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Classical sets for simplicial type theorists

av

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Abstract

In recent years, **Ho**motopy **T**ype **T**heory (**HoTT**) has emerged as an alternative formal system for mathematics, capable of expressing highly classical concepts such as homotopy theory in a type-theoretical syntax close to those found in computer science. This formal system was based in part on what is now its most archetypal model, the *simplicial model*; built in the category of simplicial sets, this model is topologically flavoured: its basic objects can be seen as spaces for instance.

Accordingly, elements of the universe of discourse of HoTT can be understood as abstract spaces, thus the notion of set can be encoded in the theory as an "abstract discrete space". The present work aims to show that properties of classical sets are preserved when those sets are interpreted as discrete spaces in the simplicial model. While the approach taken is specific to the model at hand, it lays the foundations for a more general framework which could be used in the future to answer similar questions for different models.

A notable obstruction lies with a notion needed to interpret classical existence in HoTT, propositional truncation, as it is not part of the original construction of the simplicial model. As a result, this work comprises an account of the method one can employ to truncate types down to strict propositions in the simplicial model using image factorisations.

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Contents

1 Introduction

Motivation

The recent advent of **Ho**motopy **T**ype **T**heory (**HoTT**) as an alternative foundation for mathematics has raised multiple questions on this new language's adequacy encoding classical areas of mathematics such as topology, analysis, and set theory. As the name hints at, the basic objects of HoTT, *types*, can be thought of as abstractions of topological spaces seen through the lens of homotopy theory. As such, they provide excellent models for topology and homotopy theory; classical notions from set theory have also been formalised in HoTT; as has the bedrock of every analysis textbook, the construction of the real numbers [Uni13, Part II].

However, one of the logical strengths of this new foundation has also be seen as one of its weaknesses in regard to the formalisation of classical fields. Namely, bare HoTT assumes fewer axioms than classical foundations, e.g., the law of excluded middle (LEM) can be present in the theory or absent from it according to one's needs, similarly to how the axiom of choice (AC) is optional in many Zermelo-Fraenkel-style foundations of set theory.

This is indeed a logical strength, as being able to prove a theorem with fewer assumptions is always preferable, think of how constructing an actual basis of a vector space is more useful than merely knowing one exists by AC. In particular, such a weaker collection of axioms allows HoTT to simultaneously be a language for classical mathematics and versions of intuitionistic mathematics, and its type-theoretical syntax is close enough to that of functional programming languages so as to allow the existence of effective proof checkers and proof assistants.

But there is a downside to these advantages, the problem is that it is not always clear which theorems of a classical field can be formalised in HoTT, as it is rarely apparent if all proofs of said theorems rely or not on axioms such as LEM. One blanket solution would be to add all the classical axioms to HoTT, but then the advantages described above would be, for the main part, lost.

A better solution to this issue, but one which requires more work, would be to have access to some sort of dictionary between areas of mathematics and flavours of HoTT, variations of the theory in which various axioms have been added. For example, one might expect, due to the ubiquity of Zorn's lemma therein, that many parts of algebra would be most naturally translated in a version of HoTT endowed with AC. Since it is conjectured that there exists a notion of duality between said variations of the theory and models of HoTT, the dictionary could equivalently assign models in place of axiom systems. Or, from the opposite point of view, this translation device could instead pair a model of HoTT with the mathematical fields which can be satisfactorily interpreted in those models.

Goal

The present work offers to give partial information about the entry of this dictionary corresponding to the first univalent model of HoTT, the simplicial model [KL21], pertaining to the theory of sets. In other words, it aims to provide the basis of a framework making it easier to transfer properties and constructions of classical sets to the simplicial model; for example, it outlines a general procedure one can follow to obtain results similar to the recent proof that LEM holds in the simplicial model [KL20].

Precisely, the present work aims to facilitate, in the context of the simplicial model, the comparison of the notion of sets internal to HoTT, the so-called *h-sets*, with classical sets. In bare terms, given a property satisfied by sets, the goal is to show that its translation in the language of HoTT, restricted to h-sets, and interpreted in the simplicial model is also satisfied.

Setting the stage

Suppose you are interested in a property P of sets which can be expressed in type theory, meaning that you have a specific type theory T in mind whose syntactic rules let you derive the type (of proofs) of P. In addition, suppose that P holds in the model[1] $M_{\mathbf{Set}}$ of type theory based on the category of sets; that is, suppose that the *interpretation* of the type corresponding to P in $M_{\mathbf{Set}}$ admits elements, as those elements model proofs of P.

To clarify, *interpreting* syntactic objects in a specific model such as $M_{\mathbf{Set}}$ can be done "manually" by translating the derivation of said object in the language of the framework at hand or it can be done in a more category-theoretic manner. Namely, one associates to each type theory T a *syntactic model* M_T, whose constituents all originate from the syntax, sets up the correct *category of models* $\mathrm{Mod}(T)$ of T, in which M_T is the initial object, thus obtains for each model $M \in \mathrm{Mod}(T)$ a unique map $[\![\ ]\!]_T^M : M_T \to M$, and uses it to interpret the syntax.

As an example, one might be interested in the validity of the axiom of choice in simplicial sets, expressed as follows in type theory

$$\frac{A\ \mathsf{type} \qquad A \vdash B\ \mathsf{type}}{(\Pi_{a:A}\|B_a\|) \to \|\Pi_{a:A}B_a\|}$$

where $\|A\|$ stands for the *propositional truncation* of A.

The naive diagram

The first step towards proving a version of P in the simplicial model consists in choosing a fragment T of HoTT expressive enough to formulate the property P and that of being an h-set (i.e., it should have Id-types) but also modest enough so as to be consistent with UIP, *uniqueness of identity proofs*, an axiom asserting that all types are h-sets. Indeed, restricting P to h-sets involves adding UIP to the type theory T, so one cannot simply take T to be HoTT itself and call it a day, as axioms such as univalence contradict UIP. The precise meaning of being "consistent with UIP" depends on the type and term formers at hand, but in practice it takes the form of closure properties, e.g., Π-types preserve h-sets. For the case of $P = \mathsf{AC}$, such a minimal type theory T_{AC} needs to comprise Id-types, Π-types, propositional truncation, and a universe.

Of course, the model based on sets should validate the fragment T, since the strategy is to interpret P in $M_{\mathbf{Set}}$ using T before doing so in the simplicial model $M_{\mathbf{sSet}}$. As mentioned above, this is carried out via the interpretation map $[\![\ ]\!]_T^{\mathbf{Set}} : M_T \to M_{\mathbf{Set}}$. The result can then be sent in the simplicial model via a map (of models) $D : M_{\mathbf{Set}} \to M_{\mathbf{sSet}}$ which takes a set to the corresponding discrete simplicial set. Alternatively, one can start by sending (the type of proofs of) P to M_{HoTT}, the syntactic model of HoTT, along the map $M_T \to M_{\mathsf{HoTT}}$ encoding that T is a fragment of HoTT; one can then take the result to its interpretation in the the simplicial model via $[\![\ ]\!]_{\mathsf{HoTT}}^{\mathbf{sSet}}$. Diagrammatically, the process above consists in chasing P along the two paths depicted below.

$$\begin{array}{ccc} M_T & \longrightarrow & M_{\mathsf{HoTT}} \\ {\scriptstyle [\![\,]\!]}\big\downarrow & & \big\downarrow{\scriptstyle [\![\,]\!]} \\ M_{\mathbf{Set}} & \xrightarrow{\ D\ } & M_{\mathbf{sSet}} \end{array}$$

However this is still a rather naive process, and an attentive reader could raise some objections. The following aims to identify and partially answer these obstructions, referring to the relevant sections along the way.

[1]The precise notion of models used here are *categories with attributes* but such specificity is not needed for this explanation.

Restricting to h-sets

The first major issue is that no mention of h-sets is made, even though restricting to h-sets was the main point. As mentioned above, a possible solution would be to add UIP to T, and construct a similar square with T + UIP in place of T. This raises another problem: one does not get a map $M_{T+\mathsf{UIP}} \to M_{\mathsf{HoTT}}$ "for free" as before since T + UIP is not a fragment of HoTT. One possible approach to remedy this would involve modifying the syntax of HoTT so that the result validates both T and UIP, however such an endeavour can be tricky since working directly at the level of the syntax can prove tedious and requires great bookkeeping skills. For this reason, it is advantageous to work on the semantic level instead; thus modifying M_{HoTT}, viewed as a model of T, rather than its syntax.

The objective then is to construct a map

$$M_{T+\mathsf{UIP}} \to SM_{\mathsf{HoTT}}$$

in $\mathrm{Mod}(T + \mathsf{UIP})$ from the already given map

$$M_T \to M_{\mathsf{HoTT}}$$

in $\mathrm{Mod}(T)$, with SM_{HoTT} some h-set-based variation of M_{HoTT}. It turns out that the forgetful functor $U : \mathrm{Mod}(T{+}\mathsf{UIP}) \to \mathrm{Mod}(T)$ sends $M_{T+\mathsf{UIP}}$ to M_T, which means that the above objective can reformulated so that one now searches for a map

$$\mathrm{Hom}_{\mathrm{Mod}(T)}(UM_{T+\mathsf{UIP}}, M_{\mathsf{HoTT}}) \to \mathrm{Hom}_{\mathrm{Mod}\,T+\mathsf{UIP}}(M_{T+\mathsf{UIP}}, SM_{\mathsf{HoTT}}).$$

This suggests that one might just as well directly search for a right adjoint to U. Indeed, up to some technical details, this is essentially the work carried out in Section 2. More precisely, Section 2.1 sets up the notion of models used throughout this work, Section 2.2 then constructs the desired adjoint for a minimal T, and finally Section 2.3 discusses extensions of said adjunction as well as its consequences.

Simplicial truncation

Another considerable obstruction to the approach sketched above is tied with the constructive nature of type theory. Indeed, modelling (classical) existence in HoTT involves propositional truncation $\| \ \|$, a collapsing operation which "squashes" types down to *mere propositions*, which have at most one element up to homotopy.

For example, if A is a type and $x : A \vdash P(x)$ a family of types over it, seen as a predicate over A, then terms of $\|\Sigma_{x:A} P(x)\|$ encode that there exist terms of type A satisfying P. Truncation can also be used without Σ-types and in that case providing a term of $\|A\|$ proves that the type A is inhabited (without actually having to give terms of type A), it is this version that appears above in the formulation of AC as the property that a product of non-empty types is itself not empty.

Since many choices of P are bound to be properties involving existence, AC to name but one, it may reasonably be ascertained that propositional truncation will often be among the logical rules of T. However, its case is not treated in the standard account of the simplicial model [KL21], which means that, for completeness, an interpretation of propositional truncation in simplicial sets ought to appear in the present work. And indeed it does, it is the topic of Section 3; the chosen approach is based on image factorisations, which means, on the one hand, that the associated computation rule is only modelled propositionally but, on the other hand, it does yield strict propositions.

More precisely, Section 3.1 gives a quick and terse introduction to the construction of the simplicial model and clarifies how one interprets propositional truncation therein. Because the notion of mere proposition is an important one when dealing with truncation, Section 3.2 is devoted to exploring and characterising said notion in the simplicial model. Finally, Section 3.3 introduces image factorisations, describes how they can model propositional truncation, and assembles them with what has come before to conclude.

Discrete interpretation

Lastly, there remains the matter of the map $D : M_{\mathbf{Set}} \to M_{\mathbf{sSet}}$, encoding the fact that sets can be viewed as discrete simplicial sets (those whose only non-degenerate simplices are in dimension 0). However, given the nature of the simplicial model, it is not as easy to define as one might expect; for this reason the notion of weak maps is introduced. Moreover, one needs D to preserve the logical structure, at least weakly, so that the strategy outlined above can go through.

Once a satisfactory account of D has been given, one can go back to the naive diagram and redraw it in a more rigorous manner. Actually, it is more natural to split it into two diagrams:

$$\begin{array}{ccc} M_{T+\mathsf{UIP}} & \longrightarrow & SM_{\mathsf{HoTT}} \\ {\scriptstyle [\![\,]\!]}\big\downarrow & & \big\downarrow{\scriptstyle S[\![\,]\!]} \\ M_{\mathbf{Set}} & \xrightarrow{D} & SM_{\mathbf{sSet}} \end{array} \qquad\qquad \begin{array}{ccc} USM_{\mathsf{HoTT}} & \longrightarrow & M_{\mathsf{HoTT}} \\ {\scriptstyle US[\![\,]\!]}\big\downarrow & & \big\downarrow{\scriptstyle [\![\,]\!]} \\ USM_{\mathbf{sSet}} & \longrightarrow & M_{\mathbf{sSet}} \end{array}$$

The first one commutes in a weak sense in $\mathrm{Mod}(T+\mathsf{UIP})$ while the second does so on the nose in $\mathrm{Mod}(T)$. One can thus chase the interpretation of (the type of proofs of) P along the two paths of the first square, and while the two results are not be strictly equal they are closely related enough for a term in one of the two induces a term in the other. Because P was assumed true in **Set**, this means that P is also true in $SM_{\mathbf{sSet}}$. The strict commutativity of the second square then implies that P is also true in $M_{\mathbf{sSet}}$, as desired.

In Section 4, those pieces are put together and it is explained how they can be used to satisfy the initial goal of interpreting properties of sets in **sSet**. More precisely, Section 4.1 gives a precise definition of D, while Section 4.2 gives a summary based on the two squares above.

2 Restricting to h-sets

2.1 Categories with attributes

Variants of categorical semantics for type theories [Hof97] abound in the literature, with varying degree of generality. Of those, structures known as *contextual categories* ([Car78, Section 2.2] and [Str91, Definition 1.2]) are among those most closely modelled on the syntax of type theory, and were Voevodsky's models of choice, which he called *C-systems*, when working on the *simplicial model of univalence* [KL21].

However, the constructions carried out in the present work can be streamlined when relying on the slightly more general notion of *categories with attributes* [Car78, Mog91, Pit00]. Since the standard approach used to equip simplical sets with a contextual structure [KL21, Sections 1 & 2] can easily be adapted to yield a category with attributes instead, the change of framework does not come at any consequent cost and is therefore adopted hereafter.

Definition 2.1. A **category with attributes**, often shortened to CwA, consists of:

(1) an **underlying category** $\mathbf{C}$ with a chosen terminal object 1, called the **empty context**;
(2) a functor $\mathrm{Ty} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, whose action on a morphism f is written f^* instead of $\mathrm{Ty}(f)$;
(3) for each $A \in \mathrm{Ty}(\Gamma)$, an **extension** $\Gamma.A \in \mathbf{C}$ and a **projection** $p_A : \Gamma.A \to \Gamma$;
(4) for each $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \to \Gamma$, a **connecting map** $f.A : \Delta.f^*A \to \Gamma.A$;

such that:

(5) for each $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \to \Gamma$, the diagram

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{f.A} & \Gamma.A \\ {\scriptstyle p_{f^*A}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_A} \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

is a pullback square (the **canonical pullback** of A along f);
(6) these canonical pullbacks are functorial: for each A and f as above and each map $g : E \to \Delta$, the identities

$$\mathrm{id}_\Gamma.A = \mathrm{id}_{\Gamma.A} \quad \text{and} \quad (f \circ g).A = (f.A) \circ (g.f^*A)$$

hold.

Remark 2.2. The second clause above, which asks for a presheaf Ty on $\mathbf{C}$, may arguably be the most important point of the definition. However, its formulation favours conciseness over transparency. It might therefore be worth, if only to introduce some terminology, to unpack the data of the functor Ty by giving an equivalent formulation of said clause. One can replace the point (2) of the last definition by:

(a) for each $\Gamma \in \mathbf{C}$, a set $\mathrm{Ty}(\Gamma)$ of **types over** Γ;
(b) for each $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \to \Gamma$, a type $f^*A \in \mathrm{Ty}(\Delta)$, the **reindexing** of A along f;

such that

(c) the mapping $f \mapsto f^*$ is functorial: for each A and f as above and each map $g : E \to \Delta$, the identities

$$\mathrm{id}_\Gamma^* A = A \quad \text{and} \quad (f \circ g)^* A = g^* f^* A$$

hold.

Categories with attributes can easily be viewed as the models of an essentially algebraic theory [AR94, 3.D] with three sorts: one for objects, one for morphisms, and one for types. In general, the canonical notion of homomorphism between models of an e.a.t. is that of set maps between the corresponding sorts which commute with all the operations of the theory; the next definition reformulates this for the case at hand.

Definition 2.3. An **CwA map** $F : \mathbf{C} \to \mathbf{D}$ between categories with attributes consists of an underlying functor F between the underlying categories and a natural transformation $F^{\mathrm{Ty}} : \mathrm{Ty}_{\mathbf{C}} \to \mathrm{Ty}_{\mathbf{D}} \circ F^{\mathrm{op}}$ which strictly preserve the CwA structure, i.e., such that

$$F(\Gamma.A) = F(\Gamma).F^{\mathrm{Ty}}(A), \qquad F(p_A) = p_{F^{\mathrm{Ty}}(A)},$$
$$F(f.A) = F(f).F^{\mathrm{Ty}}(A), \qquad F(1_{\mathbf{C}}) = 1_{\mathbf{D}}$$

hold for any choice of variables of the right sorts.

The category of categories with attributes and CwA maps, denoted CwA, is therefore the category of models of an essentially algebraic theory. Note that if there is no risk of confusion, the supscript Ty of the natural transformation F^{Ty} may be omitted.

Much of the intuition and terminology of categories with attributes stems from the following archetypal example.

Example 2.4 (Syntactic CwA)**.** Let T be a type theory as mentioned in the introduction, it induces a category with attributes $\mathbf{C}_T$ in which:

(1) objects are contexts $[x_1 : A_1, \ldots, x_n : A_n]$ of T, up to definitional equality and renaming of free variables;
(2) maps are *context morphisms*, once again considered up to definitional equality and renaming of free variables, this means that a map $f : [y_1 : B_1, \ldots, y_m : B_m] \to [x_1 : A_1, \ldots, x_n : A_n]$ in $\mathbf{C}_T$ is an equivalence class $[f_1, \ldots, f_n]$ of sequences of terms

$$y_1 : B_1, \ldots, y_m : B_m \vdash f_1 : A_1,$$
$$y_1 : B_1, \ldots, y_m : B_m \vdash f_2 : A_2[f_1/y_1],$$
$$\vdots$$
$$y_1 : B_1, \ldots, y_m : B_m \vdash f_n : A_n[f_1/y_1, \ldots, f_{n-1}/y_{n-1}],$$

where two such sequences $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are equivalent precisely when, for each $i \leq n$,

$$y_1 : B_1, \ldots, y_m : B_m \vdash f_i \equiv g_i : A_i[f_1/y_1, \ldots, f_{i-1}/y_{i-1}];$$

(3) the composition $f \circ g$ of two maps f and g is obtained by substituting the terms of g for the variables of the terms of f, and the identity map at Γ is given by the sequence consisting of the variables of Γ, seen as terms in context Γ;
(4) the elements of $\mathrm{Ty}([x_1 : A_1, \ldots, x_n : A_n])$ are types

$$x_1 : A_1, \ldots, x_n : A_n \vdash A \text{ type},$$

considered up to definitional equality and renaming of free variables, with substitution as reindexing;
(5) extensions are induced by syntactic context extensions, and the projection $p_A : \Gamma.A \to A$ corresponding to a type $A \in \mathrm{Ty}(\Gamma)$ is also given by the variables of Γ, this time viewed as weakened terms in the context $\Gamma.A$;
(6) the connecting map $f.A$ of a map $f : \Delta \to \Gamma$ and a type $A \in \mathrm{Ty}(\Gamma)$ results from weakening the terms of a sequence defining f to the context $\Delta.f^*A$ and adding the last variable of said context to that sequence.

The category of sets can also be given the structure of a category with attributes, resulting in a more concrete example.

Example 2.5 (CwA of sets)**.** The category with attributes based on **Set**, written $\mathbf{C}_{\mathbf{Set}}$, is defined as follows:

(1) the underlying category is **Set**, the category of sets, with an arbitrarily chosen one-element set $\{*\}$ as the terminal object;
(2) for each set Γ, a type A over Γ is a Γ-indexed family of sets $A = (A_\gamma)_{\gamma\in\Gamma}$, and if $f : \Delta \to \Gamma$ is a map of sets then the reindexing of A along f is the Δ-indexed family

$$f^*A = (A_{f(\delta)})_{\delta\in\Delta};$$

(3) for $A \in \mathrm{Ty}(\Gamma)$, the extension $\Gamma.A$ is the disjoint union of the A_γ, with $p_A : \Gamma.A \to \Gamma$ being the evident projection, that is,

$$\Gamma.A = \{(\gamma, a) : \gamma \in \Gamma, a \in A_\gamma\} \quad \text{and} \quad p_A : (\gamma, a) \mapsto \gamma;$$

(4) for $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \to \Gamma$, the connecting map $f.A$ is

$$f.A : \Delta.(f^*A) \to \Gamma.A : (\delta, a) \mapsto (f(\delta), a).$$

When writing elements of iterated extensions in $\mathbf{C}_{\mathbf{Set}}$, it is useful to view nested ordered pairs as ordered tuples, so that elements of $\Gamma.A.B$, say, are of the form (γ, a, b) and not $((\gamma, a), b)$.

