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Coxeter groups, Hecke algebras and Kazhdan-Lusztig cells

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Abstract

Coxeter groups arise in many different areas of mathematics and are extensively studied in algebra, geometry, and combinatorics. One of their important areas of applications is by means of their associated Hecke algebras, which are certain deformations of the group algebras of Coxeter groups that play an important role in representation theory. A turning point in the study of representations of Hecke algebras was the celebrated paper *"Representations of Coxeter groups and Hecke algebras"* by Kazhdan and Lusztig where the notions of left, right and two-sided cells of an arbitrary Coxeter group, now called Kazhdan-Lusztig cells, were first introduced. Their definition incorporates a new canonical basis of the Hecke algebra, the Kazhdan-Lusztig basis, and they give rise to representations of both the Coxeter group itself and the associated Hecke algebra.

In this thesis we start with an introduction of Coxeter groups, focusing on some structural aspects of its rich theory that are of combinatorial, algebraic and geometric interest. We then move on to study their associated Hecke algebras, starting from a more general construction of associative algebras over a commutative ring, leading towards the construction of the Kazhdan-Lusztig basis and the study of the action of the canonical basis of the Hecke algebra on the Kazhdan-Lusztig basis, which turns out to be key in the determination of the partition of the Coxeter group into Kazhdan-Lusztig cells as well as the properties of the partition. We then focus on the study of Kazhdan-Lusztig cells and discuss several tools that allow us to deduce deep properties about Kazhdan-Lusztig cells as well as a series of related conjectures by Lusztig.

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Contents

1 Introduction

This thesis is divided into three main parts, and each part builds on the previous ones. We now give a brief outline of the content of this thesis.

Chapter 2 is intended as an introduction to the rich theory of Coxeter systems so that the reader gets a feeling of the different areas where one encounters these. In particular, we define it as a group defined by generators and relations, introduce the length function and the Bruhat ordering. We also present a characterization of Coxeter systems as discrete reflection groups and establish the relation between Coxeter systems, Coxeter graphs and Coxeter matrices. We also study a special class of subgroups, and give a brief introduction to the classification of the finitely generated Coxeter groups.

In Chapter 3, we start by constructing general associative algebras over a commutative ring with unity, which we then use to obtain the Hecke algebra associated to a Coxeter system together with its canonical basis, by choosing a particular set of parameters. We also study the relation between the Hecke algebra and the Bruhat ordering, which requires the study of 'R-polynomials as well as the inverse of the canonical basis elements of the Hecke Algebra. We then introduce the bar involution which is used to define Kazhdan-Lusztig basis of the Hecke algebra together with a set of polynomials, called the Kazhdan-Lusztig polynomials, which we compute explicitly from scratch. We also study the action of the canonical basis on the Kazhdan-Lusztig basis which allows us to derive a crucial property about the Kazhdan-Lusztig polynomials. We then introduce a 'dual' basis for the Kazhdan-Lusztig basis.

Finally, in Chapter 4, we get to my favourite part of the thesis. We introduce the original setting for Kazhdan-Lusztig cells and explain their relevance in that they give rise to representations of both the Coxeter system and its associated Hecke algebra and explore some of its most important properties. We then explore a series of tools such as Lusztig's **a**-function and the set of distinguished involutions, which turn out to be key in the proof of many of the results known today about the partition of the group into Kazhdan-Lusztig cells. We also consider the Weyl group of type A_2 and the affine Weyl group of type $\widetilde{\mathrm{A}}_2$ to illustrate some of the results we present. We end with a thorough account of a series of 15 conjectures by Lusztig' which seem to control the behaviour of the **a**-function, as well as a detailed discussion about the relations between these. We also present a quick survey on the cases for which the partition into Kazhdan-Lusztig cells is known.

2 Coxeter Systems

In this section, we introduce the theory of Coxeter groups following [5] and [21]. We fix a group W, written multiplicatively and with identity element 1, together with a generating subset S of W consisting of elements of order 2. If s, s' are elements of S, we denote by $m_{ss'}$ the order of the ss' in W. Note that S is not assumed to be finite and that $m_{ss'}$ might be infinite.

2.1 Coxeter systems

In this section we define what does it mean for the pair (W, S) to be a Coxeter system.

Definition 2.1.1. We say that the pair (W, S) is a *Coxeter system* if W admits a presentation of the form

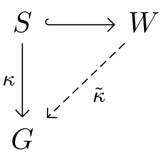
$$W = \langle S \mid (ss')^{m_{ss'}} = 1 \text{ for all pairs } (s, s') \in S_F \rangle, \tag{2.1.1}$$

where S_F is the set of all pairs $(s, s') \in S \times S$ such that $m_{ss'}$ is finite. The group W is called the *Coxeter group* and S is the set of *Coxeter generators*. The cardinality of S is called the *rank* of (W, S).

Remark 2.1.1. The fact that the group W admits a presentation of the form in (2.1.1), means that (W, S) satisfies the following universal property: if G is a group and $\kappa : S \to G$ is a map from S to G satisfying

$$\big(\kappa(s)\kappa(s')\big)^{m_{ss'}} = 1 \quad \text{for all } (s, s') \in S_F,$$

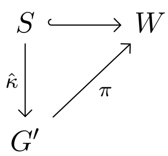
there exists a unique homomorphism $\tilde{\kappa} : W \to G$ from W to G that is an extension of κ.



The uniqueness of the homomorphism $\tilde{\kappa} : W \to G$ follows from the fact that S generates W. Equivalently, if G' is a group, $\pi : G' \to W$ a homomorphism from G' to W and $\hat{\kappa} : S \to G'$ a map from S to G' such that

$$(\pi \circ \hat{\kappa})(s) = s \quad \text{and} \quad \big(\hat{\kappa}(s)\hat{\kappa}(s')\big)^{m_{ss'}} = 1 \text{ for all } (s, s') \in S \times S,$$

and such that the $\hat{\kappa}(s)$ for $s \in S$ generate G', then π is injective and hence an isomorphism from G' to W.



Definition 2.1.2. If s, s' are two elements of S and $r \geq 0$ is an integer, we may define an element $\mathsf{prod}(r; s, s')$ of W inductively as follows:

$$\mathsf{prod}(0; s, s') = 1,$$
$$\mathsf{prod}(r + 1; s, s') = \mathsf{prod}(r; s', s) \cdot s',$$

with the convention that $\mathsf{prod}(\infty; s, s') = 1$. With this notation in mind, we may write a presentation of W as follows:

$$W = \left\langle \, S \;\middle|\; s^2 = 1 \text{ for all } s \in S,\ \mathsf{prod}(m_{st}; s, s') = \mathsf{prod}(m_{ss'}; s', s) \text{ for all } s, s' \in S \, \right\rangle.$$

2.2 The length function

Note that since every element of S is of order 2, every element of S is in particular its own inverse, and thus, since S generates W, every non-identity element $w \in W$ can be written as a product $w = s_1 s_2 \cdots s_r$ of elements $s_1, \ldots, s_r \in S$. In this section we define the length of a typical element $w \in W$ and the concept of reduced decomposition of w. We also prove some elementary properties about the length function.

Definition 2.2.1. Let $w \in W$ be an element of W. The *length of w with respect to S*, denoted by $\ell(w)$, is the smallest integer $r \geq 0$ such that w can be written as the product of a sequence of r elements of S. The map $w \mapsto l(w)$ defines a function $\ell : W \to \mathbb{N}_0$, called the *length function of W with respect to S.*

Remark 2.2.1. Clearly, 1 is the unique element of length 0 in W, and S is precisely the set of elements of length 1.

Definition 2.2.2. Let $w \in W$. A *reduced decomposition* of w with respect to S is any sequence $(s_1, s_2, \ldots, s_r)$ of elements of S such that $w = s_1 s_2 \cdots s_r$ and $r = l(w)$. We say that $w = s_1 \cdots s_r$ is a *reduced expression of w.*

Remark 2.2.2. Since the elements of S are involutions, if $w = s_1 \cdots s_r$ is a reduced expression of w, then clearly

$$\ell(ws_r) = l(s_1 \cdots s_{r-1}) = r - 1,$$
$$\ell(s_1 w s_r) = \ell(s_2 \cdots s_{r-1}) = r - 2,$$

and so on. However, since an element of W may have more than one reduced decomposition with respect to S, the length function has its subtleties. Nevertheless, we can still derive some elementary properties of the length function.

Proposition 2.2.1. *Let $w, w' \in W$ be arbitrary. Then*

$$\ell(ww') \leq \ell(w) + \ell(w'), \tag{2.2.1}$$
$$\ell(w^{-1}) = \ell(w), \tag{2.2.2}$$
$$\big|\ell(w) - \ell(w')\big| \leq \ell(ww'^{-1}). \tag{2.2.3}$$

Proof. Let $w = s_1 \ldots s_r$ and $w' = s'_1 \ldots s'_q$ be reduced expressions of w and w', respectively. Then $\ell(w) = r$, $\ell(w') = q$ and $ww' = s_1 \cdots s_r s'_1 \cdots s'_q$. Hence ww' can be writen as the product of a sequence of $r + q$ elements of S, and thus $\ell(ww') \leq r + q$, which proves (2.2.1). Also, since $s_i^{-1} = s_i$ for each $i = 1, \ldots, r$, it follows that

$$w^{-1} = (s_1 \cdots s_r)^{-1} = s_r^{-1} \cdots s_1^{-1} = s_r \cdots s_1,$$

and thus $\ell(w^{-1}) \leq r = l(w)$. Moreover, since $w = \left(w^{-1}\right)^{-1}$, the same argument shows that $\ell(w) \leq \ell(w^{-1})$, which proves (2.2.2). Finally, replacing w by ww'^{-1} in (2.2.1) and (2.2.2) yields the relations

$$\ell(w) - \ell(w') \leq \ell(ww'^{-1}), \tag{2.2.4}$$
$$\ell(ww'^{-1}) = \ell(w'w^{-1}). \tag{2.2.5}$$

Exchanging w and w' in (2.2.4) and using (2.2.5) gives also

$$\ell(w') - \ell(w) \leq \ell(ww'^{-1}),$$

which proves (2.2.3). □

Corollary 2.2.1. *Let* $\mathbf{s} = (s_1, \ldots, s_r)$ *and* $\mathbf{s}' = (s'_1, \ldots, s'_q)$ *be two sequences of elements of* S *such that* $w = s_1 \cdots s_r$ *and* $w' = s'_1 \cdots s'_q$*. If the sequence* $(s_1, \ldots, s_r, s'_1, \ldots, s'_q)$ *is a reduced decomposition of* ww'*, then* $\mathbf{s}$ *is a reduced decomposition of* w *and* $\mathbf{s}'$ *is a reduced decomposition of* w'*.*

Proof. Since $w = s_1 \cdots s_r$ and $w' = s'_1 \cdots s'_q$, we have that $\ell(w) \leq r$ and $\ell(w') \leq q$, and since $ww' = s_1 \cdots s_r s'_1 \cdots s'_q$ is a reduced expression of ww', we have that $\ell(ww') = r + q$. Combining these with (2.2.1), we must have $l(w) = r$ and $l(w') = q$, showing that $\mathbf{s}$ and $\mathbf{s}'$ are reduced decompositions of w and w', respectively. □

Remark 2.2.3. If (W, S) is a Coxeter system, we may define a homomorphism $\varepsilon : \mathrm{F}(S) \to \{1, -1\}$ from the free group $\mathrm{F}(S)$ on the set S onto the multiplicative group $\{1, -1\}$ by sending each element of S to -1. By the universal property, this induces an epimorphism $\tilde{\varepsilon} : W \to \{1, -1\}$ sending each $s \in S$ to -1. Moreover note that if $w = s_1 \cdots s_r$ is any reduced expression of the element $w \in W$, then

$$\tilde{\varepsilon}(w) = \tilde{\varepsilon}(s_1 \cdots s_r) = \tilde{\varepsilon}(s_1) \cdots \tilde{\varepsilon}(s_r) = (-1)^r = (-1)^{\ell(w)}, \tag{2.2.6}$$

which shows that the epimorphism $\tilde{\varepsilon} : W \to \{1, -1\}$ in Remark 2.2.3 is given in terms of the length function by

$$\tilde{\varepsilon}(w) = (-1)^{\ell(w)} \quad \text{for any } w \in W.$$

This is very useful to prove the following result.

Lemma 2.2.1. *Let* $w, w' \in W$ *be arbitrary elements of* W *and let* $s \in S$ *be an arbitrary element of* S*. We have:*

(a) $\ell(ws) = \ell(w) \pm 1.$

(b) $\ell(sw) = \ell(w) \pm 1.$

(c) $\ell(ww') \equiv \ell(w) + \ell(w') \mod 2.$

Proof. Since

$$(-1)^{\ell(ws)} = \tilde{\varepsilon}(ws) = \tilde{\varepsilon}(w)\tilde{\varepsilon}(s) = -\tilde{\varepsilon}(w) = -(-1)^{\ell(w)}$$

it follows that $\ell(ws) \neq \ell(w)$. The inequalities in (2.2.1) and (2.2.3) then give

$$\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$$

This together with $\ell(ws) \neq \ell(w)$ gives (a). The proof of (b) is similar. Finally, since

$$(-1)^{\ell(ww')} = \tilde{\varepsilon}(ww') = \tilde{\varepsilon}(w)\tilde{\varepsilon}(w') = (-1)^{\ell(w)+\ell(w')},$$

the equivalence in (c) follows. □

2.3 Characterization of Coxeter systems

In this section, our aim is to characterize Coxeter systems as discrete reflection groups. In order to do this, we first need to study the relation between the set T of conjugates in W of elements of S and the reduced decompositions of elements of W with respect to S.

Definition 2.3.1. Define

$$T := \bigcup_{w \in W} wSw^{-1}. \tag{2.3.1}$$

The elements of S are called the *simple reflections of* W and the elements of T are called the *reflections of* W. Moreover, define

$$\Phi := \{1, -1\} \times T.$$

The elements of Φ are called the *roots of* W.

Definition 2.3.2. For any finite sequence $\mathbf{s} = (s_1, \ldots, s_r)$ of elements of S, denote by $\Psi(\mathbf{s})$ the sequence $(t_1, \ldots, t_r)$ of elements of T defined by

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1} \quad \text{for } 1 \leq j \leq r. \tag{2.3.2}$$

Remark 2.3.1. Note that $t_1 = s_1$ and $s_1 \cdots s_r = t_r t_{r-1} \cdots t_1$.

Definition 2.3.3. For any element $t \in T$, let $n(\mathbf{s}, t)$ denote the number of integers $1 \leq j \leq r$ such that $t_j = t$.

Definition 2.3.4. For each $s \in S$, define a map $U_s : \Phi \to \Phi$ by

$$U_s(\epsilon, t) := \left(\epsilon \cdot (-1)^{\delta_{s,t}}, sts^{-1}\right) \quad \text{for all } (\epsilon, t) \in \Phi, \tag{2.3.3}$$

where $\delta_{s,t}$ is the Kronecker symbol.

Lemma 2.3.1. *We have $U_s^2 = \mathrm{id}_\Phi$. In particular, U_s is a permutation of Φ, i.e., $U_s \in \mathsf{Perm}(\Phi)$.*

Proof. For any $(\epsilon, t) \in \Phi$, we have that

$$\begin{aligned} U_s^2(\epsilon, t) &= U_s \left(\epsilon \cdot (-1)^{\delta_{s,t}}, sts^{-1}\right) \\ &= \left(\epsilon \cdot (-1)^{\delta_{s,t}} \cdot (-1)^{\delta_{s,sts^{-1}}}, s(sts^{-1})s^{-1}\right) \\ &= \left(\epsilon \cdot (-1)^{\delta_{s,t}} \cdot (-1)^{\delta_{s,sts^{-1}}}, t\right). \end{aligned} \tag{2.3.4}$$

Since the elements of S are of order 2, we have that $s = t$ if and only if $s = sts^{-1}$, and thus

$$\delta_{s,t} + \delta_{s,sts^{-1}} \equiv 0 \mod 2.$$

Combining this with (2.3.4) yields

$$U_s^2(\epsilon, t) = (\epsilon, t) = \mathrm{id}_\Phi(\epsilon, t),$$

which shows that $U_s^2 = \mathrm{id}_\Phi$. □

Lemma 2.3.2. $(U_{s'} U_s)^{m_{ss'}} = \mathrm{id}_\Phi$ *for all $s, s' \in S$.*

Proof. Let $\mathbf{s} := (s_1, \ldots, s_r)$ be a sequence of elements of S. Set $w := s_r \cdots s_1$ and $U_\mathbf{s} := U_{s_r} \cdots U_{s_1}$. We show, by induction on $r \geq 0$, that

$$U_\mathbf{s}(\epsilon, t) = \left(\epsilon \cdot (-1)^{n(\mathbf{s},t)}, wtw^{-1}\right) \quad \text{for any } (\epsilon, t) \in \Phi. \tag{2.3.5}$$

For $r = 0$, the claim is clear, and for $r = 1$ we have that $\delta_{s,t} = n(\mathbf{s}, t)$ and the claim follows from the definition of U_s in (2.3.3). If $r > 1$, set $\mathbf{s}' := (s_1, \ldots, s_{r-1})$ and $w' := s_{r-1} \cdots s_1$. By the induction hypothesis, we have that for any $(\epsilon, t) \in \Phi$,

$$U_{\mathbf{s}'}(\epsilon, t) = \left(\epsilon \cdot (-1)^{n(\mathbf{s}',t)}, w'tw'^{-1}\right).$$

We thus obtain

$$\begin{aligned} U_\mathbf{s}(\epsilon, t) = U_{s_r} U_{\mathbf{s}'}(\epsilon, t) &= U_{s_r} \left(\epsilon \cdot (-1)^{n(\mathbf{s}',t)}, w'tw'^{-1}\right) \\ &= U_{s_r} \left(\epsilon \cdot (-1)^{n(\mathbf{s}',t)} \cdot (-1)^{\delta_{s_r,w'tw'^{-1}}}, s_r w'tw'^{-1} s^{r-1}\right) \\ &= \left(\epsilon \cdot (-1)^{n(\mathbf{s}',t) + \delta_{s_r,w'tw'^{-1}}}, wtw^{-1}\right). \end{aligned} \tag{2.3.6}$$

But

$$\Psi(\mathbf{s}) = \big(\Psi(\mathbf{s}'), t_r\big) = \Big(\Psi(\mathbf{s}'), w' s_q w'^{-1}\Big)$$

and

$$n(\mathbf{s}, t) = n(\mathbf{s}', t) + \delta_{w'^{-1} s_r w', t} = n(\mathbf{s}', t) + \delta_{s_r, w' t w'^{-1}},$$

so (2.3.6) becomes

$$U_{\mathbf{s}}(\epsilon, t) = \Big(\epsilon \cdot (-1)^{n(\mathbf{s},t)}, wtw^{-1}\Big),$$

which proves the claim.

Now, take any $s, s' \in S$ and let $\mathbf{s} := (s_1, \ldots, s_{2m_{ss'}})$ be the sequence of elements of S defined by $s_j = s$ for j odd and $s_j = s'$ for j even. Then

$$s_1 \cdots s_{2m_{ss'}} = (ss')^{m_{ss'}} = 1$$

and (2.3.2) implies that

$$t_j = (ss')^{j-1} s \quad \text{for all } 1 \le j \le 2_{m_{ss'}}.$$

Since $m_{ss'}$ is the order of ss', the elements $t_1, \ldots, t_{m_{ss'}}$ are all distinct and, for each $1 \le j \le m_{ss'}$, we have $t_{j+m} = t_j$. Therefore, for any given $t \in T$, the number of integers j such that $t_j = t$ is equal to 0 or 2, i.e. $n(\mathbf{s}, t) = 0$ or $n(\mathbf{s}, t) = 2$. Combining this with (2.3.5), we obtain

$$(U_{s'} U_s)^{m_{ss'}}(\epsilon, t) = (\epsilon, t) = \mathrm{id}_\Phi(\epsilon, t) \quad \text{for any } (\epsilon, t) \in \Phi,$$

which shows that $(U_{s'} U_s)^{m_{ss'}} = \mathrm{id}_\Phi$. □

Remark 2.3.2. In view of Lemmas 2.3.1 and 2.3.2, it follows, by the definition of Coxeter systems, that the map $s \mapsto U_s$ extends uniquely to a homomorphism from W to the group of permutations of Φ given by

$$\begin{aligned} U : W &\to \mathsf{Perm}(\Phi) \\ w &\mapsto U_w, \end{aligned} \tag{2.3.7}$$

where $U_w = U_{\mathbf{s}}$ for every sequence $\mathbf{s} = (s_1, \ldots, s_r)$ such that $w = s_r \cdots s_1$. In particular, it follows from (2.3.5) that $(-1)^{n(\mathbf{s},t)}$ has the same value for all sequences $\mathbf{s} = (s_1, \ldots, s_r)$ such that $w = s_1 \cdots s_r$. For each w in W and each t in T, we denote such a value by $\eta(w;t)$. Moreover, note that defining the homomorphism U in (2.3.7) is equivalent to having defined an action of W on the set of roots Φ.

Lemma 2.3.3. *Let* $\mathbf{s} = (s_1, \ldots, s_r)$, $\Psi(\mathbf{s}) = (t_1, \ldots, t_r)$ *and* $w = s_1 \cdots s_r$. *Let* T^w *be the set of elements* t *in* T *such that* $\eta(w;t) = -1$, *and let* card (T^w) *denote the cardinality of* T^w. *Then* $\mathbf{s}$ *is a reduced decomposition of* w *if and only if the* t_j *are all distinct. In that case we have* $T^w = \{t_1, \ldots, t_r\}$ *and* card $(T^w) = \ell(w)$.

Proof. First assume that the t_j are not all distinct. Then there exists a pair of indices i, j with $1 \le i < j \le r$ such that $t_i = t_j$, which implies that $s_i = (s_{i+1} \cdots s_{j-1}) s_j (s_{i+1} \cdots s_{j-1})^{-1}$. Hence, since $w = s_1 \cdots s_r$, we have

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_r,$$

which shows that $\mathbf{s}$ is not a reduced decomposition of w.

Conversely, assume that the t_j are all distinct. First note that if t belongs to T^w, then

$$-1 = \eta(w;t) = (-1)^{n(\mathbf{s}',t)}$$

for any sequence $\mathbf{s}' = (s'_1, \ldots, s'_q)$ such that $w = s'_1 \cdots s'_q$. It follows that $n(\mathbf{s}', t) \ge 1$ and thus, by definition, we have that $t = t'_i$ for an odd number of j's with $1 \le i \le q$, where $\Psi(\mathbf{s}') = (t'_1, \ldots, t'_q)$.

This shows that $t \in \{t'_1, \ldots, t'_q\}$. Hence $T^w \subseteq \{t'_1, \ldots, t'_q\}$ and $\operatorname{card}(T^w) \leq q$. In particular, if we chose $\mathbf{s}'$ such $\mathbf{s}'$ is a reduced decomposition of w, it follows that

$$\operatorname{card}(T^w) \leq \ell(w). \tag{2.3.8}$$

Now, since the t_j are all distinct, we have $n(\mathbf{s}, t_j) = 1$ for each $1 \leq j \leq r$, which gives $\eta(w; t_j) = -1$. This shows that $t_j \in T^w$ and thus $\{t_1, \ldots, t_r\} \subseteq T^w$. Then, by the previous paragraph $T^w = \{t_1, \ldots, t_r\}$ and $\ell(w) \leq r = \operatorname{card}(T^w)$. This together with the inequality in (2.3.8) implies that $r = \ell(w)$, which shows that $\mathbf{s}$ is a reduced decomposition of w. □

Lemma 2.3.4. *Let $w \in W$ and $s \in S$ be elements such that $\ell(sw) \leq \ell(w)$. For any sequence $\mathbf{s} = (s_1, \ldots, s_r)$ of elements of S with $w = s_1 \cdots s_r$, there exists an integer i such that $1 \leq i \leq r$ and*

$$ss_1 \cdots s_{i-1} = s_1 \cdots s_{i-1}s_i. \tag{2.3.9}$$

Proof. Let $q = \ell(w)$ and set $w' = sw$. By (c) in Lemma 2.2.1, we have that

$$\ell(w') \equiv q + 1 \mod 2.$$

Thus, combining the hypothesis $\ell(w') \leq \ell(w)$ with the relation

$$\left|q - \ell(w')\right| \leq \ell(ww'^{-1}) = \ell(s) = 1$$

yields $\ell(w') = q - 1$. Now, if $(s'_1, \ldots, s'_{q-1})$ is any reduced decomposition of w', set

$$\mathbf{s}' := \left(s, s'_1, \ldots, s'_{q-1}\right) \quad \text{and} \quad \Psi(\mathbf{s}') := (t'_1, \ldots, t'_q).$$

Then $\mathbf{s}'$ is a reduced decomposition of w and $t'_1 = s$, so by Lemma 2.3.3, the t'_j are all distinct, and we have $n(\mathbf{s}', s) = 1$. Since w is the product of the elements in the sequence $\mathbf{s}$, we have $n(\mathbf{s}, s) \equiv n(\mathbf{s}', s) \mod 2$, which implies that $n(\mathbf{s}, s) \neq 0$. Therefore s must be one of the elements t_i in the sequence $\Psi(\mathbf{s})$. Multiplying both sides of the equality $s = t_i$ by $s_1 \cdots s_{i-1}$ gives the equality in (2.3.9). □

Remark 2.3.3. For the rest of Section 2.3, let (W, S) be a pair with the properties described in the introduction of Section 2. Note that at this point we are no longer requiring that (W, S) is a Coxeter system. The following statement about the pair (W, S) is known as the *Exchange condition.*

Exchange condition. *Let $w \in W$ and $s \in S$ be elements such that $\ell(sw) \leq \ell(w)$. For any reduced decomposition $\mathbf{s} = (s_1, \ldots, s_r)$ of w, there exists an integer i such that $1 \leq i \leq r$ and*

$$ss_1 \cdots s_{i-1} = s_1 \cdots s_{i-1}s_i. \tag{2.3.10}$$

From now until the end of Section 2.3 we only assume that the pair (W, S) satisfies the Exchange condition. Note that by Lemma 2.3.4, if the pair (W, S) is a Coxeter system, then (W, S) satisfies the Exchange condition. Therefore, all the following results apply to Coxeter systems.

