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A Study on Domination in Graphs

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A Study on Domination in Graphs

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Abstract

A subset $S \subseteq V$ is a dominating set in a graph $G = (V, E)$ if every vertex $v \in V \setminus S$ is adjacent to at least one vertex $v \in S$. The minimum cardinality of a dominating set is called the domination number of G and it is denoted by $\gamma(G)$. In this thesis, we study the domination number for standard graphs and derive some upper and lower bounds for $\gamma(G)$. The main object of this thesis to study dominating set.

Keywords: Dominating set, Minimal dominating set, Domination number, Independent set.

Preface

This work constitutes a degree project of 15 credits at the Department of Mathematics at Stockholm University.

This thesis is intended to encourage students who wish to gain knowledge about the subject of domination in graphs. Domination in graphs is one of the main emerging concepts in graph theory. It has a variety of applications in fields such as linear algebra and optimization, online social networks, design and analysis of communication network, algorithm design and computational complexity.

Acknowledgements

I would like to thank my supervisor Gregory Arone for guiding me through the work and for his patience and kindness. He spent countless hour to help me with this thesis and I am really grateful for that. I would also like to thank Annemarie Luger who gave suggestions to modify the work. Finally, I would like to thank Pavel Kurasov who reviewed the work and gave good feedback.

Contents

1

Introduction

Graph theory is an important field of mathematics. It has applications in areas such as cognitive science, physics, architecture, transportation networks and biological networks.

In this paper, we focus on an invariant of graphs called domination number. A set of vertices S in a graph G is called dominating set, if every vertex outside S is adjacent to a vertex in S. The dominating number of a graph is the smallest possible number of vertices in a dominating set.

The paper is organized follows:
In Chapter 2, we review basic concepts and definitions in graph theory that are used later in this thesis.

In Chapter 3, we define the domination number and calculate it for a number of standard examples.

In Chapter 4, we present bounds on domination number and prove some theorems about them. We also illustrate the results on some examples.

1.1 History of domination in graphs

As written in the book by Haynes et al. [4], the idea of domination came into existence in 1850 when Chess freaks in Europe came up with the idea of finding the smallest number of queens that can be placed on a chessboard with a goal so that all squares are either attacked by a queen or inhibited by a queen. It was believed in the 1850s, that five is the smallest number of queens that can dominate all of the squares in an 8×8 chessboard. In 1862, De Jaenisch tried to determine the minimum number of queens required to cover an $n \times n$ chess board, for $n \leq 8$ (Bozoki et al. [2]).

However, the study of domination in graphs developed from the 1950s onwards. As mentioned in the Sugumaran and Jayachandran's article [9], Claude Berge [1] defined the concept of domination number of a graph in 1958, called as "coefficient of external stability", which is now called by the domination number for a graph. Oystein Ore [7] introduced the name "dominating set" and "domination number" for the same concept in his book on *Theory of Graphs* which was published in 1962.

As described by Sukumaran and Jayachandran [9], Cockayne and Hedetniemi [3] published a study in the article *Towards a theory of domination in graphs*, in which the notation $\gamma(G)$ for the domination number of a graph G was first used, which has become very popular. Since the publication of this article, the domination in graphs has been studied in detail and several additional research articles on this topic have been published.

2

Basic Definitions concerning Graphs

We cover some basic definitions here in graph theory. We will define others when needed. Most of the terms and notations used may be found in Haynes [4].

Definition 2.1. A *graph* $G = (V, E)$ consists of two sets V and E, where V is non-empty set and E is a set of unordered pairs of distinct elements of V. The element of V are called vertices of G, and the elements of E are called edges.

Remark. We will only consider finite graphs, which means that we will always assume that V is finite.

It is common to represent a graph with a figure where vertices are represented by points or nodes, while the edges are similarly represented by lines between pairs of points.

This can be best illustrated with the following example found in Figure 1 below.
As per figure, the set $V = \{A, B, C, D, E, F\}$ and $E = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, D\}, \{D, E\}, \{E, F\}\}$ define a graph with 6 vertices (V) and 6 edges (E).

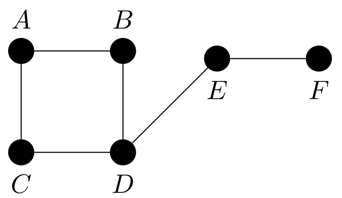


Figure 1

In general, the sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. However, in the current discussion, an edge is represented by uv instead of $\{u, v\}$.

Definition 2.2. If two vertices are joined by an edge, they are called the *endpoints* or *endvertices* for the edge.

Definition 2.3. Let u, v be two vertices of a graph G. If uv is an edge of G, then u and v are said to be *adjacent* in G. It can also be said that u is connected to v or u is a neighbour of v.

Definition 2.4. If $e = uv$ be an edge of graph G, then the edge e is said to be *incident* to the vertices u and v.

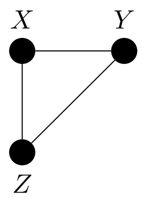


Figure 2

Example 2.5. In the above Figure 2, vertices X and Y are endpoints of edge e_1, also e_1 is incident to X and Y. We also say that X and Y are adjacent since they are endpoints of e_1.

Definition 2.6. The set of neighbours of a vertex v is called the *neighbourhood* of v and is denoted by $N(v) = \{u \in V(G) \mid vu \in E(G)\}$. Simultaneously, the closed neighbourhood of v is denoted by $N[v] = N(v) \cup \{v\}$.

Definition 2.7. The number of vertices in G is called the *order* of G, which is denoted by $|V|$. Similarly, the number of edges in G is called the *size* of G, which is denoted by $|E|$.

Definition 2.8. The *degree* of v in $V(G)$, is the number of edges incident to v and is denoted by $deg_G(v)$ or $deg(v)$. Alternatively, $deg(v)$ is the number of elements in the neighbourhood of v, i.e. $deg(v) = |N(v)|$.

A vertex v is said be an *isolated vertex* if and only if $deg(v) = 0$.

Definition 2.9. The *minimum degree* of a graph G, is the minimum degree among all the vertices of G and is denoted by $\delta(G)$. Formally we defined it as follows

$$\delta(G) = min\{\deg(v)|v \in V(G)\}$$

Definition 2.10. The *maximum degree* of a graph G, is the maximum degree among all the vertices of G and is denoted by $\Delta(G)$. Formally we defined it as follows

$$\Delta(G) = max\{\deg(v)|v \in V(G)\}$$

Example 2.11. From the illustration given in Figure 3 below, G has number of vertices $n = 8$ and number of edges $m = 9$; vertices a and e are adjacent while vertices a and f are non-adjacent. Also, $N(g) = \{b, d, h\}$, $N[g] = \{b, d, g, h\}$; $\Delta(G) = 3$ and $\delta(G) = 1$.

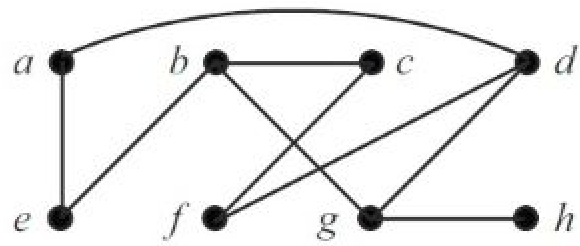


Figure 3: Visual Presentation of Graph G. *Source: Knor et al.* [5]

The following elementary result establishes a connection between the size of the graph and the degrees of its vertices.

Theorem 2.12. *For a graph G of size $|E| = m$,*

$$\sum_{v \in V} deg(v) = 2m.$$

Proof. The sum $\sum_{v \in V} deg(v)$ gives the number of edges at all vertices, where each edge is counted twice because each edge is incident to exactly 2 vertices, which means that $\sum_{v \in V} deg(v) = 2m$. □

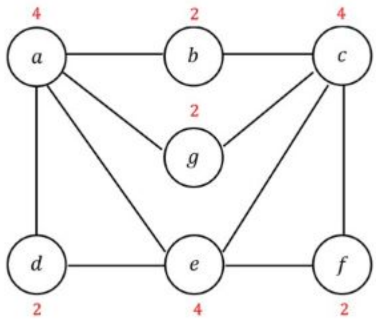


Figure 4

Example 2.13. In the above graph (Figure 4), the red values show how many neighbours each vertex has and $|E| = 10$, i.e. $E = \{ab, ad, ae, ag, bc, ce, cf, cg, de, ef\}$. So,

$$\sum_{v \in V} deg(v) = 20.$$

Definition 2.14. A graph $H = (V_1, E_1)$ is called a *subgraph* of graph $G = (V, E)$, which is written as $H \subseteq G$, if all the vertices and the edges of H are in G, i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

It can be mentioned that G is said to contain H as a subgraph or H is contained in G.

A subgraph is obtained by deleting some vertices and edges from G.

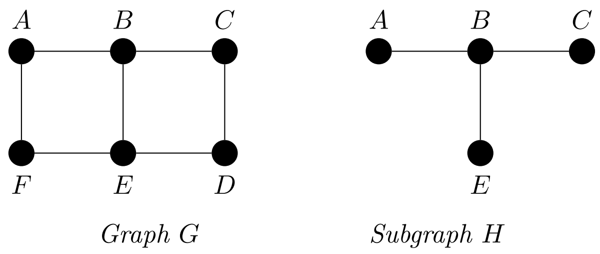


Figure 5

Definition 2.15. If the subgraph $F = (V_1, E_1)$ of a graph $G = (V, E)$ has the same vertex set as in G (i.e. $V_1 = V$), then F is called a *spanning subgraph* of G.

In other words, a spanning subgraph is obtained by deleting only some edges from G.

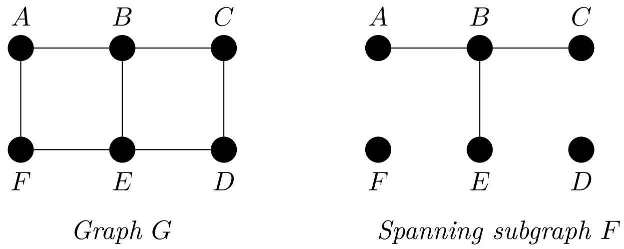


Figure 6

Definition 2.16. If $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$ and for all u and v in $V(G_1)$, uv in $E(G_1)$ if and only uv is in $E(G)$ then G_1 is called an *induced subgraph* of G.

