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G-Zips and Global Section Cones

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Abstract

The topic of this thesis is the stack of G-Zips and what we can say about the geometry of G-Zip schemes, that is, schemes with a nice map into this stack. In particular, it treats the cone conjecture of Goldring and Koskivirta. The cone conjecture states that the global sections of certain vector bundles on a G-Zip scheme are determined by the global sections of a related bundle over the stack of G-Zips. An exposition of a basic strategy of proof is given, followed by an application of this strategy to the case where G is of Dynkin type C_2. We conclude with a discussion of applications to good reductions of Shimura varieties.

Contents

Introduction

The idea to write this thesis surfaced during a talk held by my supervisor Wushi Goldring. The theme of the talk was using group theory as a tool in geometry, a theme aptly named *geometry-by-groups* in [1]. A classical example of this theme is the description of Grassmanian manifolds/varieties and more generally flag spaces as a quotient of a reductive algebraic group by a parabolic subgroup. The Grassmanian $\mathrm{Gr}(n,r) \cong \mathrm{GL}(n)/H$ over a field k where H is the stabilizer in $\mathrm{GL}(n)$ of some r-dimensional subspace of k^n, can also be viewed as a moduli space. Namely, giving a map $X \to \mathrm{Gr}(n,r)$, where X is a k-scheme, is the same as giving quotient bundle of rank r of the vector bundle $k^n \times_k X$. Equivalently it corresponds to choosing a rank $n-r$ subbundle of $k^n \times_k X$ (the kernel of the natural projection to the quotient), which we can view as choosing a filtration of $k^n \times_k X$. As we will see in section 2.1.1, this can be re-interpreted as giving a pair of torsors on X, with one contained in the other.

Two influential papers using *geometry-by-groups* are Deligne's [2] and in Griffiths and Schmid's [3]. A further unifying theme of these is that the central geometric objects are defined in terms of a cocharacter datum, namely a pair (G,μ) where G is a reductive group and μ is a cocharacter. Another is that these objects both parametrize families of Hodge structures, that is, they are period domains. The stack of G-zips is also constructed from a cocharacter datum (G,μ), and as is shown in [4], when the Hodge-de Rahm spectral sequence of a map $X \to S$ of schemes in positive characteristic degenerates at page E_1, the Hodge and conjugate filtrations on the relative cohomology gives rise to a G-Zip. Hence, we can view the stack of G-Zip as a characteristic p period domain. In [5], Zhang redefines the Ekedahl-Oort stratification of Shimura varieties of PEL type, and generalizes it to Shimura varieties of Hodge type. He does this by giving a smooth map to the stack of G-Zips, and then apply the combinatorial description of the stack of G-Zips in [6]. Further establishing the connection between G-Zips and Shimura varieties.

Similarily to the Grassmanians, the stack of G-Zips can be described both as a quotient and as a moduli space for certain tuples of torsors. When $G = \mathrm{GL}(n)$, these torsors can be viewed as giving a vector bundle over the scheme mapping into the stack of G-Zips together with a pair of filtrations with some compatibility requirements.

Just like flag varieties can be used to compute the cohomology of other varieties, the cone conjecture, when true, gives us a way to compute cohomology of other schemes when they carry a nice map to the stack of G-zips. Specifically, it allows us to compute the global sections of certain vector bundles pulled back from the stack of G-Zips, the automorphic vector bundles. When we have a map from the reduction of a Shimura variety to the stack of G-Zips, these bundles are the usual automorphic vector bundles (see [7] and [8]).

The original intent with this project was to, in addition to this thesis, write a program implementing the basic strategy given in [7], then apply it to the case where G is of type $C_2 \times C_2$, with non-trivial Galois action.[1] However, time ran out and I decided to instead include a road-map of the algorithm implicit in [7] to make its implementation easier for any interested party, see section 4.5. Anyone interested in implementing this algorithm after reading this thesis or just this introduction is welcome to contact me[2] if any questions come up.

Outline

In section 1 we fix some notation and discuss the basic tools that are used in the thesis. Some familiarity with schemes and algebraic groups is assumed.

In section 2 we discuss quotient stacks, first introducing torsors and then using these to give the basic definitions of the quotient stacks of a group scheme acting on a scheme. We then look at some properties of quotient stacks with finite underlying topological spaces, in particular the properties that can be used to define stratifications on schemes mapping into these quotient stacks.

In section 3 we give the definition of the stacks of G-Zips and G-Zip flags together with a proof that the stack of G-Zips is a quotient stack, due to Pink-Wedhorn-Ziegler [9]. Then we give the definition of the Schubert stack of G, which under one presentation can be viewed as having Schubert cells of G as points, hence the name. We describe a smooth map from the stack of Zip flags to the Schubert stack, thus getting a stratification on the stack of Zip flags using the results of section 2.

In section 4, we discuss automorphic vector bundles on the three stacks introduced in section 3 and review the cone conjecture (Conjecture C in [7]). We then discuss a strategy for proving this conjecture for G of a given Dynkin type, together with a cocharacter. This is the basic strategy given in [7], with some small modifications and additions.

In section 5, we give a proof (due to Goldring & Koskivirta, [7]) of the conjecture in the case when G is of Dynkin type C_2.

[1] This would, if the conjecture was proven in that case, compute the cone of characters with non-trivial automorphic forms defined on the reduction of Shimura varieties associated to the Weil restriction of GSp(4) from a quadratic extension of $\mathbb{Q}$ back to $\mathbb{Q}$.

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Section 6 is dedicated to an introduction to Shimura varieties and their reduction modulo p (for Shimura varieties of Hodge type), and a construction of a G-Zip on this reduction, giving a map to the stack of G-Zips which is smooth by [5]. It ends with a discussion on the implications of the existence of this map, in particular on what the cone conjecture tells us about the cohomology of these varieties. This concludes the thesis.

Acknowledgments

I wish to thank Wushi Goldring for the countless enlightening mathematical discussions we have had the last few years. They have inspired me to keep learning new mathematics and consider different perspectives on topics I thought I already knew. Without these discussions, this thesis would have been very different.

I want to thank my father for reading this manuscript and catching many of my typos.

Finally, I want to thank all my friends and family for their patience with me and their support when work with this project have taken all my time.

1 Notations and conventions

1.1 Commutative Algebra

Throughout this thesis, k will denote an algebraic closure of $\mathbb{F}_p$ if nothing else is mentioned.

Given a commutative ring R, an R-algebra A and a R-module M, we use the notation $M_A := M \otimes_R A$.

We denote by $\mathbb{A}_f$ the ring of finite adeles of $\mathbb{Q}$, that is $\mathbb{A}_f \subset \prod_{l \text{ prime}} \mathbb{Q}_l$ with all but finitely many components lying in the l-adic integers (l varying), and $\mathbb{A}_f^p$ denotes the subring with trivial p-adic component.

1.2 Algebraic Geometry

Given a scheme $f : X \to S$ over S and a morphism $g : T \to S$, we denote analogously to the algebra situation $X_T := X \times_S T$. If $g : T \to S$ comes from a morphism of rings $g^\# : A \to B$ where $T = \text{Spec } B$ and $S = \text{Spec } A$, we write both X_T and X_B for the scheme $X \times_S T$.

For any scheme X, X_{red} denotes the associated reduced subscheme (see lemma 26.12.4 in [10] for a construction).

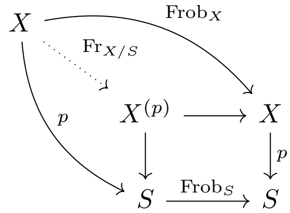
1.2.1 Frobenius morphims

If R is a commutative ring of characteristic $p > 0$, then it follows from the binomial theorem that the map $x \mapsto x^p$ is a homomorphism, this is called the Frobenius morphism of the ring and is denoted by Fr.

Lemma 1.2.1. *The Frobenius morphism commutes with localization.*

Given a $\mathbb{F}_p$-scheme S and an open cover $S = \bigcup_i \operatorname{Spec} R_i$ the absolute Frobenius morphism of S is defined by $\operatorname{Frob} : (S, \mathcal{O}_S) \to (S, \mathcal{O}_S)$, locally as Spec(Fr) on each Spec R_i. As Fr commute with localization, this is a well defined endomorphism of S.

If X is a scheme over S, where S is a characteristic p scheme as above, we get diagram



where p is the structure morphism of X and $X^{(p)}$ is the pullback of the absolute Frobenius of S along p. $\operatorname{Fr}_{X/S}$ is called the relative Frobenius of X over S.

Similarly, given an $\mathcal{O}_S$-module $\mathcal{F}$, denote by $\mathcal{F}^{(p)}$ the pullback of $\mathcal{F}$ along the absolute Frobenius of S.

When we only mention the Frobenius morphism it is assumed that it is clear from the context which one we mean.

1.3 Sections of line bundles

Given a line bundle $\mathscr{L}$ on a scheme X and a section $s \in H^0(X, \mathscr{L}) = \operatorname{Hom}(\mathscr{O}_X, \mathscr{L})$, let $\operatorname{div}(s)$[3] denote the associated Cartier divisor, $Z(s)$ the subscheme of X defined by the exact sequence

$$\mathscr{L}^\vee \xrightarrow{s^\vee} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0$$

with $\mathscr{L}^\vee = \underline{hom}(\mathscr{L}, \mathcal{O}_X)$ the dual of $\mathcal{L}$ and $s^\vee(\phi) = \phi \circ s$. This is called the scheme of zeroes of s. Let $\operatorname{nonvanish}(s) := X - Z(s)$.[4]

1.4 Algebraic Groups and Root Data

Let G be an algebraic group over k, then $X^*(G) := \operatorname{Hom}_{k-groups}(G, \mathbb{G}_m)$ and $X_*(G) := \operatorname{Hom}_{k-groups}(\mathbb{G}_m, G)$ denotes the group of characters and cocharacters of G. When $G = T$ is a torus[5] let $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ denote the perfect pairing defined by $\lambda(\mu(t)) = t^{\langle \lambda, \mu \rangle}$.

If G is a connected reductive group over k, and $T \subset G$ a maximal torus, let $\Phi := \Phi(G, T)$ denote the set of T-roots in G and $\Phi^\vee := \Phi^\vee(G, T)$ the T-coroots in G. The root datum of (G, T) is denoted $\mathcal{RD}(G, T) := (X^*(T), \Phi, X_*(T), \Phi^\vee)$. For a Levi subgroup $L \subset G$ containing T, let $\Phi_L := \Phi(L, T) \subset \Phi$ denote the T-roots in L (similarly for the coroots).

Given a Borel subgroup B of G with $T \subset B$, let

$$\Phi^+ := \{\alpha \in \Phi \big| U_{-\alpha} \subset B\}$$

[3] In most cases below X will be normal and thus we consider these divisors as Weil divisors.

[4] We will use "$-$" for set difference as "$\backslash$" will be reserved for quotients of left actions.

[5] Isomorphic to a product of multiplicative groups as k is algebraically closed.

where $U_\beta \cong \mathbb{G}_a$ is the root group of β in G (see Theorem 8.1 in [11]), let also $\Phi^- := -\Phi^+$. The (unique, see Theorem 8.2.8 in [11]) set of simple roots of Φ^+ is denoted Δ and the set of simple coroots by $\Delta^\vee$. We write

$$X_+^*(T) := \{\chi \in X^*(T) \big| \langle \chi, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

for the cone of Δ-dominant characters.

The quadruple $(X^*(T), \Delta, X_*(T), \Delta^\vee)$ is the based root datum associated to B. Similarly, for a Levi subgroup L containing T, we denote by $\Delta_L := \Delta \cap \Phi_L$ and $\Delta_L^\vee := \Delta^\vee \cap \Phi^\vee$, and $X_{+,L}^*$ for the cone of Δ_L-dominant characters, $\Phi_L^+ := \Phi^+(L,T)$ and $\Phi_L^- := \Phi^-(L,T)$.

We denote by $W := W(G,T)$ the Weyl group of T in G, that is the normalizer of T modulo its center or equivalently the group of automorphisms of $X^*(T)$ generated by simple reflections $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ where α ranges over Φ. Given $w \in W$, $\dot{w} \in N_G(T)$ denotes a representative, i.e. an element in the fiber of w under the natural projection $N_G(T) \to W$.

The Weyl group is a finite Coxeter group described by the Coxeter system $(W, \{s_\alpha\}_{\alpha\in\Delta})$, and thus comes equipped with a length function $l : W \to \mathbb{Z}_{\geq 0}$ mapping w to the length of a minimal word in the generators $\{s_\alpha\}_{\alpha\in\Delta}$ equalling w in W. Thus we also have a longest element $w_0 \in W$. For a Levi subgroup $L \subset G$ containing T, let W_L be the Weyl group of L, and $w_{0,L} \in W_L$ the longest element (with respect to simple root reflections of roots contained in Δ_L). Given a parabolic subgroup $P \subset G$ containing B, with Levi subgroup L, its type is Δ_L, which we also denote by $\mathrm{type}(P) = \Delta_L$. We call $P^{\mathrm{op}} := \dot{w}_0 P \dot{w}_0{}^{-1}$ the opposite parabolic of P. If $P = \{g \in G \big| \lim_{t\to 0} \mu(t^{-1}) g \mu(t^{-1})^{-1} \text{ exist}\}$,[6] then $P^{\mathrm{op}} = \{g \in G \big| \lim_{t\to 0} \mu(t) g \mu(t)^{-1} \text{ exist}\}$.

For $J \subset \Delta$, $W_J \subset W$ denotes the subgroup generated by $\{s_\alpha\}_{\alpha\in J}$, and its longest element (with respect to J) is denoted $w_{0,J}$. Similarly, W^I denotes the set of coset representatives of W/W_J with minimal length (with respect to Δ). The longest element of W^J is $w_0 w_{0,J}$. We have the following inclusions for a Levi subgroup of type J (containing T)

$$w_{0,L}\Phi_L^+ \subset \Phi^-$$

and

$$w(\Phi^+ \setminus \Phi_L^+) \subset \Phi^+ \text{ for all } w \in W_L$$

to see the second formula, notice that $w_{0,J}$ is a product of simple reflections s_α with $\alpha \in J$, so it is enough to prove it for these, which follows directly from lemma 8.2.7 (i) in [11].

If G and T are defined over $\kappa \subset k$ a subfield, the root datum and Weyl group of G, T are then the corresponding gadgets of G_k and T_k. The group $\mathrm{Gal}(k/\kappa)$ naturally acts on both W and the root datum of G, T. On the root datum through Dynkin diagram automorphisms and on W by conjugation (viewing W as automorphisms of the root datum). The actions are related by the formula

$$^\sigma(w\lambda) = ^\sigma w ^\sigma \lambda$$

[6] The limit is taken in the sense of [11]. Precisely, the limit exist if the morphism $\mathbb{G}_m \to G$ given by $t \mapsto \mu(t^{-1}) g \mu(t^{-1})^{-1}$ extends to $\mathbb{A}^1$.

for all $w \in W$, $\lambda \in X^*(T)$ and $\sigma \in \mathrm{Gal}(k/\kappa)$. Similarly as for the root datum, given a Borel subgroup $T \subset B \subset G$ defined over some finite extension of κ, the associated based root datum is the one defined by (G_k, B_k, T_k).

1.4.1 Some useful notions

Definition 1.4.1. *Let G be an affine algebraic group G defined over a field k*

- *G is quasi-split if there is a Borel subgroup $B \subset G$, also defined over k*
- *G is split if it contains a maximal torus T which is isomorphic to $\mathbb{G}_m^n$ over k for some $n \geq 1$.*

Definition 1.4.2. *Let G be a connected reductive algebraic group defined over a global field K (number field or function field), given a prime v of the ring of integers, we say that G is unramified at v if G_{K_v} (the base change to the v-adic completion of K) is quasi-split over K_v and split over some unramified extension of K_v (see e.g [12] for an introduction to extensions of local fields).*

Definition 1.4.3. *A cocharacter $\mu \in X_*(T)$ is miniscule if for every $\alpha \in \Phi$*

$$\langle \alpha, \mu \rangle \in \{1, -1, 0\}.$$

2 Quotient Stacks

This section is devoted to describe the basic properties of quotient stacks. The description of the stack of G-Zips as a quotient stack, see section 3.2, is essential to deduce facts about a G-Zip scheme from the root system of G.

2.1 Torsors

Given a group scheme $G \to S$, a G-torsor is a scheme $I \to S$ with an action $\alpha : I \times G \to I$[7] such that whenever $I(U) \neq \emptyset$,

$$\alpha_U : I(U) \times_S G(U) \to I(U)$$

is simply transitive, and such that there is a covering $\{U_i \to S\}$[8] with $I(U_i) \neq \emptyset$ for all i. A morphism of G torsors is a G-equivariant morphism of schemes. It follows from the first axiom that if $I(U) \neq \emptyset$, then $I_U \cong G_U$, since we can pick $f \in I(U)$ and define a morphism $G \times_S U \to I \times_S U$ by

$$G_U(U') \ni g \to f.g \in I_U(U')$$

for any scheme $U' \to U$, which is an isomorphism as it is objectwise by the torsor axioms. If we have an equivariant morphism $\phi : I \to I'$ of torsors, choose a covering $\{U_i \to S\}$ such that $I(U_i) \neq \emptyset \neq I'(U_i)$. Then by the above there exist equvariant $\alpha : I_{U_i} \cong G_{U_i}$ and $\beta : I'_{U_i} \to G_{U_i}$, each of these pairs induce a equivariant morphism $\beta \circ \phi_{U_i} \circ \alpha^{-1} : G_{U_i} \to G_{U_i}$, which must be given by left multiplication of some element of $G(U_i)$,[9] in particular the pullback of ϕ to this cover is an isomorphism and hence f itself is an isomorphism. Thus the category of torsors with equivariant maps as morphisms over S is a groupoid.

[7] Alternatively $\alpha : G \times I \to I$.

[8] in the fpqc site, that is; fatihfully flat, quasi-compact morphisms are covers.

[9] since $\phi(g'g) = \phi(g').g$ for all g, g' and the action is simply transitive so $\phi(g) = \phi(e)g$

2.1.1 The torsor related to a vector bundle with tensors and a filtration

Given a vector bundle $\mathscr{V}$ of rank n over S, we define an associated $\mathrm{GL}(n)$-torsor. This is done in the following way. Let Λ denote the constant sheaf associated to a free abelian group of rank n and define

$$I := \mathit{Isom}_S(\mathscr{V}, \Lambda)$$

which as a pre-sheaf on $\mathrm{Sch}_{/S}$ takes $U \to S$ to the set of $\mathcal{O}_U$-module isomorphisms from $\mathscr{V}_U$ to $\Lambda \otimes \mathcal{O}_U$. This is a $\mathrm{GL}(n)$ torsor under the action $\theta.g(x) = g^{-1} \circ \theta(x)$ where $\theta \in I(U)$, $x \in \mathscr{V}_U(U')$ for any open $U' \subset U$ and $g \in \mathrm{GL}(n)(U')$.

To see that this is a torsor, first note that a local trivialization of $\mathscr{V}$ over U' yields an isomorphism $\mathscr{V}_{U'} \cong \mathcal{O}^n_{U'} \cong \Lambda \otimes \mathcal{O}_{U'}$, so there is a covering with the $I(U') \neq \emptyset$ for each U' in the covering. Also, if $\phi, \psi \in I(U)$, we may choose bases of $\mathscr{V}_U$ and Λ_U such that ϕ is represented by the identity matrix and ψ by some invertible matrix and we may take g to be this matrix, so $\mathrm{GL}(n)(U)$ acts transativly. Furthermore, if $\phi.g = \phi.h$, the same choice of basis as above yields $g = h$ so the action is simple and transitive, i.e. I is a torsor.