Proposition 2.3.1. *Let (W, S) be a pair satisfying the Exchange condition. Let $w \in W$ and $s \in S$ be any elements and let $\mathbf{s} = (s_1, \ldots, s_r)$ be a reduced decomposition of w. Then either*

(a) $\ell(sw) = \ell(w) + 1$ and $(s, \mathbf{s})$ is a reduced decomposition of sw; or

(b) $\ell(sw) = \ell(w) - 1$ and there exists an integer $i \in \mathbb{Z}$ such that $1 \leq i \leq r$, the sequence $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_r)$ is a reduced decomposition of sw and $(s, s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_r)$ is a reduced decomposition of w.

Proof. Set $w' := sw$. Then $ww'^{-1} = s$, so by (2.2.3), we have

$$\left|\ell(w) - \ell(w')\right| \leq \ell(s) = 1,$$

and we distinguish two cases: either $\ell(w') > \ell(w)$ or $\ell(w') \leq \ell(w)$.

So suppose that $\ell(w') > \ell(w)$. Then $w' = ss_1 \cdots s_r$ and $\ell(w') = r + 1$, which shows that $(s, s_1, \ldots, s_r)$ is a reduced decomposition of w' and proves (a).

On the other hand, if $\ell(w') \leq \ell(w)$, then, by the Exchange condition, there exists an integer $j \in \mathbb{Z}$ such that $1 \leq i \leq r$ and (2.3.10) holds. Then $w = ss_1 \cdots s_{i-1}, s_{i+1} \cdots s_r$ and thus

$$w' = s_1 \cdots s_{i-1} s_{i+1} \cdots s_r.$$

Therefore, since $r - 1 \leq \ell(w') < r$, we have that $\ell(w') = r - 1$ and $(s_1, \ldots, s_{i-1} s_{i+1} \ldots, s_r)$ is a reduced decomposition of w'. Finally, by (a) just proven, it follows that the sequence $(s, s_1, \ldots, s_{i-1}, s_{i+1} \ldots, s_r)$ is a reduced decomposition of w. □

Lemma 2.3.5. *Let (W, S) be a pair satisfying the Exchange condition. Let $w \in W$ be an element of length $r \geq 1$, let $\mathsf{Red}(w)$ be the set of reduced decompositions of w, and let $F : \mathsf{Red}(w) \to X$ be a map from $\mathsf{Red}(w)$ to a set X. Assume that $F(\mathbf{s}) = F(\mathbf{s}')$ if the elements $\mathbf{s} = (s_1, \ldots, s_r)$ and $\mathbf{s}' = (s'_1, \ldots, s'_r)$ of $\mathsf{Red}(w)$ satisfy one of the following hypothesis:*

(A) $s_1 = s'_1$ or $s_r = s'_r$.

(B) There exist s, s' in S such that $s_i = s'_j = s$ and $s_j = s'_i = s'$ for all i odd and all j even.

Then F is constant on $\mathsf{Red}(w)$.

Proof. Let $\mathbf{s}, \mathbf{s}' \in \mathsf{Red}(w)$ and $F : \mathsf{Red}(w) \to X$ be as in the statement of the lemma. Set $\mathbf{u} := (s'_1, s_1, \ldots, s_{r-1})$. We first show that if $F(\mathbf{s}) \neq F(\mathbf{s}')$, then $\mathbf{u}$ is an element of $\mathsf{Red}(w)$ and $F(\mathbf{u}) \neq F(\mathbf{s})$. Indeed, since $w = s'_1 \cdots s'_r$, we have that $s'_1 w = s'_2 \cdots s'_r$, so we see that $s'_1 w$ is of length strictly less that r. Hence, by Proposition 2.3.1(b), since $w = s_1 \cdots s_r$ is also a reduced expression of w, there exists an integer i such that $1 \leq i \leq r$ and the sequence $\mathbf{u}' := (s'_1, s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_r)$ is a reduced decomposition of w, and thus belongs to $\mathsf{Red}(w)$. In particular, $\mathbf{s}'$ and $\mathbf{u}'$ satisfy the condition in Hypothesis *(A)*, so we have $F(\mathbf{u}') = F(\mathbf{s}')$. Now, if $i \neq r$, the last term of the sequence $\mathbf{s}$ and the last term of the sequence $\mathbf{u}'$ are the same, so $\mathbf{s}$ and $\mathbf{u}'$ satisfy the condition in Hypothesis *(A)* and thus $F(\mathbf{u}') = F(\mathbf{s}')$. But then,

$$F(\mathbf{s}) = F(\mathbf{u}') = F(\mathbf{s}'),$$

which contradicts our assumption $F(\mathbf{s}) \neq F(\mathbf{s}')$. It thus follows that $i = r$ and hence $\mathbf{u} = \mathbf{u}'$ belongs to $\mathsf{Red}(w)$ and $F(\mathbf{u}) = F(\mathbf{s}') \neq F(\mathbf{s})$, as claimed.

Now, for any integer i with $0 \leq i \leq r + 1$, define a sequence $\mathbf{s}_i$ of r elements of S as follows:

$$\begin{aligned} \mathbf{s}_0 &= (s'_1, \ldots, s'_r), \\ \mathbf{s}_1 &= (s_1, \ldots, s_r), \\ \mathbf{s}_{r+1-j} &= (s_1, s'_1, \ldots, s_1, s'_1, s_1, s_2, \ldots, s_j) \quad \text{for } r - j \text{ even and } 0 \leq j \leq r, \\ \mathbf{s}_{r+1-j} &= (s'_1, s_1, \ldots, s_1, s'_1, s_1, s_2, \ldots, s_j) \quad \text{for } r - j \text{ odd and } 0 \leq j \leq r. \end{aligned} \tag{2.3.11}$$

For any given $0 \leq i \leq r+1$, denote by (A_i) the assertion "$\mathbf{s}_i, \mathbf{s}_{i+1} \in \mathsf{Red}(w)$ and $F(\mathbf{s}_i) \neq F(\mathbf{s}_{i+1})$". By the previous paragraph, (A_i) implies (A_{i+1}) for i with $0 \leq i \leq r - 1$. Moreover, by the condition in Hypothesis (B), (A_r) does not hold. Hence (A_0) is not satisfied. Since $\mathbf{s}_0 = \mathbf{s}'$ and $\mathbf{s}_1 = \mathbf{s}$, it follows that $F(\mathbf{s}) = F(\mathbf{s}')$. □

Proposition 2.3.2. *Let M be a monoid with unit element 1_M and $\psi : S \to M$ a map from S to M. For any $s, s' \in S$, let $m_{ss'}$ be the order of ss' and put*

$$a(s,s') = \begin{cases} \big(\psi(s)\psi(s')\big)^m & \text{if } m_{ss'} = 2m, \ m \text{ finite} \\ \big(\psi(s)\psi(s')\big)^m \psi(s) & \text{if } m_{ss'} = 2m+1, \ m \text{ finite} \\ 1 & \text{if } m_{ss'} = \infty. \end{cases} \tag{2.3.12}$$

If $a(s,s') = a(s',s)$ whenever $s \neq s'$ are distinct elements in S, there exists a map $\tilde{\psi} : W \to M$ from W to M such that

$$\tilde{\psi}(w) = \psi(s_1) \cdots \psi(s_r) \tag{2.3.13}$$

for all w in W and any reduced decomposition $(s_1, \dots, s_r)$ of w.

Proof. For any $w \in W$, let $F_w : \mathsf{Red}(w) \to M$ be the map defined by

$$F_w(s_1, \dots, s_r) = \psi(s_1) \cdots \psi(s_r).$$

We show, by induction $\ell(w)$, that F_w is constant. The case $\ell(w) = 0$ is trivial. The case $\ell(w) = 1$ is also trivial because in this case, we have $\mathsf{Red}(w) = \{s\}$ for some $s \in S$. So let $w \in W$ be an element such that $\ell(w) = r \geq 2$ and assume that the assertion is true for all the elements $w' \in W$ such that $\ell(w') < r$. Now let $\mathbf{s}, \mathbf{s}' \in \mathsf{Red}(w)$. By Lemma 2.3.5, it suffices to show that $F(\mathbf{s}) = F(\mathbf{s}')$ in the cases in Hypothesis (A) and (B) of the lemma. So assume that we are in the case in Hypothesis (A). We then have

$$F_w(s_1, \dots, s_r) = \psi(s_1) F_{w''}(s_2, \dots, s_r) = F_{w'}(s_1, \dots, s_{r-1}) \psi(s_r)$$

for $w' := s_1 \cdots s_{r-1}$ and $w'' := s_2 \cdots s_r$. Moreover, by the induction hypothesis, $F_{w''}$ and $F_{w'}$ are both constant. Since by hypothesis we have $s_1 = s'_1$ or $s_r = s'_r$, it follows that $F_w(\mathbf{s}) = F_w(\mathbf{s}')$.

On the other hand, suppose that we are in the case in Hypothesis (B), so that there exist two elements $s, s' \in S$ such that $s_i = s'_j = s$ and $s_j = s'_i = s'$ for i odd and j even. Note that it $s = s'$, then $w = s_1 \cdots s_r$ is not a reduced expression of w, so it suffices to consider the case when $s \neq s'$ are distinct. Now, when $s \neq s'$ are distinct, the sequences $\mathbf{s}$ and $\mathbf{s}'$ are two distinct reduced decompositions of w and both belong to the dihedral group generated by s and s'. Moreover, the order $m_{ss'}$ of ss' must be necessarily finite, since otherwise, if $m_{ss'} = \infty$, every element of of the subgroup $W_{\{s,s'\}}$ of W generated by s and s' has a unique reduced decomposition as $\mathsf{prod}(r; s, s')$ and $\mathsf{prod}(r; s', s)$ are distinct for every integer $r \geq 0$. Consequently, $F_w(\mathbf{s}) = a(s,s')$ and $F_w(\mathbf{s}') = a(s',s)$, and hence $F_w(\mathbf{s}) = F_w(\mathbf{s}')$, since by assumption, $a(s,s') = a(s',s)$. This shows that F_w is constant. The claim in Proposition 2.3.2 then follows. □

Remark 2.3.4. We are now in possession of all the necessary tools to prove the following characterization of Coxeter systems:

Theorem 2.3.1. *The pair (W,S) is a Coxeter system if and only if it satisfies the Exchange condition.*

Proof. If (W,S) is a Coxeter system, then Lemma 2.3.4 shows that (W,S) satisfies the Exchange condition.

Conversely, assume that the pair (W,S) satisfies the Exchange condition. Let G be a group and let $\psi : S \to G$ be a map from S to G satisfying

$$\big(\psi(s)\psi(s')\big)^{m_{ss'}} = 1 \quad \text{for every } (s,s') \in S_F.$$

By Proposition 2.3.2, there exists a map $\tilde{\psi} : W \to G$ from W to G such that

$$\tilde{\psi}(w) = \psi(s_1) \cdots \psi(s_r)$$

whenever $w = s_1 \cdots s_r$ is a reduced expression of w. Therefore, to prove that (W, S) is a Coxeter system, it suffices to prove that $\tilde{\psi}$ is a homomorphism. In order to do this, we use the fact that S generates W. So take any $s \in S$ and any w be in W, and let $\mathbf{s} = (s_1, \ldots, s_r)$ be a reduced decomposition of w. By Proposition 2.3.1, only two cases are possible: either $\ell(sw) = l(w) + 1$ or $\ell(sw) = l(w) - 1$. If $\ell(sw) = \ell(w) + 1$, then $(s, s_1, \ldots, s_r)$ is a reduced decomposition of sw, and hence

$$\tilde{\psi}(sw) = \psi(s)\psi(s_1) \cdots \psi(s_r) = \psi(s)\tilde{\psi}(w).$$

If on the contrary, $\ell(sw) = \ell(w) - 1$, we set $w' := sw$, so that $sw' = w$ and $\ell(sw') = \ell(w') + 1$. Notice that we are back to the first case, where we have already shown that

$$\tilde{\psi}(sw') = \psi(s)\tilde{\psi}(w') = \psi(s)\tilde{\psi}(sw).$$

Since $\psi(s)\psi(s) = 1$, it follows that $\psi(s)\tilde{\psi}(w) = \tilde{\psi}(sw)$. In both cases multiplication is preserved by $\tilde{\psi}$, proving that $\tilde{\psi} : W \to G$ is indeed a homomorphism. $\square$

2.4 Families of partitions

In this section we continue to assume the pair (W, S) only has the properties in the introduction of Section 2. In this section, we are interested in studying the elements $w \in W$ such that $\ell(sw) > \ell(w)$ for a fixed given $s \in S$.

Proposition 2.4.1. *Let (W, S) be a Coxeter system. For any element $s \in S$, let P_s^+ be the set of elements $w \in W$ such that $\ell(sw) > \ell(w)$, that is,*

$$\mathrm{P}_s^+ = \{w \in W \mid \ell(sw) > \ell(w)\}. \tag{2.4.1}$$

The collection of sets $(\mathrm{P}_s^+)_{s \in S}$ has the following properties:

(a) $\bigcap_{s \in S} \mathrm{P}_s^+ = \{1\}$.

(b) For any element $s \in S$, the sets P_s^+ and $s\mathrm{P}_s^+$ form a partition of W.

(c) Let $s, s' \in S$ and let $w \in W$. If $w \in \mathrm{P}_s^+$ and $ws' \in s\mathrm{P}_s^+$, then $sw = ws'$.

Proof. (a) First, recall that if $w \in W$ is a non-identity element and $\mathbf{s} = (s_1, \ldots, s_r)$ is a reduced decomposition of w, then $r \geq 1$ and $(s_2, \ldots, s_r)$ is a reduced decomposition of $s_1 w$, so $\ell(w) = r$ and $\ell(s_1 w) = r - 1$. Hence $w \notin \mathrm{P}_{s_1}^+$, which shows that

$$\bigcap_{s \in S} \mathrm{P}_s^+ \subseteq \{1\}.$$

Conversely, since $\ell(s \cdot 1) > \ell(1)$ for any $s \in S$, we have that $1 \in \mathrm{P}_s^+$ for any $s \in S$, and thus

$$\{1\} \subseteq \bigcap_{s \in S} \mathrm{P}_s^+.$$

The equality in (a) follows.

(b) Now, let $w \in W$ and $s \in S$. By Proposition 2.3.1, we have two possibilities: either $\ell(sw) = \ell(w) + 1$, in which case $w \in \mathrm{P}_s^+$, or $\ell(sw) = \ell(w) - 1$. In the latter case, if we set $w' := sw$ so that $sw' = w$, then $\ell(w') < \ell(sw')$, which implies that $w' \in \mathrm{P}_s^+$, and thus $w \in s\mathrm{P}_s^+$. This proves (b).

(c) Finally, take any $s, s' \in S$ and any $w \in W$ such that $\ell(w) = r$ and $w \in \mathrm{P}_s^+$ and $ws' \in s\mathrm{P}_s^+$. The fact that $w \in \mathrm{P}_s^+$ implies that $\ell(sw) = r + 1$, and the fact that $ws'\mathrm{P}_s^+$ implies that $\ell(sws') = \ell(ws') - 1 \leq r$. Therefore, since $\ell(sws') = \ell(sw) \pm 1$, we conclude that $\ell(ws') = r + 1$

and $\ell(sws') = r$. Now, let $(s_1, \ldots, s_r)$ be a reduced decomposition of w. Then $(s_1, \ldots, s_r, s')$ is a reduced decomposition of ws', which is an element of length $r + 1$. By the Exchange condition, there exists an integer i such that $1 \leq i \leq r + 1$ and

$$ss_1 \cdots s_{i-1} = s_1 \cdots s_{i-1} s_i. \tag{2.4.2}$$

Note that if $1 \leq i \leq r$, then $sw = s_1 \cdots s_{i-1} s_{i+1} \cdots s_r$, contradicting the fact that $\ell(sw) = r + 1$. Thus $i = r + 1$ and (2.4.2) reads $sw = ws'$. □

Proposition 2.4.2. *Let* $(\mathrm{P}_s)_{s \in S}$ *be a family of subsets of* W *satisfying Property* (c) *in Proposition 2.4.1 together with the following conditions:*

(a') The identity element 1 *in* W *belongs to* P_s *for all* s *in* S*.*

(b') The sets P_s *and* $s\mathrm{P}_s$ *are disjoint for all* s *in* S*.*

Then (W, S) *is a Coxeter system and* P_s *consists of the elements* $w \in W$ *such that* $\ell(sw) > \ell(w)$*, i.e.,* $\mathrm{P}_s = \mathrm{P}_s^+$ *for every* $s \in S$*, where* P_s^+ *is defined as in* (2.4.1)*.*

Proof. Take any $s \in S$ and any $w \in W$, and let $(s_1, \ldots, s_r)$ be a reduced decomposition of w. Also, for each integer i with $1 \leq i \leq r$, let $w_i := s_1 \cdots s_i$, and set $w_0 := 1$. First, if $w \notin \mathrm{P}_s^+$, then $w_r = w \notin \mathrm{P}_s^+$. Also note that $w_0 = 1 \in \mathrm{P}_s^+$ by (a'). Therefore, there exists an integer j with $1 \leq j \leq r$ such that $w_{j-1} \in \mathrm{P}_s^+$ but $w_j = w_{j-1} s_j \notin \mathrm{P}_s^+$. Then, by (c) in Proposition 2.4.1 $sw_{j-1} = w_{j-1} s_j$, which proves the formula

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_{j-1} s_j.$$

Such formula implies that $sw = s_1 \cdots s_{j-1} s_{j+1} \cdots s_r$ and $\ell(sw) < \ell(w)$.

On the other hand, if $w \in \mathrm{P}_s^+$, set $w' := sw$ so that $w' \notin \mathrm{P}_s^+$ by (b'). From the previous paragraph, we then have $\ell(sw') < \ell(w')$, which is equivalent to $\ell(w) < \ell(sw)$. Since the first paragraph of the proof proves that if $\ell(sw) > \ell(w)$, then $w \in \mathrm{P}_s^+$, it follows that $w \in \mathrm{P}_s^+$ if and only if $\ell(sw) > \ell(w)$. Finally, as we have seen in the first paragraph, the Exchange condition follows from this, and thus, by Theorem 2.3.1, the pair (W, S) is a Coxeter system. □

Remark 2.4.1. Note that in Proposition 2.4.2, we are not assuming that the pair (W, S) is a Coxeter system. In particular, Proposition 2.4.2 is the converse of Proposition 2.4.1.

2.5 Parabolic Subgroups

In this section, we continue to assume that (W, S) is a Coxeter system, and we study a class of special subgroups of W generated by subsets of S. We show that the set of letters appearing in a reduced decomposition of $w \in W$ is independent of the particular choice of reduced decomposition of w.

Definition 2.5.1. Let $X \subseteq S$ be any subset of S, and denote by W_X the subgroup of W generated by the the elements in X. Then W_X is called a *standard parabolic subgroup of* W, and for any $w \in W$, the subgroup wW_Xw^{-1} of W is called a *parabolic subgroup of* W.

Example 2.5.1. If $X = \emptyset$, then $W_\emptyset = \{1\}$, and if $X = S$, then $W_S = W$.

Proposition 2.5.1. *Let* $w \in W$*. There exists a subset* $S_w \subseteq S$ *such that* $\{s_1, \ldots, s_r\} = S_w$ *for any reduced decomposition* $(s_1, \ldots, s_r)$ *of* w*.*

Proof. Denote by $\mathscr{P}(S)$ the set of all subsets of S. Recall that $\mathscr{P}(S)$ is a monoid with binary operation given by

$$\begin{aligned} \mathscr{P}(S) \times \mathscr{P}(S) &\to \mathscr{P}(S) \\ (A, B) &\mapsto A \cup B, \end{aligned}$$

and identity element $\emptyset$. Now, let $\psi : S \to \mathscr{P}(S)$ be the map defined by $\psi(s) = \{s\}$ for s in S. Using (2.3.13), note that if s, s' in S are such that $m_{ss'}$ is finite, then $a(s, s') = a(s', s)$. Hence, by Proposition 2.3.2, there exists a map $\tilde{\psi} : W \to \mathscr{P}(S)$ from W to $\mathscr{P}(S)$ that maps w to S_w such that

$$\tilde{\psi}(w) = \psi(s_1) \cup \cdots \cup \psi(s_r).$$

In other words, $S_w = \{s_1, \dots, s_r\}$ for any w in W and any reduced decomposition $(s_1, \dots, s_r)$ of w. □

Remark 2.5.1. Note that we have just shown that for any $w \in W$, the set S_w does not depend on the choice of reduced decomposition of W.

Corollary 2.5.1. *For any subset $X \subseteq S$, the subgroup W_X of W consists of the elements $w \in W$ such that $S_w \subseteq X$.*

Proof. Let

$$U_X := \{z \in W \mid S_z \subseteq X\},$$

that is, U_X is the set of elements $z \in W$ such that $S_z \subseteq X$, and let $w, w' \in W$ be arbitrary elements in W. We first prove, by induction on the length of w, the containment

$$S_{ww'} \subseteq S_w \cup S_{w'}. \tag{2.5.1}$$

If the length of w is 0, then $w = 1$ and $S_{ww'} = S_{w'}$, so the containment in (2.5.1) is clear. If the length of w is 1, then $w = s$ for some s in S. By Proposition 2.3.1, we know that $S_{sw'}$ is a subset of $\{s\} \cup S_{w'}$, so the containment in (2.5.1) holds too. Now, suppose that the length of w is equal to $r > 1$, and assume that the containment in (2.5.1) holds for any y in W with $\ell(y) < r$. Now, if $w = s_1 \cdots s_r$ is a reduced expression of w, then $s_2 \cdots s_r$ is a reduced expression of $s_1 w$ which is of length $r - 1$, so by the induction hypothesis, we have

$$S_{s_1 ww'} \subseteq S_{s_1 w} \cup S_{w'}. \tag{2.5.2}$$

But $S_{s_1 w}$ is a subset of $\{s_1\} \cup S_w$, and, by Proposition 2.5.1, we have $S_w = \{s_1, \dots, s_r\}$. Hence $\{s_1\} \cup S_w = S_w$ and (2.5.2) becomes $S_{s_1 ww'} \subseteq S_w \cup S_{w'}$. Then, since $ww' = s_1 \cdot s_1 ww'$, it follows that

$$S_{ww'} \subseteq \{s_1\} \cup S_{s_1 ww'} \subseteq \{s_1\} \cup S_w \cup S_{w'} = S_w \cup S_{w'},$$

completing the proof of the containment in (2.5.1). Moreover, since $w^{-1} = s_r \cdots s_1$, it follows, by Proposition 2.5.1, that

$$S_w = S_{w^{-1}}. \tag{2.5.3}$$

Combining (2.5.1) and (2.5.3), wee see that the set U_X is a subgroup of W. Moreover, we have $X \subseteq U_X \subseteq W_X$. Therefore, as X generates W_X, it follows that $U_X = W_X$, and it is, in particular the smallest subgroup of W containing X. □

Corollary 2.5.2. *For any subset $X \subseteq S$, we have $W_X \cap S = X$.*

Proof. The containment of X in $W_X \cap S$ is clear. Conversely, if $s \in W_X \cap S$, then, by Corollary 2.5.1, we have that $S_s \subseteq X$. But $s \in S$, so we have $S_s = \{s\}$, and thus $s \in X$. This shows that $W_X \cap S \subseteq X$, completing the proof. □

Corollary 2.5.3. *The set S is a minimal generating set of W.*

Proof. If $X \subseteq S$ is a subset of S generating W, then $W = W_X$, and hence, by Corollary 2.5.2, we have

$$X = W_X \cap S = S.$$

□

Corollary 2.5.4. *For any subset X and any $w \in W_X$, the length $\ell_X(w)$ of w with respect to the generating set X of the subgroup W_X of W is equal to $\ell_S(w)$.*

Proof. Let $w \in W_X$ and let $(s_1, \dots, s_r)$ be a reduced decomposition of w with respect to S considered as an element of W. We have $w = s_1 \cdots s_r$ and, by Corollary 2.5.1, we know that $s_i \in X$ for each $1 \leq i \leq r$. Moreover, the element w cannot be a product of less than r elements of X, which is a subset of S, as $\ell_S(w) = r$. □

Theorem 2.5.1. *Let (W, S) be a Coxeter system. Then:*

(a) For any subset X, the pair (W_X, X) is a Coxeter system.

(b) Let $(X_i)_{i \in I}$ be a family of subsets of S. If

$$X = \bigcap_{i \in I} X_i,$$

then

$$W_X = \bigcap_{i \in I} W_{X_i}.$$

(c) Let $X, X' \subseteq S$ be two subsets of S. Then $W_X \subseteq W_{X'}$ if and only if $X \subseteq X'$.