Example 2.6 (Trivial CwA)**.** Every category $\mathbf{C}$ with a terminal object 1 can be equipped with the structure of a trivial category with attributes. Namely, the type presheaf is the initial object in $\mathrm{Hom}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$, i.e., the one taking each object of $\mathbf{C}$ to the empty set; the choice of extensions, projections, and connecting maps is then vacuous as there are no types.

The third clause of Definition 2.1 is equivalent to choosing an object in the slice $\mathbf{C}/\Gamma$ for each $A \in \mathrm{Ty}(\Gamma)$. This suggests that the notion of slice categories is relevant in the study of CwAs. Indeed, the notion of *fibrant slice* [KL18, Sections 2.1 & 4.1] is a useful tool, both a notational and mathematical one, when dealing with CwAs.

Definition 2.7. A **context over** an object Γ of a CwA $\mathbf{C}$ is a (possibly empty) sequence $(A_1, \dots, A_n)$ with

$$A_1 \in \mathrm{Ty}(\Gamma),\ A_2 \in \mathrm{Ty}(\Gamma.A_1),\ \dots,\ A_n \in \mathrm{Ty}(\Gamma.A_1.\cdots.A_{n-1}).$$

The **length** of such a context is the length of the corresponding sequence, i.e., the number of types in Δ.

Even though such contexts are not strictly speaking types of $\mathbf{C}$, they can be seen as generalisations thereof; the next definition extends the CwA structure accordingly.

Definition 2.8. Let $\Delta = (A_1, \dots, A_n)$ be a context over an object Γ of a CwA $\mathbf{C}$. The **extension** $\Gamma.\Delta \in \mathbf{C}$ of Γ by Δ and the corresponding **projection** $p_\Delta : \Gamma.\Delta \to \Gamma$ are

$$\Gamma.\Delta = \Gamma.A_1.\cdots.A_n \quad \text{and} \quad p_\Delta = p_{A_n} \circ \cdots \circ p_{A_1}.$$

In the presence of a map $f : \Gamma' \to \Gamma$ of $\mathbf{C}$, the **reindexing** $f^*\Delta$ of Δ along f, which is a context over Γ', and the corresponding **connecting map** $f.\Delta : \Gamma'.f^*\Delta \to \Gamma.\Delta$ are

$$f^*\Delta = (f^*A_1, (f.A_1)^*A_2, \dots, (f.A_1.\cdots.A_{n-1})^*A_n) \quad \text{and} \quad f.\Delta = f.A_1.\cdots.A_n.$$

Definition 2.9 ([KL18, Definition 4.13])**.** Let $\mathbf{C}$ be a category with attributes, and Γ an object of $\mathbf{C}$. The **fibrant slice** of $\mathbf{C}$ over Γ, written $\mathbf{C}/\!\!/\Gamma$, is the category with attributes specified by the following:

(1) objects are contexts over Γ;
(2) maps of $\mathbf{C}/\!\!/\Gamma$ between Δ and Δ' are maps $g : \Gamma.\Delta \to \Gamma.\Delta'$ of $\mathbf{C}$ such that $p_{\Delta'} \circ g = p_\Delta$, with composition and identity morphisms inherited from $\mathbf{C}$;

(3) the chosen terminal object is the empty context over Γ, corresponding to the empty sequence of types;
(4) the set of types over $\Delta \in \mathbf{C} /\!\!/ \Gamma$ is $\mathrm{Ty}_{\mathbf{C}}(\Gamma.\Delta)$, and the extension $\Delta.A$ of Δ by such a type A is the result of adding A at the end of the sequence defining Δ;
(5) reindexing of types, projections, and connecting maps are all inherited from $\mathbf{C}$ in the evident ways;

Proposition 2.10. *Let* $\mathbf{C}$ *be a category with attributes, and* $f : \Gamma' \to \Gamma$ *a map of* $\mathbf{C}$*. Sending* $\Delta \in \mathbf{C} /\!\!/ \Gamma$ *to* $f^*\Delta \in \mathbf{C} /\!\!/ \Gamma'$ *defines a CwA map* $f^* : \mathbf{C} /\!\!/ \Gamma \to \mathbf{C} /\!\!/ \Gamma'$*, acting on morphisms via the universal property of pullbacks and on types via reindexing.*

Moreover, the mapping $f \mapsto f^*$ *is functorial, hence defines a functor* $\mathbf{C} \to \mathrm{CwA} : \Gamma \mapsto \mathbf{C} /\!\!/ \Gamma$*.*

Proof. A map $g : \Delta' \to \Delta$ of $\mathbf{C} /\!\!/ \Gamma$ is sent to the unique map $g' : f^*\Delta' \to f^*\Delta$ of $\mathbf{C} /\!\!/ \Gamma'$ such that $f.\Delta \circ g' = g \circ f.\Delta'$, this is clearly a functorial assignment. A type $A \in \mathrm{Ty}_{\mathbf{C}/\!\!/\Gamma}(\Delta)$ is sent to $(f.\Delta)^*A \in \mathrm{Ty}_{\mathbf{C}/\!\!/\Gamma'}(f^*\Delta)$, and the definition of f^* on morphisms ensures that $A \mapsto (f.\Delta)^*A$ is natural. Showing that f^* preserves the remainder of the CwA structure is a straightforward exercise. The last claim is clear. □

So far, both types and contexts have been given semantic counterparts, as well as have various other syntactic notions such as context extensions. However, one notable omission still remains: terms. In Example 2.4, the syntactic CwA, maps of the underlying category are context morphisms, i.e., sequences of terms. In particular, terms of

$$x_1 : A_1, \ldots, x_n : A_n \vdash A \text{ type}$$

correspond, up to definitional equality and renaming of free variables, to maps

$$[x_1 : A_1, \ldots, x_n : A_n] \to [x_1 : A_1, \ldots, x_n : A_n, x : A]$$

which leave the first n variables unchanged. Bearing this in mind, one should not be surprised by the definition in a general category with attributes.

Definition 2.11. In a category with attributes $\mathbf{C}$, a **term** of a type $A \in \mathrm{Ty}(\Gamma)$ is a section of the corresponding projection p_A, i.e., a map $a : \Gamma \to \Gamma.A$ such that $p_A \circ a = \mathrm{id}_\Gamma$. The set of terms of A is written $\mathrm{Tm}(\Gamma, A)$, or $\mathrm{Tm}(A)$ if no confusion can arise from leaving Γ implicit.

For example, elements of $\mathrm{Tm}(\Gamma, A)$ in $\mathbf{C}_{\mathbf{Set}}$ are Γ-indexed families of elements $(a_\gamma)_{\gamma\in\Gamma}$ such that $a_\gamma \in A_\gamma$ for all $\gamma \in \Gamma$.

As is the case with types, terms can be reindexed. Namely, a term in $\mathrm{Tm}(\Gamma, A)$ of a CwA $\mathbf{C}$ is precisely a map in the fibrant slice $\mathbf{C} /\!\!/ \Gamma$ from the empty context to A, viewed as a context over Γ; hence the pullback functors of Proposition 2.10 can be applied to terms. Concretely, given a map $f : \Gamma' \to \Gamma$ of $\mathbf{C}$ and a term $a \in \mathrm{Tm}(\Gamma, A)$, the image f^*a of a under f^* is a map of $\mathbf{C} /\!\!/ \Gamma'$ from the empty context over Γ' to f^*A, i.e., a term of the latter, henceforth known as the **reindexing** of a along f. It is worth noting that reindexing of terms is also functorial.

As one might expect, CwA maps play well with terms.

Proposition 2.12. *Let* $F : \mathbf{C} \to \mathbf{D}$ *be a CwA map. For each type* $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$*, the CwA map* F *induces natural maps*

$$\mathrm{Tm}(\Gamma, A) \to \mathrm{Tm}(F\Gamma, FA) : a \mapsto Fa,$$

natural in the sense that $Ff^*a = (Ff)^*Fa$ *for each* $a \in \mathrm{Tm}(\Gamma, A)$ *and* $f : \Delta \to \Gamma$*.*

Proof. Since F preserves projections, the image of a section of p_A is a section of p_{FA}, hence the induced maps are well-defined. The claimed naturality follows from F's preserving of canonical pullback squares, as that implies that F also preserves pullbacks of morphisms. □

One of the reasons for introducing fibrant slices is that they provide a formal setting in which to extend the reindexing notation f^*, as was already hinted at with the case of terms.

Notation. Let $\mathbf{C}$ be a CwA, and $f : \Gamma' \to \Gamma$ a map of $\mathbf{C}$. In the remainder of the present work, each time an expression f^*X is encountered, with X a type, map, or object of $\mathbf{C}$, then X can be understood as a type, map, or object of the fibrant slice $\mathbf{C}/\!\!/\Gamma$. Accordingly, the expression f^*X is to be read in the sense of Proposition 2.10, as the image of X under the reindexing functor f^*. Note that this does not clash with the previous meaning of f^*A for types $A \in \mathrm{Ty}(\Gamma)$.

2.2 Adjunction from h-sets

The previous section laid out the foundations of categories with attributes, the bedrock of all semantic considerations to follow. Those models encode precisely the basic judgements and structural rules of type theory. In order to interpret more complex types, such as products types, Π-types, or identity types, it is necessary to endow CwAs with extra structure, corresponding to said types.

The main types of interest here are identity types. The following is the semantic translation of their defining properties as laid out in the syntax.

Definition 2.13. An Id-**structure** on a CwA $\mathbf{C}$ consists of:

(1) for each type $A \in \mathrm{Ty}(\Gamma)$, a type
$$\mathsf{Id}_A \in \mathrm{Ty}(\Gamma.A.p_A^*A);$$
(2) for each type $A \in \mathrm{Ty}(\Gamma)$, a map
$$r_A : \Gamma.A \to \Gamma.A.p_A^*A.\mathsf{Id}_A$$
such that $p_{\mathsf{Id}_A} \circ r_A = (\mathrm{id}_{\Gamma.A}, \mathrm{id}_{\Gamma.A})$;
(3) for each type $A \in \mathrm{Ty}(\Gamma)$, type $C \in \mathrm{Ty}(\Gamma.A.p_A^*A.\mathsf{Id}_A)$, and map $d : \Gamma.A \to \Gamma.A.p_A^*A.\mathsf{Id}_A.C$ such that $p_C \circ d = r_A$, a term
$$J_{C,d} \in \mathrm{Tm}(\Gamma.A.p_A^*A.\mathsf{Id}_A, C)$$
such that $J_{C,d} \circ r_A = d$;

under the the additional condition that the above data is stable under reindexing, that is, for each map $f : \Delta \to \Gamma$ and A, C, d as above the following identities hold:
$$f^*\mathsf{Id}_A = \mathsf{Id}_{f^*A}, \quad f^*r_A = r_{f^*A}, \quad f^*J_{C,d} = J_{f^*C,f^*d}.$$

A CwA **with** Id-**types** is one that is equipped with an Id-structure.

As usual, adding structure warrants giving a name to those maps which preserve said structure.

Definition 2.14. A CwA map $F : \mathbf{C} \to \mathbf{D}$ between CwAs with Id-types is said to **preserve** Id-**types** if, for any A, C, d as above, the following holds:
$$F\mathsf{Id}_A = \mathsf{Id}_{FA}, \quad Fr_A = r_{FA}, \quad FJ_{C,d} = J_{FC,Fd}.$$

The category of CwAs with Id-types and CwA maps preserving Id-types is denoted by $\mathsf{CwA}_{\mathsf{Id}}$.

Example 2.15. The CwA of sets $\mathbf{C}_{\mathbf{Set}}$ admits a canonical choice of Id-types. Indeed, given an indexed family of sets $(A_\gamma)_{\gamma\in\Gamma}$, there is a prevalent choice of family, indexed by the extension
$$\Gamma.A.p_A^*A = \{(\gamma, a, a') : \gamma \in \Gamma, \ \ a, a' \in A_\gamma\},$$

which models equality:

$$(\mathsf{Id}_A)_{(\gamma,a,a')} = \begin{cases} \{*\} & \text{if } a = a', \\ \emptyset & \text{if } a \neq a'. \end{cases}$$

In other words, the set $(\mathsf{Id}_A)_{(\gamma,a,a')}$ contains at most one element and is inhabited precisely when $a = a'$. The extension of Id_A is therefore the set of tuples $(\gamma, a, a, *)$ with $a \in A_\gamma$, which is isomorphic to $\Gamma.A$ via the map

$$r_a : \Gamma.A \to \Gamma.A.p_A^*A.\mathsf{Id}_A : (\gamma, a) \mapsto (\gamma, a, a, *).$$

The above clearly satisfies $p_{\mathsf{Id}_A} \circ r_A = (\mathrm{id}_{\Gamma.A}, \mathrm{id}_{\Gamma.A})$, so it can be chosen to model reflexivity. Lastly, if C is a family of sets indexed by the extension of Id_A and $d : \Gamma.A \to \Gamma.A.p_A^*A.\mathsf{Id}_A.C$ a map such that $p_C \circ d = r_A$ then one can simply let $J_{C,d}$ be $d \circ r_A^{-1}$.

Syntactically, an h-set is type A together with a proof that, for any two elements $x, y : A$, any two equalities between x and y are themselves equal. CwAs with Id-types provide the right setting for a semantic translation of this definition.

Definition 2.16. Let $\mathbf{C}$ be a CwA with Id-types, and $\Gamma \in \mathbf{C}$. An **h-set over** Γ is a pair (A, w) with A a type in $\mathrm{Ty}(\Gamma)$ and w a term in $\mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$.[2] The set of h-sets over Γ is written $\mathrm{Ty}_{\leq 0}(\Gamma)$.[3]

Stability of Id-types under reindexing ensures that if (A, w) is an h-set over $\Gamma \in \mathbf{C}$, as in the definition above, and $f : \Delta \to \Gamma$ is any map of $\mathbf{C}$ then (f^*A, f^*w) is an h-set over Δ, called the reindexing of (A, w) along f and denoted by $f^*(A, w)$. This shows that $\mathrm{Ty}_{\leq 0}$ is actually a presheaf, with which one can replace the type presheaf Ty and obtain a new CwA as a result.

Definition 2.17. Let $\mathbf{C}$ be a CwA with Id-types, its **CwA of h-sets** $S\mathbf{C}$ is the CwA defined as follows:

(1) it has the same underlying category as $\mathbf{C}$;
(2) types of $S\mathbf{C}$ over Γ are h-sets of $\mathbf{C}$ over Γ, meaning that $\mathrm{Ty}_{S\mathbf{C}}(\Gamma) = (\mathrm{Ty}_{\mathbf{C}})_{\leq 0}(\Gamma)$, with reindexing as defined above;
(3) extensions, projections, and connecting maps are inherited from $\mathbf{C}$ by forgetting the second components of h-sets, signifying that

$$\Gamma.(A, w) = \Gamma.A, \quad p_{(A,w)} = p_A, \quad f.(A, w) = f.A$$

for each h-set $(A, w) \in \mathrm{Ty}_{S\mathbf{C}}(\Gamma)$ and map $f : \Delta \to \Gamma$.

The **stripping map** $s_{\mathbf{C}} : S\mathbf{C} \to \mathbf{C}$ is the CwA map whose underlying functor is the identity and which acts on types by $(A, w) \mapsto A$.

One of the first facts one proves when learning about h-sets in HoTT is that if A is an h-set then so is the corresponding identity type Id_A. Syntactic proofs of this fact translate to particularly stable terms.

Proposition 2.18. *There exists a family of terms*

$$\ell_{\mathbf{C},A,w} \in \mathrm{Tm}_{\mathbf{C}}(\mathsf{Id}_{\mathsf{Id}_{\mathsf{Id}_A}}),$$

indexed by CwAs with Id*-types* $\mathbf{C}$*, types* A *of* $\mathbf{C}$*, and terms* w *in* $\mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$ *such that*

$$f^*\ell_{\mathbf{C},A,w} = \ell_{\mathbf{C},f^*A,f^*w} \quad \textit{and} \quad F\ell_{\mathbf{C},A,w} = \ell_{\mathbf{D},FA,Fw}$$

for each map $f : \Delta \to \Gamma$ *of* $\mathbf{C}$ *and* $\mathrm{CwA}_{\mathsf{Id}}$*-map* $F : \mathbf{C} \to \mathbf{D}$.

[2]In full, the term w belongs to $\mathrm{Tm}(\Gamma.A.p_A^*A.\mathsf{Id}_A.p_{\mathsf{Id}_A}^*\mathsf{Id}_A, \mathsf{Id}_{\mathsf{Id}_A})$.

[3]In the more general context of n-types, h-sets correspond to 0-types, hence the notation.

Disclaimer. As currently stated, the above proposition is not exactly true, at least it seems difficult to prove. This is because the standard syntactic proof (or variants thereof) of the fact that mere propositions are also h-sets relies on path composition, which itself cannot be defined using only the version of identity types considered here. Indeed, to set up the double identity elimination used to define composition one usually relies on Π-types, which are not available here.

Undoubtedly, the constructions and definitions of this section can be adapted if one were to add (extensional) Π-types to the picture; however, the present goal is to give a minimal working example of restriction to h-sets, in which the only assumption on CwAs is that they carry an Id-structure.

This goal can still be achieved if one is willing to adopt a stronger notion of Id-types: Frobenius identity types, which allow a version of identity elimination with extra assumptions. The only difference with Definition 2.13 is that the elimination rule is stronger: for each

- object $\Gamma \in \mathbf{C}$,
- type $A \in \mathrm{Ty}(\Gamma)$,
- context $\Delta = (B_1, \ldots, B_n)$ over $\Gamma.A.p_A^*A.\mathsf{Id}_A$,
- type $C \in \mathrm{Ty}(\Gamma.A.p_A^*A.\mathsf{Id}_A.\Delta)$, and
- map $d : \Gamma.A.r_A^*\Delta \to \Gamma.A.p_A^*A.\mathsf{Id}_A.\Delta.C$ such that $p_C \circ d = r_A.\Delta$

one obtains a section $J_{\Delta,C,d}$ of p_C, stable over Γ, such that $J_{\Delta,C,d} \circ (r_A.\Delta) = d$. As soon as one works with more fully fledged type theories, as opposed to one with only Id-types, the distinction between plain identity types and Frobenius identity types is blurred since the latter can be defined from the former and Π-types, as explained in the start of section 3.4.3 of [LW15].

The proof given below relies on Frobenius identity types, while the remainder of the section stays ambiguous as to exactly which notion of identity types is used. This seemingly faux-pas in terms of rigour has the advantage that the section can either be read with Frobenius identity types in mind, thus providing a minimal working example albeit with a different notion of Id-structure; or it can be read under the hidden assumption that (extensional) Π-types are also considered.

Proof of Proposition 2.18. This is based on the syntactic proof from the HoTT book [Uni13, 3.1.8].

First note that it suffices to define stable (in the sense defined above) terms $\ell'_{\mathrm{C},A,w} \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$ where $w \in \mathrm{Tm}(\mathsf{Id}_A)$, and to let $\ell_{\mathrm{C},A,w} = \ell'_{\mathrm{C},\mathsf{Id}_A,w}$. Indeed, this corresponds to applying a proof that mere propositions are h-sets to the type Id_A of an h-set A.

Using Frobenius identity types, one can define the semantic counterpart of path composition. Indeed, let Γ be an object of a CwA $\mathbf{C}$ and A a type over it, denote by A' and A'' respectively the weakened types p_A^*A and $p_{A'}^*A'$. Since $\Gamma.A.A'.A''$ and $\Gamma.A.A'$ are respectively isomorphic to $A \times_\Gamma A \times_\Gamma A$ and $A \times_\Gamma A$, there is a map $q : \Gamma.A.A'.A'' \to \Gamma.A.A'$ corresponding to the projection $(\pi_1, \pi_3) : A \times_\Gamma A \times_\Gamma A \to A \times_\Gamma A$ which forgets the middle component. Consider the type

$$p^*_{p^*_{\mathsf{Id}'_A} p^*_{A''} \mathsf{Id}_A} p^*_{\mathsf{Id}_{A'}} q^* \mathsf{Id}_A \in \mathrm{Ty}(\Gamma.A.A'.A''.\mathsf{Id}_{A'}.p^*_{\mathsf{Id}'_A} p^*_{A''} \mathsf{Id}_A)$$

One can construct a section of said type using two identity eliminations. This section corresponds to the operation $(p, q) \mapsto p \cdot q$ where $p : x = y$ and $q : y = z$.