If G is an algebraic group embedded into $\mathrm{GL}(n)$ given as the stabilizer of a collection of tensors $\{s_\alpha\} \subset \Lambda^\otimes$, then if there are associated tensors $\{s_{\alpha,\mathscr{V}}\} \subset \mathscr{V}^\otimes$ we can form the sheaf

$$\mathit{Isom}_S((\mathscr{V}, \{s_{\alpha,\mathscr{V}}\}), (\Lambda, \{s_\alpha\}))$$

of isomorphisms mapping $s_{\alpha,\mathscr{V}}$ to s_α, which by identical considerations as above are G torsors.

If we have filtrations Fil_1 of $\mathscr{V}$ and Fil_2 of Λ, with the same jumps in rank, we can get torsors for the subgroup of G stabilizing the filtration of Λ. To define these torsors, let

$$\mathit{Isom}_S((\mathscr{V}, \{s_{\alpha,\mathscr{V}}\}, \mathrm{Fil}_1), (\Lambda, \{s_\alpha, \mathrm{Fil}_2\}))$$

be the subsheaf of the torsors above consisting of isomorphisms preserving the filtrations.

2.2 The quotient of an action

Given a group scheme $G \to S$ acting on a scheme $X \to S$ on the left, we define a category fibered in groupoids $[G \setminus X]$, over $\mathrm{Sch}_{/S}$ by the following. The objects in $[G \setminus X](U)$[10] consist of pairs (I, f). Here $I \to U$ is a left G_U-torsor over U and $f : I \to X$ is a G-equivariant morphism over S. Given two objects $(I, f), (I', f') \in [G \setminus X]$, a morphism is a morphism of G-torsors over S, $\alpha : I \to I'$ such that $f = f' \circ \alpha$. As this is a subcategory of torsors over U, it is indeed a groupoid by section 2.1. If $g : U \to V$ is a morphism of S-schemes,

[10]the fiber over U

then a morphism over g is a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{f} & I' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{g} & V \end{array}$$

with $(I, \alpha) \in [G \setminus X](U)$, $(I', \beta) \in [G \setminus X](V)$ where π is the projection map i.e. the functor mapping an object in $[G \setminus X]$ to the S-scheme it is defined over, and f is G-equivariant and the diagram

$$\begin{array}{ccc} I & \xrightarrow{f} & I' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow{\text{id}} & X \end{array}$$

commutes. Note that the morphisms in $[G \setminus X](U)$ are the morphisms defined above lying over the identity of U.

This stack also carries a natural projection $p : X \to [G \setminus X]$, which to $x \in X(U)$ assigns the trivial torsor G_U with the map $G_U \to X$ defined by mapping $g \in G_U(U')$ to $g.x_{U'}$.

If G is separated, flat and of finite presentation over S, then it follows from proposition 10.13.1 in [13] that $[G \setminus X]$ is an algebraic stack. This will always be the case in the examples of this thesis.

2.3 The topological space underlying a quotient stack

Given an algebraic stack $\mathcal{X}$, the underlying topological space $|\mathcal{X}|$ is defined as the set of equivalence classes of morphisms $\text{Spec}(K) \to \mathcal{X}$ for fields K. Here two morphisms $\text{Spec}(K_1) \to \mathcal{X}$ and $\text{Spec}(K_2) \to \mathcal{X}$ are equivalent if there is a common field extension K of K_1 and K_2 such that the two morphisms $\text{Spec}(K) \to \text{Spec}(K_i) \to \mathcal{X}$ are isomorphic in the groupoid $\mathcal{X}(K)$. A subset $|\mathcal{U}| \subset |X|$ is open if there is an open immersion of algebraic stacks[11] $\mathcal{U} \hookrightarrow \mathcal{X}$ with $|\mathcal{U}|$ the set defined above. If $\mathcal{X}$ is a scheme, then $|\mathcal{X}|$ is naturally homeomorphic to the topological space underlying $\mathcal{X}$ as a locally ringed space.

2.3.1 Finite T_0-spaces

A topological space is T_0 if for each pair of points, there is an open set containing one but not the other. Equivalently X is T_0 if for all $x, y \in X$, $x = y$ if and only if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. If X is a finite T_0 space, we say that $x \leq y$ if and only if $x \in \overline{\{y\}}$, which by the above is a partial order. On the other hand, if $(X, \leq)$ is a finite partially ordered set, we say a subset of X is closed if it is the union of sets of the form $\{x | x \leq y\}$ for some $y \in X$, this defines a topology on X. The open sets of this space has a basis consisting of the sets of the form $\{x | y \leq x\}$.

A map between topological spaces is continuous if and only if pre-images of closed sets are closed. This happens precisely when the map preserves the

[11] see 98.3 in [10] for a precise definition of open immersions of stacks

orders given above, when the spaces are finite and T_0. So the categories of finite T_0 spaces and finite partially ordered sets are isomorphic.

Suppose G is an algebraic group over a field K[12], with absolute Galois group $\Gamma = \mathrm{Gal}(\overline{K}/K)$, acting on the left on a scheme X of finite type over K. Each $G(\overline{K})$ orbit $\mathcal{O}$ is a locally closed subset of $X(\overline{K})$, as it is open in its closure by lemma 2.3.3 in [11]. The closure of $\mathcal{O}$ is stable under $G(\overline{K})$,[13] so $\overline{\mathcal{O}}$ is a union of orbits. It follows that if $\mathcal{O}'$ is another orbit with $\mathcal{O}' \cap \overline{\mathcal{O}} \neq \emptyset$, then $\mathcal{O}' \subset \overline{\mathcal{O}}$ and $\dim \mathcal{O}' < \dim \mathcal{O}$.

Hence the orbit set $\Theta := G(\overline{K}) \setminus X(\overline{K})$, is T_0 under the quotient topology. To see this, suppose $\mathcal{O}_1 \neq \mathcal{O}_2$ and $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$. Then $\dim \mathcal{O}_2 < \dim \mathcal{O}_1$ and hence two points are equal if and only if each point is contained in the others closure.

Assume Θ is finite, then Θ is a partially ordered set under the order induced by its topology defined by $\mathcal{O} \leq \mathcal{O}' :\Leftrightarrow \mathcal{O} \subset \overline{\mathcal{O}'}$. The Galois group of $\overline{K}/K$, Γ, acts continuously on $X(\overline{K})$[14], which descends to a continuous action on Θ. Taking the quotient, we obtain another finite T_0 space $\Gamma \setminus \Theta$.

By II 3.2(ii) in [14] together with proposition 11.2.8 in [11], there is a natural bijection between $\Gamma \setminus \Theta$ and the set of locally closed reduced G-stable subschemes of X that contain no nontrivial locally closed G-stable subschemes. In other words, $\Gamma \setminus \Theta$ corresponds to the set of algebraic orbits of G in X.

We have the following

Proposition 2.3.1. *In the set up as above, if Θ is finite, then $|[G \setminus X]|$ is homeomorphic to $\Gamma \setminus \Theta$.*

Proof. A point $p \in |G \setminus X|$ is represented by a map $x_p : \mathrm{Spec}\, L \to [G \setminus X]$ for some field L/K. This map corresponds to a torsor I_p over $\mathrm{Spec}\, L$ and a G-equivariant map $f_p : I_p \to X$. A torsor over an algebraically closed field is trivial, and by the definition of the underlying topological space of a stack we can allways assume the field defining our point is algebraically closed. Hence p is determined by $f_p : G_L \to X$. Two such maps, f and h, comes from the same point precisely when their scheme theoretic image are the same locally closed G-stable subschemes of X, since then there is a $g \in G(L)$ with $f = g.h$. Hence $|G \setminus X|$ corresponds bijectively with the set of algebraic orbits of G in X. By the discussion above this defines a bijection $\theta : |[G \setminus X]| \to \Gamma \setminus \Theta$.

By Corollary 5.6.1(i) in [13], the topology on $|[G \setminus X]|$ is the quotient topology induced by the map $|p| : |X| \to |[G \setminus X]|$. Let U be a subset of $\Gamma \setminus \Theta$ such that $V = |p|^{-1}(\theta(U))$ is open in X. There is an induced G-action on V, as it is stable under the action on X by definition. Let S be the set of closed G orbits in V. By II 3.3 in [14], $S \neq \emptyset$ and U is the union of all sets of the form $\{\xi \in \Gamma \setminus \Theta | \xi \geq s\}$ for $s \in S$. Hence U is open.

On the other hand, if $U \subset \Gamma \setminus \Theta$ is open, then $V := |p|^{-1}(\theta(U))$ is the union of finitely many locally closed subsets, i.e. it is constructible. As U is open, it is

[12] in particular it is separated, flat and of finite presentation over K, see 39.8.2 in [10]

[13] Otherwise, there is a $y \in \overline{\mathcal{O}}$ such that $g.y \in U = X \setminus \overline{\mathcal{O}}$. Then $g^{-1}U \cap g^{-1}\mathcal{O} = \emptyset$ as $U \cap \mathcal{O} = \emptyset$. However, $g^{-1}\mathcal{O} = \mathcal{O}$ is dense in $\overline{\mathcal{O}}$ and $g^{-1}U \cap \overline{\mathcal{O}} \neq \emptyset$ as it contains y and it is open in $\overline{\mathcal{O}}$, by the definition of the subspace topology. Hence, $g^{-1}U \cap g^{-1}\mathcal{O} \neq \emptyset$, contradiction

[14] By pulling back $\sigma : \mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\overline{K})$ by the structure morphism of $X_{\overline{K}}$ for all $\sigma \in \Gamma$.

a union of sets of the form $\{\xi \in \Gamma \setminus \Theta | \xi \geq s\}$, and therefore V is stable under generization, and therefore it is open.[15] So θ is a homeomorphism. $\square$

2.3.2 Stratification induced by an action

Using the above, we can get a nice decomposition of $[G \setminus X]$ into disjoint locally closed reduced substacks. Namely, if Y is an algebraic G-orbit of X, then $[G \setminus Y]$ is a locally closed reduced substack of $[G \setminus X]$. Thus we get a jointly surjective family of substacks $[G \setminus Y] \hookrightarrow [G \setminus X]$, where Y ranges over $\Gamma \setminus \Theta$ such that for any scheme S and morphism $S \to [G \setminus X]$, S decomposes into disjoint locally closed subschemes $S \times_{[G \setminus X]} [G \setminus Y]$.

If we assume that K is perfect, with X, Y, G as above, then $Y_{\overline{K}}$ is reduced. Pick any $y \in Y(\overline{K})$, as $Y_{\overline{K}}$ is reduced, it is the disjoint union of Γ-conjugates of $G.y$. Furthermore, $G.y$ is smooth over $\overline{K}$ as it is a reduced orbit by theorem 4.3.7 in [11]. Hence $Y_{\overline{K}}$ is smooth as well. Thus $[G \setminus Y]$ is a smooth algebraic stack over K[16]. As $X \to [G \setminus X]$ is smooth too, it preserves codimension and

$$\operatorname{codim}([G \setminus Y], [G \setminus X]) = \operatorname{codim}(Y, X) = \operatorname{codim}(G.y, X_{\overline{K}}).$$

2.3.3 Automorphisms

Let again G be a group scheme over K and X a scheme over K with G acting on the left on X. Let S be a scheme over K, $x \in X(S)$ and $\bar{x} \in [G \setminus X]$ the image of x under the natural projection $p : X \to [G \setminus X]$. Define $\underline{\operatorname{Aut}}(\bar{x})$ as the sheaf of groups on the category of schemes over S with

$$\underline{\operatorname{Aut}}(\bar{x})(S') = \operatorname{Aut}_{[G \setminus X](S')}(\bar{x}_{S'})$$

and let $\operatorname{Stab}_{G_S}(x)$ denote the closed subgroup scheme of G_S with

$$\operatorname{Stab}_{G_S}(x)(S') = \{g \in G_S(S') | g.x_{S'} = x_{S'}\}.$$

Proposition 2.3.2. *There is a natural isomorphism* $\underline{Aut}(\bar{x}) \cong Stab_{G_S}(x)$.

Proof. The S-point $\bar{x}$ of $[G \setminus X]$ is represented by the trivial G-torsor $S \times_k G$ together with the map $G_S(S') \ni g \mapsto g.x_{S'}$, as $\bar{x} = p \circ x$.

If $\phi \in \underline{\operatorname{Aut}}(\bar{x})(S')$, it is $G(S')$ equivariant, and $\phi(g) = g.\phi(e)$ for all $g \in G_S(S')$ with $\phi(e) \in G_S(S')$. But we also have that $\phi(g).x_{S'} = g.(\phi(e).x_{S'}) = g.x_{S'}$, which implies that $\phi(e)$ lies in $\operatorname{Stab}_{G_S}(x)(S')$. Hence the $\phi \mapsto \phi(e)$ defines a natural isomorphism (with inverse $g \mapsto (h \mapsto h.g)$). $\square$

2.4 Bundles associated to representations

Given an algebraic group G over K acting on a K-scheme X and a n-dimensional algebraic representation over K $\rho : G \to \mathrm{GL}_{n,K}$ of G, there is an associated

[15] Constructible and stable under generization implies open by 5.19.10 in [10], as finite T_0 spaces are Noetherian and sober.

[16] see definition 4.14 in [13] for defining properties of stacks

vectorbundle $\mathscr{V}(\rho)$ on the quotient stack $[G \setminus X]$. As a sheaf it is defined by

$$\mathscr{V}(\rho)(s : U \to [G \setminus X]) := \{\, U \times_{[G\setminus X],s,\pi} X \xrightarrow{f} \mathbb{A}^n \,\big|\, f(g.x) = \rho(g)f(x)$$
$$\text{for all } x \in U \times_{[G\setminus X],s,\pi} X(A) \text{ all } g \in G(A) \text{ and all affine schemes } A\}$$

for any algebraic stack $s : U \to [G \setminus X]$, so in particular, the global sections are given by

$$H^0([X \setminus G], \mathscr{V}(\rho)) = \{f : X \to \mathbb{A}^1 \big| f(g.x) = \rho(g)f(x),$$
$$\forall g \in G(A),\ x \in X(A) \text{ and all } A\}.$$

This is the associated sheaf construction[17] adapted to the situation when the quotient, $[G \setminus X]$, isn't necessarily a scheme. Namely, if $s : U \to X/G$ is an open immersion of schemes, then $U \times_{X/G,s,\pi} X = \pi^{-1}(U)$.

2.5 Maps between quotient stacks

One way to produce representable maps between quotient stacks is by giving maps between the schemes involved in the definition of the stacks. Namely, let G, H be group schemes acting on schemes X, Y respectively and a pair of morphisms

$$f : X \to Y$$

and

$$\alpha : G \times X \to H$$

such that

$$f(g.x) = \alpha(g, x).f(x)$$

for all $(g, x) \in G \times X$ and the cocycle condition

$$\alpha(gg', x) = \alpha(g, g'.x)\alpha(g', x)$$

is satisfied for all $g, g' \in G$ and all $x \in X$. Then (f, α) induces a morphism between the groupoids associated the actions (3.4.3 in [13]) which then induce a morphism $\tilde{f} : [G \setminus X] \to [H \setminus Y]$ (76.20.1, [10]) such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ [G \setminus X] & \xrightarrow{\tilde{f}} & [H \setminus Y] \end{array}$$

commutes.[18]

A particularly useful α is the following: Given a homomorphism $\phi : G \to H$ let $\alpha(g, x) = \phi(g)$, which trivially satisfy the cocycle condition. In these cases we will use the identification $\alpha = \phi$.

[17]See I.5.8 in [15].

[18]As above, p denotes the natural projection to the quotient.

3 The stacks of G-zips and G-zip flags

3.1 Cocharacter data and G-Zips

Let G be a reductive algebraic group over $\mathbb{F}_p$ and $\mu \in X_*(G_k)$[19] a cocharacter. Define a parabolic subgroup $P \subset G_k$ by

$$P(S) := \{g \in G(S) | \lim_{t \to 0} \mu(t^{-1}) g \mu(t^{-1})^{-1} \text{ exist}\}$$

for any k-scheme S, Q as the Frobenius pullback of the opposite parabolic[20], $L := \mathrm{Cent}_G(\mu)$ and $M := L^{(p)}$, the associated Levi subgroups of P and Q respectively.

Cocharacter data as the pair (G, μ) above form a category, with morphisms $f : (G_1, \mu_1) \to (G_2, \mu_2)$ being homomorphisms of algebraic groups $f : G_1 \to G_2$ such that $f \circ \mu_1 = \mu_2$.

Let $\varphi : G_k \to G_k$ denote the Frobenius morphism, given a cocharacter and the groups defined above, the Zip datum of type μ is the following

Definition 3.1.1 (Zip-data)**.** *The Frobenius Zip datum associated to (G, μ) is the tuple $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$*

Letting U and V be the unipotent radicals of P and Q respectively, we get through the Levi projections that $P^{(p)}/U^{(p)} \cong L^{(p)} = M \cong Q/V$.

Fix a G-Zip datum and let U, V denote the unipotent radicals of P and Q respectively.

Definition 3.1.2. *A G-zip of type μ over a k-scheme S is a tuple $\underline{I} = (I, I_P, I_Q, \iota)$ of torsors over S. Here I is a right G_k-torsor, $I_P \subset I$ a P-torsor, $I_Q \subset I$ a Q-torsor, and $\iota : (I_P)^{(p)}/U^{(p)} \to I_Q/V$ an isomorphism of M-torsors.*

The category, $G\text{-Zip}^\mu$ of G-zips over S, with morphisms being compatible isomorphisms of torsors is a groupoid, and hence this gives rise to a category fibered in groupoids, $G\text{-Zip}^\mu$ over the category of k-schemes. It is in fact a smooth algebraic stack of dimension 0, which we will see below.

If $T \subset B \subset P$ are a maximal torus and a Borel subgroup, we define:

Definition 3.1.3. *A G-zip flag of type μ is a tuple $(\underline{I}, J)$ where $\underline{J}$ is a G-zip of type μ and $J \subset I_P$ is a B-torsor.*

The terminology of Zip flags can be motivated in the following way, suppose $G = \mathrm{GL}(n)$ and $P \subset G$ a parabolic given as the stabiliser of some flag $V_0 \subset \ldots \subset k^n$, which is not full. Then, fixing a Borel subgroup contained in P corresponds to refining the flag that P is the stabiliser of to a full flag, which has B as its stabilizer.

Just like G-zips form an algebraic stack over k-schemes, so does the category of G-zip flags, which we denote $G\text{-ZipFlag}^\mu$.

[19] Recall that k is an algebraic closure of $\mathbb{F}_p$.
[20] The opposite parabolic is defined by the existence of $\lim_{t \to 0} \mu(t) g \mu(t)^{-1}$.

3.2 Quotient stack description of G-Zip$^\mu$

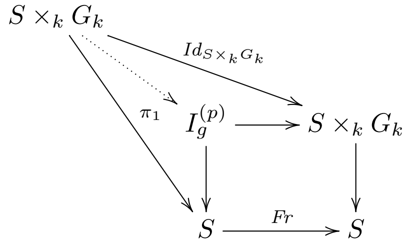
The first step to show that we have a quotient stack is to construct a local model of G-Zips, allowing us to work with them more concretely.

Construction 3.2.1. *Let $S \in Sch_{/Spec\ k}$. Given $g \in G(S)$, the following defines a G-zip $\underline{I}_g = (I_g, I_{g,-}, I_{g,+}, \iota_g)$ of type μ: Let*

$$I_g := S \times_k G_k$$

$$I_{g,-} := S \times_k P \subset I_g$$

which are trivial torsors for G_k and P respectively. The Frobenius morphism on S induces the commutative diagram



which shows that $I_g^{(p)} \cong S \times_k G_k$ canonically. Using this, define $I_{g,+}$ as the image of $S \times_k Q \subset S \times_k G_{\bar{k}}$ under left multiplication by g. Left multiplication by g then induces an isomorphism of M-torsors

$$\iota_g : I_{g,-}^{(p)}/U^{(p)} = S \times_k P^{(p)}/U^{(p)} \cong S \times_k Q/V \to g(S \times_k Q)/V = I_{g,+}/V$$

so $\underline{I}_g = (I_g, I_{g,-}, I_{g,+}, \iota_g)$ is indeed a G-zip of type μ.

