq f^{-1}(\overline{H}).$$

After taking images we have

$$f(\overline{H \times H}) \subseteq \overline{H}.$$

If (x, y) is an element in $\overline{H} \times \overline{H}$ then for any open $V \subseteq G$ such that $H \cap V = \varnothing$ it follows that neither x nor y is an element of V. Now if $C \subseteq G \times G$ is a closed set containing $H \times H$ then $C = G \times G - \cup_i V_i \times U_i$ for some collection of open sets $(V_i), (U_i)$ each having empty intersection with H. It follows that (x, y) is not an element of $V_i \times U_i$ for any i and it follows that $(x, y) \in C$. Since C was an arbitary closed set containing $H \times H$ it follows that $(x, y) \in \overline{H \times H}$ hence

$$f(\overline{H} \times \overline{H}) \subseteq f(\overline{H \times H}) \subseteq \overline{H}.$$

□

Definition 3. *Let X be a Hausdorff space. We say that X is locally compact if any $x \in X$ has an open neighbourhood V containing x such that the closure of V in X, $\overline{V}$ is compact.*

Proposition 5. *If X is a subspace of a Hausdorff space Y such that X is dense in Y and X is locally compact then X is open in Y.*

Proof. Pick $x \in X$ then by local compactness of X there exists an open set V in X containing x such that the closure of V in X, $K := \mathrm{cl}_X(V)$ is compact in X. Since the inclusion map $\iota : X \to Y$ is continuous it follows that K is compact in Y. Since compact subsets of Hausdorff spaces are closed we know that K is closed in Y. Since the set V is open in X there exists an open U in Y such that $V = X \cap U$. Since X is dense in Y and $U \subseteq Y$ is open it follows that $\mathrm{cl}_Y(U) = \mathrm{cl}_Y(X \cap U)$. This is true since if z is an element of $\mathrm{cl}_Y(U)$ and W is any open neighbourhood of z then $U \cap W$ is open and by density of X it follows that $(X \cap U) \cap W \neq \varnothing$ which is equivalent to saying that $z \in \mathrm{cl}_Y(X \cap U)$. It follows that

$$x \in U \subseteq \mathrm{cl}_Y(U) = \mathrm{cl}_Y(X \cap U) = \mathrm{cl}_Y(V) \subseteq K \subseteq X.$$

Hence X is open in Y. □

Using the commutative group operation for abelian groups we can also introduce the concept of uniform continuity.

Definition 4. *If f is a function from an abelian group G into a metric space M with metric d and E is a subset of G then f is uniformly continuous on E if for every $\epsilon > 0$ there exists an open set V of G containing 0 such that $x - y \in V$ implies $d(x, y) < \epsilon$ whenever $x, y \in E$.*

Proposition 6. *If f is a continuous function defined on the abelian group G into a metric space M and K is a compact subset of the G then f is uniformly continuous on K.*

Proof. Choose $\epsilon > 0$. For each y in K, there exists by continuity of f an open set V_y of 0 such that $V_y + y$ contains y and $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $x \in V_y + y$. By Proposition 2 we can find, for each y in K an open set A_y containing 0 such that $A_y + A_y \subseteq V_y$. The set

$$\bigcup_{y \in K} (A_y + y)$$

is an open cover of K and by compactness of K it has a finite subcover

$$K \subseteq \bigcup_{n=1}^{N} A_{y_n + y_n}.$$

Note also that if we define

$$V := \bigcap_{n=1}^{N} A_{y_n}$$

then V is open since each A_n is open and is non-empty since it contains 0. If $x - y$ is an elements of V then there exists n such that $y = y_n + A_{y_n} \subseteq y_n + V_{y_n}$ and

$$x \in y + A_{y_n} \subseteq y_n + A_{y_n} + A_{y_n} \subseteq y_n + V_{y_n}.$$

Hence

$$|f(x) - f(y)| \leq |f(x) - f(y_n)| + |f(y_n) - f(y)| < \epsilon.$$

$\square$

2.2 The Haar-measure

Let G be an abelian group. In order to define a Fourier transform on G we first need to be able integrate complex functions defined on G and for that we need a measure on G. Since we are working in the context of topological groups it is reasonable to demand that our measure interacts nicely with the topological group structure of G. More precisely we want to define a non-trivial positive measure, $m : \mathfrak{M} \to [0, +\infty]$ where $\mathfrak{M}$ is the σ-algebra of Borel subsets of G. Recall that the σ-algebra of Borel sets is the minimal σ-algebra containing all the open sets. Furthermore, another property we would like is that m is translation invariant, that is, if $x \in G$ and $E \in \mathfrak{M}$ then $x + E \in \mathfrak{M}$ and $m(E) = m(x + E)$. Other useful properties to have would be that $m(K) < +\infty$ for any compact K and that $m(E) = \sup_{K \subseteq E} m(K)$ holds true (atleast when E is open or of finite measure) where the supremum is over compact K. Similarly we would like that $m(E) = \inf_{V \supseteq E} m(V)$ for any $E \in \mathfrak{M}$ where the infimum is over open supersets of E.

It turns out that if we put some restrictions on G we can always find such a measure. The restrictions on G we need for this to hold true are the following: we need G to be a locally compact, abelian topological Hausdorff Group. We call this measure a Haar-measure and define it formally below.

Definition 5. *Let G be a locally compact abelian Hausdorff group. We say that a non-trivial positive measure m on G is a Haar-measure if its corresponding σ-algebra, $\mathfrak{M}$ consists of the Borel sets of G and that m satisfies the following:*
i) $x + E \in \mathfrak{M}$ and $m(x + E) = m(E)$ whenever $x \in G$ and $E \in \mathfrak{M}$.
ii) $m(K) < +\infty$ for any compact K.
iii) $m(E) = \inf_{E \subseteq V} m(V)$ for any $E \in \mathfrak{M}$ where the infimum is over open supersets of E.
iv) $m(E) = \sup_{K \subseteq E} m(K)$ for any $E \in \mathfrak{M}$ which is either open or of finite measure. The supremum is over compact subsets of E.

Remark. *Property i) is called translation invariance. Property iii) and iv) are called outer- and inner regularity respectively.*

Theorem 1. *If G is a locally compact abelian Hausdorff group then G admits a Haar-measure.*

Proof. We do not sketch the entire proof since it is quite lengthy and technical. Various references for this construction are given in Section 1.1.1 in [3]. The idea of the construction of m is to construct a positive linear functional, Λ on $C_c(G)$, (the set of complex valued continuous functions defined on G with compact support) which also satisfies that $\Lambda f_y = \Lambda f$ for any $f \in C_c(G)$ and $y \in G$ where f_y is defined by $f_y(x) = f(x - y)$. Since G is locally compact and Hausdorff the Riesz representation Theorem for positive linear functionals on $C_c(G)$ can be applied (see, Theorem 2.14 in [1]) from which we get a measure m defined on the Borel sets such that $\Lambda f = \int_G f(x)dm(x)$. The measure m also satisfies property ii), iii) and iv) of Definition 5.

Lastly we prove property i), namely that $x + E \in \mathfrak{M}$ and $m(x + E) = m(E)$ whenever $x \in G$ and $E \in \mathfrak{M}$. Let $\mathfrak{M}$ denote the Borel sets and fix an $x \in G$. Then since the function $T_{-x}(y) = -x + y$ is a homeomorphism it follows that $T_{-x}(\mathfrak{M})$ contains any open set. Since T_{-x} is a bijection it commutes with unions and intersections and it follows that T_{-x} is a σ-algebra containing the open sets. By definition of the Borel sets it follows that $\mathfrak{M} \subseteq T_{-x}(\mathfrak{M})$. Taking T_x-images on both sides yields $T_x(\mathfrak{M}) \subseteq T_x(T_{-x}(\mathfrak{M})) = \mathfrak{M}$. This clearly shows that $x + E \in \mathfrak{M}$ whenever $x \in G$ and $E \in \mathfrak{M}$.

We now show that $m(x + E) = m(E)$ for any $x \in G$ and $E \in \mathfrak{M}$. Let $x \in G$ and assume first that $E = K \in \mathfrak{M}$ is compact. Since T_x is homeomorphism it follows that $x + K$ is compact hence $x + K \in \mathfrak{M}$ since compact subsets of Hausdorff spaces are closed. Proving $m(x + K) = m(K)$ is straightforward if we can find a sequence $f_n \in C_c(G)$ such that each f_n is dominated by an $L^1(G)$ function and that $\lim_{n\to+\infty} f_n = \chi_K$ a.e where χ_K denotes the characteristic function on K. Since if we construct such a sequence (f_n) we also get that $\lim_{n\to+\infty}(f_n)_x = \chi_{x+K}$ a.e and using that $\Lambda f_n = \Lambda(f_n)_x$ we get:

$$
\begin{aligned}
m(K) &= \int_G \chi_K(y)dm(y) = \lim_{n\to+\infty}\int_G f_n(y)dm(y) = \lim_{n\to+\infty}\Lambda f_n \\
&= \lim_{n\to+\infty}\Lambda(f_n)_x = \lim_{n\to+\infty}\int_G (f_n)_x(y)dm(y) \\
&= \int_G \chi_{x+K}(y)dm(y) = m(x+K).
\end{aligned}
$$

Note that the dominated convergence theorem was applied twice. Since property i) is then true for any compact set it follows from inner regularity that property i) is also true for any open set V since $x+V$ is open and hence is member of $\mathfrak{M}$. By outer regularity it then follows that property i) holds for any Borel set.

The only thing that remains is to construct the sequence (f_n) in $C_c(G)$ which converges to χ_K a.e and each f_n is dominated by an $L^1(G)$ function. Let $K \subseteq G$ be a compact set. Since $m(K) = \inf_{K\subseteq V} m(V)$ there exists for each positive integer n an open set V_n such that $m(K) \leq m(V_n) \leq m(K) + \frac{1}{n}$. After considering $U_n = \bigcap_{i=1}^n V_i$ we can assume that $V_{n+1} \subseteq V_n$ for all n. By Urysohn's Lemma (see Theorem 2.12 in [1]) we know that for each V_n there exists a function $f_n \in C_C(G)$ such that $0 \leq f_n \leq 1$, $f_n = 1$ on K and f_n is supported in V_n. Consider the function $g(x) = \lim_{n\to+\infty} f_n(x)$ for any $x \in G$ where it is well-defined. Clearly $g(x) = 1$ when $x \in K$ and $g(x) = 0$ when $x \in G - \bigcap_{n=1}^\infty V_n$ hence $g(x) = \chi_K(x)$ except possibly on the set $(\bigcap_{n=1}^\infty V_n) - K$. But since $m(K) < +\infty$ we have

$$
\begin{aligned}
m(\bigcap_{i=1}^\infty V_i - K) &= m(\bigcap_{i=1}^\infty V_i) - m(K) \\
&\leq m(V_n) - m(K) \\
&< m(K) + \frac{1}{n} - m(K) = \frac{1}{n}
\end{aligned}
$$

for any n. Since n is an arbitrary positive integer it follows that $g = \chi_K$ a.e. Furthermore $f_n \leq \chi_{V_1}$ for each n and $\chi_{V_1} \in L^1(G)$. □

Remark. *Since the Haar-measure is fundamental for the Fourier transform we will after this always let G denote a locally compact abelian Hausdorff group and m a Haar-measure of G with the Borel σ-algebra $\mathfrak{M}$. Furthermore, in order to be able to apply Fubini's theorem without any technicalities we also always assume that G is σ-finite with respect to m which means that G is a countable union of sets $(X_i)_i$ where $m(X_i) < +\infty$ for all i.*

Another benefit of assuming that G is σ-finite is that m is inner regular on any Borel set.

Theorem 2. *$m(E) = \sup_{K \subseteq E}$ for any $E \in \mathfrak{M}$ where the supremum is over compact subsets of E*

Proof. Since X is σ-finite there exists a sequence of measurable sets $(X_i)_{1 \leq i}$ such that $X = \bigcup_{1 \leq i} X_i$ and $m(X_i) < +\infty$ for each i. We can also assume that $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Pick $E \in \mathfrak{M}$ and $\epsilon > 0$ then, since $m(E \cap X_i) < +\infty$ there exists for each i a compact set $K_i \subseteq E \cap X_i$ such that

$$m(E \cap X_i) \geq m(K_i) > m(E \cap X_i) - \frac{\epsilon}{2^i}.$$

Define $C_n = \bigcup_{i=1}^{n} K_i$, it follows that C_n is compact and

$$\begin{aligned} m(E) = \sum_{i=1}^{+\infty} m(E \cap X_i) &\leq \sum_{i=1}^{+\infty} (m(K_i) + \frac{\epsilon}{2^i}) \\ &\leq \epsilon + \sum_{i=1}^{+\infty} m(K_i) = \epsilon + \lim_{n \to +\infty} m(C_n). \end{aligned}$$

It follows that

$$m(E) - \epsilon \leq \lim_{n \to +\infty} m(C_n) \leq \sup_{K \subseteq E} m(K) \subseteq m(E).$$

Since ϵ was arbitrary it follows that

$$m(E) = \sup_{K \subseteq E} m(K).$$

□

Proposition 7. *$m(V) > 0$ whenever $V \subseteq G$ is open and non-empty.*

Proof. Assume that V is open, non-empty and $m(V) = 0$. Let $K \subseteq G$ be a compact set and pick a $v \in V$. Since $k = k - v + v$ for any $k \in K$ it follows that $k \in (k-v)+V$ hence $\cup_{k \in K}(k-v)+V$ is an open cover of K. By outer regularity and translation invariance it follows that $m(K) = 0$. Since K was an arbitrary compact set it follows that any compact set has measure 0. By inner regularity it follows that any open set has measure 0 and lastly by outer regularity it follows that $m(E) = 0$ for any $E \in \mathfrak{M}$ so m is the trivial measure. □

Remark. *The proof above shows that we don't need to assume that the Haar measure is inner regular on all sets for Proposition 7 to hold, that is we don't need to assume that G is σ-finite with respect to the Haar-measure. We will assume as said in the above remark that G is σ-finite with respect to m.*

Remark. *The uniqueness Theorem below of the Haar measure comes from Theorem 1.1.3 in [3].*

Theorem 3. *If m and μ are two Haar-measure of G then there exists a constant $c > 0$ such that $m(E) = c\mu(E)$ for any Borel set $E \subseteq G$.*

Proof. By outer regularity it will follow that there exists a $c > 0$ such that $m(E) = c\mu(E)$ for any Borel set E if $m(V) = c\mu(V)$ for any open set V. Pick a positive $g \in C_c(G)$ such that $\int_G g d\mu = 1$ and pick any $f \in C_c(G)$. Using Fubini's theorem, (see Theorem 8.8 in [1]) twice and translation invariance we get the following:

$$\begin{aligned}\int_G f dm &= \int_G (\int_G g d\mu) f dm = \int_G \int_G g(x) d\mu(x) f(y) dm(y) \\ &= \int_G \int_G g(x-y) d\mu(x) f(y) dm(y) = \int_G \int_G g(x-y) f(y) d\mu(x) dm(y) \\ &= \int_G \int_G g(x-y) f(y) dm(y) d\mu(x) = \int_G \int_G g(-y) f(y+x) dm(y) d\mu(x) \\ &= \int_G \int_G g(-y) f(y+x) d\mu(x) dm(y) = \int_G g(-y) \int_G f(y+x) d\mu(x) dm(y) \\ &= \int_G g(-y) \int_G f(x) d\mu(x) dm(y) = \int_G g(-y) dm(y) \int_G f(x) d\mu(x).\end{aligned}$$

Letting $c = \int_G g(-y) dm(y)$ we see that $c > 0$ and $\int_G f dm = c \int_G f d\mu$ for any $f \in C_c(G)$. So if Λ and Λ' are the functionals on $C_c(G)$ corresponding to m and μ respectively it follows that $\Lambda = c\Lambda'$. Since the Haar-measure of an open set is defined completely in terms of the corresponding linear functional (see the proof of Theorem 2.14 in [1]) it follows that $m(V) = c\mu(V)$.

Note that the function $F(x, y) = g(x - y)f(y)$ is continuous since if we define the continuous functions $\phi(x, y) = x - y$ and $\pi(x, y) = y$ on $G \times G$ then $F(x, y) = g(\phi(x, y))f(\pi(x, y))$. Since $g \circ \phi$ and $f \circ \pi$ are continuous and since the product of two continuous functions is continuous it follows that F is continuous and hence measurable. It follows that $F \in C_c(G \times G)$ and therefore $\int_G \int_G |g(x - y)f(y)| d\mu(x) dm(y) < +\infty$ hence the first application of Fubini's theorem was legitimate. A similar argument shows that it was also legitimate for the function given by $G(x, y) = g(-y)f(y + x)$.

$\square$

Remark. *Since the Haar measure is unique up to multiplicative constant we shall often write $L^1(G)$ instead of $L^1(m)$ and $\int_G f(x)dx$ instead of $\int_G f(x)dm(x)$.*

Proposition 8. *Let $-E$ denote the set of inverses of elements of E where $E \in \mathfrak{M}$ then $m(E) = m(-E)$.*

Proof. Define $\mu(E) = m(-E)$ for any $E \in \mathfrak{M}$ then

$$\mu(x + E) = m(-(x + E)) = m(-x + (-E)) = m(-E) = \mu(E)$$

and it follows by checking the rest of the Haar-measure properties that μ is another Haar-measure on G. By uniqueness it follows that $m = c\mu$ for some

$c > 0$. If K is compact and chosen such that $m(K) > 0$ then $-K$ is compact and

$$\begin{aligned} m(K \cup -K) &= c\mu(K \cup -K) \\ &= cm(-(K \cup -K)) \\ &= cm(K \cup -K) \end{aligned}$$

hence $c = 1$ since $0 < m(K \cup -K) < +\infty$. □

Corollary 1. *If $f \in L^1(G)$ then*

$$\int_G f(x)dx = \int_G f(-x)dx.$$

Proof. Let $s(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}(x)$ be a simple function then

$$\begin{aligned} \int_G s(-x)dx &= \int_G \sum_{i=1}^n \alpha_i \chi_{E_i}(-x)dx = \int_G \sum_{i=1}^n \alpha_i \chi_{-E_i}(x)dx \\ &= \sum_{i=1}^n \alpha_i m(-E_i) = \sum_{i=1}^n \alpha_i m(E_i) = \int s(x)dx. \end{aligned}$$

By definition of the Lebesgue integral we have for positive f

$$\int_G f(x)dx = \sup_{s \le f} \int_G s(x)dx = \sup_{s \le f} \int_G s(-x)dx = \int_G f(-x)dx$$

where s denotes a simple function. If f maps into $\mathbb{C}$ the proposition follows after writing f as a sum of its real positive, real negative, imaginary positive and imaginary negative part. □

The following proposition shows that equivalent topological groups will have the same Haar-measures.

Proposition 9. *Let G be an abelian topological group with Haar-measure m and let H be a topological group. If there exists a function $\Phi : G \to H$ which is an isomorphism of groups and a homeomorphism then $m_H(E) = m(\Phi^{-1}(E))$ is a Haar measure on H. Furthermore, by uniqueness any Haar measure μ on H will be of the form $\mu = cm_H$ for some real positive constant c.*

Proof. Define $m_H(E) = m(\Phi^{-1}(E))$. The set function m_H is clearly not trivial since m is not trivial. It is easy to check that m_H is a measure and that its domain of definition is the Borel sets since homeomorphisms maps Borel sets to Borel sets. Since Φ is an isomorphism the function Φ^{-1} is a homomorphism and it follows that

$$\begin{aligned} m_H(x + E) &= m(\Phi^{-1}(x + E)) = m(\Phi^{-1}(x) + \Phi^{-1}(E)) \\ &= m(\Phi^{-1}(E)) = m_H(E). \end{aligned}$$

For any Borel set E in H and x in H so m_H is translation-invariant. The measure m_H is finite on compact subsets of H since m is finite on compact subsets of G and Φ^{-1} takes compact sets to compact sets. Since Φ is a homeomorphism between G and H the open subsets of G are in bijection with the open subsets of H, hence for any Borel set E

$$\begin{aligned}\inf_{V \subseteq E} m_H(V) &= \inf_{\Phi(W) \subseteq E} m_H(\Phi(W)) = \inf_{\Phi(W) \subseteq E} m(\Phi^{-1}(\Phi(W))) \\ &= \inf_{\Phi(W) \subseteq E} m(W) = \inf_{W \subseteq \Phi^{-1}(E)} m(W) \\ &= m(\Phi^{-1}(E)) = m_H(E).\end{aligned}$$

Using outer regularity of m. Thus we have proved outer regularity of m_H. A similar argument shows the inner regularity of m_H. $\square$

3 The Dual Group and the Fourier Transform

In this chapter we will define the Fourier transform but in order to do that we first need to define the dual group of G. We will then proceed to show the important property that the dual group of G is itself a locally compact abelian Hausdorff group. Many of the theorems in this section can be found in section 1.2 in [3]. The theory of Banach Algebras will be used in several theorems in this chapter as well as in subsequent chapters and for this I refer to Chapter 18 in [1] for an introduction or Chapter 10 and 11 in [2] for a more complete treatment.

3.1 Basic Definitions

Definition 6. *Let $\mathbb{T} \subseteq \mathbb{C}$ denote the unit circle. A group homomorphism $\gamma : G \to \mathbb{T}$ is called a character of G. The set Γ of all continuous characters of G is called the dual group of G.*

Proposition 10. *Γ is an abelian group under pointwise multiplication of functions*

Proof. The group operation is clearly associative since multiplication in $\mathbb{C}$ is associative. If $\gamma, \delta \in \Gamma$ then for any $x, y \in G$ we have

$$(\gamma\delta)(x+y) = \gamma(x+y)\delta(x+y) = \gamma(x)\gamma(y)\delta(x)\delta(y) = (\gamma\delta)(x)(\gamma\delta)(y)$$

so $\gamma\delta$ is homomorphism and $\gamma\delta \in \Gamma$ since $\gamma\delta$ is continuous. Clearly our group operation is commutative since $\mathbb{C}$ is commutative under multiplication. The function $1 : G \to \mathbb{T}$ defined by $1(x) = 1$ for all $x \in G$ is the identity element of Γ. Lastly if $\gamma \in \Gamma$ then we can define $\gamma^{-1}(x) = \overline{\gamma(x)}$ where the overline is the complex conjugate. Since $|\overline{\gamma}| = |\gamma| = 1$ we know that γ^{-1} maps G into $\mathbb{T}$ and $\gamma\gamma^{-1} = 1$. For any $x, y \in G$ we have

$$\overline{\gamma(x+y)} = \overline{\gamma(x)\gamma(y)} = \overline{\gamma(x)\gamma(y)}$$

and it follows that γ^{-1} is a homomorphism. Since $|z - w| = |\overline{z} - \overline{w}|$ for all $z, w \in \mathbb{C}$ it follows that complex conjugation is continuous hence $\gamma^{-1} \in \Gamma$ so Γ is an abelian group. $\square$

Remark. *We will often denote the inverse of $\gamma \in \Gamma$ by $\overline{\gamma}$ since $\gamma^{-1}(x) = \overline{\gamma(x)}$ for any $x \in G$.*

We are now able to define the Fourier transform.

Definition 7. *Pick an $f \in L^1(G)$ and define the function*

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)dx$$

for $\gamma \in \Gamma$. We call this function the Fourier transform of f and we also define $A(\Gamma) = \{\hat{f}; f \in L^1(G)\}$, the set of all fourier transforms defined on Γ.

3.2 The Weak Topology of the Dual Group

We want to give Γ a topology which turns it into a locally compact Hausdorff group. A natural candidate is a topology which makes any $\hat{f} \in A(\Gamma)$ a continuous function.

Definition 8. *Let X be any set, Y be a topological space and let M be any set of functions defined on X which maps into Y. Define $\mathscr{B}$ to be the set of all elements of the form*

$$\bigcap_{n=1}^{N} f_n^{-1}(V_n)$$

where $f_n \in M$ and $V_n \subseteq Y$ is open for each n and $N \in \mathbb{N}$. This $\mathscr{B}$ will be shown to be the basis of a topology on X and this topology is called the weak topology on X induced by M.