Similarly, if $s : \Gamma.A.A' \to \Gamma.A.A'$ designs the map which swaps components, then the type

$$p^*_{\mathsf{Id}_A} s^* \mathsf{Id}_A \in \mathrm{Ty}(\Gamma.A.A'.\mathsf{Id}_A)$$

admits the section $J_{p^*_{\mathsf{Id}_A} s^* \mathsf{Id}_A, (r_A, r_A)}$. This section corresponds to the operation $p \mapsto p^{-1}$.

The remainder of the proof could also be carried out in the language of CwAs but for the sake of clarity it is given in a more syntactic language. Using another instance of identity elimination one can obtain a proof of $p \cdot p^{-1} = \mathsf{refl}$. Now, given a term w such that $w(x, y) : x = y$ for all $x, y : A$, one can construct a term $w'(p) : p = w(x, y) \cdot w(y, y)^{-1}$ for each $p : x = y$ by identity elimination; indeed, in the case $p \equiv \mathsf{refl}_x$ the above proof of $p \cdot p^{-1} = \mathsf{refl}$ can be used to define w'. To show that any two $p, q : x = y$ are themselves equal to one another, it suffices to take the term $w'(p) \cdot w'(q)^{-1} : p = q$.

In the world of CwAs, this last term would have been defined in terms of sections $J_{\Delta,C,d}$ and would therefore be stable both under reindexing and under CwA maps which preserve the identity structure. Hence, the conclusion follows. ◻

Fix such a family of terms $\ell = (\ell_{\mathbf{C},A,w})$ for the remainder of this section. Such a choice induces an ℓ-canonical h-set structure on identity types of h-sets.

Proposition 2.19. *Let* $\mathbf{C}$ *be a CwA with* Id*-types. For* $(A, w) \in \mathrm{Ty}_{\leq 0}(\Gamma)$*, setting*

$$\mathsf{Id}_{(A,w)} = (\mathsf{Id}_A, \ell_{\mathbf{C},A,w})$$

and, for each suitable choice of h-set (C, w') *and map* d*,*

$$r_{(A,w)} = r_A, \quad J_{(C,w'),d} = J_{C,d}$$

defines an Id*-structure on* $S\mathbf{C}$*, which is preserved by* $s_{\mathbf{C}} : S\mathbf{C} \to \mathbf{C}$*.*

Proof. By definition of ℓ, the pair $(\mathsf{Id}_A, \ell_{\mathbf{C},A,w})$ is an h-set over $\Gamma.(A,w).p^*_{(A,w)}(A,w)$, so it is a suitable candidate for Id-types. And since the objects, maps, extensions, and projections of $S\mathbf{C}$ are inherited from $\mathbf{C}$, the proposed $r_{(A,w)}$ and $J_{(C,w'),d}$ satisfy their respective conditions.

All that remains is to check stability under reindexing, that of r and J follows directly from them being part of an Id-structure on $\mathbf{C}$. As for $\mathsf{Id}_{(A,w)}$, it is due in part to the stability properties of ℓ under reindexing: for any map $f : \Delta \to \Gamma$,

$$f^*\mathsf{Id}_{(A,w)} = (f^*\mathsf{Id}_A, f^*\ell_{\mathbf{C},A,w}) = (\mathsf{Id}_{f^*A}, \ell_{\mathbf{C},f^*A,f^*w}) = \mathsf{Id}_{f^*(A,w)}.$$

Finally, checking that $s_{\mathbf{C}}$ preserves Id-types is straightforward. ◻

The idea that proofs of equality should be propositionally indistinguishable from each other is known as *uniqueness of identity proofs*, UIP for short; this is an equivalent way of saying that all types should be h-sets. Since the concept of h-sets was translated in CwAs with Id-types, so can that of UIP.

There is a slight complication however. One would expect a UIP-structure to consist of a family of terms $w_A \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$ indexed by types A and stable under reindexing, but with such a definition the following property fails: if $\mathbf{C}$ is a CwA with Id-types equipped with such a UIP-structure then the canonical map $\mathbf{C} \to S\mathbf{C}$, which takes objects and maps to themselves but act on types by $A \mapsto (A, w_A)$, might fail to respect Id-types, as there is no reason that $w_{\mathsf{Id}_A} = \ell_{\mathbf{C},A,w_A}$ should hold.

The fix might seem a simple one but it works for the present needs, the missing equation is simply added to the definition. Of course this depends upon the choice of ℓ, and as to avoid any confusion this dependence is recorded in the name.

Definition 2.20. Let $\mathbf{C}$ be a CwA with Id-types. A UIP_ℓ**-structure** on $\mathbf{C}$ is a family of terms

$$w_A \in \mathrm{Tm}(\mathrm{Id}_{\mathrm{Id}_A}),$$

indexed by the types $A \in \mathrm{Ty}(\Gamma)$ of $\mathbf{C}$, such that

(1) they agree with the canonical h-set structure on identity types: for each type $A \in \mathrm{Ty}(\Gamma)$,

$$w_{\mathsf{Id}_A} = \ell_{\mathbf{C},A,w_A};$$

(2) they are stable under reindexing: for each map $f : \Delta \to \Gamma$ of $\mathbf{C}$,

$$f^*w_A = w_{f^*A}.$$

A **CwA with** UIP_ℓ is one equipped with a UIP_ℓ-structure. For $\mathbf{C}$ and $\mathbf{D}$ two CwAs with UIP_ℓ, a CwA map $F : \mathbf{C} \longrightarrow \mathbf{D}$ preserving Id-types is said to further preserve UIP_ℓ if

$$Fw_A = w_{FA}$$

for each type A. The corresponding category is called $\mathsf{CwA}_{\mathsf{UIP}_\ell}$.

Example 2.21. Since h-sets are meant to model sets, verifying that $\mathbf{C}_{\mathbf{Set}}$ supports UIP is a good sanity check, a straightforward one at that. Recall that, for an indexed family of sets $(A_\gamma)_{\gamma\in\Gamma}$, there is at most one section of p_{Id_A}, which exists precisely each A_γ has at most one element.Applying this to the family Id_A directly shows that $p_{\mathsf{Id}_{\mathsf{Id}_A}}$ has a section since the sets $(\mathsf{Id}_A)_{(\gamma,a,a')}$ have at most one element by definition; call that section w_A. Moreover, there being at most one section directly guarantees that the chosen family of terms is stable under reindexing and respect the ℓ-induced h-set structure on identity types.

Proposition 2.22. *Let* $\mathbf{C}$ *be a CwA with* Id*-types. For* $(A, w) \in \mathrm{Ty}_{\leq 0}(\Gamma)$*, letting*

$$w_{(A,w)} = w$$

defines an UIP_ℓ*-structure on* $S\mathbf{C}$*.*

Proof. By definition, terms of $\mathsf{Id}_{\mathsf{Id}_{(A,w)}}$ in $S\mathbf{C}$ are precisely terms of $\mathsf{Id}_{\mathsf{Id}_A}$ in $\mathbf{C}$, hence w is a term of the right sort. Moreover, this choice of terms is compatible with the canonical h-set structure of identity types: for each h-set (A, w) of $\mathbf{C}$,

$$w_{\mathsf{Id}_{(A,w)}} = w_{(\mathsf{Id}_A, \ell_{\mathbf{C},A,w})} = \ell_{\mathbf{C},A,w} = \ell_{\mathbf{C},A,w_{(A,w)}}.$$

Finally, they are also stable under reindexing as

$$f^* w_{(A,w)} = f^* w = w_{(f^*A, f^*w)} = w_{f^*(A,w)}$$

for each h-set $(A, w) \in \mathrm{Ty}_{\leq 0}(\Gamma)$ and map $f : \Delta \longrightarrow \Gamma$. □

In summary, every $\mathbf{C} \in \mathsf{CwA}_{\mathsf{Id}}$ induces an object $S\mathbf{C}$ of $\mathsf{CwA}_{\mathsf{UIP}_\ell}$ together with a CwA map $s_{\mathbf{C}} : S\mathbf{C} \longrightarrow \mathbf{C}$ preserving Id-types. In other words, if U denotes the forgetful functor $\mathsf{CwA}_{\mathsf{UIP}_\ell} \longrightarrow \mathsf{CwA}_{\mathsf{Id}}$, then $s_{\mathbf{C}}$ is a map $US\mathbf{C} \longrightarrow \mathbf{C}$ in $\mathsf{CwA}_{\mathsf{Id}}$. This is actually a universal construction, as shown below.

Proposition 2.23. *For each* $\mathbf{C} \in \mathsf{CwA}_{\mathsf{Id}}$*, the pair* $(S\mathbf{C}, s_{\mathbf{C}})$ *is a universal arrow from the forgetful functor* $U : \mathsf{CwA}_{\mathsf{UIP}_\ell} \longrightarrow \mathsf{CwA}_{\mathsf{Id}}$ *to* $\mathbf{C}$*.*

That is, for each $\mathbf{D} \in \mathsf{CwA}_{\mathsf{UIP}_\ell}$ *and map* $F : U\mathbf{D} \longrightarrow \mathbf{C}$ *in* $\mathsf{CwA}_{\mathsf{Id}}$ *there exists a unique map* $G : \mathbf{D} \longrightarrow S\mathbf{C}$ *in* $\mathsf{CwA}_{\mathsf{UIP}_\ell}$ *such that* $F = s_{\mathbf{C}} \circ UG$*.*

Proof. Suppose that such a map G exists. Since the stripping map $s_{\mathbf{C}}$ acts as the identity on objects and morphisms, the underlying functor of G must coincide with that of F. If A is a type of $\mathbf{D}$ then

$$s_{\mathbf{C}}(GA) = FA,$$

hence the first component of GA has to be FA, meaning that $GA = (FA, w)$ for some w. But G preserves UIP_ℓ, thus

$$Gw_A = w_{GA} = w_{(FA,w)} = w.$$

This means that w must be equal to Fw_A, because $Gw_A = Fw_A$, as the underlying functors of G and F agree. The above shows that if such a CwA map G exists then its action on objects and morphisms coincide with that of F and it sends a type A of $\mathbf{D}$ to the h-set (FA, Fw_A). This settles uniqueness.

Conversely, it is easy to check that the actions on objects, morphisms, and types above yield a well defined CwA map $G : \mathbf{D} \longrightarrow S\mathbf{C}$ preserving Id-types and UIP_ℓ, and satisfying the desired equality. □

Corollary 2.24. *The mapping* $\mathbf{C} \mapsto S\mathbf{C}$ *is the object function of a right adjoint* S *to* $U : \mathsf{CwA}_{\mathsf{UIP}_\ell} \to \mathsf{CwA}_{\mathsf{Id}}$. *The action of* S *on morphisms takes a map* $F : \mathbf{C} \to \mathbf{D}$ *in* $\mathsf{CwA}_{\mathsf{Id}}$ *to* $SF : S\mathbf{C} \to S\mathbf{D}$, *defined as* F *on the underlying categories and on types by*

$$(A, w) \in \mathrm{Ty}_{S\mathbf{C}}(\Gamma) \mapsto (FA, Fw) \in \mathrm{Ty}_{S\mathbf{D}}(F\Gamma).$$

Proof. Applied to the last proposition, the characterisation of adjoints by universal arrows [Mac98, Theorem IV.1.2] directly shows that there exists a right adjoint S to U whose object function is given by $\mathbf{C} \mapsto S\mathbf{C}$, and whose action on morphisms is the unique one making the maps $s_\mathbf{C} : US\mathbf{C} \to \mathbf{C}$ into a natural transformation $US \to \mathrm{id}_{\mathsf{CwA}_{\mathsf{Id}}}$.

In other words, the image SF of a morphism $F : \mathbf{C} \to \mathbf{D}$ of $\mathsf{CwA}_{\mathsf{Id}}$ is the unique map $S\mathbf{C} \to S\mathbf{D}$ such that $s_\mathbf{D} \circ USF = F \circ s_\mathbf{C}$, which results from the universality of $s_\mathbf{D}$. The proof of the previous proposition showed that SF and $F \circ s_\mathbf{C}$ have the same underlying functor, i.e., the same as F; and that SF sends a type (A, w) of $S\mathbf{C}$ to

$$\big((F \circ s_\mathbf{C})(A, w), (F \circ s_\mathbf{C})w_{(A,w)}\big) = (FA, Fw).$$

□

2.3 Extensions and consequences

In the previous section, a right adjoint to the forgetful functor $U : \mathsf{CwA}_{\mathsf{UIP}_\ell} \to \mathsf{CwA}_{\mathsf{Id}}$ was constructed, thus answering the problem of "restricting to h-sets" posed in the corresponding paragraph of the introduction. However, it constitutes a minimal answer, where the type theory at hand comprises only the structural rules and those governing identity types. Ideally, one would be able to extend said adjunction to more general type theories T, assumed to have Id-types.

The first half of the present section informally discusses what form could take conditions on a type theory T which ensure that the forgetful functor $U_T : \mathsf{CwA}_{T+\mathsf{UIP}_\ell} \to \mathsf{CwA}_T$ admits a similarly defined right adjoint S_T. Consequences of such adjunctions are then discussed in the second half of the section.

In the definition below, the notation $w : \mathrm{isSet}\, A$ is shorthand for the syntactic meaning of $w \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$.

Definition 2.25. A type former

$$\frac{\Gamma_1 \vdash A_1 \ \mathsf{type} \quad \dots \quad \Gamma_n \vdash A_n \ \mathsf{type} \qquad \Gamma_{n+1} \vdash b_1 : B_1 \quad \dots \quad \Gamma_{n+m} \vdash b_m : B_m}{\Gamma \vdash H(A_1, \dots, A_n, b_1, \dots, b_m) \ \mathsf{type}}$$

is said to **preserves h-sets** in a type theory T, which has at least Id-types, if the rule below can be derived in T.

$$\frac{\begin{array}{ccc} \Gamma_1 \vdash A_1 \ \mathsf{type} & \Gamma_1 \vdash w_1 : \mathrm{isSet}\, A_1 & \Gamma_{n+1} \vdash b_1 : B_1 \\ \vdots & \vdots & \vdots \\ \Gamma_n \vdash A_n \ \mathsf{type} & \Gamma_n \vdash w_n : \mathrm{isSet}\, A_n & \Gamma_{n+m} \vdash b_m : B_m \end{array}}{\Gamma \vdash w(w_1, \dots, w_n) : \mathrm{isSet}\, H(A_1, \dots, A_n, b_1, \dots, b_m)}$$

If all the type formers a given family of rules preserve h-sets in T then the family is also said to preserve h-sets in T. If all the type formers of a type theory T' contained in T preserve h-sets in T then T' is said to preserve h-sets in T; saying that a type theory with Id-types preserves h-sets means that it does so in itself.

Remark 2.26. If $n = 0$ in the above definition, meaning that F does not depend on any type A_i, then it preserves h-sets precisely if $H(b_1, \dots, b_n)$ is an h-set for any choice of b_i's.

Example 2.27. The family of rules defining identity types preserves h-sets in any type theory with Id-types or, in slogan form, Id-types preserve h-sets. Indeed, it is precisely upon a proof of that fact that Proposition 2.18 relies to provide a family of terms $\ell_{\mathbf{C},A,w}$.

Example 2.28. Section 7.1 of the HoTT book [Uni13] shows that most of the rule families defining logical structures preserve h-sets in HoTT, though said proofs could be adapted to work in a smaller type theory. It is worth noting that function extensionality appears in the proof that Π-types preserve h-sets.

As mentioned in the first example, preservation of h-sets under identity types was crucial in obtaining a stable family of terms; Proposition 2.18 can be generalised.

Proposition 2.29. *If*

$$\frac{\Gamma_1 \vdash A_1 \text{ type} \quad \ldots \quad \Gamma_n \vdash A_n \text{ type} \qquad \Gamma_{n+1} \vdash b_1 : B_1 \quad \ldots \quad \Gamma_{n+m} \vdash b_m : B_m}{\Gamma \vdash H(A_1, \ldots, A_n, b_1, \ldots, b_m) \text{ type}}$$

is a type former preserving h-sets in T, then there exists a stable family of terms $\ell^H_{\mathbf{C},A_i,b_j,w_k}$ indexed by

- ◇ *categories with attributes* $\mathbf{C}$ *in* CwA_T,
- ◇ *types* $A_1 \in \mathrm{Ty}(\Gamma_1), \ldots, A_n \in \mathrm{Ty}(\Gamma_n)$,
- ◇ *terms* $b_1 \in \mathrm{Tm}(B_1), \ldots, b_m \in \mathrm{Tm}(B_m)$, *and*
- ◇ *terms* $w_1 \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_{A_1}}), \ldots, w_n \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_{A_n}})$

such that

$$\ell^H_{\mathbf{C},A_i,b_j,w_k} \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_{H(A_1,\ldots,A_n,b_1,\ldots,b_m)}});$$

here, the adjective "stable" means that

$$f^* \ell^H_{\mathbf{C},A_i,b_j,w_k} = \ell^H_{\mathbf{C},f^*A_i,f^*b_j,f^*w_k} \quad \text{and} \quad F\ell^H_{\mathbf{C},A_i,b_j,w_k} = \ell^H_{\mathbf{D},FA_i,Fb_j,Fw_k}$$

for each map $f : \Delta \to \Gamma$ of $\mathbf{C}$ *and* CwA_T*-map $F : \mathbf{C} \to \mathbf{D}$.*

□

Suppose now that T is a type theory which preserve h-sets. According to the proposition above, each type former of T can be given a stable family of terms; if they are bundled together, they can be regarded as a single family ℓ^T indexed by type formers. Fix such an ℓ for the remainder of the section.

Let $\mathbf{C}$ be a model of T, i.e., an object of CwA_T, one can endow its CwA of h-sets $S\mathbf{C}$ with a T-structure preserved by the stripping map. Namely, the missing term part of the interpretation of each type former in $S\mathbf{C}$ is given by ℓ^T, while the structure corresponding to term formers can be inherited from $\mathbf{C}$ since it and $S\mathbf{C}$ share the same underlying category. Denote the obtained element of CwA_T by $S_T\mathbf{C}$.

As before, the chosen ℓ^T also induces an ℓ^T-canonical notion of UIP. Namely, an UIP_{ℓ^T}-structure on a $\mathbf{C} \in \mathsf{CwA}_A$ consists of a choice of a family of terms $w_A \in \mathrm{Tm}(\mathsf{Id}_{\mathsf{Id}_A})$ which both respects ℓ^T and is stable under reindexing. As expected, the CwA of h-sets $S_T\mathbf{C}$ naturally carries a UIP_{ℓ^T}-structure and Proposition 2.23 can be generalised.

Proposition 2.30. *For each* $\mathbf{C} \in \mathsf{CwA}_T$*, the pair* $(S_T\mathbf{C}, s_\mathbf{C})$ *is a universal arrow from the forgetful functor* $U_T : \mathsf{CwA}_{T+\mathsf{UIP}_{\ell^T}} \to \mathsf{CwA}_T$ *to* $\mathbf{C}$. □

Corollary 2.31. *The mapping* $\mathbf{C} \mapsto S_T\mathbf{C}$ *is the object function of a right adjoint S_T to the forgetful functor* $U_T : \mathsf{CwA}_{T+\mathsf{UIP}_{\ell^T}} \to \mathsf{CwA}_T$. □

Before moving to consequences of the above, a word should be said on the notion of *universe* in type theory. A universe can be thought of as a "type of types", and indeed the crudest kind of universe is just that: a closed type U with a dependent family of types

$$A : U \vdash \mathrm{El}(A) \text{ type}.$$

In most cases, one requires of U a number of closure properties such as asking for a term constructor

$$\frac{\Gamma \vdash A : U \quad \Gamma, x : \mathrm{El}(A) \vdash B(x) : U}{\Gamma \vdash \pi(A, B) : U}$$

together with a judgemental equality $\mathrm{El}(\pi(A, B)) \equiv \Pi_{a:\mathrm{El}(A)} \mathrm{El}(B(a))$, in which case U is said to be closed under Π-types. What, however, is *not* a standard condition on universes is that they and the types they index be h-sets. Therefore, universes, as usually encountered in type theory, do *not* preserve h-sets.

As a result, the above framework applies to a universe U only if it is explicitly requested that U and each $A : U \vdash \mathrm{El}(A)$ is an h-set. In particular, it seems difficult to adapt the above for a univalent universe. While a universe of h-sets which is itself an h-set is not commonplace in most descriptions of HoTT, adding one does not lead to an inconsistent theory, at least in the case of a universe without closure properties. Indeed, any family of sets indexed by a set induces such a universe in the simplicial model.