Similarly for G-Zip flags

Definition 3.2.1. *The standard zip-flag of type μ associated to $(g, r) \in (G \times P)(S)$ is a tuple $\underline{I}_{g,r} = (\underline{I}_g, J_{(g,r)})$ where $\underline{I}_g$ is the standard G-zip of type μ associated to $g \in G(S)$ and $J_{(g,r)}$ is the image of $B \times S \subset P \times S$ under left multiplication by r.*

These are indeed local models of G-Zips in the sense of the following lemma.

Lemma 3.2.1. *Every G-zip of type μ is étale locally isomorphic to a standard G-zip $\underline{I}_g$.*

Proof. Let $\underline{I} = (I, I_-, I_+, \iota)$ be a G-zip of type μ. We first note that taking fpqc-local trivialisations of I, I is fqpc locally affine and faithfully flat over S (as $G \times_k S$ is), and hence I is affine and faithfully flat, by faithfully flat descent[21]. As k is perfect, G is smooth over k, by [10] 39.8.4. Hence any G-torsor I is fpqc-locally smooth, therefore I is smooth over S by 35.20.27 in [10]. By Corrollarie 17.16.3 in [16], there is a surjective étale morphism $S' \to S$ such that $I(S) \neq \emptyset$, and not only a fpqc one. Similarly for I_- and I_+.

Hence, after replacing S by an étale covering, we have sections $i_\pm \in I_\pm$. These induces sections i_+V and $\iota(i_-^{(p)}U^{(p)})$ in $I_+/V(S)$. As I_+/V is a M-torsor, there is

[21] Combining 35.20.18, 36.20.15 and 30.20.7 in [10]

a unique section $l \in M(S)$ such that $i_+ V.l = \iota(i_-^{(p)} U^{(p)})$. Hence we may replace i_+ with $i_+ l$, and assume that i_- and i_+ induce the same section of I_+/V. By composing with the embeddings $I_\pm \subset I$, i_- and i_+ induce two sections of I. As I is a G-torsor, there is a unique $g \in G(S)$ such that $i_+ = g.i_-$.

Now, we have a trivialisation of I and I_- over S in i_-. As these trivialisations yields $I \cong I_g$, and $I_- \cong I_{g,-}$ we may assume that $I_- = I_{g,-} \subset I = I_g$ and that i_- is the identity section. Then $i_+ = i_- g = g \in I_g(S)$, so $I_+ = g(P^{(p)}) = I_{g,+}$. Since the M-equivariant isomorphism ι satisfies $\iota(U^{(p)}) = \iota(i_-^{(p)} U^{(p)}) = i_+ V = gV$, it follows that $\iota = \iota_g$. So $\underline{I}_g = \underline{I}$. □

A similar proof, given in [1], shows

Lemma 3.2.2. *Every G-zip flag of type μ is étale locally isomorphic to a standard G-zip flag $\underline{I}_{g,r}$.*

For $a \in P$ and $b \in Q$, let $\overline{a} \in L$ and $\overline{b} \in M$ denote their image under the Levi projection.

Definition 3.2.2. *The zip group, E[22], of (G, μ) is defined by*

$$E(S) = \{(a, b) \in (P \times Q)(S) | \varphi(\overline{a}) = \overline{b}\}$$

for all k-schemes S.

Let E act on G by

$$(a, b).g = agb^{-1}$$

for $(a, b) \in E$ and $g \in G$.

Lemma 3.2.3. *For all pairs of sections $g, g' \in G(S)$ there is a natural bijection between the transporter*

$$\mathit{Transp}_{E(S)}(g, g') := \{(a, b) \in E(S) \big| agb^{-1} = g'\}$$

and the set of morphisms of G-zips $\underline{I}_g \to \underline{I}_{g'}$ over S, in which $(a, b) \in E(S)$ corresponds to the morphism of G-Zips given by $I_g \to I_{g'}$ and $I_{g,-} \to I_{g,-}$ defined by left multiplication with a, and $I_{g,+} \to I_{g',+}$ defined by left multiplication with $g'bg^{-1}$.

Proof. A morphism $\underline{f} : \underline{I}_g \to \underline{I}_{g'}$ is given by equivariant isomorphisms $f : I_g \to I_{g'}$ and $f_\pm : I_{g,\pm} \to I_{g',\pm}$. By definition, $I_{g,-} = S \times_k P = I_{g',-}$, so f_- is given by left multiplication by a unique $a \in P(S)$. Together with $I_g = S \times_k G = I_{g'}$, this implies that f is given by left multiplication with a as well, since $i' \circ f_- = f \circ i$. Here i and i' are the respective inclusions $I_{g,-} \subset I_g$ and $I_{g',-} \subset I_{g'}$.

Similarly, we have that $I_{g,+} = g(S \times_k Q)$ and $I_{g',+} = g'(S \times_k Q)$, so f_+ is given by left multiplication with $g'bg^{-1}$ for a unique $b \in Q(S)$. As $j' \circ f_+ = f \circ j$, with j and j' the respective inclusions $I_{g,+} \subset I_g$ and $I_{g',+} \subset I_{g'}$, we must have

[22] E_μ if we want to emphasize which cocharacter it is associated to.

$g'bg^{-1} = a$. These isomorphisms also need to be compatible with ι_g and $\iota_{g'}$, that is the diagram

$$\begin{array}{ccc} I^{(p)}_{g,-}/U^{(p)} & \xrightarrow{a} & I^{(p)}_{g',-}/U^{(p)} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ I_{g,+}/V & \xrightarrow{g'bg^{-1}} & I_{g,+}/V \end{array}$$

is commutative, where an arrow labeled h is left multiplication with $h \in G(S)$. In other words $g'\varphi(\bar{a}) = g'\bar{b} \in M$, equivalently $\varphi(\bar{a}) = \bar{b}$, so $(a, b) \in E(S)$. Thus $(a, b) \in \mathrm{Transp}_{E(S)}(g, g')$. Conversely, all the compatibilities above are satisfied for any $(a, b) \in \mathrm{Transp}_{E(S)}(g, g')$, which proves the lemma. □

We are now ready to prove that G-Zip$^\mu$ is a quotient stack.

Proposition 3.2.1. *The fibered category G-Zip$^\mu$ is equivalent to $[E_\mu \setminus G] = [E \setminus G]$. In particular, the isomorphism classes of G-Zips of type μ over an algebraically closed field K are in bijection with the $E(K)$ orbits in G.*

Proof. Let $\mathcal{X}$ denote the category fibered in groupoids with objects of $\mathcal{X}(S)$ being elements of $G(S)$, and morphisms between g and g' given by $\mathrm{Transp}_{E(S)}(g, g')$ where composition is given by multiplication in $E(S)$. Given a morphism $f : S' \to S$ over k, the pullback in $\mathcal{X}$ is defined by $G(f) : G(S) \to G(S')$ for objects and $E(f) : E(S) \to E(S')$ for morphisms. As E is a scheme, it is in particular a sheaf, so $\mathcal{X}$ satisfies effective descent for morphisms and $\mathcal{X}$ is a pre-stack. The stackification [23] of $\mathcal{X}$ is $[E \setminus G]$ by 3.4.3 in [13].

A diagram chase shows that the bijection $\mathrm{Transp}_{E(S)}(g, g') \leftrightarrow \mathrm{Hom}_{G\text{-Zip}^\mu}(\underline{I}_g, \underline{I}_{g'})$ given in the lemma above is compatible with pullbacks. As composition on both sides is given by multiplication in E, the bijection is compatible with composition as well and $1 \in \mathrm{Transp}_E(g, g)$ corresponds to $1_{\underline{I}_g} \in \mathrm{Hom}_{G-Zip}(I_g, I_g)$.

Hence we can define a morphism of fibered categories $\mathcal{X} \to G$-Zip$^\mu$ mapping $g \in \mathcal{X}(S) = G(S)$ to $\underline{I}_g$ acting by the correspondence $\mathrm{Transp}_{E(S)}(g, g') \leftrightarrow \mathrm{Hom}_{\text{-Zip}^\mu}(\underline{I}_g, \underline{I}_{g'})$ on morphisms. Since the action on morphisms is a bijection, this morphism is fully faithfull on fibers. We saw above that every G-zip of type μ étale locally is isomorphic to one of the form $\mathcal{I}_g$. This implies that the induced map morphism from the stackification of $\mathcal{X}$, $[E \setminus G] \to G$-Zip$^\mu$ is essentially surjective on fibers, hence an equivalence, and by Proposition 3.1.10 in [17], it is an equivalence of fibered categories, so the proposition follows.

The second part follows from the earlier discussion of the underlying topological space of an algebraic stack. □

Corollary 3.2.1. *The stack G-Zip$^\mu$ is a smooth algebraic stack of dimension 0.*

Proof. The quotient stack $[E \setminus G]$ is algebraic by Proposition 10.13.1 in [13]. It is smooth as k is perfect so the action of E and its orbits are smooth. We have $\dim E = \dim G$, which can be seen by the following way, the subgroups

[23]See 3.2 in [13] or 8.8 in [10]

$U_\alpha \times \{e\}$, $\{e\} \times U_\beta$ and $\{(l, \varphi(l)) | l \in L\}$ where α ranges over roots of P which are not roots of Q and the other way around for β are disjoint and generate E. Similarly U_α, U_β and L generate G by proposition 8.1.1 in [11].

The natural projection $G \to [E_{\mathcal{Z}} \setminus G]$ is an E-torsor, so given a geometric point $x : \operatorname{Spec} K \to [E \setminus G]$, the fiber G_x is isomorphic to E. Hence the relative dimension of the projection is $\dim E = \dim G$ and thus $\dim[E \setminus G]_x = \dim G - \dim G = 0$ for all geometric points x[24]. By the proposition above, the corollary follows. □

3.2.1 Conjugation

Given two Frobenius zip data

$$\mathcal{Z} = (G, P, Q, L, M, \varphi)$$

and

$$\mathcal{Z}_g = (G, gPg^{-1}, \varphi(g)Q\varphi(g^{-1}), gLg^{-1}, \varphi(g)M\varphi(g^{-1}), \varphi)$$

where $g \in G(k)$, we can define a morphism $[E_{\mathcal{Z}} \setminus G] \to [E_{\mathcal{Z}_g} \setminus G]$ by the maps $f : G \to G$ defined by $x \mapsto gx\varphi(g)^{-1}$ and $\alpha : E_{\mathcal{Z}} \to E_{\mathcal{Z}_g}$ defined by $(a, b) \mapsto (gag^{-1}, \varphi(g)b\varphi(g)^{-1})$ which is an isomorphism, as both f and α are isomorphisms. We conclude that conjugate Zip data yields isomorphic stacks. This together with the corollary of Langs theorem that every smooth, connected algebraic group over a finite field is quasi-split justifies the following

Assumption 3.2.1. *From now on we assume that $B \subset P$ is a Borel subgroup of G, defined over $\mathbb{F}_p$ containing T.*

3.2.2 The stack of Zip flags as a quotient

Let $E \times B$ act on $G \times P$ by

$$((a, b), c).(g, r) = (agb^{-1}, arc^{-1})$$

for $((a, b), c) \in E \times B$ and $(g, r) \in G \times P$.

A very similar argument to the one for G-Zips yields

Proposition 3.2.2. *The the fibered category G-ZipFlag$^\mu$ is equivalent to $[(E \times B) \setminus (G \times P)]$.*

There is a natural morphism $\pi : G\text{-ZipFlag}^\mu \to G\text{-Zip}^\mu$ given by forgetting the B-torsor. We also have $\pi_{\text{quot}}[(E \times B) \setminus (G \times P)] \to [E \setminus G]$ defined by the projections onto the first coordinates in $E \times B$ and $G \times P$. These fit into the commutative diagram

$$\begin{array}{ccc} G\text{-ZipFlag}^\mu & \xrightarrow{\pi} & G\text{-Zip}^\mu \\ \cong \downarrow & & \cong \downarrow \\ [(E_{(G,\mu)} \times B) \setminus (G \times P)] & \xrightarrow{\pi_{\text{quot}}} & [E_{(G,\mu)}] \end{array}$$

So we may work solely with the quotient stack description and from now on we use the identification $\pi_{quot} = \pi$.

[24] See 98.12 in [10] for a definition and discussion of dimension of algebraic stacks.

3.2.3 Alternate quotient descriptions of G-ZipFlag$^\mu$

Define the Weyl group element $z := w_0 w_{0,J} \in W$, where J is the type of the parabolic Q.

Lemma 3.2.4.

$$^zB \subset Q$$

and

$$\varphi(B \cap L) = B \cap M =^z B \cap M.$$

Proof. By the discussion in section 1.4

$$\begin{aligned} ^zB &= \langle -z.\Delta\rangle = \langle -z.(J \cup (\Delta \setminus J))\rangle \\ &= \langle -w_0(w_{0,J}J) \cup -w_0 w_{0,J}(\Delta \setminus J))\rangle \subset \langle -w_0 J \cup \Delta\rangle \\ &= \langle -\sigma I \cup \Delta\rangle \subset Q \end{aligned}$$

proving the first statement.

For the second statement, recall that the type of a parabolic not containing our chosen Borel subgroup B, such as P^{op}, is the type of $gP^{\mathrm{op}}g^{-1}$ containing B for some $g \in G_k$[25]. The roots of P^{op} are the negatives of the roots of P, due to the limits defining these groups. So one way to get $B \subset gP^{\mathrm{op}}g^{-1}$ is to choose $g = \dot{w_0}$, which takes positive roots to negative ones and vice versa. Let S be the set of roots of P, then $-S$ is the roots of P^{op} and $-w_0 S$ the roots of $w_0 P^{\mathrm{op}} w_0^{-1}$. Hence type$(P^{\mathrm{op}}) = -w_0 S \cap \Delta$ [26]. As $-w_0\Delta = \Delta$, type$(P^{\mathrm{op}}) = -w_0(S \cap \Delta)$ and $S \cap \Delta = $ type(P) we have type$(P^{\mathrm{op}}) = -w_0.$type$(P)$.

Given a subset $S \subset \Phi$ of roots let $\langle S\rangle$ denote the subgroup of G generated by our maximal torus T and the root groups U_α.

With our conventions, $B = \langle -\Delta\rangle$, $P = \langle -\Delta, I\rangle$, $L = \langle \pm I\rangle$. We get

$$Q = (P^{\mathrm{op}})^{(p)} = \langle \Delta, -w_0.I\rangle^{(p)} = \langle \sigma\Delta, -\sigma w_0 I\rangle = \langle \sigma\Delta, -w_0\sigma I\rangle,$$

where the last equality follows from $^\sigma(w\lambda) =^\sigma w^\sigma\lambda$ and w_0 being fixed by any Galois action as is the unique element of W of its length. As B is assumed split, we get that Δ is Galois invariant. So $Q = \langle \Delta, -w_0\sigma I\rangle$. We have, using $\Delta = J \cup (\Delta \setminus J)$[27] and $w_{0,J}J = -J$ that

$$\begin{aligned} ^zB &= \langle -z.\Delta\rangle = \langle -z.(J \cup (\Delta \setminus J))\rangle = \langle -z.J \cup -z.(\Delta \setminus J))\rangle \\ &= \langle w_0.J \cup -z.(\Delta \setminus J))\rangle = \langle -\sigma.I \cup -z.(\Delta \setminus J))\rangle. \end{aligned}$$

As

$$M = \langle \pm\sigma I\rangle$$

we get

$$^zB \cap M = \langle -\sigma I\rangle$$

since if $-z.\alpha \in \pm\sigma I$ for $\alpha \in \Delta \setminus J$, $w_0 z.\alpha = w_{0,J}\alpha \in \pm - w_0\sigma I = \pm J$. This is a contradiction as $w_0 z.\alpha = w_0^2 w_{0,J}.\alpha = w_{0,J}.\alpha$ which lie in $\pm J$ if and only if α does. As B is split $\varphi(B \cap L) = B \cap M = \langle -\sigma I\rangle$, proving the second statement. □

[25] The definition is independent of which g we choose.
[26] Recall our convention that Δ is the set of roots with $U_{-\alpha} \subset B$.
[27] Here we make an exception to our convention and use \ for set difference to avoid confusion

Let $E' = E \cap (B \times G) = \{(a,b) \in E \big| a \in B\}$. Then we claim that $E \subset B \times^z B$. The unipotent radical V of Q is generated by U_α where α ranges over positive roots with respect to ${}^zB$ which are not roots of M[28] since ${}^zB \subset Q$ is a Borel subgroup, so the preimage of $\varphi(l)$ under the Levi projection $Q \to M$ is $\varphi(l)V$ where $V \subset^z B$, and as $\varphi(B \cap L) =^z B \cap M$ the lemma above, $\varphi(l)V \subset^z B$ if $l \in B \cap L$ which proves the claim.

Lemma 3.2.5. *The morphism $[E' \setminus G] \to [(E \times B) \setminus (G \times P)]$ induced by the maps $E' \to E \times B$ defined by $(a,b) \mapsto ((a,b),a)$ and $G \to G \times P$ defined by $g \mapsto (g,e)$ is an isomorphism of stacks.*

Proof. By the universal property of stackifickation, and 3.4.3 in [13], it is enough to show that the quotient prestacks of the respective actions are equivalent, we denote these as well (abusing notation) by $[H \setminus X]$ for a group H acting on a scheme X.

By Proposition 3.1.10 in [17],it is enough to show that the categories $[E' \setminus G](S)$ and $[(E \times B) \setminus (G \times P)](S)$ are equivalent for any k-scheme S. These categories have objects $G(S)$ and $G \times P(S)$ respectively and morphisms

$$\mathrm{Transp}_{E'(S)}(g, g')$$

and

$$\mathrm{Transp}_{(E \times B)(S)}((g,p),(g',p'))$$

respectively.

Two objects in the image of this map will be pairs $(g,e),(g',e)$ and by definition

$\mathrm{Transp}_{(E\times B)(S)}((g,e),(g',e)) = \{((a,b),c) \in E \times B \big| agb^{-1} = g' \text{ and } aec^{-1} = e\}$.

This implies that $a = c \in B(S)$, so $(a,b) \in E \cap B \times E = E'$ and $((a,b),c) = ((a,b),a)$ is the image of $(a,b) \in \mathrm{Transp}_{E'(S)}(g,g')$, so the map is fully faithful. For any $(g,p) \in (G \times P)(S)$, $((p^{-1}, \varphi(\bar{p}^{-1})),1) \in (E \times B)(S)$ and

$$((p^{-1}, \varphi(\bar{p}^{-1})),1).(g,p) = (p^{-1} g \varphi(\bar{p}), 1) \in (G \times 1)(S).$$

So the map is also essentially surjective, which proves that the two stacks are isomorphic.