We recall that a basis for a topology, $\mathscr{B}$ is a set of open sets such that any open set can be written as a union of elements of $\mathscr{B}$

Proposition 11. *$\mathscr{B}$ is a basis of a topology, τ. Furthermore this topology is the weakest topology which makes any $f \in M$ a continuous function. In other words: if $\tau' \subseteq \tau$ is another topology on X and any $f \in M$ is continuous with respect to τ' then $\tau = \tau'$.*

Proof. Pick an $x \in X$ and let $B_1, B_2 \in \mathscr{B}$ be two basis elements which contains x. In order to show that $\mathscr{B}$ is a basis we need to find a $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$. Since $x \in B_1$ there exists functions $f_n \in M$ and open sets $V_n \subseteq Y$ and a natural number N such that $x \in \bigcap_{n=1} f_n^{-1}(V_n) = B_1$. Similarly we can find functions $g_n \in M$, open sets $W_n \subseteq Y$ and a natural number M such that $x \in \bigcap_{n=1}^{M} g_n^{-1}(W_n) = B_2$. If

$$B = \bigcap_{n=1}^{N} f_n^{-1}(V_n) \cap \bigcap_{m=1}^{M} g_m^{-1}(W_m).$$

Then it follows that $B \in \mathscr{B}$ and $x \in B \subseteq B_1 \cap B_2$ so $\mathscr{B}$ is the basis of a topology τ. Now let $\tau' \subseteq \tau$ be another topology on X such that any $f \in M$ is continuous with respect to τ'. If τ' is a proper subset of τ there exists functions $f_n \in M$ and open sets $V_n \subseteq Y$ and a natural number N such that $\bigcap_{n=1}^{N} f_n^{-1}(V_i) \notin \tau'$. But since an intersection of finitely many open sets is open it follows that there exists an n such that $f_n^{-1}(V_n) \notin \tau'$ which contradicts continuity of f_n hence $\tau = \tau'$. $\square$

Definition 9. *We give Γ the weak topology induced by $A(\Gamma)$, that is, the weakest topology which makes any Fourier transform $\hat{f}$ a continuous function on Γ for any $f \in L^1(G)$.*

Proposition 12. *The dual group Γ is locally compact when equipped with the weak topology*

Proof. We will show in Proposition 15 that the set of non-zero complex multiplicative linear functionals on $L^1(G)$ (independently of this proof) is in bijection with functionals of the form

$$\Lambda_\gamma(f) = \hat{f}(\gamma).$$

Let $D = \{\Lambda_\gamma : \gamma \in \Gamma\}$ and give D the weak topology induced by functions of the form $e_f : D \to \mathbb{C}$ where $e_f(\Lambda_\gamma) = \Lambda_\gamma(f)$ for $f \in L^1(G)$. To show that Γ is homeomorphic to D define $\Phi : \Gamma \to D$ by $\Phi(\gamma) = \Lambda_\gamma$, pick $f \in L^1(G)$ and an open set $B \subseteq \mathbb{C}$ then we get

$$\begin{aligned}\Phi^{-1}(e_f^{-1}(B)) &= \{\gamma \in \Gamma : \Lambda_\gamma \in e_f^{-1}(B)\} \\ &= \{\gamma \in \Gamma : B \ni e_f(\Lambda_\gamma) = \Lambda_\gamma(f) = \hat{f}(\gamma)\} \\ &= \hat{f}^{-1}(B)\end{aligned}$$

the last set is open in Γ and it follows that Φ is continuous. The same type of argument shows that Φ^{-1} is continuous. Since

$$|\Lambda_\gamma(f)| = |\hat{f}(\gamma)| \leq ||f||_1$$

it follows that $D \subseteq S^* \subseteq L^1(G)^*$ where $L^1(G)^*$ denotes the space of bounded linear functionals on $L^1(G)$ and S^* is the unit ball in $L^1(G)^*$ with respect to operator norm. If we also give $L^1(G)^*$ the weak topology induced by the collection of functions e_f, $f \in L^1(G)$ then it is clear that the weak topology on D coincides with the subspace topology inherited from $L^1(G)^*$. By the Banach-Alaoglu theorem S^* is compact in $L^1(G)^*$ with the weak topology, see Theorem 3.15 in [2]. According to corollary D4 in [3] the set $D \cup \{0\}$ is closed hence compact since it is a subspace of S^*. If Λ_1, Λ_2 are distinct elements of $L^1(G)^*$ then there exists f such that

$$\Lambda_1(f) \neq \Lambda_2(f)$$

hence $e_f(\Lambda_1) \neq e_f(\Lambda_2)$ and from continuity of e_f it follows that we can separate Λ_1 and Λ_2 with open sets and it follows that $L^1(G)^*$ is Hausdorff.

Pick $\Lambda_\gamma \in D$. Since $L^1(G)^*$ is Hausdorff it follows that we can find open sets $\Lambda_\gamma \in V_\gamma$ and $0 \in W_\gamma$ with empty intersection. It follows that the closure of V_γ in $D \cup \{0\}$ which is compact does not contain 0 hence this closure coincides with the closure of V_γ in D and it follows that D is locally compact. Since D homeomorphic to Γ it follows that Γ is locally compact. □

Equipped with this topology the Fourier transform has the following properties, where i) to iv) and vi) can be found in Theorem 1.2.4 in [3] with less details and v) can also be found in Theorem 1.2.6 in [3].

Proposition 13. *The following properties hold*
i) if $f \in L^1(G)$, $x_0 \in G$ and $\gamma \in \Gamma$ then $\hat{f}_{x_0}(\gamma) = \gamma(-x_0)\hat{f}(\gamma)$.

ii) The Fourier transform seen as a map taking elements of $L^1(G)$ to elements of $A(\Gamma)$ maps into $C_0(\Gamma)$.
iii) The Fourier transform seen as a map from $L^1(G)$ to $C_0(\Gamma)$ is norm decreasing, $||\hat{f}||_\infty \leq ||f||_1$ and therefore continuous.
iv) The Fourier transform separates points on Γ, that is for distinct $\gamma, \delta \in \Gamma$ there exists an $\hat{f} \in A(\Gamma)$ such that $\hat{f}(\gamma) \neq \hat{f}(\delta)$.
v) The function defined on $G \times \Gamma$ which takes (x, γ) to $\gamma(x)$ is continuous.
vi) $A(\Gamma)$ is dense in $C_0(\Gamma)$

Proof. i)
Pick $f \in L^1(G)$, $x_0 \in G$ and $\gamma \in \Gamma$ then

$$\begin{aligned}\widehat{f_{x_0}}(\gamma) &= \int_G f_{x_0}(x)\gamma(-x)dx = \int_G f(x - x_0)\gamma(-x)dx \\ &= \int_G f(x)\gamma(-x - x_0)dx = \gamma(-x_0)\int_G f(x)\gamma(-x)dx \\ &= \gamma(-x_0)\hat{f}(\gamma)\end{aligned}$$

ii) Recall that F is an element of $C_0(\Gamma)$ if F is a complex valued continuous function and for any $\epsilon > 0$ there exists a compact set $K \subseteq \Gamma$ such that $|F(x)| \leq \epsilon$ whenever x lies in the complement of K. The space $C_0(\Gamma)$ is normed by the supremum norm. If $f \in L^1(G)$ then $\hat{f}$ is continuous by definition of the topology on Γ. Pick $\epsilon > 0$ and consider the set

$$K = \{\gamma \in \Gamma : |\hat{f}(\gamma)| \geq \epsilon\}.$$

Using the notation and the homeomorphism from Proposition 12 K is mapped into

$$\begin{aligned}K' &= \{\Lambda_\gamma \in D : |\hat{f}(\gamma)| \geq \epsilon\} \\ &= \{\Lambda_\gamma \in D \cup \{0\} : |e_f(\Lambda_\gamma)| \geq \epsilon\}.\end{aligned}$$

It follows that K' is a closed subset of the compact set $D \cup \{0\}$ and is therefore compact. Hence K is compact which shows that $\hat{f} \in C_0(\Gamma)$.

iii) If $f \in L^1(G)$ then

$$|\hat{f}(\gamma)| = |\int_G f(x)\gamma(-x)dx| \leq \int_G |f(x)||\gamma(-x)|dx \leq ||f||_1$$

hence $||\hat{f}||_\infty \leq ||f||_1$

iv) Let $\gamma, \delta \in \Gamma$ be distinct characters. If $\hat{f}(\gamma) = \hat{f}(\delta)$ for all $\hat{f} \in A(\Gamma)$ then by definition of the Fourier transform it follows that

$$\int_G f(x)(\gamma(-x) - \delta(-x))dx = 0$$

for all $f \in L^1(G)$. Since $\gamma \neq \delta$ there exists a $x_0 \in G$ such that $\gamma(-x_0) \neq \delta(-x_0)$ and by local compactness of G there exists an open set $V \subseteq G$ such that $-x_0 \in V$ and $K = \overline{V}$ is compact. Let $f(x) = \chi_K(x)\overline{(\gamma(-x) - \delta(-x))}$ then $f \in L^1(G)$ since

$$\int_G |\chi_K(x)\overline{(\gamma(-x) - \delta(-x))}|dx \leq \int_K |\gamma(-x)|dx + \int_K |\delta(-x)|dx = 2m(K) < \infty$$

using the fact that K is compact. It follows that

$$\int_K |(\gamma(-x) - \delta(-x))|^2 dx = 0.$$

Hence $\gamma(-x) - \delta(-x) = 0$ for all $x \in K$ since $m(K) > m(V) > 0$. In particular it follows that $\gamma(-x_0) = \delta(-x_0)$ hence $\gamma = \delta$ and the Fourier transform separates points.

v) Pick a point $(x_0, \gamma_0) \in G \times \Gamma$ By i) we have that

$$\hat{f}_{-x_0}(\gamma_0) = \gamma_0(x_0)\hat{f}(\gamma_0)$$

for any $f \in L^1(G)$. If $f(x) = g(x)\gamma_0(x)$ where g is a continuous, non-negative and non-zero $L^1(G)$-function it follows that

$$\hat{f}(\gamma_0) = ||f||_1 > 0.$$

Hence

$$\gamma_0(x_0) = \frac{\hat{f}_{-x_0}(\gamma_0)}{\hat{f}(\gamma_0)}.$$

Since $\hat{f}$ is continuous and non-zero at γ_0 it follows that $\gamma(x)$ is continuous at (x_0, γ_0) if $\hat{f}_x(\gamma)$ is continuous at (γ_0, x_0). Since the mapping taking $x \in G$ to $f_x \in L^1(G)$ is continuous there exists an open set $V \subseteq G$ containing x_0 such that

$$||f_x - f_{x_0}||_1 < \frac{\epsilon}{2}$$

whenever $x \in V$. By continuity of the Fourier transform there exists an open set $U \subseteq \Gamma$ containing γ_0 such that

$$|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < \frac{\epsilon}{2}$$

whenever $\gamma \in U$. If $(x, \gamma) \in V \times U$ it follows that

$$\begin{aligned} |\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma_0)| &\leq |\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma)| + |\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| \\ &\leq ||f_x - f_{x_0}||_1 + |\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < \epsilon \end{aligned}$$

vi) We know by iv) that $A(\Gamma)$ separates points on Γ and by ii) we know that $A(\Gamma) \subseteq C_0(\Gamma)$. In the next section an operation in $L^1(G)$ called convolution will be defined which satisfies $\widehat{f * g}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$ where $f, g \in L^1(G)$ and $*$ denotes convolution, see Proposition 14. This shows that $A(\Gamma)$ is multiplicatively closed set under pointwise multiplication of functions. It is easy to check that $A(\Gamma)$ is closed under conjugation since given $f \in L^1(G)$ we can define the function $g(x) = \overline{f(-x)}$ and it is easy to show that $\hat{g} = \overline{\hat{f}}$ hence $\overline{\hat{f}} \in A(\Gamma)$. Lastly we prove that for any $\gamma_0 \in \Gamma$ there exists a Fourier transform that does not evaluate to 0 at γ_0. Pick a function $f \in L^1(G)$ such that $\int_G f dx \neq 0$ then it follows that $\widehat{f\overline{\gamma}_0}(\gamma_0) = \int_G f dx \neq 0$. It follows from the Stone-Weierstrass Theorem, (see Appendix A.14 in [3]) that $A(\Gamma)$ is dense in $C_0(\Gamma)$. □

A different proof of Theorem 4 for is given in Theorem 1.2.6 in [3].

Theorem 4. *If we give Γ the weak topology induced by $A(\Gamma)$ then Γ is a locally compact, abelian, Hausdorff topological group.*

Proof. The only things that remains to be shown is that Γ is Hausdorff and that multiplication and inversion are continuous group operations.

To show that Γ is Hausdorff, let $\gamma, \delta \in \Gamma$ be distinct characters. Since the Fourier transform separates points there exists an $\hat{f} \in A(\Gamma)$ such that $\hat{f}(\gamma) \neq \hat{f}(\delta)$. Since $\mathbb{C}$ is Hausdorff there exists $V_1, V_2 \subseteq \mathbb{T}^1$ both open such that $\hat{f}(\gamma) \in V_1$, $\hat{f}(\delta) \in V_2$ and $V_1 \cap V_2 = \emptyset$. By continuity of $\hat{f}$ the sets $\hat{f}^{-1}(V_1)$ and $\hat{f}^{-1}(V_2)$ are both open, they contain γ, δ respectively and they are disjoint since V_1, V_2 are disjoint and it follows that Γ is Hausdorff.

We now show that inversion, $i : \Gamma \to \Gamma$ is continuous. By definition of continuity we need to show that $i^{-1}(W)$ is open whenever $W \subseteq \Gamma$ is open. It is sufficient to show it when W is a basis element so pick functions $\hat{f}_n \in A(\Gamma)$ and open sets $V_n \subseteq \mathbb{C}$. We then need to show that

$$U = i^{-1}\Big(\bigcap_{n=1}^{N} \hat{f}_n^{-1}(V_n)\Big) \subseteq \Gamma$$

is open. We have

$$\begin{aligned} U &= \{\gamma \in \Gamma; \gamma^{-1} \in \hat{f}_n^{-1}(V_n)\ \forall n\} \\ &= \{\gamma \in \Gamma; \hat{f}_n(\gamma^{-1}) \in V_n\ \forall n\} \\ &= \bigcap_{n=1}^{N} \{\gamma \in \Gamma; \hat{f}_n(\gamma^{-1}) \in V_n\}. \end{aligned}$$

For any n we have

$$\begin{aligned}\hat{f}_n(\gamma^{-1}) &= \int_G f_n(x)\gamma^{-1}(-x)dx = \int_G f_n(x)\overline{\gamma(-x)}dx \\ &= \overline{\int_G \overline{f_n(x)}\gamma(-x)dx} = \overline{\widehat{\overline{f_n}}(\gamma)}.\end{aligned}$$

Hence if $\mathrm{conj}(E) \subseteq \mathbb{C}$ denotes the set of complex conjugates of elements of $E \subseteq \mathbb{C}$ then

$$\begin{aligned}U &= \bigcap_{n=1}^{N} \{\gamma \in \Gamma; \overline{\widehat{\overline{f_n}}(\gamma)} \in V_n\} \\ &= \bigcap_{n=1}^{N} \{\gamma \in \Gamma; \widehat{\overline{f_n}}(\gamma) \in \mathrm{conj}(V_n)\} \\ &= \bigcap_{n=1}^{N} \widehat{\overline{f_n}}^{-1}(\mathrm{conj}(V_n))\end{aligned}$$

which is open since $\mathrm{conj}(V_n)$ is open and $\hat{\overline{f_n}} \in A(\Gamma)$ is continuous.

We now show that multiplication, $m : \Gamma \times \Gamma \to \Gamma$ is continuous. Pick a point $(\gamma_0, \delta_0) \in \Gamma \times \Gamma$, continuity of m is equivalent to showing that for any open $T \subseteq \Gamma$ such that $\gamma_0\delta_0 \in T$ there exists open subsets V, W of Γ such that $(\gamma_0, \delta_0) \in V \times W$ and $m(V, W) \subseteq T$. By our basis of Γ we know that there exists functions $f_n \in L^1(G)$ and open sets $U_n \subseteq \mathbb{C}$ such that

$$\delta_0\gamma_0 \in \bigcap_{n=1}^{N} \hat{f}_n^{-1}(U_n) \subseteq T.$$

We are done if we can find V and W such that $\gamma_0 \in V$, $\delta_0 \in W$ and for each n, $1 \leq n \leq N$ we have that $\hat{f}_n(\delta\gamma) \in U_n$ whenever $\gamma \in V$ and $\delta \in W$. Since U_n is open there exists for each n an $\epsilon_n > 0$ such that the ball centered at $\hat{f}_n(\gamma_0\delta_0)$ with radius ϵ_n is contained in U_n. Define

$$\epsilon = \min_{1 \leq n \leq N} \epsilon_n > 0.$$

Pick an n, $1 \leq n \leq N$. Since $C_c(G)$ is dense in $L^1(G)$, (see Theorem 3.14 in [1]) there exists a function $g_n \in C_c(G)$ supported in the compact set $K_n \subseteq G$ such that $||f_n - g_n||_1 < \frac{\epsilon}{4}$. Pick a point $x_0 \in K_n$ then the inequality

$$\begin{aligned}|\gamma(-x) - \gamma_0(-x)| &\leq |\gamma(-x) - \gamma_0(-x_0)| + |\gamma_0(-x_0) - \gamma_0(-x)| \\ &= |\overline{\gamma(x)} - \overline{\gamma_0(x_0)}| + |\overline{\gamma_0(x_0)} - \overline{\gamma_0(x)}| \\ &= |\gamma(x) - \gamma_0(x_0)| + |\gamma_0(x_0) - \gamma_0(x)|\end{aligned}$$

holds for any $x \in G$. Combining this inequality with Propostition 13 v) and continuity of γ_0 at x_0 shows that there exists open sets $N_{x_0} \subseteq G$ and $M_{x_0} \subseteq \Gamma$ containing x_0 and γ_0 respectively such that

$$|\gamma(-x) - \gamma_0(-x)| < \frac{\epsilon}{4||g_n||_1}$$

whenever $(x, \gamma) \in N_{x_0} \times M_{x_0}$. Since

$$K_n \subseteq \bigcup_{x \in K_n} N_x$$

is an open cover and K_n is compact there exists $x_1, ..., x_M$ such that

$$K_n \subseteq \bigcup_{m=1}^{M} N_{x_m}.$$

If we define

$$V := \bigcap_{m=1}^{M} M_{x_m}$$

then V is a non-empty open set of Γ since it contains γ_0 and

$$|\gamma(-x) - \gamma_0(-x)| < \frac{\epsilon}{4||g_n||_1}$$

for any $x \in K_n$ whenever $\gamma \in V$. Defining W analogously for δ_0 we get that

$$\begin{aligned}
|\hat{f}_n(\gamma\delta) - \hat{f}_n(\gamma_0\delta_0)| &\leq \int_G |f_n(x)||\gamma(-x)\delta(-x) - \gamma_0(-x)\delta_0(-x)|dx \\
&\leq \int_G |f_n(x) - g_n(x) + g_n(x)||\gamma(-x)\delta(-x) - \gamma_0(-x)\delta_0(-x)|dx \\
&\leq 2||f_n - g_n||_1 + \int_{K_n} |g_n(x)||\gamma(-x)\delta(-x) - \gamma_0(-x)\delta_0(-x)|dx \\
&\leq \frac{\epsilon}{2} + \int_{K_n} |g_n(x)||\delta(-x) - \delta_0(-x)|dx + \int_{K_n} |f_K(x)||\gamma(-x) - \gamma_0(-x)|dx \\
&< \epsilon \\
&\leq \epsilon_n
\end{aligned}$$

whenever $(\gamma, \delta) \in V \times W$. Which shows that $\hat{f}_n(\gamma\delta) \in U_n$ whenever $(\gamma, \delta) \in V \times W$ which proves that multiplication is continuous in Γ. $\square$

4 Convolutions and Algebras

4.1 Convolution of Functions

A very useful operation in Fourier analysis is convolution. Convolution takes two functions in $L^1(G)$ and produces a third function $f*g$ which can be thought of as a weighted average of translates of f with weight function g. Parts of this section can also be found in sections 1.1.6, 1.2.2 and 1.3 in [3] with a somewhat different exposition and parts are left to the reader. A great resource for complex measures is Chapter 6 in [1].

Definition 10. *Let $f, g \in L^1(G)$ the convolution of f and g, $f * g$ is given by*

$$(f * g)(x) \int_G f(x-y)g(y)dy$$

Provided that

$$\int_G |f(x-y)g(y)|dy < +\infty.$$

As the next theorem shows, when $f, g \in L^1(G)$ then $\int_G |f(x-y)g(y)|dy < +\infty$ for almost all $y \in G$ and $f * g \in L^1(G)$ and $L^1(G)$ equipped with $*$ is a commutative Banach algebra. This proof of Theorem 5 is found in 1.1.6 in [3]

Theorem 5. *if f and g are in $L^1(G)$ then*

$$\int_G |f(x-y)g(y)|dy < +\infty$$

for almost all $x \in G$ and $f * g \in L^1(G)$. Furthermore the following properties hold
i) $||f * g||_1 \leq ||f||_1||g||_1$.
ii) $f * g = g * f$.
iii) $(f * g) * h = f * (g * h)$ whenever $h \in L^1(G)$.

Proof. The proof proceeds by Fubini's theorem (see Theorem 8.8 in [1]) so we begin by proving that $H(x, y) = f(x-y)g(y)$ is a (Borel) measurable function on $G \times G$. Define $\phi(x, y) = x - y$ and $\pi(x, y) = y$ then it is enough to prove (Borel) measurability of $f \circ \phi$ and $g \circ \pi$. Let $\mathfrak{M}$ denote the Borel sets in G, $\mathfrak{M} \otimes \mathfrak{M}$ denote the Borel sets of $G \times G$ and let V be an open subset of $\mathbb{C}$. It follows that $g^{-1}(V) \in \mathfrak{M}$ since g is Borel measurable. Recall that $\mathfrak{M} \otimes \mathfrak{M}$ is the smallest σ-algebra that contains all sets of the form $E_1 \times E_2$ for any $E_1, E_2 \in \mathfrak{M}$ (see Definition 8.1 in [1]) hence

$$\pi^{-1}(g^{-1}(V)) = G \times g^{-1}(V) \in \mathfrak{M} \otimes \mathfrak{M}$$

and it follows that $g \circ \pi$ is measurable. Similarly since f is Borel measurable we have that $f^{-1}(V) \in \mathfrak{M}$. It is clear that

$$\phi^{-1}(f^{-1}(V)) = \{(x, y) \in G \times G; x - y \in f^{-1}(V)\}.$$

If we define the function $F : G \times G \to G \times G$ given by $F(x, y) = (x + y, y)$ then it is clear that F is a homeomorphism and therefore maps Borel sets to Borel sets. It follows that

$$\phi^{-1}(f^{-1}(V)) = F(f^{-1}(V) \times G) \in \mathfrak{M} \otimes \mathfrak{M}$$

hence $f \circ \phi$ is measurable and therefore H is measurable. By Fubini's theorem (see Theorem 8.8 in [1]) we then get that

$$\begin{aligned}
\int_G \int_G |f(x-y)g(y)|dydx &= \int_G \int_G |f(x-y)g(y)|dxdy \\
&= \int_G |g(y)| \int_G |f(x-y)|dxdy \\
&= \int_G |g(y)| \int_G |f(x)|dxdy \\
&= ||f||_1 ||g||_1 < +\infty.
\end{aligned}$$

Hence

$$\int_G |f(x-y)g(y)|dy < \infty \text{ a.e}$$

and it follows that $f * g$ is well defined as a function for almost all $x \in G$. From above we see that

$$||f * g||_1 \leq \int_G \int_G |f(x-y)g(y)|dydx = ||f||_1 ||g||_1$$

and i) follows.
For ii) we have that

$$\begin{aligned}
(f * g)(x) &= \int_G f(x-y)g(y)dy \\
&= \int_G f(x-(y+x))g(y+x)dy \\
&= \int_G f(-y)g(y+x)dy \\
&= \int_G f(-(-y))g(-y+x)dy = (g * f)(x).
\end{aligned}$$

For $f, g \in L^1(G)$. Note that Corollary 1 was used above. For iii) we have that

$$\begin{aligned}
((f * g) * h)(x) &= \int_G (f * g)(x-y)h(y)dy = \int_G (\int_G (f(x-y-z)g(z)dz)h(y)dy \\
&= \int_G \int_G f(x-y-z)g(z)dzh(y)dy = \int_G \int_G f(x-z)g(z-y)dzh(y)dy \\
&= \int_G \int_G f(x-z)g(z-y)h(y)dzdy = \int_G f(x-z) \int_G g(z-y)h(y)dydz \\
&= \int_G f(x-z)(g * h)(z)dz = (f * (g * h))(x).
\end{aligned}$$

Whenever $f, g, h \in L^1(G)$. Where the usage of Fubini's theorem is justified by the discussion in i) $\square$

Two useful properties of convolution are the following

Proposition 14.
i) If $f, g \in L^1(G)$ then $\widehat{(f * g)} = \hat{f}\hat{g}$.
ii) If $f, g \in L^2(G)$ then $f * g$ is well-defined and continuous

Proof. i) Let $f, g \in L^1(G)$ then

$$\begin{aligned}
\widehat{(f * g)}(\gamma) &= \int_G (f * g)(x)\gamma(-x)dx = \int_G \int_G f(x-y)g(y)\gamma(-x)dydx \\
&= \int_G \int_G f(x-y)g(y)\gamma(-x)dxdy \\
&= \int_G \int_G f(x-y)g(y)\gamma(-x+y-y)dxdy \\
&= \int_G g(y)\gamma(-y) \int_G f(x-y)\gamma(-x+y)dxdy \\
&= \int_G g(y)\gamma(-y) \int_G f(x)\gamma(-x)dxdy = \hat{f}(\gamma)\hat{g}(\gamma)
\end{aligned}$$

ii) Let $f, g \in L^2(G)$ and pick $y \in G$ then Hölder's inequality gives

$$|(f * g)(y)| \leq \int_G |f(y-x)||g(x)|dx \leq ||f||_2^2 ||g||_2^2 < +\infty$$

which shows that $f * g$ is well defined. Define f^- by $f^-(x) = f(-x)$. Then by continuity of the map that sends $x \in G$ to $f_x^- \in L^2(G)$, the translate of f^-, there exists an open set V containing y such that $z \in V$ implies

$$||f_z^- - f_y^-||_2 < \frac{1}{||g||_2}\sqrt{\epsilon}$$

hence if $z \in V$ it follows by Hölder's inequality that

$$|(f * g)(z) - (f * g)(y)| \leq ||f_z^- - f_y^-||_2^2 ||g||_2^2 < \epsilon$$

which shows that $f * g$ is continuous. $\square$

Remark. *If A is complete Banach space that has an operator $\cdot : A \times A \to A$ called multiplication that is associative and satisfies*

$$\begin{aligned}
\alpha(xy) &= (\alpha x)y = x(\alpha y) \\
(x+y)z &= xz + yz \\
x(y+z) &= xy + xz \\
||xy|| &\leq ||x|| ||y||
\end{aligned}$$

For any $x, y, z \in A$ and $\alpha \in \mathbb{C}$. Then A is called a Banach Algebra. Note that $L^1(G)$ is a Banach Algebra with convolution. If B is another Banach Algebra and $\Lambda : A \to B$ is a linear functional that is also multiplicative,

$$\Lambda(xy) = \Lambda(x)\Lambda(y)$$

Then Λ is called a Banach Algebra homomorphism.