In summary, giving a more satisfactory account of universes within this framework requires more work; fortunately, the method does apply to simple universes, which is sufficient to model a limited form of quantification over h-sets. For example, a set-based universe is enough to interpret the formulation of AC, as given in the introduction, in the CwA of h-sets of the simplicial model; however, the obtained interpretation is a weak one in that the types quantified over are required to be *strict* h-sets, those h-sets which are isomorphic a set rather than just homotopy equivalent to one.

The first direct consequence is one that was mentioned in the introduction without justification.

Proposition 2.32. *Let T be a type theory, with* Id*-types, preserving h-sets. The forgetful functor* $U_T : \mathsf{CwA}_{T+\mathsf{UIP}_{\ell^T}} \to \mathsf{CwA}_T$ *sends the syntactical CwA* $\mathbf{C}_{T+\mathsf{UIP}_{\ell^T}}$ *of* $T + \mathsf{UIP}_{\ell^T}$ *to (an isomorphic copy of)* $\mathbf{C}_T$.

Proof. Since U_T is a left adjoint, it preserves colimits; in particular it preserve initial objects. Both $\mathbf{C}_{T+\mathsf{UIP}_{\ell^T}}$ and $\mathbf{C}_T$ are initial objects in their respective categories, hence the claim holds. □

Another direct consequence of the adjunction can be viewed as a sanity check with regards to the interpretation of the logic in CwA of h-sets of the simplicial model $\mathbf{C}_{\mathbf{sSet}}$.

Proposition 2.33. *Let T be a type theory, with* Id*-types and contained in HoTT, preserving h-sets. The diagram*

$$\begin{array}{ccc} U_T S_T \mathbf{C}_{\mathsf{HoTT}} & \longrightarrow & \mathbf{C}_{\mathsf{HoTT}} \\ {\scriptstyle U_T S_T [\![\,]\!]}\big\downarrow & & \big\downarrow{\scriptstyle [\![\,]\!]} \\ U_T S_T \mathbf{C}_{\mathbf{sSet}} & \longrightarrow & \mathbf{C}_{\mathbf{sSet}} \end{array}$$

commutes in CwA_T*, where the horizontal maps are stripping maps.*

Proof. This is just a naturality square of the counit $s : U_T S_T \to \mathrm{id}_{\mathsf{CwA}_T}$. □

In practice, this means that after restricting a rule of T via S_T, whether one first forgets the restriction and then interprets in $\mathbf{C}_{\mathbf{sSet}}$, or first interprets the rule in the CwA of h-sets of $\mathbf{C}_{\mathbf{sSet}}$ then forgets the restriction, the result is the same. In other words, the functor S_T fulfils its role of restricting to h-set, though it does so at the cost of relying on a choice of ℓ.

Fortunately, this dependence is not as large an issue as one might first think. For one, it can be disregarded in the case of a CwA $\mathbf{C}$ with Id-types which model the reflection principle, meaning that the projection $\mathsf{Id}_A \to \Gamma.A.p_A^* A$ is isomorphic to the diagonal map $\Gamma.A \to (\Gamma.A) \times_\Gamma (\Gamma.A)$ as is the case in $\mathbf{C}_{\mathbf{Set}}$. Indeed, any UIP_ℓ-structure on such a $\mathbf{C}$, for any choice of ℓ, is precisely what a standard UIP-structure would be, since the ℓ-dependent condition (1) of Definition 2.20 is vacuously satisfied.

Even when the CwA at hand does not have reflective Id-types, the composite $U_T S_T : \mathsf{CwA}_T \to \mathsf{CwA}_T$ remains unchanged for different choices of ℓ^T. Which is to say that it is a valid means of restricting to h-sets, independently of ℓ^T. Though the intermediate category $\mathsf{CwA}_{T+\mathsf{UIP}_{\ell^T}}$ used in the process might not be the first choice of someone solely wanting to study models of UIP.

In conclusion, this approach seems to be a reasonable middle ground between striving for a more satisfactory semantic notion of UIP and keeping things simple; since it was the strictness of CwA maps that raised the need for such a “fix”, it may be conjectured that switching to the 2-category of CwAs with weak maps instead of strict ones might yield a more elegant solution but that, indeed, complicates things.

3 Simplicial truncation

3.1 Coherence and universes

When seen from afar, some categories appear to carry a loose CwA structure, in which the structure is only defined up to isomorphism for example, but on close inspection it is not always clear how to translate this intuition to a strict CwA. One of the most recurring obstructions is known as the *coherence problem*, which comes in two flavours.

The first stems from the fact that part of the data of a CwA is essentially a functorial choice of pullbacks, which is quite an unnatural condition in many categories where such a choice can typically only be made up to isomorphisms. The second flavour is also linked to pullbacks, because all the extra logical structure a CwA might carry is supposed to strictly commute with them; even when the extra structure is defined by way of category-theoretic notions, e.g., by universal properties, this rarely holds without some careful and clever choices.

In other words, endowing a category with a CwA structure requires coherent choices of pullbacks and additional coherent choices of representatives for the logical structure, depending on the previous selections. While this can be done by hand in simple examples such as $\mathbf{C}_{\mathbf{Set}}$, it can quickly turn into a bookkeeping nightmare for more complex categories.

The solution to the coherence problem used to construct the simplicial model [KL21], due to Voevodsky, is based on the following observation: on-the-nose equality of objects is rarely a tractable notion in category theory, while strict equality of morphisms is much more amenable. Accordingly, one of the main insight of the solution is to shift one's view and think of types over Γ not as objects of the slice $\mathbf{C}/\Gamma$ but as morphisms from Γ into a fixed object of $\mathbf{C}$.

Definition 3.1 ([Voe15, Definition 2.1]). A (Voevodsky) *universe* in a category $\mathbf{C}$ is an object U together with a morphism $p : \tilde{U} \to U$ and, for each map $f : X \to U$, an object $E(f)$ as well as two maps $P(f) : E(f) \to X$ and $Q(f) : E(f) \to \tilde{U}$ such that

$$\begin{array}{ccc} E(f) & \xrightarrow{Q(f)} & \tilde{U} \\ {\scriptstyle P(f)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

is a pullback square.

Notation. Let U be a universe in a category $\mathbf{C}$. If $\ulcorner A \urcorner : X \to U$ is a map of $\mathbf{C}$ with A a symbol or expression, then one can simply write A for $E(\ulcorner A \urcorner)$, and p_A for $P(\ulcorner A \urcorner)$.

Definition 3.2 ([Voe15, Construction 2.12]). Let $\mathbf{C}$ be a category, U a universe in $\mathbf{C}$, and 1 a terminal object of $\mathbf{C}$. The **CwA induced by** U, denoted by $\mathbf{C}_U$, is defined as follows:

(1) the underlying category is $\mathbf{C}$ itself, and the chosen terminal object is 1;

(2) the type presheaf is the representable presheaf $\mathbf{y}U$, i.e., types over an object Γ are maps $\Gamma \to U$, with reindexing modelled by precomposition;

(3) extensions and projections are inherited from the universe structure of U, that is, if $f : \Gamma \to U$ is a type over Γ then

$$\Gamma.f = E(f) \quad \text{and} \quad p_f = P(f);$$

(4) in the presence of a type $f : \Gamma \to U$ and a map $g : \Delta \to \Gamma$, the connecting map $g.f$ is induced by

the universal properties of pullbacks:

$$\begin{array}{ccccc} E(f\circ g) & \overset{g.f}{\dashrightarrow} & E(f) & \xrightarrow{Q(f)} & \tilde{U} \\ {\scriptstyle P(f\circ g)}\downarrow & \lrcorner & {\scriptstyle P(f)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{g} & \Gamma & \xrightarrow{f} & U \end{array} \qquad Q(f\circ g) : E(f\circ g) \to \tilde{U}$$

Proposition 3.3 ([Voe15, Construction 2.12, Example 4.9])**.**

(1) The data above define a category with attributes $\mathbf{C}_U$.

(2) If U *and* U' *are two universes in* $\mathbf{C}$ *with* $p = p'$*, then the induced CwAs* $\mathbf{C}_U$ *and* $\mathbf{C}_{U'}$ *are locally isomorphic, i.e., for each* $\Gamma \in \mathbf{C}$ *the fibrant slices* $\mathbf{C}_U /\!\!/ \Gamma$ *and* $\mathbf{C}_{U'} /\!\!/ \Gamma$ *are isomorphic; the fibrant slices* $\mathbf{C}_U /\!\!/ 1$ *and* $\mathbf{C}_{U'} /\!\!/ 1'$ *are also isomorphic.* □

The above offers a method with which to tackle the first part of the coherence problem. Namely, if one chooses a universe structure encoding the desired notion of types, then the induced CwA is a strict model whose notion of type is in accordance with the initial intuition. As an example, consider the case of the simplicial model.

Definition 3.4. For a regular cardinal α, a simplicial map $f : Y \to X$ is α**-small** if, for each simplex x of X, the fibre $f^{-1}(x) = Y_x$ has cardinality strictly less than α.

Theorem 3.5 ([KL21, 2.1.10 & 2.1.12])**.** *For each regular cardinal* α*, there exists a weakly universal fibration* $p_\alpha : \tilde{U_\alpha} \to U_\alpha$ *among* α*-small fibrations; that is, a simplicial map* $p : E \to B$ *is an* α*-small fibration if and only if it can be expressed as a pullback of* p_α*.*

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U_\alpha} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p_\alpha} \\ B & \longrightarrow & U_\alpha \end{array}$$

□

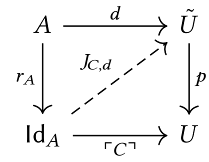
Definition 3.6. Fix a choice of pullbacks of $p_\alpha : \tilde{U_\alpha} \to U_\alpha$ and a terminal object in **sSet**, the **simplicial model** is then defined as the induced CwA $\mathbf{sSet}_{U_\alpha}$.

The above yields a strict CwA structure on **sSet** in which types over a simplicial set Γ are (up to isomorphism) α-small fibrations with base Γ. At this point, $\mathbf{sSet}_{U_\alpha}$ is a plain CwA without any additional structure, while the real simplicial model interprets most of Martin-Löf type theory and a univalent universe. The remainder of this section uses the specific cases of identity types and propositional truncation to illustrate how to tackle the second flavour of the coherence problem, i.e., how to add logical structures to universe-induced CwAs such as $\mathbf{sSet}_{U_\alpha}$.

First, consider the case of identity types, as defined in Definition 2.13. For each type $\ulcorner A \urcorner : \Gamma \to U$ of $\mathbf{C}_U$, parts (1) and (2) of said definition ask for a type $\ulcorner \mathsf{Id}_A \urcorner : A \times_\Gamma A \to U$ together with a map $r_A : A \to \mathsf{Id}_A$ such that

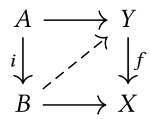
$$\begin{array}{ccc} A & \xrightarrow{r_A} & \mathsf{Id}_A \\ & {\scriptstyle \Delta_A}\searrow \quad \swarrow{\scriptstyle p_{\mathsf{Id}_A}} & \\ & A \times_\Gamma A & \end{array}$$

commutes. As for part (3), it assumes a type $\ulcorner C \urcorner : \mathsf{Id}_A \to U$ together with a map $d : A \to C$ such that $p_C \circ d = r_A$ and asks for $J_{C,d}$, a section of p_C such that $J_{C,d} \circ r_A = d$. Since p_C is a pullback of p, one can view d as a map $d : A \to \tilde{U}$ making the solid square displayed below commute, and $J_{C,d}$ as a diagonal filler of said square.



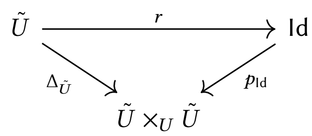
In other words, condition (3) precisely asks for fillers witnessing that r_A has the left lifting property against p, as defined below.

Definition 3.7. Let $i : A \to B$ and $f : Y \to X$ be two maps in a category **C**. The map f has the **right lifting property** against i, or equivalently, the map i has the **left lifting property** against f, if any solid square in **C** as below admits at least one diagonal filler.



One writes $i \boxslash f$ to signify that i and f have this property.

Suppose now that the "universal" type $\mathrm{id}_U : U \to U$ has been given such a structure. That is, assume a type $\ulcorner \mathsf{Id} \urcorner : \tilde{U} \times_U \tilde{U} \to U$, a map $r : \tilde{U} \to \mathsf{Id}$ such that



commutes, and fillers witnessing that $r \boxslash p$. These data *almost* induce an identity structure for each type of $\mathbf{C}_U$ via pullbacks. Namely, since each type $\ulcorner A \urcorner : \Gamma \to U$ induces a map $A \times_\Gamma A \to \tilde{U} \times_U \tilde{U}$, one can define $\ulcorner \mathsf{Id}_A \urcorner$ as that map followed by Id. Similarly, the reflexivity map $r_A : A \to \mathsf{Id}_A$ can be obtained as the pullback of $r : \tilde{U} \to \mathsf{Id}$, seen as a map in the slice $\mathbf{C}/U$, along $\ulcorner A \urcorner : \Gamma \to U$. These two satisfy $p_{\mathsf{Id}_A} \circ r_A = \Delta_A$, hence they are candidates for parts (1) and (2) of Definition 2.13.

What is more, these choices of Id_A and r_A are strictly stable under reindexing. Indeed, for r_A, e.g., this is because

$$f^* r_A = f^* \ulcorner A \urcorner^* r = (f \circ \ulcorner A \urcorner)^* r = (\ulcorner f^* A \urcorner)^* r = r_{f^* A}$$

for any type $\ulcorner A \urcorner : \Gamma \to U$ and map $f : \Delta \to \Gamma$.

However, it turns out that the condition $r \boxslash p$ on its own is not sufficient to model the elimination rule of identity types in a way that is stable under pullbacks. The correct structure can be expressed in terms of internal lifting operations, which are themselves related to a stable version of lifting properties.

Definition 3.8 ([KL21, Definition 1.4.5])**.** Let i and f be two maps of a category **C** with products, the map i is said to have the **stable left lifting property** against f if $(C \times i) \boxslash f$ for all $C \in \mathbf{C}$.

Remark 3.9. In the case where **C** has pullbacks, if i can be seen as a map in the slice $\mathbf{C}/\Gamma$ for some $\Gamma \in \mathbf{C}$, then i has the stable left lifting property against $f \times \Gamma$ in $\mathbf{C}/\Gamma$ if and only if the pullback of i along any map $g : \Delta \to \Gamma$ has the left lifting property against f in **C**.

Definition 3.10 ([KL21, Definition 1.4.6]). Let $i : A \to B$ and $f : Y \to X$ be two maps in a cartesian closed category. An **internal lifting operation** for i against f is a section of the map $(Y^i, f^B) : Y^B \to Y^A \times_{X^A} X^B$.

Proposition 3.11 ([KL21, Proposition 1.4.7]). *Let $i : A \to B$ and $f : Y \to X$ be two maps in a cartesian closed category. There exists an internal lifting operation for i against f if and only if i has the stable left lifting property against f.* □

It is now possible to express the structure a universe U corresponding to an Id-structure in $\mathbf{C}_U$.

Definition 3.12. A Id-structure on a universe U in a locally cartesian closed category $\mathbf{C}$ consists of two maps

$$\ulcorner \mathsf{Id} \urcorner : \tilde{U} \times_U \tilde{U} \to U \qquad \text{and} \qquad r : \tilde{U} \to \mathsf{Id}$$

such that the triangle

$$\begin{array}{ccc} \tilde{U} & & \\ & \overset{r}{\searrow} & \\ {\scriptstyle p}\downarrow & & \mathsf{Id} \\ & \underset{p_{\mathsf{Id}}}{\swarrow} & \\ U & & \end{array}$$

commutes, together with an internal lifting operation for r against $p \times U$ in $\mathbf{C}$.

Theorem 3.13 ([KL21, Theorem 1.4.15]). *If U is a universe in a locally cartesian closed category $\mathbf{C}$, then an Id-structure on U induces an Id-structure on $\mathbf{C}_U$.*

Proof. The idea is to define the Id-structure of a type $\ulcorner A \urcorner : \Gamma \to U$ from the Id-structure of U via precomposition and pullbacks, as was above for Id_A and r_A. The case of $J_{C,d}$ demands slightly more work but goes through as well, for more details see [KL21, Section 1.4]. □

Example 3.14 (Id-structure on U_α). The idea consists in seeing identities as paths, where the space of paths in a simplicial set X is modelled by the internal hom X^{Δ^1}. Note that for any simplicial set X, the two vertices of Δ^1 induce *source* and *target* maps $s_X, t_X : X^{\Delta^1} \to X^{\Delta^0} \cong X$, meaning that X^{Δ^1} can always be seen as living over $X \times X$.

Intuitively, the space $P_B(E)$ of paths in E over B, for a given simplicial map $p : E \to B$, consists of paths in E which become constant once projected down to B. It is defined by the pullback square below, where the lowest horizontal map is induced by the canonical map $E \times_B E \to E \times E$ and the composite $E \times_B E \to B \cong B^{\Delta^0} \xrightarrow{c_B} B^{\Delta^1}$, with c_B being the "constant path" map $B \to B^{\Delta^1}$, the transpose of the projection $B \times \Delta^1 \to B$.

$$\begin{array}{ccc} P_B(E) & \longrightarrow & E^{\Delta^1} \\ {\scriptstyle (s_p, t_p)}\downarrow & \lrcorner & \downarrow{\scriptstyle ((s_E, t_E), p^{\Delta^1})} \\ E \times_B E & \longrightarrow & (E \times E) \times_{B \times B} B^{\Delta^1} \end{array}$$

The reflexivity term $r_E : E \to P_B(E)$ is then induced by $c_E : E \to E^{\Delta^1}$, and indeed gives a factorisation of $\Delta_E : E \to E \times_B E$, as needed.

Moreover, it can be showed [KL21, Proposition 2.3.3] that if $p : E \to B$ is a fibration, then so is $(s_p, t_p) : P_B(E) \to E \times_B E$ while $r_E : E \to P_B(E)$ is a trivial cofibration, meaning it has the left lifting property against all fibrations. Since this construction is stable under pullbacks, up to isomorphism, the reflexivity maps r_E all have the stable left lifting property against each fibration, in particular against p_α. By Proposition 3.11, this means that applying the above to the fibration $p_\alpha : \tilde{U}_\alpha \to \tilde{U}$ itself yields an Id-structure on U_α.

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type}}{\begin{array}{c}\Gamma, a_1 : A, a_2 : A, e : \mathsf{Id}_A(a_1, a_2), b_1 : B[a_1/x], b_2 : B[a_2/x] \vdash \mathsf{Id}^e_B(b_1, b_2) \text{ type} \\ \Gamma, a : A, b : B[a/x] \vdash \mathsf{refl}'_b : \mathsf{Id}^{\mathsf{refl}_a}_B(b, b)\end{array}}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B \text{ type} \\ \Gamma, x : A, y : A, e : \mathsf{Id}_A(x, y), u : B, v : B[y/x], d : \mathsf{Id}^e_B(u, v) \vdash C \text{ type} \\ \Gamma, x : A, u : B \vdash c : C[x/y, \mathsf{refl}_x/e, \mathsf{refl}'_u/d]\end{array}}{\begin{array}{c}\Gamma, a_1 : A, a_2 : A, p : \mathsf{Id}_A(a_1, a_2), b_1 : B[a_1/x], b_2 : B[a_2/x], q : \mathsf{Id}^p_B(b_1, b_2) \\ \vdash \mathsf{J}'(xyeuvd.C, xu.c, a_1, a_2, p, b_1, b_2, q) : C[a_1/x, a_2/y, p/e, b_1/u, b_2/v, q/d] \\ \Gamma, a : A, b : B[a/x] \vdash \mathsf{J}'(xyeuvd.C, xu.c, a, a, \mathsf{refl}_a, b, b, \mathsf{refl}'_b) \equiv c[a/x, b/u]\end{array}}$$

Figure 1: Dependent identity types

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \|A\| \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash \mathsf{tr}(x) : \|A\|}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : \|A\|, y : \|A\| \vdash \mathsf{treq}(x, y) : \mathsf{Id}_{\|A\|}(x, y)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \qquad \Gamma, z : \|A\| \vdash C \text{ type} \qquad \Gamma, x : A \vdash c : C[\mathsf{tr}(x)/z] \\ \Gamma, x : \|A\|, y : \|A\|, u : C[x/z], v : C[y/z] \vdash d : \mathsf{Id}^{\mathsf{treq}(x,y)}_{z.C}(u, v)\end{array}}{\begin{array}{c}\Gamma, w : \|A\| \vdash \mathsf{trrec}(z.C, x.c, xyuv.d, w) : C[w/z] \\ \Gamma, a : A \vdash \mathsf{trreceq}(z.C, x.c, xyuv.d, a) : \mathsf{Id}_{C[\mathsf{tr}(a)/z]}(\mathsf{trrec}(z.C, x.c, xyuv.d, \mathsf{tr}(a)), c[a/x])\end{array}}$$

Figure 2: Propositional truncation

A very similar treatment can be given for propositional truncation, albeit after its syntactic rules have been slightly reworked as to allow for an easier semantic translation.