□

Another useful way to describe the stack of Zip-flags is as

$$[E \setminus (G \times P/B)]$$

where E acts on $G \times P/B$ by

$$(a,b).(g, rB) := (agb^{-1}, arB)$$

for all $(a,b) \in E$ and $(g,rB) \in G \times P/B$. Under this description, the morphism $G\text{-ZipFlag}^\mu \to G\text{-Zip}^\mu$ is given by mapping an object $f : I \to G \times P/B$ of $[E \setminus (G \times P/B)](S)$ to $\pi_1 \circ f$ where π_1 is projection to the first coordinate. Thus the fiber of this morphism is P/B, so the stack of ZipFlags can be viewed as P/B-bundle over the stack of Zips. [29]

[28] This follows by proposition 8.4.3(ii) in [11]

[29] An equivariant morphism into P/B is determined by where it maps one point, that is given $s \in I(S)$, f is determined by $f(s)$ which is equivalent to choosing a section of $P/B(S)$

3.3 The flag stratification of G-ZipFlag$^\mu$

3.3.1 The Schubert stack

Let $B \times B$ act on G by
$$(a,b).g = agb^{-1}$$
for $(a,b) \in B$ and $g \in G$.

Definition 3.3.1. *The Schubert stack of G is defined by*
$$Sbt := [(B \times B) \setminus G].$$

Its points corresponds to Schubert cells of G/B, i.e. to the Bruhat decomposition of $G(k)$[30], which in particular implies that the number of points is finite. This implies that $|\mathrm{Sbt}| \cong W = W(G_k, T_k)$ where the topology on the Weyl group is the one induced by the Bruhat order[31], see section 2.3.1. Hence, by section 2.3.2, Sbt carries a stratification indexed by W. The strata and their closures are the substacks
$$\mathrm{Sbt}_w := [(B \times B) \setminus C_w]$$
and
$$\overline{\mathrm{Sbt}_w} := [(B \times B) \setminus \overline{C_w}]$$
where $C_w = B\dot{w}B$, and $\dot{w} \in G$ is a representative of $w \in W$.

3.4 The flag stratification

Define $\beta : [E' \setminus G] \to [B \times^z B \setminus G]$ as the morphism induced by the inclusion of E' into $B \times^z B$ and the identity on G. Let $\alpha : [(B \times^z B) \setminus G]$ be the isomorphism induced by the map $G \to G$ defined by $g \mapsto g\dot{z}$ and the map $B \times^z B \to B \times B$ defined by $(a,b) \mapsto (a, \dot{z}^{-1}b\dot{z})$. Composing these two gives us a smooth morphism
$$\psi := \alpha_z \circ \beta : G\text{-ZipFlag}^\mu \to \mathrm{Sbt}.$$

Using this we define the flag stratification on G-ZipFlag$^\mu$. Given $w \in W$ let $\mathcal{Y}_w := \psi^{-1}(\mathrm{Sbt}_w)$, and $\overline{\mathcal{Y}}_w := \psi^{-1}(\overline{\mathrm{Sbt}_w})$ endowed with the reduced structure. We call these the flag strata and closed flag strata associated to w respectively. As ψ is smooth, the flag strata $\mathcal{Y}_w$ are smooth and locally closed.

Let $\tilde{\psi} : G \times P \to G$ be defined by $(g,h) \mapsto h^{-1}g\varphi(\bar{h})\dot{z}$ and define for $w \in W$ the locally closed subvariety of $G \times P/B$
$$H_w := \{(g, hB) \in G \times P/B \big| \tilde{\psi}(g,h) \in C_w\}.$$

The action of E on $G \times P/B$ restricts to an action on H_w[32]. We can identify $\mathcal{Y}_w = [E \setminus H_w]$.

[30] See chapter 8.3 in [11]

[31] See [11], in particular $a \leq b$ implies $l(a) \leq l(b)$

[32] Essentially as raising the entries of a matrix to the p'th power commutes with matrix multiplication over $\mathbb{F}_p$ together with the defining relation for E and theorem 8.4.3 in [11]

Lemma 3.4.1. *1. The closed flag strata are normal and irreducible.*

2. The closed flag strata coincide with the closures of the flag strata.

3. For all $w \in W$, one has $\dim(H_w) = l(w) + \dim(P)$.

Proof. As $\overline{C}_w$ is normal and ψ is smooth, $\overline{\mathcal{Y}}_w$ is normal by lemma 10.164.3 in [10]. The morphism $\tilde{\psi} : G \times P \to G$ is smooth with fibers isomorphic to P, in particular its fibers are irreducible. As C_W is irreducible and $\tilde{\psi}$ is open with irreducible fibers, if $\tilde{\psi}^{-1}(C_w)$ wasn't irreducible, $\tilde{\psi}^{-1}(C_w) = U \cup V$ for some distinct non-empty open subsets, then the same holds for C_w, so $\tilde{\psi}^{-1}(C_w) = H_w$ is irreducible. Hence $[E \setminus H_w]$ is irreducible as well. The closure the flag strata coincide with the closed flag strata, as ψ is smooth. The last statement follows from ψ being smooth of relative dimension $\dim P/B = \dim P - \dim B$ and $\dim C_w = l(w) + \dim B$ by lemma 8.3.6 in [11] so $\dim H_w = l(w) + \dim P$. $\square$

3.5 The automorphic vector bundles

Given a character $\lambda : T \to \mathbb{G}_m$, we get a P-equivariant line bundle on P/B by 8.5.7 in [11]. Pulling back the global sections of this bundle via the projection $E \to P$ yields an $E-$module. Using the construction of bundles on quotient stacks in section 2.4 yields a vector bundle $\mathscr{V}(\lambda)$ on $[E \setminus G] \cong G\text{-Zip}^{\mu}$. Similarly, the first projection $E' \to B$ yields a line bundle $\mathscr{L}(\lambda)$ on $[E' \setminus G] \cong G\text{-ZipFlag}^{\mu}$.

Given a point $x : \text{Spec } K \to [E \setminus G]$,

$$H^0(\text{Spec } K, x^*\mathscr{V}(\lambda)_{[E \setminus G]}) = H^0(P/B, \mathscr{L}_{P/B}(\lambda))$$

and as the fiber of x in $[E' \setminus G]$ is P/B,

$$H^0(\text{Spec } K \times_{[E \setminus G]} [E' \setminus G], \mathscr{L}_{[E' \setminus G]}(\lambda) = H^0(P/B, \mathscr{L}_{P/B}(\lambda))$$

as well. Hence we have the following direct image formula

$$(\pi_{Y/X})_* \mathscr{L}(\lambda) = \mathscr{V}(\lambda).$$

3.6 G-Zips of Hodge type

Let (W, ψ) be a non-degenerate symplectic space over $\mathbb{F}_p$ and $\text{GSp}(W, \psi)$ the associated group of symplectic similitudes i.e.

$$\text{GSp}(W, \psi)(A) = \{g \in \text{GL}(W)(A) \big| \psi(gv, gw) = \lambda \psi(v, w) \text{ for some } \lambda \in \mathbb{G}_m(A)\}$$

for all $v, w \in W \otimes_{\mathbb{F}_p} A$. We have the following decomposition

$$\mathcal{D} : W \otimes k = W_+ \oplus W_-$$

where W_+ and W_- are maximal isotropic subspaces.[33]

Define $\mu_{\mathcal{D}} : \mathbb{G}_{m,k} \to \text{GSp}(W, \psi)$ by letting $x \in \mathbb{G}_{m,k}$ act trivially on W_- and by scalar multiplication by x on W_+. Then $(\text{GSp}(W, \psi), \mu_{\mathcal{D}})$ is a cocharacter

[33] This means that they are maximal with respect to the property $\psi(x, y) = 0$ for all $x, y \in W_+$ and $x, y \in W_-$ respectively.

datum. Cocharacter datum arising in this way are called Siegel-type cocharacter data. Given a Siegel-type cocharacter datum $\mathrm{GSp}(W, \psi)$, we define the groups

$$P^+_{\mathcal{D}} := \mathrm{Stab}_{\mathrm{GSp}(W,\psi)}(W_+), \qquad P_{\mathcal{D}} = P^-_{\mathcal{D}} := \mathrm{Stab}_{\mathrm{GSp}(W,\psi)}(W_-),$$

$$Q_{\mathcal{D}} := \left(P^+_{\mathcal{D}}\right)^{(p)}, \qquad L_{\mathcal{D}} := P^+_{\mathcal{D}} \cap P^-_{\mathcal{D}}, \qquad M_{\mathcal{D}} := \left(L_{\mathcal{D}}\right)^{(p)}.$$

These together with the Frobenius morphism gives us a zip datum. As $\mathrm{GSp}(W, \psi)$ acts transitively on the set of decompositions like $\mathcal{D}$ above,[34], all cocharacters defined from such decompositions are conjugate, and thus the stacks of $\mathrm{GSp}(W, \psi)$-zips we get from them are all isomorphic.

The adjoint group, that is $G/Z(G)$, of $\mathrm{GSp}(W, \psi) \cong \mathrm{GSp}(2g)$ for some $g \geq 1$, is isomorphic to $\mathrm{PGSp}(2g)$, and the composition of $\mu_{\mathcal{D}}$ with the natural projection yields a miniscule cocharacter μ_g of $\mathrm{PGSp}(2g)$[35]. Similarly as for $\mathrm{GSp}(W, \psi)$, we can associate a Zip datum.

Definition 3.6.1. *Let (G, μ) be a cocharacter datum.*

1. *We say that (G, μ) is of Hodge-type if there is a Siegel-type cocharacter datum $(GSp(W, \psi), \mu_{\mathcal{D}})$ and an embedding $\iota : (G, \mu) \to (GSp(W, \psi), \mu_{\mathcal{D}}$*
2. *We say that (G, μ) is of connected Hodge-type if there is a $g \in \mathbb{Z}_{\geq 1}$ and a morphism of cocharacter data $\rho : (G, \mu) \to (PGSp(2g), \mu_g)$ such that $\rho : G \to PGSp(2g)$ has central kernel.*

Hodge-type implies connected Hodge type, as we can take $\rho = \pi \circ \iota$ where π denotes the natural projection $\mathrm{GSp}(2, g) \to \mathrm{PGSp}(2g)$. This morphism has central kernel as ι is an embedding and π is defined by having the center of $\mathrm{GSp}(2g)$ as kernel. Working with the more general notion of connected Hodge type in the context of the cone conjecture can be motivated, for example, by the change of center lemma for the global sections cones (see below). A zip datum $\mathcal{Z}$ is of Hodge type or connected Hodge type if there is a cocharacter datum (G, μ) of Hodge type or connected Hodge type such that $\mathcal{Z}$ is isomorphic to the G-zip datum associated to (G, μ).

4 The Cone Conjecture

Throughout this chapter $\mathcal{X} = G\text{-Zip}^{\mathcal{Z}}$ and $\mathcal{Y} = G\text{-ZipFlag}^{\mathcal{Z}}$ for some Zip datum $\mathcal{Z}$.

A G-zip scheme is a morphism

$$X \xrightarrow{\quad \zeta \quad} \mathcal{X}$$

where X is a scheme. We call the product $Y = X \times_{\mathcal{X}} \mathcal{Y}$ the flag space of X. It

[34]By conjugating the standard symplectic form $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

[35]The unique one, corresponding to the root $2e_g$ in the notation of [18]

is a P/B-bundle over X as $\mathcal{Y}$ is a P/B-bundle over $\mathcal{X}$. We have the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\zeta_{\mathcal{Y}}} & \mathcal{Y} & \xrightarrow{\psi} & \mathrm{Sbt} \\ \downarrow{\scriptstyle \pi_{Y/X}} & & \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{\zeta} & \mathcal{X} & & \end{array}$$

Define the saturated[36] global sections cones of $\mathcal{X}$ and X respectively by

$$C_{\mathcal{X}} = \{\lambda \in X^*(T) | H^0(\mathcal{X}, \mathscr{V}(N\lambda)) \neq 0 \text{ for some } N \in \mathbb{Z}_{\geq 1}\},$$

$$C_X = \{\lambda \in X^*(T) | H^0(X, \zeta^*\mathscr{V}(N\lambda)) \neq 0 \text{ for some } N \in \mathbb{Z}_{\geq 1}\}.$$

We will drop the adjective saturated in the following.

One could hope that given a sufficiently nice ζ, these two cones are the same. This is indeed what is expected. Goldring and Koskivirta put forward the following conjecture in [7]

Conjecture 4.0.1 (The Cone Conjecture)**.** *Assume that*

1. *The zip datum $\mathcal{Z}$ is of connected Hodge type.*
2. *For all connected components $X^\circ \subset X$, the map $\zeta : X^\circ \to \mathcal{X}$ is smooth and surjective*
3. *All length one strata closures in Y are pseudo-complete.*[37]

Then $C_{\mathcal{X}} = C_X$.

Fix the notation

$$\mathcal{Y}_w = \psi^{-1}(\mathrm{Sbt}_w)$$
$$\overline{\mathcal{Y}}_w = \psi^{-1}(\overline{\mathrm{Sbt}}_w)$$
$$Y_w = \zeta_{\mathcal{Y}}^{-1}(\mathcal{Y}_w)$$
$$Y_w^* = \zeta_{\mathcal{Y}}^{-1}(\overline{\mathcal{Y}}_w)$$

all endowed with the reduced structure as subschemes/substacks.

By lemma 3.4.1 above, all $\overline{\mathcal{Y}}_w$ are normal, and hence the same holds for Y_w^* if ζ satisfy the conditions of the conjecture. The following definitions are crucial in the basic strategy of proof.

Definition 4.0.1. *Let $w \in W$ and $\lambda \in X^*(T)$*

1. $C_{\mathcal{Y},w} := \{\lambda \in X^*(T) | H^0(\overline{\mathcal{Y}_w}, \mathscr{L}_{\mathcal{Y}}(N\lambda)) \neq 0$ *for some* $N \in \mathbb{Z}_{\geq 1}\}$
2. $C_{Y,w} := \{\lambda \in X^*(T) | H^0(Y_w^*, \zeta^*\mathscr{L}_{\mathcal{Y}}(N\lambda)) \neq 0$ *for some* $N \in \mathbb{Z}_{\geq 1}\}$

[36] Many of the good properties of the saturated cones are due to the possibility of taking positive multiples as you will see in the proofs below. So the unsaturated variant where we don't allow multiples are not as stable.

[37] This is a weaker assumption than in the original paper, but it is all that is needed to prove the relations of the cones in the same paper.

3. *A partial Hasse invariant of* $\mathscr{L}_{\mathcal{Y}}(\lambda)$ *on* $\overline{\mathcal{Y}_w}$ *is a section* $s \in H^0(\overline{\mathcal{Y}}_w, \mathscr{L}_{\mathcal{Y}}(\lambda))$ *which is pulled back from the Schubert stratum* $\overline{Sbt}_w$*.*
4. *The Schubert cone* $C_{Sbt,w} \subset C_{\mathcal{Y},w}$ *of* $w \in W$ *is the cone of* $\lambda \in X^*(T)$ *such that* $\mathscr{L}_{\mathcal{Y}}(n\lambda)$ *admits a partial Hasse invariant for some* $N \geq 1$*.*

4.1 Partial Hasse invariants

To start the investigation on line bundles related to characters, we notice that the presentation $G \to \mathrm{Sbt}$ factors through the flag space G/B. This allows us to use the rich classical theory of Schubert cells in our study of automorphic bundles. Specifically, we have

Proposition 4.1.1. *Let* $w \in W$*. The following is true*

1. $H^0(Sbt_w, \mathscr{L}_{Sbt}(\lambda, \mu)) \neq 0 \Leftrightarrow \mu = -w^{-1}\lambda$.
2. $\dim_k H^0(Sbt_w, \mathscr{L}_{Sbt}(\lambda, -w^{-1}\lambda)) = 1$.
3. *For any nonzero* $f \in H^0(Sbt_w, \mathscr{L}_{Sbt}(\lambda, -w^{-1}\lambda))$ *viewed as a rational function on* $\overline{C}_w$*, one has*

$$div(f) = -\sum_{\alpha \in E_w} \langle \lambda, w\alpha^\vee \rangle \overline{C}_{ws_\alpha}$$

Proof. Suppose $f \in H^0(\mathrm{Sbt}_w, \mathscr{L}_{\mathrm{Sbt}}(\lambda, \mu)) - \{0\}$. Identify f with a rational function $f : C_w \to \mathbb{A}^1$ descending to f under the presentation morphism. This has to be non-vanishing since C_w is irreducible and we assumed it descends to a nonzero section. This function satisfy the relation

$$f(bxb') = \lambda(b)\mu(b')^{-1}f(x)$$

for all $(b, b') \in B \times B$ and all $x \in C_w$. It follows that, for $t \in T$

$$f(\dot{w}t) = \mu(t)^{-1}f(\dot{w}) = \lambda(\dot{w}t\dot{w}^{-1})f(\dot{w})$$

as $\dot{w}t = \dot{w}t\dot{w}^{-1}\dot{w}$. Since this holds for all $t \in T$, we get that $\mu(t)^{-1} = \lambda(\dot{w}t\dot{w}^{-1})$, equivalently $\mu = -w^{-1}\lambda$ proving (1).

As C_w/B is an open B orbit in $\overline{C_w/B}$, which is an irreducible B-variety, (2) follows from (1) together with proposition 1.18 in the second version of [19].

Part (3) is Chevalley's formula. For the original proof, see proposition 10 in [20] or [21] for an approach using K-theory in the complex case. Note that there is a sign difference due to our convention on positive roots. Given this formula for G/B the same follows for quotient stacks by the definition of the line-bundles. Namely that global sections are the B-invariants of the corresponding global sections over G/B. □

Lemma 4.1.1. *For* $\lambda, \nu \in X^*(T)$*, we have*

$$\psi^*\mathscr{L}_{Sbt}(\lambda, \nu) = \mathscr{L}(\lambda + p^\sigma(z\nu)),$$

where $\sigma : k \to k$ *denotes the inverse of the map* $x \mapsto x^p$*.*

Proof. Recall that $\psi : \mathcal{Y} \to \mathrm{Sbt}$ was defined as the composition of the natural projection $[E' \backslash G] \to [B \times^z B \backslash G]$ follows by the isomorphism $\alpha_z : [B \times^z B \backslash G] \to [B \times B \setminus G]$. Thus $\mathscr{L}_{\mathrm{Sbt}}(\lambda, \nu)$ is given by the restriction[38] of (λ, ν) along the composition

$$E' \subset B \times^z B \cong B \times B$$

defined by $\beta : (a, b) \mapsto (a, z^{-1}bz)$ for $(a, b) \in E'$. Hence $\beta^*(\lambda, \nu)(a, b) = \lambda(a)\nu(z^{-1}bz) = \lambda(a)(z\nu)(b)$ [39]. Furthermore, since $(a, b) \in E' \subset E$, we have that $\varphi(\bar{a}) = \bar{b}$. Hence, using that ${}^\sigma\nu(t) = \sigma(\nu(\sigma^{-1}(t))) = \sigma(\nu(t^p))$ for all $t \in T$[40], we have

$$\lambda(a)(z\nu)(b) = \lambda(\bar{a})^\sigma(z\nu)^p(\bar{a})$$

which proves the lemma[41]. ☐

By the above, we get that $\psi^*\mathscr{L}_{\mathrm{Sbt}}(\lambda, -w^{-1}\lambda) = \mathscr{L}(\lambda - p^\sigma(zw^{-1}\lambda))$. We are really interested in the sections of $\mathscr{L}(\lambda - p^\sigma(zw^{-1}\lambda))$ but have a formula for the vanishing of sections of $\mathscr{L}_{\mathrm{Sbt}}(\lambda, -w^{-1}\lambda)$. Thus we consider the map

$$D_w : X^*(T) \to X^*(T)$$

defined by

$$\lambda \mapsto \lambda - p^\sigma(zw^{-1}\lambda).$$

In order to study this map, we need the following operation on the Weyl group of G. Namely for $w \in W$, define $w^{(0)} := e$ and $w^{(r)} :=^\sigma (w^{(r-1)}w)$, for all $r \in \mathbb{Z}_{\geq 1}$.