In other words, an *induced subgraph* is obtained by deleting some vertices from G, along with their adjacent edges.

The induced subgraph $\langle G_1 \rangle$ on the right can be denoted as $G - \{u_1, u_6\}$ as illustrated in the Figure 7.

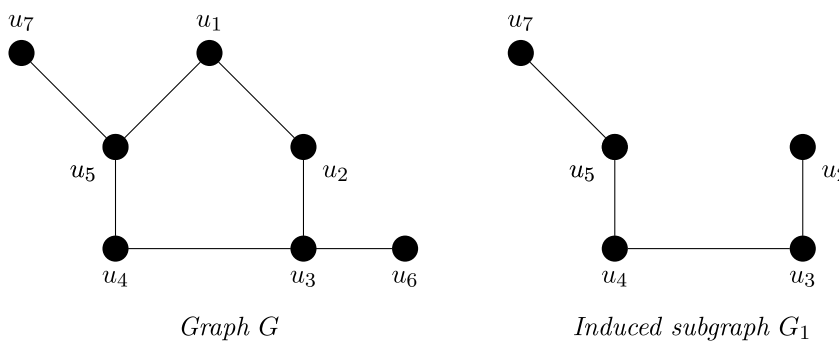
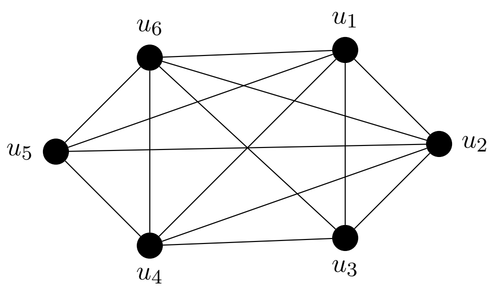


Figure 7

Definition 2.17. A graph G is said to be a *complete graph* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n, it has $\frac{n(n-1)}{2}$ edges.

This is illustrated in Figure 8 below wherein $|V| = n = 6$ and $|E| = m = 6(6-1)/2 = 15$.

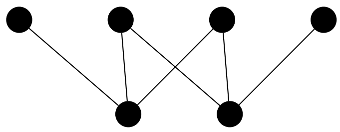


K_6

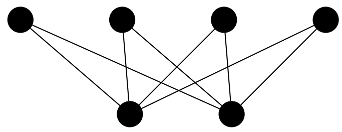
Figure 8

Definition 2.18. A graph $G = (V, E)$ is called a *bipartite graph* if the vertex set of G can be divided into two disjoint parts namely V_1 and V_2, and $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that every edge is of the form ab where $a \in V_1$ and $b \in V_2$.

If every vertex in V_1 is adjacent to every vertex in V_2, and there are no edges within V_1 and V_2, it results a *complete bipartite graph.* This is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$.



Bipartite graph



Complete bipartite graph $K_{2,4}$

Figure 9

Definition 2.19. A $u - v$ *walk* W in G is a sequence of vertices and edges existing in a graph G, which begins with u and ends at v such that consecutive vertices in the sequence are adjacent to each other. More formally, a $u - v$ walk W is a sequence of the following form:

$$W : u = v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k = v,$$

where each v_i is a vertex of G and $e_i = v_{i-1}v_i$ for all i. Sometimes we will omit e_i from the notation. Vertex and edge can be repeated.

In a walk, if the starting and ending vertices are different, then it is said to be an *open walk.* Meanwhile, if the starting and ending vertices are the same, a walk is said to be a *closed walk.*

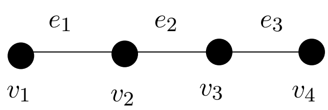


Figure 10

Example 2.20. As depicted in the above graph (Figure 10), $v_1e_1v_2e_2v_3e_3v_4$ is a walk (also it is an open walk).

Definition 2.21. A *trail* is a walk with no repeated edges.

Definition 2.22. A closed trail is called a *circuit.*

Definition 2.23. A *path* is a walk with no repeated vertices.

Definition 2.24. A closed path is called a *cycle.*

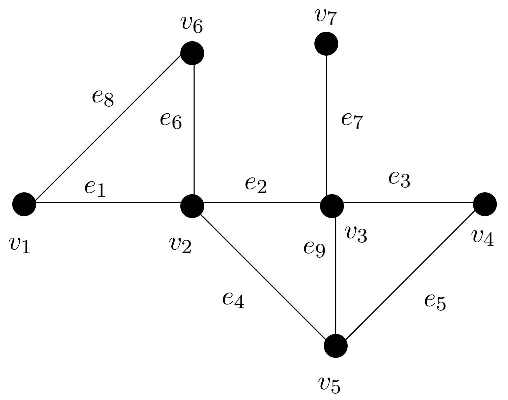


Figure 11

In the above graph (Figure 11):
$v_3e_3v_4e_5v_5e_4v_2e_1v_1$ is an open walk and $v_2e_2v_3e_3v_4e_5v_5e_4v_2$ is a closed walk.
$v_2e_1v_1e_8v_6e_6v_2e_2v_3$ is a trail.
$v_2e_1v_1e_8v_6e_6v_2e_2v_3e_9v_5e_4v_2$ is a circuit.
$v_1e_1v_2e_2v_3e_3v_4$ is a path but $v_1e_1v_2e_4v_5e_5v_4e_3v_3e_2v_2$ is not a path.
$v_1e_8v_6e_6v_2e_1v_1$ is a cycle.

Definition 2.25. A *path graph* is a graph with vertices $v_1, v_2, ...v_n$ where there is an edge connecting v_i and v_{i+1} for each $i = 1, 2, ..., n-1$ and no other edges. It is denoted by P_n.

Definition 2.26. A graph with n vertices for $n \geq 3$ and n edges forming a cycle with all its n edges is called a *cycle graph.* It is denoted by C_n.

Cycle graphs are $2-$ regular, it means the every vertex in a cycle graph has degree 2.

For instance, we can see cycle graph of C_4 and C_5 in Figure 12.

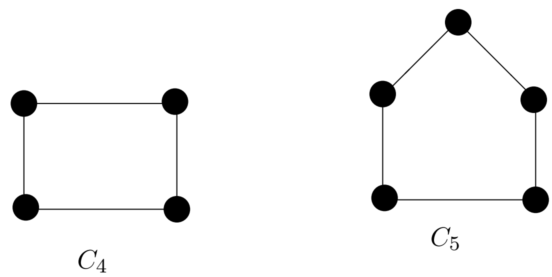


Figure 12

Definition 2.27. A graph is said to be *acyclic* if it does not contain a cycle. If a graph contains at least one cycle, it is called as *cyclic graph.*

Definition 2.28. The length of a walk, a path, a trail or a cycle is its number of edges.

Thus, a path of n vertices has length $n-1$, while a cycle of n vertices has length n. For example, the length of path is 3 in the path $v_1e_1v_2e_2v_3e_3v_4$, i.e. $P_4 = 3$.

Definition 2.29. If there is at least one path between every two vertices in a graph G, it is said be a *connected graph*. Otherwise, it is a *disconnected graph*.

Definition 2.30. Let G be a connected graph of order n and $u, v \in V(G)$. Hence, the *distance* between u and v is the length of a shortest/minimal path from u to v in G. This minimal length of a path is denoted by $d_G(u, v)$ or $d(u, v)$.

Definition 2.31. The maximum *distance* between two vertices of G is called by *diameter* for a connected graph G, which is denoted by $diam(G)$.

We will now review some standard ways to construct new graphs out of given graphs.

Definition 2.32. A set of graphs is said to be *vertex disjoint* if no two of them have any vertex in common (Wang et al. [11]).

Definition 2.33. The *union*, $G \cup H$ is deemed to be a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ for two graphs G and H.

Example 2.34. We can see in Figure 13, the union for two vertex-disjoint graphs G and H.

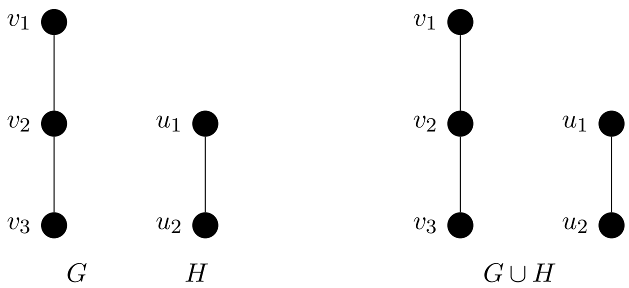


Figure 13

Example 2.35. We can see in Figure 14, the union of for two graphs H_1 and H_2 that are not vertex-disjoint.

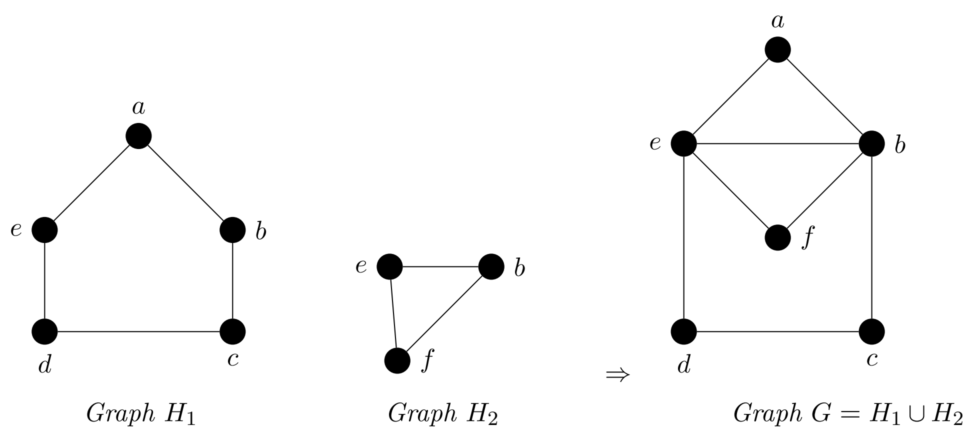


Figure 14

Definition 2.36. The *join* of vertex-disjoint graphs $G + H$ consists of $G \cup H$ and all edges joining every vertex of G to every vertex of H.