Remark. *The proof of Proposition 15 can be found in section 1.2.2 in [3].*

Proposition 15. *For any fixed $\gamma \in \Gamma$ the Fourier transform which takes $f \in L^1(G)$ to $\hat{f}(\gamma) \in \mathbb{C}$ is a complex valued Banach Algebra homomorphism that is not the zero homomorphism. Conversely any complex valued Banach Algebra homomorphism of $L^1(G)$ that is not the zero homomorphism is of the form $h(f) = \hat{f}(\gamma)$ for some $\gamma \in \Gamma$. Furthermore different elements of Γ corresponds to distinct homomorphisms.*

Proof. Using Proposition 14 it is clear that any function of the form $\Lambda_\gamma(f) = \hat{f}(\gamma)$ is a complex Banach Algebra homomorphism. Given γ we can pick an $f \in L^1(G)$ that is not 0 and then we have

$$\Lambda_\gamma(|f|\overline{\gamma}) = ||f||_1 > 0$$

so Λ_γ is not the zero functional. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then

$$\int_G f(x)(\gamma_1(-x) - \gamma_2(-x))dx = 0$$

for any $f \in L^1(G)$ from which it follows that $\gamma_1 = \gamma_2$.

Conversely, suppose that Λ is a complex Banach Algebra homomorphism that is not the zero functional. Since the kernel of Λ_γ is a maximal ideal (see Theorem 11.5 in [2]) and maximal ideals are closed it follows that the kernel is closed. Since linear functionals are continuous if and only if their kernel is closed it follows that Λ is a bounded linear functional. By Theorem 11.9 in [2] we know that $||\Lambda|| \leq 1$. Since G is assumed to be σ-finite it follows from Theorem 6.16 in [1] that there exists a unique $\lambda \in L^\infty(G)$ such that $||\lambda||_\infty = ||\Lambda|| \leq 1$ and

$$\Lambda(f) = \int_G f(x)\lambda(x)dx.$$

If f and g are in $L^1(G)$ then

$$\begin{aligned}\int_G \Lambda(f)g(x)\lambda(x)dx &= \Lambda(f)\Lambda(g) = \Lambda(f*g)\\ &= \int_G (f*g)(x)\lambda(x)dx\\ &= \int_G\int_G f(x-y)g(y)dy\lambda(x)dx\\ &= \int_G\int_G f(x-y)\lambda(x)dxg(y)dy\\ &= \int_G \Lambda(f_y)g(y)dy\end{aligned}$$

Note that Fubinis Theorem, (see Theorem 8.8 in [1]) is applicable since

$$\begin{aligned}\int_G\int_G |f(x-y)\lambda(x)g(y)|dxdy &\leq ||\lambda||_\infty \int_G |g(y)| \int_G |f(x-y)|dxdy\\ &= ||\lambda||_\infty \int_G |g(y)| \int_G |f(x)|dxdy\\ &= ||f||_1||g||_1||\lambda||_\infty \leq +\infty\end{aligned}$$

It follows that

$$\Lambda(f)\lambda(x) = \Lambda(f_x) \tag{1}$$

for almost all $x \in G$. By continuity of translation and continuity of Λ it follows that the right hand side of above is continuous. Since $\Lambda \neq 0$ we can choose an f such that $\Lambda(f) \neq 0$ and conclude that $\lambda(x)$ coincides with a continuous function almost everywhere on G. We can redefine λ as $\lambda(x) = \frac{\Lambda(f_x)}{\Lambda(f)}$ on this set of measure 0 and it follows that we can assume that λ is continuous on G. Substitute x with $x+y$ in (1) to obtain

$$\begin{aligned}\Lambda(f)\lambda(x+y) &= \Lambda(f_{x+y}) = \Lambda((f_x)_y)\\ &= \Lambda(f_x)\lambda(y) = \Lambda(f)\lambda(x)\lambda(y).\end{aligned}$$

Hence

$$\lambda(x+y) = \lambda(x)\lambda(y).$$

Lastly, since $|\lambda(x)| \leq 1$ for all $x \in G$ and $\lambda(-x) = \lambda(x)^{-1}$ it follows that $|\lambda(x)| = 1$ for all $x \in G$ hence $\lambda \in \Gamma$. □

4.2 Convolution of Measures

It will be shown later that $L^1(G)$ does not have a unit in most cases. However $L^1(G)$ can always be embedded in a commutative Banach algebra with a unit,

$M(G)$, the algebra of complex regular Borel measures on G. We begin with some definitions.

Definition 11. *We say that a complex Borel measure λ is regular if*

$$|\lambda|(E) = \inf_{E\subseteq V} |\lambda|(V) = \sup_{K\subseteq E} |\lambda|(K)$$

for any Borel set E where $|\lambda|$ is the total variation measure of λ.

By defining $(\lambda + \mu)(E) := \lambda(E) + \mu(E)$ and $(c\lambda)(E) := c\lambda(E)$ for any Borel set E and any complex Borel measures μ, λ and any $c \in \mathbb{C}$ it is easy to show that the set of complex Borel measures is a vector space. It is also easy to show that this addition and multiplication preserves regularity so the set of complex regular Borel measures $M(G)$ is also vector space. If we also define $||\lambda|| = |\lambda|(G)$ for any $\lambda \in M(G)$ it follows that $M(G)$ is a normed vector space and by Theorem 6.4 in [1] we have $\mu(G) < +\infty$ for any $\mu \in M(G)$ so $||\mu|| < +\infty$ for any $\mu \in M(G)$. We summarize the results below.

Proposition 16. *Let $M(G)$ be the set of all complex regular Borel measures on G and define the function $||\mu|| = |\mu|(G)$ for any $\mu \in M(G)$. Then the function $||\cdot||$ defines a norm on $M(G)$ so $M(G)$ is a normed space*

Proposition 17. *$M(G)$ is complete and hence a Banach space.*

Proof. Let $C_0(G)^*$ be the dual space of $C_0(G)$. That is the the set of all bounded linear functionals on $C_0(G)$ normed by

$$||\Lambda|| = \sup_{||f||\leq 1} |\Lambda f|.$$

Then it is well known that the dual of a normed space is a Banach space so $C_0(G)^*$ is a Banach space. By the Riesz representation theorem for bounded linear functionals (see Theorem 6.19 in [1]) there is a bijective correspondence between $C_0(G)^*$ and $M(G)$ which we will denote by $\Phi : C_0(G)^* \to M(G)$ which also satisfies $||\Phi(\Lambda)|| = ||\Lambda||$ for any $\Lambda \in C_0(G)^*$. Recall that integration with respect to a complex measure μ is defined as

$$\int_G f d\mu = \int_G f h d|\mu|$$

where the right hand side is the polar decomposition of μ (see Theorem 6.12 in [1]) where $|\mu|$ is the total variation measure and $|h(x)| = 1$ for all $x \in G$. From this it follows that

$$\int_G \chi_E d\mu = \int_G \chi_E h d|\mu| = \int_E h d|\mu| = \mu(E)$$

for any Borel set $E \subseteq G$. Hence if λ also is a complex measure and $E \subseteq G$ is a Borel set we have

$$\begin{aligned}\int_G \chi_E d(\mu+\lambda) &= (\mu+\lambda)(E) = \mu(E)+\lambda(E)\\ &= \int_G \chi_E d\mu + \int_G \chi_E d\lambda.\end{aligned}$$

By a standard approximation argument with simple functions it follows that

$$\int_G f d(\mu+\lambda) = \int_G f d\mu + \int_G f d\lambda$$

for any $f \in L^1(\mu)$. With a similar argument it is easy to show that

$$\int_G f d(\alpha\mu) = \alpha \int_G f d\mu$$

For any $\alpha \in \mathbb{C}$ and $f \in L^1(\mu)$. Using these two relations it follows that

$$\begin{aligned}\int_G f d\Phi(\alpha\Lambda_1+\beta\Lambda_2) &= (\alpha\Lambda_1+\beta\Lambda_2)f = \alpha\Lambda_1 f + \beta\Lambda_2 f\\ &= \alpha\int_G f d\Phi(\Lambda_1) + \beta\int_G f d\Phi(\Lambda_2)\\ &= \int_G f d\alpha\Phi(\Lambda_1) + \int_G f d\beta\Phi(\Lambda_2)\\ &= \int_G f d(\alpha\Phi(\Lambda_1 + \beta\Phi(\Lambda_2))\end{aligned}$$

for any $\Lambda_1, \Lambda_2 \in C_0(G)^*$ and $\alpha, \beta \in \mathbb{C}$. It follows by uniqueness of the Riesz representation theorem (see Theorem 6.19 in [1]) that Φ is linear and hence an isometric isomorphism between $C_0(G)^*$ and $M(G)$. Since $C_0(G)^*$ is complete and isometric isomorphisms preserve completeness it follows that $M(G)$ is complete and hence a Banach space. □

We now give $M(G)$ a multiplicative structure.

Definition 12. *Pick two measures $\lambda, \mu \in M(G)$ and consider the linear functional Λ defined on $C_0(G)$ by*

$$\Lambda(f) = \int_G \int_G f(x+y) d\lambda(x) d\mu(y).$$

Since

$$\begin{aligned}|\Lambda(f)| &\leq \int_G \int_G |f(x+y)| d\lambda(x) d\mu(y)\\ &\leq \int_G \int_G ||f||_\infty d\lambda(x) d\mu(y)\\ &= ||f||_\infty |\lambda(G)||\mu(G)|\\ &= ||f||_\infty ||\lambda|| ||\mu||\end{aligned}$$

We see that Λ is a bounded linear functional of norm less than $||\lambda||||\mu||$. By the Riesz representation theorem for bounded linear functionls there corresponds unique measure denoted $\lambda * \mu \in M(G)$ such that $||\lambda * \mu|| \leq ||\lambda||||\mu||$ and

$$\int_G f d(\lambda * \mu) = \int_G \int_G f(x+y) d\lambda(x) d\mu(y)$$

for any $f \in C_0(G)$. We call $\lambda * \mu$ the convolution of λ and μ.

By the theorem below it follows that $M(G)$ equipped with $*$ is a unital commutative Banach algebra.

Theorem 6. *Let λ, μ and ν be arbitrary elements of $\in M(G)$ then*
i) $\lambda * \mu \in M(G)$
ii) $||\lambda * \mu|| \leq ||\lambda||||\mu||$
iii) $\lambda * \mu = \mu * \lambda$
iv) $(\lambda * \mu) * \nu = \lambda * (\mu * \nu)$
v) there exists $e \in M(G)$ such that $e * \mu = \mu$

Proof. i) and ii) was shown above. For iii) we note that if $\lambda, \mu \in M(G)$ then they are finite measures and by Fubini's theorem (see Theorem 8.8 in [1]) we get that

$$\int_G \int_G f(x+y) d\lambda(x) d\mu(y) = \int_G \int_G f(y+x) d\mu(y) d\lambda(x)$$

for any $f \in C_0(G)$. The left hand side functional of above correspond to the unique measure $\lambda * \mu$. Similarly the right hand side functional corresponds uniquely to the measure $\mu * \lambda$ and since the two functionals are equal it follows that the corresponding measures are equal aswell which shows iii). For iv) we pick $f \in C_0(G)$ and see that

$$\begin{aligned}
\int_G f d((\lambda * \mu) * \nu) &= \int_G \int_G f(x+y) d(\lambda * \mu)(x) d\nu(y) \\
&= \int_G \int_G \int_G f(x+y+z) d\lambda(x) d\mu(z)(x) d\nu(y) \\
&= \int_G \int_G \int_G f_{-x}(z+y) d\mu(z)(x) d\nu(y) d\lambda(x) \\
&= \int_G \int_G f_{-x}(y) d(\mu * \nu)(y) d\lambda(x) \\
&= \int_G \int_G f(x+y) d\lambda(x) d(\mu * \nu)(y) \\
&= \int_G f d(\lambda * (\mu * \nu))
\end{aligned}$$

and by the Riesz representation theorem (see Theorem 6.19 in [1]) it follows that $(\lambda * \mu) * \nu = \lambda * (\mu * \nu)$. For v) we can define $e := \delta_0$ where δ_0 is the measure defined as $\delta_0(E) = 1$ if $0 \in E$ and $\delta_0(E) = 0$ otherwise. It is clear that $\delta_0 \in M(G)$, inner regularity follows from that any set containing 0 has the same measure as any compact set containing 0 and any set not containing 0 has the same measure as any compact set not containing 0, outer regularity has the same argument. It is easy to show that

$$\int_G f(x)d\delta_0(x) = f(0)$$

for any $f \in C_0(G)$ hence

$$\begin{aligned} \int_G \int_G f(x+y)d\delta_0 d\mu(y) &= \int_G \int_G f_{-y}(x)d\delta_0(x)d\mu(y) \\ &= \int_G f_{-y}(0)d\mu(y) \\ &= \int_G f(y)d\mu(y). \end{aligned}$$

The left hand side functional corresponds to $\delta_0 * \mu$ while the right hand side corresponds to μ and v) is proved. $\square$

Proposition 18. *If $f \in L^1(G)$ and the complex Borel measure μ_f is defined by*

$$\mu_f(E) = \int_E f(x)dx$$

then $\mu_f \in M(G)$.

Proof. For any integer $n > 0$ let V_n be an open set containing E such that $m(E) \leq m(V_n) < m(E) + \frac{1}{n}$. We can also freely assume that $V_{n+1} \subseteq V_n$. Then $\lim_{n\to+\infty} \chi_{V_n}(x) = \chi_E(x)$ for any $x \in G$ except possibly on the set $\cap_n V_n - E$ but $m(\cap_n V_n - E) = 0$. From Theorem 6.6 in [1] we know that the total variation is given by

$$|\mu_f|(A) = \int_A |f(x)|dx$$

for any Borel set $A \subseteq G$. By the dominated convergence theorem it follows that

$$\begin{aligned} \inf_{E\subseteq V} |\mu_f|(V) &\leq \lim_{n\to+\infty} |\mu_f|(V_n) = \lim_{n\to+\infty} \int_{V_n} |f|dm \\ &= \lim_{n\to+\infty} \int_G \chi_{V_n}|f|dm = \int_G \chi_E|f|dm = |\mu_f|(E). \end{aligned}$$

So $|\mu|(E) \geq \inf_{E\subseteq V} |\mu|(V)$ and it follows that μ_f is inner regular. Since G is assumed to be σ-finite we know that m is inner regular on any Borel set E. It follows that for any $n > 0$ we can find a compact set $K_n \subseteq E$ such that $m(E) - \frac{1}{n} < m(K_n) \leq m(E)$. We can without loss of generality assume that

$K_n \subseteq K_{n+1}$ then as above $\lim_{n\to+\infty} \chi_{K_n} = \chi_E$ except on a set of measure zero and by the monotone convergence theorem it follows that

$$\begin{aligned}
\sup_{K\subseteq E} |\mu_f|(K) &\geq \lim_{n\to+\infty} |\mu_f|(K_n) = \lim_{n\to+\infty} \int_{K_n} |f|dm \\
&= \lim_{n\to+\infty} \int_G \chi_{K_n}|f|dm = \int_G \chi_E|f|dm = |\mu_f|(E).
\end{aligned}$$

Hence $|\mu|(E) \leq \sup_{K\subseteq E} |\mu|(K)$ and it follows that μ_f is outer regular which proves that $\mu_f \in M(G)$. $\square$

We now come to to the embedding of $L^1(G)$. Define the mapping $\iota : L^1(G) \to M(G)$ given by $\iota(f) = \mu_f$. The following properties show us that ι is an embedding of $L^1(G)$ into $M(G)$.
i) If $\mu_f = \mu_g$ then $f = g$ in $L^1(G)$
ii) If $f, g \in L^1(G)$ and $c \in \mathbb{C}$ then $\mu_{f+g} = \mu_f + \mu_g$ and $\mu_{cf} = c\mu_f$
iii) $\mu_{f*g} = \mu_f * \mu_g$
iv) $||\mu_f|| = ||f||_1$

The proof of i) amounts to showing that if $\int_E (f - g)dm = 0$ for any E then $f = g$ a.e which is a well known property of measurable functions. Property ii) is trivial and iv) is true since

$$||\mu_f|| = |\mu_f|(G) = \int_G |f|d\mu = ||f||_1$$

using Theorem 6.13 in [1]. For part iii) we pick $h \in C_0(G)$ then

$$\begin{aligned}
\int_G hd(\mu_f * \mu_g) &= \int_G \int_G h(x+y)d\mu_f(x)d\mu_g(y) \\
&= \int_G \int_G h(x+y)f(x)dxg(y)dy \\
&= \int_G \int_G h(x)f(x-y)dxg(y)dy \\
&= \int_G h(x) \int_G f(x-y)g(y)dydx \\
&= \int_G h(x)(f*g)(x)dx \\
&= \int_G hd\mu_{f*g}
\end{aligned}$$

and by the Riesz representation theorem (see Theorem 6.19 in [1]) it follows that $\mu_f * \mu_g = \mu_{f*g}$. Note that Fubinis theorem could be applied since

$$\begin{aligned}\int_G \int_G |h(x)f(x-y)g(y)|dxdy &\leq \int_G \int_G ||h||_\infty |f(x-y)|dx|g(y)|dy \\ &= ||h||_\infty \int_G \int_G |f(x-y)|dx|g(y)|dy \\ &= ||h||_\infty \int_G \int_G |f(x)|dx|g(y)|dy \\ &= ||h||_\infty ||f||_1 ||g||_1 < +\infty\end{aligned}$$

4.3 The Fourier-Stieltjes Transform

Since $L^1(G)$ can be embedded in $M(G)$ the Fourier-Stieltjes transform is natural to consider.

Definition 13. *The Fourier-Stieltjes transform. If μ is in $M(G)$ the function $\hat{\mu} : \Gamma \to \mathbb{C}$ given by*

$$\hat{\mu}(\gamma) = \int_G \gamma(-x)d\mu$$

is called the Fourier-Stieltjes Transform of μ.

Remark. *If $\mu_f \in M(G)$ is the embedding of $f \in L^1(G)$ then*

$$\hat{\mu}_f(\gamma) = \int_G \gamma(-x)d\mu_f(x) = \int_G f(x)\gamma(-x)dx = \hat{f}(\gamma)$$

The following uniqueness theorem will be used in the proof of Pontryagins duality theorem. Pontryagins theorem in turn will show that applying this uniqueness theorem to Γ in place of G will give us the uniqueness theorem for the Fourier-Stieltjes transform.

Proposition 19. *If $\mu \in M(\Gamma)$ and*

$$\int_\Gamma \gamma(x)d\mu(\gamma) = 0$$

for any $x \in G$ then $\mu = 0$.

Proof. Asssume that μ is as above and pick $f \in L^1(G)$, then it follows that

$$\begin{aligned}\int_\Gamma \hat{f}d\mu &= \int_\Gamma \int_G f(x)\gamma(-x)dxd\mu(\gamma) \\ &= \int_G \int_\Gamma f(x)\gamma(-x)d\mu(\gamma)dx \\ &= \int_G f(x) \int_\Gamma \gamma(-x)d\mu(\gamma)dx = 0.\end{aligned}$$

Since $||\mu||$ is finite we can interchange integration above. If we define the linear transformation B on $C_0(\Gamma)$ as the integral $\int_\Gamma g d\mu$ we see that $|B(g)| \leq ||g||_\infty ||\mu||$ hence B is a bounded linear transformation. Pick $g \in C_0(\Gamma)$. By Proposition 13, $A(\Gamma)$ is dense in $C_0(\Gamma)$ hence there exists a sequence $\hat{f}_n \in A(\Gamma)$ such that $||g - \hat{f}_n||_\infty \to 0$ and therefore

$$\begin{aligned}|B(g)| = |\int_\Gamma g d\mu| &\leq |\int_\Gamma g - \hat{f}_n d\mu| + |\int_\Gamma \hat{f}_n d\mu| \\ &\leq \int_\Gamma |g - \hat{f}_n| d\mu \leq ||\mu|| ||g - \hat{f}_n||_\infty \to 0.\end{aligned}$$

Thus $B(g) = 0$ for any $g \in C_0(\Gamma)$ and by the Riesz representation theorem for bounded linear functionals on $C_0(\Gamma)$ (see Theorem 6.19 in [1]) it follows that $\mu = 0$. $\square$

5 Bochner's Theorem and the Inverse Fourier Transform

5.1 Positive Definite Functions

We will first introduce the positive-definite functions. The reason why we choose to study such functions is that for functions which are $L^1(G)$ and positive-definite the Fourier inversion theorem holds. Bochner's Theorem and the Fourier inversion theorem are the highlights of the section and the proofs follow the proofs in 1.4.3 and 1.5.1 respectively in [3] but with more detail.

Definition 14. *Let f be a complex valued function defined on G. We say that f is positive-definite if for any complex sequence $c_1, ..., c_N$ and sequence $x_1, ..., x_N$ in G the inequality*

$$\sum_{n,m=1}^{N} c_n \overline{c}_m f(x_n - x_m) \geq 0$$

holds.

Remark. *The proof of the following can also be seen in 1.4.1 in [3].*

Proposition 20. *If f is a positive definite function on G then*
i) $f(0) \geq 0$
ii) $f(-x) = \overline{f(x)}$
iii) $|f(x)| \leq f(0)$
iv) $|f(x) - f(y)|^2 \leq 2f(0)\Re(f(0) - f(x - y))$ hence f is uniformly continuous if it is continuous at 0.

Proof. If we let $N = 1$ and $x_1 = 0$ and $c = 1$ in the definition we get $f(0) \geq 0$. Now let $N = 2$ and set $c_1 = 1, c_2 = c, x_1 = 0, x_2 = x$ then

$$(1 + |c|^2)f(0) + \overline{c}f(-x) + cf(x) \geq 0. \tag{1}$$

Letting $c = 1$ in (1) we see by i) that $f(-x) + f(x)$ from which it follows that $\Im f(x) = -\Im f(-x)$. Letting $c = i$ in (1) we get that $i(f(x) - f(-x))$ is real hence $\Re f(x) = \Re f(-x)$ from which it follows that $\overline{f(x)} = f(-x)$ which proves ii). Part iii) is trivially true at the point x if $f(x) = 0$ so assume that $f(x) \neq 0$ and set $c = -|f(x)|/f(x)$ in (1) to get

$$2f(0) + \overline{\frac{-|f(x)|}{f(x)}}f(-x) + \frac{-|f(x)|}{f(x)}f(x) \geq 0.$$

After simplifying the second term with ii) and rearranging we get

$$f(0) \geq |f(x)|$$

which proves iii).