Lemma 4.1.2. *1. For $r, s \geq 1$ and $w \in W$, we have ${}^{\sigma^s}(w^{(r)})w^{(s)} = w^{(r+s)}$*

2. The set $R := \{r \geq 0 | w^{(r)} = e\}$ forms a non-trivial submonoid of $\mathbb{Z}_{\geq 0}$ (with addition as operation).

3. If $w^{(r)} = e$ for $r \geq 1$, then $w^{(r-1)} = w^{-1}$.

Proof. For $s = 1$, $w^{(r+1)} =^\sigma (w^{(r)}w) =^\sigma (w^r)^\sigma(w) =^\sigma (w^r)w^{(1)}$. If (1) holds for $s = k - 1$, then

$$w^{(r+k)} =^\sigma (w^{(r+k-1)}w) =^\sigma ({}^{\sigma^{k-1}}(w^r)w^{(k-1)})w^{(1)}$$

$$=^{\sigma^k} (w^{(r)})^\sigma(w^{(k-1)})w^{(1)} =^{\sigma^k} (w^{(r)})w^{(k)}$$

by induction, proving (1). It follows that R is stable under addition. As W is finite, there are (by the pigeonhole principle) $r > s \geq 0$ such that $w^{(r)} = w^{(s)}$, and therefore $w^{(r-s)} = e$ and $r - s \in R$, proving (2). The last part follows by combining (1) and (2), as

$$e = w^{(r)} = w^{(r-1+1)} =^\sigma (w^{(r-1)})w^{(1)} =^\sigma (w^{(r-1)}w)$$

using that $w^{(1)} =^\sigma w$ and that the Galois action commute with multiplication and inverses. ☐

[38]See Jantzen I.3.1 for the relation between restriction and pullback.

[39]$w \in W$ acts on $\lambda \in X^*(T)$ by $w\lambda(t) = \lambda(w^{-1}tw)$ where w acts on T through some representative in $N(T)$

[40]see 13.1.3 in [11]

[41]${}^\sigma(z\nu)^p(\bar{a}) = \sigma(z\nu(\sigma^{-1}(\bar{a})))^p = (zv)(\bar{a}^p) = (zv)(\bar{a}^p)$ as σ is defined as the inverse of Frobenius on k

Lemma 4.1.3. *Let $w \in W$*

1. *The map D_w is a $\mathbb{Q}$-linear automorphism of $X^*(T)_{\mathbb{Q}}$*
2. *The inverse of D_w is given as follows: Let $\chi \in X^*(T)$. Fix $r \geq 1$ such that $(zw^{-1})^{(r)} = e$ and $m \geq 1$ such that χ is defined over $\mathbb{F}_{p^m}$. Then*

$$D_w^{-1}(\chi) = -\frac{1}{p^{rm}-1} \sum_{i=0}^{rm-1} p^i (zw^{-1})^{(i)}({}^{\sigma^i}\chi).$$

Proof. As D_w is the identity modulo p, it must preserve the rank of $X^*(T)$, hence it is a $\mathbb{Q}$-linear isomorphism.

For the second statement, suppose $D_w(\lambda) = \chi$. Then, use induction on j together with Lemma 5.1.2(1) to prove

$$\sum_{i=1}^{j-1} p^i (zw^{-1})^{(i)}({}^{\sigma^i}\chi) = \lambda - p^j (zw^{-1})^{(j)}({}^{\sigma^j}\lambda).$$

For $j = rm$, $(zw^{-1})^{(j)} = e$ by assumption and Lemma 5.1.2(2), and ${}^{\sigma^j}\lambda = \lambda$ as λ is defined over $\mathbb{F}_{p^m}$ and thus fixed by the m'th power of Frobenius (and thus also its inverse). Hence

$$\sum_{i=1}^{rm-1} p^i (zw^{-1})^{(i)}({}^{\sigma^i}\chi) = \lambda - p^{rm}\lambda$$

proving the formula. □

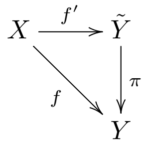
Combining the above with the Chevalley formula of proposition 4.1.1 gives us a way to compute which characters admit non-trivial partial Hasse-invariants on a given flag stratum.

4.2 Reduction to semisimple groups

Here we show that to prove the conjecture for a cocharacter pair (G, μ), it is enough to prove it for the pair $(G/Z(G), \pi \circ \mu)$ with $\pi : G \to G/Z(G)$ the natural projection.

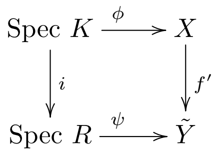
Lemma 4.2.1 (Descent Lemma)**.** *Let $f : X \to Y$ be a proper surjective morphism of integral schemes of finite type over k. Let $\mathscr{L}$ be a line bundle on Y. Let $U \subset Y$ be a normal open subset and $h \in \mathscr{L}(U)$ a non-vanishing section over U. Assume that the section $f^*(h) \in H^0(f^{-1}(U), f^*\mathscr{L})$ extends to X with non-vanishing locus $f^{-1}(U)$. Then there exists $d \geq 1$ such that h^d extends to Y, with non-vanishing locus U.*

Proof. Let $\nu : \tilde{X} \to X$ and $\pi : \tilde{Y} \to Y$ denote the normalization of X and Y respectively. As $(f \circ \nu)^*(h) = h \circ f \circ \nu$ is non-vanishing on exactly $(f \circ \nu)^{-1}(U)$, we may assume $X = \tilde{X}$ and $f = f \circ \nu$. We get the commutative diagram

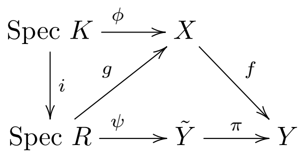


where $f' : X \to \tilde{Y}$ is the unique morphism yielded by the universal property of normalizations,[42] as ν is dominating and f is surjective.

Consider $i^* : R \to K$, the inclusion of a local ring R into its field of fractions K and a commutative diagram



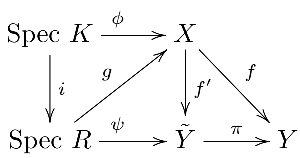
As f is proper, we get by the valuative criteria of properness the commutative diagram



where g is uniquely determined. Let $U_i = \text{Spec } A_i$ be an affine open cover of X. Then g is determined by $g_i^* : A_i \to R$ such that $\phi^* = i^* \circ g_i^*$ on $\mathcal{O}_X(U_i) = A_i$. The diagrams above yield that

$$i^* \circ \psi^* = \phi^* \circ f'^* = i \circ g_i^* \circ f'^*$$

and since i^* is injective, $g_i^* \circ f'^* = \psi^{*}$[43]. Thus we can fill the diagram in to



locally in a unique way, and by gluing also globally. Thus f' is proper, again by the valuative criterion.

Let η be the generic point of Y and $\tilde{\eta} = \pi^{-1}(\eta)$ the generic point of $\tilde{Y}$. As normalization maps are surjective, $f \circ \nu$ is surjective and hence $\eta \in \nu \circ f(\widetilde{(X)})$. As $\nu \circ f = \pi \circ \tilde{f}$, we get that $\tilde{\eta} \in \tilde{f}(\tilde{X})$, but then $\tilde{f}(\tilde{X})$ is dense in $\tilde{Y}$, and thus all of $\tilde{Y}$ as f' is proper. It follows that f' is surjective,

[42] See lemma 29.54.5 (4) in [10]

[43] Here we have implicitly restricted to an affine cover of $\tilde{Y}$.

Let $\mathrm{div}(\pi^*(h)) = \sum_{i=1}^r n_i Z_i$ where $n_i \in \mathbb{Z}$ and $Z_i \subset \tilde{Y} \setminus \pi^{-1}(U)$ are codimension one irreducible subvarieties. If $n_i < 0$ for some i, then by surjectivity of f', $f^*(h)$ would have a pole which it can't have by assumption. Hence $\pi^*(h)$ extends to $\tilde{Y}$ with non-vanishing locus $\pi^{-1}(U)$. Thus we may assume without loss of generality that $X = \tilde{Y}$.

As we can cover Y with affines we can use lemma 29.54.3 in [10] to reduce to $Y = \mathrm{Spec}\ A$ and $X = \mathrm{Spec}\ B$. Here B is the integral closure of the integral domain A in its field of fractions and π^* the inclusion. After potentially localizing further, we may also assume that $\mathscr{L} = \mathcal{O}_Y$, so $h \in B$ as it extends to X by assumption. Denote by I the ideal sheaf of $Z := Y \setminus U$ with the reduced structure. It follows that $U = D(I) = \{\mathfrak{p} \in \mathrm{Spec}\ A | I \not\subset \mathfrak{p}\}$. By assumption, IB is the ideal sheaf defined by vanishing of h, hence h has finite order in B/IB. Thus, after taking powers we may assume $h \in IB$.

We get that $h = \sum_{i=1}^l g_i x_i$ where $g_i \in I$ and $x_i \in B$. Since U is normal, $f : f^{-1}(U) \to U$ is an isomorphism, so for every $g \in I$, the map $A_g \to B_g$ is an isomorphism, as $D(g) \subset D(I) = U$. So there is a $m \geq 1$ such that $g_i^m x_i \in A$. We have that $A[x_1, \ldots, x_l]$ is generated as an A-module by a finite set S of monomials in the x_i. Hence we may increase m so that $g_i^m s \in A$ for all $s \in S$, in particular $g_i^m x_i^d \in A$ for all $d \geq 0$. We may increase m further so that $m = p^n$, and we get that

$$h^{p^n} = \left(\sum_{i=1}^l g_i x_i \right)^{p^n} = \sum_{i=1}^l g_i^{p^n} x_i^{p^n} \in A$$

as the p^n-power map is additive in characteristic p. If $V \subset Y = \mathrm{Spec}\ A$ is the non-vanishing locus of h^{p^n}, then $f^{-1}(V) = f^{-1}(U)$, so $U = V$. This finishes the proof.

□

Let $\tilde{G}$ denote the product of the simply connected cover of the derived subgroup of G with the center of G and $\iota : \tilde{G} \to G$ the induced isogeny.[44] This induces a morphism of zip data, which gives a homeomorphism of stacks, also denoted $\iota : \tilde{\mathcal{X}} \to \mathcal{X}$. Thus if we have a G-zip stack $\zeta : X \to \mathcal{X}$ we get a cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\zeta_{\tilde{\mathcal{X}}}} & \tilde{\mathcal{X}} \\ \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota} \\ X & \xrightarrow{\zeta} & \mathcal{X} \end{array}$$

Proposition 4.2.1 (Change of center lemma)**.** *If $\zeta : X \to \mathcal{X}$ is a smooth surjective morphism of stacks, then for all $w \in W$, one has $\iota^* C_{Y,w} = C_{\tilde{Y},w}$. In particular, $C_{\mathcal{Y},w} = C_{Y,w} \Leftrightarrow C_{\tilde{\mathcal{Y}},w} = C_{\tilde{Y},w}$.*

Proof. By pullback of sections, we have $\iota^* C_{Y,w} \supset C_{\tilde{Y},w}$ for all $w \in W$.

For the converse, ι induces an inclusion of the character group $X^*(T)$ of maximal torus of G into the corresponding character group of the maximal torus, $\tilde{T}$, of

[44]That this morphism is an isogeny follows from $G^{\mathrm{der}} \times Z(G) \to G$ defined by multiplying the arguments being one.

$\tilde{G}$. So we may view $X^*(T)$ as a submodule of $X^*(\tilde{T})$, which has finite index as ι is an isogeny.

The induced morphisms between the groups in the Zip-data of $\tilde{G}$ to the data of G will also be isogenies. Hence the all induced morphisms of stacks are proper and surjective as well.[45]

We have by definition, presentations $BwB/B \to \mathrm{Sbt}_w$ and $\overline{BwB}/B \to \overline{\mathrm{Sbt}}_w$, where BwB/B. It is shown in II.13.3 [15], that $BwB/B \cong \mathbb{A}^{l(w)}$, which in particular shows that it is a normal subscheme in $\overline{BwB}/B$. Using that the morphism from zip-flag stacks into the Schubert stack is smooth, we get the same properties for the induced presentations of $\mathcal{Y}_w$ and $\tilde{\mathcal{Y}}_w$.

Suppose $\chi \in C_{\tilde{\mathcal{Y}},w}$, then as $X^*(T) \subset X^*(\tilde{T})$ with finite index, there is a $n \geq 0$ such that $n\chi \in X^*(T)$. The vanishing of sections for $n\chi$ is completely determined by the Chevalley formula, which is only dependent on the root system, which is the same for G and $\tilde{G}$. Hence $n\chi$ admits a section over $\mathcal{Y}_w$, and by the Descent Lemma, there is a power,m, of this section which extends to $\mathcal{Y}_w^*$. This power is a section for $nm\chi$, and hence $\chi \in C_{\mathcal{Y},w}$ proving the reverse inclusion. □

Since ι restricted to the simply connected cover followed by the projection to the adjoint group of G is an isogeny, we are reduced to prove the conjecture for adjoint groups.

4.3 Relations between the strata cones

Proposition 4.3.1. *Suppose $\zeta : X \to \mathcal{X}$ satisfies the assumptions of the conjecture, then if $w \in W$ with $l(w) = 1$, $C_{\mathcal{Y},w} = C_{Y,w}$.*

Proof. If $\lambda \in C_{Y,w} - C_{\mathcal{Y},w}$, then $-\lambda \in C_{\mathcal{Y},w}$, as $l(w) = 1$.[46] So there is a $N \geq 1$ such that $H^0(\overline{\mathcal{Y}}_w, \mathscr{L}(-N\lambda)) \neq 0$ and $H^0(Y_w^*, \mathscr{L}(N\lambda)) \neq 0$.

Pick $h \in H^0(\overline{\mathcal{Y}}_w, \mathscr{L}(-N\lambda))$ and $f \in H^0(Y_w^*, \mathscr{L}(N\lambda)$, both not identically 0. Since ζ is smooth, Y_w^* is reduced.[47] Hence there is an irreducible component $Y_w' \subset Y_w$ with $f|_{Y_w'} \neq 0$.[48]

As h nontrivial and $\overline{\mathcal{Y}}_w$ irreducible, it is non-zero on $\mathcal{Y}_w$. So $\zeta_Y^*(h)$ is non-zero on Y_w. Hence $\zeta_Y^*(h)f \in H^0(Y_w', \mathcal{O}_{Y_w'})$ is non-zero too. As we have assumed all length one strata closures are pseudo-complete and Y_w' is irreducible, this means that $\zeta_Y^*(h)f$ is nowhere vanishing on Y_w'. But then as $\zeta|_{Y_w'}$ is surjective by assumption, we must have $h \in H^0(\overline{\mathcal{Y}}_w, \mathscr{L}(-N\lambda))$ nowhere vanishing and thus $1/h \in H^0(\mathcal{Y}, \mathscr{L}(N\lambda))$ a well defined non-zero section, contradicting $\lambda \notin C_{\mathcal{Y},w}$.

□

[45]See 76.20 and 99.37 in [10].
[46]This follows from Chevally's formula as w only has one lower neighbour.
[47]Since $\overline{\mathcal{Y}}_w$ is smooth over k, so is Y_w^* hence it is reduced by lemma 33.25.3 in [10].
[48]Otherwise f would be 0 on all of Y_w^*, contrary to our assumption.

Definition 4.3.1. *Suppose $w \in W$, let $\leq$ denote the Bruhat-Chevalley order on W.The set of lower neighbours of w is defined as*

$$E_w = \{s \in W | s \leq w \text{ and } l(s) = l(w) - 1\}.$$

Definition 4.3.2. *A separating system of $\mathcal{Y}_w$ is a set $\{(s_v, \lambda_v)\}_{v \in E_w}$ where $\lambda_v \in X^*(T)$ and*

1. *$s_v \in H^0(\overline{\mathcal{Y}}_w, \mathscr{L}_{\mathcal{Y}}(\lambda_v))$ is a partial Hasse invariant*
2. *$div(s_v) = \overline{\mathcal{Y}}_v$*

Proposition 4.3.2. *Let $w \in W$ with lower neighbours $\{w_i\}_{i=1}^n$. Assume that:*

1. *There exists a separating system of partial Hasse invariants for $\mathcal{Y}_w$.*
2. *$\bigcap_{i=1}^n C_{\mathcal{Y},w_i} \subset C_{\mathcal{Y},w}$*
3. *For all $i \in 1, \ldots, n$, one has $C_{Y,w_i} = C_{\mathcal{Y},w_i}$.*

Then w satisfies $C_{Y,w} = C_{\mathcal{Y},w}$.

Proof. Suppose $\lambda \in C_{Y,w} - C_{\mathcal{Y},w}$. Pick a nontrivial section $f \in H^0(Y_w^*, \mathscr{L}_Y(N\lambda))$ for some $N \geq 1$. Let $\{s_i \in H^0(\mathcal{Y}_w^*, \mathscr{L}(\lambda_i))\}_{i=1}^n$ be a separating system of partial Hasse invariants for $\mathcal{Y}_w$. As we assumed $\bigcap_{i=1}^n C_{\mathcal{Y},w_i} \subset C_{\mathcal{Y},w}$, there is a $i_0 \in \{1, \ldots, n\}$ with $\lambda \notin C_{\mathcal{Y},w_{i_0}}$. By (c), $\lambda \notin C_{Y,w_{i_0}}$, so by definition, $H^0(Y_{w_{i_0}}^*, \mathscr{L}(N\lambda)) = 0$.

We have for all $i \in \{1, \ldots, n\}$ that multiplication by $\zeta_Y^*(s_i)$ induces an exact sequence

$$0 \longrightarrow H^0(Y_w, \mathscr{L}(N\lambda - \lambda_i)) \longrightarrow H^0(Y_w^*, \mathscr{L}(N\lambda)) \longrightarrow H^0(Y_{w_i}^*, \mathscr{L}(N\lambda))$$

as given a section $f \in H^0(Y_w, \mathscr{L}(N\lambda - \lambda_i))$, we get that for any $g \in G$,

$$g.(\zeta^*(s_i)f) = (\lambda_i(g)\zeta^*(s_i))(\lambda(g)^N \lambda_i(g)^{-1} f) = \lambda(g)^N \zeta^*(s_i) f.$$

The cokernel of this morphism is $H^0(Y_{w_i}^*, \mathscr{L}(N\lambda))$ since s_i is a partial Hasse invariant of $Y_{w_i}^*$, and therefore vanishes exactly on $Y_{w_i}^*$. Letting $i = i_0$, we get that $H^0(Y_w, \mathscr{L}(N\lambda - \lambda_{i_0})) \cong H^0(Y_w^*, \mathscr{L}(N\lambda))$, and thus $N\lambda - \lambda_{i_0} \in C_{Y,w}$. As $\lambda_i \in C_{\mathcal{Y},w}$, so would $N\lambda = (N\lambda - \lambda_i) + \lambda_i$ if $N\lambda - \lambda_i \in C_{\mathcal{Y},w}$, so $N\lambda - \lambda_i \notin C_{\mathcal{Y},w}$. Hence we may repeat the procedure above for $N\lambda - \lambda_i$, replacing λ_i with some $i_1 \in \{1, \ldots, n\}$[49], to get $N\lambda - \lambda_0 - \lambda_1 \in C_{Y,w}$. Iterating we get a sequence $(i_d)_{d \geq 0}$ with values in $\{1, \ldots, n\}$ such that $N\lambda - \sum_{d=0}^m \in C_{Y,w}$ for all $m \geq 0$ and multiplication by $\prod_{d=0}^m \zeta_Y^*(s_{i_d})$ produces an isomorphism

$$H^0(Y_w^*, \mathscr{L}(N\lambda - \sum_{d=0}^m \lambda_{i_d})) \to H^0(Y_w^*, \mathscr{L}(N\lambda)).$$

As $\{1, \ldots, n\}$ is finite, there is a j in this set such that $i_d = j$ for infinitely many d. Hence by the ismorphism above, f is divisible by $\zeta^*(s_j)$ infinitely many times, but then $\zeta^*(s_j)$ must be non-vanishing, contradicting that s_i is a partial Hasse invariant for $\mathcal{Y}_{w_j}$. In particular it contradicts that $div(s_i) = \overline{\mathcal{Y}}_{w_i}$. Hence $C_{Y,w} = C_{\mathcal{Y},w}$. □

[49] Possibly $i_1 = i_0$.