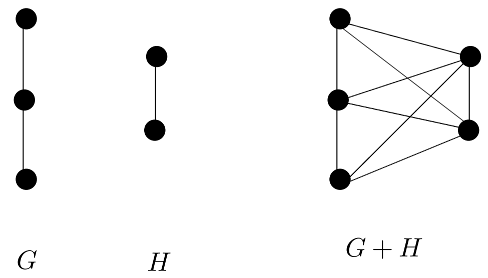


Figure 15

Definition 2.37. A *Fan* graph denoted by F_n, is defined as the graph join of a single vertex and a path graph. That is, $F_n = K_1 + P_{n-1}$, where K_1 is a single vertex.

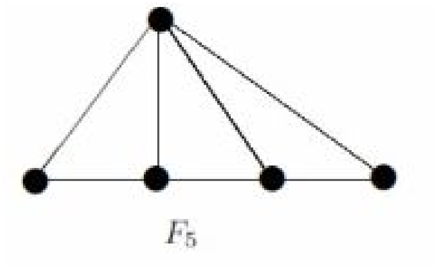


Figure 16

Definition 2.38. A *wheel* graph W_n is the graph join of a single vertex and a cycle graph C_{n-1}. One can say that $W_n = K_1 + C_{n-1}$.

The number of edges in a wheel graph is $m = 2n - 2$. For instance, in the Figure 17 below, $m = (2 \times 6) - 2 = 10$; $diam(W_6) = 2$ and $d(W_6) = 1$.

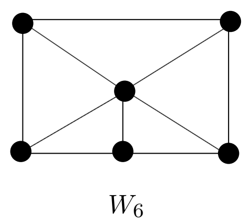


Figure 17

Definition 2.39. A *Star* is a graph in which $n - 1$ vertices have degree 1 and a single vertex has degree $n - 1$. It is denoted by S_n or $K_{1,n-1}$ with n vertices.

Example 2.40. For instance, in star graph S_5 (Figure 18) it can be observed that, $n - 1$ vertices have degree 1 and other one vertex has degree $n - 1$. That is, $deg(a) = deg(b) = deg(c) = deg(d) = 1$ and $deg(e) = 4$.

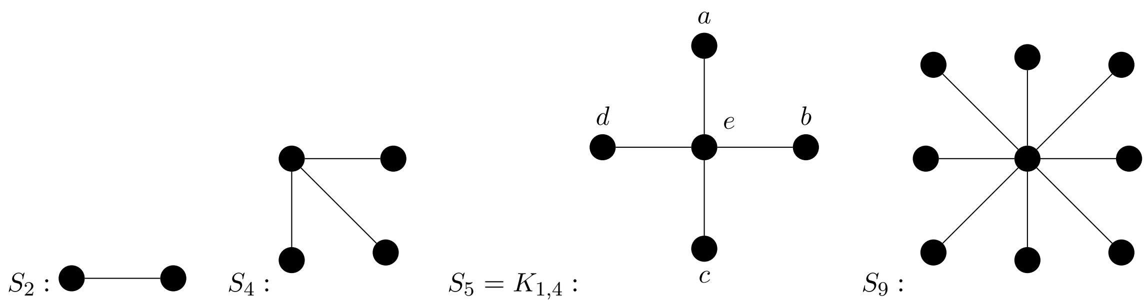


Figure 18: Star graph of order-$n(S_n)$

3

A DOMINATING SET AND DOMINATION NUMBER

Having defined basic notions in graph theory, we are now going to define and discuss the concept of a dominating set in a graph.

Definition 3.1. A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is said to be a *dominating set* if every vertex of $V(G) \setminus S$ (i.e. every vertex not in S) is adjacent to at least one vertex in subset S.

Example 3.2. In the graph illustrated in Figure 19, $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and let $S = \{u_3, u_5\}$, so $V \setminus S = \{u_1, u_2, u_4, u_6\}$.
The vertices u_1, u_4 are adjacent to vertex u_5 and vertices u_2, u_6 are adjacent to vertex u_3. Therefore, $\{u_3, u_5\}$ is a dominating set of G.

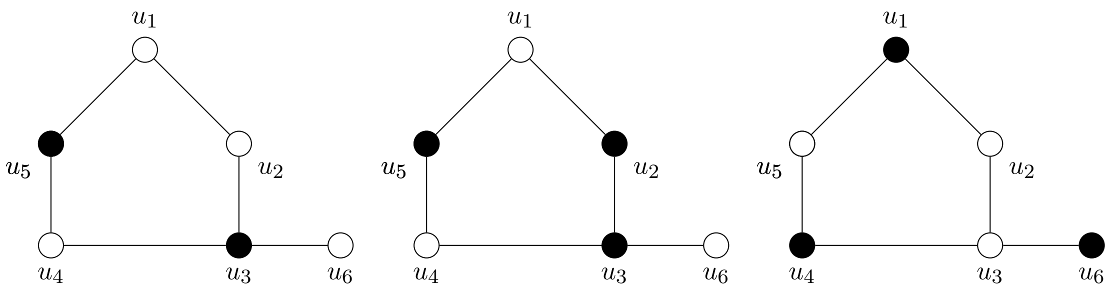


Figure 19

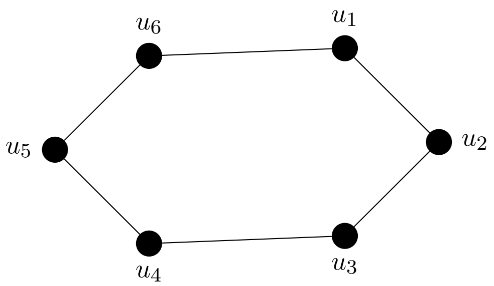
In Figure 19, $\{u_3, u_5\}$, $\{u_2, u_3, u_5\}$ and $\{u_1, u_4, u_6\}$ are dominating sets of G.

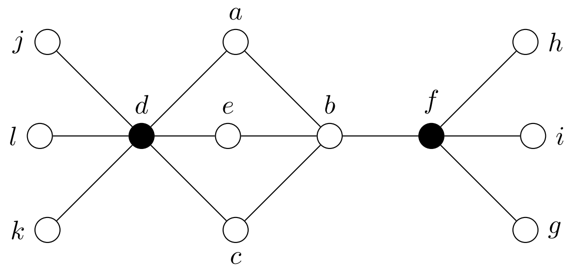
Definition 3.3. A dominating set with minimum cardinality is called *minimum dominating set* in a graph G.

A minimum dominating set is usually not necessarily unique dominating set. However there are graphs that have a unique dominating set.

For instance, C_6 shown in Example 1 below has possible minimum dominating sets $\{u_1, u_4\}, \{u_2, u_5\}, \{u_3, u_6\}$. So, a minimum dominating sets of a graph is not a unique in this case.

The graph shown in Example 2 has the unique dominating set $\{d, f\}$.





Example 1 Example 2

Definition 3.4. The *domination number* of the graph G is the cardinality of a minimum dominating set, which is denoted by $\gamma(G)$.

For the graph G in above Example 3.2, it can be seen that, $\gamma(G) = 2$ for minimum dominating set $\{u_3, u_5\}$.

In general, the domination number is defined for every graph because every vertex set dominates itself, so it is a dominating set.

Example 3.5. In the below Figure 20, $\{u_3, u_5\}$, $\{u_2, u_5, u_6\}$ and $\{u_1, u_3, u_7\}$ are dominating sets of G and $\gamma(G) = 2$.

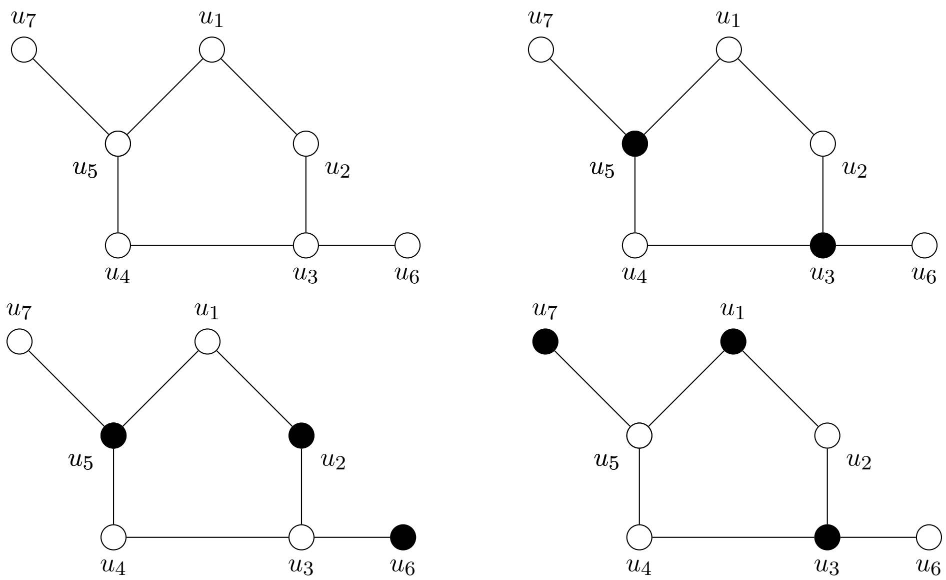


Figure 20

3.1 The domination number $\gamma(G)$ for standard graphs

Observation 1: If G is a graph of order n, then $1 \leq \gamma(G) \leq n$.

Lemma 3.6. *A graph G has a vertex of degree $n-1$ if and only if $\gamma(G) = 1$.*

Proof. Let u be a vertex of degree $n-1$. Since u is adjacent to all the vertices of G, $\{u\}$ is a dominating set of G. Thus, $\gamma(G) = 1$.

Conversely, suppose that $\gamma(G) = 1$. Let $\{u\}$ be a dominating set of G. Since $\{u\}$ is adjacent to all the vertices of G, $deg(u) = n - 1$. □

For example, it is easy to see in the below Figure 21: $n = 5$, then $deg(u) = 4$ and $\gamma(G) = 1$.