Now for iv), if $f(x) = f(y)$ then iv) is clearly true so assume that $f(x) \neq f(y)$. Let $N = 3$, $x_1 = 0$, $x_2 = x$, $x_3 = y$, $c_1 = 1$ and

$$c_2 = \frac{t|f(x) - f(y)|}{f(x) - f(y)}$$

for some real t and lastly let $c_3 = -c_2$. Then the following relations follows

$$|c_2|^2 + |c_3|^2 = 2t^2.$$
$$c_2 f(x) + c_3 f(y) = t|f(x) - f(y)|.$$

Using ii) we also get that

$$\overline{c_2} f(-x) + \overline{c_3} f(-y) = t|f(x) - f(y)|.$$

and lastly we have

$$\begin{aligned} c_2\overline{c_3} f(x-y) + c_3\overline{c_2} f(y-x) &= -t^2(f(x-y) + f(y-x)) \\ &= -t^2(f(x-y) + \overline{f(x-y)} \\ &= -2t^2\Re(f(x-y)). \end{aligned}$$

By applying the definition of positive definite and the relations above it follows that

$$(1 + 2t^2)f(0) + 2t|f(x) - f(y)| - 2t^2\Re(f(x-y)) \geq 0.$$

Using i) this can be rewritten as

$$f(0) + 2t|f(x) - f(y)| + 2t^2\Re(f(0) - f(x-y)) \geq 0.$$

This is a quadratic polynomial in t which must be positive for all $t \in \mathbb{R}$. Which happens if and only if the quadratic polynomial does not have two distinct real roots which happens if and only if the discrimant is less than or equal to 0. In our case this is equivalent to

$$4|f(x) - f(y)|^2 - 4f(0)2\Re(f(0) - f(x-y)) \leq 0.$$

From which we see that

$$|f(x) - f(y)|^2 \leq 2f(0)\Re(f(0) - f(x-y))$$

□

An example of a positive definite function is the following

Proposition 21. *If $\mu \in M(\Gamma)$ is a positive measure on the dual group we can define*

$$f(x) = \int_\Gamma \gamma(x) d\mu(\gamma)$$

then $f(x)$ is positive-definite and continuous.

Proof. We have

$$\begin{aligned}\sum_{n,m=1}^{N} c_n\overline{c_m} f(x_n - x_m) &= \sum_{n,m=1}^{N} c_n\overline{c_m} \int_\Gamma \gamma(x_n - x_m) d\mu(\gamma) \\ &= \int_\Gamma \sum_{n,m=1}^{N} c_n\overline{c_m} \gamma(x_n)\overline{\gamma(x_m)} d\mu(\gamma) \\ &= \int_\Gamma |\sum_{n=1}^{N} c_n \gamma(x_n)|^2 d\mu(\gamma) \geq 0.\end{aligned}$$

Hence f is positive definite. The function f is also continuous since if we pick $\epsilon > 0$ then by inner regularity of μ there exists a compact $K \subseteq \Gamma$ such that

$$\mu(\Gamma) - \frac{\epsilon}{4} < \mu(K) \leq \mu(\Gamma)$$

since μ is assumed to be real. It follows that

$$\mu(\Gamma - K) < \frac{\epsilon}{4}.$$

An easy computation then gives

$$|f(x) - f(x_0)| \leq \max_{\gamma \in K} |\gamma(x - x_0) - 1|\mu(K) + \frac{\epsilon}{2}. \tag{1}$$

By continuity of $\gamma(x)$ on $\Gamma \times G$ there exists for each γ_0 in K an open set V_{γ_0} of Γ containing γ_0 and an open set W_{γ_0} of G containing $x - x_0$ such that $\gamma \in V_{\gamma_0}$ and $y \in W_{\gamma_0}$ implies that

$$|\gamma(y) - 1| < \frac{\epsilon}{2\mu(K)}.$$

The union $\bigcup_{\gamma \in K} V_\gamma$ is an open cover of the compact set K hence K has the finite subcover $V_{\gamma_1} \cup ... \cup V_{\gamma_m}$. The set $W = W_{\gamma_1} \cap ... \cap W_{\gamma_m}$ is a non-empty open subset of G and if x is in $W + x_0$ then (1) shows us that $|f(x) - f(x_0)| < \epsilon$ hence f is continuous. ☐

5.2 Bochner's Theorem

We have just shown that any function of the form $f(x) = \int_\Gamma \gamma(x) d\mu(\gamma)$ with a non-negative μ in $M(G)$ is continuous and positive-definite. The content of Bochner's theorem is that the converse is also true, any continuous positive definite function is of that form.

Theorem 7. *Bochner's Theorem. The function f is a continuous positive-definite function if and only if there exists a unique postive measure μ in $M(\Gamma)$ such that*

$$f(x) = \int_\Gamma \gamma(x) d\mu(\gamma) \tag{1}$$

Proof. The easy part, that any function of the form (1) is continuous and positive-definite was shown above. Assume that $f : G \to \mathbb{C}$ is a continuous and positive-definite function. From propostion 10 we see that we can without loss of generality assume that $f(0) = 1$. The idea of the proof is to construct a bounded linear functional Λ on $C_0(\Gamma)$ which can then be rewritten as an integral with respect to a complex measure with the Riesz representation theorem. This measure will be the measure in (1). To do that we first need to define the functional T on $L^1(G)$ given by

$$T(g) = \int_G g(x)f(x)dx.$$

Since $|f(x)| \leq f(0)$ for all x in G the functional is well defined. The goal with the remainder of the proof is to show that T can be used to define a linear functional $C_0(\Gamma)$ which we will call Λ. We begin by proving the inequality

$$|T(g)|^2 \leq T(g * \tilde{g}) \tag{2}$$

where $\tilde{g}(x) = \overline{g(-x)}$. If we define for g and h in $L^1(G)$ the function

$$[g, h] = T(g * \tilde{h}).$$

Using Fubini's theorem (see Theorem 8.8 in [1]) we see that

$$\begin{aligned}
[g, h] &= \int_G (g * \tilde{h})(x)f(x)dx = \int_G (\tilde{h} * g)(x)f(x)dx \\
&= \int_G \int_G \overline{h(-(x-y))}g(y)f(x)dydx \\
&= \int_G \int_G \overline{h(y)}g(y+x)f(x)dydx \\
&= \int_G \int_G g(y+x)\overline{h(y)}f(x)dxdy \\
&= \int_G \int_G g(x)\overline{h(y)}f(x-y)dxdy.
\end{aligned}$$

From this expression we see that $[g, h]$ is linear in g and $[g, h] = \overline{[h, g]}$. It follows that

$$[g, \alpha h_1 + \beta h_2] = \overline{\alpha}[g, h_1] + \overline{\beta}[g, h_2]$$

whenever α, β are complex. Assume for the moment that

$$[g, g] \geq 0$$

for any g in $L^1(G)$. Then we can show that

$$|[g, h]|^2 \leq [g, g][h, h] \tag{3}$$

holds for any g and h in $L^1(G)$. Since if $[g,h] \neq 0$ we can let $\alpha = \frac{[g,h]}{|[g,h]|}$ then $|\alpha = 1|$ and for any real r we have

$$\begin{aligned}0 &\leq [g - r\alpha h, g - r\alpha h] \\ &= [g, g - r\alpha h] - r\alpha[h, g - r\alpha h] \\ &= [g,g] - r\overline{\alpha}[g,h] - r\alpha[h,g] + r^2|\alpha|^2[h,h] \\ &= [g,g] - 2r|[g,h]| + r^2[h,h].\end{aligned}$$

This is a positive quadratic in r. We know by assumption that $[h,h] \geq 0$ and if $[h,h] = 0$ we must have $[g,h] = 0$ since otherwise the quadratic is negative for large enough r. If $[h,h] > 0$ then we can let $r = \frac{|[g,h]|}{[h,h]}$ and obtain

$$0 \leq [g,g] - 2\frac{|[g,h]|^2}{[h,h]} + \frac{|[g,h]|^2}{[h,h]}.$$

Hence

$$|[g,h]|^2 \leq [g,g][h,h].$$

We now show that $[g,g] \geq 0$ which amounts to showing that

$$\int_G \int_G g(x)\overline{g(y)} f(x-y) dx dy \geq 0$$

for any $g \in L^1(G)$. Pick a g in $C_c(G)$ with support in the compact set K and note that the continuous function $\phi(x,y) = g(x)\overline{g}(y)f(x-y)$ is uniformly continuous on $K \times K$ since $K \times K$ is compact. By uniform continuity there exists an open set $V \subseteq K \times K$ containing the identity such that $(x_1,y_1) - (x_2,y_2) \in V$ implies $|\phi(x_1,y_1) - \phi(x_2,y_2)| < \epsilon$. By using the basis of the product topology there exists B_1 and B_2 open in G such that $0 \in B_1 \times B_2 \subseteq V$. Let $B = B_1 \cap B_2$ then $0 \in B \times B \subseteq V$ hence

$$|\phi(x_1,y_1) - \phi(x_2,y_2)| < \epsilon$$

whenever $(x_1,y_1) - (x_2,y_2) \in B \times B$. Since B is open and

$$K \subseteq \bigcup_{x \in K} (x + B)$$

is an open cover of K it follows that there exists points $x_1, ..., x_N$ such that

$$K \subseteq \bigcup_{n=1}^{N} (x_n + B)$$

Let $E_1 = K \cap (x_n + B)$ and define for $1 < n \leq N$

$$E_n = ((x_n + B) - E_1) \cap K.$$

It is then clear that the sets $(E_n)_n$ are mutually disjoint and

$$K = \bigcup_{n=1}^{N} E_n$$

Furthermore $(y_n, y_m) \in E_n \times E_m$ implies that

$$|\phi(y_n, y_m) - \phi(x_n, x_m)| < \epsilon.$$

Since sets of the form E_n are mutually disjoint and cover K it follows that

$$m(K)^2 = \sum_{n,m=1}^{N} m(E_n)m(E_m).$$

Using these properties we get

$$\begin{aligned}
&|\int_G \int_G \phi(x,y)dxdy - \sum_{n,m=1}^{N} m(E_n)m(E_m)\phi(x_n, x_m)| \\
&= |\int_K \int_K \phi(x,y)dxdy - \sum_{n,m=1}^{N} \int_{E_n} \int_{E_m} \phi(x_n, x_m)dxdy| \\
&= |\sum_{n,m=1}^{N} \int_{E_n} \int_{E_m} \phi(x,y)dxdy - \sum_{n,m=1}^{N} \int_{E_m} \int_{E_n} \phi(x_n, x_m)dxdy| \\
&\leq \sum_{n,m=1}^{N} \int_{E_n} \int_{E_m} |\phi(x,y) - \phi(x_n, x_m)|dxdy \\
&< \sum_{n,m=1}^{N} \int_{E_n} \int_{E_n} \epsilon dxdy \\
&= \sum_{n,m=1}^{N} m(E_n)m(E_m)\epsilon = \epsilon m(K)^2.
\end{aligned}$$

So the difference betweem the double integral of $\phi(x,y)$ and the sum

$$\sum_{n,m=1}^{N} m(E_n)m(E_m)\phi(x_n, x_m) = \sum_{n,m=1}^{N} m(E_n)g(x_n)\overline{m(E_m)g(x_m)}f(x_n - x_m) \tag{4}$$

can be made arbitrary small. Since f is positive-definite it follows that the sum in (4) is always positive and therefore the integral is always positive,

$$\int_G \int_G g(x)\overline{g}(y)f(x-y)dxdy \geq 0 \tag{5}$$

if $g \in C_c(G)$. Assume now that g is in $L^1(G)$. Since $C_c(G)$ is dense in $L^1(G)$ we can find a sequence of functions (g_n) in $C_c(G)$ converging to g in $L^1(G)$. This implies that

$$\lim_{n\to+\infty}\int_G\int_G g_n(x)\overline{g_n(y)}f(x-y)dxdy = \int_G\int_G g(x)\overline{g(y)}f(x-y)dxdy. \tag{6}$$

Since

$$\begin{aligned}
&|\lim_{n\to+\infty}\int_G\int_G g_n(x)\overline{g_n(y)}f(x-y)dxdy - \int_G\int_G g(x)\overline{g(y)}f(x-y)dxdy| \\
&\leq \lim_{n\to+\infty}\int_G\int_G |g_n(x)\overline{g_n(y)} - g(x)\overline{g(y)}||f(x-y)|dxdy \\
&\leq \lim_{n\to+\infty} f(0)\int_G\int_G |g_n(x)\overline{g_n(y)} - g(x)\overline{g_n(y)} + g(x)\overline{g_n(y)} - g(x)\overline{g(y)}|dxdy \\
&\leq \lim_{n\to+\infty} f(0)(||g_n-g||_1||g_n||_1 + ||g||_1||g_n-g||_1) = 0.
\end{aligned}$$

From (5) and (6) it follows that

$$[g,g] = \int_G\int_G g(x)\overline{g(y)}f(x-y)dxdy \geq 0$$

for any g in $L^1(G)$.

Pick $\epsilon > 0$. By uniform continuity of f there exists an open set V of 0 such that $x-y \in V$ implies that $|f(x)-f(y)| < \epsilon$. By continuity of multiplication m and definition of the product topology there exists an open set $B_1 \times B_2$ of $G \times G$ which contains $(0,0)$, let $U = B_1 \cap B_2$ then U is an open subset of G which contains 0 such that $U + U \subseteq V$. After considering $W = U \cap (-U)$ we can assume that $U = -U$. Now let $h = \frac{\chi_U}{m(U)}$ then

$$\begin{aligned}
[h,h] - 1 &= \int_G\int_G h(x)\overline{h(y)}f(x-y)dxdy - \frac{1}{m(U)^2}\int_U\int_U dxdy \\
&= \frac{1}{m(U)^2}\int_U\int_U (f(x-y)-f(0))dxdy < \epsilon.
\end{aligned}$$

Therefore

$$[h,h] - 1 < \epsilon. \tag{7}$$

Also, for any g in $L^1(G)$

$$\begin{aligned}
[g,h] - T(g) &= \int_G\int_G g(x)\overline{h(y)}f(x-y)dxdy - \int_G g(x)f(x)dx \\
&= \int_G \frac{g(x)}{m(U)}\int_U f(x-y)dydx - \int_G \frac{g(x)}{m(U)}\int_U f(x)dydx \\
&= \int_G \frac{g(x)}{m(U)}\int_U (f(x-y)-f(x))dydx.
\end{aligned}$$

If $y \in U$ then $x - y - x = -y \in U \subseteq V$ hence

$$|[g, h] - T(g)| < \epsilon||g||_1.$$

It follows by the reverse triangle inequality that

$$|T(g)| < |[g, h]| + \epsilon||g||_1. \tag{8}$$

Using (3), (7) and (8) we get

$$\begin{aligned}
|T(g)|^2 &< |[g, h]|^2 + 2|[g, h]|\epsilon||g||_1 + \epsilon^2||g||_1^2 \\
&\leq [g, g][h, h] + 2\sqrt{[g, g][h, h]}\epsilon||g||_1 + \epsilon^2||g||_1^2 \\
&< [g, g](1 + \epsilon) + 2\sqrt{[g, g](1 + \epsilon)}\epsilon||g||_1 + \epsilon^2||g||_1^2 \\
&= [g, g] + \epsilon([g, g] + 2\sqrt{[g, g](1 + \epsilon)}||g||_1 + \epsilon||g||_1^2).
\end{aligned}$$

Letting $\epsilon \to 0$ we see

$$|T(g)|^2 \leq [g, g] = T(g * \tilde{g})$$

which gives us (2). We now have what we need to show T can be used to define a bounded linear functional Λ on $C_0(\Gamma)$.

Define $k = g * \tilde{g}$ and $k^n = k^{n-1} * k$ for any integer $n > 1$. By commutativity of convolution we get that $k^n = k * k^{n-1}$. Using Corollary 1 we see that for any pair of functions X, Y in $L^1(G)$ we have

$$\begin{aligned}
(\widetilde{X * Y})(x) &= \overline{\int_G X(-x - y)Y(y)dy} = \int_G \overline{X(-x - y)Y(y)}dy \\
&= \int_G \overline{X_x(-y)Y(y)}dy = \int_G \overline{X_x(y)Y(-y)}dy \\
&= \int_G \overline{X(y - x)Y(-y)}dy = \int_G \tilde{X}(x - y)\tilde{Y}(y)dy \\
&= (\tilde{X} * \tilde{Y})(x).
\end{aligned}$$

Applying this to the case $X = g$ and $Y = \tilde{g}$ we get

$$k * \tilde{k} = (g * \tilde{g}) * (\tilde{g} * g) = (g * \tilde{g}) * (g * \tilde{g}) = k * k = k^2$$

By induction on n and associativity of convolution we have

$$\begin{aligned}
k^n * \widetilde{(k^n)} &= (k * k^{n-1}) * (\widetilde{k^{n-1} * k}) \\
&= k * k^{n-1} * \widetilde{(k^{n-1})} * \tilde{k} = k * k^{2(n-1)} * \tilde{k} \\
&= k^2 * k^{2(n-1)} = k^{2n}
\end{aligned}$$

hence

$$k^n * (\tilde{k^n}) = k^{2n}. \tag{9}$$

By combining (9) and (2) repeatedly we get the expression

$$\begin{aligned}|T(g)|^2 &\leq T(g*\tilde{g}) = T(k)\\ &\leq T(k*\tilde{k})^{1/2} = T(k^2)^{1/2}\\ &\leq T(k^2*\tilde{k}^2)^{1/4} = T(k^4)^{1/4}\\ &\leq ... \leq T(k^{2^n})^{2^{-n}}\end{aligned}$$

for any positive integer n. Since $||f||_\infty = 1$ it follows that $||T|| \leq 1$ hence $|T(k^{2^n})| \leq ||k^{2^n}||_1$ and therefore

$$|T(g)|^2 \leq T(k^{2^n})^{2^{-n}} \leq ||k^{2^n}||_1^{2^{-n}}.$$

Recall that $L^1(G)$ is a sub-Banach-Algebra of the unital Banach Algebra $M(G)$ hence if we let $n \to +\infty$ then

$$|T(g)|^2 \leq ||k^{2^n}||_1^{2^{-n}} \to \rho(k) \tag{10}$$

by the spectral radius formula, see Theorem 18.9 in [1] where

$$\rho(k) = \sup\{|\lambda|; k - \lambda e \text{ is not invertible}\}$$

and $e = \delta_0$ is the unit element in $M(G)$. Note that if λ_0 is such that $k - \lambda_0 e$ is not invertible then by Theorem 18.17 b in [1] there exists a complex linear functonal h_0 defined on $M(G)$ which is not 0 everywhere, $h_0(xy) = h_0(x)h_0(y)$ for any $x, y \in M(G)$ and $h_0(k - \lambda_0 e) = 0$. It follows that $h_0(k) = \lambda_0 h_0(e)$ and since $h_0 \neq 0$ it follows that $h_0(k) = \lambda_0$. Let $\triangle$ be the set of all complex linear functionals defined on $M(G)$ which are also multiplicative, then it follows that

$$\sup_{h\in\triangle} |h(k)| \geq |h_0(k)| = \lambda_0.$$

Since λ_0 was an arbitrary element of the set of all elements $\lambda \in \mathbb{C}$ such that $k - \lambda e$ is not invertible it follows that

$$\begin{aligned}\rho(k) &\leq \sup_{h\in\triangle} |h(k)| \leq \sup_{h\in\triangle} |h(g*\tilde{g})|\\ &\leq \sup_{h\in\triangle} |h(g^2)| \leq (\sup_{h\in\triangle} |h(g)|)^2\end{aligned}$$

and combining this with (10) we get

$$|T(g)| \leq \sup_{h\in\triangle} |h(g)|. \tag{11}$$

We know that the supremum is not achieved at $h = 0$. If we let $\triangle' - \{0\}$ be the set of all function restrictions to $L^1(G) \subseteq M(G)$ of elements in $\triangle - \{0\}$ then $\triangle' - \{0\}$ is in correspondence with Γ by Proposition 15 hence

$$|T(g)| \leq \sup_{h\in\triangle} |h(g)| = \sup_{\gamma\in\Gamma} |\hat{g}(\gamma)| = ||\hat{g}||_\infty. \tag{12}$$

If $\hat{g}$ is in $A(\Gamma)$ (the set of all Fourier transforms) then we can define $\Lambda : A(\Gamma) \to \mathbb{C}$ by

$$\Lambda(\hat{g}) = T(g). \tag{13}$$

If $\hat{g}_1 = \hat{g}_2$ then $||\hat{g}_1 - \hat{g}_2||_\infty = 0$ and therefore (12) shows us that $T(\hat{g}_1) = T(\hat{g}_2)$ so it follows that Λ is well defined. The expression (12) also shows us that Λ is a bounded linear function of norm $||\Lambda|| \leq 1$. Since $A(\Gamma)$ is a subspace of the Banach space $C_0(\Gamma)$ the Hahn-Banach Theorem (see Theorem 5.16 in [1]) allows us to extend Λ to a bounded linear functional on $C_0(\Gamma)$ of norm less than 1 (which we will also call Λ). By the Riesz Representation Theorem of bounded linear functionals (see Theorem 6.19 in [1]), there exists a unique regular complex Borel measure ν defined on Γ such that

$$\Lambda g = \int_\Gamma g(\gamma) d\nu(\gamma)$$

for any $g \in C_0(\Gamma)$ and $|\nu|(\Gamma) = ||\nu|| = ||\Lambda|| \leq 1$. If we define $\mu(E) = \nu(-E)$ (where $-E$ is the inverses of E in Γ) then it is easy to show that μ is a complex regular Borel measure such that $|\mu|(\Gamma) = |\nu|(\Gamma) \leq 1$. We also have for any Borel set E the formula

$$\begin{aligned}\int_\Gamma \chi_E(-\gamma) d\mu(\gamma) &= \int_\Gamma \chi_{-E}(\gamma) d\mu(\gamma) \\ = \mu(-E) = \nu(E) &= \int_\Gamma \chi_E(\gamma) d\nu(\gamma).\end{aligned}$$

By linearity we get

$$\int_\Gamma s(-\gamma) d\mu(\gamma) = \int_\Gamma s(\gamma) d\nu(\gamma)$$

for any simple function s in $L^1(\mu)$. Since those simple functions are dense in $L^1(\mu)$ and $C_0(\Gamma) \subseteq L^1(\mu)$ we get that

$$\int_\Gamma G(-\gamma) d\mu(\gamma) = \int_\Gamma G(\gamma) d\nu(\gamma)$$

for any G in $C_0(\Gamma)$. If g is in $L^1(G)$ it follows that

$$T(g) = \Lambda(\hat{g}) = \int_\Gamma \hat{g}(-\gamma) d\mu(\gamma)$$

and by Fubini's Theorem (see Theorem 8.8 in [1]) we have

$$\begin{aligned}\int_G g(x) f(x) dx = T(g) &= \Lambda(\hat{g}) \\ &= \int_\Gamma \hat{g}(-\gamma) d\mu(\gamma) \\ &= \int_G g(x) \int_\Gamma \gamma(x) d\mu(\gamma) dx\end{aligned}$$

and therefore

$$\int_G g(x)(f(x) - \int_\Gamma \gamma(x)d\mu(\gamma))dx = 0.$$

If we let $g = \chi_E$ where E is any Borel set of finite measure we see that

$$f(x) = \int_\Gamma \gamma(x)d\mu(\gamma) \text{ a.e on } E \tag{14}$$

Since G is locally compact any $x \in G$ has an open neighbourhood V_x with finite Haar-measure. If we restrict the left hand side and the right hand side of (14) to V_x then we know that $f|_{V_x}$ is continuous since f is continuous by assumption. The proof in proposition 21 transfers directly to arbitrary elements of $M(G)$ by using the polar decomposition $d\mu = hd|\mu|$ which shows that the right hand side (14) is continuous when restricted to V_x. It follows that

$$f(y) = \int_\Gamma \gamma(y)d\mu(\gamma) \tag{15}$$

for all $y \in V_x$. Since sets of the form V_x are an open cover of G it follows that (15) holds for all $y \in G$.

If we let $y = 0$ in (15) yields

$$1 = f(0) = \int_\Gamma d\mu = \mu(\Gamma) \leq |\mu|(\Gamma) \leq 1.$$

It follows that $||\mu - |\mu||| = 0$ hence $\mu = |\mu|$ and therefore $\mu \geq 0$. □

5.3 The Fourier Inversion Theorem

Definition 15. *Let $B(G)$ be the set of all functions $f : G \to \mathbb{C}$ such that there exists a measure $\mu \in M(G)$ that satisfies*

$$f(x) = \int_\Gamma \gamma(x)d\mu(\gamma)$$

for any x in G.

Remark. *If μ is in $M(G)$ it can clearly be written on the form $\mu = \mu_1 + i\mu_2$ where μ_1, μ_2 are both real measures in $M(G)$. Using the Hahn Decomposition we can decompose both μ_1 and μ_2 as the difference of two positive measures in $M(G)$ respectively. Therefore we have*

$$\mu = \mu_1^+ - \mu_1^- + i\mu_2^+ - i\mu_2^-$$

and Bochner's Theorem implies that $B(G)$ is the set of all linear combinations of positive definite functions.

Theorem 8. *The Fourier Inversion Theorem. If f is in $L^1(G) \cap B(G)$ then $\hat{f}$ is in $L^1(\Gamma)$ and if the Haar measure on Γ is suitably normalized the inversion formula*

$$f(x) = \int_\Gamma \hat{f}(\gamma)\gamma(x)d\gamma$$

holds for any x in G.

