# SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

## Hilbert's Axiom System

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Rikard Bögvad och Torbjörn Tambour

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**Abstract**
Greek mathematician Euclid wrote The Elements (Elementa) about 2300 years ago. This book covers many areas including geometry. After 22 centuries, David Hilbert reformulated the geometry results in Euclid's Elements in his work "The Foundations of Geometry". The aim of this thesis is to understand why David Hilbert needed to rewrite the theorems. In particular, we seek to answer the following questions: What were the deficiencies in Euclid's treatment and how did Hilbert ultimately achieve his goal?

Keywords: Hilbert's Axiom, Elementa, The Foundations of Geometry

**Contents**

## 1.INTRODUCTION TO EUCLIDEAN GEOMETRY

Geometry and arithmetics were the two primary fields at the origin of mathematics.

The term ‘geometry’ is derived from the word ‘*land’* and *‘measurement’.* According to ancient Greek geographer, writer, and historian Herodotus (c. 484 – c. 425 BC), both the word and the concept of geometry originated from Egypt.

Geometry gained immense prominence over the years. So much so that the ancient Greek philosopher Plato (c. 428 - 348 BC), who founded the Platonic Academy, placed the inscription “Let no one ignorant of geometry enter here” on the entrance of his Academy. Eudemus of Rhodes (c.370 – c.300 BC), one of the first notable historians of mathematics and one of Aristotle’s two followers, wrote his book “History of Geometry” so already at that time, geometry had a long history, and there was a book describing this history.

Following the death of Alexander, the Great in 323 B.C., his empire was split into three parts. In the Egyptian section there were developments that ensured the reintroduction of science during the reign of the pharaoh Ptolemy I. Schools providing completely free education were established in Alexandria.

The greek mathemathician Euclid of Alexandria wrote 13 books (published c. 300 B.C.) in which all except two (the 5$^{th}$ and 7$^{th}$ book) consisted of geometry, despite this he chose to name the collection of books ‘The Elements’. Euclidean geometry deals with space and shape using a system of logical deductions, with theorems and proofs.

The *Elements* consist of thirteen books / chapters. The content is as follows[1]:

Book 1: Fundamentals of Plane Geometry Involving Straight Lines
Book 2: Fundamentals of Geometric Algebra
Book 3: Fundamentals of Plane Geometry Involving Circles
Book 4: Construction of Rectilinear Figures In and Around Circles
Book 5: Proportion
Book 6: Similar Figures
Book 7: Elementary Number Theory
Book 8: Continued Proportion
Book 9: Applications of Number Theory
Book 10: Incommensurable Magnitudes

[1] [1]

Book 11: Elementary Stereometry
Book 12: Proportional Stereometry
Book 13: The Platonic Solids

In more detail the contents are as follows:

- Book 1 contains 5 postulates (including the famous parallel postulate), 5 common notions, 23 definitions, and 48 propositions. It covers the fundamental propositions of plane geometry such as the Pythagorean theorem, equality of angles and areas, parallelism, the sum of the angles in a triangle, and the construction of various geometric figures.

- Book 2 deals with “geometric algebra” as it contains some lemmas concerning the equality of rectangles and squares.

- Book 3 investigates circles and their properties: finding the center, inscribed angles, theorems on tangents, the power of a point, Thales' theorem.

- Book 4 is concerned with inscribed and circumscribed triangles together with regular polygons with 4, 5, 6, and 15 sides.

- Book 5 studies the arithmetic theory of the proportion.

- Book 6 applies proportions to plane geometry containing theorems on recognition of similar figures.

- Book 7 studies the elementary number theory: divisibility, prime numbers, composite numbers, the greatest common divisor, and the least common multiple.

- Book 8 is concerned with the construction and existence of geometric series.

- Book 9 applies the results of Book 7 and Book 8 including theorems on the infinitude of prime numbers, construction of perfect numbers in addition to geometric series.

- Book 10 classifies the commensurable and incommensurable lines and introducing the term irrational.

- Book 11 generalizes the fundamental propositions of three-dimensional geometry studied in the book 6: perpendicularity, parallelism, volumes, and similarity of parallelepipeds.

- Book 12 studies the volumes of cones, pyramids, cylinders and spheres by using the method of exhaustion, which is a method of approximating its volume by inscribing a union of many pyramids.

- Book 13 investigates the five regular solids which are known as Platonic solids inscribed in a sphere and compares the ratios of their edges to the radius of the sphere.

Euclidean geometry is the basic form of geometry still taught in secondary schools and it has countless practical applications. The general topics are triangles, calculations with angles, circles, squares, polygons and polyhedra, similar figures and congruences. As well as plane geometry, Euclid also included three-dimensional objects and their volumes.

Euclid's method consists in assuming a small set of axioms and deriving many other theorems from these. Although many of Euclid's results had been stated by earlier mathematicians, Euclid was the first to show how these theorems could fit into a comprehensive deductive and logical system. It became a fundamental part of science and mathematics.

## 2.The Axioms of Euclid

Definition of the term "axiom" is needed in order to understand the topic. An axiom is a proposition that cannot be proved but is assumed correct so that it does not need to be proved and later statements can build on the axiom as a starting point. In Euclidean geometry, all theorems are deduced from a small number of simple axioms.

**The Elemen**t is translated to English by various mathematicians, one of the most well-known version's author is J.L. Heiberg. The modern English translation which is used in this thesis was done by Richard Fitzpatrick.

### 2.1. How did Euclid describe the basis of his geometry?

Euclid starts his geometry by giving *definitions*. We include the following definitions which are relevant for our subject.

**Definitions:**[2]

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.

[2] [1] p.6, 7

9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right angle.
12. And an acute angle (is) one less than a right angle.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line which is called a circumference], (such that) all of the straight lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half (*This should really be counted as a postulate, rather than as part of a definition*).
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
23. Parallel lines are straight lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

Later, he continued with his axioms called *postulates* and here is the translation of the Five Axioms (by Fitzpatrick)

**Postulates:**[3]

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

Finally, he gives additional postulates which are called *common notions*. He probably thought that these are more general than above mentioned 5 axioms and could be used in other parts of science and philosophy.

**Common Notions:**[4]

[3] [1] p.6
[4] [1] p.7

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Having described the basis of his geometry, Euclid starts proving theorems in Elementa, in the manner which we still do today. We will however discuss this basis and see in what sense it actually is sound.

## 2.2. What do the axioms and the definitions explain?

**The Five Axioms** *(The word ”axiom” instead of postulate will be used in the rest of the article)***:**

*1. Let it have been postulated to draw a straight-line from any point to any point.*[5]
Axiom 1 says that between two points, you can draw one and only one straight line. In other words, with a straight line segment, you may join two distinct points.