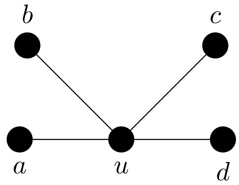


Figure 21

Corollary 3.7. $\gamma(K_n) = 1$ *for* $n \geq 2$.

Proof. For any complete graph $K_n (n \geq 2)$, $\delta(K_n) = \Delta(K_n) = n - 1$. Therefore, each vertex of K_n is a dominating set (by Lemma 3.6). Thus, $\gamma(K_n) = 1$. □

Example 3.8. In the below Figure 22, $\{u_1\}$ is a dominating set of K_4 and so $\gamma(K_4) = 1$.

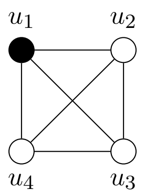


Figure 22

Lemma 3.9. $\gamma(K_{m,n}) = 2$, *where* $2 \leq m \leq n$.

Proof. By definition, $V(K_{m,n}) = (V_1 \cup V_2)$, V_1 and V_2 are independent sets and $V_1 \cap V_2 = \emptyset$. That is, $V_1 = \{v_1, v_2, ..., v_m\}, V_2 = \{u_1, u_2, ..., u_n\}$, and $E(K_{m,n}) = \{v_i u_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.

At least one vertex from V_2 is needed if V_1 is to be dominated; likewise, vertex from V_1 is needed if V_2 is to be dominated. In these conditions, the minimal dominating set is $S = \{v_i, u_i\}$, where $v_i \in V_1$ and $u_i \in V_2$. So, $\gamma(K_{m,n}) = |S|$. Thus, $\gamma(K_{m,n}) = 2$. □

Example 3.10. In the below Figure 23, $\{u_1, v_1\}$ is a dominating set of $K_{2,3}$ and also $\gamma(K_{2,3}) = 2$.

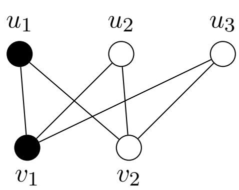


Figure 23

Corollary 3.11. $\gamma(W_n) = 1$ *for all* $n \geq 4$.

Proof. Since wheel graph $W_n = K_1 + C_{n-1}$, let $V(W_n) = \{u, v_i \mid 1 \leq i \leq n-1\}$, and $E(W_n) = \{uv_i \mid 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} \mid 1 \leq i \leq n-2\} \cup \{v_{n-1}v_1\}$.
Also, $\Delta(W_n) = n - 1$.
By lemma 3.6, {u} is a dominating set. Thus, $\gamma(W_n) = 1$ for all $n \geq 4$. □

Example 3.12. In the Figure 24, $\{u\}$ is a dominating set of W_5 and $\Delta(W_5) = 4$. Hence $\gamma(W_5) = 1$.

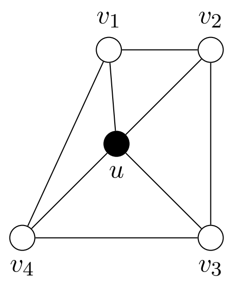


Figure 24

Corollary 3.13. $\gamma(F_n) = 1$ *for all* $n \geq 3$.

Proof. Since fan $F_n = K_1 + P_{n-1}$, let $V(F_n) = \{u, v_i \mid 1 \leq i \leq n-1\}$, $E(F_n) = \{uv_i \mid 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} \mid 1 \leq i \leq n-2\}$.

Since $\{u\}$ is a dominating set, $\Delta(F_n) = n - 1$ by lemma 3.6. Thus, $\gamma(F_n) = 1$ for all $n \geq 3$. □

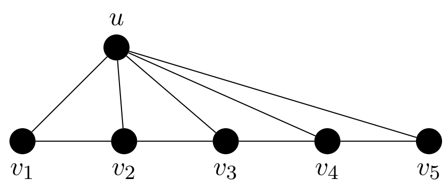


Figure 25

Example 3.14. In Figure 25, $\{u\}$ is a dominating set of F_6 and $\Delta(F_6) = 5$. Thus, $\gamma(F_6) = 1$.

Theorem 3.15. $\gamma(P_n) = \lceil n/3 \rceil$ *for* $n \geq 2$.

Proof. Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$, $E(P_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$.
Maximum degree of path graph is 2, i.e. $\Delta(P_n) = 2$. By Theorem 4.6, which is proved that $\gamma(P_n) \geq \lceil \frac{n}{1+\Delta(P_n)} \rceil$. Hence, $\gamma(P_n) \geq \lceil \frac{n}{3} \rceil$.

Also, if we starting with the second vertex of the path graph and selecting every third vertex then gives a dominating set of size $\lceil \frac{n}{3} \rceil$. It means, we starting with $v_2, v_5, v_8, v_{11}, ...$ (the last vertex depends on the value of n, i.e. when $n \not\equiv 0 \pmod 3$).
So it can be concluded that, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. □

For instance, let $n = 8$, then P_8 :

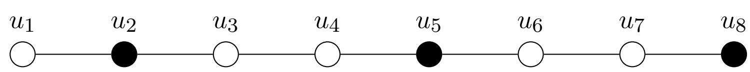


Figure 26

From the above Figure 26, $\{u_2, u_5, u_8\}$ is a dominating set of P_8 and $\gamma(P_8) = 3$, which is satisfied our theorem, i.e. $\gamma(P_8) = \lceil \frac{8}{3} \rceil = 3$.

Theorem 3.16. $\gamma(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$.

The theorem is proved similarly to Theorem 3.15.

Remark. (Sugumaran [9]) The floor function for a real number k is the largest integer less than or equal to k and it is denoted by $\lfloor k \rfloor$ while the ceiling function for a real number k is the lowest integer greater than or equal to k and it is denoted by $\lceil k \rceil$.

3.2 A minimal dominating set and an independent set

Definition 3.17. A dominating set S is considered as a minimal dominating set if no proper subset of S is a dominating set.

Example 3.18. In the Figure 27 below, $\{u_2, u_5\}, \{u_1, u_4, u_5\}, \{u_1, u_3, u_5\}, \{u_1, u_3, u_6, u_7\}, and \{u_1, u_4, u_6, u_7\}$ are minimal dominating sets.

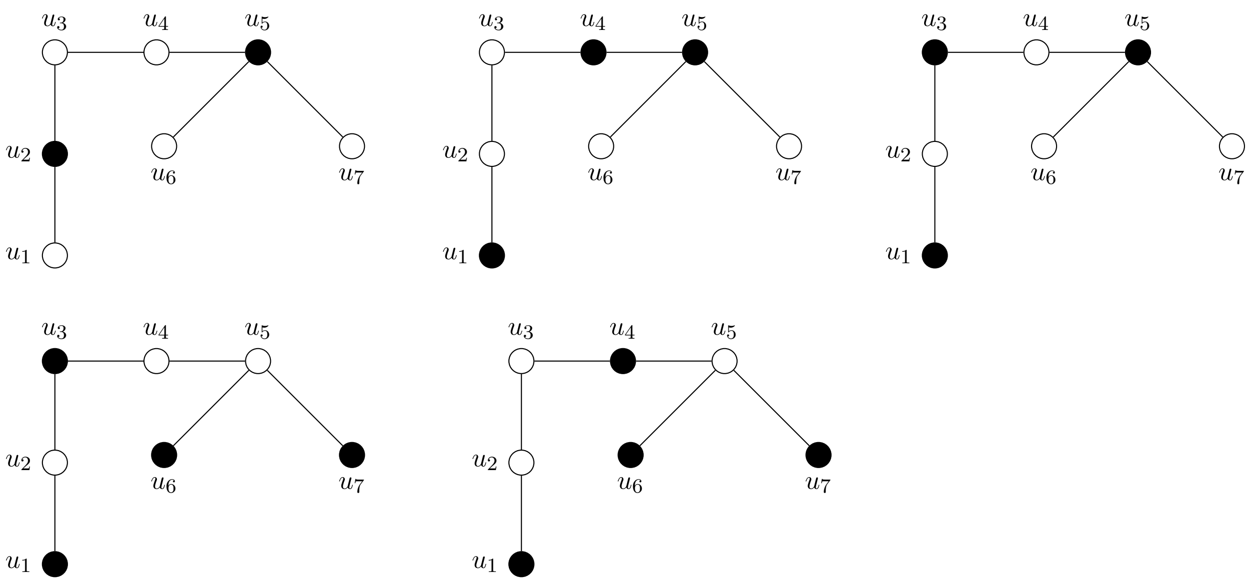


Figure 27

Theorem 3.19. *(Theorem 1.1, [4]) A dominating set S is a minimal dominating set if and only if for each vertex v in S, one of the following two conditions holds.*
(i) v is an isolated vertex of the subgraph induced by S.
(ii) there exists a vertex u in $V \setminus S$ such that $N(u) \cap S = \{v\}$.

Proof. Assuming that S is a minimal dominating set of G, it leads to the fact that for every vertex v in S, $S \setminus \{v\}$ not a dominating set. Consequently, there is a vertex $u \in (V \setminus S) \cup \{v\}$ that is not adjacent to any vertex in $S \setminus \{v\}$. So, now either $u = v$ or $u \in V \setminus S$. But if $u = v$, then v is an isolated vertex of S. Much similar to this, supposing that $u \in V \setminus S$, and u is not dominated by

$S \setminus \{v\}$, but is dominated by S, then it is clear that u is adjacent only to vertex $v \in S$, that is $N(u) \cap S = \{v\}$.

Conversely, suppose S is a non-minimal dominating set. Then there exists a vertex v in S in such that $S \setminus \{v\}$ is a dominating set. Therefore, v is adjacent to at least one vertex in $S \setminus \{v\}$. Thus, condition (i) does not hold for this assertion. Further, in case of $S \setminus \{v\}$ being a dominating set, then every vertex in $V \setminus S$ will be adjacent to at least one vertex in $S \setminus \{v\}$. As a result, for v, condition (ii) does not hold, thereby implying that neither of the conditions holds. □

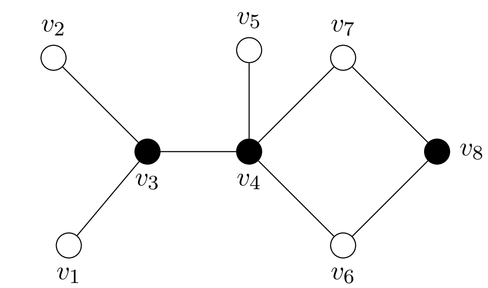


Figure 28

Example 3.20. If we consider a dominating set $S_1 = \{v_3, v_4, v_8\}$ in Figure 28, then $V \setminus S_1 = \{v_1, v_2, v_5, v_6, v_7\}$.