(d) Let $X, X' \subseteq S$ be two subsets of S. Then $W_X = W_{X'}$ if and only if $X = X'$.

Proof. (a) First, every element of X is an element of S, so it is of order 2. Moreover, X generates W_X. Now, let $x \in X$ and $w \in W_X$ be elements such that $\ell_X(xw) \leq \ell_X(w) = r$. By Corollary 2.5.4, we have $\ell_S(xw) \leq \ell_S(w) = r$. Let $x_1, \dots, x_r$ be elements of X such that $w = x_1 \cdots x_r$. Since (W, S) is a Coxeter system, it satisfies, by Theorem 2.3.1, the Exchange condition, so there exists an integer $j \in \mathbb{Z}$ such that $1 \leq j \leq r$ and

$$x x_1 \cdots x_{j-1} = x_1 \cdots x_{j-1} x_j.$$

But this shows that (W_X, X) satisfies the Exchange condition too, and thus, by Theorem 2.3.1 again, it is a Coxeter system. This proves (a).

(b) Let $(X_i)_{i \in I}$ be a family of subsets of S satisfying

$$X = \bigcap_{i \in I} X_i, \tag{2.5.4}$$

and let $w \in W_X$. By Corollary 2.5.1, we know that $S_w \subseteq X$, and thus, from the equality in (2.5.4), we see that $S_w \subseteq X_i$ for each $i \in I$. Another application of Corollary 2.5.1 then gives that $w \in W_{X_i}$ for each $i \in I$, which shows that

$$W_X \subseteq \bigcap_{i \in I} W_{X_i}$$

Conversely, if

$$w \in \bigcap_{i \in I} W_{X_i},$$

then, by Corollary 2.5.1 again, $S_{w'} \subseteq X_i$ for each $i \in I$, and thus, from the equality in (2.5.4), we see that $S_{w'} \subseteq X$. One last application of Corollary 2.5.1 then gives that $w' \in W_X$, which shows that

$$\bigcap_{i \in I} W_{X_i} \subseteq W_X.$$

We have hence shown that

$$W_X = \bigcap_{i \in I} W_{X_i},$$

proving (b).

(c) Now, let $X, X' \subseteq S$ be two subsets of S. If $W_X \subseteq W_{X'}$, then $W_X \cap S \subseteq W_{X'} \cap S$. But, by Corollary 2.5.2, we now that $X = W_X \cap S$ and $X' = W_{X'} \cap S$, so we have hence shown that $X \subseteq X'$.

Conversely, suppose that $X \subseteq X'$, and let $w \in W_X$. Then, by Corollary 2.5.1, we know that $S_w \subseteq X$, so in particular $S_w \subseteq X'$, and thus, by Corollary 2.5.1 again, $w \in W_{X'}$. This shows that $W_X \subseteq W_{X'}$ and completes the proof of (c).

(d) Finally, we have $W_X = W_{X'}$ if and only if $W_X \subseteq W_{X'}$ and $W_{X'} \subseteq W_X$, and by (c) just proven, this is the case if and only if $X \subseteq X'$ and $X' \subseteq X$, which in turn is the case if and only if $X = X$. This proves (d) and completes the proof. □

2.6 Coxeter matrices and Coxeter graphs

Let I be a set. In this section we define Coxeter matrices and Coxeter graphs and study how they are related. We also start our discussion of their relation with Coxeter systems.

Definition 2.6.1. A *Coxeter matrix of type* I is a symmetric square matrix $\mathrm{M} = (m_{ij})_{i,j \in \mathrm{I}}$ whose entries are integers $m_{ij} \in \mathbb{Z}$ or $m_{ij} = +\infty$ satisfying

$$\begin{aligned} m_{ii} &= 1 \quad \text{for all } i \in I; \\ m_{ij} &\geq 2 \quad \text{for } i, j \in I \text{ with } i \neq j. \end{aligned} \tag{2.6.1}$$

Definition 2.6.2. A *Coxeter graph of type* I is a pair consisting of a graph $\mathcal{G}$ having I as its set of vertices and a map $\psi : \mathrm{E}(\mathcal{G}) \to \mathbb{Z}_{\geq 3} \cup \{\infty\}$ from the set of edges $\mathrm{E}(\mathcal{G})$ of the graph $\mathcal{G}$ to the set $\mathbb{Z}_{\geq 3} \cup \{\infty\}$, where $\mathbb{Z}_{\leq 3}$ is the set of integers greater or equal to 3. In particular, $\mathcal{G}$ is called the *underlying graph of the Coxeter graph* $(\mathcal{G}, \psi)$.

Remark 2.6.1. Note that we may associate, to any Coxeter matrix M of type I, a Coxeter graph $(\mathcal{G}, \psi)$, since the graph $\mathcal{G}$ has I as set of vertices and the set pairs $\{i, j\}$ of elements of I such that $m_{ij} \geq 3$ as edges, and the map ψ associates to the edge $\{i, j\}$ the corresponding entry m_{ij} of M. This gives a bijection between the set of Coxeter matrices of type I and the set of Coxeter graphs of type I.

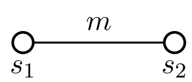
Remark 2.6.2. We represent a Coxeter graph $(\mathcal{G}, \psi)$ of type I by the diagram used to represent its underlying graph $\mathcal{G}$, and write above each edge $\{i, j\}$ the number $\psi(\{i, j\})$, omitting these numbers if they are equal to 3.

Remark 2.6.3. Now, if (W, S) is a Coxeter system, the matrix $\mathrm{M} := (m_{ss'})_{s,s' \in S}$ is a Coxeter matrix of type S, since $m_{ss} = 1$ as $s^2 = 1$ for all s in S and $m_{ss'} = m_{s's} \geq 2$ if $s \neq s'$ as $ss' = (s's)^{-1} \neq 1$. Such a matrix is called the *Coxeter matrix of* (W, S), and the Coxeter graph $(\mathcal{G}, \psi)$ associated to M is called the *Coxeter graph of* (W, S), and is denoted by $\mathsf{Cox}(W, S)$. Note that any two vertices s, s' of Γ are *linked* if and only if s and s' do not commute.

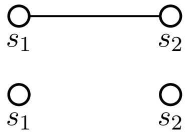
Example 2.6.1. The Coxeter matrix of a dihedral group $\mathcal{D}_m$ of order $2m$ is

$$\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$$

and its Coxeter graph $\mathsf{Cox}(\mathcal{D}_m, S)$, where $S = \{s_1, s_2\}$, is represented by:



(when $m > 3$)



(when $m = 3$)

(when $m = 2$)

Example 2.6.2. The $\mathcal{S}_n$ is generated by the the set of adjacent transpositions $s_i := (i, i+1)$, where $i \in Z$ such that $1 \leq i \leq n-1$. Therefore, the Coxeter graph $\mathsf{Cox}(\mathcal{S}_n, S)$ of the symmetric groups $\mathcal{S}_n$ with respect to the generation set $S = \{s_1, s_2, \ldots, s_{n-1}\}$ is represented by:

Definition 2.6.3. A Coxeter system (W, S) is said to be *irreducible* if the underlying graph $\mathcal{G}$ of its Coxeter graph $\mathsf{Cox}(W, S)$ is non-empty and connected in the graph-theoretic sense.

Remark 2.6.4. Note that the Coxeter system (W, S) is irreducible if the generating set S is non-empty and there exists no partition of S into two distinct subsets $S, S' \subseteq S$ of S such that every element of S' commutes with every element of S'. More generally, let $(\mathcal{G}_i)_{i\in I}$ be the family of connected components of $\mathcal{G}$ in the graph-theoretic sense, let $\mathrm{V}(\mathcal{G}_i)$ be the set of vertices of $\mathcal{G}_i$ and let $W_{\mathrm{V}(\mathcal{G}_i)}$ be the subgroup of W generated by $\mathrm{V}(\mathcal{G}_i)$. Then, by Theorem 2.5.1(a), the pairs $\big(W_{\mathrm{V}(\mathcal{G}_i)}, \mathrm{V}(\mathcal{G}_i)\big)$ are all irreducible Coxeter systems called the *irreducible components of* (W, S).

Proposition 2.6.1. *Let $(X_i)_{i\in I}$ be a partition of S such that every element of X_i commutes with every element of X_j if $i \neq j$. Then, for any finite subset $J \subseteq I$, the subgroup W_J of W generated by the W_{X_j} for $j \in J$ is the direct product of the W_{X_j} for $j \in J$ and*

$$W = \bigcup_{\substack{J \subseteq I \\ J \text{ finite}}} W_J.$$

Proof. For every $i \in I$, the subgroup W'_{X_i} generated by the union of the all the W_{X_j} for $j \neq i$ is also generated by

$$X'_i := \bigcup_{\substack{j \in I \\ i \neq j}} X_j,$$

and thus, by *(a)* in Theorem 2.5.1,

$$W_{X_i} \cap W'_{X_i} = W_\emptyset = \{1\}.$$

Since W is generated by the union of the W_{X_i}, the proposition is proved. ☐

Remark 2.6.5. From Proposition 2.6.1, we see that every element of $w \in W$ can be written uniquely as a product

$$w = \prod_{i \in I} w_i$$

with $w_i \in W_{X_i}$ and $w_i = 1$ for all but finately many indices $i \in I$.

2.7 Geometric representation of W

Let S be a set and $\mathrm{M} = (m_{ss'})_{s,s' \in S}$ be a Coxeter matrix of type S. So far we have established the following relations:

$$\left\{\begin{matrix}\text{Coxeter} \\ \text{systems } (W,S)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Coxeter matrices} \\ \text{of type } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Coxeter graphs} \\ \text{of type } S\end{matrix}\right\}$$

In Section 2.7, we show that, conversely, any Coxeter matrix is the matrix of a Coxeter system, and we are thus able to close the circle and conclude that up to isomorphism, there is a one-to-one correspondence between Coxeter graphs, Coxeter matrices and Coxeter systems. In order to do this, let $\mathrm{E} := \mathbb{R}^S$ be a real vector space and let $(e_s)_{s\in S}$ be the canonical basis of E indexed by S. We first associate a symmetric bilinear form $\mathrm{B_M}$ on E to the Coxeter matrix M, and use this to define, for each $s \in S$, a linear automorphism σ_s of the vector space E that fixes a hyperplane in E pointwise and sends some non-zero vector to its negative. We then study the restriction of the bilinear form $\mathrm{B_M}$ to plane $\mathrm{E}_{s,s'} := \mathbb{R}e_s \oplus \mathbb{R}e_{s'}$ where $s, s' \in S$ are two distinct elements of S, as well as the group generated by the two automorphisms σ_s and $\sigma_{s'}$, and use this to prove the existence a uniqueness of a particular representation of the group $W(\mathrm{M})$ associated to the Coxeter matrix M of type S, called the *geometric representation of* $W(\mathrm{M})$, and show that the pair $(W(\mathrm{M}), S)$ is in fact a Coxeter system.

Definition 2.7.1. Let $\mathrm{B_M} : \mathrm{E} \times \mathrm{E} \to \mathbb{R}$ be the bilinear for defined by

$$\mathrm{B_M}(e_s, e_{s'}) := -\cos\frac{\pi}{m_{ss'}} \quad \text{for all } s, s' \in S. \tag{2.7.1}$$

The bilinear form $\mathrm{B_M}$ is called the *associated bilinear form of the Coxeter matrix M of type S.*

Remark 2.7.1. Recall, from Definition 2.6.1, that the Coxeter matrix $\mathrm{M} = (m_{ss'})_{s,s'\in S}$ is a square symmetric matrix, and so we have that $m_{ss'} = m_{s's}$ for every $s, s' \in S$. We then see from Definition 2.7.1, that the associated bilinear form of the Coxeter matrix M is in fact a symmetric bilinear form, that is,

$$\mathrm{B_M}(x, y) = \mathrm{B_M}(y, x) \quad \text{for any } x, y \in \mathrm{E}.$$

Remark 2.7.2. Note that since $m_{ss} = 1$ and $m_{ss'} \geq 2$ for any $s, s' \in S$ such that $s \neq s'$, we have that

$$\cos\frac{\pi}{m_{ss}} = -1 \quad \text{and} \quad 0 \leq \cos\frac{\pi}{m_{ss'}} \leq 1 \text{ if } s \neq s', \tag{2.7.2}$$

and thus

$$\mathrm{B_M}(e_s, e_s) = 1 \quad \text{and} \quad \mathrm{B_M}(e_s, e_{s'}) \leq 0 \quad \text{if } s \neq s',$$

Definition 2.7.2. Let $s \in S$, and let e_s^* be the linear form on E given by

$$\begin{aligned} e_s^* : \mathrm{E} &\to \mathbb{R} \\ x &\mapsto 2\mathrm{B_M}(e_s, x) \quad (x \in \mathrm{E}), \end{aligned} \tag{2.7.3}$$

and let $\langle \cdot, \cdot \rangle$ be the pairing given by

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathrm{E} \times \mathrm{E}^\vee &\to \mathrm{E} \\ \left(x, f^\vee\right) &\mapsto f^\vee(x) \quad \left(x \in \mathrm{E}, f^\vee \in \mathrm{E}^\vee\right), \end{aligned} \tag{2.7.4}$$

where $\mathrm{E}^\vee$ denotes the algebraic dual of E. We then denote by σ_s be the automorphism $\sigma_s : \mathrm{E} \to \mathrm{E}$ of E given by

$$\sigma_s(x) = x - \langle x, e_s^* \rangle\, e_s \quad (x \in \mathrm{E}). \tag{2.7.5}$$

Remark 2.7.3. Note that for any $x \in \mathrm{E}$, we have that

$$\sigma_s(x) = x - e_s^*(x) = e_{s'} - 2\mathrm{B_M}(e_s, x)e_s.$$

In particular, for any $s, s' \in S$, we have that

$$\sigma_s(e_{s'}) = e_{s'} - e_s^*(e_{s'}) = e_{s'} - 2\mathrm{B_M}(e_s, e_s')e_s = e_{s'} + 2\cos\frac{\pi}{m_{ss'}}e_s,$$

and thus

$$\sigma_s^2 = (\sigma_s \circ \sigma_s)(e_{s'}) = \sigma_s\left(e_{s'} + 2\cos\frac{\pi}{m_{ss'}}e_s\right) = \sigma_s(e_{s'}) + 2\cos\frac{\pi}{m_{ss'}}\sigma_s(e_s) = e_{s'}.$$

Since $(e_{s'})_{s'\in S}$ is a basis of E, it follows that $\sigma_s^2 = 1$.

Remark 2.7.4. Let $\mathrm{id_E} : \mathrm{E} \to \mathrm{E}$ denote the identity automorphism on E. Then for any $x \in \mathrm{E}$ and any $s \in S$, we have that

$$(\sigma_s - \mathrm{id_E})(x) = \sigma_s(x) - x = x - \langle x, e_s^* \rangle e_s - x = - \langle x, e_s^* \rangle e_s,$$

which shows that the image of $\sigma_s - \mathrm{id_E}$ on E is generated by e_s. Moreover, since e_s is a non-zero vector in E, we have that $x \in \mathrm{E}$ belongs to $\ker(\sigma_s - \mathrm{id_E})$ if and only if $\langle x, e_s^* \rangle = 0$. Also note that for any $x \in \mathrm{E}$ and any $s \in S$, we have that

$$(\sigma_s + \mathrm{id_E})(x) = \sigma_s(x) + x - = x - \langle x, e_s^* \rangle e_s + x = 2x - \langle x, e_s^* \rangle e_s,$$

and thus $x \in \ker(\sigma_s + \mathrm{id_E})$ if and only if $2x = \langle x, e_s^* \rangle e_s$. Therefore, if $x \in \ker(\sigma_s - \mathrm{id_E}) \cap \ker(\sigma_s + \mathrm{id_E})$, it follows that $x = 0$. Now, for any vector $v \in \mathrm{E}$, let $v' := \sigma_s(v) + v$ and $v'' := \sigma_s(v) - v$. Then, since $\sigma_s^2 = \mathrm{id_E}$, we have that

$$(\sigma_s - \mathrm{id_E})(v') = \sigma_s^2(v) + \sigma_s(v) - \sigma_s(v) - v = v + \sigma_s(v) - \sigma_s(v) - v = 0$$

and

$$(\sigma_s + \mathrm{id_E})(v'') = \sigma_s^2(v) - \sigma_s(v) + \sigma_s(v) - v = v - \sigma_s(v) + \sigma_s(v) - v = 0,$$

which show that $v' \in \ker(\sigma_s - \mathrm{id_E})$ and $v'' \in \ker(\sigma_s + \mathrm{id_E})$, respectively. Since we $2v = v' - v''$, it follows that E is the direct sum of $\ker(\sigma_s - \mathrm{id_E})$ and $\ker(\sigma_s + \mathrm{id_E})$. Moreover, since $\sigma_s^2 = \mathrm{id_E}$, we have that $\ker(\sigma_s - \mathrm{id_E})$ is a hyperplane in E, and thus $\ker(\sigma_s + \mathrm{id_E})$ must necessarily be of dimension 1. More precisely, we have $\ker(\sigma_s + \mathrm{id_E}) = \mathbb{R}e_s$.

Remark 2.7.5. For any fixed $s \in S$, let $\mathrm{Z}_s := \ker(\sigma_s - \mathrm{id_E})$, so that E is the direct sum of the line $\mathbb{R}e_s$ and the hyperplane Z_s. Now, take any $x, y \in \mathrm{E}$, and write $x = v_x + u_x$ and $y = v_y + u_y$ where $v_x, v_y \in \mathbb{R}e_s$ and $u_x, u_y \in \mathbb{Z}_s$. Since we have that

$$\sigma_s(v_x) = -v_x, \quad \sigma_s(v_y) = -v_y, \quad \sigma_s(u_x) = u_x \quad \text{and} \quad \sigma_s(u_y) = u_y,$$

it follows that σ_s preserves $\mathrm{B_M}$.

Remark 2.7.6. When S is finite and $\mathrm{B_M}$ is non-degenerate, then σ_s is in fact an orthogonal reflection.

Proposition 2.7.1. *The restriction of* $\mathrm{B_M}$ *to* $\mathrm{E}_{s,s'}$ *is positive, and it is non-degenerate if and only if* $m_{ss'} = m_{s's}$ *is finite.*

Proof. Let $x = \lambda_s e_s + \lambda_{s'} e_{s'} \in \mathrm{E}_{s,s'}$ where $\lambda_s, \lambda_{s'} \in \mathbb{R}$. Using the bilinearity and the symmetry of $\mathrm{B_M}$ together with the fact that $\mathrm{B_M}(e_s, e_s) = \mathrm{B_M}(e_{s'}, e_{s'}) = 1$, we have

$$\mathrm{B_M}(x, x) = \lambda_s^2 + 2\lambda_s \lambda_{s'} \mathrm{B_M}(e_s, e_{s'}) + \lambda_{s'}^2.$$

Then, substituting (2.7.1) into the above, completing the square and using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ where $\theta \in \mathbb{R}$ is a real number with $0 \leq \theta \leq 2\pi$, gives

$$\mathrm{B_M}(x, x) = \left(\lambda_s - \lambda_{s'} \cos\frac{\pi}{m_{ss'}}\right)^2 + \lambda_{s'}^2 \sin^2\frac{\pi}{m_{ss'}}, \tag{2.7.6}$$

which shows that $\mathrm{B_M}(x, x) \geq 0$. Since x is an arbitrary element of $\mathrm{E}_{s,s'}$, it follows that $\mathrm{B_M}$ is positive on $\mathrm{E}_{s,s'}$. Moreover, suppose that $m_{ss'} = \infty$, then $\mathrm{B_M}(x, x) = (\lambda_s - \lambda_{s'})^2$. Therefore, if $\lambda_s, \lambda_{s'} \in \mathbb{R}$ are non-zero real numbers with $\lambda_s = \lambda_{s'}$, we have that x is a non-zero element of $\mathrm{E}_{s,s'}$ with $\mathrm{B_M}(x, x) = 0$, which shows that $\mathrm{B_M}$ is degenerate.

Conversely, suppose that $\mathrm{B_M}$ is degenerate, so that, since $\mathrm{E}_{s,s}$ is finite dimensional, there exists a non-zero element $x = \lambda_s e_s + \lambda_{s'} e_s \in \mathrm{E}_{s,s'}$, with $\lambda_s, \lambda_{s'} \in \mathbb{R}$ real numbers, such that $\mathrm{B_M}(x, y) = 0$ for all $y \in \mathrm{E}_{s,s}$. In particular $\mathrm{B_M}(x, x) = 0$, so by (2.7.6) we must have

$$\left(\lambda_s - \lambda_{s'} \cos\frac{\pi}{m_{ss'}}\right)^2 = -\lambda_{s'}^2 \sin^2\frac{\pi}{m_{ss'}}. \tag{2.7.7}$$

But for $x \in \mathrm{E}_{s,s'}$ to be non-zero, at least one of λ_s or $\lambda_{s'}$ must be non-zero, so the left hand side of (2.7.7) is at least zero and the right hand side of (2.7.7) is at most zero. Therefore, the only way we can have equality is if both sides are equal to zero. Now note that if $\lambda_{s'} = 0$, we must have $\lambda_s \neq 0$, since otherwise $x = 0$ contradicting the fact that $x \neq 0$. But then the left hand side of (2.7.7) is equal to $\lambda_s^2 > 0$ which is not possible. Hence $\lambda_{s'}$ must be non-zero, and thus $-\lambda_s^2 \sin^2 \frac{\pi}{m_{ss'}} = 0$ implies that $\sin^2 \frac{\pi}{m_{ss'}} = 0$, which is the case of and only if $\sin \frac{\pi}{m_{ss'}} = 0$. Since $m_{ss'} \geq 2$, this implies that $m_{ss'} = \infty$, since otherwise, $0 < \sin \frac{\pi}{m_{ss'}} \leq 1$. We have hence proven that $\mathrm{B_M}$ is non-degenerate on $\mathrm{E}_{s,s'}$ if and only if $m_{ss'}$ is finite, completing the proof of the proposition. □

Remark 2.7.7. Proposition 2.7.1 describes how precisely how $\mathrm{B_M}$ behaves on $\mathrm{E}_{s,s'}$. In particular note that if we take any element $x \in \mathrm{E}_{s,s'}$ and write it as in the proof of Proposition 2.7.1, we get

$$\sigma_s(x) = x - 2\lambda_s \mathrm{B_M}(e_s, e_s)e_s - 2\lambda_{s'}\mathrm{B_M}(e_s, e_{s'})e_s = -(\lambda_s + 2\lambda_{s'}\mathrm{B_M}(e_s, e_{s'}))e_s + \lambda_{s'}e_{s'},$$

which shows that $\mathrm{E}_{s,s'}$ is stable under σ_s. The same is true for $\sigma_{s'}$. It thus seems reasonable to determine the order of the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$. In order to do this we must distinguish two cases: the case $m_{ss'} = \infty$ and the case $m_{ss'}$ is finite.

Proposition 2.7.2. *The order of the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ is:*

(a) infinite, if $m_{ss'} = \infty$;

(b) $m_{ss'}$, if $m_{ss'}$ is finite.

Proof. *(a)* Suppose that $m_{ss'} = \infty$, so that $\cos \frac{\pi}{m_{ss'}} = 1$ and

$$\mathrm{B_M}(e_s, e_{s'}) = \mathrm{B_M}(e_{s'}, e_s) = -1,$$

and let $u \in \mathrm{E}_{s,s}$ be the element of $\mathrm{E}_{s,s}$ given by $u = e_s + e_{s'}$. For such an element u we see, using the linearity in the first argument and the symmetry of $\mathrm{B_M}$, that $\mathrm{B_M}(u, e_s) = 0 = \mathrm{B_M}(u, e_{s'})$, so that

$$\sigma_s(u) = u - 2\mathrm{B_M}(e_s, u)e_s = u = u - 2\mathrm{B_M}(e_{s'}, u)e_{s'} = \sigma_{s'}(u),$$

which shows that u is fixed by both σ_s and $\sigma_{s'}$. Moreover, using the linearity of σ_s and the fact that $\sigma_s(e_{s'}) = e_{s'} + 2e_s$ and $\sigma_{s'}(e_s) = e_s + 2e_{s'}$ for $m_{ss'} = \infty$, we see that

$$\sigma_s\sigma_{s'}(e_s) = \sigma_s(e_s + 2e_{s'}) = \sigma_s(u) + \sigma_s(e_{s'}) = u + e_{s'} + 2e_s = 2u + e_s,$$

and thus, by induction,

$$(\sigma_s\sigma_{s'})^n(e_s) = 2nu + e_s \quad \text{for all integers } n.$$

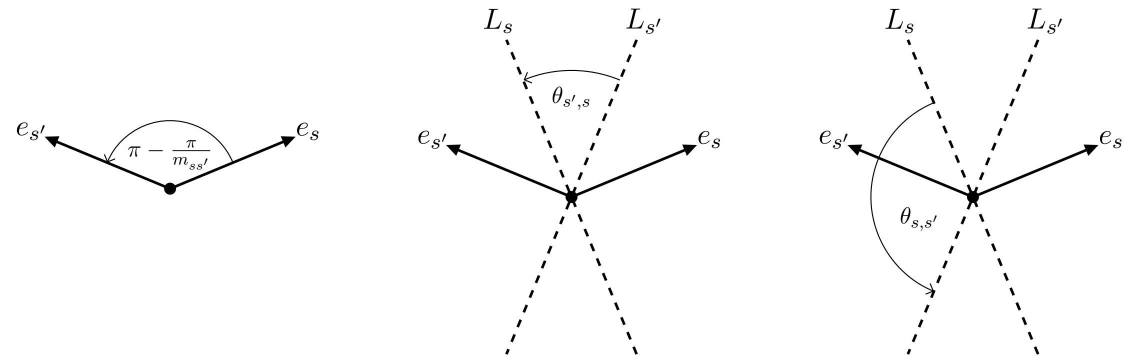
It thus follows that the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ has infinite order.