*2. And to produce a finite straight-line continuously in a straight-line.*[6]
Any line segment can be extended infinitely in a long straight line.

*3. And to draw a circle with any center and radius.* [7]
Given two distinct points P and Q, there exists a circle centered at P with radius PQ (radius).

*4. And that all right angles are equal to one another.* [8]
All right angles are congruent to each other.

*5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side)* [9]

[5] [1] p.6
[6] [1] p.6
[7] [1] p.6
[8] [1] p.6
[9] [1] p.6

**The parallel axiom:** There are two straight lines which are intersected by a third straight line, in a way that, the interior angles on the same side are less than two right angles. If produced indefinitely, these two lines would meet on that side on which the angles are less than two right angles.

**Definitions:**

Let's take the definition of **a point** and **a line** and compare them to the modern definitions.

In the Elements, Euclides defines points and lines respectively; (Def.1) a point is that of which there is no part and and (Def.2) a line is a length without breadth. (Def.3) And the extremities of a line are points.[10]

These definitions can not be taken as mathematical definitions because they are dependent on other concepts which are breadth, width and/or length. The phrase "has no part" does not give us a clear explanation about **a point**. What did Euclid mean when he mentioned "no parts"? A modern definition might be that they are points in a coordinate system given by two real numbers. But coordinate systems is a modern invention so perhaps the idea of Euclid is to define concepts by giving some (but not all) of their important properties.

**Line:** The definition of lines is open to the same criticism as above. In a coordinate system $R^2$, the definitions would be as the set of solutions to equations $ax+by=c$.

### 2.3. "What are the problems of Euclid's Axiom?

Aside from the problem with non-precise definitions, which we already discussed, there are problems with Euclid's axioms in that not everything that is assumed is explicitly described.

Let us give an example.

#### 2.3.1 Equilateral Triangles

In Book Prop 1, Euclid wants to prove that equilateral triangles do exist. Here is his result and his proof.

[10] [1] p.6

Here is his result and his proof. The following description of the first proposition of Elementa and its proof is taken (and translated by me) from Tengstrand.[11]

***Book 1 Proposition 1:*** To construct an equilateral triangle on a finite straight line.

***Proof of Prop.1:***
Let the given finite straight line be AB.
In order to construct the required triangle, point A is taken as the midpoint and line AB as the radius *(Postulate 3: To describe a circle with any center and radius).*
Point B is taken as the midpoint of the other circle with radius BA. (*Postulate 3* is used again.)

<u>Created circles intersect at point C.</u>

In the next step, point C is connected with point A and point B. *(Postulate 1: To draw a straight line from any point to any point.)*
A triangle is constructed as a result of joining point C with point A and B and it is needed to prove that the sides are equal.
Circle with midpoint A has segment AB as radius. Segment AC is also the radius.

Circle with midpoint B has segment BA as radius. Segment BC is also the radius.

Circle with midpoint A has segment AB and AC as radius which leads to the fact that segment AB = segment AC. Again, circle with midpoint B has segment BA and BC as radius which means that BA= BC.
If AB is equal to AC and BC, then AC = BC (Notion *1: Things which equal the same thing also equal to each other).*
The three sides of the constructed triangle are equal; therefore, it is an equilateral triangle. ■ [12]

Tengstrand summarizes:
The above mentioned proof is a typical form of construction of proofs in the Elementa. As we see, Euclid first formulates a problem then he specifies what is given and what is searched. Then, the problem is solved and ended with

[11] [2] p.53
[12] [2] p.53

the words "which would be proven". In this way, the presentation becomes very clear.[13]

**Deficits in Euclid's Proof:**

Euclid assumes that the two circles intersect in the point C. There is not any clear argument using the axioms that two circles intersect. In other words, these two circles may not necessarily meet in a point, therefore Euclid's proof is not complete. We will show this in an example.

Let us interpret points and lines, differently from Euclid, in $Q^2$, that is the set of points in a coordinate system with rational coordinates. The plane consists of points whose coordinates are rational numbers, lines are lines through these different points in $Q^2$, circles with center in a poinR in $Q^2$ and radius PR where $R \in Q^2$.

**All of the axioms are true**

We know that the axioms of Euclid as well as all theorems are true in $R^2$, with the usual interpretation of points, lines, circles etc. In our interpretation we have the following table.

| | **How we interpret:** | **Usual Interpretation** |
|---|---|---|
| points | Points in $Q^2$ : coordinates are arbitrary rational numbers | Points in $R^2$ : coordinates are arbitrary real numbers. |
| lines | Lines through two points in $Q^2$, that is lines given by equations ax+by+c=0 where $a,b,c \in Q$, by Lemma 1 below. | Lines through 2 points in $R^2$ that is lines given by equations ax+by+c=0, a, b, $c \in R$ |
| circles | Circles with center in a point P in $Q^2$ and radius which is a distance PR where $R \in Q^2$. | Circles with center in a point in $R^2$ and radius an arbitrary real number |
| segment | If P and S are points in $Q^2$ and $\ell$, the line through P and S, then the segment consists of all points in $Q^2$ between P and S on the line that lie between P and S on $\ell$. | If P and S are points in $R^2$ and $\ell$, the line through them, then the segment consists of all points in $R^2$ between P and S on the line that lie between P and S on $\ell$. |

[13] [2] p.52,53

**Lemma 1:** Suppose ℓ goes through P = (α, β) ∈ $Q^2$ and S = (γ, δ) ∈ $Q^2$. Then ℓ has an equation of the form ax + by – c = 0 where a, b, c ∈ Q.

**Proof:** By the 2-point formula (a and b), ℓ has an equation of the form

$$\frac{(\delta-\beta)}{(\gamma-\beta)} \cdot (x - \alpha) + (\gamma - \beta) = 0, \text{ where}$$

$\frac{(\delta-\beta)}{(\gamma-\beta)}$ is in Q. We can write this as;

$$\frac{(\delta-\beta)}{(\gamma-\beta)} \cdot x + y - \left( \alpha \cdot \frac{(\delta-\beta)}{(\gamma-\beta)} + \beta \right) = 0$$

and since $a = \frac{(\delta-\beta)}{(\gamma-\beta)}$ is a rational number,

as well as $c = \alpha \cdot \frac{(\delta-\beta)}{(\gamma-\beta)} + \beta$,

we are done. If γ= α this does not work, but then the equation of the line is x= α, so again an equation with rational coefficients.

**Lemma 2:** Suppose that $l_1$ and $l_2$ are two lines in our model of Euclid in $Q^2$. If they intersect in a point M as lines in $R^2$, then m ∈ $Q^2$.