It can be seen that vertices v_3 and v_4 do not satisfy the condition (i), but vertex v_8 satisfy the condition (i). Likewise, vertices v_3 and v_4 satisfy the condition (ii), but vertex v_8 does not satisfy the condition (ii). So, S_1 is a minimal dominating set since each vertex in $\langle S_1 \rangle$ satisfies either condition (i) or condition (ii).

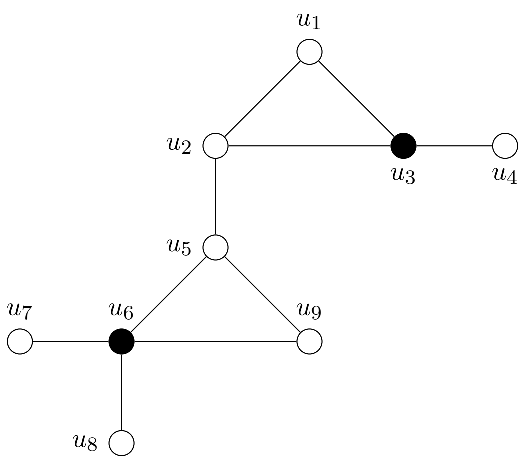
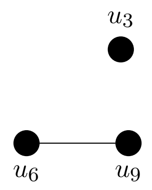


Figure 29

Example 3.21. In this example, we can see that both conditions holds.
In Figure 29, a dominating set is taken for consideration.
$S' = \{u_3, u_6\}$ then $V \setminus S' = \{u_1, u_2, u_4, u_5, u_7, u_8, u_9\}$. Also, every vertex in $\langle S' \rangle$ is an isolated vertex, i.e. there is no edge connectivity between them. Therefore, condition (i) holds.
Hence, S' is a minimal dominating set in a graph G.

Example 3.22. Now, as seen in the Figure 29, a dominating set $S'' = \{u_3, u_6, u_9\}$ be considered. Then $V \setminus S'' = \{u_1, u_2, u_4, u_5, u_7, u_8\}$. Also, $\langle S'' \rangle$ is,



It can be seen that vertices u_6, u_9 are connected, justifying that every vertex in $\langle S'' \rangle$ is not an isolated vertex. Hence, condition (i) does not hold.
In addition, vertex $u_5 \in V \setminus S''$ has two neighbours u_6 and u_9. Therefore, condition (ii) also does not hold. So, S'' is not a minimal dominating set of G.

Theorem 3.23. *(Theorem 1.3, [4]) If G is a graph with no isolated vertices, and if S is a minimal dominating set, then the complement $V \setminus S$ is a dominating set.*

Proof. Consider that S is a minimal dominating set of G and assume that vertex u in S is not adjacent to any vertex in $V \setminus S$. So, u must be adjacent to at least one vertex in $S \setminus \{u\}$ since G has no isolated vertices. It follows that, $S \setminus \{u\}$ is a dominating set, which is contradicting to the minimality of S. □

For instance, consider the graph in Figure 30, where $V = \{a, b, c, d, e, f\}$. Let $S' = \{a, c\}$ be one of the minimal dominating sets of G, then $V \setminus S' = \{b, d, e, f\}$ also is a dominating set.

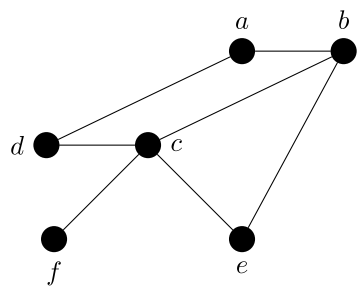
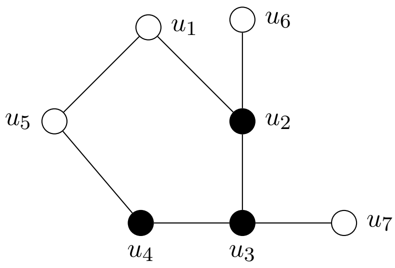


Figure 30

Conversely, a dominating sets $S_1 = \{u_2, u_3, u_4\}$ and $S_2 = \{u_2, u_3, u_4, u_7\}$ are taken for consideration in Example 3 below.
But, S_2 is not a minimal dominating set since $S_1 \subset S_2$. Also, $V \setminus S_2 = \{u_1, u_5, u_6\}$. The vertex $\{u_7\}$ is not dominated by any of the vertices in $V \setminus S_2$. Therefore, $V \setminus S_2$ is not a dominating set.



Example 3

Definition 3.24. A set S of vertices in a graph G is called an *independent set* if no two vertices in S are adjacent in G.

Definition 3.25. An independent set S is called a *maximal independent* set when no proper superset of S is independent.

Definition 3.26. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G.

Definition 3.27. An independent set S of a graph G such that $|S| = \beta_0(G)$ is called a *maximum independent* set of G.

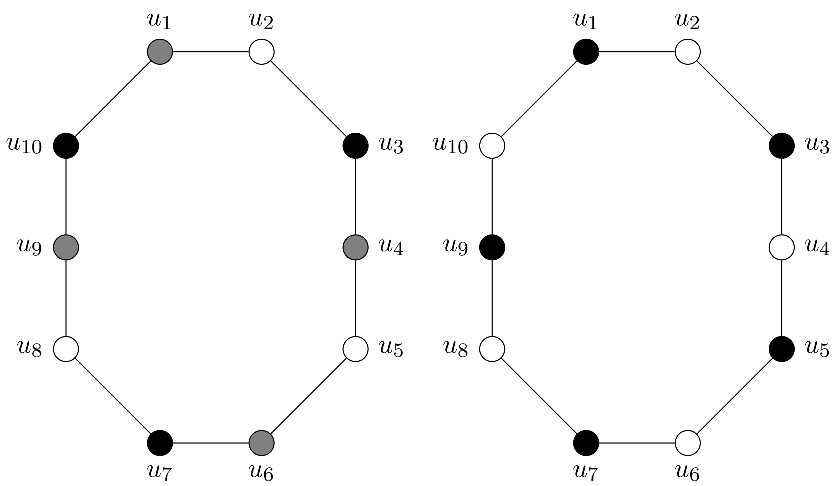


Figure 31

Example 3.28. In the cycle C_{10} of Figure 31,
(i) the sets $\{u_3, u_7, u_{10}\}$, $\{u_1, u_4, u_6, u_9\}$, $\{u_1, u_3, u_5, u_7, u_9\}$ are independent sets.
(ii) the sets $\{u_1, u_3, u_5, u_7, u_9\}$,$\{u_2, u_4, u_6, u_8, u_{10}\}$ are maximal independent sets.
For the cycle C_{10}, $\beta_0(C_{10}) = 5$.

Theorem 3.29. *(Proposition 3.5, [4]) An independent set S is maximal independent set if and only if it is independent and dominating.*

Proof. Let an independent set S be maximal. This means that the set $S \cup \{u\}$ for every vertex u in $V \setminus S$ can not be independent. This means that there is a vertex v in S for every vertex u in

$V \setminus S$ in such a way that u is adjacent to v. Therefore, it can be asserted that S is a dominating and independent set.

Now suppose S is both independent and dominating. We want to prove that S is maximal independent. Conversely, suppose S is not maximal independent, there is a vertex u in $V \setminus S$ exists in such a way that $S \cup \{u\}$ is independent, then S will be dominating set. But when no vertex exists adjacent to u in S, even $S \cup \{u\}$ is independent. Hence S cannot be a dominating set, which is a contradiction. Therefore S is a maximal independent set. □

Example 3.30. From the above Figure 31, an independent set $S = \{u_1, u_3, u_5, u_7, u_9\}$ is maximum independent. Then, $V \setminus S = \{u_2, u_4, u_6, u_8, u_{10}\}$. Now, let v in $V \setminus S$. Thus, the set $S \cup \{v\}$ is not independent sets since every vertex v in $V \setminus S$ adjacent to at least one vertex in S. By definition 3.1, S is a dominating set. Thus, S is both independent and dominating set.

Theorem 3.31. *(Proposition 3.6, [4]) Every maximal independent set in a graph G is a minimal dominating set of G.*

Proof. Let S be a maximal independent set in a graph G, then S is dominating set by Theorem 3.29. Assuming that S is not a minimal dominating set, there is an obvious existence of at least one vertex v in S in such a way that $S \setminus \{v\}$ is a dominating set. Similarly, at least one vertex in $S \setminus \{v\}$ is adjacent to v, when $S \setminus \{v\}$ dominates $V \setminus (S \setminus \{v\})$. This is a contradiction to the assertion that S is an independent set in G, whereby it can be determined that S will be a minimal dominating set. Therefore, every maximal independent set in a graph G is a minimal dominating set of G. □

For instance, in the above Figure 31, let $S_1 = \{u_2, u_5, u_8, u_{10}\}$ is a maximal independent set. Then, every vertex in $V \setminus S_1$ is adjacent to at least one vertex in S_1. Thus, S_1 is a dominating set. But, $S_1 \setminus \{u_2\} = \{u_5, u_8, u_{10}\} = S_2$ is not a dominating set since at least one vertex in $V \setminus S_2$ is not adjacent to vertex in S_2. Therefore, S_1 is both a maximal independent set and a minimal dominating set of graph G.