Recall that $\Gamma \vdash Q$ type is a *mere proposition* if for each pair of terms $q_1, q_2 : Q$ there exists an identity term $w(q_1, q_2) : \mathsf{Id}_Q(q_1, q_2)$. The intuition behind the *propositional truncation* of $\Gamma \vdash A$ type is that it is supposed to be a mere proposition $\Gamma \vdash \|A\|$ type generated by the elements of A; in other words, it records whether A is inhabited or not without actually requiring terms of type A. More precisely, the propositional truncation of a type A is the higher inductive type $\|A\|$ generated by the elements of A and identity terms $\mathsf{treq}(x, y)$ between any two terms $x, y : \|A\|$.

Accordingly, the elimination property of $\|A\|$ is most clearly expressed using the notion of dependent identity types, whose rules are given in Figure 1 [LS20, Figure 2]. The rules defining propositional truncation induced by its description as a higher inductive type are displayed in Figure 2 [LS20, Figure 7], note that the equality in the computation rule is propositional and not judgemental. As they currently stand, giving semantic translations of the elimination and computation rules may be somewhat lengthy; fortunately, there exists a more amenable presentation of propositional truncation.

Lemma 3.15. *For types* $\Gamma \vdash A$ type *and* $\Gamma, x : A \vdash B$ type, *the judgement*

$$\Gamma, a : A, u : B[a/x], v : B[a/x] \vdash \mathsf{Id}_{B[a/x]}(u, v) \cong \mathsf{Id}^{\mathsf{refl}_a}_B(u, v)$$

is derivable.

Proof. One can easily define a map $\mathsf{Id}_{B[a/x]}(u, v) \to \mathsf{Id}^{\mathsf{refl}_a}_B(u, v)$ via J-elimination, by sending refl_u to

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \|A\| \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash \mathsf{tr}(x) : \|A\|}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : \|A\|, y : \|A\| \vdash \mathsf{treq}(x, y) : \mathsf{Id}_{\|A\|}(x, y)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \qquad \Gamma, z : \|A\| \vdash C \text{ type} \\ \Gamma, x : A \vdash c : C[\mathsf{tr}(x)/z] \\ \Gamma, x : \|A\|, u : C[x/z], v : C[x/z] \vdash d' : \mathsf{Id}_{C[x/z]}(u, v)\end{array}}{\Gamma, w : \|A\| \vdash \mathsf{trrec}'(z.C, x.c, xuv.d', w) : C[w/z]}$$

Figure 3: Alternative propositional truncation

refl'_u. Conversely, consider the term

$$\begin{aligned}\Gamma, a_1 : A, a_2 : A, p : \mathsf{Id}_A(a_1, a_2), b_1 : B[a_1/x], b_2 : B[a_2/x], q : \mathsf{Id}^p_B(b_1, b_2) \\ \vdash t(a_1, a_2, p, b_1, b_2, q) : \mathsf{Id}_{B[a_2/x]}(p_* b_1, b_2),\end{aligned}$$

where p_* is transport along p, defined via J'-elimination by

$$t(a, a, \mathsf{refl}_a, b, b, \mathsf{refl}'_b) \equiv \mathsf{refl}_b.$$

This term then defines a map $\mathsf{Id}^{\mathsf{refl}_a}_B(u, v) \to \mathsf{Id}_{B[a/x]}(u, v)$ sending q to $t(a, a, \mathsf{refl}_a, u, v, q)$. Finally, it is easy to check that those two maps are inverses of one another, thus they induce the desired equivalence. □

Proposition 3.16. *If any of the two sets of rules for propositional truncation displayed in Figure 2 and Figure 3 is added to the theory then the other becomes admissible.*

Proof. Assume first that the rules of Figure 2 are part of the theory. To derive the alternative elimination rule displayed in Figure 3, it is actually easier to show that the term d' induces another term

$$\Gamma, x : \|A\|, y : \|A\|, p : \mathsf{Id}_{\|A\|}(x, y), u : C[x/z], v : [y/z] \vdash d''(x, y, p, u, v) : \mathsf{Id}^p_{z.C}(u, v)$$

and then let $\mathsf{trrec}'(z.C, x.c, xuv.d', w)$ be $\mathsf{trrec}(z.C, x.c, xyuv.d''[\mathsf{treq}(x, y)/p], w)$. By J-elimination, it is enough to define d'' in the case $x \equiv y$ and $p \equiv \mathsf{refl}_x$. In other words, it suffices to provide a term of $\mathsf{Id}^{\mathsf{refl}_x}_{z.C}(u, v)$; by Lemma 3.15, one can instead give a term of $\mathsf{Id}_{C[x/z]}(u, v)$ such as $xuv.d'$.

Conversely, both trrec and trreceq can be defined in terms of trrec′. Indeed, by definition $\|A\|$ is a mere proposition, so it is also an h-set. Hence, there exists an identity term from $\mathsf{treq}(x, x)$ to refl_x; by transporting along it, one obtains a map $\mathsf{Id}^{\mathsf{treq}(x,y)}_{z.C}(u, v) \to \mathsf{Id}^{\mathsf{refl}_x}_{z.C}(u, v)$. This map, applied to d, and Lemma 3.15 yield a term d''' of the same type as d'. The result of substituting d''' for d' in trrec′ is a derivation of trrec assuming trrec′, and trreceq is a special case of d'''. □

The above shows that one can choose any of the two displayed presentations of propositional truncation. The latter, in Figure 3, has the advantage of being easier to model semantically, and is therefore chosen for the remainder of the section. The corresponding structure in a CwA is given below, after a quick piece of terminology used to simplify the definition.

Definition 3.17. Let C be a CwA with Id-types. A **mere proposition** over $\Gamma \in \mathbf{C}$ is a pair (P, w) where P is a type over Γ and w a section of p_{Id_P}. When w is not needed for the discussion at hand, one usually suppresses it and says instead simply that P is a mere proposition.

Definition 3.18. Let C be a CwA with Id-types, an $\| \ \|$**-structure** on C consists of:

(1) for each type $A \in \mathrm{Ty}(\Gamma)$, a mere proposition $\|A\|$ over Γ;
(2) for each type $A \in \mathrm{Ty}(\Gamma)$, a map $\mathsf{tr}_A : \Gamma.A \to \Gamma.\|A\|$ such that $p_{\|A\|} \circ \mathsf{tr}_A = p_A$;
(3) for each type $A \in$ type(Γ), mere proposition C over $\|A\|$, and map $c : \Gamma.A \to \Gamma.\|A\|.C$ such that $p_C \circ c = \mathsf{tr}_A$, a section $\mathsf{trrec}_{C,c}$ of p_C;

under the additional condition that the above data is stable under reindexing, that is, for each map $f : \Delta \to \Gamma$ and A, C, c as above the following identities hold:

$$f^*\|A\| = \|f^*A\|, \quad f^*\mathsf{tr}_A = \mathsf{tr}_{f^*A}, \quad f^*\mathsf{trrec}_{C,c} = \mathsf{trrec}_{f^*C,f^*c}.$$

Example 3.19. Propositional truncation can also be interpreted in $\mathbf{C}_{\mathbf{Set}}$. Namely, the truncation of an indexed family of sets $(A_\gamma)_{\gamma\in\Gamma}$ is the indexed family $(\|A\|_\gamma)_{\gamma\in\Gamma}$ where

$$\|A\|_\gamma = \begin{cases} \{*\} & \text{if } A_\gamma \neq \emptyset, \\ \emptyset & \text{if } A_\gamma = \emptyset. \end{cases}$$

Mere propositions in $\mathbf{C}_{\mathbf{Set}}$ are indexed families of sets such that each indexed set has at most one element, hence $(\|A\|_\gamma)_{\gamma\in\Gamma}$ is a mere proposition. The corresponding map $\mathsf{tr}_A : \Gamma.A \to \Gamma.\|A\|$ sends (γ, a) to $(\gamma, *)$. Finally, if C is a mere proposition over $\Gamma.\|A\|$ and c a map $\Gamma.A \to \Gamma.\|A\|.C$ such that $p_C \circ c = \mathsf{tr}_A$, then there is a unique section of p_C, call it $\mathsf{trrec}_{C,c}$; indeed, there can be at most one because C is a mere proposition, and there is at least one because c guarantees that each $C_{(\gamma,*)}$ is inhabited, for $(\gamma, *) \in \Gamma.\|A\|$. In this case, stability under reindexing is straightforward to check manually.

In a CwA induced by a universe, the notion of mere proposition can be nicely encoded.

Proposition 3.20. *Let U be a universe in a locally cartesian closed category* $\mathbf{C}$*, together with an* Id*-structure for U. There exists a map $U_{-1} \to U$ of* $\mathbf{C}$ *such that, for each type $\ulcorner A \urcorner : \Gamma \to U$, the sections of p_{Id_A} correspond bijectively with factorisations of $\ulcorner A \urcorner$ through $U_{-1} \to U$.*

Proof. Let $U_{-1} \to U$ be the dependent product $\Pi_{\tilde{U}\times_U\tilde{U}\to U}\mathsf{Id}$ in $\mathbf{C}/U$. For any type $\ulcorner A \urcorner : \Gamma \to U$, one has

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}/U}(\ulcorner A \urcorner, U_{-1} \to U) &\cong \mathrm{Hom}_{\mathbf{C}/\tilde{U}\times_U\tilde{U}}(A \times_\Gamma A, \mathsf{Id}) \\ &= \mathrm{Hom}_{\mathbf{C}/\tilde{U}\times_U\tilde{U}}(\Sigma_{A\times_\Gamma A\to\tilde{U}\times_U\tilde{U}}\mathrm{id}_{A\times_\Gamma A}, \mathsf{Id}) \\ &\cong \mathrm{Hom}_{\mathbf{C}/A\times_\Gamma A}(\mathrm{id}_{A\times_\Gamma A}, \mathsf{Id}_A). \end{aligned}$$

Hence, the conclusion follows. □

Corollary 3.21. *In the same conditions as above, the object U_{-1} represents mere propositions, in the sense that maps $\Gamma \to U_{-1}$ are in bijection with mere propositions over Γ.* □

Definition 3.22. In the same conditions as above, let $p_{-1} : \tilde{U}_{-1} \to U_{-1}$ be the pullback (given by the universe structure) of $p : \tilde{U} \to U$ along the map $U_{-1} \to U$ defined above, as displayed below.

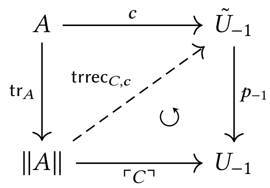
$$\begin{array}{ccc} \tilde{U}_{-1} & \longrightarrow & \tilde{U} \\ {\scriptstyle p_{-1}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ U_{-1} & \longrightarrow & U \end{array}$$

In the case of a locally cartesian closed category $\mathbf{C}$ with a universe U carrying an Id-structure, one can use this universe of mere propositions to adapt Definition 3.18 to $\mathbf{C}_U$, a $\| \ \|$-structure becomes:

(1) for each type $\ulcorner A \urcorner : \Gamma \to U$, a mere proposition $\ulcorner \|A\| \urcorner : \Gamma \to U_{-1}$;

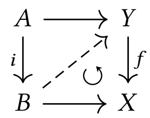
(2) for each type $\ulcorner A \urcorner : \Gamma \to U$, a map $\mathrm{tr}_A : A \to \|A\|$ such that $p_{\|A\|} \circ \mathrm{tr}_A = p_A$;
(3) for each type $\ulcorner A \urcorner : \Gamma \to U$, mere proposition $\ulcorner C \urcorner : \|A\| \to U_{-1}$, and map $c : A \to C$ such that $p_C \circ c = \mathrm{tr}_A$, a section $\mathrm{trrec}_{C,c}$ of p_C.

As was the case with identity types, clause (3) can be restated in terms of lifting problems. Indeed, the assumed $c : A \to C$ can be viewed as a map $A \to \tilde{U}_{-1}$ making the solid square below commute, while the section $\mathrm{trrec}_{C,c}$ is equivalent to a diagonal $\|A\| \to \tilde{U}_{-1}$ such that the lower of the triangles it defines commutes.

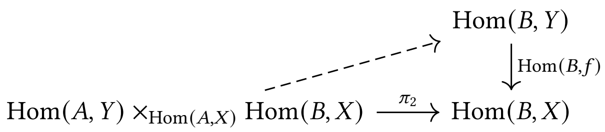


In other words, condition (3) expresses that tr_A has a very weak left lifting property against p_{-1}.

Definition 3.23. Let $i : A \to B$ and $f : Y \to X$ be two maps of a category **C**, the map i has the **lower-only left lifting property** against f, written $i \boxslash_l f$, if any solid commutative square



admits a lower filler, i.e., a diagonal map $B \to Y$ making the lower triangle commute. Equivalently, it means that $i \boxslash_l f$ holds precisely when there exists a dashed map making



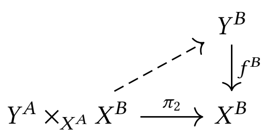
commute.

Remark 3.24. If $\|\ \|$-structures were directly modelled on the rules of Figure 2, then the term trreceq would correspond to a homotopy between the two maps of the upper triangle induced by the filler $\mathrm{trrec}_{C,c}$. In other words, the lifting property corresponding to those rules would be one where the upper triangle commutes up to homotopy while the lower one does so on the nose, similar to that considered in [CMS20, Definition 29].

As before, there are stable and internal variants.

Definition 3.25. Let $i : A \to B$ and $f : Y \to X$ be two maps of a category **C** with products, the map i has the **stable lower-only left lifting property** against f if $(C \times i) \boxslash_l f$ holds for all $C \in \mathbf{C}$.

Definition 3.26. Let $i : A \to B$ and $f : Y \to X$ be two maps of a cartesian closed category **C**, an **internal lower-only lifting operation** for i against f is an arrow $Y^A \times_{X^A} X^B \to Y^B$ of **C** making the diagram



commute.

Proposition 3.27. *Let $i : A \to B$ and $f : Y \to X$ be two maps in a cartesian closed category. There exists an internal lower-only lifting operation for i against f if and only if i has the stable lower-only left lifting property against f.*

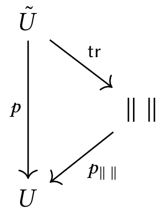
Proof. The proof of Proposition 3.11 is easily adapted to the case at hand. □

As was the case with Id-types, internal lifting operations are the right tools to express the elimination rules of propositional truncation. Hence, it is finally time for the main definition of the section.

Definition 3.28. A $\| \ \|$-structure on a universe U in a locally cartesian closed category $\mathbf{C}$ consists of two maps

$$\ulcorner \| \ \| \urcorner : U \to U_{-1} \qquad \text{and} \qquad \mathsf{tr} : \tilde{U} \to \| \ \|$$

such that the triangle



commutes, together with an internal lower-only lifting operation for tr against $p_{-1} \times U$ in $\mathbf{C}/U$.

Proposition 3.29. *If U is a universe in a locally cartesian closed category $\mathbf{C}$, then an $\| \ \|$-structure on U induces an $\| \ \|$-structure on $\mathbf{C}_U$.*

Proof. The techniques used in the proof of [KL21, Theorem 1.4.15] can be adapted for the case at hand. □

This concludes the present section, the next logical step would be to give such an $\| \ \|$-structure on the universe U_α inducing the simplicial model. However, this requires some deeper knowledge of the guise taken by mere propositions in the simplicial model; this is precisely the topic of the following section, which acts as a stepping stone towards the interpretation of propositional truncation based on epi-mono factorisations presented in the subsequent section.

3.2 Unwinding mere propositions

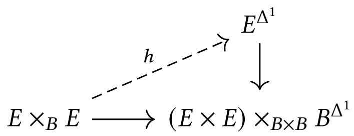
Similarly to how h-sets are types in which proofs of equality are propositionally unique, *mere propositions*, also known as *h-props*, are types whose elements are propositionally equal to one another. In other words, to say that $\Gamma \vdash A$ type is a a mere proposition is to give a term

$$\Gamma, x_1 : A, x_2 : A \vdash w(x_1, x_2) : \mathsf{Id}_A(x_1, x_2).$$

In a CwA, such terms are sections of the projection p_{Id_A}; for a fibration $p : E \to B$ in the simplicial model, these correspond to sections of $(s_p, t_p) : \mathrm{P}_B(E) \to E \times_B E$.

Lemma 3.30. *Given a fibration $p : E \to B$, sections of $(s_p, t_p) : \mathrm{P}_B(E) \to E \times_B E$ correspond bijectively to homotopies between the two projections $\pi_1, \pi_2 : E \times_B E \to E$ over B.*

Proof. From its definition as a pullback, sections of $\mathrm{P}_B(E) \to E \times_B E$ correspond to maps h making the diagram



commute. Under the adjunction $-\times\Delta^1 \dashv -^{\Delta^1}$, the transpose of such an h is a homotopy $H : (E\times_B E)\times\Delta^1 \to E$ from π_1 to π_2 such that $p \circ H$ factors through the projection $(E \times_B E) \times \Delta^1 \to E \times_B E$, i.e., an homotopy over B. □

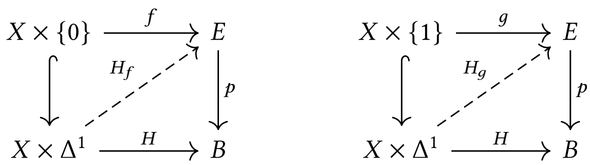
The above reformulates the property of being a mere proposition to a more homotopical condition, but it can be strengthened, as shown below.

Lemma 3.31. *Let $p : E \to B$ be a fibration. The two projections $E \times_B E \to E$ are homotopic over B if and only if each pair of maps $f, g : X \to E$ and homotopy H from pf to pg induce a homotopy H' from f to g such that $p \circ H' = H$.*

Proof. Clearly, the second claim implies the first.

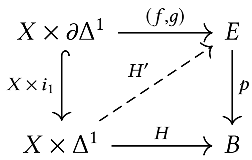
As for the other implication, consider the intermediate claim: if the two projections $E \times_B E \to E$ are homotopic over B, then any two maps $f, g : X \to E$ satisfying $pf = pg$ are homotopic over B. This is true because precomposing any homotopy between the two projections with $(f, g) : X \to E \times_B E$ yields a homotopy between f and g.

Assume now that f, g, and H satisfy the conditions of the second claim. Consider the two solid squares below, they admit fillings H_f and H_g because p is a fibration and the maps on the left sides are anodyne extensions, meaning that they have the left lifting property against all fibrations [GJ09, Corollary I.4.6].



By definition, the maps H_f and H_g satisfy the conditions of the intermediate claim, hence there exists a homotopy $\mathcal{H} : (X \times \Delta^1) \times \Delta^1 \to A$ from H_f to H_g. Finally, one need only choose $H' = \mathcal{H} \circ (X \times d)$, where $d : \Delta^1 \to \Delta^1 \times \Delta^1$ is the diagonal map. □

The second condition of the lemma above can be reformulated as a filling problem: the two maps f and g can be bundled into one $(f, g) : X \times \partial\Delta^1 \to E$ and the desired H' is then a diagonal filler of the square below, which commutes precisely because H is a homotopy from pf to pg.

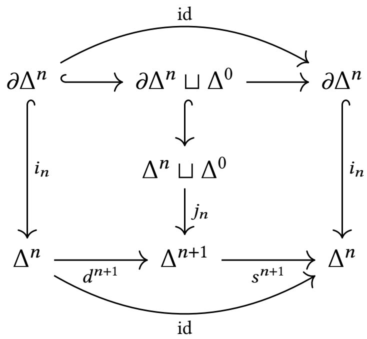


In other words, the two preceding lemmas show that a fibration $E \to B$ is a mere proposition if and only if it has the right lifting property against $X \times i_1$ for all simplicial sets X, where i_1 is the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$. Informally, this means that lifting the extremities of a prism in B to E induces a lift of the whole prism. As shall be seen below, this nearly makes $E \to B$ into a trivial fibration.

Notation. For each $n \geq 0$, let i_n denote the inclusion of the simplicial circle $\partial\Delta^n$ in the standard simplex Δ^n.