(b) On the other hand, suppose that $m_{ss'}$ is finite, and let $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the set of non-negative and non-positive real numbers, respectively. By Proposition 2.7.1, the restriction of the symmetric $\mathrm{B_M}$ to $\mathrm{E}_{s,s'}$ is positive and non-degenerate, so it provides $\mathrm{E}_{s,s'}$ with the structure of a Euclidean plane and represents the scalar product on $\mathrm{E}_{s,s}$. In particular, since $m_{ss'}$ finite implies that $0 < \frac{\pi}{m_{ss'}} \leq \pi$, the scalar product of the basis vectors e_s and $e_{s'}$ is given by

$$\mathrm{B_M}(e_s, e_{s'}) = -\cos \frac{\pi}{m_{ss'}} = \cos\left(\pi - \frac{\pi}{m_{ss'}}\right),$$

and we can orient $\mathrm{E}_{s,s'}$ so that the angle between the half lines $\mathbb{R}^+e_s$ and $\mathbb{R}^+e_{s'}$, is equal to $\pi - \frac{\pi}{m_{ss'}}$ (see Figure 2.1a). Then, if L_s and $L_{s'}$ denote the lines orthogonal to e_s and $e_{s'}$, respectively, and $\theta_{s,s'}$ denotes the angle between the lines L_s and $L_{s'}$ (see Figure 2.1c), we have

$$\theta_{s',s} = \pi - \theta_{s,s'} = \frac{\pi}{m_{ss'}}.$$



(a) Angle between e_s and $e_{s'}$ (b) Angle between $L_{s'}$ and L_s (c) Angle between L_s and $L_{s'}$

Figure 2.1: $\mathrm{E}_{s,s'}$ as a euclidean plane

This is depicted in Figure 2.1 above.

Now, the restrictions of σ_s and $\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ are orthogonal symmetries with respect to L_s and $L_{s'}$, respectively. Moreover, since $\mathrm{E}_{s,s'}$ is finite dimensional and the restrictions of σ_s and $\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ are reflections, they both have determinant -1, so the determinant of the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ is 1, showing that the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ is a rotation. Moreover, since $\sigma_{s'}$ is linear and $\sigma_{s'}(e_{s'}) = -e_{s'}$, the angle between the half-line $\mathbb{R}^+e_{s'}$ and the half-line $\sigma_s\sigma_{s'}(\mathbb{R}^+e_{s'})$ is equal to the angle between the half-line $\mathbb{R}^+e_{s'}$ and the half-line $\sigma_{s'}(\mathbb{R}^+e_{s'}) = \mathbb{R}^-e_{s'}$, which is π, plus the angle between the half-line $\mathbb{R}^-e_{s'}$ and the half-line $\sigma_s(\mathbb{R}^-e_{s'})$, which is equal to the angle between the half-line $\mathbb{R}^+e_{s'}$ and the half-line $\sigma_s(\mathbb{R}^+e_{s'})$ and thus equal to $2\pi - 2\left(\frac{\pi}{2} - \theta_{s',s}\right)$. Hence, the angle between the half-line $\mathbb{R}^+e_{s'}$ and the half-line $\sigma_s\sigma_{s'}(\mathbb{R}^+e_{s'})$ is equal to

$$\pi + 2\pi - 2\frac{\pi}{2} + 2\theta_{s',s} \equiv 2\theta_{s',s} \equiv \frac{2\pi}{m_{ss'}} \mod 2\pi,$$

and thus the restriction of $\sigma_s\sigma_{s'}$ to $\mathrm{E}_{s,s'}$ is a rotation with angle $\frac{2\pi}{m_{ss'}}$. In particular we see that it is of order $m_{ss'}$. □

Remark 2.7.8. Having determined the order of $\sigma_s\sigma_{s'}$ viewed as an operator on $\mathrm{E}_{s,s'}$, we return to E, and use this to determine the order of $\sigma_s\sigma_{s'}$ viewed as an operator on E.

Proposition 2.7.3. *The subgroup of* $\mathsf{GL}(\mathrm{E})$ *generated by* σ_s *and* $\sigma_{s'}$ *is a dihedral group of order* $2m_{ss'}$.

Proof. Since σ_s and $\sigma_{s'}$ are distinct reflections, they are in particular distinct involutions, so it suffices to show that their product $\sigma_s\sigma_{s'}$ is of order $m_{ss'}$ on E. Again, we must the distinguish between the case when $m_{ss'} = \infty$ and the case when $m_{ss'}$ finite.

So suppose that $m_{ss'} = \infty$. By *(a)* in Proposition 2.7.2, we know that $\sigma_s\sigma_{s'}$ has infinite order on $\mathrm{E}_{s,s'}$, and therefore also on E.

On the other hand, if $m_{ss'}$ is finite, it follows, by Proposition 2.7.1, that E is the direct sum of $\mathrm{E}_{s,s'}$ and its orthogonal complement $\mathrm{E}_{s,s'}^{\perp}$. Since both σ_s and $\sigma_{s'}$ fix $\mathrm{E}_{s,s'}^{\perp}$, that is, both σ_s and $\sigma_{s'}$ are the identity on $\mathrm{E}_{s,s'}^{\perp}$, and since $\sigma_s\sigma_{s'}$ has finite order $m_{ss'}$ on $\mathrm{E}_{s,s'}$, it follows that the order of $\sigma_s\sigma_{s'}$ on E is equal to $m_{ss'}$. □

Definition 2.7.3. Let $W(\mathrm{M})$ be the group defined by the family of generators $(g_s)_{s\in S}$ and the relations

$$(g_s g_{s'})^{m_{ss'}} = 1 \quad \text{for } (s,s') \in S_F. \tag{2.7.8}$$

The group $W(\mathrm{M})$ is called the *Coxeter group associated to* M.

Remark 2.7.9. Note that Definition 2.7.3 just means that $W(\mathrm{M})$ is the quotient of the free group $\mathrm{F}(S)$ on S by the smallest normal subgroup of $\mathrm{F}(S)$ containing the elements $(ss')^{m_{ss'}}$ for $(s, s') \in S_F$.

Proposition 2.7.4. *There exists a unique homomorphism $\sigma : W \to \mathsf{GL}(\mathrm{E})$ such that $\sigma(g_s) = \sigma_s$ for all s in S. Moreover, the elements of $\sigma(W)$ preserve the bilinear form $\mathrm{B_M}$.*

Proof. To prove the existence and the uniqueness of σ, it suffices to show that $(\sigma_s\sigma_s)^{m_{ss'}} = 1$ if $m_{ss'}$ is finite. But if $s = s'$, this follows from the fact that σ_s is an involution, and if $s \neq s'$, this follows from the proof of Proposition 2.7.3. Finally, since we showed that the reflections σ_s preserve $\mathrm{B_M}$, so do the elements of $\sigma(W)$. □

Definition 2.7.4. The homomorphism in Proposition 2.7.4 is called the *geometric representation of* $W(\mathrm{M})$.

Remark 2.7.10. We note that this is not the only way we may represent W as a group generated by reflections (see Vinberg [37], [38], [39] and [40]).

Proposition 2.7.5. *Let $\kappa : S \to W(\mathrm{M})$ denote the map from S to W that maps $s \mapsto g_s$ for each $s \in S$. Then:*

- *(a) The map $\kappa : S \to W$ is injective.*
- *(b) For each $s \in S$, the generator g_s of $W(\mathrm{M})$ is of order 2 in $W(\mathrm{M})$.*
- *(c) If $s, s' \in S$, then $g_s g_{s'}$ is of order $m_{ss'}$ in $W(\mathrm{M})$.*

Proof. (a) Since for every pair of distinct elements $s, s' \in S$ we have that σ_s and $\sigma_{s'}$ are distinct in $\mathsf{GL}(\mathrm{E})$, it follows that the map $s \mapsto \sigma_s$ is injective. Hence, the composite map

$$\begin{aligned} \sigma \circ \kappa : S \to W \to \mathsf{GL}(\mathrm{E}) \\ s \mapsto g_s \mapsto \sigma_s \end{aligned}$$

from S to $\mathsf{GL}(\mathrm{E})$ is injective, and thus $\kappa : S \to W$ must also be injective, as required.

(b) Since for each $s \in S$ we have $g_s^2 = 1$, we know that the order of g_s is at most 2. Since we have that the order of σ_s in $\mathsf{GL}(\mathrm{E})$ is exactly 2, we conclude that the order of g_s in W is exactly 2 and (b) follows.

(c) Similarly, since for any distinct elements $s, s' \in S$, we have that the order of $g_s g_{s'}$ is at most $m_{ss'}$ and the order of $\sigma_s\sigma_{s'}$ is exactly $m_{ss'}$, we conclude that the order of $g_s g_{s'}$ in W is exactly $m_{ss'}$ and (c) follows. □

Remark 2.7.11. Note that (a) in Proposition 2.7.5 implies that S can be identified with a subset of W by means of the map $\kappa : s \to g_s$. Then (b) and (c) in Proposition 2.7.5 together with the definition of $W(M)$ give the following result.

Corollary 2.7.1. *The pair $(W(\mathrm{M}), S)$ is a Coxeter system with matrix* M.

Remark 2.7.12. Note that we have in fact shown that to every Coxeter matrix M one can associate a Coxeter group, namely $W(\mathrm{M})$.

2.8 Contragradient representation of W

In this section we keep the notation of Section 2.7. The aim of this section is to show that the geometric representation $\sigma : W(\mathrm{M}) \to \mathsf{GL}(\mathrm{E})$ is a faithful representation of the group $W(\mathrm{M})$ as a group generated by reflections. In order to prove this we use some sort of "chamber geometry". Since the bilinear form $\mathrm{B_M}$ may well be degenerate, we do not have an euclidean inner product to work with, so there is no such thing as positive and negative half-spaces defined by a reflecting hyperplane as in the case when $m_{ss'}$ is finite in Proposition 2.7.2. The natural place to look for is thus the algebraic dual $\mathrm{E}^\vee$ of the vector space E. We start by describing a representation of the group $W(\mathrm{M})$ in $\mathrm{E}^\vee$ as well as a collection of subsets in $E^\vee$ that play the role of positive half-spaces defined by a reflecting hyperplane. Next we state the main result of this section, a result by Tits, and before proving it, we give some relevant consequences if it. Next, we focus on the proof of Tits' Theorem, for which we first need to prove some lemmas. This will provide some intuition for the choice of term "chamber geometry", so we end this section by formalizing what its components.

Remark 2.8.1. Since $W(\mathrm{M})$ acts on E via $\sigma : W(\mathrm{M}) \to \mathsf{GL}(\mathrm{E})$, by transport of structure, $W(\mathrm{M})$ also acts on $\mathrm{E}^\vee$ via the representation $\sigma^\vee : W(\mathrm{M}) \to \mathsf{GL}(\mathrm{E}^\vee)$, called the *contragradient representation of* σ, and is given by

$$\sigma^\vee(w) = {}^t\sigma\left(w^{-1}\right) \quad \text{for all } w \in W(\mathrm{M}). \tag{2.8.1}$$

To ease notation, if $f^\vee \in \mathrm{E}^\vee$ is an element of $\mathrm{E}^\vee$ and $w \in W(\mathrm{M})$, we denote by $w\left(f^\vee\right)$ the transform of $f^\vee$ by $\sigma^\vee(w)$, i.e.,

$$w\left(f^\vee\right) = \sigma^\vee(w)\left(f^\vee\right) = {}^t\sigma\left(w^{-1}\right)(x^\vee).$$

Therefore, for any $x \in \mathrm{E}$, we have that

$$w\left(f^\vee\right)(x) = \sigma^\vee(w)\left(f^\vee(x)\right) = {}^t\sigma\left(w^{-1}\right)\left(f^\vee(x)\right) = f^\vee\left(\sigma(w^{-1})(x)\right).$$

Definition 2.8.1. For any $s \in S$, let A_s denote the set of all $f^\vee \in \mathrm{E}^\vee$ such that $f^\vee(e_s) > 0$, that is,

$$A_s := \left\{f^\vee \in \mathrm{E}^\vee \;\middle|\; f^\vee(e_s) > 0\right\}.$$

Also, let $A_\circ$ be the intersection of the A_s where s runs through S, that is

$$A_\circ := \bigcap_{s\in S} A_s = \left\{f^\vee \in \mathrm{E}^\vee \;\middle|\; f^\vee(e_s) > 0 \text{ for all } s \in S\right\}.$$

Remark 2.8.2. When S is finite, $A_\circ$ is a simplicial cone in $E^\vee$.

Theorem 2.8.1 (Tits). *If $w \in W(\mathrm{M})$ is an element of $W(\mathrm{M})$ and $A_\circ \cap w(A_\circ) \neq \emptyset$, then $w = 1$.*

Corollary 2.8.1. *The group $W(\mathrm{M})$ acts simply transitively on the set of $w(A_\circ)$ where w runs through $W(\mathrm{M})$.*

Proof. This follows immediately from Theorem 2.8.1. □

Corollary 2.8.2. *The representations σ and $\sigma^\vee$ are faithful.*

Proof. Let $w \in W(\mathrm{M})$ be an element of $W(\mathrm{M})$ contained in the kernel of the representation $\sigma^\vee$. Then, by definition of the kernel of a homomorphism, we have $\sigma^\vee(w) = \mathbb{1}_{\mathrm{E}^\vee}$, where $\mathbb{1}_{\mathrm{E}^\vee}$ denotes the identity in $\mathsf{GL}(\mathrm{E}^\vee)$, which implies that $w(A_\circ) = A_\circ$, and thus, by Theorem 2.8.1, $w = 1$. This shows that

$$\ker\sigma^\vee \subseteq \{1\} \tag{2.8.2}$$

Since the $1 \in W(\mathrm{M})$ is trivially an element of the kernel of σ, it follows that

$$\{1\} \subseteq \ker\sigma^\vee. \tag{2.8.3}$$

The inclusions in (2.8.2) and (2.8.2) imply that $\ker\sigma = \{1\}$ proving that $\sigma^\vee : W(\mathrm{M}) \to \mathsf{GL}(\mathrm{E}^\vee)$ is faithful. Now, let $w \in W(\mathrm{M})$ be instead an element of $W(\mathrm{M})$ belonging to the kernel of the geometric representation σ of $W(\mathrm{M})$, so that $\sigma\left(w^{-1}\right) = \sigma(w)^{-1} = \mathbb{1}_{\mathrm{E}}$, where $\mathbb{1}_{\mathrm{E}}$ denotes the identity in $\mathsf{GL}(\mathrm{E})$. Then, for any $f^\vee \in \mathrm{E}^\vee$, we have

$$\sigma^\vee(w)\left(f^\vee(x)\right) = f^\vee\left(\sigma(w^{-1})(x)\right) = f^\vee(x) \quad \text{for any } x \in \mathrm{E},$$

which shows that $\ker\sigma \subseteq \ker\sigma^\vee = \{1\}$. Hence since $1 \in W(\mathrm{M})$ is trivially an element of the kernel of σ, it follows that

$$\ker\sigma = \ker\sigma^\vee = \{1\},$$

proving the injectivity of σ. □

Corollary 2.8.3. *If S is finite, then $\sigma\big(W(\mathrm{M})\big)$ is a discrete subgroup of $\mathsf{GL}(\mathrm{E})$, and similarly, $\sigma^\vee(W(\mathrm{M}))$ is a discrete subgroup of $\mathsf{GL}(\mathrm{E}^\vee)$.*

Proof. Assume that S is finite, let $f^\vee \in \mathrm{E}^\vee$ be an element such that $f^\vee \in A_\circ$. Note that the orbit map

$$\begin{aligned} \mathsf{GL}\big(\mathrm{E}^\vee\big) &\to \mathrm{E}^\vee \\ B &\mapsto B \cdot f^\vee \end{aligned} \tag{2.8.4}$$

is continuous since it is given, in coordinate form, by linear polynomials. Then, the set $C_\circ$ of elements $B \in \mathsf{GL}(\mathrm{E}^\vee)$ such that $B \cdot f^\vee \in A_\circ$ is an open neighbourhood of the identity element $\mathbb{1}_{\mathrm{E}^\vee}$ in $\mathsf{GL}(\mathrm{E}^\vee)$. Note that the openness is due to the continuity of the orbit map and Remark 2.8.2. Then, by Theorem 2.8.1,

$$\sigma^\vee\big(W(\mathrm{M})\big) \cap C_\circ = \{\mathbb{1}_{\mathrm{E}^\vee}\}.$$

This shows that $\sigma^\vee\big(W(\mathrm{M})\big)$ is a discrete subgroup of $\mathsf{GL}(\mathrm{E}^\vee)$. Moreover, by transport of structure, it follows that $\sigma(W(\mathrm{M}))$ is a discrete subgroup of $\mathsf{GL}(\mathrm{E})$. □

Lemma 2.8.1. *Let $s, s' \in S$ be distinct elements of S, and let $u' \in W_{s,s'}$, where $W_{s,s'}$ denotes the subgroup of $W(\mathrm{M})$ generated by s and s'. Then the set $u'\left(A_s \cap A_{s'}\right)$ is contained in either A_s or in $s(A_s)$, and in the latter case, $\ell(su') = \ell(u') - 1$.*

Proof. Let $\mathrm{E}^\vee_{s,s'}$ denote the algebraic dual of the plane $\mathrm{E}_{s,s'} = \mathbb{R}e_s \oplus \mathbb{R}e_{s'}$. The transpose of the injection $\iota : \mathrm{E}_{s,s} \hookrightarrow \mathrm{E}$ given by inclusion is a surjection

$$\begin{aligned} \iota^\vee : \mathrm{E}^\vee &\to \mathrm{E}^\vee_{s,s'} \\ f^\vee &\mapsto f^\vee \circ \iota \quad (f^\vee \in \mathrm{E}^\vee) \end{aligned}$$

that commutes with the action of the group $W_{s,s}$. Now, for s and s', let

$$A'_s := \left\{g^\vee \in \mathrm{E}^\vee_{s,s'} \;\middle|\; g^\vee(e_s) > 0\right\} \quad \text{and} \quad A'_{s'} := \left\{g^\vee \in \mathrm{E}^\vee_{s,s'} \;\middle|\; g^\vee(e_{s'}) > 0\right\}.$$

Then, if $g^\vee \in A'_s$, the inverse image of $g^\vee$ under $\iota^\vee$ is equal to the set

$$\left\{f^\vee \in \mathrm{E}^\vee \;\middle|\; f^\vee \circ \iota = g^\vee\right\},$$

that is, the set of $f^\vee \in \mathrm{E}^\vee$ such that the restriction of $f^\vee$ to $\mathrm{E}_{s,s}$ is equal to $g^\vee$, and thus $f^\vee(e_s) > 0$, which implies that $f^\vee \in A_s$. Conversely, for any element $p^\vee \in A_s$ we have $p^\vee(e_s) > 0$, and thus

$$\iota^\vee\left(p^\vee\right)(e_s) = (p^\vee \circ \iota)(e_s) = p^\vee(e_s) > 0,$$

which shows that $\iota^\vee\left(p^\vee\right) \in A'_s$. We have hence shown that A_s is the inverse image of the set A'_s under $\iota^\vee$. By the same argument, $A_{s'}$ and $A_s \cap \mathrm{A}_{s'}$ are the inverse image of $A'_{s'}$ and $A'_s \cap A'_{s'}$, respectively. Moreover, since by Corollary 2.5.4, the length of any element of $W_{s,s'}$ is the same with respect to $\{s, s'\}$ and with respect to S, we are reduced to the case $S = \{s, s'\}$, where $\mathrm{E} = \mathrm{E}_{s,s'}$. Recall that in this case, $W(\mathrm{M})$ is a dihedral group of order $2m_{ss'}$. As usual, we distinguish two cases: the case $m_{ss'} = \infty$ and the case $m_{ss'}$ is finite.

Suppose $m_{ss'} = \infty$, and let $\left(e_s^\vee, e_{s'}^\vee\right)$ be the dual basis of the basis $(e_s, e_{s'})$ of E. Recall that the two dual basis elements $e_s^\vee$ and $e_{s'}^\vee$ are the two linear maps on E defined by the relations

$$e_s^\vee\left(\lambda_s e_s + \lambda_{s'} e_{s'}\right) = \lambda_s \quad \text{and} \quad e_{s'}^\vee\left(\lambda_s e_s + \lambda_{s'} e_{s'}\right) = \lambda_{s'}$$

for any choices of real numbers $\lambda_s, \lambda_{s'} \in \mathbb{R}$. Then, since $s = s^{-1}$, we have

$$s\left(e_s^\vee\right) = {}^t\sigma\left(s^{-1}\right)\left(e_s^\vee\right) = {}^t\sigma_s\left(e_s^\vee\right) = e_s^\vee \circ \sigma_s,$$

and so, since $e_s^\vee : \mathrm{E}_{s,s'} \to \mathbb{R}$ is a linear map, and $\mathrm{B_M}(e_s, e_{s'}) = -1$, it follows that

$$s\left(e_s^\vee\right)(e_s) = \left(e_s^\vee \circ \sigma_s\right)(e_s) = e_s^\vee\big(\sigma_s(e_s)\big) = e_s^\vee\left(-e_{s'}\right) = -1$$

and

$$s\left(e_s^\vee\right)(e_{s'}) = \left(e_s^\vee \circ \sigma_s\right)(e_{s'}) = e_s^\vee\left(e_{s'} + 2e_s\right) = e_s^\vee(e_{s'}) + 2e_s^\vee(e_s) = 0 + 2 = 2.$$

Hence

$$s\left(e_s^\vee\right) = -e_s^\vee + 2e_{s'}^\vee. \tag{2.8.5}$$

Similarly, we have

$$s\left(e_{s'}^\vee\right) = e_{s'}^\vee \circ \sigma_s, \quad s'\left(e_s^\vee\right) = e_s^\vee \circ \sigma_{s'} \quad \text{and} \quad s'(e_{s'}^\vee) = e_{s'}^\vee \circ \sigma_{s'},$$

and so, as we have just done, since

$$\sigma_s(e_s) = -e_s, \quad \sigma_s(e_{s'}) = e_{s'} + 2e_s \quad \text{and} \quad \sigma_{s'}(e_s) = e_s + 2e_{s'},$$

we have

$$\begin{aligned} s\left(e_{s'}^\vee\right)(e_s) = e_{s'}^\vee\left(-e_s\right) = 0 \quad &\text{and} \quad s\left(e_{s'}^\vee\right)(e_{s'}) = e_{s'}^\vee\left(e_{s'} + 2e_s\right) = 1, \\ s'\left(e_s^\vee\right)(e_s) = e_s^\vee\left(e_s + 2e_{s'}\right) = 1 \quad &\text{and} \quad s'\left(e_s^\vee\right)(e_{s'}) = e_s^\vee\left(-e_{s'}\right) = 0, \\ s'\left(e_{s'}^\vee\right)(e_s) = e_{s'}^\vee\left(e_s + 2e_{s'}\right) = 2 \quad &\text{and} \quad s'\left(e_{s'}^\vee\right)(e_{s'}) = e_{s'}^\vee\left(-e_{s'}\right) = -1, \end{aligned}$$

and thus

$$s\left(e_{s'}^\vee\right) = e_{s'}^\vee, \quad s'\left(e_s^\vee\right) = e_s^\vee, \quad \text{and} \quad s'\left(e_{s'}^\vee\right) = 2e_s^\vee - e_{s'}^\vee. \tag{2.8.6}$$

From (2.8.5) and (2.8.6), if L is an affine line of $\mathrm{E}^\vee$ containing both $e_s^\vee$ and $e_{s'}^\vee$, we see that L is stable under s and s' and that the restriction of s and s' to L is the reflection with respect to the point $e_{s'}^\vee$ and $e_s^\vee$ of $E^\vee$, respectively. Let

$$\begin{aligned} \vartheta : \mathbb{R} &\to L \\ \lambda &\mapsto \lambda e_s^\vee + (1-\lambda)e_{s'}^\vee, \end{aligned}$$

let $\vartheta(n, n+1)$ denote the image of the open interval $(n, n+1)$ in $\mathbb{R}$, where $n \in Z$, under ϑ, and let

$$\Theta_{(n,n+1)} := \bigcup_{\alpha \in \mathbb{R}^+} \alpha \cdot \vartheta(n, n+1)$$