**Proof:**
The point of intersection (x , y) satisfies the equation of both lines

$$a_1x + b_1y - c_1 = 0$$
$$a_2x + b_2y - c_2 = 0 \qquad \text{(by Lemma 1)}$$

And if we solve this equation system, for x, y ; the solution will be given as a rational expression in $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$ , that is a rational number. ■

So we know what points, lines and circles are!

Let us check the axioms in our model.

Axiom 1: If P, S are two pints in $Q^2$ then, there is a segment between them, as in the table.

Axiom 2: Given a segment between P and S there is obviously by our definition of what a line is a line through P and S meaning that we can continue the segment.

Axiom 3: Obvious by our definition of a circle.

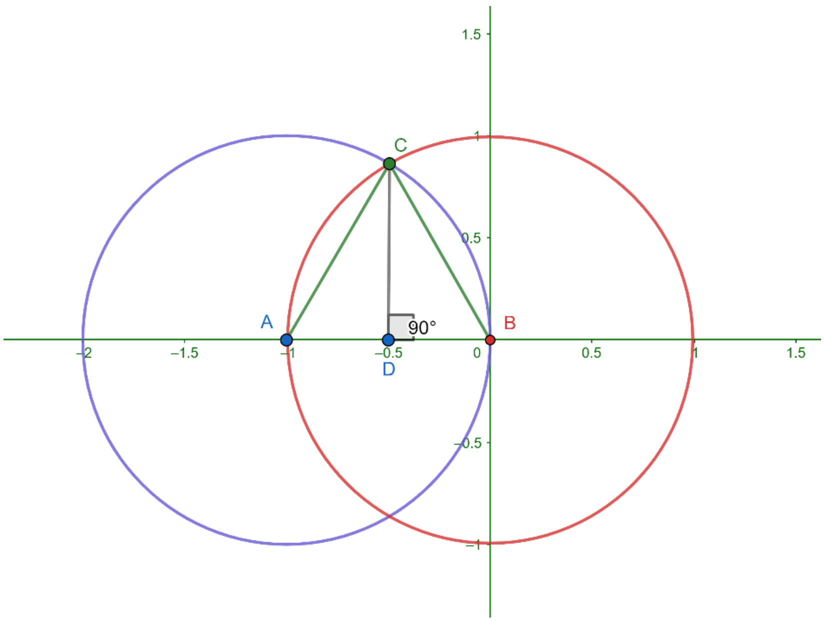
Axiom 4: Obvious.

Axiom 5: This is true in $R^2$, so there is an intersection point m, but according to Lemma 2, this intersection point has to have rational coordinates (since the lines we are considering have to be given by equations with rational coefficients by Lemma 1)

Hence all axioms are true, not only in the usual model of Euclidean geometry, in $R^2$ but in our model as well. But does Proposition 1 hold?

We will see that the coordinates of the equilateral triangle ABC can not be represented in $Q^2$ in a plane since the coordinates of point C are non-rational numbers ($Q^2$).

**Lemma:** There is no equilateral triangle with corners $Q^2$ in which two of the corners are respectively point A (-1,0) and point B (0,1).
(see the following figure)



**PROOF:**

As we see here, segments AB, AC abd BA are all equal to 1. Let CD be the height of the triangle ABC. In order to find the length of the height, CD, the Pythagoras Theorem is used.

$|CD|^2 + (1/2)^2 = 1 \qquad \Leftrightarrow$

$|CD|^2 = 1 - 1/4 = 3/4 \qquad \Leftrightarrow$

$|CD| = \sqrt{3/4} \qquad \Leftrightarrow$

$|CD| = \frac{\sqrt{3}}{2}$.

Point C is not a point in $Q^2$ because its coordinates are ( -0.5, $\frac{\sqrt{3}}{2}$), and $\frac{\sqrt{3}}{2}$ is an irrational number which does not belong to the rational numbers Q but above mentioned $R^2$. □

*Irrational numbers are real numbers that can not be expressed as the fraction of two integers.* [14]

**But why does Euclid's argument not function now?** Euclid uses the two circles with centers in A and B with the same radius and claims that they are intersecting each other. This is not true in our model as we see from the above theorem and thus indicates that Euclid uses more assumptions, postulates and "common notions" than he has written down. Thus, we must supplement Euclid's axiom with more assumptions. This was what Hilbert did.

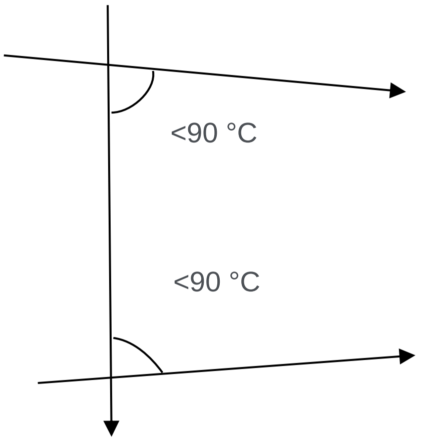
### 2.3.2 Parallel Axiom

There is another type of criticism one may make against Euclides, which refers to the last axiom.

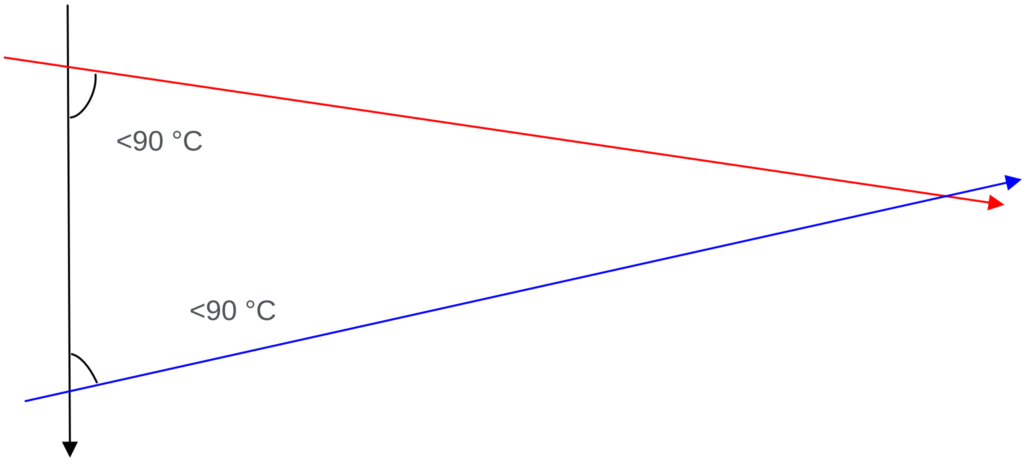
The parallel axiom differs from the other axioms because it is not easy to formulate and the definition is not considered by everyone to be as obvious as the other four postulates. The question was raised early on on whether it was possible to prove it with the help of the first four (thus, whether it could be removed from the set of axioms). Already the Greek philosopher Proclus, who lived in the 5th century, is known to have made the earliest attempts on

[14] [3]

Euclid's Elements in this context illustrating this type of criticism of Euclid. We illustrate it by a picture.