Proposition 3.32. *Let $p : E \to B$ be a fibration. If $(X \times i_1) \boxslash p$ for all $X \in$ **sSet**, then $i_n \boxslash p$ for all $n \geq 1$.*

Proof. Let $j_n : \Delta^n \sqcup \Delta^0 \to \Delta^{n+1}$ be the map induced by the face $d^{n+1} : \Delta^n \to \Delta^{n+1}$ and the map $v^{n+1} : \Delta^0 \to \Delta^{n+1}$ singling out the vertex $n + 1$ of Δ^{n+1}. For $n \geq 1$, the inclusion i_n is a retract of $j_n \circ (i_n \sqcup \Delta_0)$ in the arrow category $\mathbf{sSet}^{\to}$, as shown in the diagram below, the map $\partial\Delta^n \sqcup \Delta^0 \to \partial\Delta^n$ in the top-right corner is induced by the identity $\mathrm{id}_{\partial\Delta^n}$ and the restriction of $v^n : \Delta^0 \to \Delta^n$ through the boundary.

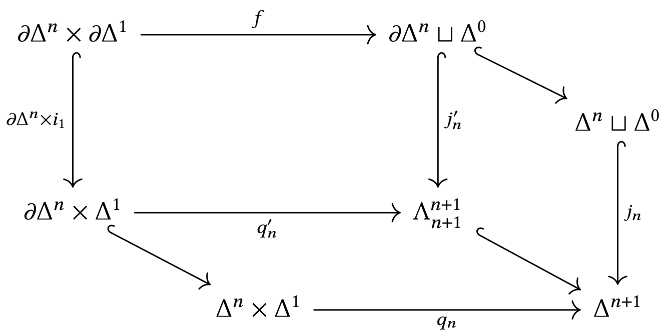


Since the set of maps which have the left lifting property against p is stable under retracts, it suffices to show that $(j_n \circ (i_n \sqcup \Delta_0)) \boxslash p$.

The diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & \dots & \longrightarrow & n-1 & \longrightarrow & n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ n+1 & \longrightarrow & n+1 & \longrightarrow & n+1 & \longrightarrow & \dots & \longrightarrow & n+1 & \longrightarrow & n+1 \end{array}$$

defines a map $q^n : \mathbf{n} \times \mathbf{1} \to \mathbf{n+1}$ in the simplex category, and it itself induces another map $q_n : \Delta^n \times \Delta^1 \to \Delta^{n+1}$. Geometrically, this map q_n collapses the lower base $\Delta^n \times \{1\}$ of the prism $\Delta^n \times \Delta^1$ down to a single vertex. Let j_n' and q_n' be the restrictions of j_n and q_n, defined by the two parallelograms in the diagram below, where the map f is defined on $\partial\Delta^n \times \{0\}$ as the inclusion $\partial\Delta^n \hookrightarrow \partial\Delta^n \sqcup \Delta^0$, and on $\partial\Delta^n \times \{1\}$ as the composite $\partial\Delta^n \to \Delta^0 \hookrightarrow \partial\Delta^n \sqcup \Delta^0$.



Lemma 3.33 below shows that the top-left rectangle of the diagram not only commutes but is also a pushout square.

It has thus been showed that $j_n \circ (i_n \sqcup \Delta_0)$ factors as a pushout of $\partial\Delta^n \times i_1$ followed by a horn inclusion. By assumption, both of these maps have the left lifting property with respect to p, and the set of maps with that property is closed under composition hence the conclusion follows. $\square$

Lemma 3.33. *The diagram*

$$\begin{array}{ccc} \partial\Delta^n \times \partial\Delta^1 & \xrightarrow{f} & \partial\Delta^n \sqcup \Delta^0 \\ {\scriptstyle \partial\Delta^n \times i_1} \big\downarrow & & \big\downarrow {\scriptstyle j_n'} \\ \partial\Delta^n \times \Delta^1 & \xrightarrow[q_n']{} & \Lambda^{n+1}_{n+1} \end{array}$$

defined in Proposition 3.32 is a pushout square.

Proof. This proof relies on the following two observations: a map $\mathbf{m} \to \mathbf{n}$ belongs to $(\partial\Delta^n)_m$ if and only if it is not surjective, and a map $\mathbf{m} \to \mathbf{n+1}$ belongs to $(\Lambda^{n+1}_{n+1})_m$ if and only if $\{0, 1, \dots, n\}$ is not contained in its image.

Evaluating this square of presheaves at the ordinal $\mathbf{m}$, one obtains the diagram of sets displayed below, where the right-most vertical map sends $*$ to the constant map of value $n+1$ and sends $f : \mathbf{m} \to \mathbf{n}$ to $d^{n+1} \circ f$, while the left-most vertical map pairs f and g into a single map $\mathbf{m} \to \mathbf{n} \times \mathbf{1}$.

$$\begin{array}{ccc}
\left\{ \begin{pmatrix} f : \mathbf{m} \to \mathbf{n}, \\ g : \mathbf{m} \to \mathbf{1} \end{pmatrix} \,\middle|\, \begin{matrix} \mathrm{im} f \neq \mathbf{n}, \\ g = 0 \text{ or } g = 1 \end{matrix} \right\} & \xrightarrow[\substack{(f,0) \mapsto f, \\ (f,1) \mapsto *}]{} & \{ f : \mathbf{m} \to \mathbf{n} \mid \mathrm{im} f \neq \mathbf{n} \} \sqcup \{*\} \\
\downarrow & & \downarrow \\
\{ h : \mathbf{m} \to \mathbf{n} \times \mathbf{1} \mid \mathrm{im}(\pi_{\mathbf{n}} \circ h) \neq \mathbf{n} \} & \xrightarrow{q^n \circ -} & \{ k : \mathbf{m} \mapsto \mathbf{n+1} \mid \mathbf{n} \nsubseteq \mathrm{im} f \}
\end{array}$$

For each $\mathbf{m}$, the above is is a pushout square; and since colimits can be computed pointwise in **sSet**, this is enough. ◻

It is useful to give a name to maps with such lifting properties against boundary inclusions, for they will remain important in the rest of this section.

Definition 3.34. A map of simplicial sets is **truncated** if it has the right lifting property against all the boundary inclusions $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ with $n \geq 1$.

All the above has shown that a fibration $p : E \to B$ encoding a mere proposition is in particular a truncated fibration. The remainder of the present section builds towards a proof of the converse fact, that truncated fibrations are mere propositions.

Definition 3.35. If $u : A \to B$ and $v : X \to Y$ are two simplicial maps, then their *pushout-product* $u \mathbin{\widehat{\times}} v$ is the map out of the pushout $B \times X \sqcup_{A \times X} A \times Y$ induced by the square below.

$$\begin{array}{ccc}
A \times X & \xrightarrow{A \times v} & A \times Y \\
{\scriptstyle u \times X} \downarrow & & \downarrow {\scriptstyle u \times Y} \\
B \times X & \xrightarrow[B \times v]{} & B \times Y
\end{array}$$

Definition 3.36. If $v : X \to Y$ and $w : C \to D$ are two simplicial maps, then their *Leibniz exponential* $v \rhd w$ is the map into the pullback $C^X \times_{D^X} D^Y$ induced by the square below.

$$\begin{array}{ccc}
C^Y & \xrightarrow{w^Y} & D^Y \\
{\scriptstyle C^v} \downarrow & & \downarrow {\scriptstyle D^v} \\
C^X & \xrightarrow[w^X]{} & D^X
\end{array}$$

Proposition 3.37. *For any simplicial maps u, v, w, the following bi-implication holds:*

$$(u \mathbin{\widehat{\times}} v) \boxslash w \iff u \boxslash (v \rhd w).$$

Proof. See [JT07, Proposition 7.6] for a proof in a more general setting. In any case, a standard diagram chase suffices to show this claim. ◻

Why those concepts are relevant to the problem at hand is partly answered by the following proposition.

Proposition 3.38. *If p is a truncated fibration, then it has the right lifting property against $i_n \widehat{\times} i_1$ for all $n \geq 0$.*

The idea of the proof, which is delayed for the moment, is to construct a subcomplex P of $\Delta^n \times \Delta^1$ containing the image of $i_n \widehat{\times} i_1$, i.e., the boundary of the prism; showing that P can be obtained from said boundary by filling a sequence of horn inclusions; and finally that the whole prism is just one boundary filling away from P. In that regard, it is quite similar to standard proofs which give equivalent generating sets for anodyne extensions, see [GZ67, IV.2.1] and [GJ09, Proposition I.4.2].

The methods used in those proofs would undoubtedly also work for the case at hand, however they can be slightly heavy in notational and technical details. For that reason, switching instead to the approach of [Mos20], which introduces tools able to streamline such arguments, is beneficial.

Notation. For an inclusion $m : A \hookrightarrow B$ of simplicial sets, write N_m for the set of non-degenerate simplices of B not in $m(A)$.

Definition 3.39. Let $m : A \hookrightarrow B$ be an inclusion of simplicial sets. A pair (x, y) of N_m is a **filling pair** of m if y is a face of codimension 1 of x, i.e., if $y = d_i(x)$ for some (necessarily unique) i. If this is the case, then x is the called **simplex of type I** of the pair, while y is the **simplex of type II**. A partition of N_m into filling pairs is called a **proper pairing** of m.

Definition 3.40. Let P be a proper pairing of an inclusion $m : A \hookrightarrow B$ in **sSet**. The **ancestral relation** $\prec$ of P is the relation on filling pairs given by $(x, y) \prec (x', y')$ if and only if $(x, y) \neq (x', y')$ and x is a face of y'. A proper pairing of m whose ancestral relation is well-founded is called a **regular pairing** of m.

Theorem 3.41 ([Mos20, Propositions 2.10 & 2.12])**.** *Let $m : A \hookrightarrow B$ be an inclusion in* **sSet**. *If m admits a regular pairing then m is an anodyne extension.*

More precisely, the map m is a strong *anodyne extension, in the sense of [Mos20, Definition 2.1], if and only if m admits a regular pairing.*

Proof. The idea is that in the presence of a filling problem for m against a fibration, one can fill the filling pairs one by one since they encode horn-filling problems, and the well-foundedness condition ensures that one can carry out those fillings in a recursive manner. See the proof of [Mos20, Proposition 2.12] for more details. ◻

The next application provides a good example of this tool, and is also a fact that will be needed in a later section.

Proposition 3.42. *For any $0 \leq k \leq n$, the map $\Delta^0 \to \Lambda^n_k$ which picks out the vertex labelled k is an anodyne extension.*

Proof. Let ℓ be the inclusion $\Delta^0 \hookrightarrow \Lambda^n_k$ at hand. By definition, elements of N_ℓ are injective maps $f : \mathbf{m} \to \mathbf{n}$ such that $\operatorname{im} f$ does not contain $\{0, \ldots, n\} \setminus \{k\}$, except for the constant map $x \in \mathbf{m} \mapsto k \in \mathbf{n}$.

The members of N_ℓ can be separated into two sets, a map $f \in N_\ell$ is of type I if k appears in the image of f, and of type II if k does not. To each map of type I corresponds a unique map of type II, and vice versa, obtained by removing k from, and adding it to, the image of the map. Since the face maps act on elements of N_ℓ precisely by removing elements from images, the pairs defined by this correspondence are filling pairs, hence a proper pairing of ℓ was just defined.

Moreover, it is easy to see that the ancestral relation of this pairing is isomorphic to the strict subset relation on a set of n elements, and is therefore well-founded. Hence, the conclusion follows from Theorem 3.41. ◻

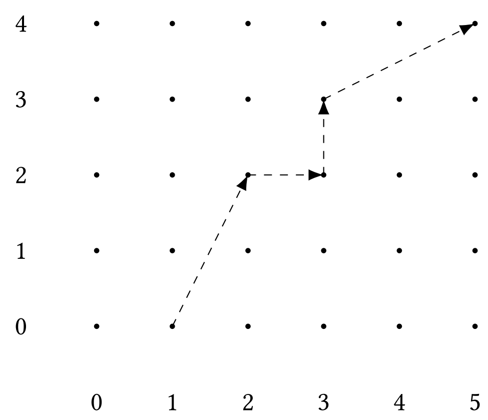


Figure 4: The path $(1, 0), (2, 2), (3, 2), (3, 3), (5, 4)$ in $L_{5,4}$

To characterise the non-degenerate simplices of $\Delta^n \times \Delta^1$, as is done during the proof of Proposition 3.38, it is useful to introduce some terminology.

Given two non-negative integers m and n, the **integer lattice** $L_{m,n}$ is the set of pairs (a, b) in $\{0, \dots, m\} \times \{0, \dots, n\}$. A **path** in $L_{m,n}$ is a sequence of pairs (a_i, b_i) in the lattice such that $a_i < a_{i+1}$ or $b_i < b_{i+1}$ for each i. In Figure 4, an example for $m = 5$ and $n = 4$ is displayed.

Equivalently, a path can be defined by its **start**, the first pair in the sequence, and its **moves**, where a move is a pair $(+p, +q)$ which indicates that the path will go p units rightwards and q unit upwards to reach the next pair from the current one. For example, the start of the path displayed in Figure 4 is $(1, 0)$, while the associated sequence of moves is $(+1, +2)$, $(+1, 0)$, $(0, +1)$, and $(+2, +1)$.

Proof of Proposition 3.38. Let ℓ denote the pushout-product $i_n \widehat{\times} i_1$. If $n = 0$, then ℓ is isomorphic to i_1 and thus the claim is satisfied; assume $n \geq 1$ in what follows.

By definition, an m-simplex of $\Delta^n \times \Delta^1$ is a pair of order-preserving maps $\mathbf{m} \to \mathbf{n}$ and $\mathbf{m} \to \mathbf{1}$; they correspond to maps $f : \mathbf{m} \to \mathbf{n} \times \mathbf{1}$, on which the face and degeneracy maps act by precomposition. Such an f is non-degenerate if and only if it is injective, which is the case precisely when, for all $i < m$, the respective images (a, b) and (a', b') of i and $i + 1$ under f satisfy $a < a'$ or $b < b'$. Clearly, those are exactly the paths in the integer lattice $L_{n,1}$, on which the kth face map d_k acts by removing the $(k + 1)$th pair of the path (so that d_0 removes the first, d_1 the second, and so on).

By definition, the simplex $f : \mathbf{m} \to \mathbf{n} \times \mathbf{1}$ belongs to $\partial\Delta^n \times \Delta^1$ if and only if $\pi_\mathbf{n} \circ f : \mathbf{m} \to \mathbf{n} \times \mathbf{1} \to \mathbf{n}$ is not surjective; under the identification with paths, this translates to the path 'missing' at least one column. Similarly, the non-degenerate simplices of $\Delta^n \times \partial\Delta^1$ may be identified with paths avoiding one of the two rows. Therefore, the paths representing elements of N_ℓ are precisely those that visit each row and each column. Here, and displayed in Figure 5 for $n = 3$, are examples of such paths:

- For $0 \leq i \leq n$, the path starting from $(0, 0)$, ending at $(n, 1)$, and in which every move is $(+1, 0)$ except for a single move $(0, +1)$ to link $(i, 0)$ and $(i, 1)$ together; call the simplex it represents x_i.
- For $0 \leq i < n$, the path starting from $(0, 0)$, ending at $(n, 1)$, and in which every move is $(+1, 0)$ except for a single move $(+1, +1)$ to link $(i, 0)$ and $(i + 1, 1)$ together; call the simplex it represents y_i.

One can easily see that, in fact, those are the only paths with that property, i.e.,

$$N_\ell = \{x_0, \dots, x_n, y_0, \dots, y_{n-1}\}.$$

Note that $d_{i+1}(x_i) = y_i$ for $0 \leq i < n$, and $d_i(x_i) = y_{i-1}$ for $0 < i \leq n$. Moreover, those are the only face relations between members of N_ℓ, as indicated in Figure 5 for $n = 3$.

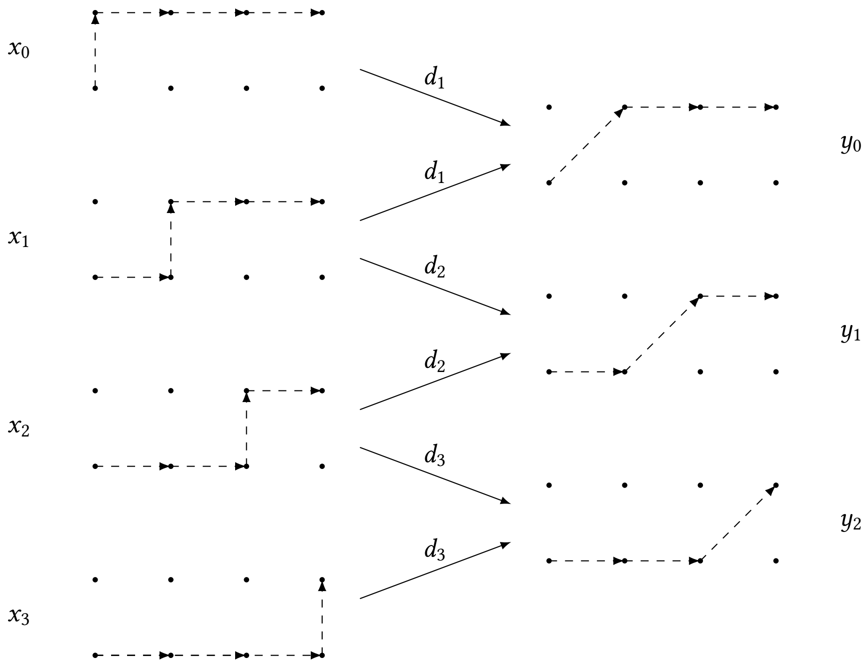


Figure 5: The paths and face relations of N_ℓ for $n = 3$

In particular, if ℓ' is the factorisation of ℓ through P, the smallest subcomplex of $\Delta^n \times \Delta^1$ containing the simplices in $N_\ell \setminus \{x_n\}$ and the image of ℓ, then $N_{\ell'} = \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$ and Theorem 3.41 can be used to show that ℓ' is an anodyne extension. Indeed, each (x_i, y_i) is a filling pair of ℓ and they define a proper pairing of ℓ whose ancestral relation is the empty relation, which is a well-founded relation.

Finally, if ι_P is the inclusion $P \hookrightarrow \Delta^n \times \Delta^1$, then it is clear that $N_{\iota_P} = \{x_n\}$; this means that ι_P is a pushout of i_n, against which p has the right lifting property. By stability under composition, the pushout-product $i_n \widehat{\times} i_1 = \ell = \ell' \circ \iota_P$ also has the left lifting property against p, as desired. □

A final observation is needed before proving the main result of this section.

Lemma 3.43. *For any simplicial set X, the map $\emptyset \hookrightarrow X$ can be obtained as a countable composition of pushouts of direct sums of boundary inclusions; in particular, any map which has the right lifting property against all boundary inclusions also has it against inclusions of the empty simplicial set.*

Proof. Each simplicial set X is a colimit of the diagram

$$\mathrm{sk}_{-1} X \hookrightarrow \mathrm{sk}_0 X \hookrightarrow \mathrm{sk}_1 X \hookrightarrow \mathrm{sk}_2 \hookrightarrow \dots$$

where $\mathrm{sk}_{-1} X = \emptyset$. The countable composition of $(\mathrm{sk}_{n-1} X \hookrightarrow \mathrm{sk}_n X)_{n \geq 0}$ is therefore precisely the inclusion of $\emptyset$ in X. Moreover, for each $n \geq 0$, the diagram below, in which NX_n is the set of non-degenerate simplices of X of dimension n, is a pushout square.

$$\begin{array}{ccc} \coprod_{x \in NX_n} \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \big\downarrow & & \big\downarrow \\ \coprod_{x \in NX_n} \Delta^n & \longrightarrow & \mathrm{sk}_n X \end{array}$$

Hence, the conclusion follows. □

Finally, the result below sums up all the above and can be said to ‘unwind mere propositions’.

Proposition 3.44. *Given a fibration $p : E \to B$ in* **sSet***, the following are equivalent.*

(a) The map $\mathsf{Id}_E \to E \times_B E$ admits a section.

(b) The two projections $\pi_1, \pi_2 : E \times_B E \to E$ are homotopic over B.

(c) Given any $X \in$ **sSet***, maps $f, g : X \to E$, and homotopy H between pf and pg over B, there exists a lift of H along p which is a homotopy between f and g.*

(d) The fibration p has the right lifting property against $X \times i_1 : X \times \partial\Delta^n \hookrightarrow X \times \Delta^n$ for all $X \in$ **sSet***.*

(e) The Leibniz exponential $i_1 \rhd p$ is a trivial fibration, i.e., it has the right lifting property against all the boundary inclusions $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$.

(f) The fibration p has the right lifting property against all the pushout products $i_n \widehat{\times} i_1$ for $n \geq 0$.

(g) The fibration p is truncated, i.e., it has the right lifting property against all the boundary inclusions $i_n : \partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 1$.

Proof. The equivalences $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$ are Lemmas 3.30 and 3.31, respectively. The paragraph just below those two lemmas explained how (c) and (d) are reformulations of one another, so $(c) \Leftrightarrow (d)$ as well. Proposition 3.32 showed $(d) \Rightarrow (g)$, while $(g) \Rightarrow (f)$ is the precise statement of Proposition 3.38. A simple application of Proposition 3.37 yields $(f) \Leftrightarrow (e)$, so that it only remains to prove $(e) \Rightarrow (d)$.