Then $\vartheta(0,1)$ is the convex open set containing all the $f^\vee \in L$ such that that $f^\vee = \alpha e_s^\vee + (1-\alpha)e_{s'}^\vee$ for some $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, and thus, any element $g^\vee \in \Theta_{(0,1)}$ is of the form

$$g^\vee = \lambda f^\vee = \lambda\alpha e_s^\vee + \lambda(1-\alpha)e_{s'}^\vee$$

for some $\lambda, \alpha \in \mathbb{R}$ with $0 < \alpha < 1$. Then

$$g^\vee(e_s) = \lambda\alpha e_s^\vee(e_s) + \lambda(1-\alpha)e_{s'}^\vee(e_s) = \lambda\alpha > 0$$

and

$$g^\vee(e_{s'}) = \lambda\alpha e_s^\vee(e_{s'}) + \lambda(1-\alpha)e_{s'}^\vee(e_{s'}) = \lambda(1-\alpha) > 0,$$

which shows that $g^\vee \in A_s \cap A_{s'}$. Conversely, if $q^\vee \notin \Theta_{(0,1)}$, then $q^\vee$ is of the form

$$q^\vee = \lambda f^\vee = \lambda'\alpha' e_s^\vee + \lambda'(1-\alpha')e_{s'}^\vee$$

for some $\lambda', \alpha' \in \mathbb{R}$ with $\alpha > 1$. In this case we have

$$q^\vee(e_s) = \lambda'\alpha' e_s^\vee(e_s) + \lambda'(1-\alpha')e_{s'}^\vee(e_s) = \lambda'\alpha' > 0$$

and

$$q^\vee(e_{s'}) = \lambda'\alpha' e_s^\vee(e_{s'}) + \lambda'(1-\alpha')e_{s'}^\vee(e_{s'}) = \lambda'(1-\alpha') < 0,$$

which shows that $q^\vee \notin A_{s'}$, and in particular $q^\vee \notin A_\circ = A_s \cap A_{s'}$. We have hence shown that $\Theta_{(0,1)} = A_\circ$. Moreover, since

$$s's\left(e_s^\vee\right) = s'\left(-e_s^\vee + 2e_{s'}^\vee\right) = 3e_s^\vee - 2e_{s'}^\vee \quad \text{and} \quad s's\left(e_{s'}^\vee\right) = s\left(e_{s'}^\vee\right) = 2e_s^\vee - e_{s'}^\vee,$$

we have, for any $p^\vee = \lambda\beta e_s^\vee + \lambda(1-\beta)e_{s'}^\vee \in L$, with $\lambda, \beta \in \mathbb{R}$ and $\lambda > 0$, that

$$s's\left(p^\vee\right) = \lambda\left(\beta+2\right)e_s^\vee + \lambda\left(-\beta-1\right)e_{s'}^\vee = \lambda\left(\beta+2\right)e_s^\vee + \lambda\left(1-\left(\beta+2\right)\right)e_{s'}^\vee,$$

so we see that $p^\vee \in A_\circ = \Theta_{(0,1)}$ if and only if $s's\left(p^\vee\right) \in \Theta_{(2,3)}$, which implies that

$$s's(A_\circ) = s's\left(\Theta_{(0,1)}\right) = \Theta_{(2,3)}.$$

Hence, by induction on $n \geq 1$, we obtain

$$(s's)^n(A_\circ) = (s's)^n\left(\Theta_{(0,1)}\right) = s's\left(\Theta_{(2(n-1),2(n-1)+1)}\right) = \Theta_{(2n,2n+1)}.$$

Also, since

$$s\left(p^\vee\right) = \lambda\beta\left(-e_s^\vee + 2e_{s'}^\vee\right) + \lambda(1-\beta)e_{s'}^\vee = -\lambda\beta e_s^\vee + \lambda\left(1-(-\beta)\right)e_{s'}^\vee,$$

we see that $p^\vee \in A_\circ = \Theta_{(0,1)}$ if and only if $s\left(p^\vee\right) \in \Theta_{(-1,0)}$, which implies that

$$s(A_\circ) = s\left(\Theta_{(0,1)}\right) = \Theta_{(-1,0)}.$$

Hence, again by induction on $n \geq 1$, we obtain

$$(s's)^n s(A_\circ) = (s's)^n s\left(\Theta_{(0,1)}\right) = (s's)^n\left(\Theta_{(-1,0)}\right) = \Theta_{(2n-1,2n)}.$$

It thus follows that the dihedral group $W(\mathrm{M})$ generated by s and s' permutes the $\Theta_{(n,n+1)}$ $(n \in \mathbb{Z})$ simply-transitively. Note that, by taking $\lambda = 1$ in the argument for $p^\vee$ above, we have also shown that $W(\mathrm{M})$ permutes the $\vartheta(n, n+1)$ $(n \in \mathbb{Z})$ simply-transitively. Therefore, if $u' \in W(\mathrm{M})$, we have that $u'(A_\circ) = \Theta_{(n,n+1)}$ for some $n \in \mathbb{Z}$, and thus

$$u'(A_\circ) \subseteq \begin{cases} A_s & \text{if } n \geq 0 \\ s(A_s) & \text{if } n < 0. \end{cases}$$

In the latter case, $\vartheta(0,1)$ and $\vartheta(n, n+1)$ are on opposite sides of the point $e_{s'}^\vee \in \mathrm{E}^\vee$, and hence $\ell(su') = \ell(u') - 1$, as claimed.

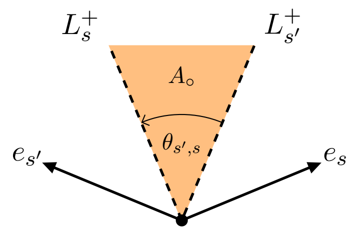


Figure 2.2: The half-lines L_s^+ and $L_{s'}^+$ of E as a euclidean plane with basis $(e_s, e_{s'})$

On the other hand, suppose now that $m_{ss'}$ is finite. Then, by Proposition 2.7.1, the symmetric bilinear form $\mathrm{B_M}$ is non-degenerate, so we may identify $\mathrm{E}^\vee$ with E. Recall from the proof of Proposition 2.7.1, that E can be oriented so that the angle between the half-lines $\mathbb{R}^+e_s$ and $\mathbb{R}^+e_{s'}$ is equal to $\pi - \frac{\pi}{m_{ss'}}$. Now, for any $\theta \in \mathbb{R}$ with $0 \leq \theta \leq 2\pi$, let ρ_θ denote a rotation of θ and let L_s^+ and $L_{s'}^+$ be the half-lines corresponding to $A_s \cap L_s$ and $A_{s'} \cap L_{s'}$, respectively. Then L_s^+ and $L_{s'}^+$ are the the half-lines obtained from $\mathbb{R}^+e_s$ and $\mathbb{R}^+e_{s'}$ by a rotation of $\frac{\pi}{2}$ and a rotation of $-\frac{\pi}{2}$, respectively (see Figure 2.2), i.e.,

$$L_s^+ = \rho_{\frac{\pi}{2}}\left(\mathbb{R}^+e_s\right) \quad \text{and} \quad L_{s'}^+ = \rho_{-\frac{\pi}{2}}\left(\mathbb{R}^+e_{s'}\right).$$

Now let $\Theta_{(n,n+1)}$ be the set of open half-lines L' such that the angle $\theta_{L_{s'}^+,L'}$ between $L_{s'}^+$ and L' satisfies

$$n\theta_{s',s} < \theta_{L_{s'}^+,L'} < (n+1)\theta_{s',s}.$$

Note that the $\Theta_{(n,n+1)}$ for $n \in \mathbb{Z}$ with $-m_{ss'} \leq n \leq m_{ss'}$ are connected open subsets forming a partition of the complement of

$$\bigcup_{-m_{ss'} \leq n \leq m_{ss'}} \rho_{n\theta_{s',s}}\left(L_{s'}^+\right)$$

in E. Since $A_\circ = A_s \cap A_{s'}$ is the set of x in E whose scalar product with e_s and $e_{s'}$ is greater than zero, it is the open angular sector with origin $L_{s'}^+$ and extremity L_s^+ (see Figure 2.2). But this is of course the set $\Theta_{(0,1)}$, so $A_\circ = \Theta_{(0,1)}$. Since for any half-line L' and any $n \in \mathbb{Z}$ such that $1 \leq n \leq m_{ss'}$, the angle between the half-line $L_{s'}^+$ and the half-line $\rho_{n\theta_{s,s'}}(L')$ is equal to the angle between $L_{s'}^+$ and L' plus $n\theta_{s',s}$, it follows that L' belongs to $A_\circ$ if and only if $\rho_{n\theta_{s',s}}(L')$ belongs to the open angular sector with origin $\rho_{n\theta_{s',s}}\left(L_{s'}^+\right)$ and extremity $\rho_{(n+1)\theta_{s',s}}\left(L_s^+\right)$, that is, $\Theta_{(n,n+1)} = \rho_{n\theta_{s',s}}(A_\circ)$. In particular, we have that

$$\Theta_{(2k,2k+1)} = \rho_{2k\theta_{s',s}}(A_\circ) \quad \text{and} \quad \Theta_{(2k-1,2k)} = \rho_{2k\theta_{s',s}}s'(A_\circ)$$

for any $k \in \mathbb{Z}$, and $\Theta_{(n,n+1)} = A_\circ$ if and only if $n \in 2m_{ss'}\mathbb{Z}$. This shows that the group $W(\mathrm{M})$ permutes the sets $\Theta_{(n,n+1)}$ $(n \in \mathbb{Z})$ simply-transitively, and thus every element $u' \in W(\mathrm{M})$ transforms $A_\circ$ into an open angular sector that is either on the same side of the line L_s as $A_\circ$, in which case $u' \in A_s$, or on the opposite side of the line L_s as $A_\circ$, in which case $u' \in s(A_s)$. In the latter case, we have that $u'(A_\circ) = \Theta_{(n,n+1)}$ for some $n \in \mathbb{Z}$ with $0 < n \leq m_{ss'}$. If $n = 2k$ for some $k \in \mathbb{N}$, we have

$$u' = (ss')^k \quad \text{and} \quad su' = s'(ss')^{k-1},$$

and thus

$$\ell(u') = 2k \quad \text{and} \quad \ell(su') = 2k - 1 = \ell(u') - 1.$$

If $n = 2k - 1$ for some $k \in \mathbb{N}$, we have

$$u' = (ss')^k s' \quad \text{and} \quad su' = s(ss')^k s' = (s's)^{k-1}$$

and thus

$$\ell(u') = 2k-1 \quad \text{and} \quad \ell(su') = 2k-2 = \ell(u')-1,$$

completing the proof of the lemma. $\square$

Lemma 2.8.2. *Let $w \in W$, and let $r \in \mathbb{N}_0$, where $\mathbb{N}_{\{}0,1,2,3,4,\dots\}$. Then the following statements are true:*

(P_r) If $\ell(w) = r$ and $s \in S$, then either $w(A_\circ) \subseteq A_s$ or $w(A_\circ) \subseteq s(A_s)$ and $\ell(sw) = \ell(w)-1$.

(Q_r) If $\ell(w) = r$ and $s, s' \in S$ are such that $s \neq s'$, then there exists an element $u \in W_{s,s'}$, such that

$$w(A_\circ) \subseteq u\left(A_s \cap A_{s'}\right) \quad \textit{and} \quad \ell(w) = \ell(u) + \ell(u^{-1}w).$$

Proof. First consider the case when $r = 0$. In this case, *(P_0)* holds since $\ell(w) = 0$ if and only if $w = 1$, and thus $w(A_\circ) = A_\circ$ is, by definition of $A_\circ$, a subset of A_s. Similarly, if we take $u = 1$, which is an element of $W_{s,s'}$, we see that

$$w(A_\circ) = A_\circ \subseteq u(A_s \cap A_{s'}) = A_s \cap A_{s'}.$$

Also, $\ell(u) = 0$ and $\ell(u^{-1}w) = \ell(1) = 0$, so we have

$$\ell(w) = 0 = (u) + \ell(u^{-1}w),$$

showing that *(Q_0)* also holds. To prove the lemma, we will by induction on $r \geq 0$, that *(P_r)* and *(Q_r)* imply *(P_{r+1})*, and that *(P_{r+1})* and *(Q_r)* imply *(Q_{r+1})*.

We start buy showing that *(P_r)* and *(Q_r)* imply *(Q_{r+1})*. So let $w \in W(\mathrm{M})$ such that $\ell(w) = r+1$, let $s \in S$ and assume that *(P_r)* and *(Q_r)* hold. Recall that we may write $w = s'w'$ with $s' \in S$ and $w' \in W$ such that $\ell(w') = r$. If $s' = s$, then $w'(A_\circ) \subseteq A_s$ by *(P_r)*, and thus

$$w(A_\circ) = s'w'(A_\circ) = sw'(A_\circ) \subseteq s(A_s).$$

Moreover, we have $sw = ss'w' = s^2w' = w'$, and thus $\ell(sw) = \ell(w') = r = \ell(w)-1$. On the contrary, if $s' \neq s$, there exists, by *(Q_r)*, $u \in W_{s,s'}$ such that

$$w'(A_\circ) \subseteq u\left(A_s \cap A_{s'}\right) \quad \text{and} \quad \ell(w') = \ell(u) + \ell(u^{-1}w'), \tag{2.8.7}$$

and thus

$$w(A_\circ) = s'w'(A_\circ) \subseteq s'u\left(A_s \cap .A_{s'}\right)$$

Lets now apply Lemma 2.8.1 to the element $u' := s'u$. Note that we have two possibilities: either $u'\left(A_s \cap A_{s'}\right) \subseteq A_s$, in which case $w(A_\circ) \subseteq A_s$, or $u'\left(A_s \cap A_{s'}\right) \subseteq s(A_s)$, in which case $w(A_\circ) \subseteq s(A_s)$. Moreover, in the latter case, we have $\ell(su') = \ell(u')-1$, and hence, since

$$sw = ss'w' = ss'uu^{-1}w' = su'u^{-1}w',$$

we have

$$\ell(sw) = \ell(su'u^{-1}w') \leq \ell(su') + \ell(u^{-1}w') = \ell(u') - 1 + \ell(u^{-1}w'), \tag{2.8.8}$$

where the first inequality follows from (2.2.1) in Proposition 2.2.1. Then, since multiplying u' on the left by s reduces the length by one and $u \in W_{s,s'}$, multiplying u' on the left by s' increases the length by one so that $\ell(u) = \ell(s'u') = \ell(u')+1$, and (2.8.8) becomes

$$\ell(sw) \leq \ell(u) - 1 + \ell(u^{-1}w'). \tag{2.8.9}$$

Then, substituting into (2.8.9) the equality in (2.8.7), yields

$$\ell(sw) \leq \ell(w') - 1 \leq \ell(w) - 1,$$

which implies that $\ell(sw) = \ell(w) - 1$, and the first part of the proof is complete.

We now show that *(P_{r+1})* and *(Q_r)* imply *(Q_{r+1})*. Again, let $w \in W(\mathrm{M})$ such that $\ell(w) = r+1$, let $s \neq s'$ be distinct elements of S and assume that *(P_{r+1})* and *(Q_r)* hold. If $w(A_\circ) \subseteq A_s \cap A_{s'}$, then *($Q_{r+1}$)* holds with $u = 1$. If on the contrary $w(A_\circ) \nsubseteq A_s \cap A_{s'}$, then $w(A_\circ)$ is not contained in at least one of A_s and $A_{s'}$. So suppose, for example that $w(A_\circ) \nsubseteq A_s$. Then $w(A_\circ) \subseteq s(A_s)$ by *(P_{r+1})*, and $\ell(sw) = \ell(w) - 1 = r$. Then, since *($Q_r$)* holds, there exists $v \in W_{s,s'}$ such that

$$sw(A_\circ) \subseteq v\,(A_s \cap A_{s'}) \quad \text{and} \quad \ell(sw) = \ell(v) + \ell(v^{-1}sw).$$

Then $w(A_\circ) = ssw(A_\circ) \subseteq sv\,(A_s \cap A_{s'})$ and,

$$\ell(w) = \ell(sw) + 1 = \ell(v) + \ell(v^{-1}sw) + 1 \geq \ell(sv) + \ell\left((sv)^{-1}w\right) \geq \ell(w), \tag{2.8.10}$$

where the last inequality follows from (2.2.1) in Proposition 2.2.1. We thus see that the inequalities in (2.8.10) are all in fact equalities. It follows that *(Q_{r+1})* holds with $u = sv$. A similar argument shows that *(Q_{r+1})* holds in the case that $w(A_\circ) \nsubseteq A_{s'}$, and the second part of the proof is now complete. □

Proof of Theorem 2.8.1. Let $w \in W$ such that $w \neq 1$. Recall that we may write w in the form $w = sw'$ with $s \in S$ and $w' \in W$ such that $n := \ell(w') = \ell(w) - 1$. By *($P_n$)* applied to w', we must have that $w'(A_\circ) \subseteq A_s$ since $\ell(sw') = \ell(w) = \ell(w') + 1$ excludes the possibility that $w(A_\circ) \subseteq s(A_s)$. Hence

$$w(A_\circ) = sw'(A_s) \subseteq s(A_s),$$

and thus, since A_s and $s(A_s)$ are disjoint, it follows that $A_\circ \cap w(A_\circ) = \emptyset$, as required. □

Definition 2.8.2. Let H_s be defined by

$$H_s := \{f^\vee \in \mathrm{E}^\vee \mid f^\vee(e_s) = 0\} = \{f^\vee \in E^\vee \mid \langle e_s, f^\vee \rangle = 0\},$$

that is, H_s is the hyperplane of $E^\vee$ orthogonal to e_s with respect to the pairing $\langle \cdot, \cdot \rangle : \mathrm{E} \times \mathrm{E}^\vee \to \mathrm{E}$ in (2.7.4). Then the hyperplanes $w(H_s)$ ($w \in W$, $s \in S$) are called *walls*, and the convex cones $w(A_\circ)$ in $E^\vee$ are called *chambers* and in particular $A_\circ$ is called the *fundamental chamber*. Let $\mathcal{W}$ and $\mathcal{C}$ denote the collection of walls and chambers, respectively.

Remark 2.8.3. It is important to note that the walls and chambers are in $E^\vee$, and, by Theorem 2.8.1, W acts simply transitively on $\mathcal{C}$. Moreover, if S is finite, by Remark 2.8.2, the chambers are actually simplicial cones in $\mathrm{E}^\vee$.

Remark 2.8.4. From Lemmas 2.8.1 and 2.8.2 we see that every chamber $w(A_\circ)$ ($w \in W(M)$) lies on one side of each wall.

Remark 2.8.5. Note that if we set

$$\mathring{A}_s := \left\{f^\vee \in \mathrm{E}^\vee \mid \langle e_s, f^\vee \geq 0 \rangle\right\},$$

and

$$\mathring{A}_\circ := \bigcap_{s \in S} \mathring{A}_s,$$

then, for the weak topology on E with respect to the canonical pairing $\langle \cdot, \cdot \rangle : \mathrm{E} \times \mathrm{E}^\vee \to \mathrm{E}$ in (2.7.4), the sets $\mathring{A}_s$ ($s \in S$) are closed half-spaces, and $\mathring{A}_\circ$ is a closed convex cone. In particular, since for any $f^\vee \in A_\circ$ and any $g^\vee \in \mathring{A}_\circ$, we have $f^\vee + \lambda g^\vee \in A_\circ$ and any $\lambda \in \mathbb{R}$ with $\lambda > 0$ and

$$f^\vee = \lim_{\lambda \to 0} f^\vee + \lambda g^\vee,$$

we see that $\mathring{A}_\circ$ is in fact the closure of $A_\circ$.

Definition 2.8.3. For any subset $X \subseteq S$, let

$$A_X := \left(\bigcap_{s \in X} H_s \right) \cap \left(\bigcap_{s \in S \setminus X} A_s \right).$$

Remark 2.8.6. Note that $A_X \subseteq \mathring{A}_\circ$ for any $X \subseteq S$, and in particular, we have $A_\emptyset = A_\circ$ and $A_S = \{0^\vee\} \subseteq \mathrm{E}^\vee$. Also note that for any $w \in W_X$ in the standard parabolic subgroup W_X and any $f^\vee \in A_X$, we have that $w\left(f^\vee\right) = f^\vee$, since for the restriction of $\sigma^\vee$ to the Coxeter system (W_X, X) we have $s\left(f^\vee\right) = f^\vee$ for every $f^\vee$ contained in the intersection of the walls H_s $(s \in S)$.

Proposition 2.8.1. *Let $X, Y \subseteq S$ and let $w, w' \in W(\mathrm{M})$. If $w(A_X) \cap w'(A_Y) \neq \emptyset$, then*

$$X = Y, \quad wW_X = w'W_Y \quad \textit{and} \quad w(A_X) = w'(A_Y).$$

Proof. Note that by Remark 2.8.6 and Tits' Theorem (Theorem 2.8.1), the proof reduces to the case when $w' = 1$. We thus proceed by induction on the length $\ell(w) \geq 0$. If $\ell(w) = 0$, then $w = 1$ and the statement is clear. So suppose that $\ell(w) > 0$, and assume that for any $z \in W(\mathrm{M})$ such that $\ell(z) < \ell(w)$ and $z(A_X) \cap A_Y \neq \emptyset$, then

$$X = Y, \quad zW_X = W_Y \quad \text{and} \quad z(A_X) = A_Y.$$

holds. Recall that there exists some $s \in S$ such that $\ell(sw) = \ell(w) - 1$, so that, by Lemma 2.8.2, we have $w(A_\circ) \subseteq s(A_s)$, and thus $w(\mathring{A}_\circ) \subseteq s(\mathring{A}_s)$. In particular, since $\mathring{A}_\circ \subseteq \mathring{A}_s$, it follows that $\mathring{A}_\circ \cap w(\mathring{A}_\circ) \subseteq H_s$. Hence $s\left(f^\vee\right) = f^\vee$ for all $f^\vee \in \mathring{A}_\circ \cap w(\mathring{A}_\circ)$, and in particular, $s\left(f^\vee\right) = f^\vee$ holds for all $f^\vee \in w(A_X) \cap A_Y$. As a result, the relation $w(A_X) \cap A_Y \neq \emptyset$ implies that $H_s \cap A_Y \neq \emptyset$ and $A_Y \cap sw(A_X) \neq \emptyset$, and the former in turn implies that $s \in Y$. By the induction hypothesis and *(d)* in Proposition 2.5.1, it then follows that

$$X = Y, \quad swW_X = W_Y = W_X,$$

which implies, since $s \in W_X$, that $sw \in W_X$ and $w \in W_X$. This in turn implies that $wW_X = W_Y$ and $w(A_X) = A_X = A_Y$, as required. ☐

Corollary 2.8.4. *Let $X \subseteq S$ and $f^\vee \in A_X$ be arbitrary. Then $\mathsf{stab}_{W(\mathrm{M})}\left(f^\vee\right) = W_X$, where $\mathsf{stab}_{W(\mathrm{M})}$ denotes the stabilizer of $f^\vee$ in $W(\mathrm{M})$.*

Definition 2.8.4. The subset $\mathsf{Tits}(W(\mathrm{M}), S) \subseteq \mathrm{E}^\vee$ defined as

$$\mathsf{Tits}(W(\mathrm{M}), S) := \bigcup_{s \in S} w(\mathring{A}_\circ),$$

is called the *Tits' cone of* $(W(\mathrm{M}), S)$.

Remark 2.8.7. Note that by Remark 2.8.5, the set $\mathsf{Tits}(W(\mathrm{M}), S)$ is convex, and in particular, if S is finite, it is actually a simplicial cone. Moreover, by Corollary 2.8.4, the subsets $w(A_X)$ where X runs through the subsets of S form a partition of $\mathsf{Tits}(W(\mathrm{M}), S)$.

Proposition 2.8.2. *The cone $\mathring{A}_\circ$ is a fundamental domain for the action of $W(\mathrm{M})$ on the convex set $\mathsf{Tits}(W(\mathrm{M}), S)$.*

Proof. Since by definition $\mathring{A}_\circ \subseteq \mathsf{Tits}(W(\mathrm{M}), S)$, and we know $\mathring{A}_\circ$ is convex, to show that $\mathring{A}_\circ$ is actually a fundamental domain for the action of $W(\mathrm{M})$ on $\mathsf{Tits}(W(\mathrm{M}), S)$, we just need to show that for any $f^\vee, g^\vee \in \mathring{A}_\circ$ such that $w\left(f^\vee\right) = g^\vee$, we have $f^\vee = g^\vee$. Since $f^\vee, g^\vee \in \mathring{A}_\circ$, we know that there exist $X, Y \subseteq S$ such that $f^\vee \in A_X$ and $g^\vee \in A_Y$. Now, since by assumption $w\left(f^\vee\right) = g^\vee$, we have that $w(A_X) \cap C_Y \neq \emptyset$, and thus, by Proposition 2.8.1, we obtain that $X = Y$ and $w \in W_X$. Hence $f^\vee = g^\vee$, as required. ☐

2.9 Classification

The results in Section 2.6 and Section 2.7 show that Coxeter systems are classified by the equivalence classes of Coxeter matrices. In this section we present some important results about the classification of Coxeter systems so that the reader is familiar whenever we refer to a specific type of Coxeter system in later sections. We do not prove these results, but rather give some examples and provide exact references to the literature where the interested reader can find the corresponding proofs. Good sources to read more about the material presented in this section are [1], [5], [9] and [21]. We continue to use the notation of Sections 2.6-2.7. Since we have already established that the Coxeter systems are classified by the equivalence classes of Coxeter matrices, we let (W, S) be an irreducible Coxeter system with Coxeter matrix M of type S and associated symmetric bilinear form $\mathrm{B_M}$ as in (2.7.1). In this section, and only in this section, we assume that S finite. Also, for any $x \in \mathrm{E}$, let $w(x)$ denote the action of $w \in W$ via the geometric representation $\sigma : W \to \mathsf{GL}(\mathrm{E})$, i.e. $w(x) := \sigma(w)(x)$.

Theorem 2.9.1. *The following are equivalent:*

(a) The group W is finite.

(b) The bilinear form $\mathrm{B_M}$ is positive definite.