Theorem 3.32. *For any graph G, $\gamma(G) \leq \beta_0(G)$.*

Proof. By definition 3.27, $|S| = \beta_0(G)$ if S a maximum independent set of vertices in G, which leads to G having no larger independent set. Then, every vertex $v \in V \setminus S$ is adjacent to at least one vertex of S. Thus $\gamma(G) \leq |S|$, since S is a dominating set. Therefore, $\gamma(G) \leq \beta_0(G)$. □

For instance, in the Figure 32 below: let $S = \{u_1, u_4, u_6\}$, then $V \setminus S = \{u_2, u_3, u_5\}$. Thus, S is a dominating set since every vertex of $V \setminus S$ is adjacent to at least one vertex of S. So, S is an independent set since induced subgraph $\langle S \rangle$ contains isolated vertices. Also, $|S|$ is maximum and $|S| = \beta_0(G) = 3$. Thus, $\gamma(G) \leq \beta_0(G)$.

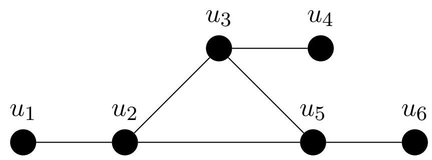


Figure 32

4

Bounds of domination number

In this chapter, we are now going to discuss upper and lower bounds for $\gamma(G)$.

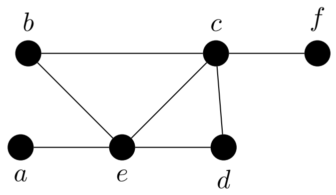


Figure 33

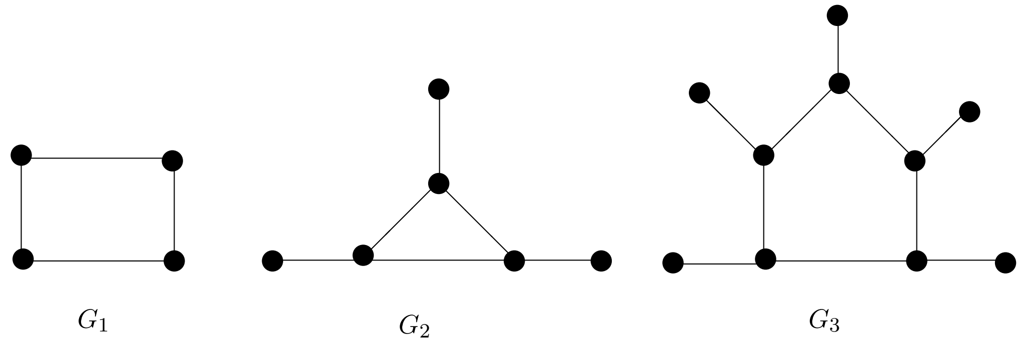
From the above Figure 33: $V = \{a, b, c, d, e, f\}$ and let $v' = \{e\}$, vertex of maximum degree $\Delta(G)$. Then $N(v') = \{a, b, c, d, \}$, so v' dominates the set $N(v') \cup \{e\}$.
And $V \setminus N(v') = \{e, f\}$, it is clearly that $V \setminus N(v')$ is a dominating set. Also, $|N(v')| = \Delta(G)$.
Similarly, if we let $v'' = \{d\}$ then we have $N(v'') = \{e, c\}$ and $V \setminus N(v'') = \{a, b, d, f\}$. Thus, v'' dominates the set $N(v'')$ and complement of $V \setminus N(v'')$ is a dominating set. Also, $\Delta(G) > |N(v'')|$.
These assertions lead to the following observations:

Observation 1: Every vertex v in $V(G)$ dominates $N(v)$ vertices in G.
Observation 2: Let v be any vertex in G, then $V \setminus N(v)$ is a dominating set of G.
Observation 3: $\Delta(G) \geq |N(v)|$ for every vertex $v \in V(G)$.

Theorem 4.1. *(Theorem 2.1, [4])(Ore's Theorem)*
If G is a graph with no isolated vertices, then

$$\gamma(G) \leq n/2.$$

Proof. Let S be a minimal dominating set of graph G. Graph G is a connected graph of order $n \geq 2$ since it has no isolated vertices. On the contrary, suppose that $\gamma(G) > n/2$. By Theorem 3.23, $V \setminus S$ is a dominating set of G. So, $|V \setminus S| < n - n/2$. Hence, we conclude that $\gamma(G) \leq min\{|S|, |V \setminus S|\} \leq n/2$. □



Connected graph G

Figure 34

Example 4.2. In the above Figure 34 it can be seen that graph G_1 has $n = 4$ and $\gamma(G_1) = 2$, and also $\gamma(G_1) = n/2$.
Similarly, it is true for $\gamma(G_2)$ and $\gamma(G_3)$.

Theorem 4.3. *(Theorem 2.11, [4]) For any graph G,*

$$\gamma(G) \leq n - \Delta(G).$$

Proof. Let S be a dominating set of G. Then by Observation 2, $V \setminus N(v)$ is a dominating set of G for any vertex v in $V(G)$. Therefore, $\gamma(G) \leq |V \setminus N(v)| = |V| - |N(v)| = n - \Delta(G)$ by Observation 3. Thus, $\gamma(G) \leq n - \Delta(G)$. □

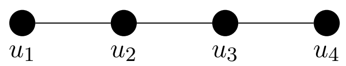


Figure 35

Example 4.4. In the above Figure 35,
$n = 4$, $\Delta(G) = 2$, $\gamma(G) = 2$ and $n - \Delta(G) = 2$, so $\gamma(G) = n - \Delta(G)$.

Example 4.5. In the Figure 36 below, $n = 5$, $\Delta(G) = 2$, $\gamma(G) = 2$ and $n - \Delta(G) = 3$, so $\gamma(G) < n - \Delta(G)$.

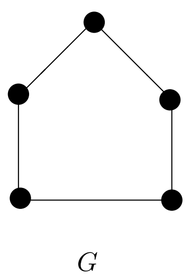


Figure 36

Theorem 4.6. *(Theorem 2.11, [4]) For any graph G,*

$$\gamma(G) \geq \lceil \frac{n}{1+\Delta(G)} \rceil.$$

Proof. Let S be a dominating set of G. Then, every vertex v in $V(G)$ can dominate at most itself and $N(v)$ vertices in G by Observation 1. Since $|N[v]| \leq \Delta(G)$, this necessitates at least $\lceil \frac{n}{1+\Delta(G)} \rceil$ closed neighbourhood, to cover every vertex of graph G. Hence the result, $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G)$. □

4.1 Bounds in term of number of vertices and edges

The following theorem establishes a bound on the number of edges in terms of the domination number.

Theorem 4.7. *(Theorem 2.20, [4]) If a graph G has domination number $\gamma(G)$, then*

$$m \leq \lfloor \frac{1}{2}(n-\gamma(G))(n-\gamma(G)+2) \rfloor,$$

wherein m represents the number of edges in G, and n refers the number of vertices.

Proof. If $\gamma(G) = 1$, then we get the inequality

$$m \leq \lfloor \frac{1}{2}(n-1)(n-1+2) \rfloor = \frac{1}{2}(n-1)(n+1).$$

This inequality always holds, because $m \leq \frac{1}{2}n(n+1)$ in every graph.

Suppose first that $\gamma(G) = 2$ (it is not logically necessary to include the case $\gamma(G) = 2$, but we do it to illustrate the idea). In this case, we want to prove the following inequality

$$m \leq \lfloor \frac{1}{2}(n-\gamma(G))(n-\gamma(G)+2) \rfloor = \frac{1}{2}n(n-2).$$

By Theorem 4.3, we have $2 \leq n - \Delta(G) \iff \Delta(G) < n-2$ which means that the degree of each vertex is at most $n-2$. And by Theorem 2.12, $m \leq \frac{1}{2}n(n-2)$.

Now suppose G be a graph of order n, size m. We use induction on the order n. The basic case $n = 1$ is obvious. We assume that the inequality holds for all graphs having order less than n. Let v be a vertex of maximum degree $\Delta(G)$. By Theorem 4.3 we know that $|N(v)| = \Delta(G) \leq n - \gamma(G)$. Thus, $|N(v)| = \Delta(G) = n - \gamma(G) - r$, where $0 \leq r \leq n - \gamma(G)$.

Let $S = V \setminus N[v]$. If S is empty, then $\gamma(G) = 1$, a case that we discussed already. We will assume that S is not empty, then (notice that in this case S is not a dominating set, but $S \cup \{v\}$ is a dominating set)

$$|S| = |V| - (|N(v)| + |\{v\}|) = n - \Delta(G) - 1 = \gamma(G) + r - 1. \tag{1}$$

Then, the m (size of G) can be divided into three parts such as,

$$m = m_1 + m_2 + m_3, \quad where$$

m_1 is the number of edges in induced graph $\langle S \rangle$
m_2 is the number of edges between $N(v)$ and S and
m_3 is the number of edges in induced graph $\langle N[v] \rangle$.

Also, suppose D is a minimum dominating set of $\langle S \rangle$, then it follows that $D \cup \{v\}$ is a dominating set of G. Therefore, $\gamma(G) \leq |D \cup \{v\}|$, by which it is implies that $\gamma(S) \geq \gamma(G) - 1$.

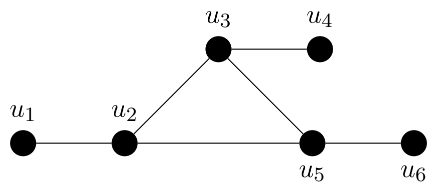
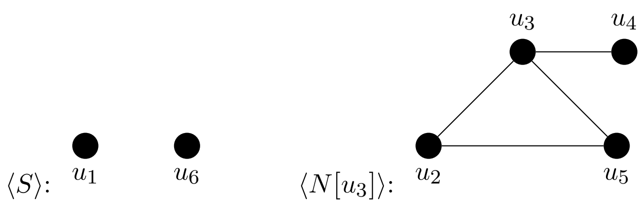


Figure 37

For instance, if a graph in Figure 37 is taken for consideration, $S = V \setminus N[u_3]$ and $N[u_3] = \{u_2, u_3, u_4, u_5\}$. Then, $S = \{u_1, u_6\}$.