Assume that $i_1 \rhd p$ is a trivial fibration, by Lemma 3.43 it has the right lifting property against $\iota_X : \emptyset \hookrightarrow X$ for any $X \in$ **sSet**. By Proposition 3.37, this means that $(\iota_X \widehat{\times} i_1) \boxslash p$. Finally, one can easily check that $\iota_X \widehat{\times} i_1$ is isomorphic to $X \times i_1$. Hence, the conclusion follows. □

3.3 Image as truncation

As mentioned at the end of Section 3.1, the present section aims to endow the universe U_α inducing the simplicial model with a $\| \ \|$-structure, building on the last section to do so.

The present approach is based on image factorisations, which roughly means that the truncation of a type $p_A : A \to \Gamma$ will be the inclusion of the image of p_A into Γ; in particular, the truncation $\|A\| \to \Gamma$ is *strict*, in the sense that it is a monomorphism, the semantic counterpart of saying that terms of $\|A\|$ are definitionally equal. Actually, most of the following is stated in terms of the the closely related notion of *epi-mono factorisations*, but that makes no difference.

Definition 3.45. An **epi-mono factorisation** of a map $f : A \to B$ in a category **C** is a factorisation $f = me$ with m a monomorphism and e an epimorphism.

The archetypal example of such a factorisation for a map of sets $f : A \to B$ is the image factorisation $A \twoheadrightarrow \mathrm{im}(f) \hookrightarrow B$. The category of simplicial sets shares many properties with that of sets, one of them is a similar image factorisation.

Example 3.46 (Canonical image factorisation)**.** For a map $f : A \to B$ in **sSet**, let M_n be the set of n-vertices of B which can be expressed as $f_n(a)$ for some $a \in A_n$. This family of subsets $M_n \subset B_n$ is stable under face and degeneracy maps, in the sense that if b is an element of M_n then, for example, its degeneracy $s_i(b)$ is an element of M_{n+1}. Indeed, there is some $a \in A_n$ such that $f_n(a) = b$ by definition, and thus

$$s_i(b) = s_i(f_n(a)) = f_{n+1}(s_i(a))$$

belongs to M_{n+1}. This ensures that the sequence of sets M_n satisfy the simplicial identities and thence induces a simplicial set $\mathrm{im}(f)$, which comes with a canonical inclusion $m_f : \mathrm{im}(f) \hookrightarrow B$. Sending $a \in A_n$ to $f_n(a) \in M_n$ then defines a simplicial map $e_f : A \to \mathrm{im}(f)$ which, being surjective at each dimension, is an epimorphism. By definition, the diagram

$$\begin{array}{ccc} A & & \\ & \overset{e_f}{\searrow\!\!\!\!\searrow} & \\ {\scriptstyle f}\Big\downarrow & & \mathrm{im}(f) \\ & \overset{m_f}{\swarrow} & \\ B & & \end{array}$$

commutes, it is called the **canonical epi-mono factorisation** of f.

The preceding example shows that each map in **sSet** admits at least one epi-mono factorisation, which settles existence considerations. It is then natural to ask if there is some order among the ways to factorise a fixed map, or even a notion of uniqueness. This is best answered via the concept of maps between such factorisations.

Definition 3.47. A **map of epi-mono factorisations** from $f = me$ to $f' = m'e'$ is a triple of maps (u, v, w) such that the diagram below commutes.

$$\begin{array}{ccccc} A & \xrightarrow{e} & M & \xhookrightarrow{m} & B \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle w} \\ A' & \xrightarrow{e'} & M' & \xhookrightarrow{m'} & B' \end{array}$$

Recall that the *arrow category* $\mathbf{C}^{\rightarrow}$ of a category $\mathbf{C}$ has as objects arrows of $\mathbf{C}$ while a map from $f : A \longrightarrow B$ to $f' : A' \longrightarrow B'$ in $\mathbf{C}^{\rightarrow}$ is a pair of maps (u, w) with $u : A \longrightarrow A'$ and $w : B \longrightarrow B'$ which satisfy $wf = f'u$.

Proposition 3.48. *If $f = me$ and $f' = m'e'$ are both epi-mono factorisations in* **sSet***, then for each map (u, w) from f to f' in* $\mathbf{sSet}^{\rightarrow}$ *there exists a unique map v such that (u, v, w) is a map from $f = me$ to $f' = m'e'$.*

Proof. This is a special case of [MM92, Proposition IV.6.2], where it is shown in arbitrary topos. ◻

Corollary 3.49. *If $f = me$ and $f = m'e'$ are two epi-mono factorisations of the same map $f : A \longrightarrow B$ in* **sSet***, then there exists a unique isomorphism of the form $(\mathrm{id}_A, v, \mathrm{id}_B)$ between the two factorisations.*

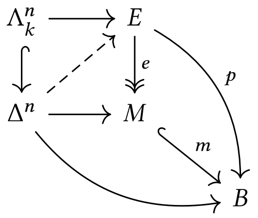
$$\begin{array}{ccccc} & & M & & \\ & \overset{e}{\nearrow\!\!\!\!\nearrow} & & \overset{m}{\searrow} & \\ A & & \Big\downarrow{\scriptstyle v} & & B \\ & \underset{e'}{\searrow\!\!\!\!\searrow} & & \underset{m'}{\nearrow} & \\ & & M' & & \end{array}$$

Proof. Since $(\mathrm{id}_A, \mathrm{id}_B)$ is a map from f to itself in $\mathbf{sSet}^{\rightarrow}$, there exist unique maps $v : M \longrightarrow M'$ and $v' : M \longrightarrow M'$ such that $(\mathrm{id}_A, v, \mathrm{id}_B)$ and $(\mathrm{id}_A, v', \mathrm{id}_B)$ are maps of epi-mono factorisations. As a result, the triple $(\mathrm{id}_A, v' \circ v, \mathrm{id}_B)$ is a map from $f = me$ to itself, but that is also the case of $(\mathrm{id}_A, \mathrm{id}_M, \mathrm{id}_B)$; uniqueness then implies that $v' \circ v = \mathrm{id}_M$. Similarly, the map $v \circ v'$ is the identity of M', which means that $(\mathrm{id}_A, v, \mathrm{id}_B)$ is an isomorphism between the two factorisations, and it is already known to be unique. ◻

The aim of this section is to model propositional truncation using epi-mono factorisations, it is therefore important that they be well-behaved with respect to fibrations.

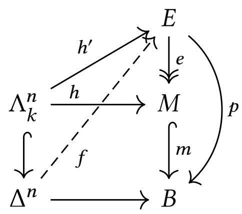
Proposition 3.50. *If $p : E \to B$ is a fibration and $p = me$ is an epi-mono factorisation of p, then both m and e are fibrations.*

Proof. Consider a lifting problem for $\Lambda^n_k \hookrightarrow \Delta^n$ against e, as in the diagram below. As can also be seen there, composing with m yields another lifting problem for the same horn inclusion against p. By assumption, this second problem admits a filler, which turns out to also solve to the first problem, because m is monic. Hence, the map e is a fibration.



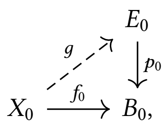
Being an epimorphism, the map e has the right lifting property against $\emptyset \hookrightarrow \Delta^0$. Recall that the inclusion $\Delta^0 \overset{k}{\hookrightarrow} \Lambda^n_k$ is an anodyne extension, as shown by Proposition 3.42. Hence, by the above, this map $\Delta^0 \overset{k}{\hookrightarrow} \Lambda^n_k$ has the left lifting property against e; as a result, so does the composite $\emptyset \hookrightarrow \Delta^0 \hookrightarrow \Lambda^n_k$. This precisely means that any map $\Lambda^n_k \to M$ admits a lift along e.

Consider a lifting problem for $\Lambda^n_k \hookrightarrow \Delta^n$ against m as below. By the above, the map $h : \Lambda^n_k \to M$ of this problem can be lifted along e to a map $h' : \Lambda^n_k \to E$. Then, replacing h by h' induces a new lifting problem for the same horn inclusion against p. By assumption, this modified square admits a filler f and a simple diagram chase then shows that $e \circ f$ is a solution to the original lifting problem (note that one uses the monicity of m once more in this diagram chase). Hence, the map m is a fibration as well.

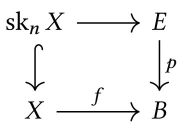


□

Proposition 3.51. *Let $p : E \to B$ and $f : X \to B$ be simplicial maps, with p truncated. Any vertex lift g, as in the diagram below, can be extended to a lift $F : X \to E$ of f along p.*



Proof. The final lift F is built up one dimension at the time, via a recursively defined sequence of partial lifts $F^n : \mathrm{sk}_n X \to E$. Here, a map $\mathrm{sk}_n X \to E$ is called a partial lift if the square



commutes. Since $\mathrm{sk}_0 X$ is freely generated by the vertices of X, the data of g is equivalent to that of a partial lift $F^0 : \mathrm{sk}_0 X \to E$.

Assuming a partial lift $F^n : \mathrm{sk}_n X \to E$, one defines the next partial lift F^{n+1} by constructing the commutative diagram displayed below.

$$\begin{tikzcd}
\coprod_{x\in NX_{n+1}} \partial\Delta^{n+1} \arrow[r] \arrow[d, hook] & \mathrm{sk}_n X \arrow[rr, "F^n"] \arrow[d, hook] \arrow[dr] & & E \arrow[d, "p"] \\
\coprod_{x\in NX_{n+1}} \Delta^{n+1} \arrow[r] & \mathrm{sk}_{n+1} X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, hook] & X \arrow[r, "f"] & B.
\end{tikzcd}$$

The map p being truncated, the outer rectangle admits a filler $h : \coprod_{x\in NX_{n+1}} \Delta^{n+1} \to E$. Since $\mathrm{sk}_{n+1} X$ is obtained by gluing the non-degenerate $(n+1)$-simplices of X to $\mathrm{sk}_n X$, the upper triangle of this filling induces a putative partial lift F^{n+1}.

$$\begin{tikzcd}
\coprod_{x\in NX_{n+1}} \partial\Delta^{n+1} \arrow[r] \arrow[d, hook] & \mathrm{sk}_n X \arrow[d, hook] \arrow[ddr, bend left, "F^n"] & \\
\coprod_{x\in NX_{n+1}} \Delta^{n+1} \arrow[r] \arrow[drr, bend right, "h"'] & \mathrm{sk}_{n+1} X \arrow[dr, dashed, "F^{n+1}"'] & \\
& & E
\end{tikzcd}$$

As for the lower triangle of the filling, it and the square associated with F^n help prove that F^{n+1} is indeed a partial lift.

$$\begin{tikzcd}
& & \mathrm{sk}_{n+1} X \arrow[d, hook] \\
& \mathrm{sk}_n X \arrow[ur, hook] \arrow[r, hook] \arrow[dl, hook] \arrow[d, "F^n"] & X \arrow[d, "f"] \\
\mathrm{sk}_{n+1} X \arrow[r, "F^{n+1}"] & E \arrow[r, "p"] & B \\
& \coprod_{x\in NX_{n+1}} \Delta^{n+1} \arrow[ul] \arrow[u, "h"] \arrow[dr] & X \arrow[u, "f"'] \\
& & \mathrm{sk}_{n+1} X \arrow[u, hook]
\end{tikzcd}$$

Indeed, the commutative diagram above shows that $p \circ F^{n+1}$ and $f \circ (\mathrm{sk}_{n+1} X \hookrightarrow X)$ are induced, as maps out of a pushout, by the same pair of maps, hence they are themselves equal.

By definition, the triangle

$$\begin{tikzcd}
\mathrm{sk}_n X \arrow[r, hook] \arrow[dr, "F^n"'] & \mathrm{sk}_{n+1} X \arrow[d, "F^{n+1}"] \\
& E
\end{tikzcd}$$

commutes for each n. As a consequence, the sequence of maps $F^n : \mathrm{sk}_n \to E$ induces a map $F : X \to E$; and their being partial lifts ensures that F is itself a lift (which extends g by construction). $\square$

Corollary 3.52. *Let $e : A \to B$ and $f : Y \to X$ be two maps in* **sSet**. *If e is an epimorphism and f is truncated, then e has the lower-only left lifting property against f.*

Proof. Since e is an epimorphism, its vertex map $e_0 : A_0 \to B_0$ is surjective and thus, by AC, admits a section $s : B_0 \to A_0$. Given any square from e to f, such as the one displayed below, one can precompose $g_0 : A_0 \to Y_0$ with s and thus obtain a vertex lift of h along f.

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{h} & X \end{array}$$

Since $f : Y \to X$ is truncated, the above proposition can be applied and yields a lift of h along f, as required to show that $e \boxslash_l f$. □

Theorem 3.53. *Any epi-mono factorisation*

$$\begin{array}{ccc} \tilde{U}_\alpha & & \\ & \overset{e}{\searrow\!\!\!\!\!\searrow} & \\ {\scriptstyle p_\alpha}\downarrow & & M \\ & \underset{m}{\hookleftarrow\!\!\swarrow} & \\ U_\alpha & & \end{array}$$

induces a $\| \ \|$-structure on U_α.

Proof. By Proposition 3.50, the map m is a fibration, and it is clearly α-small, hence there exists a map $\ulcorner M \urcorner : U_\alpha \to U_\alpha$ such that

$$\begin{array}{ccc} M & \longrightarrow & \tilde{U}_\alpha \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle p_\alpha} \\ U_\alpha & \xrightarrow{\ulcorner M \urcorner} & U_\alpha \end{array}$$

is a pullback square. Moreover, the fact that m is a monomorphism means that the diagonal map $\Delta_M : M \to M \times_{U_\alpha} M$ is an isomorphism and therefore that $r_M \circ \Delta_M^{-1}$ is a section of p_{Id_M}. This yields a factorisation of $\ulcorner M \urcorner$ through $(U_\alpha)_{-1} \to U_\alpha$, let $\ulcorner \| \ \| \urcorner$ be the map $U_\alpha \to (U_\alpha)_{-1}$ of said factorisation.

As a result, the projection $p_{\| \ \|} : \| \ \| \to U_\alpha$ is isomorphic to $m : M \hookrightarrow U_\alpha$ over U_α. Composing this isomorphism with $e : \tilde{U}_\alpha \twoheadrightarrow M$ yields the desired map $\mathsf{tr} : \tilde{U}_\alpha \to \| \ \|$, which satisfies $p_{\| \ \|} \circ \mathsf{tr} = p_\alpha$, as required.

Finally, it only remains to give an internal lower-only lifting operation for tr against $(p_\alpha)_{-1} \times U_\alpha$ in $\mathbf{sSet}/U_\alpha$. By Proposition 3.27, it suffices to show that tr has the stable lower-only left lifting property against $(p_\alpha)_{-1} \times U_\alpha$ in $\mathbf{sSet}/U_\alpha$. This is equivalent to the claim that, for each $f : X \to U_\alpha$, the pullback $f^*\mathsf{tr}$ of tr (seen as a map over U_α) along f has the lower-only left lifting property against $(p_\alpha)_{-1}$ in **sSet**. Corollary 3.52 shows that this is indeed the case since pullbacks of epis in **sSet** remain epis and that $(p_\alpha)_{-1}$ is a truncated map by definition. □

Corollary 3.54. *The simplicial model admits a $\| \ \|$-structure such that $p_A = p_{\|A\|} \circ \mathsf{tr}_A$ is an epi-mono factorisation for each type $A \to \Gamma$; in particular, the truncation of A is a* strict *proposition.*

Proof. It suffices to apply Proposition 3.29 to the last theorem, with canonical image factorisation of p_α, introduced in Example 3.46, as the chosen epi-mono factorisation; the strictness claim follows directly because pullbacks of monos are monos as well. □

4 Discrete interpretation

4.1 Discrete simplicial sets

This section focuses on the map D from **Set** to **sSet** induced by viewing sets as discrete simplicial sets. The technical definition of the simplicial model slightly complicates the form that precise definitions of D as a CwA map can take. The first aim of this section is to give a satisfactory account of a definition of D.

Properties of D are then studied, especially those expressing how and in which sense does D preserve the logical structure of $\mathbf{C}_{\mathbf{Set}}$ such as Id-types, propositional truncation. It is also explained how to modify D so that it takes its values in the CwA of h-sets of the simplicial model, as defined in Section 2, rather than in the simplicial model itself. Those are the necessary tools needed for the last part of the section.

Definition 4.1. A simplicial set X is **discrete** if its only non-degenerate simplices are of dimension 0. If A is a set, then let DA be the discrete simplicial set $\bigsqcup_{a \in A} \Delta^0$, i.e., the simplicial with one vertex for each element of A. The assignment $A \mapsto DA$ extends to a functor $D : \mathbf{Set} \to \mathbf{sSet}$ in the evident way. Note that D is injective on objects as well as fully faithful

The first aim of this section is to upgrade D to the rank of CwA map into $\mathbf{sSet}_{U_\alpha}$. However, given how extensions and projections are not specifically chosen in $\mathbf{sSet}_{U_\alpha}$, the best one can hope for is a weaker notion of CwA map which only preserves part of the structure up to isomorphism.

Definition 4.2. Let $\mathbf{C}$ and $\mathbf{D}$ be two CwAs, a **weak CwA map** between them consists of a functor $F : \mathbf{C} \to \mathbf{D}$, a natural transformation $F^{\mathrm{Ty}} : \mathrm{Ty}_{\mathbf{C}} \circ F^{\mathrm{op}} \to \mathrm{Ty}_{\mathbf{D}}$, and isomorphisms $\phi_{\Gamma,A} : F(\Gamma.A) \to F\Gamma.F^{\mathrm{Ty}}A$ for each type $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ such that the diagrams

$$
\begin{array}{ccc}
F(\Gamma.A) & \xrightarrow{\phi_{\Gamma,A}} & F\Gamma.FA \\
& {\scriptstyle Fp_A} \searrow \quad \swarrow {\scriptstyle p_{FA}} & \\
& F\Gamma &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
F(\Delta.f^*A) & \xrightarrow{F(f.A)} & F(\Gamma.A) \\
{\scriptstyle \phi_{\Delta,f^*A}} \downarrow & & \downarrow {\scriptstyle \phi_{\Gamma,A}} \\
F\Delta.Ff^*A & \xrightarrow{Ff.FA} & F\Gamma.FA
\end{array}
$$

commute for each type $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ and map $f : \Delta \to \Gamma$ in $\mathbf{C}$.

Proposition 4.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two categories with respective universes* U *and* V*. Assume a functor* $F : \mathbf{C} \to \mathbf{D}$ *which sends* U*-canonical pullback squares in* $\mathbf{C}$ *to pullback squares in* $\mathbf{D}$ *(not necessarily the* V*-canonical ones). Further assume that* F *sends* $p : \tilde{U} \to U$ *to a pullback of* $q : \tilde{V} \to V$ *in* $\mathbf{D}$*, as displayed below.*

$$
\begin{array}{ccc}
F\tilde{U} & \xrightarrow{\tilde{\eta}} & \tilde{V} \\
{\scriptstyle Fp} \downarrow & \lrcorner & \downarrow {\scriptstyle q} \\
FU & \xrightarrow{\eta} & V
\end{array}
$$

In those conditions, the functor F *together with the natural transformation*

$$A \in \mathrm{Hom}_{\mathbf{C}}(\Gamma, U) \mapsto \eta \circ FA \in \mathrm{Hom}_{\mathbf{D}}(F\Gamma, V)$$

induce a weak CwA map $\mathbf{C}_U \to \mathbf{D}_V$*, whose isomorphisms* $\phi_{\Gamma,A}$ *are induced by the uniqueness of pullbacks up unique isomorphisms.*

Proof. For each condition to check, one has only to draw the corresponding diagram to see that the condition holds. □

The strategy therefore consists in choosing a universe in **Set** to which the above proposition applies with $F = D$. Since U_α classifies fibrations whose fibres are of cardinality $< \alpha$, there is an evident choice of universe in **Set**.

Definition 4.4. Let $\mathbf{Set}_\alpha$ be the CwA induced by the universe

$$\pi_\alpha : \bigsqcup_{\beta<\alpha} \beta \to \alpha$$

in **Set**, with arbitrarily chosen pullbacks.

Remark 4.5. Note that this does not clash with the previous definitions of the CwA structure on **Set**. Indeed, the induced CwA $\mathbf{Set}_\alpha$ is morally equivalent to the CwA of sets defined in Example 2.5 with the additional restriction that the cardinality of A_γ is $< \alpha$ for each type $(A_\gamma)_{\gamma\in\Gamma}$ over Γ.

In order to apply Proposition 4.3, one has to check that D preserves α-canonical pullbacks; of course, it enjoys a stronger property.