(c) The Coxeter graph $\mathsf{Cox}(W, S)$ is of type $\mathrm{A}_n (n \geq 1)$, $\mathrm{B}_n (n \geq 2)$, $\mathrm{D}_n (n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4 or $\mathrm{I}_2(m)$ ($m = 5$ or $m \geq 7$), as given in Figure 2.3, where the index in the name of the types corresponds to the number of vertices.

Proof. See [5], Chapter V, §4.8, Theorem 2 and Chapter VI, §4.1, Theorems 1 and 2, or see [21], Chapter 6, §Theorem 6.4. □

Remark 2.9.1. In this case, (W, S) is a finite Coxeter system that is finitely generated. The symmetric bilinear form $\mathrm{B_M}$ is a scalar product on E and we identify W via $\sigma : W \to \mathsf{GL}(\mathrm{E})$ with a discrete subgroup of the orthogonal group $\mathsf{O}(\mathrm{E})$ of the finite-dimensional vector space E. In particular, W is the subgroup of $\mathsf{O}(\mathrm{E})$ generated by the reflections with respect to the hyperplanes in the family $\mathcal{Z}$ of hyperplanes $w(\mathrm{Z}_s) = w\left(\ker\left(\sigma_s - \mathrm{id_E}\right)\right)$ $(s \in S, w \in W)$ in E, where the Z_s are as defined in Remark 2.7.5. Moreover, there is no non-zero $x \in \mathrm{E}$ that is fixed by the action of W, since any fixed element would be $\mathrm{B_M}$-orthogonal to all e_s $(s \in S)$, and this is only possible if it is 0. Finally, the isomorphism $\mathrm{E} \to \mathrm{E}^\vee$ defined by $\mathrm{B_M}$ transforms the set

$$C_\circ = \left\{x \in \mathrm{E} \mid \mathrm{B_M}(x, e_s) > 0 \text{ for all } s \in S\right\}$$

of E to the fundamental chamber $A_\circ$ of $\mathrm{E}^\vee$ and Property *(P_n)* in Lemma 2.8.2 shows that for every $w \in W$ and every $s \in S$, the sets $w(C_\circ)$ of E does not intersect Z_s, and thus

$$C_\circ \subseteq \mathrm{E} - \bigcup_{\mathrm{Z} \in \mathcal{Z}} \mathrm{Z},$$

that is, $C_\circ$ is contained in the complement of the union of the hyperplanes of $\mathcal{Z}$. Since, recall $A_\circ$ a connected convex subset of E, it is a chamber of E. All the properties proved in Section 2.8 for the fundamental chamber $A_\circ$ apply to W and $A_\circ$, and in particular, $\mathring{C}_\circ$ is a fundamental domain for the action of W on E, which is equivalent to saying that the inverse image of the Tits' cone $\mathsf{Tits}(W, S)$ under the isomorphism $\mathrm{E} \to \mathrm{E}^\vee$ defined by $\mathrm{B_M}$ is the whole of E.

Example 2.9.1. (Type A_2) If (W, S) is of type A_2, then $S = \{s_1, s_2\}$ with $m_{s_1 s_2} = 3$, and in particular, the generators s_1 and s_2 are conjugate in W since

$$s_2 = s_1 s_2 s_1 s_2 s_1 = s_1 s_2 \cdot s_2 \cdot (s_1 s_2)^{-1}.$$

Moreover, W has a presentation

$$W = \left\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1\right\rangle,$$

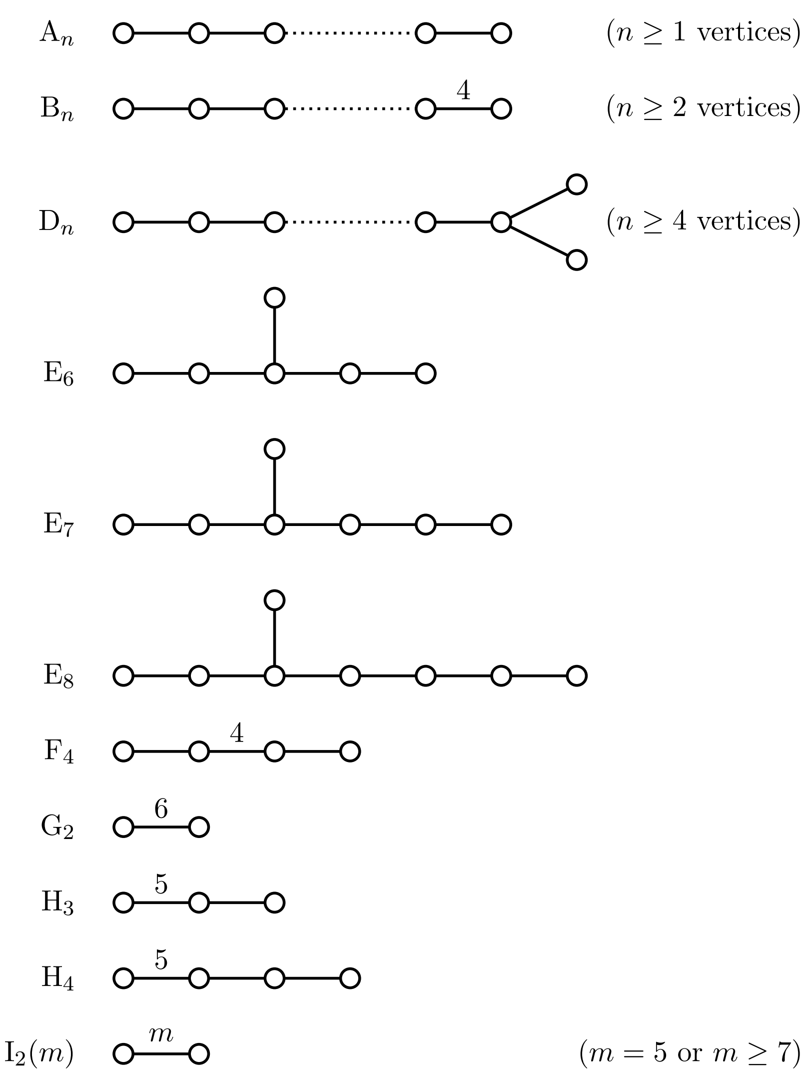


Figure 2.3: Coxeter graphs of irreducible Coxeter systems of finite type

so that

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\},$$

and we see that W is indeed finite with longest element $w_\circ = s_1s_2s_1 = s_2s_1s_2$. In the reflection representation E of (W, S), with basis (e_{s_1}, e_{s_2}), the matrices of s_1 and s_2 are given by

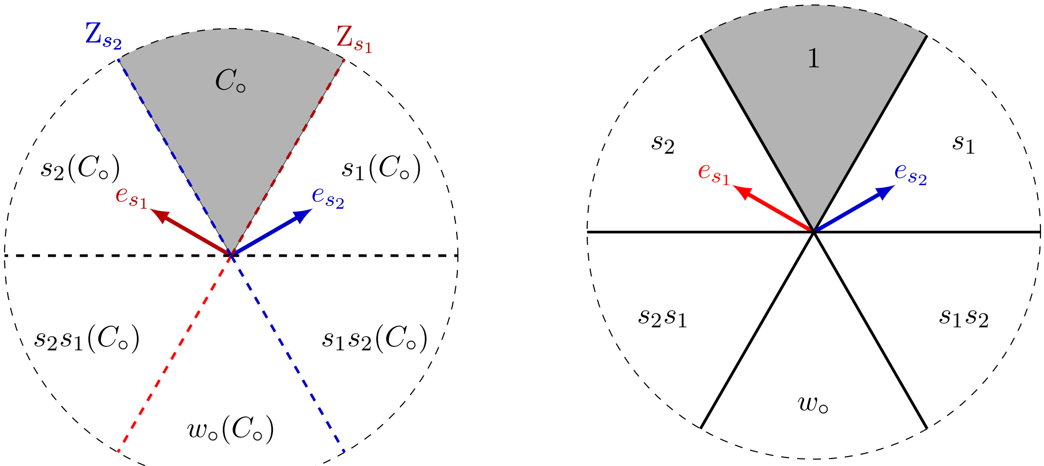
$$s_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Recall from Example 2.6.2 that W is isomorphic to the symmetric group $\mathcal{S}_3$ via the isomorphism mapping $s_1 \mapsto (12)$ and $s_2 \mapsto (23)$, and note that $Z(W) = \{1\}$.

Definition 2.9.1. We say that the Coxeter system (W, S) is *tame* if the associated bilinear form $\mathrm{B_M}(e, e) \geq 0$ for every $e \in E$. If (W, S) is tame and infinite, we day that W is an *affine Weyl group*.

Definition 2.9.2. We say that the Coxeter system (W, S) is *integral* if $m_{ss'} \in \{2, 3, 4, 6, \infty\}$ for all $s, s' \in S$. We sometimes also use the term *crystallographic* in what follows.

Example 2.9.2. If (W, S) is of finite type, then (W, S) is tame, and by Theorem 2.9.1 and Figure 2.3, we see that those for which $\mathsf{Cox}(W, S)$ is of type $\mathrm{A}_n (n \geq 1)$, $\mathrm{B}_n (n \geq 2)$, $\mathrm{D}_n (n \geq 4)$, E_6, E_7, E_8, F_4, G_2 and $\mathrm{I}_2(\infty)$ are integral and those those for which $\mathsf{Cox}(W, S)$ is of type H_3, H_4 and $\mathrm{I}_2(m)$ with $m = 5$ or $7 \leq m < \infty$ are non-integral.



(a) Basis (e_{s_1}, e_{s_2}) of E, the hyperplanes in $\mathcal{Z}$

(b) Identifying the elements of W with the chambers via $w \leftrightarrow w(A_\circ)$

Figure 2.4: Euclidean plane E and $\mathsf{Tits}(W, S)$ for (W, S) of type A_2

Theorem 2.9.2. *If (W, S) is an infinite Coxeter system, then (W, S) is tame if and only if $\mathsf{Cox}(W, S)$ is of type $\widetilde{A}_1$, $\widetilde{A}_n (n \geq 2)$, $\widetilde{B}_2$, $\widetilde{B}_n (n \geq 3)$, $C_n (n \geq 3)$, $\widetilde{D}_n (n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ or $\widetilde{G}_2$, as given in Figure 2.5, where the index in the name of the types is exactly one less than the number of vertices of the corresponding Coxeter graph.*

Proof. See [5], Chapter VI, §4.3, Theorem 4, Chapter VI, §2.5, Proposition 8 and Chapter V, §4.9, Proposition 10. □

Remark 2.9.2. In this case (W, S) is an infinite Coxeter system that is finitely generated. The symmetric bilinear form B_M is positive and degenerate. A technical result, but not difficult result in linear algebra (see, for example [5], Chapter V, §3.5 Lemma 4, or [21], Chapter 2, Proposition 2.6) tells us that the radical of the bilinear form B_M coincides with the null space of the matrix indexed by S whose entries are equal to $B_M(e_s, e_{s'})$ for $s, s' \in S$, and moreover, this null space is one-dimensional and is spanned by a vector $v_0 \in E$ of the form

$$v_0 = \sum_{s \in S} \lambda_s e_s \quad \text{with } 0 < \lambda_s \in \mathbb{R} \text{ for every } s \in S. \tag{2.9.1}$$

Such vector spans the radical $E^\perp$ of B_M, i.e.

$$E^\perp = \{x \in E \mid B_M(x, y) = 0 \text{ for all } y \in E\}.$$

The quotient space $E/E^\perp$ becomes a euclidean plane of dimension $\operatorname{card}(S) - 1$ relative to the positive definite bilinear form induced from B_M. Since $E^\perp$ is in fact the intersection of the hyperplanes Z_s where s runs through S and is therefore fixed pointwise by W, it follows that W also stabilizes the dual hyperplane

$$E^{\perp\vee} = \left\{ f^\vee \in E^\vee \mid \langle v_0, f^\vee \rangle = 0 \right\}$$

of $E^\perp$, and $E^{\perp\vee}$ identifies naturally with the dual space $\left(E/E^\perp\right)^\vee$ of the quotient space $E/E^\perp$, giving it also the structure of a euclidean plane. Notice also that W also stabilizes the translated hyperplane

$$H_0 := \left\{ f^\vee \in E^\vee \mid \langle v_0, f^\vee \rangle = 1 \right\},$$

to which we can naturally transfer the euclidean structure of $E^{\perp\vee}$ so that it becomes an affine euclidean space with translation group $E^{\perp\vee}$. It is such structure that allows us to deduce, as in

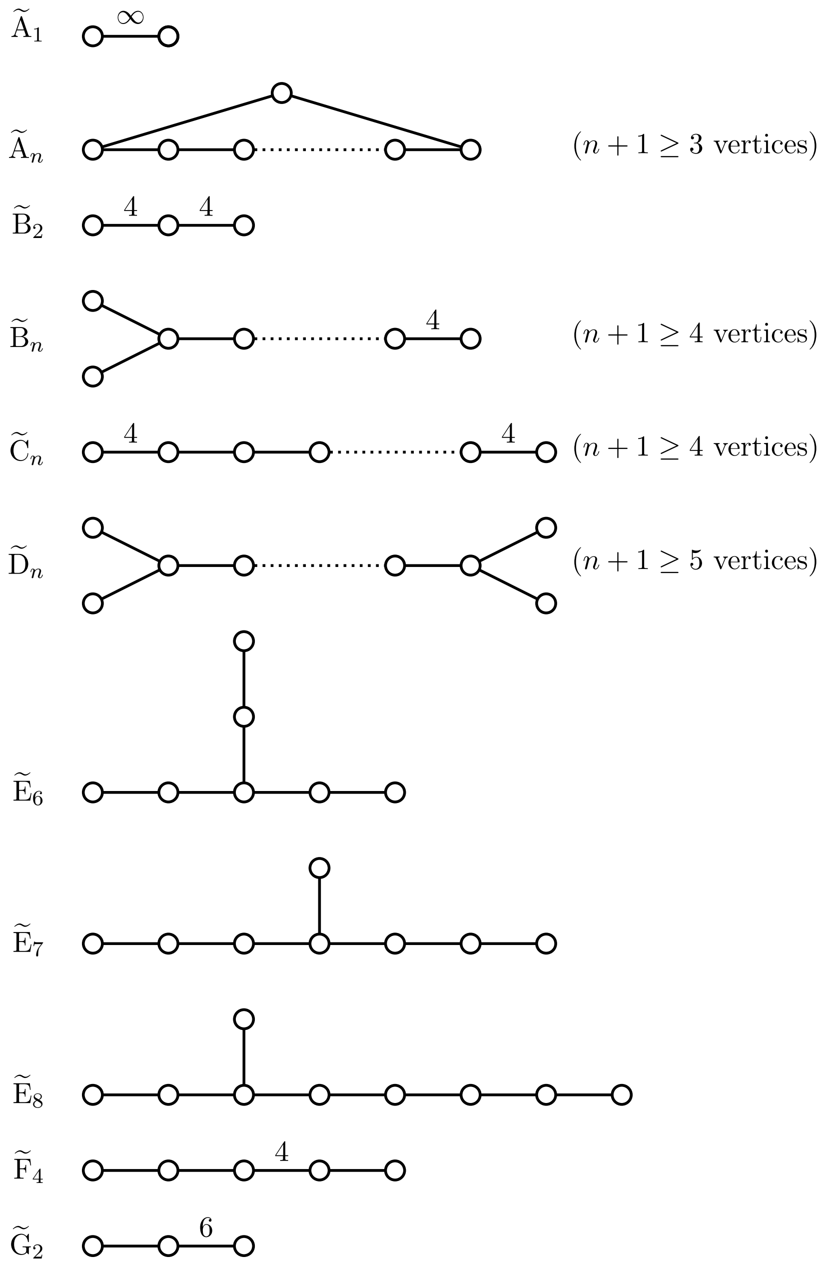


Figure 2.5: Coxeter graphs of irreducible Coxeter systems of affine type

the case of finite Coxeter groups, that the stabilizer $\mathsf{stab}_W(g^\vee)$ of any point $g^\vee \in H_0$ acts on H_0 as an orthogonal group. Also notice that since $\operatorname{card}(S) > 1$ and the coefficients $\lambda_s > 0$ in the expression of v_0 in (2.9.1) in terms of the basis $(e_s)_{s\in S}$ of E, v_0 does not lie in the linear span of any of the e_s ($s \in S$) and thus $\mathrm{E}^{\perp\vee} \neq \mathrm{Z}_s$, forcing $H_0 \cap \mathrm{Z}_s \neq \emptyset$. Such intersection is an affine hyperplane $H_{0,s}$ in H_0 that is fixed pointwise by s, and since s acts as a reflection, it acts, in particular, as an orthogonal reflection relative to $H_{0,s}$. This gives a realization of W as a subgroup of $\mathsf{GL}(H_0)$ generated by affine reflections.

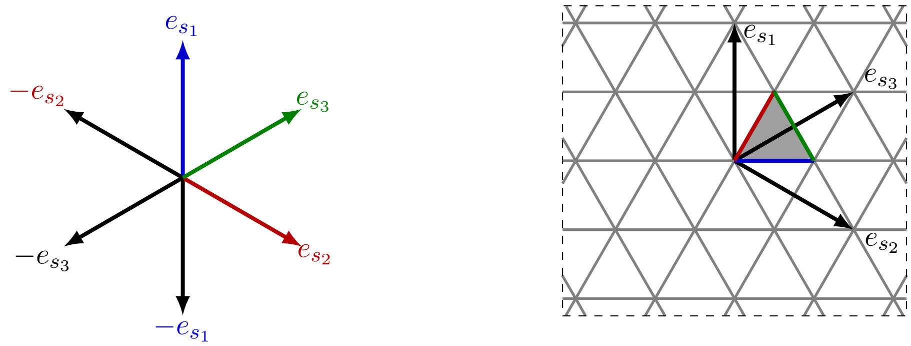
Example 2.9.3. (Type $\widetilde{A}_2$) If (W, S) is of type $\widetilde{A}_2$, then W admits a presentation of the form

$$W = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^3 = 1 \right\rangle .$$

In particular, all the elements of S are conjugate in W as

$$\begin{aligned} s_1 &= s_2 s_1 s_2 s_1 s_2 = (s_2 s_1) s_2 (s_2 s_1)^{-1} \\ s_2 &= s_3 s_2 s_3 s_2 s_3 = (s_3 s_2) s_3 (s_3 s_2)^{-1} \\ s_3 &= s_1 s_3 s_1 s_3 s_1 = (s_1 s_3) s_1 (s_1 s_3)^{-1}. \end{aligned}$$

The basis elements e_{s_1}, e_{s_2} and e_{s_3} are linearly independent (see Figure 2.6a), but the bilinear form is such that the three hyperplanes $\mathrm{Z}_{s_1}, \mathrm{Z}_{s_2}$ and Z_{s_3} intersect in a common line in E spanned by the vector $(1,1,1) \in \mathrm{E}$. Thus the corresponding elements in the dual space lie on a plane, giving a tiling of equilateral triangles (see Figure 2.6b). We call such triangles alcoves to distinguish these from the chambers of the action of W in $\mathrm{E}^{\vee}$. In particular, the fundamental alcove is $H_0 \cap A_{\circ}$ (see Figure 2.6b).

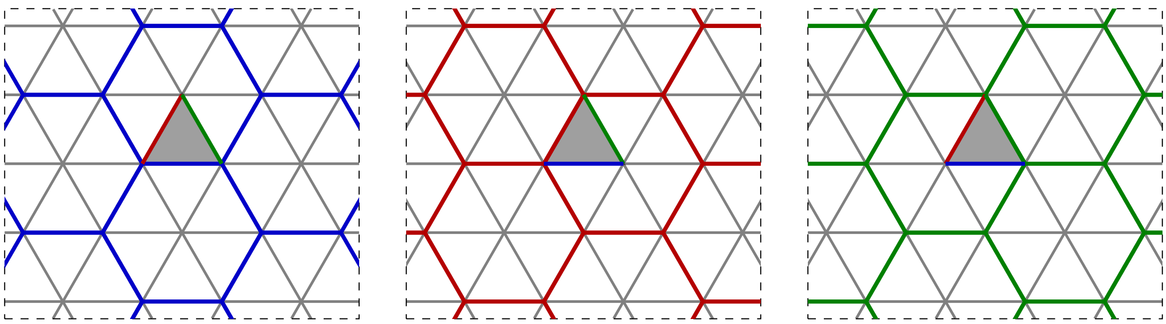


(a) Basis e_1, e_{s_2}, e_{s_3} of E corresponding to positive roots, and their negatives.

(b) Tiling of the plane and fundamental chamber $H_{\circ} \cap A_{\circ}$ shaded in gray

Figure 2.6: Geometric presentation of (W, S) of affine type $\widetilde{\mathrm{A}}_2$

We now look at the orbit of the facets of $H_0 \cap A_{\circ}$ under the orthogonal reflections. The orbits for each particular facet is given in Figure 2.7, and the complete set of orbits in the set of codimension 1 facets of alcoves is given in Figure 2.8. Notice that there is a nice correspondence between the geometry of the triangulated plane and the decompositions of the elements of W as products of the generators in S. We can get from the fundamental alcove to any other alcove by a series of crossings of walls of alcoves. The same is true for any two random alcoves. Since the orbits of the facets of the fundamental alcove has assigned a fixed type, represented by one of the colours red, blue or green in Figure 2.8, to each of the walls of the alcoves, any sequence of wall crosses corresponds to a sequence of elements of S. Therefore to such a sequence of wall crosses starting from the fundamental alcove, say $\mathbf{s} = (x_1, \ldots, x_r)$ where $x_1, \ldots, x_r \in S$, one may associate the



(a) Orbit of the blue facet

(b) Orbit of the red facet

(c) Orbit of the green facet

Figure 2.7: Orbits of facets of the alcove $H_0 \cap A_{\circ}$ under the action of W

element $w = x_1 \cdots x_r \in W$. Such a decomposition is said to be reduced if r is minimal, in which case $r = \ell(w)$. In fact, the minimal sequences of wall crosses from the fundamental alcove to any other alcove corresponds exactly to reduced decompositions of $w \in W$. Therefore, for any $w \in W$ with reduced expression $w = x_r \cdots x_1$, the alcove $w\tilde{A}_\circ$, where $\tilde{A}_\circ := H_0 \cap A_\circ$ is the alcove of the unique sequence of wall crosses from $\tilde{A}_\circ$ and of type $(x_1, \ldots, x_r)$. Various example for elements of smaller length in (W, S) are given in Figure 2.8, where we have indicated on each alcove the element $w = x_r \ldots x_1$ corresponding to the sequence of wall crosses $\mathbf{s} = (x_1, \ldots, x_r)$.

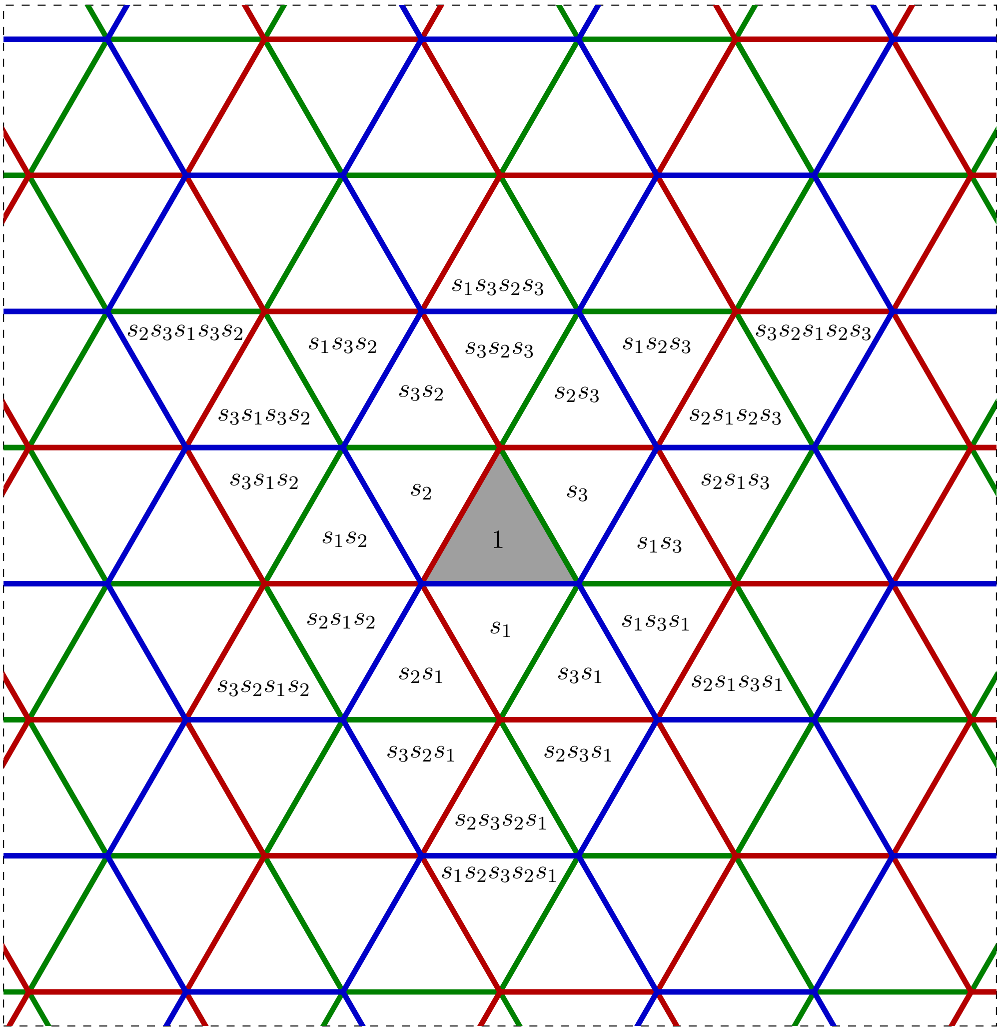


Figure 2.8: Orbits of the facets of $\tilde{A}_\circ = H_0 \cap A_\circ$, where each orbit corresponds to one of the colours red blue or green, and some elements of $w = x_r \ldots x_1$ corresponding to sequences of wall crosses $(x_1, \ldots, x_r)$.