Proof. If f is in $L^1(G) \cap B(G)$ then there exists a measure μ_f in $M(G)$ such that

$$f(x) = \int_\Gamma \gamma(x)d\mu_f(\gamma) \tag{1}$$

If h is any function in $L^1(G)$ then

$$(h * f)(0) = \int_G h(-x)f(x)dx = \int_G \int_\Gamma h(-x)\gamma(x)d\mu_f(\gamma)dx.$$

Fubini's theorem (see Theorem 8.8 in [1]) clearly applies since h is $L^1(G)$ and $|\int_\Gamma \gamma(x)d\mu_f(\gamma)| \leq ||\mu_f||$ so

$$\begin{aligned}(h * f)(0) &= \int_\Gamma \int_G h(-x)\gamma(x)dxd\mu_f(\gamma)\\ &= \int_\Gamma \int_G h(x)\gamma(-x)dxd\mu_f(\gamma)\\ &= \int_\Gamma \hat{h}(\gamma)d\mu_f(\gamma).\end{aligned}$$

Let g be in $L^1(G) \cap B(G)$ and associate $\mu_g \in M(G)$ to g as in (1). By using properties of convolution and letting $h*g$ take the role of h in the formula above we obtain

$$\begin{aligned}\int_\Gamma \hat{h}\hat{g}d\mu_f &= \int_\Gamma \widehat{(h * g)}d\mu_f\\ &= ((h * g) * f)(0)\\ &= ((h * f) * g)(0)\\ &= \int_\Gamma \hat{h}\hat{f}d\mu_g.\end{aligned}$$

From this it follows that complex measures ν_f and ν_g given by integrating $\hat{g}$ with respect to μ_f and integrating $\hat{f}$ with respect to μ_g respectively satisfy

$$\int_\Gamma \hat{h}d\nu_f = \int_\Gamma \hat{h}d\nu_g$$

for any h in $L^1(G)$. Define the linear functional $L : C_0(\Gamma) \to \mathbb{C}$ by

$$L(F) = \int_\Gamma F d(\nu_f - \nu_g).$$

The functional L is well defined on $C_0(\Gamma)$ since ν_f, ν_g are complex measures and hence $|\nu_f - \nu_g|(\Gamma) < +\infty$. The functional is also bounded since $|L(F)| \leq |\nu_f - \nu_g|(\Gamma)$ whenever $||F||_\infty \leq 1$. By construction $L(\hat{h}) = 0$ for any $h \in L^1(G)$, so L vanishes on the set $A(\Gamma) \subseteq C_0(\Gamma)$. Since $A(\Gamma)$ is dense in $C_0(\Gamma)$ it follows that $L(F) = 0$ for any $F \in C_0(\Gamma)$ and by the Riesz representation theorem for bounded linear functionals (see Theorem 6.19 in [1]) this can only happen if $\nu_f = \nu_g$.

We will now define a positive linear functional T on $C_c(\Gamma)$. Pick ψ in $C_c(\Gamma)$ and let K_ψ be the support of ψ. If $\gamma_0 \in K_\psi$ we see by picking v in $C_c(G)$, $||v||_1 \neq 0$ that the function given by $(v\gamma_0)(x) = v(x)\gamma_0(x)$ for $x \in G$ satisfies $\widehat{v\gamma_0}(\gamma_0) \neq 0$. So for any $\gamma_0 \in K_\psi$ there exists a function $u \in C_c(G)$ such that $\hat{u}(\gamma_0) \neq 0$. By defining $\widetilde{u}(x) = \overline{u(-x)}$ it follows that $\widehat{\widetilde{u}} = \overline{\hat{u}}$ hence the Fourier transform of $u * \tilde{u}$ is nowhere negative and strictly positive at γ_0.

For any γ_0 in K_ψ we have for the corresponding function $u \in C_c(G)$ that the set

$$M_{\gamma_0} = \{\gamma \in \Gamma; |\widehat{u * \widetilde{u}}| > 0\}$$

is open and therefore the family of sets $(M_{\gamma_0})_{\gamma_0 \in K_\psi}$ is an open cover of K_ψ. By compactness of K_ψ there exists functions $u_1, ..., u_n$ in $C_c(G)$ such that the continuous function

$$g = \sum_{i=1}^{n} u_i * \tilde{u_i}$$

satisfies that $\hat{g} > 0$ on K_ψ. By the definition of a positive definite function it is clear that any finite sum of positive definite functions is positive definite so g is positive definite if each $u_i * \widetilde{u}_i$ is. But $u_i * \widetilde{u}_i$ is positive definite for each i since

$$\begin{aligned}
\sum_{j,k=1}^{N} c_j \overline{c_k} (u_i * \tilde{u_i})(x_j - x_k) &= \sum_{j,k=1}^{N} c_j \overline{c_k} \int_G u_i(x_j - x_k - y)\overline{u_i(-y)} dy \\
&= \sum_{j,k=1}^{N} c_j \overline{c_k} \int_G u_i(x_j - y)\overline{u_i(x_k - y)} dy \\
&= \int_G \sum_{j,k=1}^{N} c_j \overline{c_k} u_i(x_j - y)\overline{u_i}(x_k - y) dy \\
&= \int_G |\sum_{j=1}^{N} c_j u_i(x_j - y)|^2 dy \geq 0.
\end{aligned}$$

By Bochner's theorem there exists a positive measure μ_g in $M(\Gamma)$ such that

$$g(x) = \int_\Gamma \gamma(x) d\mu_g.$$

We can now define the functional $T : C_c(\Gamma) \to \mathbb{C}$ by

$$T(\psi) = \int_{K_\psi} \frac{\psi}{\hat{g}} d\mu_g.$$

Here of course g depends implicitly on ψ by its construction. The functional T is well defined in the sense that g can be replaced with any other positive definite function $f \in C_c(G)$ such that $\hat{f}$ is strictly positive on K_ψ in the defining equation for $T(\psi)$ without changing the value of $T(\psi)$. This is true since

$$\begin{aligned}
\int_{K_\psi} \frac{\psi}{\hat{g}} d\mu_g &= \int_{K_\psi} \frac{\psi}{\hat{g}\hat{f}} \hat{f} d\mu_g = \int_{K_\psi} \frac{\psi}{\hat{g}\hat{f}} d\nu_g \\
&= \int_{K_\psi} \frac{\psi}{\hat{g}\hat{f}} d\nu_f = \int_{K_\psi} \frac{\psi}{\hat{g}\hat{f}} \hat{g} d\mu_f \\
&= \int_{K_\psi} \frac{\psi}{\hat{f}} d\mu_f.
\end{aligned}$$

From this it is also clear that T is linear. Also, since $\hat{g} > 0$ on the support of ψ and μ_g is a positive measure it follows that $T(\psi) \geq 0$ whenever $\psi \geq 0$, that is, T is a positive linear functional.

Notice that for the function g defined above we have that $\mu_g(\Gamma - K_{\hat{g}}) = 0$. If this was not the case then there exists by inner regularity of μ_g a compact set $K \subseteq \Gamma - K_{\hat{g}}$ such that $0 < \mu_g(K) \leq \mu_f(\Gamma - K_{\hat{g}})$. Construct a positive definite function f such that $\hat{f} > 0$ on K then since $\nu_g = \nu_f$ it follows that

$$\int_K \hat{f} d\mu_g = \int_K \hat{g} d\mu_f = 0.$$

Since $\hat{f} > 0$ on K it follows that $\mu_g(K) = 0$, a contradiction. Hence $\hat{g} \neq 0$ a.e. with respect to μ_g and it follows that we can instead equivalently define T by integrating over Γ instead of K_ψ:

$$T(\psi) = \int_\Gamma \frac{\psi}{\hat{g}} d\mu_g.$$

Pick a function ϕ in $C_c(\Gamma)$ that is non-negative and not the zero function and, as before pick a positive definite function f such that $\hat{f} > 0$ on K_ϕ. Then since the measure μ_f is positive by Bochners theorem we get

$$T(\phi\hat{f}) = \int_\Gamma \frac{\phi\hat{f}}{\hat{g}} d\mu_g = \int_\Gamma \phi d\mu_f > 0$$

so T is not the zero functional.

We now prove that T is translation-invariant. Pick $\psi \in C_c(\Gamma)$, $\gamma_0 \in \Gamma$ and set $\psi_0(\gamma) = \psi(\gamma\gamma_0^{-1})$. Contruct the g corresponding to ψ such that $\hat{g}$ is also strictly positive on the support of ψ_0. Define f by $f(x) = \gamma_0(-x)g(x)$, then it follows that $\hat{f}(\gamma) = \hat{g}(\gamma\gamma_0)$ and f is continuous. Associate μ_g to g with Bochners theorem,

$$g(x) = \int_\Gamma \gamma(x) d\mu_g(\gamma).$$

Then we have that

$$f(x) = \gamma_0(-x)g(x) = \int_\Gamma (\overline{\gamma_0}\gamma)(x) d\mu_g(\gamma)$$

By defining $\mu(E) = \mu_g(\gamma_0 E)$ on any Borel set $E \subseteq \Gamma$ it follows that $\mu \in M(\Gamma)$ and $\mu \geq 0$ since μ_g satisfies the same properties. By definition of μ we also have

$$\int_\Gamma \chi_E(\gamma) d\mu(\gamma) = \int_\Gamma \chi_{\gamma_0 E}(\gamma) d\mu_g(\gamma) = \int_\Gamma \chi_E(\overline{\gamma_0}\gamma) d\mu_g(\gamma)$$

for any Borel set E. By a standard approximation argument of simple functions it follows that

$$\int_\Gamma F(\overline{\gamma_0}\gamma) d\mu_g(\gamma) = \int_\Gamma F(\gamma) d\mu(\gamma)$$

for any $F \in L^1(\mu_g)$. In particular, by letting $F_x(\gamma) = \gamma(x)$ for any fixed $x \in G$ it follows that

$$f(x) = \int_\Gamma (\overline{\gamma_0}\gamma)(x) d\mu_g(\gamma) = \int_\Gamma \gamma(x) d\mu(\gamma).$$

From Böchners theorem it follows that f is positive definite and its corresponding measure μ_f is given by $\mu_f(E) = \mu(E) = \mu_g(\gamma_0 E)$. By letting

$$F(\gamma) = \frac{\psi(\gamma)}{(\hat{g})_{\overline{\gamma_0}}(\gamma)}$$

where we recall that $(\hat{g})_{\overline{\gamma_0}}(\gamma) = \hat{g}(\gamma_0\gamma)$ it follows that

$$\begin{aligned}
T(\psi_0) &= \int_\Gamma \frac{\psi_0(\gamma)}{\hat{g}(\gamma)} d\mu_g(\gamma) = \int_\Gamma \frac{\psi(\overline{\gamma_0}\gamma)}{\hat{g}(\gamma)} d\mu_g(\gamma) \\
&= \int_\Gamma \frac{\psi(\overline{\gamma_0}\gamma)}{\hat{g}(\gamma_0\overline{\gamma_0}\gamma)} d\mu_g(\gamma) = \int_\Gamma F(\overline{\gamma_0}\gamma) d\mu_g(\gamma) = \\
&= \int_\Gamma F(\gamma) d\mu_f(\gamma) = \int_\Gamma \frac{\psi(\gamma)}{\hat{g}(\gamma_0\gamma)} d\mu_f(\gamma) = \\
&= \int_\Gamma \frac{\psi(\gamma)}{\hat{f}(\gamma)} d\mu_f(\gamma) = T(\psi).
\end{aligned}$$

Hence $T : C_c(\Gamma) \to \mathbb{C}$ is a positive translation-invariant linear functional. By the Riesz representation theorem for positive linear functionals on $C_c(\Gamma)$, (see Theorem 2.14 in [1]) there corresponds a regular measure m on Γ such that

$$T(\psi) = \int_\Gamma \psi dm$$

Since T is translation invariant we see from the proof of the construction of the Haar measure, Theorem 1 that m is in fact a Haar-measure on Γ and we will denote it by $dm(\gamma) = d\gamma$. If $\psi \in C_c(\Gamma)$, $f \in L^1(G) \cap B(G)$ and g is positive definite $L^1(G)$-function such that $\hat{g} > 0$ on the support of ψ then

$$\begin{aligned}\int_\Gamma \psi \hat{f} d\gamma = T(\psi \hat{f}) &= \int_\Gamma \frac{\psi \hat{f}}{\hat{g}} d\mu_g \\ &= \int_\Gamma \frac{\psi}{\hat{g}} \hat{g} d\mu_f = \int_\Gamma \psi d\mu_f.\end{aligned}$$

Since above is true for any $\psi \in C_c(\Gamma)$ it follows that

$$\mu_f(E) = \int_E \hat{f} d\gamma.$$

Since μ_f is a finite measure it follows that $\hat{f} \in L^1(\Gamma)$ and since $f \in B(G)$ we have

$$f(x) = \int_\Gamma \gamma(x) d\mu_f(\gamma) = \int_\Gamma \gamma(x) \hat{f}(\gamma) d\gamma$$

and the theorem follows. □

Proposition 22. *If H is another topological group and the function $\Phi : G \to H$ is a group isomorphism and a homeomorphism then*

$$\int_G f(x) dm(x) = \int_H f(\Phi^{-1}(y)) dm_H(y)$$

for any $f \in L^1(G)$ where m is the Haar-measure on G and m_H is defined by $m_H(E) = m(\Phi^{-1}(E))$ for any Borel set $E \subseteq H$. Note that by Proposition 9 m_H is a Haar-measure.

Proof. Let f be in $L^1(G)$. After decomposing f into its real positive-, real negative- , imaginary positive- and imaginary negative part we can without loss of generality assume that f is positive. Since f is positive there exists a sequence of simple measurable functions $(s_n)_n$ converging to f pointwise such that $s_n(x) \leq s_{n+1}(x) \leq f(x)$ for each x and n (see Theorem 1.17 in [1]). If

$s = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$ is any simple function defined on G then

$$\begin{aligned}
\int_H s(\Phi^{-1}(y))dm_H(y) = \sum_{n=1}^{N} \alpha_n \int_H \chi_{E_n}(\Phi^{-1}(y))dm_H(y) &= \sum_{n=1}^{N} \alpha_n \int_H \chi_{\Phi(E_n)}(y)dm_H(y) \\
&= \sum_{n=1}^{N} \alpha_n m_H(\Phi(E_n)) = \sum_{n=1}^{N} \alpha_n m(\Phi^{-1}(\Phi(E_n))) \\
&= \sum_{n=1}^{N} \alpha_n m(E_n) = \int_G s(x)dm(x)
\end{aligned}$$

So the proposition holds for any simple function. After using the monotone convergence theorem twice below the theorem follows

$$\begin{aligned}
\int_H f(\Phi^{-1}(y))dm_H(y) &= \lim_{n\to+\infty} \int_H s_n(\Phi^{-1}(y))dm_H(y) \\
= \lim_{n\to+\infty} \int_G s_n(x)dm(x) &= \int_G f(x)dm(x).
\end{aligned}$$

Note that the composition $s_n \circ \Phi^{-1}$ is a simple function for each n. $\square$

6 Plancherel's Theorem

In this section we will prove Plancherels Theorem and some important Corollaries. Plancherels Theorem will allow us to define the so called Plancherel transform. Less detailed proofs can be found in Section 1.6 in [3].

6.1 Plancherel's Theorem

Theorem 9. *Plancherel's Theorem. The Fourier transform restricted to $L^1(G) \cap L^2(G)$ is an isometry with respect to L^2-norms onto a dense linear subspace of $L^2(\Gamma)$. Therefore it may be extended uniquely to an isometry of $L^2(G)$ onto $L^2(\Gamma)$. This extension is often referred to as the Plancherel transform.*

Proof. Assume that $f \in L^1(G) \cap L^2(G)$ and define $\tilde{f}(x) = \overline{f(-x)}$ and set $g = f * \tilde{f}$. Then we have that $g \in L^1(G)$ and that g is continuous by Proposition 14. We also know as shown in the inversion Theorem proof that g is positive definite. After applying the Fourier inversion theorem to g and the relation $|\hat{g}| = |\hat{f}|^2$ we get the following relation

$$\int_G |f(x)|^2 dx = \int_G f(x)\tilde{f}(0-x) = g(0) = \int_\Gamma \hat{g}(\gamma)\gamma(0)d\gamma = \int_\Gamma |\hat{f}|^2 d\gamma.$$

It follows that $||f||_2 = ||\hat{f}||_2$. Hence the Fourier transform is an L^2-norm isometry when restricted to $L^1(G) \cap L^2(G)$. If we let Θ be the set of all Fourier transforms of functions in $L^1(G) \cap L^2(G)$ then we need to show that Θ is dense in $L^2(\Gamma)$. Since the Fourier transform is an L^2-isometry of $L^1(G) \cap L^2(G)$ into $L^2(\Gamma)$ it is clear that $\Theta \subseteq L^2(\Gamma)$. Since the Fourier transform of a translate of a function is

$$\hat{f}_{x_0}(\gamma) = \gamma(-x_0)\hat{f}(\gamma)$$

it follows that $H(\gamma) = F(\gamma)\gamma(x_0)$ is in Θ whenever $F \in \Theta$ for any $x_0 \in G$. Hence if $\psi \in L^2(\Gamma)$ and

$$(\phi, \psi) = \int_\Gamma \phi\overline{\psi} d\gamma = 0$$

for any ϕ in Θ then

$$\int_\Gamma \phi(\gamma)\gamma(x)\overline{\psi(\gamma)}d\gamma = 0$$

for any $x \in G$. By defining the measure η by $d\eta = \phi(\gamma)\overline{\psi(\gamma)}d\gamma$ we get from above that

$$\int_\Gamma \gamma(x)d\eta(\gamma) = 0$$

for any $x \in G$. By Proposition 19 it follows that $\eta = 0$ since $\eta \in M(\Gamma)$. It follows that

$$\phi\overline{\psi} = 0 \text{ a.e.}$$

for any ϕ in Θ. Note that Θ is translation invariant since the Fourier transform for the function $h(x) = \delta(x)f(x)$, $\delta \in \Gamma$ is equal to $\widehat{h}(\gamma) = \hat{f}(\gamma\delta^{-1})$.

Now pick a $\gamma_0 \in \Gamma$. Since there exists a Fourier transform $\hat{F} : \Gamma \to \mathbb{C}$ which is strictly positive at $1 \in \Gamma$, the identity in Γ it follows that the function $\hat{F}$ translated by γ_0^{-1}, $\hat{F}_{\gamma_0^{-1}}$ is in Θ and $|\hat{F}_{\gamma_0^{-1}}(\gamma_0)| > 0$. By continuity of $\hat{F}_{\gamma_0^{-1}}$ there exists a neighbourhood around γ_0 such that $\hat{F}_{\gamma_0^{-1}}$ is strictly positive in this neighbourhood. By letting $\phi = \hat{F}_{\gamma_0^{-1}}$ in

$$\phi\overline{\psi} = 0 \text{ a.e.}$$

it follows that $\overline{\psi} = 0$ almost everywhere inside that neighbourhood. Since γ_0 was we see that

$$\psi = 0 \text{ a.e.}$$

It follows that the only element in $L^2(\Gamma)$ which is orthogonal to every element of Θ is 0. Any $f \in L^2(\Gamma)$ can be decomposed as $f = Pf + Qf$ where $Pf \in \overline{\Theta}$ and $(g, Qf) = 0$ for any $g \in \overline{\Theta}$ see (Theorem 4.11 in [1]). If $(g, Qf) = 0$ for any $g \in \overline{\Theta}$ then in particular $(g, Qf) = 0$ for any $g \in \Theta$ hence $Qf = 0$ and it follows that $f = Pf \in \overline{\Theta}$ so Θ is dense in $L^2(\Gamma)$.

We have that the Fourier transform is an L^2-norm isometry of $L^2(G) \cap L^1(G)$ onto a dense linear subspace of $L^2(\Gamma)$. We will now extend this to an L^2-norm isometry of $L^2(G)$ onto $L^2(\Gamma)$, the so called Plancherel transform. Since $C_c(G) \subseteq L^1(G) \cap L^2(G)$ and $C_c(G)$ is dense in $L^2(G)$ it follows that $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$. If f is in $L^2(G)$ we can take a sequence f_n in $L^1(G) \cap L^2(G)$ such that $f_n \to f$ in $L^2(G)$ and define the Plancherel transform of f, $\hat{f}$ as

$$\hat{f} = \lim_{n \to +\infty} \hat{f}_n$$

where the $\hat{f}_n$ on the right side denotes the regular Fourier transform and the limit is in $L^2(\Gamma)$. This is well-defined since if g_n is another such sequence converging to f we have by the isometry of the Fourier transform

$$\begin{aligned} \lim_{n \to +\infty} ||\hat{f}_n - \hat{g}_n||_2 &= \lim_{n \to +\infty} ||\widehat{f_n - g_n}||_2 \\ &= \lim_{n \to +\infty} ||f_n - g_n||_2 = 0. \end{aligned}$$

Hence $\lim_{n \to +\infty} \hat{f}_n - \hat{g}_n = 0$ in $L^2(\Gamma)$ and it follows that

$$\lim_{n \to +\infty} \hat{f}_n = \lim_{n \to +\infty} \hat{f}_n - \hat{g}_n + \hat{g}_n = \lim_{n \to +\infty} \hat{g}_n.$$

The Plancherel transform clearly coincides with the Fourier transform when restricted to $L^1(G) \cap L^2(G)$ and by the reverse triangle inequality we see that it is an isometry since

$$||\hat{f}||_2 = \lim_{n\to+\infty} ||\hat{f}_n||_2 = \lim_{n\to+\infty} ||f_n||_2 = ||f||_2.$$

It remains to show that the Plancherel transform maps onto $L^2(\Gamma)$. Choose a function $F \in L^2(\Gamma)$. Since Θ is dense in $L^2(\Gamma)$ there exists a sequence F_n in Θ such that $F_n \to F$ in $L^2(\Gamma)$ and by definition of Θ there exists a sequence of functions f_n in $L^1(G) \cap L^2(G)$ such that $\hat{f}_n = F_n$. Using the isometry property and the triangle inequality we get

$$||f_n - f_m||_2 \leq ||F - F_m||_2 + ||F_n - F||_2.$$

thus f_n is a Cauchy sequence in $L^2(G)$. By completeness of $L^2(G)$ the sequence f_n converges to a function $f \in L^2(G)$ and by definition of the Plancherel transform we get

$$\hat{f} = \lim_{n\to+\infty} \hat{f}_n = \lim_{n\to+\infty} F_n = F.$$

□

6.2 Parseval's Formula and other Corollaries

Corollary 2. *Parseval's formula. If $f, g \in L^2(G)$ then Parseval's formula*

$$\int_G f(x)\overline{g(x)}dx = \int_\Gamma \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\gamma$$

holds where $\hat{f}$ denotes the plancherel transform of f.