The two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.




The intersection points may be very far away, thus this is for example not reasonable to check it by paper, the intersection point may be somewhere beyond the next city.

**The problem of parallel lines**

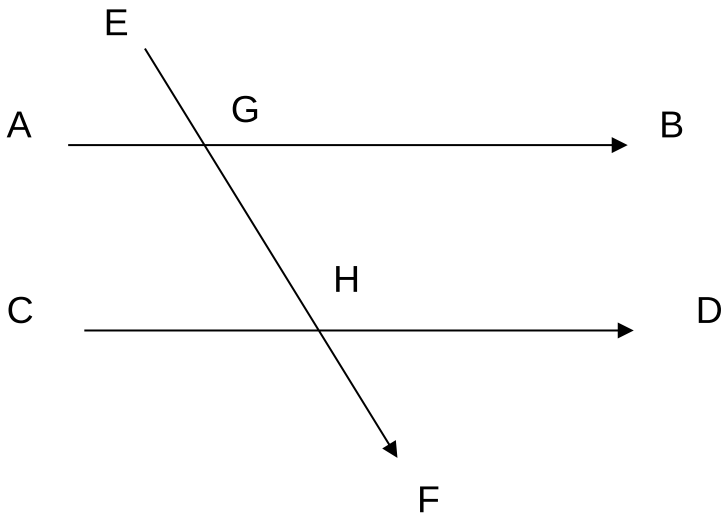
Euclid himself may possibly have doubted it because he avoided postulate 5. He uses it first in proposition 29. It may therefore be interesting to study the wordings of both propositions 28 and 29[15].

**Proposition 28**

If a straight-line falling across two straight-lines makes the external angle EGB equal to the internal and opposite angle GHD on the same side, or

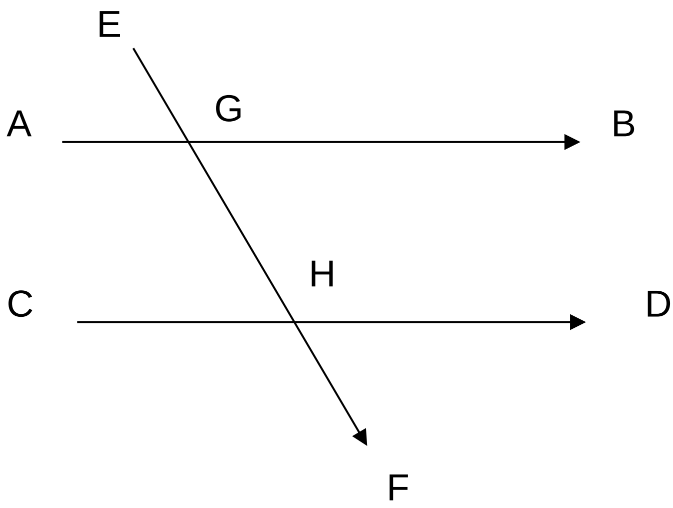
[15] [2] p.260

(makes) the (sum of the) internal (angles) on the same side (BGH and GHD) equal to two right angles, then the (two) straight-lines AB and CD will be parallel to one another[16].




**Proposition 29**

A straight-line EF falling across parallel straight lines AB and CD makes the alternate angles AGH and GHD equal to one another, the external (angle) EGB equal to the internal and opposite (angle) GHD, and the (sum of the) internal (angles) BGH and GHD on the same side equal to two right-angles[17].

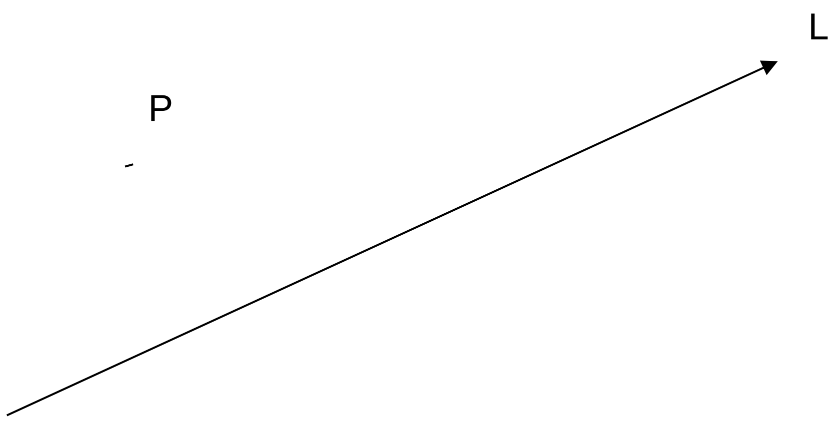


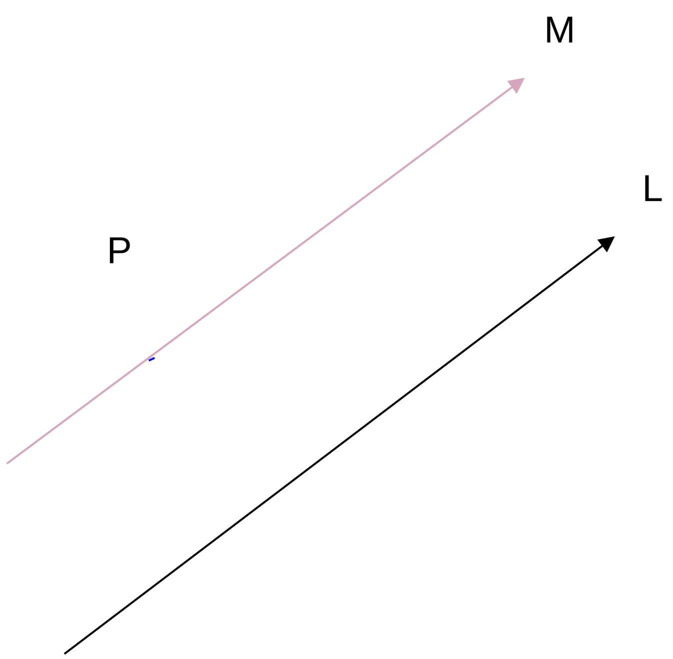

[16] [1] p.31
[17] [2] p.260

In order to prove the converse of proposition 28, Euclid must for the first time use the fifth postulate, because it can be replaced by the following statement[18]:

This statement, called Playfair's Axiom, is one that is easier to formulate:

Given a line L and a point P (which is not on line L), there is a unique line M that passes through P but does not intersect line L.