So, subgraph induced by S and $N[u_3]$:



Furthermore, $D = \{u_1, u_6\}$ is a minimum dominating set of $\langle S \rangle$, i.e. it dominates itself, then $D \cup \{u_3\} = \{u_1, u_3, u_6\}$ is a dominating set of G. Hence $\gamma(G) \leq |D \cup \{u_3\}|$, that is

$$\gamma(G) \leq |D| + 1. \tag{2}$$

By the inductive hypothesis which implies that by substitution of equation (1) and inequality (2) that (the number of edges in $\langle S \rangle$, say m_1 is),

$$m_1 \leq \lfloor \frac{1}{2} (|S| - |D|) (|S| - |D| + 2) \rfloor$$

$$\leq \lfloor \frac{1}{2} (\gamma (G) + r - 1 - (\gamma (G) - 1)) (\gamma (G) + r - 1 - (\gamma (G) - 1) + 2) \rfloor$$

$$= \frac{1}{2} r(r + 2).$$

$$m_1 \leq \frac{1}{2} r(r + 2). \tag{3}$$

It is not hard to see that if u in $N(v)$, then $(S \setminus N(u)) \cup \{u, v\}$ is a dominating set of G.

For example, if the graph G (Figure 39) is considered that u_3 in $V(G)$, and $S = V \setminus N[u_3] = \{u_1, u_6\}$, it leads that $N(u_4) = \{u_3\}$ and $N(u_3) = \{u_2, u_4, u_5\}$. Therefore,

$$(S \setminus N(u_3)) \cup \{u_3, u_4\} = \{u_1, u_6\} \cup \{u_3, u_4\} = \{u_1, u_3, u_4, u_6\}$$

is a dominating set of G.

Therefore,

$$\gamma(G) \leq |S \setminus N(u)| + |\{u, v\}| = |S| - |S \cap N(u)| + 2$$

Then, by substitution of equations (1)

$$\gamma(G) \leq ((\gamma(G) + r - 1) - |S \cap N(u)| + 2.$$

It implies that $|S \cap N(u)| \leq r + 1$, for each vertex u in $N(v)$.

From this,

$$m_2 \leq |N(v)| \cdot (r + 1) = \Delta(G) \cdot (r + 1). \tag{4}$$

As we mentioned above, $m_3 = |E(\langle N[v] \rangle)|$.

So, our next goal is to estimate $m_2 + m_3$. $m_2 + m_3$ is the number edges that are adjacent to a vertex in $N(v)$. For every vertex u of $N(v)$, let $i(u)$ be the number of edges connecting u to other vertices of $N(v)$, and let $e(u)$ be the number edges from u to S. We think of $i(u)$ and $e(u)$ as the internal and external degree of u respectively. There also is an edge from u to v, so the total degree of u is $1 + i(u) + e(u)$.

We know that $i(u) + e(u) \leq \Delta(G) - 1$, because $\Delta(G)$ is the maximal degree of a vertex.

It is easy to see that

$$m_3 = \Delta(G) + \frac{1}{2} \sum_{u \in N(v)} i(u).$$

The first summand counts the edges from v to $N(v)$, and the second summand counts the internal edges of $N(v)$.

Similarly, we have

$$m_2 = \sum_{u \in N(v)} e(u).$$

Then,

$$m_2 + m_3 = \Delta(G) + \sum_{u \in N(v)} \frac{1}{2} i(u) + e(u).$$

We can write this,

$$2(m_2 + m_3) = 2\Delta(G) + \sum_{u \in N(v)} i(u) + 2e(u). \tag{5}$$

We know that for each u, $i(u) + e(u) \leq \Delta(G) - 1$. It follows that

$$\sum_{u \in N(v)} i(u) + e(u) \leq \Delta(G)(\Delta(G) - 1). \tag{6}$$

On the other hand, we saw above that $e(u) \leq r + 1$ for all u. Therefore

$$\sum_{u \in N(v)} e(u) \leq \Delta(G)(r + 1). \tag{7}$$

Add the two inequalities (6) and (7) we obtain

$$\sum_{u \in N(v)} i(u) + 2e(u) \leq \Delta(G)(\Delta(G) + r). \tag{8}$$

If we substitute this inequality (8) in inequality (5), we get

$$2(m_2 + m_3) \leq 2\Delta(G) + \Delta(G)(\Delta(G) + r).$$

It follows that

$$m_2 + m_3 \leq \frac{1}{2}\Delta(G)(\Delta(G) + r + 2).$$

Now we substitute $r = n - \gamma(G) - \Delta(G)$, to obtain inequalities of m_2 and m_3

$$m_2 + m_3 \leq \frac{1}{2}\Delta(G)(n - \gamma(G) + 2). \tag{9}$$

If we substitute $r = n - \gamma(G) - \Delta(G)$ in inequality (3), we get

$$m_1 \leq \frac{1}{2}(n - \gamma(G) - \Delta(G))(n - \gamma(G) - \Delta(G) + 2)$$

$$m_1 \leq \frac{1}{2}[(n - \gamma(G))(n - \gamma(G) + 2) + \Delta(G)(\Delta(G) + 2\gamma(G) - 2n - 2)]. \tag{10}$$

Consequently, the size of G is

$$m = m_1 + m_2 + m_3.$$

By substitution of inequalities (9) and (10),

$$m \leq \frac{1}{2}[(n - \gamma(G))(n - \gamma(G) + 2) - \Delta(G)(n - \gamma(G) - \Delta(G))]$$

The right hand side equals to

$$\frac{1}{2}[(n - \gamma(G))(n - \gamma(G) + 2) - \Delta(G)\underbrace{(n - \gamma(G) - \Delta(G))}] = \frac{1}{2}[(n - \gamma(G))(n - \gamma(G) + 2) - \Delta(G)r]$$

We know that $0 \geq -\Delta(G) = \gamma(G) + r - n$, so we finally conclude that

$$m \leq \lfloor \frac{1}{2} \left(n - \gamma\left(G\right)\right) \left(n - \gamma\left(G\right) + 2\right) \rfloor.$$

□

Remark. If we like to know more about Theorem 4.7, and the results that come afterward, we can find it in Theorem 2.20 in Haynes [4], and also in Vizing [10].

Furthermore, we can find in Appendix A an alternative way to get m_3.

Corollary 4.8. *(Theorem 2.22, [4]) For any graph G, $\gamma(G) \leq n + 1 - \sqrt{1 + 2m}$.*

Proof. From Theorem 4.7, we have

$$2m \leq (n - \gamma(G))(n - \gamma(G) + 2) = n^2 + \gamma(G)^2 - 2n\gamma(G) - 2\gamma(G) + 2n.$$

If number 1 is added on both sides and simplify in the right side of the equation, it results with

$$1 + 2m \leq (n - \gamma(G) + 1)^2.$$

Since $n - \gamma(G) \geq 0$, it ends up with

$$\sqrt{1 + 2m} \leq n - \gamma(G) + 1.$$

Thus, $\gamma(G) \leq n + 1 - \sqrt{1 + 2m}$. □

Theorem 4.9. *(Theorem 2.24, [4]) If G is a connected graph, then*

$$\lceil \frac{diam\left(G\right) + 1}{3} \rceil \leq \gamma\left(G\right).$$

Proof. Let S be a dominating set of G. Let $P = v_1, v_2, \ldots, v_n$ be a shortest path from v_1 to v_n in G. We want to show that $n \leq 3|S|$. Since S is a dominating set, the closed stars of elements of S cover G. Since the diameter of a star is 2, we can assume that P never leaves a star and then comes back to it. Therefore, we can assume that P consists of at most $|S|$ chunks corresponding to stars it goes through, and that each chunk consists of at most 3 vertices. It follows that $n \leq 3|S|$. Remember that the length of an open path is $n - 1$ (see def 2.28). So, we prove that for any dominating set S, and any shortest path of length $l = n - 1 \Rightarrow n = l + 1 \leq 3|S|$. But, if we take the longest possible shortest path in G, its length is $l = diam$, and we have proved that $diam + 1 \leq 3|S|$. Thus, $\lceil \frac{diam(G)+1}{3} \rceil \leq \gamma\left(G\right)$. □

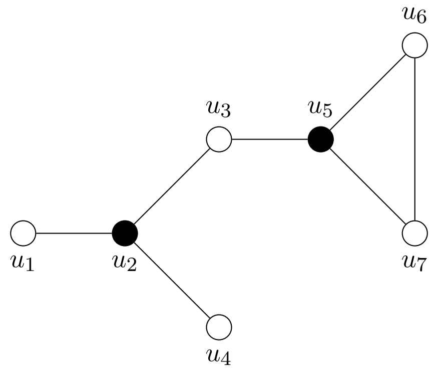


Figure 38

Example 4.10. For instance, let $S = \{u_2, u_5\}$ is a dominating set of a connected graph G in Figure 38 since every complement of set $V \setminus S$ is adjacent to at least one vertex in S. Then, it gives $diam(G) = 4$ if an arbitrary path of length, i.e. $u_1 \to u_2 \to u_3 \to u_5 \to u_6$ is considered.

This diametrical path includes a maximum of two edges from the induced subgraph $\langle N[v] \rangle$ for each $v \in S$. This can be seen in the below Figure 39, wherein the induced graph $\langle N[u_2] \rangle$ consists of u_1u_2 , u_2u_3 edges and the induced graph $\langle N[u_5] \rangle$ contains u_3u_5 , u_5u_6 edges, which includes in the diametrical path.

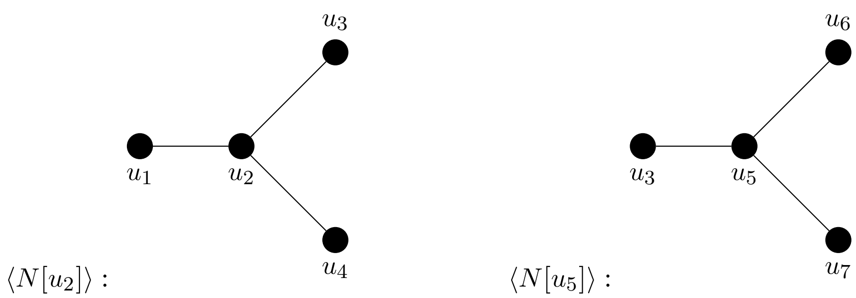


Figure 39

In addition, since S is a minimum dominating set, the diametral path comprises at most $\gamma(G) - 1$ edges joining the neighbourhoods of the vertices in S.