Proof. Let f, g be in $L^2(G)$. Using the fact that for any complex valued function f we have $|f(x)|^2 = f(x)\overline{f(x)}$, the polarization identity is easily verified

$$4f\overline{g} = |f+g|^2 - |f-g|^2 + i|f+ig|^2 - i|f-ig|^2.$$

Using Plancherel's theorem we see that

$$\begin{aligned}
&4\int_G f(x)\overline{g(x)}dx = \\
&= \int_G |f+g|^2 dx - \int_G |f-g|^2 dx + i\int_G |f+ig|^2 dx - i\int_G |f-ig|^2 dx \\
&= \int_\Gamma |\hat{f}+\hat{g}|^2 d\gamma - \int_\Gamma |\hat{f}-\hat{g}|^2 d\gamma + i\int_\Gamma |\hat{f}+i\hat{g}|^2 d\gamma - i\int_\Gamma |\hat{f}-i\hat{g}|^2 d\gamma \\
&= 4\int_\Gamma \hat{f}(\gamma)\overline{\hat{g}}(\gamma)d\gamma.
\end{aligned}$$

□

Corollary 3. *The set $A(\Gamma)$ coincides with the set of all convolutions of the form $F * G$ where $F, G \in L^2(\Gamma)$.*

Proof. Let f, g be in $L^2(G)$. If $h = \overline{g}$ is the conjugate of g then $\hat{h}(\gamma) = \overline{\hat{g}(\overline{\gamma})}$ and replacing g with h in Parseval's formula gives

$$\int_G f(x)g(x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\overline{\gamma})d\gamma.$$

If we let $k(x) = g(x)\gamma_0(-x)$ for some $\gamma_0 \in \Gamma$ then $\hat{k}(\gamma) = \hat{g}(\gamma\gamma_0)$ and replacing g with k above yields

$$\int_G f(x)g(x)\gamma_0(-x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma_0\overline{\gamma})d\gamma = (\hat{f} * \hat{g})(\gamma_0).$$

Any $H \in L^1(G)$ can be decomposed as $H = |H|P$ where $|P(x)| = 1$ for all $x \in G$ and then it follows that $H = (|H|^{\frac{1}{2}})(|H|^{\frac{1}{2}}P) := fg$ can be written as a product of $L^2(G)$-functions and from above we see that $\hat{F} = \hat{f}\hat{g}$. On the other hand, if $F, G \in L^2(\Gamma)$ then by Plancherels theorem there exists $f, g \in L^2(G)$ such that $\hat{f} = F$ and $\hat{g} = G$ and from above we know that

$$(F * G)(\gamma_0) = \int_G f(x)g(x)\gamma_0(-x)dx \in A(\Gamma)$$

. □

Corollary 4. *If $V \subseteq \Gamma$ is a non-empty and open then there exists $\hat{f} \in A(\Gamma)$ such that $\hat{f} = 0$ outside V but $\hat{f} \neq 0$.*

Proof. To simplify notation we will use additive notation for Γ. Assume that $V \subseteq \Gamma$ is a non-empty and open. By inner regularity of the Haar measure we can find a compact $K \subseteq V$ such that $m(K) > 0$. We can also find an open set W such that $K + W \subseteq V$ and $m(W) > 0$. This is true since if we pick any $\gamma \in K$ then by continuity of addition at the point $(0, \gamma)$ it follows that there exists open sets U_γ, W_γ in $\gamma \in U_\gamma$, $0 \in W_\gamma$ and $U_\gamma + W_\gamma \subseteq V$. Since K is compact the collection of all U_γ is an open cover for K hence it has a finite subcover

$$K \subseteq \bigcup_{n=1}^{N} U_{\gamma_n}.$$

Let $W = \bigcap_{n=1}^{N} W_{\gamma_n}$ then W is open and non-empty and

$$K + W \subseteq \bigcup_{n=1}^{N} (U_{\gamma_n} + W_{\gamma_n}) \subseteq V.$$

Since Γ is locally compact there exists an open neighbourhood S of 0 such that the closure of S is compact. By replacing W with $W' = W \cap S$ if needed it follows that we can assume that $0 < m(W) < +\infty$.

Let $\hat{f} = \chi_W * \chi_K$ then $\hat{f} \in A(\Gamma)$ by Corollary 3 and by definition of convolution we have

$$\begin{aligned}\hat{f}(\gamma_0) &= \int_\Gamma \chi_W(\gamma)\chi_K(\gamma_0 - \gamma)d\gamma = \int_{K\cap(\gamma_0 - W)} d\gamma \\ &= m(K \cap (\gamma_0 - W)).\end{aligned}$$

If $\gamma_0 \notin V$ then $\gamma_0 \notin W + K$ hence there exists no $w_0 \in W$ and $k_0 \in K$ such that $\gamma_0 = w_0 + k_0$ hence $K \cap (\gamma_0 - W) = \varnothing$ and it follows that $\hat{f}(\gamma_0) = 0$ for any $\gamma_0 \notin V$. Since

$$\int_\Gamma \hat{f}(\gamma)d\gamma = \widehat{\hat{f}}(0) = \chi_W \widehat{* \chi}_K(0) = \widehat{\chi}_W(0) * \widehat{\chi}_K(0) = m(W)m(K) > 0$$

It follows that $\hat{f} \neq 0$ and the Corollary follows. $\square$

7 Pontryagin's Duality Theorem

We know that the dual of the dual of G which we will call $\hat{\Gamma}$ is a locally compact abelian group since Γ is. In this section we will prove Pontryagin's duality Theorem which shows that $\hat{G}$ is in fact isomorphic to G as topological groups. As we will soon see this theorem will enable us prove many useful Corollaries, so called dual statements simply by applying previously proved Theorems to the pair $(\Gamma, \hat{\Gamma}) = (\Gamma, G)$ instead of (G, Γ). Before we prove the duality Theorem we need to prove some important topological characterizations of both Γ and G. The characterization of Γ uses the compact open topology which will be defined below

7.1 The Compact-Open Topology of the Dual Group

Definition 16. *Let X and Y be a topological spaces and let M be a set of functions mapping X to Y. Define for any compact $K \subseteq X$ and open $V \subseteq Y$ the set*

$$B_{K,V} = \{f \in M; f(K) \subseteq V\}.$$

The compact-open topology on M induced by X and Y is the topology generated by basis elements of the form

$$B = \bigcap_{n=1}^{N} B_{K_n,V_n}.$$

It is clear that such elements form a basis so the definition is well defined.

Theorem 10. *If we give Γ the compact-open topology induced by G and $\mathbb{T}$ then any subset of Γ is open in the weak topology induced by $A(\Gamma)$ if and only if it is open with respect to the compact open topology.*

Proof. We begin by showing that $\gamma_0^{-1}B_{K,V}$ is open in the compact-open topology for any $\gamma_0 \in \Gamma$. Pick $\gamma \in \gamma_0^{-1}B_{K,V}$ then it follows that

$$(\gamma\gamma_0)(K) \subseteq V$$

hence

$$\gamma(x)\gamma_0(x) \in V$$

for all $x \in K$. Since $\mathbb{T}$ is a topological group there exists for each $x \in K$ open sets S_x, T_x of $\mathbb{T}$ containing $\gamma(x)$ and $\gamma_0(x)$ respectively such that $T_xS_x \subseteq V$. Since G is locally compact and Hausdorff there exists for each $x \in K$ open sets A_x and B_x in G both containing x such that both $\overline{A_x}$ and $\overline{B_x}$ are compact and

$$A_x \subseteq \overline{A_x} \subseteq \gamma^{-1}(T_x),$$
$$B_x \subseteq \overline{B_x} \subseteq \gamma_0^{-1}(S_x),$$

see Theorem 2.7 in [1]. Since each set $A_x \cap B_x$ contains x for each $x \in K$ they form an open cover of K. By compactness of K they have a finite subcover

$$K \subseteq \bigcup_{n=1}^{N} (A_{x_n} \cap B_{x_n}).$$

The sets defined by

$$K_n = \overline{A_{x_n}} \cap \overline{B_{x_n}}$$

for $1 \leq n \leq N$ are compact. If we let

$$W = \bigcap_{n=1}^{N} B_{K_n, T_{x_n}}$$

Then it is clear that $W \subseteq \Gamma$ is open in the compact-open topology. Since $\gamma(K_n) \subseteq T_{x_n}$ for each n it follows that $\gamma \in W$. Pick $\delta \in W$ and $y \in K$. Then $y \in K_n$ for some n. Since $\delta \in W$ it follows that $\delta(y) \in T_{x_n}$. We also have that K_n is a subset of $\gamma_0^{-1}(S_{x_n})$ hence $\gamma_0(y) \in S_{x_n}$. It follows that

$$\delta(y)\gamma_0(y) \in T_{x_n} S_{x_n} \subseteq V$$

Thus W is an open subset of $\gamma_0^{-1} B_{K,V}$ containing γ and it follows that $\gamma_0^{-1} B_{K,V}$ is an open subset in the compact open topology.

We can now show that any weakly open subset in Γ is open in the compact-open topology. This will be done if we show that $\hat{f}^{-1}(B)$ is open in the compact open topology for any $f \in L^1(G)$ and $B \subseteq \mathbb{C}$ open. Pick such f and B and let γ_0 be an element of $\hat{f}^{-1}(B)$. There exists an $r > 0$ such that the open ball of radius r centered at $\hat{f}(\gamma_0)$, $B(r, \hat{f}(\gamma_0))$ is strictly contained in B. Since $C_c(G)$ is dense in $L^1(G)$ there exists a function f_K supported in the compact set K such that $||f - f_K||_1 < \frac{r}{4}$. If we let

$$V = \{\gamma \in \Gamma;\ (\gamma\gamma_0^{-1})(-K) \subseteq B(\frac{r}{2||f_K||_\infty}, 1)\}$$

then since $V = \gamma_0 B_{-K, B(\frac{r}{2||f_K||_\infty}, 1)}$ it follows that V is open by what we proved above. If $\gamma \in V$ then

$$\begin{aligned}
|\hat{f}(\gamma) - \hat{f}(\gamma_0)| &= |\int_G f(x)(\gamma(-x) - \gamma_0(-x)dx| \\
&= \int_G |f(x) - f_K(x) + f_K(x)||\gamma(-x) - \gamma_0(-x)|dx \\
&\leq 2||f - f_K||_1 + ||f_K||_\infty \int_K |\gamma(-x) - \gamma_0(-x)|dx \\
&\leq \frac{r}{2} + ||f_K||_\infty \int_K |(\gamma\gamma^{-1})(-x) - 1|dx < r.
\end{aligned}$$

thus $\gamma \in \hat{f}^{-1}(B(r, \hat{f}(\gamma_0))) \subseteq \hat{f}^{-1}(B)$ so $\hat{f}^{-1}(B)$ is open in the compact open topology.

To show that any compact-open set in Γ is open in the weak topology induced by the Fourier transforms it is enough to show that $B_{K,V}$ is open in the weak topology for any $K \in G$ compact and $V \subseteq \mathbb{T}$ open in the weak topology. Note that, in the rest of the proof, "open" in relation to Γ refers to open in the weak topology. Pick $\delta \in B_{K,V}$, by continuity of the map taking (γ, x) to $\gamma(x)$ it follows that for each $x \in K$ there exists a $V_x \subseteq G$ open and $W_x \subseteq \Gamma$ open containing x and δ respectively such that $y \in V_x$ and $\gamma \in W_x$ implies that $\gamma(y) \in V$. The sets V_x form an open cover of K hence K has a finite subcover

$$K \subseteq \bigcup_{n=1}^{N} V_{x_n}.$$

The set

$$W = \bigcap_{n=1}^{N} W_{x_n}$$

is non-empty since it contains δ and open. If $\gamma \in W$ then $\gamma(x) \in V$ for all $x \in K$ hence $\gamma \in B_{K,V}$ and it follows that $B_{K,V}$ is open. $\square$

7.2 A Characterization for the Topology on G

Corollary 5. *Sets of the form*

$$B = \bigcap_{n=1}^{N} B_{K_n, V_n}$$

is a basis for the weak topology on Γ.

The following Lemma was found in Section 1.5.2 in [3].

Lemma 1. *The collection of all sets of the form*

$$V_{K,U} = \{x \in G; \gamma(x) \subseteq U \ , \forall \gamma \in K\}$$

where U is an open subset of $\mathbb{T}$ containing 1 and K is a compact subset of Γ is a neighbourhood basis at 0 of G. Furthermore the set $V_{K,U}$ is open for any compact $K \subseteq \Gamma$ and $U \subseteq \mathbb{C}$ open.

Proof. Recall that a neighbourhood basis at 0 is a collection of open sets containing 0 such that any open subset of 0 has a basis element as a subset. We begin to show that $V_{K,U}$ is open where $K \subseteq \Gamma$ is compact and $U \subseteq \mathbb{T}$ is open. Choose $x_0 \in V_{K,U}$. By continuity of the map taking (γ, x) to $\gamma(x)$ we can find, for each $\gamma \in K$ open sets $V_\gamma \subseteq G$ and $W_\gamma \subseteq \Gamma$ containing x_0 and γ respectively such that $\delta(x) \in U$ whenever $\delta \in W_\gamma$ and $x \in V_\gamma$. The sets W_γ form an open cover of K hence

$$K \subseteq \bigcup_{n=1}^{N} W_{\gamma_n}.$$

Note that the set

$$V = \bigcap_{n=1}^{N} V_{\gamma_n}$$

is open and contains x_0 and if x is in V then $\gamma(x) \in U$ for all $\gamma \in K$ so $V_{K,U}$ is an open subset of G.

To show that any subset open subset $V \subseteq G$ which contains 0 contains a set of the form $V_{K,U}$ we pick by Proposition 2 an open set $W \subseteq G$ such that $W + (-W) \subseteq V$ and let $f = \chi_W$. It is clear that we can assume $m(W) < +\infty$ hence $f \in L^1(G) \cap L^2(G)$. Define $g = f * \tilde{f}$ where $\tilde{f}(x) = \overline{f(-x)}$ and recall (Proposition 14) that g is continuous since it is a convolution of $L^2(G)$-functions. By calculating

$$\begin{aligned} g(x) &= \int_G \chi_W(x-y)\chi_W(-y)dx \\ &= \int_G \chi_{x-W}(y)\chi_{-W}(y)dy \\ &= m((x-W) \cap (-W)) \end{aligned}$$

and using Proposition 7 it follows that $g(x) \neq 0$ if and only if

$$(x-W) \cap (-W) \neq \varnothing.$$

The set $(x-W) \cap (-W)$ is not empty if and only if there exists $w_1, w_2 \in W$ such that $x - w_1 = -w_2$ which is equivalent to saying that x is an element of $W + (-W)$. Since $W + (-W) \subseteq V$ it follows that if

$$g(x) > 0$$

then $x \in V$. By the convolution formula we know that $\hat{g} = |\hat{f}|^2 \geq 0$ and in the proof of the inverse transform theorem we saw that g is positive definite. The conditions for the inversion theorem are therefore satisfied hence

$$g(x) = \int_\Gamma \hat{g}(\gamma)\gamma(x)d\gamma$$

for all $x \in G$ where $d\gamma$ denotes integration with respect to a Haar measure on Γ. Note that $g(0) = m((0-W) \cap (-W)) > 0$ hence

$$g(0) = \int_\Gamma \hat{g}(\gamma)d\gamma > 0.$$

Using the embedding of $L^1(\Gamma)$ into $M(\Gamma)$ it follows that the measure defined by

$$\mu(E) = \int_E \hat{g}(\gamma)d\gamma$$

for any Borel set $E \subseteq \Gamma$ is an element of $M(\Gamma)$ and $\mu(E) \geq 0$ since $\hat{g} \geq 0$. It follows by inner regularity that there exists a compact set $K \subseteq \Gamma$ such that

$$g(0) = \mu(\Gamma) = \int_K \hat{g}(\gamma)d\gamma > \frac{2}{3}g(0).$$

It is then clear that

$$|\int_{\Gamma-K} \hat{g}(\gamma)\gamma(x)d\gamma| \leq \int_{\Gamma-K} \hat{g}(\gamma)d\gamma < \frac{1}{3}g(0).$$

Define the set $U \subseteq \mathbb{T}$ by

$$U = \{z \in \mathbb{T};\ \Re(z) > \frac{2}{3}\}$$

where $\Re(z)$ is the real part of z. Since $\Re$ is a continuous function on $\mathbb{C}$ and $U = \Re^{-1}((\frac{2}{3}, \infty)) \cap \mathbb{T}$ it follows that U is open in $\mathbb{T}$. If

$$x \in V_{K,U} = \{x \in G;\ \gamma(x) \in U\ \forall \gamma \in K\}$$

then since g is a real valued function we have

$$\begin{aligned}
g(x) &= \int_\Gamma \hat{g}(\gamma)\gamma(x)d\gamma = \Re \int_\Gamma \hat{g}(\gamma)\gamma(x)d\gamma \\
&= \int_K \hat{g}(\gamma)\Re\gamma(x)d\gamma + \Re \int_{\Gamma-K} \hat{g}(\gamma)\gamma(x)d\gamma \\
&\geq \int_K \hat{g}(\gamma)\Re\gamma(x)d\gamma - |\Re \int_{\Gamma-K} \hat{g}(\gamma)\gamma(x)d\gamma| \\
&> \frac{2}{3} \int_K \hat{g}(\gamma)d\gamma - |\int_{\Gamma-K} \hat{g}(\gamma)\gamma(x)d\gamma| \\
&> \frac{4}{9}g(0) - \frac{1}{3}g(0) = \frac{1}{9}g(0) > 0
\end{aligned}$$

and since $g(x) > 0$ implies that $x \in V$ it follows that $V_{K,U}$ is a subset of V. □

The following Proposition was also proved in Section 1.5.2 in [3].

Proposition 23. *If x, y are distinct elements of G then there exists $\gamma \in \Gamma$ such that $\gamma(x) \neq \gamma(y)$, that is, Γ separates points on G.*

Proof. Choose any $x_0 \in G - \{0\}$. In the second part of the proof of Lemma 1 where it was shown that any open subset of V of G contains an element of the form $V_{K,U}$ we can let V be any open subset of 0 not containing x_0 and from the proof conclude that $\gamma(x_0) \neq 1$ for some $\gamma \in \Gamma$. This is true since (using the same notation as in the proof) we have $V_{K,U} \subseteq V$ and therefore

$$\begin{aligned} x_0 \notin V_{K,U} &= \{x;\ \gamma(x) \subseteq U\ \forall \gamma \in K\} \\ &= \{x;\ \Re\gamma(x) > \frac{2}{3}\ \forall \gamma \in K\}. \end{aligned}$$

Hence there exists $\gamma \in K$ such that $\Re\gamma(x_0) \leq \frac{2}{3}$ hence $\gamma(x_0) \neq 1$. If x, y are distinct elements of G then above can be applied with $x_0 = x - y \neq 0$ and find a $\gamma \in \Gamma$ such that $\gamma(x - y) \neq 1$ hence $\gamma(x) \neq \gamma(y)$. $\square$

7.3 Pontryagin's Duality Theorem

We now prove Pontryagin's duality Theorem which says that G is isomorphic and homeomorphic to its double dual. The proof and the Corollaries can be found in Section 1.7 in [3].

Theorem 11. *Pontryagin's duality Theorem. Let $\hat{\Gamma}$ be the dual of Γ and fix an element $x \in G$. Define the map $e_x : \Gamma \to \mathbb{T}$ by $e_x(\gamma) = \gamma(x)$ then $e_x \in \hat{\Gamma}$ and the map $\Phi : G \to \hat{\Gamma}$ given by $\Phi(x) = e_x$ is an isomorphism and a homeomorphism between the two groups.*

Proof. Let $e_x(\gamma) = \gamma(x)$ for a fixed $x \in G$. Then clearly $|e_x(\gamma)| = 1$ for all $\gamma \in \Gamma$ and it follows that $e_x : \Gamma \to \mathbb{T}$. If γ and δ are two characters of Γ then

$$e_x(\gamma\delta) = (\gamma\delta)(x) = \gamma(x)\delta(x) = e_x(\gamma)e_x(\delta)$$

hence e_x is a homomorphism.

Pick $y \in G$, $\gamma_0 \in \Gamma$ and $V \subseteq \mathbb{T}$ open such that $\gamma_0(y) \in V$. By Proposition 13 the mapping defined on $G \times \Gamma$ which takes (x, γ) to $\gamma(x)$ is continuous hence the set $M = \{(x, \gamma) \in G \times \Gamma;\ \gamma(x) \in V\ \}$ is open in $G \times \Gamma$. Since $(\gamma_0, y) \in M$ it follows by the definition of the product topology that there exists an open $W \subseteq \Gamma$ and an open $U \subseteq G$ such that $(y, \gamma_0) \in U \times W \subseteq M$. Since $\{y\} \times U \subseteq M$ it follows that if $\gamma \in U$ then $e_y(\gamma) = \gamma(y) \in V$ so e_y is continuous. Hence $e_y \in \hat{\Gamma}$ for each $y \in G$.

Define $\Phi : G \to \hat{\Gamma}$ by $\Phi(x) = e_x$. The function Φ is a homomorphism since the relation

$$\begin{aligned} (\Phi(x+y))(\gamma) = e_{x+y}(\gamma) = \gamma(x+y) &= \gamma(x)\gamma(y) = e_x(\gamma)e_y(\gamma) \\ &= (\Phi(x))(\gamma)(\Phi(y))(\gamma) = (\Phi(x)\Phi(y))(\gamma) \end{aligned}$$

holds for all $x, y \in G$ and $\gamma \in \Gamma$.

If $\Phi(x) = \Phi(y)$ then $e_x(\gamma) = e_y(\gamma)$ for all $\gamma \in \Gamma$ then $\gamma(x) = \gamma(y)$ for all $\gamma \in \Gamma$ Since Γ separates points on G (Proposition 23) it follows that $x = y$ hence Φ is injective.

Since Φ is an injective homomorphism it follows that its inverse $\Phi^{-1} : \Phi(G) \to G$ is also a homomorphism. It therefore remains to prove that $\Phi(G) = \hat{\Gamma}$ and that Φ is a homeomorphism onto its image (with the subspace topology).

Note that if $\Phi : G \to \hat{\Gamma}$ is continuous then $\Phi : G \to \Phi(G)$ is continuous. For continuity of $\Phi : G \to \hat{\Gamma}$ it is enough to prove that $\Phi^{-1}(B)$ is open in G for any basis element B of $\hat{\Gamma}$. By Corollary 5 a basis for $\hat{\Gamma}$ is given by elements of the form

$$B = \bigcap_{n=1}^{N} B_{K_n,V_n}$$

for some compact sets $K_n \subseteq \Gamma$ and open sets $V_n \subseteq \mathbb{T}$. Let $B \subseteq \hat{\Gamma}$ be as above, then using the notation in Lemma 1 it follows that

$$\begin{aligned}
\Phi^{-1}(B) = \Phi^{-1}(\bigcap_{n=1}^{N} B_{K_n,V_n}) &= \bigcap_{n=1}^{N} \Phi^{-1}(B_{K_n,V_n}) \\
&= \bigcap_{n=1}^{N} \{x \in G; e_x \in B_{K_n,V_n}\} \\
&= \bigcap_{n=1}^{N} \{x \in G;\ e_x(\gamma) \in V_n \ \forall\ \gamma \in K_n\} \\
&= \bigcap_{n=1}^{N} \{x \in G;\ \gamma(x) \in V_n \ \forall\ \gamma \in K_n\} \\
&= \bigcap_{n=1}^{N} V_{K_n,V_n}
\end{aligned}$$

which is open in G by Lemma 1 and the continuity of Φ follows.

To prove that $\Phi : G \to \Phi(G)$ is an open map (where ($\Phi(G) \subseteq \hat{\Gamma}$ has the subspace topology) we recall from Lemma 1 that sets of the form $V_{K,U}$ where $K \subseteq \Gamma$ compact and $U \subseteq \mathbb{T}$ open such that $1 \in U$ is neighbourhood basis of G at 0. It follows that for any open set V and any point $x \in V$ there exists a $K_x \subseteq \Gamma$ compact and $U_x \subseteq \mathbb{T}$ open containing 1 such that $x + V_{K_x,U_x} \subseteq V$. Pick any open $V \subseteq G$ then

$$V = \bigcup_{x \in V} (x + V_{K_x,U_x}).$$

and it follows that

$$\begin{aligned}
\Phi(V) = \Phi(\bigcup_{x\in V}(x+V_{K_x,U_x})) &= \bigcup_{x\in V}\Phi(x+V_{K_x,U_x})) \\
&= \bigcup_{x\in V}\{e_{x+y}\in\Phi(G);\ y\in V_{K_x,U_x}\} \\
&= \bigcup_{x\in V}\{e_xe_y\in\Phi(G);\ y\in V_{K_x,U_x}\} \\
&= \bigcup_{x\in V}e_x\{e_y\in\Phi(G);\ y\in V_{K_x,U_x}\} \\
&= \bigcup_{x\in V}e_x\{e_y\in\Phi(G);\ \gamma(y)\in U_x\ \forall\gamma\in K_x\} \\
&= \bigcup_{x\in V}e_x\{e_y\in\Phi(G);\ e_y(K_x)\subseteq U_x\} \\
&= \bigcup_{x\in V}e_x(B_{K_x,U_x}\cap\Phi(G)).
\end{aligned}$$

Corollary 5 and Definition 16 show that B_{K_x,U_x} is a (basis) open set in $\hat{\Gamma}$ and therefore $B_{K_x,U_x}\cap\Phi(G)$ is open in $\Phi(G)$. Since $\Phi(G)\subseteq\hat{\Gamma}$ with the subspace topology is a topological group and translations are homeomorphisms (Proposition 1) it follows that

$$e_x(B_{K_x,U_x}\cap\Phi(G)) \tag{2}$$

is open in $\Phi(G)$ for any $x\in V$ and it follows that $\Phi(V)$ is open in $\Phi(G)$ and therefore $\Phi: G\to\Phi(G)$ is a homeomorphism.

Note that showing $\Phi(G)=\hat{\Gamma}$ is equivalent to showing that $\Phi(G)$ is dense in $\hat{\Gamma}$ and $\Phi(G)$ is closed in $\hat{\Gamma}$.

Clearly $\Phi(G)\subseteq\hat{\Gamma}$ is closed if and only if $\Phi(G)$ is closed in $\overline{\Phi(G)}$ but since $\overline{\Phi(G)}$ is a topological group (Proposition 4) it follows from Proposition 3 that open subgroups are closed and it is therefore sufficient to show that $\Phi(G)\subseteq\overline{\Phi(G)}$ is open. $\Phi(G)$ is locally compact since G is locally compact and Φ is a homeomorphism and clearly $\Phi(G)$ is dense in $\overline{\Phi(G)}$ hence Proposition 5 gives us that $\Phi(G)$ is open in $\overline{\Phi(G)}$. It follows that $\Phi(G)$ is closed in $\hat{\Gamma}$.