**3.Parallel Postulate**

18 [2] p.261

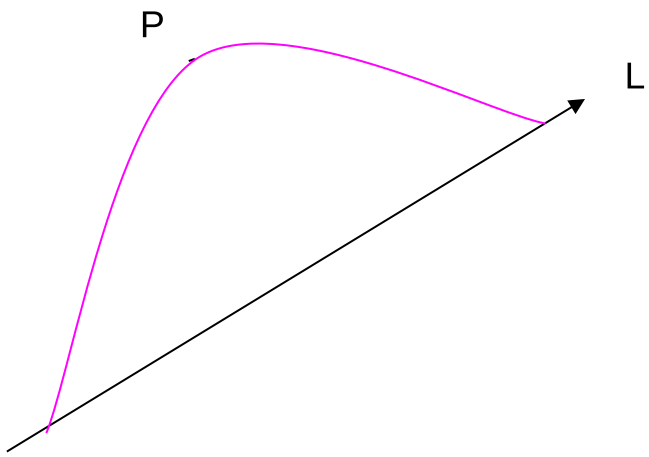
The parallel postulate was tried to be proved by several distinguished mathematicians like Proklos, Omar Khayyam, Nasir Eddin al-Tusi, John Wallis, etc. Some of them consciously or unconsciously replaced the parallel postulate with other unproven statements[19].

G. Saccheri tried to falsify the fifth postulate. The Saccheri-Legendre theorem states that in neutral geometry (using the results of the first four axioms of Euclid's Elements), the sum of the angles in a triangle is at most 180°, in other words, less or equal to 180°.

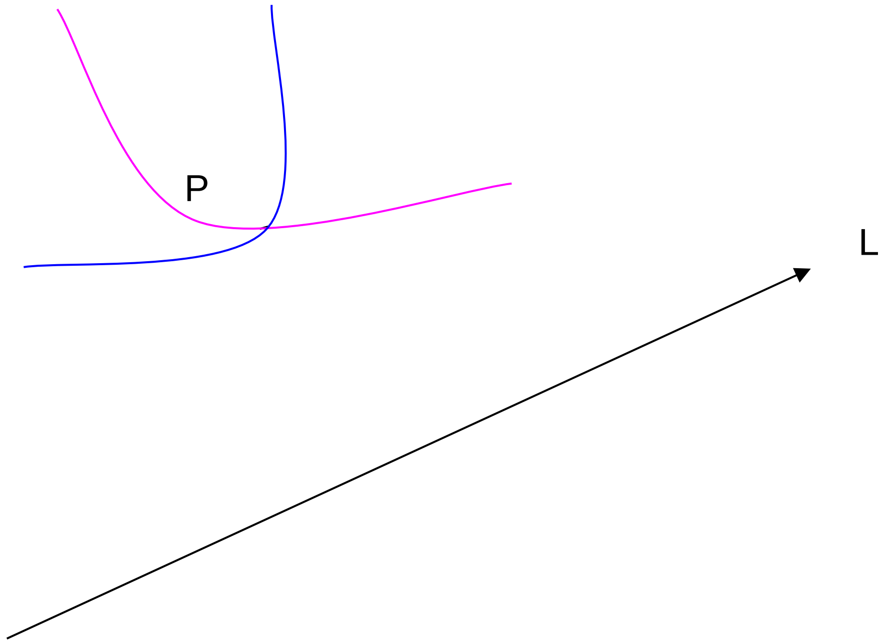
Saccheri, Lambert and Legendre had all unsuccessfully attempted to prove the parallel axiom but their failures led to the question of the parallel postulate being deepened. Instead of falsifying the case that the triangle sum is not equal to 180°, a number of mathematicians began to accept the idea that there is a new geometry in which the parallel postulate is not valid and where the sum of the angles in a triangle is more than two right angles.

If the parallel postulate is negated, then there are two alternatives.

1. Between a line L and a point P not on L, there are no lines through P that do not intersect L.




[19] [2] p.263




In the first figure, there is no parallel line going through point P. Oppositely, in the right figure, instead of one line through point P, many infinite lines can be drawn without intersecting line L.

The work with the parallel axiom was one source of inspiration for Hilbert.

## 4.Who is David Hilbert and why do we need Hilbert's Axiom System?

In 1900, David Hilbert opened the International Congress of Mathematicians in Paris and introduced 23 unsolved problems in his speech. He thought that it was crucial to solve these problems since they would be beneficial for the development of mathematics.[20] He ended his speech with these famous words which are also written on his tomb.

| | |
|---|---|
| *Wir müssen wissen.* | We must know. |
| *Wir werden wissen.* | We will know. |

After almost 120 years, 8 of them are solved, 9 of them are partially solved and the rest of these 23 major mathematical problems were remained unsolved.[21]

**4.1. David Hilbert**, (1862, Königsberg/Russia - 1943, Göttingen/Germany) is a German mathematician who influenced 19th and early 20th century with his works. One of his most important works is *The Foundations of Geometry*

[20] [4]
[21] [5]

where he reduced geometry to a series of axioms while combining both Euclidean and Non-Euclidean Geometry.

Hilbert was born to a wealthy and educated family where his father was a lawyer, his grandfather was a judge, and his mother was a merchant's daughter. His mother's passions which were philosophy, astronomy and prime numbers seems to have influenced Hilbert. He was homeschooled until the age of eight. Although he showed poor performance at school where rote learning was essential, he was exceptional in mathematics according to his school report[22] .

He began to study mathematics at the University of Königsberg. He studied under Heinrich Weber, Adolf Hurwitz and Ferdinand von Lindemann who suggested Hilbert to work in *invariant theory*[23]. He succeeded in establishing the basic laws of the theory of invariants, laying the foundations of the theory of polynomial ideals, which played an important role in algebraic geometry and modern algebra. Later, he was asked to survey algebraic numbers which is in the field of *number theory* by The German Mathematical Society. After spending five to ten years in each subject, he finally moved to a new topic; *the elements of geometry* [24].

"Euclid's Elementa based on space as he used geometric drawings with his theorems, but Hilbert was interested in the logical structure of the axioms for geometry and how they led to theorems" wrote Stewart[25]. In 1899, he published his work "Fundamentals of Geometry", which was a synthesis of his research on the basics of geometry. This led to many productive studies directed at the purpose of axiomatization in various parts of mathematics.

Avoiding resort to concrete images, Hilbert put "the system of three objects" into mathematics, which he called *points, lines* and *planes.* In contrast to Euclides, these objects, which are not exactly shown what they are, reveal some relationships explained by 21 axioms gathered in 5 groups. These belong to the axiom of *belonging, order, equality or equivalence, parallelism* and *continuity*. After that, he established geometries in which one or the other axioms were not proven. He considered the basic terms as logical entities that have no other characteristics than the attributes loaded with axioms. His debates with Brouwer to defend classical mathematics and show his clarity led to discussions on foundations of mathematics[26].