Further, $diam(G) \leq 2\gamma(G)$ because every vertex contributes two edges to this diametral path. Hence, $diam(G) \leq 2\gamma(G) + \gamma(G) - 1 = 3\gamma(G) - 1$ and the result is $\lceil \frac{diam(G)+1}{3} \rceil \leq \gamma(G)$.

For example, in the graph (Figure 38), $diam(G) = 4$ and $\gamma(G) = 2$ is observed. Then,

$$\lceil \frac{diam\,(G) + 1}{3} \rceil = \lceil \frac{4+1}{3} \rceil = \lceil 1.67 \rceil,$$

which is equal to 2. So, the desired result is true.

Definition 4.11. The length of a shortest cycle in a graph G which contains cycles is the *girth* $g(G)$. If a cycle has length $g(G)$ then it is called a g-cycle.

Theorem 4.12. *(Theorem 2.28, [4]) If a graph G has* $g(G) \geq 5$, *then*

$$\delta(G) \leq \gamma(G).$$

Proof. Let S be considered as a γ-set of a graph G. Then, the result will be true for $\delta(G) = 1$. Suppose $\delta(G) \geq 2$. By recalling Theorem 4.3, $\gamma(G) \leq n - \Delta(G) \leq n - \delta(G) \leq n - 2$, it is proved that $|V \setminus S| \geq 2$.

Let u in $V \setminus S$. A cycle C_3 or C_4 is formed if a vertex of S dominates one or more vertices in $N(u)$. Hence, there arises a contradiction with $g(G) \geq 5$. So, it can be presumed that at most one vertex in $N(u)$ is dominated by each vertex of S. Therefore, $\gamma(G) \geq |N(u)| \geq \delta(G)$. □

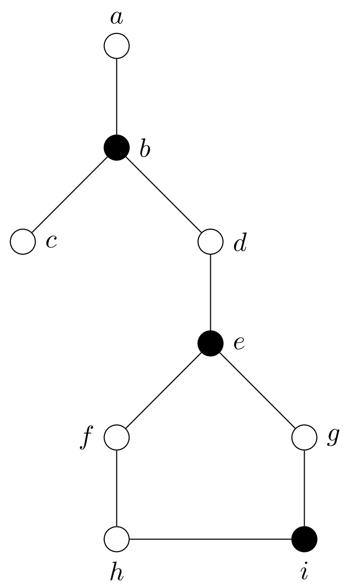


Figure 40

Example 4.13. Let $S = \{b, e, i\}$ be a dominating set of a graph G as seen in Figure 40. Then, the result is true for $\delta(G) < \gamma(G)$. That is, we have $\delta(G) = 1, \gamma(G) = 3$ and $g(G) = 5$.

If a cycle graph C_6 is considered, then it ends up with $\delta(G) = 2, \gamma(G) = 2$ and $g(G) = 6$. Thus, it is true for $\delta(G) = \gamma(G)$. Therefore $\gamma(G) \geq \delta(G)$.

Theorem 4.14. *(Theorem 2.28, [4]) For any graph G if* $g(G) \geq 6$, *then* $\gamma(G) \geq 2(\delta(G) - 1)$.

Proof. Let it be assumed that $g(G) \geq 6$. If $V \setminus S$ is found to have adjacent vertices u and v, it implies that each vertex of S is prone to dominate at most one vertex in $N(u) \cup N(v) \setminus \{u, v\}$. This leads to the conclusion that $\gamma\ (G) \geq |N\ (u)| + |N\ (v)| - 2 \geq\ 2\,(\delta\ (G) - 1)$.

Similarly, when $V \setminus S$ is an independent set while $\{u, v\} \subseteq\ V \setminus S$, it paves the way for $N(u) \subseteq S$ and $N(v) \subseteq S$. Furthermore, in terms of fact $g(G) \geq 6$, $|N(u) \cap N(v)| \leq 1$ and again $\gamma(G) \geq |N(u)| + |N(v)| - 1 \geq 2\delta(G) - 1$. □

Example 4.15. For instance, if the cycle graph C_{10} in the below Figure 41 is considered, it results with a minimal dominating set $S_1 = \{u_2, u_5, u_7, u_9\}$. Also, $g(G) = 10$, $\delta(G) = 2$ and $\gamma(G) = 4$.

So, it is true for $\gamma(G) > 2(\delta(G) - 1)$.

Similarly, when the cycle graph C_6 is considered, it results with $g(G) = 6$, $\delta(G) = 2$ and $\gamma(G) = 2$.
So, it is true for $\gamma(G) = 2(\delta(G) - 1)$.
Therefore, $\gamma(G) \geq 2(\delta(G) - 1)$.

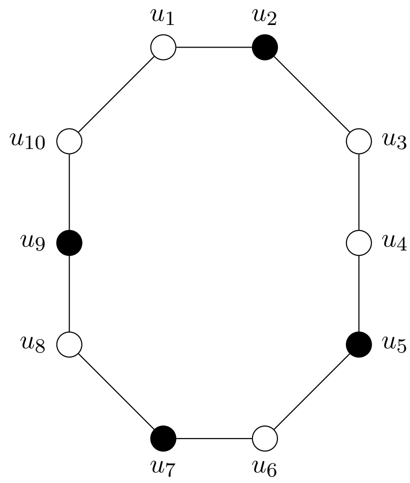


Figure 41

Remark. If we are interesting to know elementary of bounds on domination number (i.e. in chapter 4), we can look it particular of Theorem 2.11, Theorem 2.22, Theorem 2.24 and Theorem 2.28 in Haynes [4] (pp.41-57).

5

Final remarks

Graph theory is one of the thriving branches of mathematics and applications. Its growth is due to broad applications for discrete optimization problems and applications in various areas such as communication networks, coding theory, social networks, software design and computation.

Domination in graphs is an extensively researched branch of Graph Theory. In our project, we have studied domination in graphs together with bounds for $\gamma(G)$.

The idea of the domination number $\gamma(G)$ for standard graphs was written from *On the domination number of knödel graph* $W_{3,n}$ written by Xueliang et al. [12].

This thesis was written primarily from *Fundamentals of domination in graphs* written by Haynes et al. [4], pp.1-106. It may be without a comprehensive and clear overview. However, those who wish to gain knowledge in this field may be interested in exploring a more dominating set of this thesis. Even in [3], [4], [8] and [9] we can find several different types of dominating sets if someone is interesting to know about it.

A

Proof of Theorem 4.7

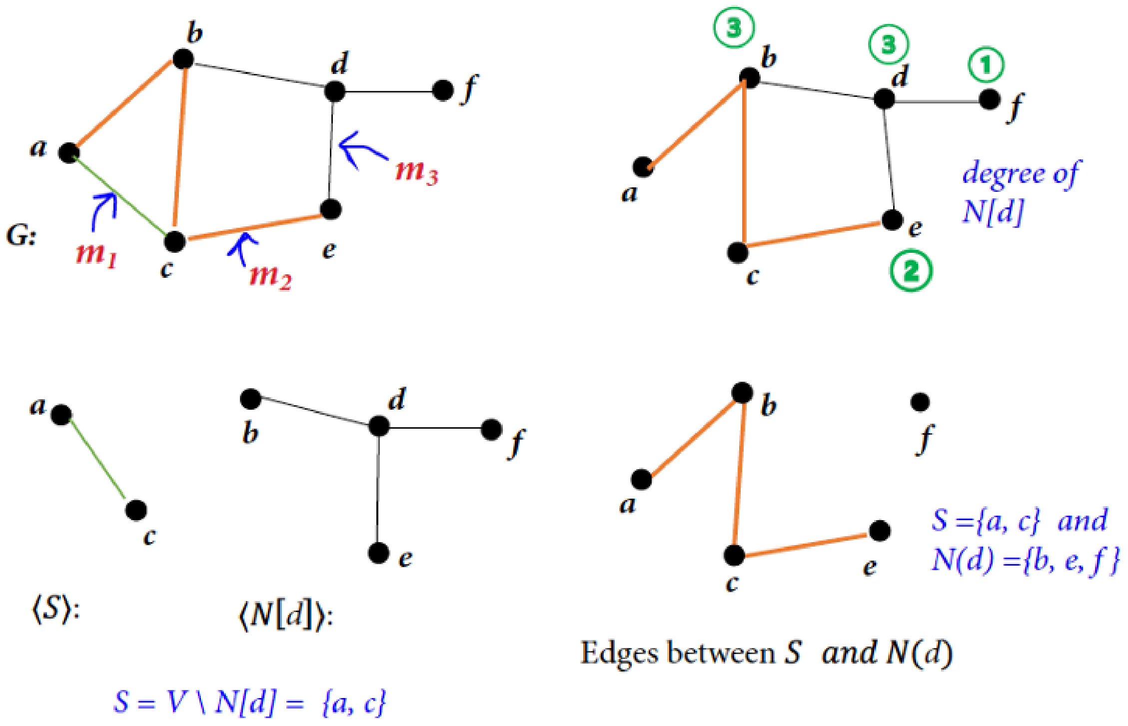
Alternative way to get m_3: As we mentioned before, $m_3 = |E(\langle N[v] \rangle)|$. By Theorem 2.12,

$$\sum_{u \in \langle N[v] \rangle} deg(u) = 2m_3.$$

Let v be a vertex of maximum degree in G. Then, $\Delta(G) + 1$ vertices are dominated by v since v dominates itself and $N(v)$. So, the sum of the degrees of the vertices of $N[v] \leq \Delta(G)(\Delta(G) + 1)$. As there m_2 edges between S and $N(v)$, we get

$$\sum_{u \in \langle N[v] \rangle} deg(u) = \sum_{u \in N[v]} deg(u) - m_2.$$

Illustration:



Therefore,

$$m_3 \leq \frac{1}{2}[\Delta(G)(\Delta(G)+1)-m_2].$$

. It follows that by adding the inequalities m_2 and m_3,

$$m_2 + m_3 \leq \frac{1}{2}\Delta(G)(\Delta(G)+r+2).$$

If we add this inequality together with inequality m_1 and if we substitute $r = n - \gamma(G) - \Delta(G)$, then we get the result finally.

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