Assume that $\Phi(G)$ is not dense in $\hat{\Gamma}$. Then there exists an open $V\subseteq\hat{\Gamma}$ such that $V\cap\Phi(G)=\varnothing$. By Corollary 4 there exists an $\hat{F}\in A(\hat{\Gamma})$ such that $\hat{F}=0$ on $\Phi(G)$ but $\hat{F}\neq 0$. For any $x\in G$ we then have

$$\begin{aligned}
0 = \hat{F}(\Phi(e_{-x})) &= \int_\Gamma F(\gamma)e_{-x}(\overline{\gamma})d\gamma \\
&= \int_\Gamma F(\gamma)\gamma(x)d\gamma
\end{aligned}$$

If we define $\mu_F \in M(\Gamma)$ by

$$\mu_F(E) = \int_\Gamma F(\gamma) d\gamma$$

then the uniqueness Theorem of proposition 19 can be applied to μ_F and it follows that $\mu_F = 0$. But this leads to the contradiction $\hat{F}(\hat{\gamma}) = 0$ for any $\hat{\gamma} \in \hat{\Gamma}$ since

$$\begin{aligned}
\hat{F}(\hat{\gamma}) &= \int_\Gamma F(\gamma)\hat{\gamma}(\overline{\gamma}) d\gamma \\
&= \int_\Gamma \hat{\gamma}(\overline{\gamma}) d\mu_F(\gamma) = 0.
\end{aligned}$$

Hence $\Phi(G)$ is dense in $\hat{\Gamma}$ and closed in $\hat{\Gamma}$ and therefore $\Phi(G) = \hat{\Gamma}$ and therefore $\Phi : G \to \hat{\Gamma}$ is an isomorphism and homemorphism of topological groups. □

Corollary 6. *If $\mu \in M(G)$ and $\hat{\mu} = 0$ then $\mu = 0$*

Proof. Applying Proposition 19 to the pair (Γ, G) yields the statement, if $\mu \in M(G)$ and

$$\int_G \gamma(x) d\mu(x) = 0$$

for all $\gamma \in \Gamma$ then $\mu = 0$. Now note that for any $\gamma \in \Gamma$ we have

$$\hat{\mu}(\gamma) = \int_G \gamma(-x) d\mu(x) = \int_G \overline{\gamma}(x) d\mu(x) = 0$$

since $\overline{\gamma} \in \Gamma$. □

Corollary 7. *If $\mu \in M(G)$ and $\hat{\mu} \in L^1(\Gamma)$ then there exists $f \in L^1(G)$ such that $d\mu = f dx$ and*

$$f(x) = \int_\Gamma \hat{\mu}(\gamma)\gamma(x) d\gamma$$

for any $x \in G$

Proof. Define

$$f(x) = \int_\Gamma \hat{\mu}(\gamma)\gamma(x) d\gamma.$$

It is easy to show that $\hat{\mu} \in B(\Gamma)$ hence applying the inversion Theorem on $\hat{\mu}$ shows that $f \in L^1(G)$ and that for any $\gamma \in \Gamma$ we have

$$\begin{aligned}
\hat{\mu}(\gamma) &= \int_G \hat{\hat{\mu}}(x)\gamma(x) dx \\
&= \int_G f(-x)\gamma(x) dx \\
&= \int_G f(x)\gamma(-x) dx.
\end{aligned}$$

But by definition of $\hat{\mu}$ it follows that

$$\int_G \gamma(-x)(f(x)dx - d\mu(x)) = 0$$

hence by Corollary 6 we see that $f(x)dx = d\mu$ $\square$

Corollary 8. *If $f \in L^1(G)$ and $\hat{f} \in L^1(\Gamma)$ then*

$$f(x) = \int_\Gamma \hat{f}(\gamma)\gamma(x)d\gamma$$

for almost all $x \in G$.

Proof. Let $\mu_f \in M(G)$ be the embedding of f. Since $\hat{\mu}_f = \hat{f}$ it follows that we can apply Corollary 7 to get a function

$$F(x) = \int_\Gamma \hat{f}(\gamma)\gamma(x)d\gamma$$

such that

$$\mu_f(E) = \int_E F(x)dx.$$

By definition of μ_f it follows that

$$\int_E (f(x) - F(x))dx = 0$$

for any Borel set $E \subseteq G$. From which it follows that

$$f(x) = \int_\Gamma \hat{f}(\gamma)\gamma(x)d\gamma$$

for almost all $x \in G$. $\square$

Corollary 9. *Every compact group is the dual of a discrete group and every discrete group is the dual of a compact group*

Proof. By Theorem 1.2.5 in [3] the dual of a compact group is discrete and the dual of a discrete group is compact. Applying Pontryagin yields the corollary. $\square$

Corollary 10. *$L^1(G) = M(G)$ if and only if G is discrete. Also, $L^1(G)$ has a unit if and only if G is discrete.*

Proof. If G is not discrete then Γ is not compact by Corollary 9 and it follows that $C_0(\Gamma)$ does not contain the constant function that equals 1 and therefore does not contain a unit. It follows that $A(\Gamma)$ does not contain a unit. Therefore $L^1(G)$ does not contain a unit since the Fourier transform of it would map to a unit in $A(\Gamma)$ which is a contradiction. Hence $L^1(G) \neq M(G)$. If G is discrete the Radon-Nikodym theorem, (see Theorem 6.10 in [2]) gives a bijective correspondence between $M(G)$ and $L^1(G)$. $\square$

8 Examples

In this section we will calculate the dual groups and Fourier transforms for the groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, and $\mathbb{Z}/n\mathbb{Z}$. The dual group of G will be denoted by $\hat{G}$.

8.1 The Real Line

We begin with $\mathbb{R}$. The Haar measure is then given by the ordinary Lebesgue measure, dx. Now let $\gamma \in \hat{\mathbb{R}}$, that is, γ is a continuous homomorphism from $\mathbb{R}$ into $\mathbb{T}$. Since $\gamma(1) \in \mathbb{T}$ there exists a $\beta \in \mathbb{R}$ such that

$$\gamma(1) = \exp(i\beta).$$

If n is an integer the homomorphism property gives us

$$\gamma(n) = \gamma(n1) = \exp(i\beta)^n = \exp(i\beta n).$$

We also have that

$$\exp(i\beta) = \gamma(\frac{n}{n}) = \gamma(\frac{1}{n})^n$$

for any non-zero integer n. Hence

$$\gamma(\frac{1}{n}) = \exp(\frac{i\beta}{n})\exp(\frac{i2\pi K_n}{n}).$$

where K_n is an integer such that $0 \leq K_n \leq n-1$ corresponding to an n:th-root of $\gamma(\frac{1}{n})$. If $\frac{p}{q}$ is a rational number the formula

$$\gamma(\frac{p}{q}) = \gamma(\frac{1}{q})^p = \exp(\frac{i\beta p}{q})\exp(\frac{i2\pi K_q p}{q})$$

follows. Pick a real x and let $(\frac{p_n}{q_n})_n$ be a sequence of rational numbers converging to x. By continuity of γ and exp we get

$$\begin{aligned}\gamma(x) &= \lim_{n\to+\infty}\gamma(\frac{p_n}{q_n}) = \lim_{n\to+\infty}\exp(\frac{i\beta p_n}{q_n})\exp(\frac{i2\pi K_{q_n}p_n}{q_n})\\ &= \exp(i\beta x)\exp(i2\pi x\lim_{n\to+\infty}K_{q_n}).\end{aligned}$$

The limit $\lim_{n\to+\infty} K_{q_n}$ exists since the left hand side is well-defined. Furthermore, $\lim_{n\to+\infty} K_{q_n}$ is an integer since $(K_{q_n})_n$ is a sequence of integers. Let x be an irrational number and pick rational sequences $(\frac{a_n}{b_n})_n$ and $(\frac{c_n}{d_n})_n$ converging to x, then

$$\exp(i2\pi x\lim_{n\to+\infty}K_{b_n}) = \exp(i2\pi x\lim_{n\to+\infty}K_{d_n})$$

hence

$$i2\pi x(\lim_{n\to+\infty}K_{b_n} - K_{d_n}) = i2\pi m(b,d,x)$$

where $m(b,d,x)$ is an integer. Thus

$$\lim_{n\to+\infty} K_{b_n} - K_{d_n} = \frac{m(b,d,x)}{x}$$

and it follows that we must have $m(b,d,x)$=0 in order for the right hand side to be an integer. Therefore

$$\lim_{n\to+\infty} K_{b_n} = \lim_{n\to+\infty} K_{d_n} = K(x).$$

That is, for any irrational x the limit of $(K_{q_n})_n$ where $(q_n)_n$ is the denominator of some Cauchy sequence converging to x depends only on x and we denote this limit as $K(x)$ (which is an integer). Note that for any irrational number x the rational sequence $(\frac{\lfloor x2^n \rfloor}{2^n})_n$ converges to x since

$$x - \frac{1}{2^n} \leq \frac{\lfloor x2^n \rfloor}{2^n} \leq x.$$

Hence

$$K(x) = \lim_{n\to+\infty} K_{2^n}$$

for any irrational x. Therefore $K(x) = k$ is a constant function on the irrationals and it follows that

$$\gamma(x) = \exp(i\beta x)\exp(i2\pi xk) = \exp(i(\beta + 2\pi k)x) = \exp(i\alpha x)$$

for any irrational x where $\alpha = \beta + 2\pi k$. Since the continuous functions γ and $exp(i\alpha x)$ agree almost everywhere in $\mathbb{R}$ they must in fact agree on all of $\mathbb{R}$. Thus any character γ is of the form

$$\gamma(x) = e^{i\alpha x}$$

for some real α and conversely any function of the form $e^{i\alpha x}$ for some $\alpha \in \mathbb{R}$ defines a character on $\mathbb{R}$. If α and β are distinct real numbers then the corresponding characters $e^{i\alpha x}$ and $e^{i\beta x}$ are also distinct (which can be seen from the fact that they have different Taylor expansions). Thus we have the following characterization for $\hat{\mathbb{R}}$

$$\hat{\mathbb{R}} = \{e^{i\alpha x}; \alpha \in \mathbb{R}\}.$$

Define the map $\Phi : \hat{\mathbb{R}} \to \mathbb{R}$ by $\Phi(e^{i\alpha x}) = \alpha$. We have already showed that Φ is well defined and surjective and it is trivially injective. If $\gamma(x) = e^{i\alpha x}$ and $\delta(x) = e^{i\beta x}$ then the expression

$$\Phi(\gamma\delta) = \Phi(e^{i\alpha x}e^{i\beta x}) = \Phi(e^{i(\alpha+\beta)x}) = \alpha + \beta = \Phi(\gamma) + \Phi(\delta)$$

shows that Φ is an isomorphism of groups between $\hat{\mathbb{R}}$ and $\mathbb{R}$. We will now show that Φ is an homeomorphism so $\hat{\mathbb{R}}$ and $\mathbb{R}$ can be identified as topological groups. We begin by showing that Φ is an open mapping which (since Φ is injective) amounts to saying that whenever $B \subseteq \mathbb{C}$ is open and $f \in L^1(\mathbb{R})$ then $\Phi(\hat{f}^{-1}(B)) \subseteq \mathbb{R}$ is open. Observe that

$$\begin{aligned}\Phi(\hat{f}^{-1}(B)) &= \Phi(\{e^{i\alpha x} \in \hat{\Gamma};\ \hat{f}(e^{i\alpha x}) \in B\ \}) \\ &= \{\alpha \in \mathbb{R};\ \int_{\mathbb{R}} f(x)e^{-i\alpha x}dx \in B\}.\end{aligned}$$

If we let α_n be a sequence of real numbers converging to α and let

$$F(y) = \int_{\mathbb{R}} f(x)e^{-iyx}dx$$

for $y \in \mathbb{R}$. The dominated convergence theorem then gives that $F(\alpha_n) \to F(\alpha)$ and therefore F is sequentially continuous and therefore continuous since $\mathbb{R}$ is a metric space. Since

$$\begin{aligned}\Phi(\hat{f}^{-1}(B)) &= \{\alpha \in \mathbb{R};\ \int_{\mathbb{R}} f(x)e^{-i\alpha x}dx \in B\} \\ &= F^{-1}(B)\end{aligned}$$

it follows that $\Phi(\hat{f}^{-1}(B))$ is open thus Φ is an open mapping.

To prove continuity it is enough to show that the set

$$U = \{e^{i\alpha x}; |\alpha| < \epsilon\}$$

is open for any $\epsilon > 0$. Pick $\epsilon > 0$ and define the sets $K = [-1, 1]$ and $V = \{z \in \mathbb{T};\ |\mathrm{Arg}(z) - \pi| < \epsilon\}$ where Arg is the principal argument function with branch cut at the non-positive real axis. By continuity of Arg outside the branch cut it follows that V is open. Since the compact open sets are open in Γ it follows that the set

$$\{e^{i\alpha x}; e^{i\alpha x} \in V\ \forall x \in [-1, 1]\}$$

is open in $\hat{\mathbb{R}}$. But $|\mathrm{Arg}(e^{i\alpha x}) - \pi| < \epsilon$ for all $x \in [-1, 1]$ if and only if $|\alpha| < \epsilon$ hence U is open and Φ is a homeomorphism and is thus an isomorphism of topological groups. It follows that

$$\hat{\mathbb{R}} = \{e^{i\alpha x} : \alpha \in \mathbb{R}\} \cong \mathbb{R}$$

and using the isomorphism we define the Fourier transform as it is commonly introduced

$$\hat{f}(\alpha) = \int_{\mathbb{R}} f(x)e^{-i\alpha x}dx$$

for $\alpha \in \mathbb{R}$.

8.2 The Integers

Now let $G = \mathbb{Z}$ and pick $\gamma \in \hat{\mathbb{Z}}$. Then $\gamma(1) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$ and since γ is a homomorphism it follows that $\gamma(n) = e^{i\alpha n}$. Define the map $\Phi : \mathbb{T} \to \hat{\mathbb{Z}}$ by

$$\Phi(e^{i\alpha}) = e^{i\alpha n}.$$

It is clear that Φ maps $\mathbb{T}$ onto $\hat{\mathbb{Z}}$. If $\Phi(e^{i\alpha}) = \Phi(e^{i\beta})$ then $e^{i\alpha n} = e^{i\beta n}$ for all $n \in \mathbb{Z}$ hence $e^{i\alpha} = e^{i\beta}$ so Φ is injective. We also have that

$$\begin{aligned}\Phi(e^{i\alpha}e^{i\beta}) &= \Phi(e^{i(\alpha+\beta)}) = e^{i(\alpha+\beta)n} \\ &= e^{i\alpha n}e^{i\beta n} = \Phi(e^{i\alpha})\Phi(e^{i\beta})\end{aligned}$$

so Φ is a homomorphism. It is clear that the counting measure on $\mathbb{Z}$ is a Haar measure hence if $f \in L^1(\mathbb{Z})$ then the function $F = \hat{f} \circ \Phi$ is given by

$$F(e^{i\alpha}) = \hat{f}(e^{i\alpha n}) = \sum_{n\in\mathbb{Z}} f(n)(e^{i\alpha})^{-n}.$$

If $e^{i\alpha_n} \to e^{i\alpha}$ then the dominated convergence Theorem shows that $F(e^{i\alpha_n}) \to F(e^{i\alpha})$ so F is continuous. Pick an element $f \in L^1(G)$ and an open set $B \subseteq \mathbb{C}$ then

$$\Phi^{-1}(\hat{f}^{-1}(B)) = \{e^{i\alpha}; F(e^{i\alpha}) \in B\} = F^{-1}(B)$$

and it follows that Φ is continuous. Since Φ is a bijective continuous function from a compact space into a Hausdorff space it follows from the closed map Lemma, (see Lemma 4.50 in [6]) that Φ is a homeomorphism. Hence the dual group of $\mathbb{Z}$ is isomorphic to $\mathbb{T}$ and

$$\hat{\mathbb{Z}} = \{e^{i\alpha n} : e^{i\alpha} \in \mathbb{T}\} \cong \mathbb{T}.$$

It follows using our isomorphism $\hat{\mathbb{Z}} \cong \mathbb{T}$ that the Fourier transform in this case is given by

$$\hat{f}(e^{i\alpha}) = \sum_{n\in\mathbb{Z}} f(n)e^{-i\alpha n}$$

for any $e^{i\alpha} \in \mathbb{T}$.

8.3 The Unit Circle

Now let $G = \mathbb{T}$. Since $\hat{\mathbb{Z}} = \mathbb{T}$ the Pontryagin duality Theorem gives that $\widehat{\mathbb{T}} \cong \hat{\hat{\mathbb{Z}}} \cong \mathbb{Z}$. Define $\Phi : \mathbb{R} \to \mathbb{T}$ by $\Phi(x) = e^{ix}$. By using the first isomorphism Theorem for groups and passing Φ to the quotient (see Theorem 3.7.3 in [6]) it follows that $\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{T}$ as topological groups. Denoting this isomorphism with $\tilde{\Phi} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{T}$ which is given by

$$\tilde{\Phi}(x + 2\pi\mathbb{Z}) = e^{ix}$$

it is straightforward to show that elements of $\hat{\mathbb{T}}$ are in bijection with elements of $\widehat{\mathbb{R}/2\pi\mathbb{Z}}$. By passing to the quotient we see that elements of $\tilde{f} \in \widehat{\mathbb{R}/2\pi\mathbb{Z}}$ are in bijection with elements of $f \in \hat{\mathbb{R}}$ that are constant on sets of the form $x + 2\pi\mathbb{Z}$ for $x \in \mathbb{R}$. Hence it follows that any $\tilde{f} \in \widehat{\mathbb{R}/2\pi\mathbb{Z}}$ has the form

$$\tilde{f}(x + 2\pi\mathbb{Z}) = e^{i\alpha x}$$

for some $\alpha \in \mathbb{R}$. In order for the function $e^{i\alpha x}$ to be constant on sets of the form $x + 2\pi\mathbb{Z}$, for $x \in \mathbb{R}$ it is necessary that

$$e^{i\alpha(x+2\pi)} = e^{i\alpha x}.$$

hence $\alpha \in \mathbb{Z}$. Conversely if $\alpha \in \mathbb{Z}$ then $e^{i\alpha x}$ is constant on sets of the form $x + 2\pi\mathbb{Z}$ for $x \in \mathbb{R}$. It follows that

$$\tilde{f}(x + 2\pi\mathbb{Z}) = e^{inx}$$

for some $n \in \mathbb{Z}$. Using the isomorphism $\tilde{\Phi}^{-1} : \mathbb{T} \to \mathbb{R}/2\pi\mathbb{Z}$ it follows that any $\gamma \in \hat{\mathbb{T}}$ has the form $\gamma = \tilde{f} \circ \tilde{\Phi}^{-1}$ for some $\tilde{f} \in \widehat{\mathbb{R}/2\pi\mathbb{Z}}$ hence

$$\gamma(e^{i\alpha}) = (\tilde{f} \circ \tilde{\Phi}^{-1})(e^{i\alpha}) = \tilde{f}(\alpha + 2\pi\mathbb{Z}) = e^{in\alpha}$$

for some $n \in \mathbb{Z}$. Hence

$$\hat{\mathbb{T}} = \{e^{in\alpha}; n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

For the Haar-measure on $\mathbb{T}$ we do the following. The restriction map $\Phi|_{[0,2\pi]}$ is clearly continuous when $[0, 2\pi] \subseteq \mathbb{R}$ has been given the subspace topology. Define the set function

$$m_{\mathbb{T}}(E) = m(\Phi|_{[0,2\pi]}^{-1}(E))$$

for any Borel set $E \subseteq \mathbb{T}$ where m denotes the ordinary Lebesgue measure on $\mathbb{R}$. Since $\Phi|_{[0,2\pi]}$ is continuous the inverse image of any Borel set is a Borel set and it follows that $m_{\mathbb{T}}$ is a well-defined function on the Borel sets. It is not hard to show that $m_{\mathbb{T}}$ is a measure. Since $m_{\mathbb{T}}(\mathbb{T}) = m([0, 2\pi]) = 2\pi$ it is clear that $m_{\mathbb{T}}$ is finite on compact sets. If $E \subseteq \mathbb{T}$ is a Borel set and $e^{i\alpha} \in \mathbb{T}$ then

$$\begin{aligned}
m_{\mathbb{T}}(e^{i\alpha}E) &= m(\Phi|_{[0,2\pi]}^{-1}(e^{i\alpha}E)) \\
&= m(\{x \in [0, 2\pi]; e^{ix} \in e^{i\alpha}E\}) \\
&= m(\{x \in [0, 2\pi]; e^{i(x-\alpha)} \in E\}) \\
&= m(\{x - \alpha + \alpha \in [0, 2\pi]; e^{i(x-\alpha)} \in E\}) \\
&= m(\{x - \alpha \in [-\alpha, -\alpha + 2\pi]; e^{i(x-\alpha)} \in E\}) \\
&= m(-\alpha + \{x - \alpha \in [0, 2\pi]; e^{i(x-\alpha)} \in E\}) \\
&= m(\{x - \alpha \in [0, 2\pi]; e^{i(x-\alpha)} \in E\}) \\
&= m(\Phi|_{[0,2\pi]}^{-1}(E)) \\
&= m_{\mathbb{T}}(E).
\end{aligned}$$

Hence $m_{\mathbb{T}}$ is translation-invariant. Let $K_n \subseteq \Phi^{-1}_{[0,2\pi]}(E) \subseteq [0, 2\pi]$ be a collection of compact sets such that $\lim_{n\to+\infty} m(K_n) = m(\Phi^{-1}_{[0,2\pi]}(E))$. Then since continuous functions preserve compactness it follows that

$$\begin{aligned}
\sup_{K\subseteq E} m_{\mathbb{T}}(K) &\geq \lim_{n\to+\infty} m_T(\Phi_{[0,2\pi]}(K_n)) \\
&= \lim_{n\to+\infty} m(\Phi^{-1}_{[0,2\pi]}(\Phi_{[0,2\pi]}(K_n))) \\
&\geq \lim_{n\to+\infty} m(K_n) \\
&= m(\Phi^{-1}_{[0,2\pi]}(E)) \\
&= m_{\mathbb{T}}(E).
\end{aligned}$$

Hence $m_{\mathbb{T}}$ is inner regular. To prove outer regularity we do the following. Use inner regularity to find compact sets $K_n \subseteq \mathbb{T} - E$ such that

$$m_{\mathbb{T}}(K_n) \to m_{\mathbb{T}}(\mathbb{T} - E)$$

as $n \to +\infty$. Then the sets $V_n = \mathbb{T} - K_n$ are open since the K_n are closed and $E \subseteq V_n$ for all n. Hence

$$\begin{aligned}
\inf_{E\subseteq V} m_{\mathbb{T}}(V) &\leq \lim_{n\to+\infty} m_{\mathbb{T}}(V_n) \\
&= \lim_{n\to+\infty} m_{\mathbb{T}}(\mathbb{T} - K_n) \\
&= \lim_{n\to+\infty} m_{\mathbb{T}}(\mathbb{T}) - m_{\mathbb{T}}(K_n) \\
&= m_{\mathbb{T}}(\mathbb{T}) - m_{\mathbb{T}}(\mathbb{T} - E) \\
&= m_{\mathbb{T}}(E)
\end{aligned}$$

since $m_{\mathbb{T}}(\mathbb{T}) = 2\pi$ is finite. It follows that $m_{\mathbb{T}}$ is a Haar-measure. We normalize $m_{\mathbb{T}}$ by replacing $m_{\mathbb{T}}$ with $\frac{1}{2\pi} m_{\mathbb{T}}$ from which it follows that $m_{\mathbb{T}}(\mathbb{T}) = 1$. This is also the proper normalization of the Haar-measure in $\mathbb{T}$ which appears in the inversion theorem formulas for the pairs $(\mathbb{T}, \mathbb{Z})$ and $(\mathbb{Z}, \mathbb{T})$. For any Borel set $E \subseteq \mathbb{T}$ we have

$$\begin{aligned}
\int_{\mathbb{T}} \chi_E dm_{\mathbb{T}} &= m_{\mathbb{T}}(E) \\
&= \frac{1}{2\pi} m(\Phi^{-1}_{[0,2\pi]}(E)) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \chi_{\Phi^{-1}_{[0,2\pi]}(E))} dx \\
&= \frac{1}{2\pi} \int_{[0,2\pi]} \chi_E(e^{ix}) dx.
\end{aligned}$$

From a standard approximation argument it is clear that

$$\int_{\mathbb{T}} f dm_{\mathbb{T}} = \frac{1}{2\pi} \int_{[0,2\pi]} f(e^{ix}) dx.$$

for any $f \in L^1(\mathbb{T})$. It follows that the Fourier transform on $\mathbb{T}$ is given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx.$$

Letting $F(x) = f(e^{ix})$ we recognize above as the commonly used definition for Fourier series.