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[22] [6] p.191
[23] [6] p.191
[24] [6] p.194
[25] [6] p.194
[26] [7]

This philosophy is what we used with our example of why Euclid's proof of propositions does not work. We interpreted points, lines, circles in a certain way, certainly not what Euclid would have accepted, and checked that relations between them, given by the axioms hold.

He worked as a professor at Göttingen University until he retired at 1929. At the beginning of the 20th century, he was considered the leader of the German mathematics school.

## 5.Hilbert's Axiom System

In 1899, David Hilbert wrote a set of 20 axioms in his book, "The Foundations of Geometry". The aim was to create a modern foundation of Euclidean geometry.

Hilbert's axiom system has six main concepts: three main terms and three main relations.
These three main terms are *points*, *straight lines* and *planes*.
These three main relations are *betweenness*, *containment* and *congruence.*

In this system, Hilbert mentioned undefined and defined terms.
Undefined terms are *point, line, plane, lie, between,* and *congruence*[27]
Defined terms are segment, inside, ray, angle, interior and triangle. In terms of the other, we can define them.

**The five axiom groups:**

**I. Axioms of Incidence / Connection**

**II. Axioms of Order**

**III. Axioms of Congruence**

**IV. Axioms of Parallels (Euclid's Axiom)**

**V. Axioms of Continuity (Archimedes' Axiom)**

### I. Axioms of Incidence / Connection

Hilbert used the word Verknüpfung which means combination, link, connection in English.
The axioms in this group indicate a connection between points, straight lines and planes.

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[27] [8]

1. Two different points A and B always establish a straight line where AB = a or BA = a.

2. Two different points of a straight line specify that line; in other words, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.
3. Three distinct points A, B, C which do not lie on the same straight line establish a plane α; ABC = α.

4. Any three points A, B, C which do not lie in the same straight line, then they are points of a unique plane.[28]

5. If two points of a straight line a lie in a plane α, then every point of line a lie in plane α.[29]

6. If two planes α, β have a point A in common, there is at least a second point B in common.[30]

7. Upon *every straight line* there exist at least two points, *in every plane* at least three points not lying in the same straight line, and *in space* there exist at least four points not lying in a plane. [31]

In the Foundations of Geometry, as 1 and 2 contain statements about points and lines only, they are called “the plane axioms of group” while 3-7 ares called “space axioms”. There are two theorems of many which could be deduced from 3 to 7 mentioned in the Foundations of the Geometry.

*Theorem 1:*
Two straight lines of a plane have either one point or no point in common; two planes have no point in common or a straight line in common;
a plane and a straight line not lying in it have no point or one point in common.[32]

*Theorem 2:*
Through a straight line and a point not lying in it, or through two different straight lines having a common point, one and only one plane may be made to pass. [33]

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[28] [9] p.3
[29] [9] p.3
[30] [9] p.3
[31] [9] p.3
[32] [9] p.3
[33] [9] p.3

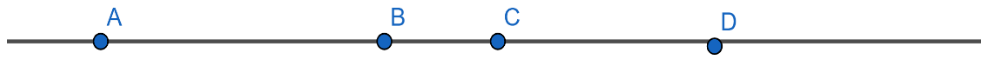
In some sources, there are 8 statements in Axioms of Incidence, but in Townsend's translations, the $7^{th}$ and $8^{th}$ were combined, therefore there are 7 statements here.

**II. Order**

This group describes the relations "between(ness)" and "at the same side".

1. If point B lies between points A and C, B lies also between C and A, then there exists a line which contains these distinct points A, B, C.


2. If A and C are on a straight line, then there exists at least one point "lying between A and C" and at least one more point D so situated that C "lies between A and D". [34]




3. Of any three points situated on a line, there is no more than one which lies between the other two.[35]

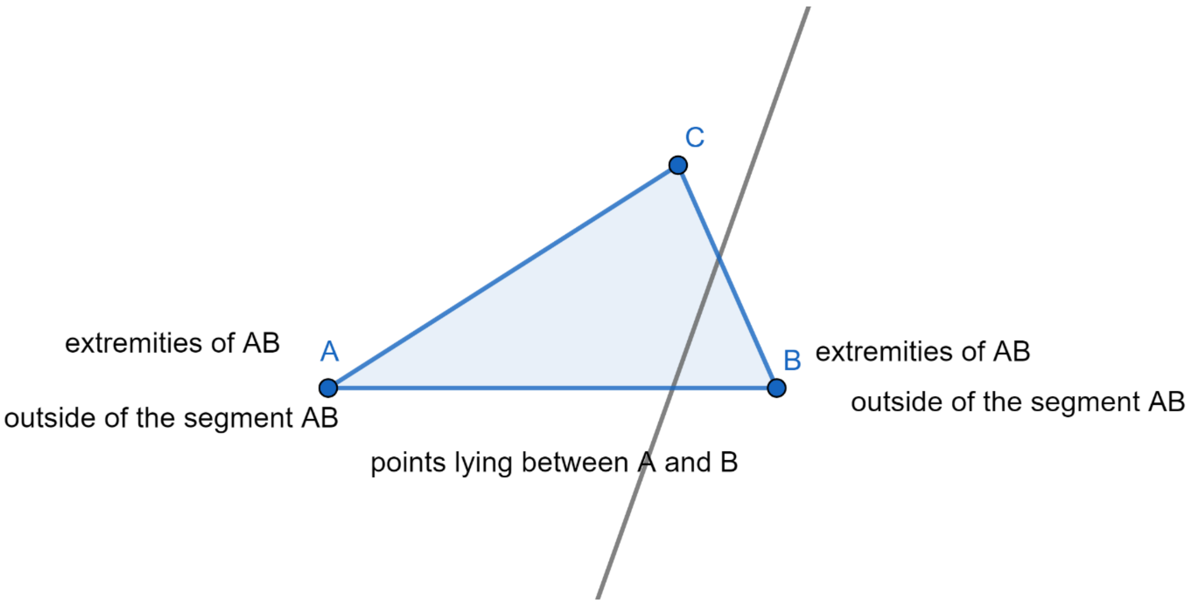
[34] [9] p.4
[35] [9] p.4


4. If there are four points A, B, C and D lie on a straight line, it can always be so situated that B shall lie between A and C and also between A and D, and , furthermore, that C shall lie between A and D and also between B and D.[36]

**DEFINITION:**

- A and B lying upon a straight line, a segment and it is indicated by AB or BA.
- The points *lying between* A and B are called the points of the segment AB or the points *lying within the segment AB*.
- All other points of the straight line are referred to as the points *lying outside the segment AB*: The points A and B are called the extremities of the segment AB.[37]

[36] [9] p.4
[37] [9] p.4




5. ***Pasch's Axiom:***
Let A, B, C be three points not lying in the same straight line
Let a be a straight line lying in the plane ABC and not passing through any of the points A, B and C.