Corollary 4.3.1. *If the assumptions of the above proposition holds for all $w \in W$, then $C_{Y,w} = C_{\mathcal{Y},w}$ for all $w \in W$. In particular, $C_Y = C_{\mathcal{Y}}$.*

Proposition 4.3.3. *Let $w \in W$ be a lower neighbour of w_0. Assume that:*

1. *The Picard group of G is trivial.*
2. *$X_{+,I}(T) \cap C_{\mathcal{Y},w} \subset C_{\mathcal{Y}}$*
3. *$C_{Y,w} = C_{\mathcal{Y},w}$*

then $C_Y = C_{\mathcal{Y}}$.

Proof. Suppose $\lambda \in C_Y \setminus C_{\mathcal{Y}}$ and $f \in H^0(Y, \mathscr{L}_Y(N\lambda))$ for some $N \geq 1$. If $\lambda \notin X^*_{+,I}(T)$, $H^0(P/B, \mathscr{L}_\lambda) = 0$, so $\mathscr{L}_Y(\lambda) = 0$, hence $\lambda \in X^*_{+,I}(T)$. By assumption, this implies that $\lambda \notin C_{w,\mathcal{Y}} = C_{w,Y}$. As $\mathrm{Pic}(G) = 0$, each line bundle on $[E' \setminus G]$ pulls back to $\mathcal{O}_G$ under the presentation morphism. Hence there is a $\mu \in X^*(T)$ and a partial Hasse invariant $s \in H^0(\mathcal{Y}, \mathscr{L}_{\mathcal{Y}}(\mu))$ with $\mathrm{div}(s) = \mathcal{Y}^*_w$ [50]. Since $\lambda \notin C_{Y,w}$, $H^0(Y^*_w, \mathscr{L}_Y(N\lambda)) = 0$, so f restricts to 0 on Y^*_w. Hence f is divisible by $\zeta^*(s)$, i.e. there is a $g \in H^0(Y, \mathscr{L}_Y(N\lambda - \mu))$ such that $f = \zeta^*(s)g$. But this shows that $N\lambda - \mu \in C_Y$, and therefore the same holds for $\lambda - \frac{\mu}{N}$. As $\mu \in C_{\mathcal{Y}}$, $\lambda - \frac{\mu}{N} \notin C_{\mathcal{Y}}$. So the argument can be repeated indefinitely, implying that f is divisible by $\zeta^*(s)^m$ for all $m \geq 1$, a contradiction. ◻

Using the change of center lemma, we get.

Corollary 4.3.2. *Let $w \in W$ be a lower neighbour of w_0. Assume that:*

1. *$X_{+,I}(T) \cap C_{\mathcal{Y},w} \subset C_{\mathcal{Y}}$*
2. *$C_{Y,w} = C_{\mathcal{Y},w}$*

then $C_Y = C_{\mathcal{Y}}$.

Proof. By the change of center lemma, we may assume that G is semisimple and simply connected. By proposition 4.6 in [22], $\mathrm{Pic}(G) = 0$ and the corollary follows. ◻

4.4 $\mathbb{Q}$-separating systems

As we will see in the C_2-case in section 5, it is not necessarily true that each strata of G-ZipFlag$^\mu$ admits a separating system of partial Hasse invariants. To remedy this, my advisor has in private communications with me given a generalization, which weakens the demand on the divisors of the partial Hasse-invariants. He further proved that the key proposition 4.3.2 on equality of strata cones still holds for this weakened notion. We will reproduce his argument in this section.

Definition 4.4.1. *A $\mathbb{Q}$-separating system of $\mathcal{Y}_w$ is a set $\{(s_v, \lambda_v)\}_{v \in E_w}$ where $\lambda_v \in X^*(T)$ and*

1. *$s_v \in H^0(\overline{\mathcal{Y}}_w, \mathscr{L}_{\mathcal{Y}}(\lambda_v))$ is a partial Hasse invariant*

[50] This follows from $\mathcal{Y}^*_w$ being a closed substack of codimension 1, so its pullback under the presentation is a closed subvariety of G of codimension 1.

2. div(s_v) = $N\overline{\mathcal{Y}}_v$ for some $N \geq 1$, equivalently $[Z(s_v)]_{red} = \overline{\mathcal{Y}}_v$

If there is such a system, we say that $\mathcal{Y}$ admits a $\mathbb{Q}$-separating system.

We assume again that $\zeta : X \to \mathcal{X}$ satisfy the assumptions of the conjecture.

Proposition 4.4.1. *Let $w \in W$ with lower neighbours $\{w_i\}_{i=1}^n$. Assume that:*

1. *$\mathcal{Y}_w$ admits a $\mathbb{Q}$-separating system.*
2. *$\bigcap_{i=1}^n C_{\mathcal{Y},w_i} \subset C_{\mathcal{Y},w}$*
3. *for all $i \in 1, \ldots, n$, one has $C_{Y,w_i} = C_{\mathcal{Y},w_i}$*

Then w satisfies equality of cones, i.e. $C_{Y,w} = C_{\mathcal{Y},w}$.

Proof. As Y_w is normal it is smooth in codimension 1. Hence the order of vanishing, $\text{ord}_Y f$, of $f \in H^0(\overline{Y}_w, \mathscr{L}(\lambda))$ along an integral codimension 1 subscheme, Z, is the valuation of f in the 1-dimensional local ring $\mathcal{O}_{\overline{Y}_w,Z}$.

For each $v \in E_w$, Y_v has finitely many irreducible components.[51] Assume $f \in H^0(\overline{Y}_w, \lambda)$ and $\lambda \notin C_{\mathcal{Y}_w}$. For $v \in E_w$ define

$$j_v := \min_{Z \subset \overline{Y}_w,\ Z \text{ irreducible}} \text{ord}_Z f.$$

If $\{(s_v, \lambda_v)\}_{v \in E_w}$ is a $\mathbb{Q}$-separating system, let

$$n_v := \text{ord}_{\overline{Y}_w} \zeta_Y^* s_v := \text{ord}_{\overline{\mathcal{Y}}_w} s_v = -\langle \lambda_v, w\alpha^\vee \rangle$$

where the first equality (after the defining one) comes from the smoothness of ζ_y together with lemma 15.108.5 [10]. These imply that the order is the same on each irreducible component of Y_v and hence that the order on all of Y_v is well defined. The second equality follows from Chevalley's formula, with $v = ws_\alpha$.

Let $N := \prod_{v \in E_w} n_v$, $N_v := N/n_v$ and $f' := \frac{f^N}{\prod_{v \in E_w} \zeta_Y^* s_v^{j_v N_v}} \in H^0(\overline{Y}_w, \lambda')$ where $\lambda' = N\lambda - \sum_{v \in E_w} j_v N_v \lambda_v$.

For $Z \in \overline{Y}_w$ irreducible, we have

$$\text{ord}_Z f' = N \text{ord}_Z f - j_v N_v n_v = N\left(\text{ord}_Z f - j_v\right) \geq 0$$

with equality being equivalent to $Z \subset Y_w$ realizing the minimum defining j_v. As the number of irreducible components is finite, this is realized for some component of each Y_v^*.

If $\lambda' \in C_{\mathcal{Y}_w}$, then so is λ since $N\lambda = \lambda' + \sum_{v \in E_w} j_v N_v \lambda_v$ and $\lambda_v \in C_{\mathcal{Y}_w}$ by assumption. Hence $\lambda' \notin C_{\mathcal{Y}_w}$. However, we have assumed $C_{Y,v} = C_{\mathcal{Y},v}$ for all $v \in E_w$, so there is a $v_0 \in E_w$ with $\lambda' \notin C_{Y,v_0}$ as $\bigcap_{v \in E_w} C_{\mathcal{Y},v} \subset C_{\mathcal{Y}_w}$. Hence $f'\big|_{Y_{v_0}^*} = 0$, but this contradicts the existence of a component Z of $Y_{v_0}^*$ realizing $\text{ord}_Z f = j_{v_0}$. □

[51] As $G \times P$ and the $E \times P$ orbits are Noetherian and $\zeta_Y : Y \to [(E \times P) \setminus (G \times P)]$ is smooth.

4.5 The basic strategy

Putting together the results above, we get the following inductive strategy for proving the conjecture

4.5.1 The strategy

First step; we can pass to the associated adjoint cocharacter datum without loss of generality by the change of center lemma.

Second step; calculate the Schubert cones at each Schubert stratum of G-ZipFlag$^\mu$. This is done by applying Chevalleys formula, looking at which weights admits sections without poles[52] at the boundary of the strata. This allow us to extend the section to the strata closure in Sbt. Then apply D_w^{-1}, to get the Schubert cone over the stratum in G-ZipFlag$^\mu$.

Third step; note that the conjecture is true for all Schubert strata corresponding to elements of length one of W by proposition 4.3.1.

Fourth step; look at strata of length 2, check if the intersections of the cones of lower neighbours is contained in the cone of this stratum. If this is true, we try finding a ($\mathbb{Q}$-)separating system for the stratum. Then we apply either proposition 4.3.2 if the separating system is integral, or proposition 4.4.1 if you only have a $\mathbb{Q}$-separating system.

Fifth step; repeat for higher strata until we can't, or the theorem is proven. An alternative way to finish is to prove the equality of cones for one w of length $l(w_0) - 1$ and then, the Picard group of the reductive group in the datum is trivial apply 4.3.3.

4.5.2 Remarks

It should be noted that the conjecture isn't necessarily false if the basic strategy fails. This is a consequence of us only using Schubert cones. It could very well happen that $C_{\mathcal{Y},w} \neq C_{\mathrm{Sbt},w}$ for some $w \in W$. As we will see in the next section however, there are cases where the strategy work. Hence, if we are interested in a specific case, applying the strategy is a reasonable way to start the attempted proof.

5 Proof in the C_2 case

The original proof of this case is contained in section 5 of [7]. Below we follow the ideas presented there but using the slightly generalized $\mathbb{Q}$-separating systems of Goldring instead of proposition 4.3.3 to deal with the fact that one of the strata doesn't admit an integral separating system.

Let G be a reductive group of Dynkin type C_2 over $\mathbb{F}_p$ with a cocharacter μ such that (G, μ) is of connected Hodge type. Let $\mathcal{Z} = (G, P, Q, L, M, \iota)$ be the induced zip datum. Fix a maximal torus $T \supset \mu(\mathbb{G}_m)$, lying in a split Borel subgroup $B \subset P$.

[52] Here a pole is a term in the divisor of the section with negative coefficient

Let e_1, e_2 be the standard basis vectors of $\mathbb{R}^2$. We follow Bourbaki and put $e_1 - e_2$ and $2e_2$ as the simple roots of G. That is $\Delta = \{e_1 - e_2, 2e_2\}$. Thus the Weyl group, W, of G is the subgroup of $\mathrm{GL}_2(\mathbb{R})$ generated by

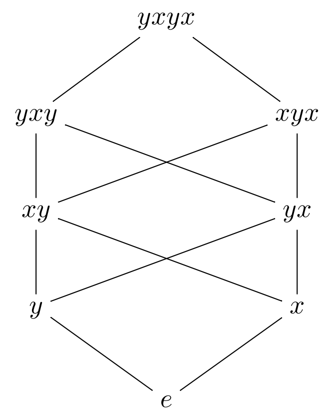
$$x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the simple reflections in the hyperplanes orthogonal to $e_1 - e_2$ and $2e_2$ respectively.

The flag stratification of G-ZipFlag$^\mu$ can be represented by the lattice



given by the elements of W under the Bruhat-Chevalley order. We note that the longest element in the Weyl group is $w_0 = yxyx = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} =: -e$.

As μ is of connected Hodge type, $\mu^{\mathrm{ad}} \in X_*(G^{\mathrm{ad}})$ is minuscule. The only miniscule, Δ-dominant cocharacter of G^{ad} is the fundamental coweight corresponding to $2e_2$, that is $\mu = e_1^* + e_2^*$ where $e_i^*(e_j) = \delta_{ij}$.

As a Dynkin diagram of type C_2 doesn't admit any non-trivial automorphisms, there cannot be a non-trivial Galois action on the roots of G. Hence $Q = P^{\mathrm{op}}$ and $D_w(\lambda) = \lambda - p(zw^{-1}\lambda)$.

The set of simple roots in J, the type of Q is determined by the inequality $\langle \lambda, \mu \rangle \leq 0$.[53] It follows that $J = I = \{e_1 - e_2\}$, thus W_J thus is cyclic of order 2 generated by its longest element $w_{0,J} = x$, so $z = w_0 w_{0,J} = yxy = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

We compute the cone at the stratum corresponding to w_0 as an example. As the Galois action is trivial, $w^{(r)} = w^r$. Hence

$$D_{w_0}(\lambda) = \lambda - p(zw_0^{-1}\lambda) = \lambda - p(x\lambda)$$

and

$$D_{w_0}^{-1}(\chi) = -\frac{1}{p^2 - 1}\left(\chi + p(x(\chi)\right)$$

[53] This is because $\lim_{t \to 0} \mu(t) u_{-\alpha}(x) \mu(t)^{-1}$ exists precisely when this inequality holds.

using the formula for the inverse of D_w in proposition 4.1.3. Acting on the standard basis, $D_{w_0}^{-1}$ is represented by the matrix

$$\frac{-1}{p^2-1}\begin{pmatrix}1 & p\\ p & 1\end{pmatrix}.$$

As we see in the lattice of W above, the lower neighbours of w_0 are

$$E_{w_0} = \{xyx, yxy\} = \{w_0y, w_0x\} = \{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\}^{54}.$$

As the fundamental weights of (G, T) span $X^*(T)$ (over $\mathbb{Z}$) it is, by Chevalley's formula enough to compute the divisors of sections of these weights to determine the cones. The fundamental weights are

$$\eta_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

and

$$\eta_2 = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Let $s_i \in H^0(\mathrm{Sbt}_{w_0}, \mathscr{L}_{\mathrm{Sbt}}(\eta_i, -w^{-1}\eta_i))$. By Chevalleys formula

$$\mathrm{div}(s_1) = -\langle \eta_1, w_0^{-1}\alpha_2^\vee\rangle \overline{C}_{w_0y} - \langle \eta_1, w_0^{-1}\alpha_1\rangle \overline{C}_{w_0x}$$
$$= -(e_1, -e_2)\overline{C}_{w_0y} - (e_1, -e_1+e_2)\overline{C}_{w_0x} = \overline{C}_{w_0x} = \overline{C}_{yxy}$$

and

$$\mathrm{div}(s_2) = -\langle \eta_2, w_0^{-1}\alpha_2^\vee\rangle \overline{C}_{w_0y} - \langle \eta_2, w_0^{-1}\alpha_1\rangle \overline{C}_{w_0x}$$
$$= -(e_1+e_2, -e_2)\overline{C}_{w_0y} - (e_1+e_2, -e_1+e_2)\overline{C}_{w_0x} = \overline{C}_{w_0y} = \overline{C}_{xyx}$$

where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$, i.e. the simple roots corresponding to x and y respectively.[55]

Thus if $\chi = \lambda_1\eta_1 + \lambda_2\eta_2$, we must have that $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ for $H^0(\mathrm{Sbt}, \mathscr{L}_{\mathrm{Sbt}}(\chi, -w_0^{-1}\chi))$[56] to be non-trivial. Otherwise, any section has singularities on some of the lower neighbours by the above.

Let $\chi = a_1e_1 + a_2e_2 \in X^*(T)$, then

$$D_{w_0}^{-1}\chi = \frac{-1}{p^2-1}\left((a_1+pa_2)e_1 + (a_2+pa_1)e_2\right)$$
$$= \frac{-1}{p^2-1}\left((a_1+pa_2)\eta_1 + (a_2+pa_1)(\eta_2-\eta_1)\right)$$
$$= \frac{-1}{p^2-1}\left((p-1)(-a_1+a_2)\eta_1 + (a_2+pa_1)\eta_2\right).$$

[54] keeping the same order in each presentation of the set

[55] We are also using that the pairing agrees with the Euclidean scalar product $(\cdot,\cdot)$ of $\mathbb{R}^2$ if we embed $X^*(T)$ into $X_*(T)$ via the pairing

[56] Here we use that Sbt_{w_0} is dense in Sbt.

Hence, for $\mathscr{L}(\chi)$ to admit a nontrivial section pulled back from Sbt, we must have $a_1 - a_2 \geq 0$ and $-a_2 - pa_1 \geq 0$. We conclude that

$$C_{\mathcal{Y},w_0} = C_{\mathcal{X}} = \{\chi = a_1e_1 + a_2e_2 \in X^*(T) | a_1 - a_2 \geq 0 \text{ and } -pa_1 - a_2 \geq 0\}.$$

To compute (partial) Hasse invariants vanishing at the lower strata, we use

$$D_{w_0}^{-1}(a_1e_1 + a_2e_2) = \frac{-1}{p^2-1}\left((p-1)(-a_1+a_2)\eta_1 + (a_2+pa_1)\eta_2\right)$$

and the fact that sections s_1 and s_2 of weight η_1 and η_2 respectively have divisors

$$\operatorname{div}(s_1) = \overline{C}_{yxy}$$

and

$$\operatorname{div}(s_2) = \overline{C}_{xyx}.$$

Thus to find a Hasse invariant with vanishing locus $\overline{C}_{yxy}$ we need to solve the equation

$$\frac{-1}{p^2-1}\left((p-1)(-a_1+a_2)\eta_1 + (a_2+pa_1)\eta_2\right) = \eta_1.$$

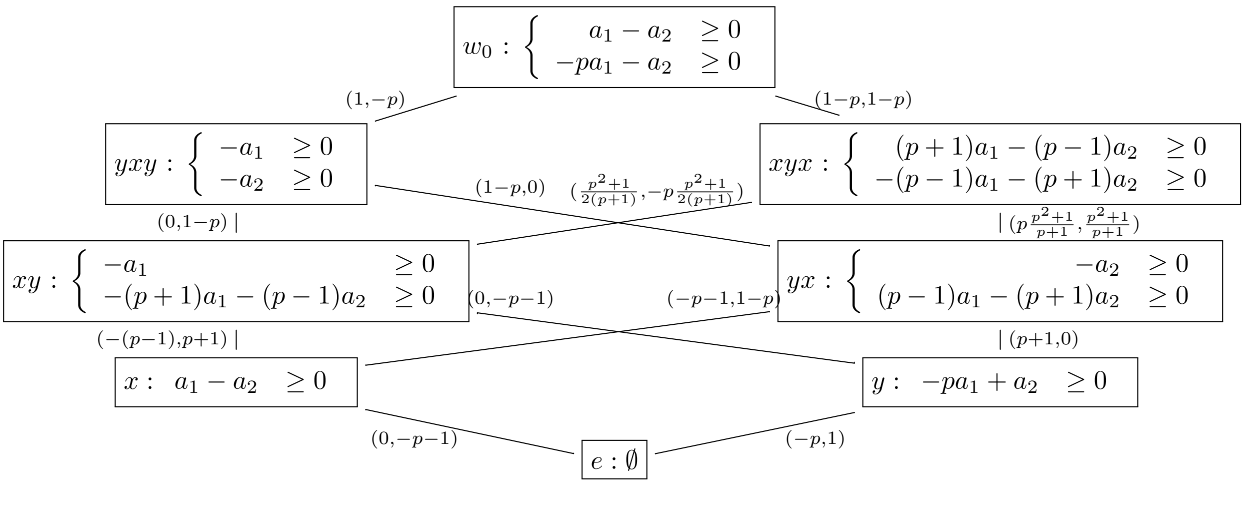
This is achieved by letting $a_1 = 1$ and $a_2 = -p$. Similarly, to find a Hasse invariant with vanishing locus $\overline{C}_{xyx}$, we need to solve

$$\frac{-1}{p^2-1}\left((p-1)(-a_1+a_2)\eta_1 + (a_2+pa_1)\eta_2)\right) = \eta_2.$$

Which is achieved by $a_1 = a_2 = 1 - p$.