Remark. *A more direct but less intuitive way of finding the Haar measure on $\mathbb{T}$ is to define the positive linear functional $\Lambda : C_c(\mathbb{T}) \to \mathbb{C}$ by*

$$\Lambda(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) dx$$

and prove that it is translation invariant. The Riesz representation Theorem (Theorem 2.14 in [1]) then yields our Haar-measure $m_{\mathbb{T}}$ on $\mathbb{T}$.

8.4 The Group of Integers Mod N

Let $G = \mathbb{Z}/N\mathbb{Z}$ and pick $\gamma \in \widehat{\mathbb{Z}/N\mathbb{Z}}$. If $q : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ is the quotient map then it follows that $\gamma \circ q \in \hat{\mathbb{Z}}$ hence there exists $e^{i\alpha} \in \mathbb{T}$ such that $\gamma([n]) = e^{i\alpha n}$, where $[n] = n + N\mathbb{Z}$. In order for $e^{i\alpha n}$ to be constant on the fibers of q we must in particular have that if $n - m = N$ then $e^{i\alpha n} = e^{i\alpha m}$ which implies that $\alpha N = 2\pi k$ for some $k \in \mathbb{Z}$ and therefore

$$\gamma([n]) = e^{i2\pi kn/N}.$$

If $k_1 \cong k_2 \mod N$ it follows that

$$e^{i2\pi k_1 n/N} = e^{i2\pi k_2 n/N}$$

for any $[n] \in \mathbb{Z}/N\mathbb{Z}$ hence it follows that

$$\widehat{\mathbb{Z}/N\mathbb{Z}} = \{e^{i2\pi kn/N} : [k] \in \mathbb{Z}/N\mathbb{Z}\}.$$

From which it is clear that $\widehat{\mathbb{Z}/N\mathbb{Z}} \cong \mathbb{Z}/N\mathbb{Z}$ since the dual of a compact group is discrete. Since $\mathbb{Z}/N\mathbb{Z}$ is discrete the Haar measure is given by the counting measure and it follows that

$$\hat{f}([k]) = \sum_{n=0}^{N-1} f([n]) e^{-i2\pi kn/N}$$

which we recognize as the discrete Fourier transform.

9 The Collatz Conjecture

9.1 Introduction to the Collatz Conjecture

Let $\mathbb{N} = \{1, 2, ...\}$ and define the function $c : \mathbb{N} \to \mathbb{N}$ by,

$$c(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Pick a number $n \in \mathbb{N}$ and associate to it its Collatz sequence $f_i(n)$ given by,

$$f_i(n) = c^i(n),$$

for $i \geq 0$ where c^i denotes i-fold function composition and $c^0(n) = n$. The Collatz conjecture can now be stated as follows: is it true that for any $n \in \mathbb{N}$ there exists an integer $j \geq 0$ such that $f_j(n) = 1$?

Note that $c(1) = 4$, $c(4) = 2$ and $c(2) = 1$ hence if $f_j(n) = 1$ for some j then the Collatz sequence will cycle through $1, 2$ and 4 indefinitely after it reaches j.

Upon inspecting the problem there are two ways for the Collatz conjecture to be false, either there exists a Collatz sequence which contains another cycle or there exists a Collatz sequence which diverges towards infinity.

In this application we will use the discrete Fourier transform to derive properties that any cycle, including the known $(1, 4, 2)$-cycle or any other potential cycle must satisfy. Finding another Collatz cycle will disprove the Collatz conjecture.

9.2 Collatz Cycles and the Discrete Fourier Transform

Definition 17. *A finite sequence $(a_n)_{0 \leq n \leq N-1}$ of mutually distinct integers is a (Collatz) cycle of order N if $c(a_{N-1}) = a_0$ and $c(a_n) = a_{n+1}$ whenever $0 \leq n < N-1$.*

Any cycle can be naturally extended to an infinite sequence by defining $a_n = c^n(a_0)$ for $n \in \mathbb{Z}$. If $n \equiv m \mod N$ then $a_n = a_m$ hence our cycle can be regarded as a function a defined on $\mathbb{Z}/N\mathbb{Z}$ by $a([n]_N) = c^n(a_0)$. The notation $a[n] = c^n(a_0)$ will be used for brevity if N is clear from the context. Throughout this application $(a_n)_n$ will denote a cycle of order N and the letter a will be reserved for the corresponding function $a : \mathbb{Z}/N\mathbb{Z} \to \mathbb{N}$.

The discrete Fourier transform of a, $\hat{a} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined by,

$$\hat{a}[k] = \sum_{n=0}^{N-1} a[n] e^{-i2\pi kn/N}.$$

Note that this expression can clearly be rewritten as,

$$\hat{a}[k] = \sum_{n=0}^{N-1} a_n e^{-i2\pi kn/N}.$$

Theorem 12. *Let a_n be a cycle of order N then the Fourier transform of a is given by*

$$\hat{a}[k] = \frac{1}{2e^{i2\pi k/N} - 1} \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (5a_n + 2)e^{-i2\pi kn/N}.$$

Proof. We derive the identity by calculating $\widehat{c \circ a}(k)$ in two ways. Firstly,

$$\begin{aligned}\widehat{c \circ a}[k] &= \sum_{n=0}^{N-1} c(a[n])e^{-i2\pi kn/N} = \sum_{n=0}^{N-1} a[n+1]e^{-i2\pi kn/N} \\ &= e^{i2\pi k/N} \sum_{n=0}^{N-1} a[n+1]e^{-i2\pi k(n+1)/N} = e^{i2\pi k/N}\hat{a}[k],\end{aligned}$$

hence

$$\widehat{c \circ a}[k] = e^{i2\pi k/N}\hat{a}[k]. \tag{1}$$

We also have,

$$\begin{aligned}\widehat{c \circ a}[k] &= \sum_{n=0}^{N-1} c(a_n)e^{-i2\pi kn/N} \\ &= \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (3a_n + 1)e^{-i2\pi kn/N} + \sum_{\substack{n=0 \\ a_n \text{ even}}}^{N-1} \frac{a_n}{2} e^{-i2\pi kn/N} \\ &= \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (\frac{5}{2}a_n + 1)e^{-i2\pi kn/N} + \frac{1}{2}\sum_{n=0}^{N-1} a_n e^{-i2\pi kn/N} \\ &= \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (\frac{5}{2}a_n + 1)e^{-i2\pi kn/N} + \frac{1}{2}\hat{a}[k],\end{aligned}$$

hence,

$$\widehat{c \circ a}[k] = \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (\frac{5}{2}a_n + 1)e^{-i2\pi kn/N} + \frac{1}{2}\hat{a}[k]. \tag{2}$$

Combining (1) and (2) gives,

$$e^{i2\pi k/N}\hat{a}[k] = \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (\frac{5}{2}a_n + 1)e^{-i2\pi kn/N} + \frac{1}{2}\hat{a}[k],$$

and if we solve for $\hat{a}[k]$ we get

$$\hat{a}[k] = \frac{1}{2e^{i2\pi k/N} - 1} \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (5a_n + 2)e^{-i2\pi kn/N}.$$

$\square$

Corollary 11. *Let l be the number of all odd terms in the cycle $(a_n)_{0\le n\le N-1}$, O be sum of all the odd terms in $(a_n)_{0\le n\le N-1}$ and E be the correspoding sum of all the evens terms then,*

$$4O + 2l = E.$$

Proof. We have by Theorem 12 and the definition of the discrete Fourier transform the equality

$$\sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (5a_n + 2) = \hat{a}_0 = \sum_{n=0}^{N-1} a_n. \tag{1}$$

Rewriting (1) in terms of O, E and l yields

$$5O + 2l = O + E,$$

hence

$$4O + 2l = E.$$

$\square$

9.3 Applying the Inversion Theorem

In this section an expression for a_n will be derived from the inverse transform. Define $b : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ and $c : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by,

$$b[k] = \frac{1}{2e^{i2\pi k/N} - 1},$$

and,

$$c[k] = \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (5a_n + 2)e^{-i2\pi kn/N}.$$

From Theorem 12 it is clear that $\hat{a}[k] = b[k]c[k]$ and by the convolution formula it follows that,

$$\widehat{(\check{b} * \check{c})}[k] = b[k]c[k] = \hat{a}, \tag{1}$$

where $\check{}$ denotes the inverse transform and $*$ denotes convolution. By uniqueness of the Fourier transform we have $a = \check{b} * \check{c}$. The inverse transform for b is given by

$$\check{b}[n] = \frac{1}{N}\sum_{k=0}^{N-1} b[k]e^{i2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i2\pi nk/N}}{2e^{i2\pi k/N} - 1}. \tag{2}$$

The inverse transform for c is,

$$\begin{aligned}
\check{c}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} c[k]e^{i2\pi nk/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\Big(\sum_{\substack{j=0 \\ a_j \text{ odd}}}^{N-1} (5a_j + 2)e^{-i2\pi kj/N}\Big)e^{i2\pi nk/N} \\
&= \frac{1}{N}\sum_{\substack{j=0 \\ a_j \text{ odd}}}^{N-1} (5a_j + 2)\sum_{k=0}^{N-1} e^{i2\pi k(n-j)/N}.
\end{aligned}$$

Fix n and j then if $n = j$ we see that,

$$\sum_{k=0}^{N-1} e^{i2\pi k(n-j)/N} = \sum_{k=0}^{N-1} 1 = N.$$

Otherwise, if $n \neq j$ the geometric series formula applies and we obtain,

$$\sum_{k=0}^{N-1} e^{i2\pi k(n-j)/N} = \frac{(e^{i2\pi(n-j)/N})^N - 1}{e^{i2\pi(n-j)/N} - 1} = 0.$$

Consider now arbitrary $n \in \mathbb{Z}$, if a_n is even then it follows that

$$\frac{1}{N}\sum_{\substack{j=0 \\ a_j \text{ odd}}}^{N-1} (5a_j + 2)\sum_{k=0}^{N-1} e^{i2\pi k(n-j)/N} = \frac{1}{N}\sum_{\substack{j=0 \\ a_j \text{ odd}}}^{N-1} (5a_j + 2)0 = 0,$$

hence $\check{c}[n] = 0$. If a_n is odd it follows that,

$$\sum_{k=0}^{N-1} e^{i2\pi k(n-j)/N} \neq 0,$$

only if $n = j$ hence,

$$\check{c}[n] = \frac{1}{N}(5a_n + 2)\sum_{k=0}^{N-1} e^{i2\pi k(n-n)/N} = 5a_n + 2.$$

Therefore,

$$\check{c}[n] = \begin{cases} 0, \text{ if } a_n \text{ is even} \\ , 5a_n + 2, \text{ if } a_n \text{ is odd.} \end{cases} \tag{3}$$

Combining (1), (2) and (3) we have,

$$\begin{aligned} a_n &= \check{b} * \check{c}(n) \\ &= \sum_{j=0}^{N-1} \check{b}[n-j]\check{c}[j] \\ &= \sum_{j=0}^{N-1} (\frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i2\pi(n-j)k/N}}{2e^{i2\pi k/N} - 1})\check{c}[j] \\ &= \sum_{\substack{j=0 \\ a_j \text{odd}}}^{N-1} (\frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i2\pi(n-j)k/N}}{2e^{i2\pi k/N} - 1})(5a_j + 2). \end{aligned}$$

Define the function

$$w(n) = \frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i2\pi nk/N}}{2e^{i2\pi k/N} - 1}. \tag{4}$$

Then we have

$$a_n = \sum_{\substack{j=0 \\ a_j \text{odd}}}^{N-1} w(n-j)(5a_j + 2). \tag{5}$$

We will now give a formula for w and after that we will summarize the results.

It is clear from (4) and the inversion theorem that

$$\hat{w}(k) = \frac{1}{2e^{i2\pi k/N} - 1}.$$

Consider now the function $x : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by first mapping $[n]_N$ to its principal remainder n and then mapping the principal remainder to $\frac{1}{1-2^{-N}}2^{-(n+1)}$ then it follows that the Fourier transform for x is

$$\begin{aligned}
\hat{x}(k) &= \sum_{n=0}^{N-1} \frac{1}{1-2^{-N}} 2^{-(n+1)} e^{-2\pi k n/N} \\
&= \frac{1}{1-2^{-N}} \frac{1}{2} \sum_{n=0}^{N-1} (\frac{e^{-2\pi k/N}}{2})^n \\
&= \frac{1}{1-2^{-N}} \frac{1}{2} \frac{1-(\frac{e^{-2\pi k/N}}{2})^N}{1-\frac{e^{-2\pi k/N}}{2}} \\
&= \frac{1}{1-2^{-N}} \frac{1}{2} \frac{1-2^{-N}}{1-\frac{e^{-2\pi k/N}}{2}} \\
&= \frac{e^{2\pi k/N}}{2e^{2\pi k/N}-1} \\
&= e^{2\pi k/N} \hat{w}(k).
\end{aligned}$$

Hence

$$\hat{w}(k) = \hat{x}(k) e^{-2\pi k/N}.$$

Note that the function $\chi_{\{1\}}$ satisfies $\hat{\chi}_{\{1\}}[k] = e^{-2\pi k/N}$ and it follows that

$$\begin{aligned}
w[n] &= (x * \chi_{\{1\}})[n] \\
&= \sum_{k=0}^{N-1} x[n-k]\chi_{\{1\}}[k] \\
&= x[n-1] \\
&= \begin{cases} \frac{1}{2^N-1}, \text{ if } [n]=0, \\ \frac{1}{1-2^{-N}} 2^{-n}, \text{ if } [n] \neq 0. \end{cases}
\end{aligned}$$

We summarize the result below.

Theorem 13. *Let $(a_n)_{0 \leq n \leq N-1}$ be a Collatz cycle. Then it follows that*

$$a_n = \sum_{\substack{j=0 \\ a_j \text{odd}}}^{N-1} w[n-j](5a_j+2)$$

where w is given by

$$w[n] = \begin{cases} \frac{1}{2^N-1}, \text{ if } [n]=0 \\ \frac{1}{1-2^{-N}} 2^{-n}, \text{ if } [n] \neq 0. \end{cases}$$

9.4 Summary of the Results and Questions for the Reader

In this section we have proved that any Collatz cycle a_n of order N has a discrete Fourier transform given by

$$\hat{a}[k] = \frac{1}{2e^{i2\pi k/N} - 1} \sum_{\substack{n=0 \\ a_n \text{ odd}}}^{N-1} (5a_n + 2)e^{-i2\pi kn/N}$$

and that if l is the number of odd elements in the cycle, O is the sum of all the odd terms in the cycle and E is the sum of all the even terms in the cycle then

$$4O + 2l = E.$$

We have furthermore also proved that a_n satisfies the following (non-linear) system of equations

$$a_n = \sum_{\substack{j=0 \\ a_j \text{odd}}}^{N-1} w[n-j](5a_j + 2)$$

where w is given by

$$w[n] = \begin{cases} \frac{1}{2^N - 1}, & \text{if } [n] = 0 \\ \frac{1}{1-2^{-N}} 2^{-n}, & \text{if } [n] \neq 0. \end{cases}$$

My question to the reader is if these relations can be used to show that there can't exists cycles of certains lengths, or to derive upper bounds of the length of a cycle. I am also interested in knowing if the relations can be used to derive upper bounds of the elements in a cycle.

10 A Formula for the Partition function

In this section we will derive an integral formula for the partition function. We define the partition function, p to be the number of ways that we can add to n using only positive integers while not caring about the order of the terms. For example, $p(4) = 5$ since we can add to 4 with only positive integer terms in the following 5 ways including the trivial way

$$1 + 1 + 1 + 1 = 2 + 1 + 1 = 3 + 1 = 2 + 2 = 4.$$

By convention we set $p(0) = 1$ and $p(n) = 0$ if n is negative. The key part of this application is Euler's Pentagonal formula, (see Section 19.10 in [4]) which states that we have the following recursion formula for p

$$p(n) = \sum_{j>0} (-1)^{j-1} (p(n - j(3j - 1)/2) + p(n - j(3j + 1)/2))$$

where $n \geq 1$. Note that the sum is finite since $p(n) = 0$ for negative n. To simplify notation we will write

$$\begin{aligned} a_j &= j(3j - 1)/2, \\ b_j &= j(3j + 1)/2. \end{aligned}$$

The famous Hardy-Ramanujan formula, (see Introduction in [5]) will also be used which asserts that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}$$

which is the same as saying

$$\lim_{n \to +\infty} \frac{p(n)}{\frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}} = 1.$$

The Hardy-Ramanujan formula gives us the following proposition.

Proposition 24. *The function* $f(n) = \frac{p(n)}{e^n}$ *is in* $L^1(\mathbb{Z})$

Proof. The proof will actually show that the function $nf(n)$ is in $L^1(\mathbb{Z})$ which will be used in the next theorem. Note that $nf(n) \in L^1(\mathbb{Z})$ implies that $f(n) \in L^1(\mathbb{Z})$. The Hardy-Ramanujan formula gives us that there exists an integer N such that

$$\frac{p(n)}{\frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}} < 2$$

whenever $n \geq N$. It follows that

$$\begin{aligned} ||nf(n)||_1 = \sum_{n \in Z} n \frac{p(n)}{e^n} &= \sum_{n=0}^{N} n \frac{p(n)}{e^n} + \sum_{n>N} n \frac{p(n)}{e^n} \\ &< \sum_{n=0}^{N} n \frac{p(n)}{e^n} + 2 \sum_{n>N} n \frac{\frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}}{e^n}. \end{aligned}$$

Note that $\pi\sqrt{\frac{2}{3}} \leq \frac{1}{2}\sqrt{n}$ whenever $n \geq c := \frac{8\pi^2}{3}$ and since

$$\begin{aligned}
\sum_{n>\max(N,c)} n\frac{\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}}{e^n} &= \frac{1}{4\sqrt{3}} \sum_{n>\max(N,c)} e^{\pi\sqrt{\frac{2n}{3}}-n} \\
&= \frac{1}{4\sqrt{3}} \sum_{n>\max(N,c)} e^{\sqrt{n}(\pi\sqrt{\frac{2}{3}}-\sqrt{n})} \\
&\leq \frac{1}{4\sqrt{3}} \sum_{n>\max(N,c)} e^{\sqrt{n}(\frac{1}{2}\sqrt{n}-\sqrt{n})} \\
&= \frac{1}{4\sqrt{3}} \sum_{n>\max(N,c)} e^{-\frac{n}{2}} < +\infty
\end{aligned}$$

the proposition follows. □

If we parameterize $\mathbb{T}$ with e^{ix}, where $x \in [0, 2\pi)$ it follows that we can denote the Fourier transform on $\mathbb{Z}$ by

$$\hat{g}(x) = \sum_{n\in\mathbb{Z}} g(n)e^{-inx}$$

for $x \in [0, 2\pi)$. The Fourier transform of the partition function p is clearly not well defined since it is not bounded but as Proposition 24 showed we can calculate the Fourier transform for $f(n) = \frac{p(n)}{e^n}$.

Theorem 14. *The Fourier transform for $f(n) = \frac{p(n)}{e^n}$ is given by*

$$\hat{f}(x) = \frac{1}{1 + \sum_{j>0}(-1)^j(e^{-j}e^{-ia_jx} + e^{-j}e^{-ib_jx})}.$$

Proof. We showed that $f \in L^1(\mathbb{Z})$ so the transform is well-defined. We can now calculate it with the Euler-Pentagonal formula and the translation invariance of the Haar-measure.

$$\begin{aligned}
\hat{f}(x) &= \sum_{n\in\mathbb{Z}} \frac{p(n)}{e^n}e^{-inx} = 1 + \sum_{n=1}^{+\infty} \frac{p(n)}{e^n}e^{-inx} \\
&= 1 + \sum_{n=1}^{+\infty} \frac{1}{e^n}(\sum_{j>0}(-1)^{j-1}(p(n-a_j) + p(n-b_j)))e^{-inx} \\
&= 1 + \sum_{n=1}^{+\infty}\sum_{j>0}(-1)^{j-1}\frac{p(n-a_j) + p(n-b_j)}{e^n}e^{-inx}.
\end{aligned}$$

Note that since $p(n)$ is increasing we have that

$$\sum_{j>0} |(-1)^{j-1}\frac{p(n-a_j) + p(n-b_j)}{e^n}e^{-inx}| \leq 2n\frac{p(n)}{e^n}.$$

Since we proved that $n\frac{p(n)}{e^n}$ is in $L^1(\mathbb{Z})$ it follows that our double sum is absolutely summable and therefore we can interchange the summation signs and get

$$\begin{aligned}\hat{f}(x) &= 1+\sum_{j>0}\sum_{n=1}^{+\infty}(-1)^{j-1}\frac{p(n-a_j)+p(n-b_j)}{e^n}e^{-inx}\\ &= 1+\sum_{j>0}\sum_{n=1}^{+\infty}(-1)^{j-1}\frac{p(n-a_j)}{e^n}e^{-inx}+\sum_{j>0}\sum_{n=1}^{+\infty}(-1)^{j-1}\frac{p(n-b_j)}{e^n}e^{-inx}.\end{aligned}$$

We begin by calculating the first sum

$$\begin{aligned}\sum_{j>0}\sum_{n=1}^{+\infty}(-1)^{j-1}\frac{p(n-a_j)}{e^n}e^{-inx} &= \sum_{j>0}\sum_{n=1}^{+\infty}(-1)^{j-1}\frac{1}{e^{a_j}}\frac{p(n-a_j)}{e^{n-a_j}}e^{-inx}\\ &= \sum_{j>0}(-1)^{j-1}\frac{1}{e^{a_j}}\sum_{n=1}^{+\infty}\frac{p(n-a_j)}{e^{n-a_j}}e^{-inx}\\ &= \sum_{j>0}(-1)^{j-1}\frac{1}{e^{a_j}}e^{-ia_jx}\sum_{n=1}^{+\infty}\frac{p(n-a_j)}{e^{n-a_j}}e^{-i(n-a_j)x}\\ &= \sum_{j>0}(-1)^{j-1}\frac{1}{e^{a_j}}e^{-ia_jx}\sum_{n\in\mathbb{Z}}\frac{p(n-a_j)}{e^{n-a_j}}e^{-i(n-a_j)x}\\ &= \sum_{j>0}(-1)^{j-1}\frac{1}{e^{a_j}}e^{-ia_jx}\hat{f}(x)\\ &= \hat{f}(x)\sum_{j>0}(-1)^{j-1}\frac{1}{e^{a_j}}e^{-ia_jx}.\end{aligned}$$

The second sum is completely analogous and it follows that

$$\hat{f}(x)(1+\sum_{j>0}(-1)^j(\frac{1}{e^{a_j}}e^{-ia_jx}+\frac{1}{e^{a_j}}e^{-ia_jx})=1$$

for any real x. It follows that

$$1+\sum_{j>0}(-1)^j(\frac{1}{e^{a_j}}e^{-ia_jx}+\frac{1}{e^{b_j}}e^{-ib_jx})\neq 0$$

hence

$$\hat{f}(x)=\frac{1}{1+\sum_{j>0}(-1)^j(e^{-a_j}e^{-ia_jx}+e^{-b_j}e^{-ib_jx})}.$$

$\square$

Theorem 15. *We have the following formula for the partition function.*

$$p(n) = \frac{e^n}{2\pi} \int_0^{2\pi} \frac{e^{inx}}{1 + \sum_{j>0} (-1)^j (e^{-a_j} e^{-ia_j x} + e^{-b_j} e^{-ib_j x})} dx.$$

Proof. If we instead regard the Fourier transform

$$\hat{f}(x) = \frac{1}{1 + \sum_{j>0} (-1)^j (e^{-a_j} e^{-ia_j x} + e^{-b_j} e^{-ib_j x})}$$

of $\frac{p(n)}{e^n}$ as a function on $\mathbb{T}$, which we for clarity call F. Then we know that $F \in C_0(\mathbb{T})$ and $\mathbb{T}$ is compact hence $F \in L^1(\mathbb{T})$. By Proposition 24 we know that $\frac{p(n)}{e^n} \in L^1(\mathbb{Z})$. Using our Haar-measure on $\mathbb{T}$ calculated in Section 8.3 and applying Corollary 8 then shows that

$$\begin{aligned} \frac{p(n)}{e^n} &= \int_{\mathbb{T}} F(e^{it})(e^{it})^n dm_{\mathbb{T}}(e^{it}) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{inx}}{1 + \sum_{j>0} (-1)^j (e^{-a_j} e^{-ia_j x} + e^{-b_j} e^{-ib_j x})} dx \end{aligned}$$

for almost all $n \in \mathbb{Z}$. But the only set of measure in $\mathbb{Z}$ is $\varnothing$ hence

$$p(n) = \frac{e^n}{2\pi} \int_0^{2\pi} \frac{e^{inx}}{1 + \sum_{j>0} (-1)^j (e^{-a_j} e^{-ia_j x} + e^{-b_j} e^{-ib_j x})} dx.$$

for all $n \in \mathbb{Z}$. $\square$

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