Definition 4.6. Say that two vertices of a simplicial set X are **connected** if they appear as faces of a common simplex of X. Let the set $\pi_0 X$ of **connected components** of X be the set X_0 of vertices of X modulo the equivalence relation generated by the pairs of connected vertices of X. A **connected** simplicial set X is one that has a single connected components.

Proposition 4.7. *The mapping $X \in$ **sSet** $\mapsto \pi_0 X \in$ **Set** extends to a functor **sSet** $\to$ **Set** which is left adjoint to D, in particular D preserves pullbacks.*

Proof. Since simplicial maps commute with face maps, they send pairs of connected vertices to pairs of connected vertices. In particular, the vertex map f_0 induced by a simplicial map f induces a map $\pi_0 f$ between the sets of connected component, which makes π_0 into a functor.

For $X \in$ **sSet** and $Y \in$ **Set**, a map from X to DY amounts to a choice of $y \in Y$ for each simplex x of X which commutes with face and degeneracy maps. In the present case, this last restriction is equivalent to asking that the y chosen for a simplex x is the same as those chosen for each of the faces of x. In other words, to define a map $X \to DY$ it is equivalent to give a map $X_0 \to Y$ which agrees on connected vertices, i.e., a map $\pi_0 X \to Y$. This correspondence is of course natural, which shows that $\pi_0 \dashv D$; and the claim about pullbacks follows directly from the fact that right adjoints preserve limits. □

Additionally, one needs to exhibit $D\pi_\alpha$ as a pullback of p_α in **sSet**. Given the nature of p_α, this amounts to showing that $D\pi_A$ is an α-small fibration. By definition, it is clear that $D\pi_A$ is α-small; as for it being a fibration, that also follows from a more general fact.

Proposition 4.8. *If $f : A \to B$ is a map of sets, then $Df : DA \to DB$ is a fibration in* **sSet**.

Proof. In general, if $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ are a pair of adjoint functors $L \dashv R$, and that f and g are maps of **C** and **D** respectively, then $Lf \boxslash g$ holds if and only if $f \boxslash Rg$ does. In the present case, one wishes to show that $(\Lambda^n_k \hookrightarrow \Delta^n) \boxslash Df$. Hence, one can equivalently check that $\pi_0(\Lambda^n_k \hookrightarrow \Delta^n) \boxslash f$, which is trivially true as $\pi_0(\Lambda^n_k \hookrightarrow \Delta^n)$ is an isomorphism. □

Corollary 4.9. *There exist maps η_α and $\tilde{\eta}_\alpha$ such that*

$$\begin{array}{ccc} D\left(\bigsqcup_{\beta<\alpha}\beta\right) & \xrightarrow{\tilde{\eta}_\alpha} & \tilde{U}_\alpha \\ {\scriptstyle D\pi_\alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle p_\alpha} \\ D\alpha & \xrightarrow{\eta_\alpha} & U_\alpha \end{array}$$

is a pullback square in **sSet**. □

Definition 4.10. Let D_α be the weak CwA map $\mathbf{Set}_\alpha \to \mathbf{sSet}_{U_\alpha}$ induced, via Proposition 4.3, by the functor $D : \mathbf{Set} \to \mathbf{sSet}$ and the pullback square of Corollary 4.9.

Since D_α is a weak CwA map, it is unreasonable to ask that D_α preserve the logical structure strictly. An alternative definition, better suited to weak maps, could be one where the structured is preserved "up to isomorphisms". In order to give a more precise statement, it is useful to make a detour through the notion of **comma CwA of isomorphisms**; this is obtained by combining comma categories with a simpler version of the concept of CwA of span-equivalences [KL18, Definition 5.5] (where isomorphisms are used instead of span-equivalences).

Namely, given two (possibly) weak CwA maps $F_0 : \mathbf{C}_0 \to \mathbf{D}$ and $F_1 : \mathbf{C}_1 \to \mathbf{D}$, the underlying category of the comma CwA of isomorphisms $F_0 \overset{\downarrow}{\cong} F_1$ is the full subcategory of the comma category $F_0 \downarrow F_1$ consisting of the triples $(\Gamma_0, \Gamma_1, \alpha)$ where Γ_0 and Γ_1 respectively belong to $\mathbf{C}_0$ and $\mathbf{C}_1$ and α is an isomorphism $F_0\Gamma_0 \xrightarrow{\cong} F_1\Gamma_1$ in $\mathbf{D}$. Types over such a context $(\Gamma_0, \Gamma_1, \alpha)$ are triples (A_0, A_1, α') where A_0 and A_1 respectively belong to $\mathrm{Ty}_{\mathbf{C}_0}(\Gamma_0)$ and $\mathrm{Ty}_{\mathbf{C}_1}(\Gamma_1)$ and α' is an isomorphism of $\mathbf{D}$ making the diagram

$$\begin{array}{ccc} F_0(\Gamma_0.A_0) & \xrightarrow[\cong]{\alpha'} & F_1(\Gamma_1.A_1) \\ {\scriptstyle F_0 p_{A_0}}\big\downarrow & & \big\downarrow{\scriptstyle F_1 p_{A_1}} \\ F_0\Gamma_0 & \xrightarrow[\cong]{\alpha} & F_1\Gamma_1 \end{array}$$

commute. By relying on the fact that F_0 and F_1 preserve (at least weakly) the CwA structures of $\mathbf{C}_0$ and $\mathbf{C}_1$, one can straightforwardly define reindexing of types, extensions, projections, and connecting maps so that the projections $F_0 \overset{\downarrow}{\cong} F_1 \to \mathbf{C}_0$ and $F_0 \overset{\downarrow}{\cong} F_1 \to \mathbf{C}_1$ become strict CwA maps.

Armed with this additional tool, one can be more specific on the extent to which D_α preserves logical structures, such as Id-types. Indeed, if one consider the variant $F_0 \overset{\downarrow}{=} F_1$ where both maps are strict CwA maps and the isomorphisms are identity maps, then it can be seen that a (strict) CwA map $F : \mathbf{C} \to \mathbf{D}$ preserves Id-types if and only if $F \overset{\downarrow}{=} \mathrm{id}_{\mathbf{D}}$ can be endowed with an Id-structure which is strictly preserved by the projections $F \overset{\downarrow}{=} \mathrm{id}_{\mathbf{D}} \to \mathbf{C}$ and $F \overset{\downarrow}{=} \mathrm{id}_{\mathbf{D}} \to \mathbf{D}$. This justifies declaring that the weak map D_α *preserves a given logical structure up to isomorphism if the CwA* $D_\alpha \overset{\downarrow}{\cong} \mathrm{id}_{\mathbf{sSet}_\alpha}$ *can be endowed with a such structure which is strictly preserved under the two projections.*

Corollary 4.13 and Proposition 4.14 below showcase what concretely has to be proven in order to show that D_α preserve Id-types and propositional truncation up to isomorphism.

Lemma 4.11. *If X is a discrete simplicial set, then the "constant path" map $c^X : X \to X^{\Delta^1}$ is an isomorphism.*

Proof. Since X is discrete, it is of the form $X = DY$ for some set Y. It follows that, for any $Z \in \mathbf{sSet}$,

$$\begin{aligned} \mathrm{Hom}(Z, X^{\Delta^1}) &\cong \mathrm{Hom}(Z \times \Delta^1, DY) \\ &\cong \mathrm{Hom}(\pi_0(Z \times \Delta^1), Y) \\ &= \mathrm{Hom}(\pi_0 Z, Y) \\ &\cong \mathrm{Hom}(Z, X). \end{aligned}$$

Indeed, the connected components of Z and $Z \times \Delta^1$ are identical. Since c^X is precisely the image of id_X under these isomorphisms, it suffices to chase $\mathrm{id}_{X^{\Delta^1}}$ in the above to obtain an inverse of c^X. □

Proposition 4.12. *If $f : Y \to X$ is a simplicial map with X and Y discrete, then the reflexivity map $r_Y : Y \to P_X(Y)$ is an isomorphism.*

Proof. By definition of $P_X(Y)$, one can equivalently show that

$$\begin{array}{ccc} Y & \xrightarrow{c^Y} & Y^{\Delta^1} \\ \downarrow{\scriptstyle \Delta_Y} & & \downarrow \\ Y \times_X Y & \longrightarrow & (Y \times Y) \times_{X \times X} X^{\Delta^1} \end{array}$$

is a pullback square, which is certainly the case if both horizontal maps are isomorphisms. By the previous lemma, the top map is an isomorphism. As for the bottom map, it can be seen to be a pullback of the isomorphism $c^X : X \to X^{\Delta^1}$ along $f \times f : Y \times Y \to X \times X$, when c^X is viewed as a map from $\Delta_X : X \to X \times X$ to $(s_X, t_X) : X^{\Delta^1} \to X \times X$ in $\mathbf{sSet}/(X \times X)$. □

Corollary 4.13. *If $f : Y \to X$ is a map in* $\mathbf{Set}$*, then $D\Delta_Y$ and $P_{DX}(DY)$ are isomorphic in* $\mathbf{sSet}^{\to}$*, as witnessed by the square below.*

$$\begin{array}{ccc} DY & \xrightarrow[\cong]{r_{DY}} & P_{DX}(DY) \\ \downarrow{\scriptstyle D\Delta_Y} & & \downarrow \\ D(Y \times_X Y) & \xrightarrow{\cong} & DY \times_{DX} DY \end{array}$$

□

Consider now the case of preservation of propositional truncation. It useful to recall from Example 3.19 that if A is a type over Γ in $\mathbf{C}_{\mathbf{Set}}$ then the projection of $\|A\|$ together with tr_A form an epi-mono factorisation of p_A. In this case, the fact that D preserves the type former $\Gamma \vdash \|A\|$ type up to isomorphism takes the following form.

Proposition 4.14. *Let $f = me$ be an epi-mono factorisation of $f : Y \to X$ in* $\mathbf{Set}$*. The image of m under D is isomorphic to m_{Df}, from Example 3.46, over DX.*

Proof. Since D preserves monos and epis, the image $Df = DmDe$ of $f = me$ under D is an epi-mono factorisation of Df. From Corollary 3.49 it then results that those two factorisations are isomorphic, which implies the stated claim. □

Indeed, the choice of $\| \ \|$-structure on U_α ensures that the projection of $\|D_\alpha A\|$, for A a type in the set model, is the monomorphism of an epi-mono factorisation of $p_{D_\alpha A}$. Note that, in particular, it also the case that the image of the projection of $p_{\|A\|}$ under D_α admits a section if and only the projection of the truncation of $D_\alpha A$ in the simplicial model does so as well.

In the above, the functor $D : \mathbf{Set} \to \mathbf{sSet}$ has been promoted to the status of CwA map, albeit a weak one which preserves the logical structure up to isomorphism, it may therefore be hoped that the techniques of Section 2 would induce a CwA map $\mathbf{Set}_\alpha \to S(\mathbf{sSet}_{U_\alpha})$, since $\mathbf{Set}_\alpha$ is a CwA with UIP. However, the framework used in said section was that of strict maps, not weak ones. While it could be conjectured that the constructions and results of that section can be adapted to the case of weak maps, it is easier for the time being to give a solution for the specific case at hand.

In Section 3.1, it was noted that any universe U carrying an Id-structure induced a universe of mere propositions U_{-1}. The same can be done for h-sets.

Proposition 4.15. *Let U be a universe in a locally cartesian closed category* $\mathbf{C}$*, together with an* Id*-structure for U. There exists a map $U_0 \to U$ of* $\mathbf{C}$ *such that, for each type $\ulcorner A \urcorner : \Gamma \to U$, the sections of $p_{\mathsf{Id}_{\mathsf{Id}_A}}$ correspond bijectively with factorisations of $\ulcorner A \urcorner$ through $U_0 \to U$.*

Proof. Let $U_0 \to U$ be the image of

$$(\mathsf{Id} \times_{\tilde{U} \times_U \tilde{U}} \mathsf{Id} \to \tilde{U} \times_U \tilde{U})^* \mathsf{Id} \in \mathbf{C}/(\mathsf{Id} \times_{\tilde{U} \times_U \tilde{U}} \mathsf{Id})$$

under the functor

$$\Pi_{\mathsf{Id} \times_{\tilde{U} \times_U \tilde{U}} \mathsf{Id} \to U} : \mathbf{C}/(\mathsf{Id} \times_{\tilde{U} \times_U \tilde{U}} \mathsf{Id}) \to \mathbf{C}/U.$$

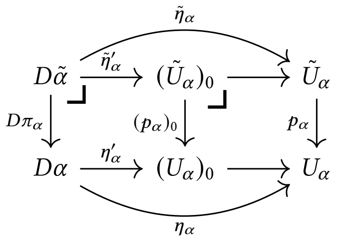
The proof of Proposition 3.20 is easily adapted. □

If one chooses pullbacks of $p_0 : \tilde{U}_0 \to U_0$ along maps $X \to U_0$ as those chosen by $\tilde{U} \to U$ along $X \to U_0 \to U$, then U_0 is also a universe in $\mathbf{C}$. Moreover, this canonical choice of pullbacks ensures that the map $\mathbf{C}_{U_0} \to \mathbf{C}_U$ induced by Proposition 4.3 is actually a strict CwA map, and not only a weak map. Since U_0 encodes h-sets, it may seem that U_0 is a different way of restricting to h-sets in CwAs induced by universes than that developed in Section 2, but that is just an impression.

Proposition 4.16. *Let U be a universe in a locally cartesian closed category $\mathbf{C}$, together with an Id-structure for U. The CwAs $\mathbf{C}_{U_0}$ and $S\mathbf{C}_U$ are isomorphic over $\mathbf{C}_U$.*

Proof. The isomorphism of underlying categories is simply the identity functor, since both CwAs have $\mathbf{C}$ as their underlying category. Recall that types over $\Gamma \in \mathbf{C}$ in $\mathbf{C}_{U_0}$ are maps $\Gamma \to U_0$, while in $S\mathbf{C}_U$ they are types over Γ in $\mathbf{C}_U$ together with a proof that said type is an h-set. The last proposition shows that those two sets of types are naturally isomorphic, and one easily checks that these isomorphisms agree with extensions, projections, and connecting maps. □

Now, Corollary 4.13 above shows in particular that $P_{DX}(DY) \to DY \times_{DX} DY$ is a monomorphism, since it is isomorphic to a diagonal map. In turn, this implies that its dependent path space $P_{DY \times_{DX} DY}(P_{DX}(DY))$ has a distinguished section, as was noted more generally in the proof of Theorem 3.53. In particular, the image of a type $A : \Gamma \to \alpha$ in $\mathbf{Set}_\alpha$ under D_α carries a canonical h-set structure and as a result the map $D_\alpha A : D\Gamma \to U_\alpha$ factors through $(U_\alpha)_0 \to U_\alpha$. Applying this to the case of α itself, one obtains the diagram displayed below.



The left-most pullback square can be used to define the desired map into the CwA of h-sets of the simplicial model.

Definition 4.17. Let D'_α be the weak CwA map $\mathbf{Set}_\alpha \to \mathbf{sSet}_{(U_\alpha)_0} \cong S(\mathbf{sSet}_{U_\alpha})$ induced, via Proposition 4.3, by the functor $D : \mathbf{Set} \to \mathbf{sSet}$ and the left-most pullback square of the previous diagram.

4.2 Assembling diagrams

Recall the original goal expressed in the introduction, that properties of classical sets are preserved under $D : \mathbf{Set} \to \mathbf{sSet}$. As explained there, if one wishes to show that a property P which holds in the set model also holds in the simplicial model using this approach, then one starts by choosing a fragment T of HoTT whose type formers preserve h-sets and in which (the type of proofs of) P can be expressed. Using the functor S defined in Section 2, one can then form the square displayed below.

$$\begin{array}{ccc} \mathbf{C}_{T+\mathsf{UIP}} & \longrightarrow & S(\mathbf{C}_{\mathsf{HoTT}}) \\ {\scriptstyle [\![\,]\!]}\downarrow & & \downarrow{\scriptstyle S[\![\,]\!]} \\ \mathbf{Set}_\alpha & \xrightarrow{D'_\alpha} & S(\mathbf{sSet}_{U_\alpha}) \end{array}$$

This square is not strictly speaking a square in the category of CwAs since D'_α is a weak map but it suffices for the current aim. Indeed, if one expresses the type of proofs of P in the syntax of T, then one can chase this element of $\mathbf{C}_{T+\mathsf{UIP}}$ along both paths of the above square.

On the one hand, going along the top-right path, the image of P under $\mathbf{C}_{T+\mathsf{UIP}} \to S(\mathbf{C}_{\mathsf{HoTT}})$ is the same type now expressed in the language of HoTT, with the additional assumption that all types are h-sets. If it is further sent down along $S(\mathbf{C}_{\mathsf{HoTT}}) \to S(\mathbf{sSet}_{U_\alpha})$ then the result is the type of proofs of P interpreted in a variant of the simplicial model where types are restricted to h-sets. On the other hand, for the bottom-left path, first sending P along $\mathbf{C}_{T+\mathsf{UIP}} \to \mathbf{Set}_\alpha$ yields its interpretation it in the set model, where it is assumed to admit a section. Finally, the map D'_α includes this set-based interpretation as discrete simplicial sets in the same variant of the simplicial model.

While the two resulting images of P are not necessarily equal on the nose, it is the case that if one of the two admits a section then so does the other one. Indeed, this results from the logical structure being preserved up to isomorphism (in the sense sketched in the previous section), since the two distinct projections are isomorphic as arrows; it was checked for identity types and propositional truncation in the previous section but those results can be extended as to include further type formers such as Π-types and Σ-types. As mentioned in Section 2.3, if T carries one or more universe, one has to interpret them in HoTT as universes of h-sets which are themselves h-sets for the current approach based on the functor S to work. Moreover, if those universes are interpreted in the simplicial model by directly lifting set-based universes in **sSet** as discrete simplicial sets, then the same commutativity up to isomorphism holds for the type formers introduced by those universes. Note that it is also possible to directly add a generic type former to T itself which, if interpreted in the simplicial model by a discrete simplicial set, fits in the current framework.

All in all, since the set-based interpretation P admits a section, so does the interpretation, through $S(\mathbf{C}_{\mathsf{HoTT}})$, of P in the modified simplicial model $S(\mathbf{sSet}_{U_\alpha})$. However, one could wonder whether the obtained type in $S(\mathbf{sSet}_{U_\alpha})$ still encodes the property P when the restriction to h-sets is stripped away. This is answered by the stripping map from Section 2.2, or more precisely by Proposition 2.33 which shows that the square

$$\begin{array}{ccc} US(\mathbf{C}_{\mathsf{HoTT}}) & \longrightarrow & \mathbf{C}_{\mathsf{HoTT}} \\ {\scriptstyle US[\![\,]\!]}\downarrow & & \downarrow{\scriptstyle [\![\,]\!]} \\ US(\mathbf{sSet}_{U_\alpha}) & \longrightarrow & \mathbf{sSet}_{U_\alpha} \end{array}$$

commutes. This means that the interpretations one obtains of P in the simplicial model by forgetting the h-set restriction from S or by directly interpreting P via $[\![\,]\!] : \mathbf{C}_{\mathsf{HoTT}} \to \mathbf{sSet}_{U_\alpha}$ are the same, as desired.

In conclusion, any property of sets is preserved in the simplicial model when those sets are seen as discrete simplicial set, as long as this property can be expressed type-theoretically with types that preserve h-sets whose structure in the set model is compatible with that of the simplicial model up to isomorphism.

5 Future directions

The work carried out in Section 2 gave a way to restrict CwAs to their CwAs of h-sets, though the interpretation of UIP took a slightly unsatisfactory form due to technical issues. Those issues arose in part from to the strictness of CwA maps, and it might be hoped that reworking said section with a weaker notion of map could yield a cleaner and more elegant version of the adjunction constructed there. Moreover, this modification could also simplify Section 4 and give a more concrete meaning to statements such as "the map D preserves Id-types up to isomorphisms".

In Section 2.3, the notion of type formers which preserve h-sets was tentatively defined. The slightly informal and imprecise tone of the section was in part due to the author's lack of knowledge of the recent frameworks which establish means of rigorously asserting statements of the form "For all type theories $T, \ldots$" or "For a arbitrary type former, …". It would therefore be fruitful to give a precise definition of h-set preservation within those paradigms, and carry out the proofs in that generality.

One could also revisit the interpretation of propositional truncation in the simplicial model. The one given here, based on epi-mono factorisations, has the advantage of providing strict propositions but the disadvantage of only modelling the computation rule propositionally. A different approach, based on a more homotopical factorisation system than the epi-mono one, might provide an alternative interpretation in which truncations are not likely to be strict but where the computation rule is modelled definitionally.

Finally, it may be hoped that the approach presented here applies, at least partly, to different models of HoTT, such as the cubical model.

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