Then, if the straight line *a* passes through point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.
Axioms II

Axioms II, 1- 4 have statements including the points of a straight line only, and in consequence, it was called *"the linear axioms of grounp II"* . On the other hand, statement 5, Pasch's axioms, was related to the elements of plane geometry so it was *called "plane axiom of group II"* [38]

*Consequences of the Axioms of Congruence and Order:*
By the help of 1-4 statements, Hilbert deduced three theorems and furthermore, two more theorems by the help of latter theorems.

*Theorem 3: There exist an unlimited number of points between any two points of a straight line.* [39]

*Theorem 4:* If there are points which take place on a straight line, they can always be situated in a way (A, B, C, D, E, ...K) that B would lie between A and C, D, E, ...,

[38] [9] p.4
[39] [9] p.4

K; C between A, B and D, E…, K, etc. Other than that, there is also on and only order which satisfies the condition, and it is K, …, E, D, C, B, A.

*Theorem 5:*

Each straight line a, lying in the plane α divides the points in that plane into two regions providing the following properties:

Each point A located in one region determine with each point B located in the other region so that segment AB contains a point from the straight line a. Also, any two points A and A' in the other region contain such a segment AA' that the straight line cross does not contain any points.

Although Euclid also used the term "between" and "at the same side" in some of his proofs and theorems, Hilbert characterized these terms by four axioms which made it easier to use definitions such as segments, rays/half-lines and angles according to Tengstrand (2005)[40]

### III. Congruence

In this group, there are six axioms, and the term *congruence* is used to describe segments, angles and triangles. This axiom defines the idea of *displacement* besides *congruence*. 1-3 are about congruence of segments and 4-6 are about congruence of angles.

1. If there are two points A and B on a straight line ℓ, and if there is a point A' on the same or another straight line ℓ', then there is always one and only one point B' so that AB or BA is congruent to segment A'B' or B'A'. This relation can be written as

$$AB \equiv A'B'$$

And furthermore, every segment is congruent to itself, and we always have AB ≡ AB.

$$A'B' \equiv A''B''.$$

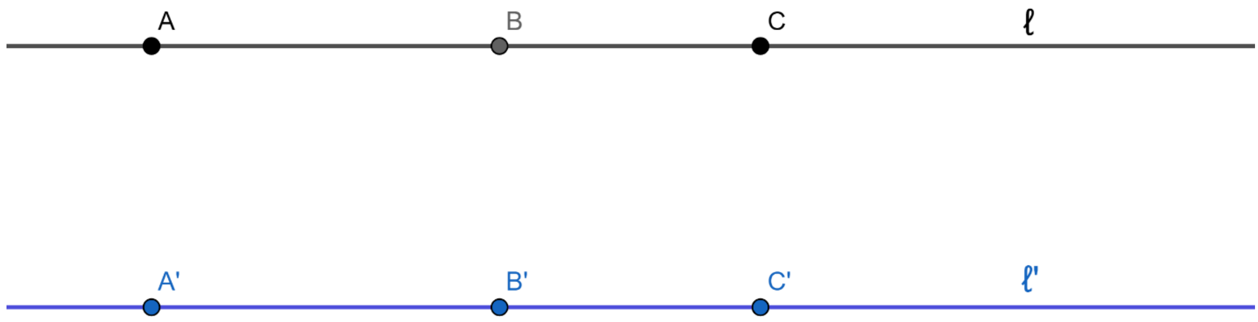
2. If a segment AB is congruent to both CD and EF, then CD is congruent to EF.

$$AB \equiv CD \text{ and } AB \equiv EF, \text{ then } CD \equiv EF$$

[40] [2] p.297

3. If AB and BC are two segments of a straight line ℓ which do not have common points other than B, and furthermore, if A'B' and C'D' are two segments of another straight line ℓ', which do not have common points, like ℓ, then

AB ≡ A'B' and BC ≡ B'C', then AC ≡ A'C'




4. In the plane α, there is a given angle (h, k) and in the plane α', there is a given straight line a'. Assume that, in the plane α, a particular side of the straight line a' was chosen

There is a half-ray h' on the straight line a' is coming out from O'. In the plane α', there is one and only one half-ray k' so that the angle (h, k) or (k, h) is congruent to the angle (h', k'). Besides all interior points of the angle (h', k') lie upon the given side of a'.41

∠ (h, k) ≡ ∠ (h′, k′)
And every angle is congruent to itself.
∠ (h, k) ≡ ∠ (h, k)

5. If ∠ (h, k) is congruent to both (h′, k′) and ∠ (h″, k''), then ∠ (h′, k′) is congruent to both (h′, k′) is congruent to ∠ (h″, k'').

If ∠ (h, k) ≡∠ (h′, k′) and ∠ (h, k) ≡∠ (h″, k″), then ∠ (h′, k′) ≡∠ (h″, k″).

6. The last is about the connection between congruence of segments and congruence of angles.
There are triangles ABC and A′B′C′ and they are congruent to each other.

41 [9] p.9

AB ≡ A′B′, AC ≡ A′C′, besides ∠BAC ≡ ∠B′A′C′ agree, then ∠ABC ≡ ∠A′B′C′ agreess and ∠ACB ≡ ∠A′C′B′ agrees.

**IV. Parallels**

***Euclid's Axiom :***
In this group, there is only one axiom.
Let ℓ is a straight line and P is a point outside of ℓ. Then there is only one straight line that passes through P which does not intersect ℓ. This axiom is the equivalent of the fifth postulate.

**V. Axiom of Continuity**

As mentioned above, this axiom does not occur in Elementa. By the help of this theorem, it may be shown that every straight line is continuous, in other words, it includes all "real numbers". Furthermore, you can reach beyond any point by stepping from a given point, with a given segment/distance with a finite number of times. [42]

By the help of this particular axiom, the geometry was introduced by the idea of continuity.