One computes the cones and partial Hasse invariants at the other strata analogously, which results in the cone diagram in Figure 1.

Figure 1: Strata cones and partial Hasse invariants for type C_2



The diagram is read in the following way: Each box corresponds to a stratum in the Schubert stratifictation of G-ZipFlag$^\mu$. The name of the strata is the leftmost text, so the top one corresponds to w_0. The inequalities to the right in each box is the cone of characters which admits non-zero partial hasse invariants on that strata closure. A line from a stratum down to another corresponds to a partial hasse invariant with the lower stratum as vanishing locus, so the lines going down from a stratum represents a ($\mathbb{Q}$-)separating system.

We note that an integral separating system of xyx is not obtainable, but we can use a $\mathbb{Q}$-separating system, as shown in the diagram, for that strata. However, due to $C_{\mathcal{Y},xy} \cap C_{\mathcal{Y},yx} \not\subset C_{xyx}$, we won't use that strata in the proof.

By proposition 4.3.1, we have that $C_{\mathcal{Y},x} = C_{Y,x}$ and $C_{\mathcal{Y},y} = C_{Y,y}$, as x, y both have length 1. We have $C_{\mathcal{Y},x} \cap C_{\mathcal{Y},y} \subset C_{\mathcal{Y},xy} \cap C_{\mathcal{Y},yx}$. Hence proposition 4.3.2 implies equality of cones at xy and yx as well, since there is a separating system as displayed in the diagram. Similarly, $C_{\mathcal{Y},xy} \cap C_{\mathcal{Y},yx} \subset C_{\mathcal{Y},yxy}$ so proposition 4.4.1 implies $C_{Y,yxy} = C_{\mathcal{Y},yxy}$ as well.

Lastly, $X_{+,I}(T) \cap C_{\mathcal{Y},yxy} \subset C_{\mathcal{Y},w_0}$, so by corollary 4.3.2, we have equality of cones at w_0 as well, which proves the theorem.

6 Shimura Varieties

One of the main applications of G-Zips is to study good reductions of Shimura varieties at primes p. In particular, the good reduction of a Shimura variety of Hodge type admits a smooth map to the stack of G-Zips, so if the cone conjecture is proven for this stack, we can use it to see which automorphic bundles on the Shimura variety admits global sections.

In this chapter, we give an introduction to Shimura varieties and their reductions. We conclude with some remarks on how to apply the cone conjecture to them.

6.1 Hodge Structures

We begin by defining the Deligne torus, which in a sentence is the group scheme

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m.$$

That is, given an $\mathbb{R}$-algebra A, $\mathbb{S}(A) = \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} A)$. In particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$. The character group

$$X^*(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{S}, \mathbb{G}_m)(\mathbb{C})$$

is generated by $z, \bar{z}$, inducing the identity and complex conjugation respectively in the composition

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \xrightarrow{i_*} \mathbb{S}(\mathbb{C}) \longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$$

where the first arrow is the map induced by the inclusion $i : \mathbb{R} \to \mathbb{C}$ and the second is z and $\bar{z}$ respectively.

There is an inclusion

$$w : \mathbb{G}_m \to \mathbb{S}$$

defined by

$$(f : t \mapsto a) \mapsto \left(w(f) : t \mapsto \frac{1}{a} \otimes 1 \right)$$

which is well defined, as $f : \mathbb{R}[t, t^{-1}] \to A$ must map t to a unit to be an algebra morphism. Note that $zw = \bar{z}w = \iota$, where ι is the inverse map of $\mathbb{G}_m$.

Definition 6.1.1. *A real Hodge structure is a homomorphism of real algebraic groups*

$$h : \mathbb{S} \to GL(V)$$

where V is finite dimensional real vector space.

We say that a Hodge structure is of weight n if for all $x \in \mathbb{G}_m(A)$ and all $v \in V \otimes_{\mathbb{R}} A$

$$h \circ w(x).v = x^n v$$

for any $\mathbb{R}$-algebra A.

Given a subring $k \subset \mathbb{R}$, a k-Hodge structure is a free k-module V, together with a real Hodge structure $h : \mathbb{S} \to GL(V_{\mathbb{R}})$.[57]

By our characterization of $X^*(\mathbb{S})$ and our definition of w, it follows that for any Hodge structure V, there is a decomposition

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} V^{pq}$$

where $\mathbb{S}(\mathbb{C})$ acts on V^{pq} by $z^{-p}\bar{z}^{-q}$[58].

Definition 6.1.2. *We call the set $\{(p,q) | V^{pq} \neq 0\}$ the type of the Hodge structure (V, h).*

As $\overline{z^{-p}\bar{z}^{-q}v} = \bar{z}^{-p}z^{-q}\bar{v}$, get that $\overline{V^{pq}} = V^{qp}$. Consequently, if (p,q) is in the type of a Hodge structure, so is (q,p), and the corresponding weight spaces has the same dimension.

A morphism of Hodge structures $(V, h) \to (W, h')$ is a linear map $f : V \to W$ with such that $f_{\mathbb{R}} : (V_{\mathbb{R}}, h) \to (W_{\mathbb{R}}, h')$ is a morphism of $\mathbb{S}$-representations.

6.1.1 Polarizations

Let $\mathbb{Q}(m)$ denote the rational Hodge structure defined by the morphism $h_m : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ which on real points is given by $h(z) = (z\bar{z}^m)$, and let $\mathbb{R}(m)$ denote the corresponding real Hodge structure.

Given a real Hodge structure (V, h) of weight n, a polarisation of (V, h) is given by a morphism of $\mathbb{S}$-representations $\psi : V \otimes V \to \mathbb{R}(-n)$ such that $\psi_{h(i)}(v, w) := \psi(v, h(i)w)$ is positive definite.

For $k \subset \mathbb{R}$ a subring, a polarisation of a k-Hodge structure (V, h) is a bilinear form $\psi : V \times V \to k$ such that $\psi_{\mathbb{R}}$ is a polarisation of the real Hodge structure $(V_{\mathbb{R}}, h)$.

A Hodge structure admitting a polarization is said to be polarizable.

[57] We will only use $k \in \{\mathbb{Z}, \mathbb{Q}\}$

[58] We are required to have $p + q = n$ as $h \circ w(x).v = z^{-p}(x^{-1})\bar{z}^{-q}(x^{-1}).v = x^{p+q}.v = x^n.v$

6.2 Shimura data and varieties

Definition 6.2.1 (Cartan Involutions)**.** *For a reductive real algebraic group G, an involution $\theta : G \to G$ is a Cartan involution if the set*

$$G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) | g = \theta(\bar{g})\}$$

is compact, where $\overline{(-)}$ denotes the morphism induced by complex conjugation of $\mathbb{C}$.

Definition 6.2.2 (Shimura data)**.** *A Shimura datum is pair (G, X) where G is a reductive group over $\mathbb{Q}$, X a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following:*

- *For all $h \in X$, the Hodge structure on Lie $G_{\mathbb{R}}$ defined by $Ad \circ h$ is of type*

$$\{(1, -1), (0, 0), (-1, 1)\}$$

- *For all $h \in X$, $ad(h(i))$ is a Cartan involution of $G^{ad}_{\mathbb{R}}$*
- *For $h \in X$, G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial*

Just like the stack of G-zips, the set X can be viewed as a parameter space of vector bundles, namely

Proposition 6.2.1 (5.9 [23])**.** *Let (G, X) be a Shimura datum. Then X has a unique structure as a complex manifold such that for every representation $\rho : G_{\mathbb{R}} \to GL(V)$, $(V, \rho \circ h)_{h \in X}$ is a holomorphic family of Hodge structures. For this complex structure $(V, \rho \circ h)_{h \in X}$ is a variation of Hodge structures and X is a disjoint union of Hermitian symmetric domains.*

For the definitions of Hermitian symmetric domains and variations of Hodge structure, see [23]. The main reason we are interested in this proposition is that it, together with what follows allow us to assign a system of quasi-projective varieties to the Shimura datum (G, X).

Let G be an algebraic group over $\mathbb{Q}$. A subgroup $\Gamma \subset G(\mathbb{Q})$ is arithmetic if it has the following property. For any faithful algebraic representation $G \hookrightarrow \mathrm{GL}(n)$, $\Gamma \cap G(\mathbb{Q}) \cap \mathrm{GL}(n)(\mathbb{Z})$ has finite index in both Γ and in $G(\mathbb{Q}) \cap \mathrm{GL}(n)(\mathbb{Z})$. By proposition 1.7 [23], the identity component of the group of holomorphic automorphisms of any connected component of X^+ of X, $\mathrm{Hol}(X^+)^+$, is the real points of some algebraic group. Thus there is a natural notion of actions of arithmetic groups on X^+.

Theorem 6.2.1 (Baily, Borel)**.** *Let $D(\Gamma) = \Gamma \backslash D$ be the quotient of a symmetric Hermitian domain D by a torsion free arithmetic subgroup Γ of $Hol(D)^+$. Then $D(\Gamma)$ has a canonical realization as a Zariski open subset of a projective variety, i.e. $D(\Gamma)$ is canonically a quasi-projective variety.*

To relate this to the Shimura datum (G, X), we consider compact open subgroups $K \subset G(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adeles and the following double quotient

$$G(\mathbb{Q})_+ \setminus G(\mathbb{A}_f)/K.$$

Here $G(\mathbb{Q})_+ = G(\mathbb{R})_+ \cap G(\mathbb{Q})$ and $G(\mathbb{R})_+$ is the pre-image of the identity component of $G^{\mathrm{ad}}(\mathbb{R})$. Also, $(a, b) \in G(Q)_+ \times K$ acts on $g \in G(\mathbb{A}_f)$ by left respectively right multiplication.[59] The quotient of this action is finite by lemma 5.12 in [23]. Consider the similar quotient,

$$G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/K$$

where $g.[a,b].k = [ga, gbk]$.

Lemma 6.2.1 (Lemma 5.13 in [23])**.** *Let $\mathcal{C}$ denote a set of representatives of $G(\mathbb{Q})_+ \setminus G(\mathbb{A}_f)/K$ and X^+ a connected component of X. There is a homeomorphism*

$$G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/K \cong \coprod_{g\in\mathcal{C}} \Gamma_g \setminus X_+$$

where $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$, which is an arithmetic subgroup.

Thus, if we choose K small enough so that Γ_g is torsion free, we get that $G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/K$ is a disjoint union of a finite set of quasi-projective varities by Baily and Borels theorem. Such K are usually called sufficiently small. [60]

Definition 6.2.3. *A Shimura variety relative to (G, X) is a quotient*

$$Sh_K(G,X) := G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/K$$

where K is a sufficiently small compact open subgroup of $G(\mathbb{A}_f$.

Given $K' \subset K$, both sufficiently small compact open subgroups of $G(\mathbb{A}_f)$. The inclusion map $i : K' \to K$ induces a regular map of varieties $i_* : \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$. This gives us an inverse system of varieties

$$\{\mathrm{Sh}_K(G,X), i_* : \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)\}$$

with $K' \subset K$ ranging over inclusions of sufficiently small compact open subgroups of $G(\mathbb{A}_f)$. There is a natural right action of $G(\mathbb{A}_f)$ on this inverse system, namely for $g \in G(\mathbb{A}_f)$, $[a,b] \in G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/K$

$$[a,b].g = [a, bg] \in G(\mathbb{Q})_+ \setminus X \times G(\mathbb{A}_f)/gKg^{-1}.$$

Definition 6.2.4 (Shimura varieties)**.** *The Shimura variety associated to a Shimura datum (G, X) is the inverse system of varieties*

$$\{Sh_K(G,X), i_* : Sh_{K'}(G,X) \to Sh_K(G,X)\}$$

endowed with the action of $G(\mathbb{A}_f)$ described above.

For each Shimura datum (G, X), there is a minimal number field $E \subset \mathbb{C}$ and an inverse system $\mathrm{Sh}_E(G,X)$ of varieties defined over E such that when we base change to $\mathbb{C}$, the system is isomorphic to $\mathrm{Sh}(G,X)$, $\mathrm{Sh}_E(G,X)$ is called the canonical model of $\mathrm{Sh}(G,X)$, for a detailed treatment, see [2].

[59] $G(\mathbb{Q})$ is embedded into $G(\mathbb{A}_f)$ through the diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_f$.

[60] If we don't choose K sufficiently small, the quotient will in general be a Deligne-Mumford stack.

6.3 Siegel Moduar Varieties

Let (W, ψ) be a symplectic space over $\mathbb{Q}$ of dimension $2g$ and $(\mathrm{GSp}(W, \psi)$ the group of symplectic similitudes. A morphism of algebraic groups $h : \mathbb{S} \to \mathrm{GSp}(W, \psi)_{\mathbb{R}}$ satisfying the first axiom of a Shimura datum[61] is one such that composing with the inclusion of $\mathrm{GSp}(W, \psi)_{\mathbb{R}}$ into $\mathrm{GL}(W_{\mathbb{R}})$ gives a Hodge structure of weight $(-1, 0), (0, -1)$ on $W_{\mathbb{R}}$. To give such an h is equivalent to giving a complex structure $J \in \mathrm{End}(W_{\mathbb{R}})$, that is an endomorphism J such that $J^2 = -\mathrm{id}$. To see this, given a complex structure J, define $h(\mathbb{R}) : \mathbb{C}^{\times} \to \mathrm{GL}(W_{\mathbb{R}})$ by $h(a + bi) = a + Ji$.

A complex structure is called positive if the bilinear form

$$\psi_J(u, v) = \psi(u, Jv)$$

is positive definite, and negative if it is negative definite. As ψ is non-degenerate, these conditions partitions the set of complex structures. Let X^+ be the set of positive complex structures and X^- the set of negative structures and $X_g := X^+ \cup X^-$ the set of all complex structures. Let $\mathrm{GSp}(W, \psi)(\mathbb{R})$ act on X by conjugation. For $g \in \mathrm{GSp}(W, \psi)$, $gh_J(a + bi)g^{-1} = a + gJg^{-1}b$, so the correspondence between Hodge structures satisfying the Shimura datum axioms and complex structures on W is equivariant with respect to conjugation. The action of $\mathrm{GSp}(W, \psi)$ on X is transitive.[62] Thus the unique Shimura datum for $\mathrm{GSp}(W, \psi)$ is given by $(\mathrm{GSp}(W, \psi), X)$.[63] We can recover an analogous cocharacter as discussed in the Siegel-type cocharacter datum for a given h by defining $\mu_h : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GSp}(W, \psi)$ by letting $x \in \mathbb{R}^{\times}$ act my multiplication on $W^{-1,0}$ and trivially on $W^{0,-1}$ in the Hodge decomposition of W defined by h.

Just like for G-Zips, a Shimura datum (G, X) is of Hodge type if there is an embedding $(G, X) \to (\mathrm{GSp}(W, \psi), X_g)$ for some symplectic space (W, ψ).

The Siegel Shimura varietes has a nice modular description, which is crucial for defining integral models of Shimura varieties of Hodge type.

6.3.1 Complex abelian varieties

An abelian variety over a field k is a projective group variety over k, with group laws given by regular maps. Abelian varieties are always commutative as group schemes, so the name is fitting in the group context as well.

When $k = \mathbb{C}$, there is a nice description of these in terms of lattices. Namely, each abelian variety over $\mathbb{C}$ is a complex torus. Its complex points are given by $\mathbb{C}^g/\Lambda$ where Λ is a lattice of $\mathbb{R}^{2g}$. The group law on the complex points is given by addition in $\mathbb{C}^g$ descended through the natural projection $\mathbb{C}^g \to \mathbb{C}^g/\Lambda$. This projection is a universal covering, and by considering covering transformations, which are given by addition of elements of Λ, we get that $H_1(\mathbb{C}^g/\Lambda, \mathbb{Z}) \cong \Lambda$.

The isomorphism $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}^g$ defines a complex structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, equivalently a Hodge structure of type $\{(-1, 0), (0, -1)\}$. The complex torus $\mathbb{C}^g/\Lambda$ is an Abelian variety if and only if the Hodge structure $H_1(\mathbb{C}^g/\Lambda, \mathbb{Z})$ is polarizable, see [23].

[61] The weight condition on $\mathrm{Ad} \circ h$.
[62] See [23] for an argument using symplectic bases
[63] to see why the other axioms are satisfied, see Milne chapter 6

Let **AV** denote the category of abelian varietes over $\mathbb{C}$ with regular maps, equivalently groups scheme homomorphisms [64], as morphisms.

Theorem 6.3.1 (Riemann's theorem)**.** *The functor $A \mapsto H_1(A, \mathbb{Z})$ from the category of complex abelian varieties, **AV**, to the category of integral polarizable Hodge structures of type $\{(-1,0),(0,-1)\}$ is an equivalence of categories.*

Proof. See Theorem 6.8 in [23]. □

The category of isogeny classes of abelian varieties $\mathbf{AV}_0$ can be defined as the category with Abelian varieties as objects and with $\mathrm{Hom}_{\mathbf{AV}_0}(A,B) = \mathrm{Hom}_{\mathbf{AV}}(A,B) \otimes \mathbb{Q}$. Tensoring with $\mathbb{Q}$ makes every morphism with finite kernel in **AV** into an isomorphism in $\mathbf{AV}_0$ as all torsion is killed.

Corollary 6.3.1. *The functor $A \mapsto H_1(A, \mathbb{Q})$ from $\boldsymbol{AV}_0$ to the category of rational polarizable Hodge structures of type $\{(-1,0),(0,-1)\}$ is an equivalence of categories.*

6.3.2 The Shimura varieties

Let $K \subset G(\mathcal{A}_f)$ and consider the small groupoid $\mathcal{H}_K$ with triples $((V,h), s, \eta K)$ as objects. Here

- (V,h) is a rational Hodge structure of type $\{(-1,0),(0,-1)\}$
- s or $-s$ is a polarisation of (V,h)
- ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $\eta : W \otimes \mathbb{A}_f \to V \otimes \mathbb{A}_f$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

Morphisms $((V,h), s, \eta K) \to ((V',h'), s', \eta K)$ are given by isomorphisms $b : (V,h) \to (V',h')$ of rational Hodge structures such that

- the induced map $b' : \mathrm{Hom}(V \otimes V, \mathbb{Q}) \to \mathrm{Hom}(V' \otimes V', \mathbb{Q})$ satisfies $b'(s) = cs'$ for some $c \in \mathbb{Q}^\times$.
- $b \circ \eta K = \eta' K$.

The fact that there are isomorphisms $\eta : W \otimes \mathbb{A}_f \to V \otimes \mathbb{A}_f$ implies that for any $((V,h), s, \eta K) \in \mathcal{H}_K$, $\dim W = \dim V$. As s corresponds to a multiple of ψ under this morphism, we have that (V,s) is a symplectic space over $\mathbb{Q}$ and h gives a complex structure on V.