Remark 2.9.3. Notice that we have found all three possible kinds of tame irreducible Coxeter groups that are finitely-generated:

1. Finite and integral, called *Weyl groups*.
2. Finite and non-integral.
3. Infinite and automatically integral, called *affine Weyl groups*.

Remark 2.9.4. We are still left to consider the case when $\mathrm{B_M}$ is non-degenerate but not positive definite, allowing us to identify E with its dual $\mathrm{E}^\vee$.

Definition 2.9.3. The Coxeter system (W, S) is called *hyperbolic* if the symmetric bilinear form $\mathrm{B_M}$ has signature $(n-1, n)$ and $\mathrm{B_M}(x, x) < 0$ for all $x \in C_\circ$, where $C_\circ$ is the fundamental chamber

$$C_\circ = \{x \in \mathrm{E} \mid \mathrm{B_M}(x, e_s) > 0 \text{ for all } s \in S\}\,.$$

Remark 2.9.5. Note that $\mathrm{B_M}(x,x) \leq 0$ for all $x \in \mathring{C}_0$ and recall that the closure $\mathring{C}_\circ$ $(w \in W)$ is a fundamental domain for the action of W on the union of all chambers $w(C_\circ)$. The following allows us to identify hyperbolic Coxeter systems via the bilinear form $\mathrm{B_M}$ and certain subgraphs of $\mathsf{Cox}(W,S)$.

Proposition 2.9.1. *The irreducible Coxeter system (W,S) is hyperbolic if and only if the following conditions hold:*

(a) The bilinear form $\mathrm{B_M}$ is non-degenerate but not positive definite.

(b) For each $s \in S$, the subgraph of $\mathsf{Cox}(W,S)$ obtained by removing s from $\mathsf{Cox}(W,S)$ is of positive type, i.e., tame.

Proof. See [21], Chapter 6, Proposition 6.8. □

2.10 The Bruhat order

Having focused on algebraic and geometric aspects of Coxeter systems for the last few sections, we now focus on a certain partial order structure on (W,S) with deep combinatorial and geometric properties. In this section we mainly focus on the combinatorial properties and in certain results that are crucial for the coming sections. The aim is to partially order W in a way compatible with the length function $l : W \to \mathbb{N}$. For this, we use the Bruhat ordering, which define in raw terms at the start of the discussion and work towards a characterization in terms of reduced expressions, which then allows us to prove several properties of the ordering and answer some natural questions about it. We finish by considering the case of dihedral groups. The main references for this section are [4] and [21].

Definition 2.10.1. Let T be the set of reflections of W defined in (2.3.1). For any w, v in W, write $v \to w$ if $w = vt$ for some t in T with $\ell(w) > \ell(v)$. Then define $v < w$ if there is a sequence $v = w_0 \to w_1 \to \cdots \to w_q = w$. If $\ell(v) < \ell(w)$ and $v^{-1}w =: t$ belongs to T, we will write $v \xrightarrow{t} w$.

Proposition 2.10.1. *The relation "$\leqslant$" resulting from Definition 2.10.1 is a partial ordering of W with the identity element 1 of W as the unique minimal element. The partial ordering "$\leqslant$" of W is called the* ***Bruhat ordering****.*

Proof. The relation "$\leqslant$" is reflexive since for any w in W, $w = w$ and thus $w \leq w$.

Now, suppose that w and w' are two elements of W such that $w' \leqslant w$ and $w \leqslant w'$. By definition, there exist two sequences

$$w' = w_0 \to w_1 \to \cdots \to w_q = w \quad \text{and} \quad w = w'_0 \to w'_1 \to \cdots \to w'_r = w'.$$

In particular, we have that $w_{i+1} = w_i t_{i+1}$ with t_{i+1} in T and $\ell(w_{i+1}) > \ell(w_i)$ for each integer i with $0 \leq i \leq q-1$, and similarly, $w'_{j+1} = w'_j t'_{j+1}$ with t'_{j+1} in T and $\ell(w'_{i+1}) > \ell(w'_i)$ for each integer j with $0 \leq i \leq r-1$. But this implies that $\ell(w') \leq \ell(w)$ and $\ell(w) \leq \ell(w')$. We hence conclude that $\ell(w) = \ell(w')$ and thus $w = w'$, proving that "$\leqslant$" is anti-symmetric.

Next, let w, w' and w'' be elements of W such that $w'' \leqslant w'$ and $w' \leqslant w$. By definition, there exists two sequences

$$w'' = w'_0 \to w'_1 \to \cdots \to w'_q = w' \quad \text{and} \quad w' = w_0 \to w_1 \to \cdots \to w_r = w.$$

Since $w'_q = w' = w_0$ and $w_1 = w_0 t_1$ for some t_1 in T, we have that $w_1 = w_0 t_1 = w'_q t_1$. Moreover, we have that $\ell(w_1) > \ell(w') = \ell(w'_q)$. This shows that

$$w'' = w'_0 \to w'_1 \to \cdots \to w'_q \to w_1 \to \cdots \to w_r = w,$$

and thus $w'' \leqslant w$, proving that the relation " $\leqslant$ " is transitive.

Finally, it remains to show that 1 in W is the unique minimal element. First, 1 satisfies $1 \leqslant w$ for all w in W since any reduced decomposition $(s_1, \ldots, s_r)$ of w induces the sequence

$$1 \to s_1 \to s_1 s_2 \to s_1 s_2 s_3 \to \cdots \to s_1 s_2 s_3 \cdots s_r = w.$$

By the anti-symmetry of " $\leqslant$ ", it follows that 1 is the unique minimal element. □

Remark 2.10.1. Even though the definition has a one-sided appearance since we have defined the arrow relation " $\to$ " by multiplying by t on the right, it can also be replaced by a left-sided version: say $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Setting $t' := w'tw'^{-1}$, we get $w = w't = t'w'$. Using this construction, we prove the following:

Proposition 2.10.2. (Exercise 5.9 in [21]) *Let $x, y \in W$. Then $x < y$ if and only if $x^{-1} < y^{-1}$.*

Proof. First note that if w and v are two elements of W such that $v \to w$, then, by definition, we have that $w = vt$ for some t in T with $\ell(w) > \ell(v)$. Since $t' := vtv^{-1}$ belongs to T and is thus of order 2, and $w = t'v$, we have that $w^{-1} = v^{-1}t'$ with $\ell\left(w^{-1}\right) = \ell(w) > \ell(v) = \ell\left(v^{-1}\right)$. Hence, by definition of the relation " $\to$ ", we have that $v^{-1} \to w^{-1}$. It thus follows that $x < y$ if and only if $x^{-1} < y^{-1}$.

$$x^{-1} = w_0^{-1} \to w_1^{-1} \to \cdots \to w_q^{-1} = q,$$

which shows that $x^{-1} < y^{-1}$. □

Remark 2.10.2. We now prove some results concerning how various elements of W are related in the Bruhat ordering, that will become useful later.

Lemma 2.10.1. *Let $s \in S$ and let $w, w' \in W$ be elements such that $w' \leqslant w$. Then either $w's \leqslant w$ or else $w's \leqslant ws$ (or both).*

Proof. Note that the claim clearly holds if $w' = w$. So assume this is not the case. Since $w' \leq w$, there exists, by definition of " $\leqslant$ ", reflections $t_1, \ldots, t_q \in T$ such that

$$w' = w_0 \xrightarrow{t_1} w_1 \xrightarrow{t_2} w_2 \xrightarrow{t_3} \cdots \xrightarrow{t_{q-1}} w_{q-1} \xrightarrow{t_q} w_q = w$$

with

$$\ell(w') = \ell(w_0) < \ell(w_1) < \ell(w_2) < \cdots < \ell(w_{q-1}) < \ell(w_q) = \ell(w).$$

In particular, we have that

$$w' = w_0 \leqslant w_1 \leqslant w_q = w \quad \text{and} \quad \ell(w') = \ell(w_0) < \ell(w_1) < \ell(w_q) = \ell(w).$$

So assume that the lemma holds for the pair $w' = w_0 \leqslant w_1$, that is either

$$w_0 s \leqslant w_1, \quad \text{or} \quad w_0 s \leqslant w_1 s \quad \text{(or both).} \tag{2.10.1}$$

If the second relation in (2.10.1) holds, then

$$w_0 s \leqslant w_1 s \leq w_q = w \quad \text{or} \quad w_0 s \leqslant w_1 s \leq w_q s = ws,$$

so the claim in the lemma holds. If the the first relation in (2.10.1) holds, then

$$w_0 s \leqslant w_1 \leq w_q = w,$$

so the claim in the lemma holds too. We are thus reduced to proving that the the lemma holds in the case when $w' \xrightarrow{t} w$ and $\ell(w') < \ell(w)$. Now, if $t = s$, then $w's = w$ and there us nothing to prove. So suppose that $t \neq s$. Note that we have two possibilities: either $\ell(w's) = \ell(w') - 1$, or $\ell(w's) = \ell(w') + 1$.

If $\ell(w's) = \ell(w') - 1$, then $w's \to w' \to w$ forces $w's \leqslant w$. If on the contrary $\ell(w's) = \ell(w') + 1$, then using the technique in Remark 2.10.1, we obtain that $w's \cdot t' = ws$, where $t' := sts \in T$, so it suffices to show that $\ell(w's) < \ell(ws)$. So suppose, for a contradiction, that this is not the case. Then, for any reduced expression $w' = s_1 \cdots s_r$ of w', the expression $w's = s_1 \cdots s_r s$ is reduced for $w's$. Then $ws = w's \cdot t'$ is obtained from $w's$ by eliminating one factor in this reduced expression. But note that such factor cannot be s, as $s \neq t$. Thus

$$ws = s_1 \cdots s_{i-1} s_{i+1} \cdots s_r s \quad \text{for some } i \in \mathbb{N} \text{ with } 1 \leq i \leq r,$$

which gives $\ell(w) < \ell(w')$, which contradicts the hypothesis $\ell(w') < \ell(w)$. The proof is now complete. □

Remark 2.10.3. Recall from *(a)* in Theorem 2.5.1 that if $X \subseteq S$, then (W_X, X) is a Coxeter system, and so it has a Bruhat ordering of its own. A natural question to ask is thus whether the Bruhat ordering of (W_X, X) agrees or not with the restriction of the Bruhat ordering of W to W_X. In order to answer this question, we describe a useful characterization of the Bruhat ordering of a Coxeter system (W, S) in terms of reduced expressions.

Definition 2.10.2. Let w be an element of W. A *subexpression* of a given reduced expression $w = s_1 \cdots s_r$ for w is a product of the form $s_{i_1} \cdots s_{i_q}$ with $1 \leq i_1 < i_2 < \cdots < i_q \leq r$.

Remark 2.10.4. Note that the product $s_{i_1} \cdots s_{i_q}$ in the Definition 2.10.2 is not necessarily reduced and may be empty.

Theorem 2.10.1. *Let $w = s_1 \cdots s_r$ be a fixed, but arbitrary, reduced expression of $w \in W$. Then $w' \leqslant w$ if and only if $w' \in W$ can be obtained as a subexpression of this reduced expression.*

Proof. First suppose that $w' \leqslant w$. If $w' = w$, then $w' = s_1 \cdots s_r$, so w' can be trivially obtained as a subexpression of the given reduced expression of w. If $w' < w$, lets start by considering the case $w' \to w$, so that $\ell(w') < \ell(w)$ and there exists $t \in T$ such that $w = w't$. Let $\eta(w;t)$ denote the equal value $(-1)^{n(\mathbf{s}',t)}$ for all sequences $\mathbf{s}' = (s'_1, \ldots, s'_r)$ such that $w = s'_r \cdots s'_1$, as introduced in Section 2.3. We claim that $\eta(w^{-1};t) = -1$. Indeed, if $\eta(w^{-1};t) = 1$, then, using equation (2.3.5) and the fact that $U_w = U_{\mathbf{s}'}$ for every sequence $\mathbf{s} = (s_1, \ldots, s_r)$ such that $w = s'_r \cdots s'_1$, we obtain, for ε in $\{1, -1\}$,

$$U_{wt}(\varepsilon, t) = U_w U_t(\varepsilon, t) = U_w(-\varepsilon, t_q) = (-\varepsilon\eta(w^{-1};t), wtw^{-1}) = (-\varepsilon, wtw^{-1}),$$

which implies that $\eta\left((wt)^{-1};t\right) = -1$. Then, if $\mathbf{s}' = (s'_1, \ldots, s'_q)$ is a sequence of elements such that $wt = s'_1 \cdots s'_q$ is a reduced expression for wt, there exists, by Lemma 2.3.3, an integer $i \in \mathbb{Z}$ such that $1 \leq i \leq q$ and $t = s'_1 \cdots s'_{i-1} s'_i s'_{i-1} \cdots s'_1$. Hence

$$\ell(w) = \ell(w^{-1}) = \ell\left(t(w't)^{-1}\right) = \ell(s'_1 \cdots s'_{i-1} s'_{i+1} \cdots s'_q) < q) < \ell(wt),$$

which contradicts the fact that $\ell(wt) = \ell(w') < \ell(w)$. This establishes the desired claim. Now, since $\eta(w^{-1};t) = -1$ and $w = s_1 \cdots s_r$ is a reduced expression of w, we know, by Lemma 2.3.3, that there exists an integer k such that $1 \leq k \leq r$ such that $t = s_r \cdots s_{k+1} s_k s_{k+1} \cdots s_r$, and thus

$$w' = wt = s_1 \cdots s_r s_r \cdots s_{k+1} s_k s_{k+1} \cdots s_r = s_1 \cdots s_{k-1} s_{k+1} \cdots s_r.$$

Iterating this process we see that if w'' is some other element of W with $w'' \xrightarrow{t'} w'$, by the same argument as above, we obtain

$$w'' = w't' = \begin{cases} s_1 \cdots s_{k-1} s_{k+1} \cdots s_{p-1} s_{p+1} \cdots s_r & \text{if } 1 \leq k < p \leq r \\ s_1 \cdots s_{p-1} s_{p+1} \cdots s_{k-1} s_{k+1} \cdots s_r & \text{if } 1 \leq p < k \leq r. \end{cases}$$

We therefore see that if $w' < w$ with

$$w' = w_0 \xrightarrow{t_1} w_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{q-1}} w_{q-1} \xrightarrow{t_q} w_q = w,$$

iterating the above process, gives an expression for w' that is a subexpression of $s_1 \cdots s_r$ with q deleted letters.

Conversely, suppose that w' is obtained as a subexpression $s_{i_1} \cdots s_{i_q}$ with $1 \leq i_1 < \cdots < i_q \leq r$ of the given reduced expression $s_1 \cdots s_r$ of w. To show that we must hence have $w' \leqslant w$, we use induction on the length $(w) = r$ of w, the case $r = 0$ being trivial. If $i_q < r$, then $s_{i_1} \cdots s_{i_q}$ is a subexpression of the reduced expression $s_1 \cdots s_{r-1}$, whose length is length than r. By the induction hypothesis,

$$s_{i_1} \cdots s_{i_q} \leqslant s_1 \cdots s_{r-1} = ws_r < w.$$

If $i_q = r$, then $s_{i_1} \cdots s_{i_{q-1}}$ is a subexpression of the reduced expression $s_1 \cdots s_{r-1}$, whose length is length than r. By the induction hypothesis,

$$s_{i_1} \cdots s_{i_{q-1}} \leqslant s_1 \cdots s_{r-1},$$

to which we can apply Lemma 2.10.1 to obtain either

$$s_{i_1} \cdots s_{i_{q-1}} s_{i_q} \leqslant s_1 \cdots s_{r-1} < w, \quad \text{or else} \quad s_{i_1} \cdots s_{i_{q-1}} s_{i_q} \leqslant s_1 \cdots s_{r-1} s_r = w.$$

In both cases we have $w' \leqslant w$, completing the proof of the theorem. □

Remark 2.10.5. Notice that the characterization in Theorem 2.10.1 in terms of subexpressions of the Bruhat ordering of a Coxeter system (W, S) answers the question about the Bruhat ordering of (W_X, X) with $X \subseteq S$ posed in Remark 2.10.3.

Corollary 2.10.1. *If $X \subseteq S$, the Bruhat ordering of W agrees on W_X with the Bruhat ordering of the Coxeter system (W_X, X).*

Lemma 2.10.2. *Let $s \in S$, $w \in W$ be such that $sw < w$. Suppose $x \in W$ with $x < w$.*

(a) If $sx < x$, then $sx < sw$.

(b) If $x < sx$, then $sx \leqslant w$ and $x \leqslant sw$.

Thus, in either case, $sx \leqslant w$.

Proof. First of all, by the Exchange condition, $sw < w$ implies that there exists some reduced expression $w = s_1 \cdots s_r$ of w with $s_1 = s$. Now, since $x < w$, we know, by Theorem 2.10.1, that x is a subexpression of w, so either x is a subexpression of the reduced expression $sw = s_2 \cdots s_r$, or there is some reduced expression of x that is also a subexpression of $w = s_1 \cdots s_r$ starting with s, in which case sx occurs as a subexpression of the reduced expression $sw = s_2 \cdots s_r$, and thus $sx \leq sw$. In either case, *(a)* follows. Similarly, in either case, *(a)* follows too. *(a)* □

Remark 2.10.6. Now, notice that when $v \to w$, the precise length difference between w and v is not given, and it is not clear at first, what this should be. Since any $t \in T$ is of the form $t = usu^{-1}$ for some $u \in W$ and some $s \in S$, we see that the length of t must be odd. But without further study, predicting what the length difference between adjacent elements in the Bruhat ordering is seems far from reach. In order to answer this question, we need to prove the following.

Lemma 2.10.3. *Let $w', w \in W$ such that $w' < w$ and $\ell(w) = \ell(w') + 1$. If there exists $s \in S$ such that $w' < w's$ and $ws' \neq w$, then both $w < ws$ and $w's < ws$.*

Proof. If $w', w \in W$ are as in the statement of the lemma and there exists $s \in S$ such that $w' < w's$ and $ws' \neq w$, then, by Lemma 2.10.2, we either have $w's \leqslant w$ or $w's \leqslant ws$ (or both). But notice that the first relation is not possible since by assumption we have $\ell(w) = \ell(w') + 1$ and $\ell(w') = \ell(w's) - 1$, and thus $\ell(w) = \ell(w's)$. Since $w \neq w'$, we must have that $w's < ws$, and thus $\ell(w) = \ell(w's) < \ell(ws)$, forcing $w < ws$. □

Proposition 2.10.3. *Let $w', w \in W$ such that $w' < w$. Then there exists $w_0, w_1, \ldots, w_n \in W$ such that $w' = w_0 < w_1 < \cdots < w_n = w$ and $\ell(w_i) = \ell(w_{i-1}) + 1$ for each integer i with $1 \leq n$.*

Proof. The proof is done by induction on $\ell(w)+\ell(w')$, and makes use of the Exchange condition as well as the characterization in Theorem 2.10.1 of the Bruhat ordering in term of subexpressions. As usual, two cases must be considered: the case when $w' < w's$ and the case when $w's < w'$. For a complete proof see [21], Chapter 5, Proposition 5.11. □

Remark 2.10.7. Note that Proposition 2.10.3 tells us that any two adjacent elements in the Bruhat ordering must differ in length by exactly 1.

Example 2.10.1. (Dihedral $\mathcal{D}_m$) Let W be a dihedral group, finite or infinite, with $S = \{s, s'\}$ where $s, s' \in W$, and let $x, w \in W$ be arbitrary elements. We show that $x < w$ if and only if $\ell(x) < \ell(w)$. First, if $x < w$, then by definition of " $\leqslant$ ", the inequality $\ell(x) < \ell(w)$ follows. Conversely, suppose that $x, w \in W$ are such that $\ell(w) - \ell(x) = n \geq 1$. We show, by induction on $\ell(w) - \ell(x)$ that $x \leqslant w$. First recall that every element of W is either of the form

$$\mathsf{prod}(r; s, s') = \cdots s'ss' \quad \text{or} \quad \mathsf{prod}(r; s', s) = \cdots ss's,$$

where the products consist of r terms, where r for any integer $r \leq m_{ss'} \in \mathbb{Z} \cup \{\infty\}$, so we may assume, without loss of generality that $w = \mathsf{prod}(r; s, s')$. Now, if $\ell(w) - \ell(x) = 1$, we have two possibilities: either $x = \mathsf{prod}(r-1; s', s)$ or $x = \mathsf{prod}(r-1; s, s')$. In the first case we have

$$xs' = \mathsf{prod}(r-1; s', s) \cdot s' = \mathsf{prod}(r; s, s') = w,$$

which shows, since $S \subseteq T$, that $x \xrightarrow{s'} w$, and thus $x < w$. In the second case, we must distinguish between the case when r is even and the case when r is odd. When r is even, we have

$$w^{-1} = \mathsf{prod}(r; s, s')^{-1} = \mathsf{prod}(r; s', s) \quad \text{and} \quad x^{-1} = \mathsf{prod}(r-1; s, s')^{-1} = \mathsf{prod}(r-1; s, s'),$$

so we see that

$$x^{-1}s = \mathsf{prod}(r-1; s, s') \cdot s = \mathsf{prod}(r; s', s) = w^{-1},$$

which shows that $x^{-1} \xrightarrow{s} w^{-1}$, and thus $x^{-1} < w^{-1}$. By Proposition 2.10.2, it follows that $x < w$. On the other hand, when r is odd, we have

$$w^{-1} = \mathsf{prod}(r; s, s')^{-1} = \mathsf{prod}(r; s, s') \quad \text{and} \quad x^{-1} = \mathsf{prod}(r-1; s, s')^{-1} = \mathsf{prod}(r-1; s', s),$$

so we see that

$$x^{-1}s' = \mathsf{prod}(r-1; s, s') \cdot s' = \mathsf{prod}(r; s, s') = w^{-1},$$

which shows that $x^{-1} \xrightarrow{s'} w^{-1}$, and thus $x^{-1} < w^{-1}$. By Proposition 2.10.2 again, it follows that $x < w$. This completes the proof.

3 Iwahori-Hecke Algebras

Throughout this section, let (W, S) be a Coxeter system.

3.1 Generic algebras

We start by constructing general associative algebras over a commutative ring R with unity. These will have a free R-basis parametrized by the elements of S with a multiplication law which reflects in a particular way the multiplication in W, and will depend on some parameters $a_s, b_s \in R$ where s runs through S, subject only to the requirement that $a_s = a_{s'}$ and $b_s = b_{s'}$ whenever s and s' are conjugate in W. We proceed following the steps in Exercise 23 in Chapter IV of [5]. The starting point is a free R-module $\mathcal{E}$ on the set W, and we let $(\varepsilon_w)_{w\in W}$ be its canonical basis elements.

Theorem 3.1.1. *Suppose we are given, for all $s \in S$, two elements $a_s, b_s \in R$ such that $a_s = a_{s'}$ and $b_s = b_{s'}$ whenever s and s' are conjugate in W. There exists a unique structure of associative R-algebra on the free R-module $\mathcal{E}$, with ε_1 acting as the identity, such that, for all $s \in S$ and all $w \in W$*

$$\varepsilon_s\varepsilon_w = \begin{cases} \varepsilon_{sw} & \text{if } \ell(sw) > \ell(w) \\ a_s\varepsilon_w + b_s\varepsilon_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases} \tag{3.1.1}$$

Definition 3.1.1. The algebra described in Theorem 3.1.1 is called a *generic algebra*, and is denoted by $\mathcal{E}_R(a_s, b_s)$.

Remark 3.1.1. Before proving the theorem, we give some consequences of it. First, one may ask about the existence of a 'right-hand' version of (3.1.1). The following gives this.

Corollary 3.1.1. *The following conditions holds for all s in s and w in W:*

$$\varepsilon_w\varepsilon_s = \begin{cases} \varepsilon_{ws} & \text{if } \ell(ws) > \ell(w) \\ a_s\varepsilon_w + b_s\varepsilon_{ws} & \text{if } \ell(ws) < \ell(w). \end{cases} \tag{3.1.2}$$

Proof. Let $s \in S$ and $w \in W$ be arbitrary. We proceed by induction on $\ell(w)$. If $\ell(ws) > \ell(w)$, we may find some $s' \in S$ such that $\ell(s'w) < \ell(w)$. Then we have

$$\ell(s'w) + 1 = \ell(w) = \ell(ws) - 1,$$

which gives $\ell(s'ws) > \ell(s'w)$, and thus, by induction

$$\varepsilon_{s'w}\varepsilon_s = \varepsilon_{s'ws}. \tag{3.1.3}$$

But by the first relation in (3.1.1),

$$\varepsilon_{s'}\varepsilon_{s'w} = \varepsilon_{s's'w} = \varepsilon_w,$$

and thus, multiplying both sides by ε_s and using (3.1.3) yields

$$\varepsilon_w\varepsilon_s = \varepsilon_{s'}\varepsilon_{s'w}\varepsilon_s = \varepsilon_{s'}\varepsilon_{s'ws} = \varepsilon_{ws}.$$

If on the contrary $\ell(ws) < \ell(w)$, the first relation in (3.1.2) just proved gives

$$\varepsilon_{ws}\varepsilon_s = \varepsilon_{wss} = \varepsilon_w.$$

Multiplying both sides and using the second relation in (3.1.1) to compute ε_s^2, we obtain

$$\varepsilon_w\varepsilon_s = \varepsilon_{ws}\varepsilon_s\varepsilon_s = \varepsilon_{ws}(a_s\varepsilon_s + b_s\varepsilon_1) = a_s\varepsilon_{ws}\varepsilon_s + b_s\varepsilon_{ws} = a_s\varepsilon_w + b_s\varepsilon_{ws}.$$

□

Remark 3.1.2. It is also handy to express the relations in (4.1.3) in a way that will become handy later.