Actually, this axiom was divided into two axioms:

**V.1.** ***Axiom of measure*** **or** ***Archimedes' Axiom***
**V.2.** ***Axiom of line completeness***

**V.1.** ***Axiom of measure*** **or** ***Archimedes' Axiom***
This axiom which is also known as Eudoxus Axiom, had survived to us in Eudoxus writings according to Boyer and Merzbach (1991)[43]. It is said that Eudoxus was the first one to understand the importance of this assumption and Otto Stolz was the first one who coined the term in the 1880s. [44]

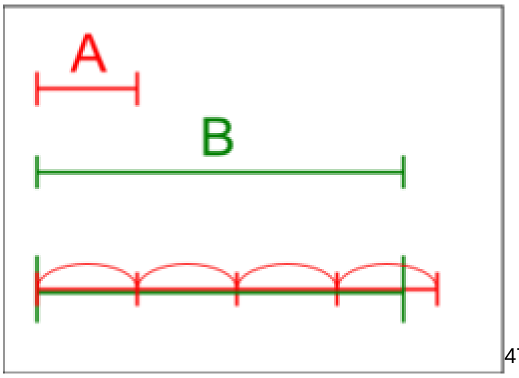
The Archimedean axiom appears in Euclid's Elements (Book V – Def 4)

*Def 4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.*[45]

The axiom says that there exist two distinct segments; A and B, then there must exist a natural number n such that if the segment B is plotted n times

[42] [2] p.299
[43] [10] p.89, 129
[44] [11]
[45] [1] p.130

perpetually from A along the ray towards B, then it comes to a point on the other side B. Archimedes' axiom can also be used for numbers, angles, surfaces, etc. [46]



[47]

As shown in the figure, if we construct segment A several times, in other words, if we multiply the smaller magnitude (segment A) with a certain ratio then it would pass beyond the point B.

Archimedes found out to solve problems of area and volume by the help of this axiom which was the foundation for the method of exhaustion.[48] *Method of exhaustion* is the method of finding the area or volume of a shape that cannot be easily identified through conventional shapes. It is based on obtaining a reasonable approximation of an area or volume by combining known shapes, for example, inscribing the unknown form with a number of polygons. This estimate can be improved by increasing the number and size of the polygons.[49]

**V.2. (*Axiom of line completeness*)**

To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.[50]

This axiom says there exists a system which includes a straight line which includes points lying on it. It is not possible to extend points on a straight line which would follow the axioms I, II, III and V.I and without violating the relations between order and congruence properties. [51]

Here is a sketch of a proof of Proposition 1:
The key is the axiom V.2 :

[46] [12] p.78,79
[47] [13]
[48] [14]
[49] [15]
[50] [9] p.15
[51] [12] p.79

The axiom is in a way cheating because in practice it says that given a segment CD on a line there are points on the line, E such that CE is any real number times CD. We thus see that Hilbert's axiom system incorporates the fact that we can parameterize each line with R, and that we can then in principle go over to making plane geometry in $R^2$.

Now for the argument:

We assume, as is reasonable, that it is known that points and lines in $R^2$ satisfy all Hilbert's axiom. Also we assume that Pythagoras theorem holds (which needs a proof).
Assume that AB is a given segment. It determines a line l (Axiom 1.1) and by axiom V.2 there is a point D on this line such that AD is one half of AB. Also there is a right angle, and we can move angles (Axiom III.4), so in particular we may assume that there is a right angle at D. This angle again determines a line n, which is normal to l. On this line we may find a point E such that DE = AB (Axiom III.1) and then we can finally use Axiom V.2 again, to conclude that there is a point C on n such that $DC=\frac{\sqrt{3}}{2}AB$. Then ABC is an equilateral triangle---this follows by Pythagoras theorem.

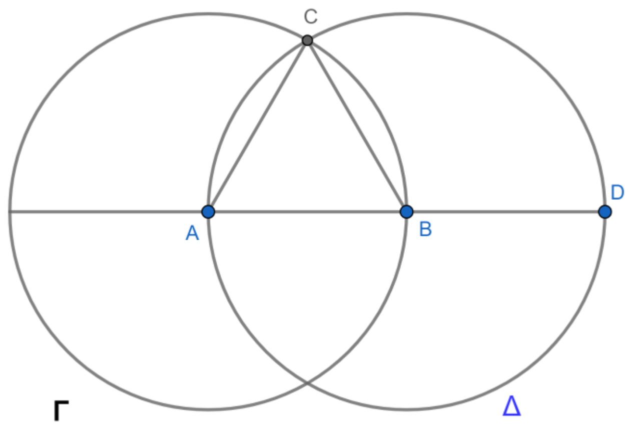
Earlier we criticized Euclid for having an axiom system that was imprecise. The same can be said about axiom V.2. It refers to the impossibility of any extension of a model, and also to the other axioms. If you think this is cheating, you can alternatively use the axiom E that Hartshorne uses in Euclidean geometry and beyond.

**Axiom E (Circle-circle intersection property):**

Given two circles Γ, Δ, if Δ contains at least one point inside Γ, and Δ contains at least one point outside Γ, then Γ and Δ will meet. [52]

Using the new axiom (E) we can now justify Euclid's first construction (prop I.l), the equilateral triangle. Given the segment AB, let Γ be the circle with center A and radius AB. Let Δ be the circle with center B and radius BA. Then A is on the circle Δ, and it is inside Γ because it is the center of Γ. The line AB meets Δ in another point D, such that A * B * D. Hence AD > AB, so D is outside Γ.
Thus Δ contains a point inside Γ and a point outside Γ, so it must meet Γ in a point C. From here, Euclid's proof shows that ΔABC is an equilateral triangle[53]

[52] [17] p.108
[53] [17] p.110




After adopting all of the Hilbert Axioms except V.2 and adding a new Axiom E, Hartshorne repaired Euclid's proof for Prop 1.

**SUMMARY:**

As a summary, what Hilbert contributed by writing The Foundations Geometry can be summarized as follows:

- As Euclides did, he gathered the related axioms of the mathematicians in order to form a well-arranged and detailed axiomatic system. He used two thousand years of discussion and criticism.

- He made clear the relationships between objects by adding new concepts that was missed in The Elements.

- The most important contribution was the idea that you do not need to explain what the objects really are, just give their behaviour by certain axioms. This is the mathematical philosophy called formalism.

- He created five groups of axioms in a way that can be proven is such that none of them are contradictory to one another; or in other words, by using a logical proof or reasoning process, it is impossible to deduce a proposition which is contradictory to any of them.

At the end of this article, I would like to say that we can advise the authors of the textbook not to blindly take everything Euclid wrote. Euclid's Elementa is long and poorly understood, some of its postulates may be inconclusive, and some of its proofs may be found incomplete. Nevertheless, we should be grateful that he collected his and other scientists' knowledge 2300 years ago, brought out a systematic and fundamental collection of works to humanity and created the basis and inspiration for many scientists.

In the same way, David Hilbert, one of the leading mathematicians of his age and who did not leave the subjects he studied completely unsolved. He created an understandable axiom system in a way followed Euclid's footsteps by combining the works of Euclid and a few other scientists.

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