For any isomorphism $a : V \to W$ under which ψ corresponds to cs, where $c \in \mathbb{Q}^\times$

$$ah := \left(z \mapsto ah(x)a^{-1}\right)$$

lies in X and $a \circ \eta \in \mathrm{End}(W \otimes \mathbb{A}_f)$ lies in $(GSp)(W,\psi)$. For any other isomorphism $a' : V \to W$ with the same property, there is a $g \in G(\mathbb{Q})$ such that $a' = g \circ a$. Hence replacing a with a' replaces $(ah, a \circ \eta)$ with $(ga, g \circ a \circ \eta)$, and similarly replacing η with ηk (for $k \in K$), replaces $(ah, a \circ \eta)$ with $(ah, a \circ \eta k)$.

If we identify all objects corresponding to each other under these relations we get a map

$$\mathrm{Ob}_{\mathcal{H}_K} \to \mathrm{Sh}_K(\mathrm{GSp}(W,\psi), X).$$

[64]See Milne Proposition 6.5 and Aside 6.6.

Proposition 6.3.1. *The map above makes precisely the same identifications as the natural map from $\mathcal{H}_K$ to the set of isomorphism classes $\mathcal{H}_K/\cong$, and it induces a bijection $\mathcal{H}_K/\cong\leftrightarrow Sh_K(GSp(W,\psi),X)$.*

Proof. See proposition 6.3 in [23]. □

Combining this proposition with Riemann's theorem implies that $\mathrm{Sh}_K(\mathrm{GSp}(W,\psi),X_g)$ is the moduli space for the classification of triples $(A,s,\eta K)$, where

- A is a complex abelian variety of dimension $\dim W$
- s or $-s$ is a polarization of $H_1(A,\mathbb{Z})$
- ηK is a K-orbit of isomorphisms $\eta : W\otimes\mathbb{A}_f \to H_1(A,\mathbb{Z})\otimes\mathbb{A}_f$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

Given a $\mathbb{Z}$-lattice $\Lambda\subset W$ consider $\mathrm{Sh}_K(\mathrm{GSp}(W,\psi),X_g)$ into locally symmetric spaces for K of the form

$$K(N) := \{g\in G(\mathbb{A}_f)\big|g.\Lambda\otimes\hat{\mathbb{Z}}\subset\lambda\otimes\hat{\mathbb{Z}} \text{ and } g \text{ acts as the identity on } (\Lambda\otimes\hat{\mathbb{Z}})/(N\Lambda\otimes\hat{\mathbb{Z}})\}.$$

Using the decomposition of $\mathrm{Sh}_K(\mathrm{GSp}(W,\psi),X_g)$ into locally symmetric spaces, we have that $\mathrm{Sh}_K(\mathrm{GSp}(W,\psi),X)$ parametrizes triples (A,λ,η_N) where (A,λ) is a principally polarised variety of dimension $\frac{1}{2}\dim W$ and $\eta_N : W\otimes\mathbb{Z}/N\mathbb{Z}\to H_1(A,\mathbb{Z}/N\mathbb{Z})$ is an isomorphism under which ψ corresponds to a $(\mathbb{Z}/N\mathbb{Z})^\times$-multiple of the bilinear form on $H_1(A,\mathbb{Z}/N\mathbb{Z})$ induced by λ.[65]

The Shimura varieties of Hodge type also have an interpretation as a moduli of Abelian varieties, however this is much more involved to describe. The interested reader can consult [23] or [2].

6.4 Canonical Integral Models and special fibers

Let (G,X) a Shimura datum of Hodge type with a fixed embedding $\phi : (G,X)\hookrightarrow(\mathrm{GSp}(W,\psi),X_g)$ and reflex field E. Let $p\geq 3$ be a prime with G unramified at p. Let $G_{\mathbb{Z}_p}$ be a reductive group over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$.

Suppose $K=K_pK^p$ where $K_p=G(\mathbb{Q}_p)$ and $K^p\subset G(\mathbb{A}_f^p)$ is a compact open subgroup, which is sufficiently small. Let v be a prime in $\mathcal{O}_E$ lying over p and denote by $\mathcal{O}_{E,v}$ its v-adic completion and E_v the v-adic completion of E.

Theorem 1 in [24] implies that the subsystem of $\mathrm{Sh}(G,X)$ with sufficiently small K as above has a $G(\mathbb{A}_f^p)$-equivariant model, in the following sense:

There is a projective system of $\mathcal{O}_{E,v}$-varieties $\mathscr{S}_K(G,X)$, with K ranging over sufficiently small compact open subgroups $K_pK^p\subset G(\mathbb{A}_f)$. There is a K such that $\mathscr{S}_{K'}(G,X)$ smooth over $\mathcal{O}_{E,v}$ for all $K'\subset K$ and $\mathscr{S}_{K''}(G,X)$ étale over $\mathscr{S}_{K'}(G,X)$ for all $K''\subset K'\subset K$. The generic fiber of these are $\mathrm{Sh}_K(G,X)_{E_v}$, and similarly for K' and K''. The maps $i_* : \mathscr{S}'_K(G,X)\to\mathscr{S}_K(G,X)$ are the ones induced by the inclusions $i : K'\to K$.

[65] See [23] for details.

Just like for the original Shimura varieties, there is a right action of $G(\mathbb{A}_f^p)$[66] on this inverse system. Namely for $g \in G(\mathbb{A}_f^p)$, there is a map $g : \mathscr{S}_K(G,X) \to \mathscr{S}_{gKg^{-1}}(G,X)$ which commutes with the projection maps of the system.

These models also satisfy the following extension property, which is the criterion Milne gave in [25]: Let $\mathscr{S} := \varprojlim_K \mathscr{S}_K$. For every regular scheme Y over $\mathcal{O}_{E,v}$ with Y_{E_v} dense in Y, any morphism $Y_{E_v} \to \mathscr{S}_{E_v}$ extends uniquely to a morphism $Y \to \mathscr{S}$.

By section 4.1.4 in [1], there is a $\mathbb{Z}_p$-lattice $\Lambda \subset W \otimes \mathbb{Q}_p$ such ψ restricts to a perfect pairing $\psi : \Lambda \times \Lambda \to \mathbb{Z}_{(p)}$, and that $\phi : G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GSp}(W \otimes \mathbb{Q}_p, \psi)$ is induced by an embedding of $\mathbb{Z}_{(p)}$-group schemes $\phi : G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GSp}(\Lambda, \psi)$. Given such a pair (Λ, ϕ), $\tilde{K}_p := \mathrm{GSp}(\Lambda, \psi)(\mathbb{Z}_p)$ and $\tilde{K}^p \subset \mathrm{GSp}(W, \psi)(\mathbb{A}_f^p)$ a compact open, let $\tilde{K} := \tilde{K}_p \tilde{K}^p$ and $\mathscr{S}_{g,\tilde{K}}$ the integral canonical model of $\mathrm{Sh}_{\tilde{K}}(\mathrm{GSp}(2g, X_g))$ over $\mathbb{Z}_{(p)}$. Then for $K^p \subset G(\mathbb{A}_f^p)$ small enough, there is a compact open $\tilde{K}^p \subset \mathrm{GSp}(2g, \mathbb{A}_f^p)$ such that $\phi(K^p) \subset \tilde{K}^p$ and a finite morphism of $\mathcal{O}_{E,(v)}$-schemes

$$\phi^{\mathrm{Sh}} : \mathscr{S}_K \to \mathscr{S}_{g,\tilde{K}} \times_{\mathbb{Z}(p)} \operatorname{Spec} \mathcal{O}_{E,(v)}.$$

6.4.1 The G-zip over the special fiber

Pick any $h \in X$ and let $\mu := h_{\mathbb{C}} \circ \mu_0 \in X_*(G)$. Here $\mu_0 : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ is the cocharacter induced by $x \mapsto (x,1) \in \mathbb{G}_m \times \mathbb{G}_m$ followed by the isomorphism $\mathbb{G}_m \times \mathbb{G}_m \cong \mathbb{S}_{\mathbb{C}}$ discussed in section 6.1.

The reflex field, E, is the field of definition of the $G(\mathbb{C})$-conjugacy class $[\mu]$ of μ, see [2] for details. Base changing to an algebraic closure $\overline{E_v}$ of E_v gives a unique conjugacy class $[\mu]_{\overline{E_v}}$ of cocharacters of $G_{\overline{E_v}}$, which has a representative $\mu \in X_*(G_{E_v})$ defined over E_v.[67]

Let $h_g := \phi \circ h \in X_g$ and $\mu_g := \phi \circ \mu \in X_*(\mathrm{GSp}(2g))$. The cocharacters μ and μ_g determine the following groups

1. A pair of opposite parabolic subgroups (P^-, P^+) in G_{E_v} defined by μ_{E_v} and a common Levi subgroup $L := P^- \cap P^+ = \mathrm{Cent}(\mu_{E_v})$. Let $P := P^-$
2. A pair of opposite parabolic subgroups (P_g^-, P_g^+) in $\mathrm{GSp}(2g)_{E_v}$ defined by μ_{g,E_v} and a common Levi subgroup $L_g := P_g^- \cap P_g^+ = \mathrm{Cent}(\mu_{g,E_v})$. Let $P_g := P_g^-$.

Using the embedding ϕ to identify G with a subgroup of $\mathrm{GSp}(2g)$, it follows that $P^{\pm} = P_g^{\pm} \cap G$ and $L_g \cap G = L$.

Fix a Borel pair (B,T) in $G_{\mathbb{Q}_p}$ such that $B_{E_p} \subset P$. Let $I \subset \Delta(B,T)$ denote the type of P and define $B_L := B \cap L$. By section 4.1.7 of [1], we may assume that all of the above is defined integrally, i.e. over $\mathcal{O}_{E,v}$, with the cocharacter μ_g defining a $\mathbb{Z}$-grading $\Lambda \otimes \mathcal{O}_{E,v} = \Lambda_0 \oplus \Lambda_{-1}$.[68]

Using the interpretation of the Siegel modular variety as a moduli of abelian varieties, which still holds for the integral model,[69] we get a universal abelian

[66] Notice the restriction to the p-adicaly trivail adelic points however.
[67] This is a consequence of G being unramified over p and therefore $G_{\mathbb{Q}_p}$ is quasi-split, see [1].
[68] Where $\mathbb{G}_m$ acts by $x \mapsto x^{-i}$ on Λ_i.
[69] See [24] for details

scheme $f : \mathcal{A}_{g,K} \to \mathscr{S}_{g,\tilde{K}}$.

By proposition 1.3.2 [24], $G_{\mathbb{Z}_p}$ is defined as the pointwize stabilizer of a set of tensors $\{s_\alpha\} \subset \Lambda^{\otimes}$, and by 2.3.9 in [24], there are associated tensors $\{s_{\alpha,dR}\} \subset H^0(\mathscr{S}_K, \phi^* H^1_{dR}(\mathcal{A}_{g,\tilde{K}}/\mathscr{S}_{g,\tilde{K}})^{\otimes})$. [70]

Let $S_K := (\mathscr{S}_K)_\kappa$, with $\kappa = \mathcal{O}_v/(v)$, be the special fiber of $\mathscr{S}_K$, set $G = G_{\mathbb{Z}_p} \otimes \mathbb{F}_p$ and $A = (\mathscr{S}_K \times_{\mathscr{S}_{g,\tilde{K}}} \mathcal{A}_g)_\kappa$. Write $\mu : \mathbb{G}_{m,\kappa} \to G_\kappa$ for the reduction of μ. Define $P, Q \subset G_\kappa$ as the stabilizers of $\mathrm{Fil}_P := \Lambda_{0,\kappa}$ and $\mathrm{Fil}_Q :=^\sigma \Lambda_{-1,\kappa}$ respectively. Both of which are parabolic subgroups.

Let $\overline{s_\alpha}$ and $\overline{s_{\alpha,dR}}$ be the reduction of s_α and $s_{\alpha,dR}$ respectively.

By section 1.11 in [4], there are two filtrations of the de Rahm cohomology sheaf $H^1(A/S_K) := H^1_{dR}(\mathcal{A}_{g,K}/\mathscr{S}_{g,\tilde{K}}) \otimes \kappa$. The Hodge filtration Fil_H and the conjugate filtration $\mathrm{Fil}_{\mathrm{conj}}$[71] related by Cartier isomorphisms

$$\iota_0 : \mathrm{Fil}_H^{(p)} \cong H^1_{dR}(A/S_K)/\mathrm{Fil}_{\mathrm{conj}} \quad \iota_1 : (H^1_{dR}(A/S_K)/\mathrm{Fil}_H)^{(p)} \cong \mathrm{Fil}_{\mathrm{conj}}.$$

Let

$$I := \mathit{Isom}_{S_K}((H^1_{dR}(A/S_K), \bar{s}_{dR}), (\Lambda, \bar{s}) \otimes \mathcal{O}_{S_K})$$
$$I_P := \mathit{Isom}_{S_K}((H^1_{dR}(A/S_K), \bar{s}_{dR}, \mathrm{Fil}_H), (\Lambda, \bar{s}, \mathrm{Fil}_P) \otimes \mathcal{O}_{S_K})$$
$$I_Q := \mathit{Isom}_{S_K}((H^1_{dR}(A/S_K), \bar{s}_{dR}, \mathrm{Fil}_{\mathrm{conj}}), (\Lambda, \bar{s}, \mathrm{Fil}_Q) \otimes \mathcal{O}_{S_K})$$

which are respectively G, P and Q torsors. See section 2.1.1.

The isomorphisms ι_0 and ι_1 induces an isomorphism between the Frobenius pullback of the associated graded of the de Rahm cohomology with the Hodge filtration and the associated graded of the de Rahm cohomology with the conjugate filtration. The subgroup of G stabilizing the associated graded of Fil_H is L and $L^{(p)}$ is the stabilizer of the associated graded of $\mathrm{Fil}_{\mathrm{conj}}$. Thus ι_0 and ι_1 induces an isomorphism of $L^{(p)}$-torsors $\iota : (I_P)^{(p)}/U^{(p)} \to I_Q/V$ with U, V the unipotent radicals of P and Q respectively.[72]

This data precisely gives a G-zip of type μ over S_K, and consequently a morphism

$$\zeta_K : S_K \to G\text{-Zip}^\mu.$$

Zhang showed, in [5], that ζ_K is smooth.

This morphism is compatible with the projections between different levels[73] and with the $G(\mathbb{A}_f^p)$-action.[74] For $K' \subset K$ and $g \in G(\mathbb{A}_f^p)$, this is summarised by

[70] Here $H^1_{dR}(\mathcal{A}_{g,\tilde{K}}/\mathscr{S}_{g,\tilde{K}})$ is the first de Rahm cohomology. See [4] for details

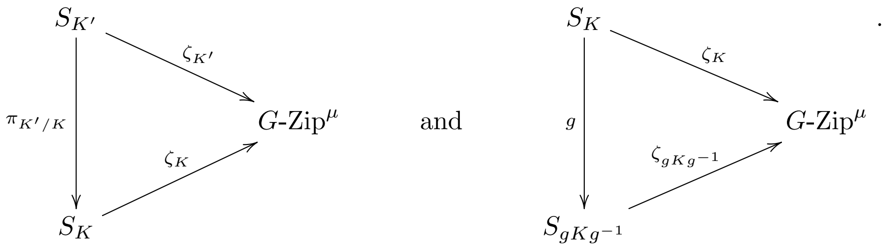
[71] Both with one non-trivial step, denoted by the name of the filtration.

[72] Recall that $L \cong P/U$.

[73] That between different K so we can view it as a morphism ζ from the inverse system of reductions of Shimura varieties

[74] For more details on this, see 4.1.10 and 4.2 in [1].

the commutative diagrams



6.5 Implications of the cone conjecture for Shimura varieties

The previous section shows that S_K is a G-zip scheme which satisfies condition (1) of the cone conjecture as the Shimura datum is of Hodge type, and condition (2) as ζ_K is smooth.

In the Siegel case, S_K is the moduli of principally polarised abelian varieties with level K structure. The flag strata corresponding to $w \in W$ of length 1 are pseudo complete, so S_K satisfies condition condition (3). Here is an argument, due to W.Goldring, showing this.

By lemma 2.4.1 in [1], the image of a flag stratum corresponding to s_α with $\alpha \in \text{type}(P)$ is the unique closed Ekedahl-Oort stratum of S_K. By lemma 6.4.1 in [1], this stratum doesn't intersect the boundary in a toroidal compactification, S_K^Σ, of S_K. So it is a closed set of a proper scheme and hence proper, which is stronger than being pseudo-complete.

For s_α with $\alpha \in \Delta \setminus \text{type}(P)$,[75] proposition 2.4.3 in [1] implies that the image of the Bruhat strata is precisely the Ekedahl Oort stratum corresponding to the same Weyl group element.[76] Then, using [26], one gets that an abelian variety A, corresponding to a point of this stratum, $S_{K,\alpha}$, has p-rank 0. That is $\dim \text{Hom}_{k-\text{Groups}}(\mathbb{Z}/p\mathbb{Z}, A[p]) = 0$[77]. Then, using the proof of lemma 6.4.1 in [1], one gets that the multiplicative rank of a semi-abelian variety lying in the strata S^Σ_{K,s_α}[78] is constant. The multiplicative rank of A is defined as $\text{mult rk}(A) := \dim \text{Hom}_{k-\text{Groups}}(\mu_p, A)$. By duality of principally polarized abelian varieties, see §15 in [27], the p-rank and the multiplicative rank of an abelian variety A corresponding to a point in S_{K,s_α}, are the same. Hence the multiplicative rank of any semi-abelian variety corresponding to a point of S^Σ_{K,s_α} is 0. If A is a semi-abelian variety with a non-trivial torus part, then there is an embedding $\mu_p \hookrightarrow \mathbb{G}_m \hookrightarrow A$, so the multiplicative rank of A is non-zero. Hence one concludes that $S_{K,s_\alpha} = S^\Sigma_{K,s_\alpha}$, with closure $S^\Sigma_{K,s_\alpha} \cup S^\Sigma_{K,e} = S_{K,s_\alpha} \cup S_{K,e}$, which is proper as a closed subscheme of a proper scheme.

Hence for every $g \geq 2$, a proof of the cone conjecture for G of type C_2 yields a vanishing result on the automorphic vector bundles of the reduction of the

[75] This implies $\alpha^\vee = \mu$ in our case.

[76] s_α corresponds to an Ekedahl-Oort stratum as it is minimal and cominimal, see [1].

[77] $A[p] := \ker(A \ni x \mapsto px \in A)$

[78] Where we're using that the extension of σ to S_K^Σ defined in chapter 6 of [1] induces an Ekedahl-Oort stratification of S_K^Σ as well.

Shimura variety $\mathrm{Sh}(\mathrm{GSp}(2g), X_g)$. Moreover, as the ζ_K are booth $G(\mathbb{A}_f)$ equivariant and compatible with the projections $\pi_{K/K'}$, this result is independent of the level structure.

For example, the proof in the C_2 case implies that the saturated global sections cone of the reduction of the Siegel modular threefold, of any level, is given by

$$\begin{aligned} a_1 - a_2 &\geq 0 \\ -pa_1 - a_2 &\geq 0 \end{aligned}$$

with the same conventions as in section 5.

Similar arguments using the relations of the different stratifications are possible to use to show that other Shimura varieties of Hodge type satisfy the pseudo-completeness in the length one strata as well. Currently there is, to my knowledge, no argument showing this for all Shimura varieties of Hodge type.

In [28], Andreatta showed that the extension of ζ_K to a toroidal compactification, S_K^{Σ}, of S_K is smooth as well. As this compactification is proper, all strata closures are proper as well. Hence a proof of the cone conjecture in any case where G is of a type admitting a Hodge-type Shimura datum, we get a vanishing result on the cohomology of the toroidal compactifications of the varieties S_K. This is again independent of the level by theorem 6.2.1 in [1].

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