Corollary 3.1.2. *The following set of relations is equivalent to* (3.1.1)*:*

$$\varepsilon_s \varepsilon_w = \varepsilon_{sw} \quad \text{if } \ell(sw) > \ell(w) \tag{3.1.4}$$
$$\varepsilon_s^2 = a_s \varepsilon_s + b_s \varepsilon_1 \tag{3.1.5}$$

Proof. Suppose that both relations in (3.1.1) hold. The first condition is the same as (3.1.4). Moreover, using the second relation in (3.1.1), we obtain

$$\varepsilon_s^2 = a_s \varepsilon_s + b_s \varepsilon_{ss} = a_s \varepsilon_s + b_s \varepsilon_1.$$

Conversely, suppose that the R-algebra structure on $\mathcal{E}$ satisfies the relations (3.1.4) and (3.1.5). Since (3.1.4) is exactly the same as the first relation in (3.1.1), we only need to check the second relation in (3.1.1), which applies to the case when $\ell(sw) < \ell(w)$. Note that we have

$$\ell(ssw) = \ell(w) > \ell(sw),$$

so by (3.1.4), we obtain

$$\varepsilon_s \varepsilon_{sw} = \varepsilon_{ssw} = \varepsilon_w.$$

Then, by (3.1.5), we get

$$\varepsilon_s \varepsilon_w = \varepsilon_s^2 \varepsilon_{sw} = (a_s \varepsilon_s + b_s \varepsilon_1)\varepsilon_{sw} = a_s \varepsilon_{ssw} + b_s \varepsilon_{sw} = a_s \varepsilon_w + b_s \varepsilon_{sw},$$

as required. □

Remark 3.1.3. In order to prove the existence part of Theorem 3.1.1, we need to somehow introduce into the R-module $\mathcal{E}$ the additional structure required. It is not clear how to this directly, but if $\mathcal{E}$ does have an algebra structure, the left multiplication operators corresponding to elements of $\mathcal{E}$ will generate an isomorphic copy of such algebra inside the algebra $\mathsf{End}\,\mathcal{E}$ of all R-module endomorphisms of $\mathcal{E}$ of which we can exploit its ring structure. Therefore, our aim is to locate the correct subalgebra of $\mathsf{End}\,\mathcal{E}$ by choosing suitable left multiplication operators λ_s, corresponding to the elements ε_s with $s \in S$, that behave according to the relations in (3.1.1). Similarly, given the symmetric relations in (3.1.2), we should also be able to give right multiplication operators ρ_t, corresponding to the elements ε_s with $s \in S$, that behave according to such relations.

Proof of Theorem 3.1.1. First, taking account the discussion in Remark 3.1.3, we define, for each $s \in S$, the left and right endomorphisms λ_s and ρ_s, respectively, by their action of the basis $(\varepsilon_w)_{w \in W}$, which can then be extended by linearity to the whole of $\mathcal{E}$. To agree with the relations in (3.1.1), the left and right multiplication operators λ_s and ρ_s corresponding to the element ε_s for $s \in S$, must be given by

$$\lambda_s(\varepsilon_w) = \begin{cases} \varepsilon_{sw} & \text{if } \ell(sw) > \ell(w) \\ a_s \varepsilon_w + b_s \varepsilon_{sw} & \text{if } \ell(sw) < \ell(w), \end{cases} \tag{3.1.6}$$

and

$$\rho_s(\varepsilon_w) = \begin{cases} \varepsilon_{ws} & \text{if } \ell(ws) > \ell(w) \\ a_s \varepsilon_w + b_s \varepsilon_{ws} & \text{if } \ell(ws) < \ell(w). \end{cases} \tag{3.1.7}$$

For now, assume that the operators λ_s and $\rho_{s'}$ commute for every $s, s' \in S$, and let $\mathcal{L}$ be the subalgebra of $\mathsf{End}\,\mathcal{E}$ generated by $(\lambda_s)_{s \in S}$, which is a ring with $1_{\mathcal{L}}$. Our next aim is transfer the algebra structure of $\mathcal{L}$ to $\mathcal{E}$. For this, define a map

$$\begin{aligned} \Upsilon : \mathcal{L} &\to \mathcal{E} \\ \lambda &\mapsto \lambda(\varepsilon_1), \end{aligned}$$

that is, Υ maps $1_{\mathcal{L}}$ to ε_1 and λ_s to $\lambda_s(\varepsilon_1) = \varepsilon_s$ for each s in S, and the map is well-defined. Moreover, since any element of $\mathcal{L}$ is an endomorphism of the free module $\mathcal{E}$, we have that for any $p_1, p_2 \in R$ and any $\lambda_1, \lambda_2 \in \mathcal{L}$,

$$\Upsilon\left(p_1\lambda_1 + p_2\lambda_2\right) = \left(p_1\lambda_1 + p_2\lambda_2\right)(\varepsilon_1) = p_1\lambda_1(\varepsilon_1) + p_2\lambda_2(\varepsilon_1)$$

which shows that $\Upsilon : \mathcal{L} \to \mathcal{E}$ is an R-module map. Also note that for any free basis element ε_w of $\mathcal{E}$ with w in W, if $w = s_1 \dots, s_r$ is a reduced expression of w, then

$$\ell(s_1 \cdots s_{i-1}) < \ell(s_1 \cdots s_{i-1}s_i) \quad \text{for each integer } i \text{ with } 2 \leq i \leq r,$$

and thus, by the definition of λ_s for s in S in (3.1.6), we have

$$\Upsilon\left(\lambda_{s_1} \cdots \lambda_{s_r}\right) = \lambda_{s_1} \cdots \lambda_{s_r}(\varepsilon_1) = \lambda_{s_1} \cdots \lambda_{s_{r-1}}(\varepsilon_{s_r}) = \cdots = \varepsilon_{s_1 s_2 \cdots s_r} = \varepsilon_w.$$

This shows that ε_w lies in the image $\Upsilon(\mathcal{L})$ of Υ. Since e_w is an arbitrary free basis element of $\mathcal{E}$, it follows that $\Upsilon : \mathcal{L} \to \mathcal{E}$ is a surjective R-module map. Now, let $\lambda \in \mathcal{L}$ be such that λ lies in the kernel $\ker \Upsilon$ of the map $\Upsilon : \mathcal{L} \to \mathcal{E}$. Then $\lambda(\varepsilon_1) = 0$, and we show, by induction on the length $\ell(w)$ of w in W, that $\lambda(\varepsilon_w) = 0$ for all w in W, which then implies that λ is the zero map. If $\ell(w) = 1$, then $w = s$ for some s in S and $\ell(ws) < \ell(w)$. Therefore, by (3.1.7), we have that $\rho_s(\varepsilon_1) = \varepsilon_s$ and so

$$\lambda(\varepsilon_s) = \lambda\left(\rho_s(\varepsilon_1)\right) = \lambda\rho_s(\varepsilon_1). \tag{3.1.8}$$

Since by assumption ρ_s commutes with the generators $\lambda_{s'}$ with s in S of the subalgebra $\mathcal{L}$ of $\mathcal{E}$, it commutes, in particular, with λ, so (3.1.8) becomes

$$\lambda(\varepsilon_s) = \rho_s\lambda(\varepsilon_1) = \rho_s(0) = 0.$$

So assume that $\lambda(\varepsilon_{w'}) = 0$ for all w in W such that $\ell(w) > \ell(w') \geq 1$, and choose $s \in S$ such that $\ell(ws) < \ell(w) = \ell(wss)$. Then, by (3.1.7), we have that $\rho_s(\varepsilon_{ws}) = \varepsilon_w$ and so

$$\lambda(\varepsilon_w) = \lambda\left(\rho_s(\varepsilon_{ws})\right) = \lambda\rho_s(\varepsilon_{ws}). \tag{3.1.9}$$

Again by the commutativity of ρ_s and λ, equation (3.1.8) becomes

$$\lambda(\varepsilon_w) = \rho_s\lambda(\varepsilon_{ws}) = \rho_s(0) = 0,$$

where the second to last equality follows by the induction hypothesis as $\ell(ws) < \ell(w)$, and the induction is complete. We hence have that λ is the zero endomorphism, and the injectivity of $\Upsilon : \mathcal{L} \to \mathcal{E}$ follows.

Now, in order to conclude that $\Upsilon : \mathcal{L} \to \mathcal{E}$ is indeed an R-module isomorphism, we still need to prove our assumption that the operators λ_s and $\rho_{s'}$ commute for every $s, s' \in S$, and in order to do this, we will compare, for any $s, s' \in S$ and any $w \in W$, the actions of $\lambda_s\rho_{s'}$ and $\rho_s\lambda_{s'}$ on ε_w. Since multiplying by s and s' changes the length by 1, note that it is not possible for the length of the four elements w, sw, ws' and sws' to be equal. In particular, we have the following possibilities:

$$\ell(sw) < \ell(w) < \ell(ws'); \tag{3.1.10}$$

$$\ell(sw) = \ell(ws') < \ell(w); \tag{3.1.11}$$

$$\ell(sw) = \ell(ws') > \ell(w); \tag{3.1.12}$$

$$\ell(sw) > \ell(w) > \ell(ws'). \tag{3.1.13}$$

Now, the cases in (3.1.10) and in (3.1.13) both force $\ell(w) = \ell(sws')$. In the case in (3.1.11) we have two possibilities: either forces $\ell(sws') < \ell(sw)$ or $\ell(sws') = \ell(w)$. Similarly, in the the case in (3.1.12), we have two possibilities: either forces $\ell(sws') > \ell(sw)$ or $\ell(sws') = \ell(w)$. We therefore have six different possibilities for the relative lengths of w, sw, ws' and sws'. We consider each of them individually, using the definitions of the operators λ_s and $\rho_{s'}$ in (3.1.6) and (3.1.7), respectively.

(1) $\ell(sw) < \ell(w) = \ell(sws') < \ell(ws')$.
In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(\varepsilon_{ws'}) = a_s\varepsilon_{ws'} + b_s\varepsilon_{sws'} = \rho_{s'}(a_s\varepsilon_w + b_s\varepsilon_{sw}) = \rho_{s'}\lambda_s(\varepsilon_w).$$

(2) $\ell(ws') < \ell(w) = \ell(sws') < \ell(sw)$.
In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(a_{s'}\varepsilon_w + b_{s'}\varepsilon_{ws'}) = a_s\varepsilon_{sws'} + b_s\varepsilon_{sws'} = \rho_{s'}(\varepsilon_{sw}) = \rho_{s'}\lambda_s(\varepsilon_w).$$

(3) $\ell(w) < \ell(ws') = \ell(sw) < \ell(sws')$. In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(\varepsilon_{ws'}) = \varepsilon_{sws'} = \rho_{s'}(\varepsilon_{sw}) = \rho_{s'}\lambda_s(\varepsilon_w).$$

(4) $\ell(w) = \ell(sws') < \ell(sw) = \ell(ws')$. In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(\varepsilon_{ws'}) = a_s\varepsilon_{ws'} + b_s\varepsilon_{sws'}, \tag{3.1.14}$$

and

$$\rho_{s'}\lambda_s(\varepsilon_w) = \rho_{s'}(\varepsilon_{sw}) = a_{s'}\varepsilon_{sw} + b_{s'}\varepsilon_{sws'}. \tag{3.1.15}$$

Note that since $\ell(w) = \ell(sws') < \ell(sw)$, the Exchange condition applied to the pair sw, s' implies that there exists some integer i such that $1 \leq i \leq \ell(sw) = r + 1$ and

$$s_{i+1}\cdots s_r s' = s_i s_{i+1}\cdots s_r \tag{3.1.16}$$

for any reduced expression $sw = ss_1\cdots s_r$ of sw or

$$s_1\cdots s_r s' = ss_1\cdots s_r. \tag{3.1.17}$$

But the case in (3.1.16) is not possible since this would imply

$$wt = s\cdot sw\cdot s' = s\cdot ss_1\cdots s_{i-1}s'_{i+1}\cdot s' = s_1\cdots s_{i-1}s_{i+1}\cdots sr,$$

which contradicts the hypothesis $\ell(w) < \ell(ws')$. Therefore (3.1.17) must hold and we have $sw = ws'$ and, moreover, $a_s = a_{s'}$ and $b_s = b_{s'}$. This together with the equalities (3.1.14) and (3.1.15) gives $\lambda_s\rho_{s'} = \rho_{s'}\lambda_s$.

(5) $\ell(sws') < \ell(ws') = \ell(sw) < \ell(w)$.
In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(a_{s'}\varepsilon_w + b_{s'}\varepsilon_{ws'}) = a_s a_{s'}\varepsilon_w + b_s a_{s'}\varepsilon_{sw} + a_s b_{s'}\varepsilon_{ws'} + b_s b_{s'}\varepsilon_{sws'}$$

and

$$\rho_{s'}\lambda_s(\varepsilon_w) = \lambda_s(a_s\varepsilon_w + b_s\varepsilon_{sw}) = a_{s'}a_s\varepsilon_w + b_{s'}a_s\varepsilon_{ws'} + a_{s'}b_s\varepsilon_{sw} + b_{s'}b_s\varepsilon_{sws'},$$

so we see that indeed $\lambda_s\rho_{s'} = \rho_{s'}\lambda_s$.

(6) $\ell(ws') = \ell(sw) < \ell(sws') = \ell(w)$.
In this case we have

$$\lambda_s\rho_{s'}(\varepsilon_w) = \lambda_s(a_{s'}\varepsilon_w + b_{s'}\varepsilon_{ws'}) = a_s a_{s'}\varepsilon_w + b_s a_{s'}\varepsilon_{sw} + b_{s'}\varepsilon_{sws'} \tag{3.1.18}$$

and

$$\rho_{s'}\lambda_s(\varepsilon_w) = \rho_{s'}(a_s\varepsilon_w + b_s\varepsilon_{sw}) = a_{s'}a_s\varepsilon_w + b_{s'}a_s\varepsilon_{ws'} + b_s\varepsilon_{sws'}. \tag{3.1.19}$$

Since $\ell(sw) < \ell(w) = \ell(ssw)$, the same argument as in Case (4) with sw in place of w gives $w = ssw = sws'$ and, moreover, $a_s = a_{s'}$ and $b_s = b_{s'}$. This together with the equalities (3.1.18) and (3.1.19) gives $\lambda_s\rho_{s'} = \rho_{s'}\lambda_s$.

Since $\lambda_s\rho_{s'} = \rho_{s'}\lambda_s$ in all six possible cases and s, s' are arbitrary elements of S, the proof of the commutativity of λ_s and $\rho_{s'}$ is now complete.

Now, as we have shown that $\Upsilon : \mathcal{L} \to \mathcal{E}$ is an R-module ismorphism, it follows that $(\lambda_w)_{w\in W}$ is a free basis of $\mathcal{L}$, where

$$\lambda_w := \lambda_{s_1} \cdots \lambda_{s_r},$$

for any reduced expression $w = s_1 \cdots s_r$. Note that the endomorphism λ is independent of the choice of reduced expression of w, and λ_1 denotes the endomorphism of $\mathcal{E}$ that is the identity on $\mathcal{E}$. Moreover, $\Upsilon : \mathcal{L} \to \mathcal{E}$ transfers the algebra structure of $\mathcal{L}$ to $\mathcal{E}$, so to complete the existence part of the proof of the theorem, it only remains for us to prove that such structure satisfies the relations in (3.1.1). Given the set of equivalent relations (3.1.4) and (3.1.5), we will actually show that these are satisfied.

So suppose that $s \in S$ and $w \in W$ are such that $\ell(sw) > \ell(w)$, and let $w = s_1 \cdots s_r$ be any reduced expression of w. Given the assumption on the lengths, $ss_1 \cdots s_r$ is a reduced expression of sw. Hence, by definition of λ_{sw} and λ_w, we obtain

$$\lambda_{sw} = \lambda_s\lambda_{s_1} \cdots \lambda_{s_r} = \lambda_s\lambda_w,$$

which verifies the relation in (3.1.4). To verify the remaining relation, we study how each side of (3.1.5) acts on some arbitrary free basis element ε_w of $\mathcal{E}$. If $\ell(sw) > \ell(w)$, we have

$$\lambda_s^2(\varepsilon_w) = \lambda_s(\varepsilon_{sw}) = a_s\varepsilon_{sw} + b_s\varepsilon_w = a_s\lambda_s(\varepsilon_w) + b_s\lambda_1(\varepsilon_w) = (a_s\lambda_s + b_s\lambda_1)(\varepsilon_w).$$

If on the contrary, $\ell(sw) < \ell(w)$, we have

$$\lambda_s^2(\varepsilon_w) = \lambda_s(a_s\varepsilon_w + b_s\varepsilon_{sw}) = a_s\lambda_s(\varepsilon_w) + b_s\varepsilon_w = a_s\lambda_s(\varepsilon_w) + b_s\lambda_1(\varepsilon_w) = (a_s\lambda_s + b_s\lambda_1)(\varepsilon_w),$$

which completes the verification of the relation in (3.1.4), and the existence part of the proof is now complete.

Now, if we iterate the first relation in (3.1.1), we see that $\varepsilon_w = \varepsilon_{s_1} \cdots \varepsilon_{s_r}$ whenever $w = s_1 \cdots s_r$ is a reduced expression of w, which shows that $\{\varepsilon_1\} \cup (\varepsilon_s)_{s\in S}$ generates $\mathcal{E}$ as an R-algebra. As a consequence, iteration of the relations in (3.1.1) yields the complete multiplication table for the free basis elements ε_w of $\mathcal{E}$ and the uniqueness of the algebra structure follows. □

Example 3.1.1. Let R be any commutative ring with unity. Then group algebra $R[W]$ is a generic algebra. It corresponds to the generic algebra $\mathcal{E}_R(0, 1)$, that is, $a_s = 0$ and $b_s = 1$ for all $s \in S$.

Definition 3.1.2. Let $\flat$ be the R-module automorphims of $\mathcal{E}$ defined

$$\begin{aligned} \flat : \mathcal{E} &\to \mathcal{E} \\ \varepsilon_w &\mapsto \varepsilon_{w^{-1}}, \end{aligned} \tag{3.1.20}$$

and extended linearly.

Proposition 3.1.1. (Exercise 1 in 7.3 of [21]) *The R-module automorphism of $\mathcal{E}$ described in* (3.1.20) *is an anti-automorphism of any generic algebra $\mathcal{E}_R(a_s, b_s)$ based on $\mathcal{E}$.*

Proof. We have to show that, for any free basis elements $\varepsilon_w, \varepsilon_v \in \mathcal{E}$ of $\mathcal{E}$, we have

$$\flat(\varepsilon_w\varepsilon_v) = \flat(\varepsilon_v) \cdot \flat(\varepsilon_w). \tag{3.1.21}$$

We show this by induction on $\ell(w) + \ell(v)$.

If $\ell(w) + \ell(v) = 0$, then $w = v = 1$, and (3.1.21) trivially holds. If $\ell(w) + \ell(v) = 1$, then one of w or v must be an element s of S and the other must be 1. If $w = s$ and $v = 1$, then

$$\flat(\varepsilon_s\varepsilon_1) = \flat(\varepsilon_s) = \varepsilon_s = \varepsilon_1\varepsilon_s = \flat(\varepsilon_1) \cdot \flat(\varepsilon_s),$$

and if $w = 1$ and $v = s$, then

$$\flat(\varepsilon_1\varepsilon_s) = \flat(\varepsilon_s) = \varepsilon_s = \varepsilon_s\varepsilon_1 = \flat(\varepsilon_s)\cdot\varepsilon_1,$$

which shows that (3.1.21) holds in this case. If $\ell(w) + \ell(v) = 2$, then either one of w or v is the identity and the other is of length 2 or $w = s$ and $w = s'$ for some elements s and s' of S. In the first case (3.1.21) clearly holds since we are multiplying either on the left or on the right by the identity. So suppose we are on the second case. If $\ell(ss') > \ell(s')$, then by the first relation in (3.1.1), we have that

$$\flat(\varepsilon_s\varepsilon_{s'}) = \flat(\varepsilon_{ss'}) = \varepsilon_{s's} = \varepsilon_{s'}\varepsilon_s = \flat(\varepsilon_{s'})\cdot\flat(\varepsilon_s).$$

If $\ell(ss') < \ell(s')$, then by the second relation in (3.1.1), we have that, we have that

$$\flat(\varepsilon_s\varepsilon_{s'}) = \flat(a_s\varepsilon_{s'} + b_s\varepsilon_{ss'}) = a_s\varepsilon_{s'} + b_s\varepsilon_{s's} = \varepsilon_{s'}\varepsilon_s = \flat(\varepsilon_{s'})\cdot\flat(\varepsilon_s),$$

which shows that (3.1.21) holds in the case $\ell(w) + \ell(v) = 2$.

Now, let $w = s_1\cdots s_r$ and $v = s'_1\cdots s'_q$ be reduced expressions for w and v respectively with $r + q > 2$, and assume that (3.1.21) holds for any $x, y \in W$ such that $\ell(x) + \ell(y) < \ell(w) + \ell(v)$. Again, since the case when one of w or v is the identity and the other is of length strictly greater than two (3.1.21) clearly holds since we are multiplying either on the left or on the right by the identity, we assume that $\ell(w) \geq 1$ and $\ell(v) \geq 1$. By (2.2.1) of Proposition 2.2.1, the length of the product wv is at most $r + q$, and so we may distinguish between two cases: either $\ell(wv) = r + q$ or $\ell(wv) < r + q$.

If $\ell(wv) = r + q$, then $wv = s_1\cdots s_r s'_1\cdots s'_q$ is a reduced expression of wv, so, iterating the first relation in (3.1.1), we see that

$$\varepsilon_{wv} = \varepsilon_{s_1\cdots s_r s'_1\cdots s'_q} = \varepsilon_{s_1\cdots s_r}\varepsilon_{s'_1\cdots s'_q} = \varepsilon_w\varepsilon_v,$$

and thus

$$\flat(\varepsilon_w\varepsilon_v) = \flat(\varepsilon_{wv}) = \varepsilon_{(wv)^{-1}} = \varepsilon_{v^{-1}w^{-1}} = \flat(\varepsilon_v)\cdot\flat(\varepsilon_w).$$

On the other hand, if $\ell(wv) < r + q$, write $w = us_r$ where $u := s_1\cdots s_{r-1}$. Since $\ell(us_r) > \ell(u)$, the second relation in (3.1.2) gives $\varepsilon w = \varepsilon_u\varepsilon_{s_r}$. Now, if $\ell(s_r v) < \ell(v)$, then by the second relation in (3.1.1), we have that

$$\varepsilon_{s_r}\varepsilon_v = a_{s_r}\varepsilon_v + b_{s_r}\varepsilon_{s_r v},$$

and thus

$$\flat(\varepsilon_w\varepsilon_v) = \flat(a_{s_r}\varepsilon_u\varepsilon_v + b_{s_r}\varepsilon_u\varepsilon_{s_r v}) = a_{s_r}\flat(\varepsilon_u\varepsilon_v) + b_{s_r}\mathrm{inv}(\varepsilon_u\varepsilon_{s_r v}). \tag{3.1.22}$$

Then, by the induction hypothesis, since

$$\ell(u) + \ell(v) < \ell(w) + \ell(v) \quad \text{and} \quad \ell(u) + \ell(s_r v) < \ell(w) + \ell(v),$$

the equation in (3.1.22) becomes

$$\flat(\varepsilon_w\varepsilon_v) = a_{s_r}\varepsilon_{v^{-1}}\varepsilon_{u^{-1}} + b_{s_r}\varepsilon_{v^{-1}s_r}\varepsilon_{u^{-1}} = (a_{s_r}\varepsilon_{v^{-1}} + b_{s_r}\varepsilon_{v^{-1}s_r})\varepsilon_{u^{-1}}. \tag{3.1.23}$$

But

$$\ell(v^{-1}s_r) = \ell(s_r v) < \ell(v) = \ell(v^{-1}),$$

so by the second relation in (3.1.2), we have

$$\varepsilon_{v^{-1}}\varepsilon_{s_r} = a_{s_r}\varepsilon_{v^{-1}} + b_{s_r}\varepsilon_{v^{-1}s_r},$$

and thus (3.1.23) becomes

$$\flat(\varepsilon_w\varepsilon_v) = \varepsilon_{v^{-1}}\varepsilon_{s_r}\varepsilon_{u^{-1}} = \varepsilon_{v^{-1}}\varepsilon_{s_r u^{-1}} = \flat(\varepsilon_v)\cdot\flat(\varepsilon_w),$$

as required. On the other hand, if $\ell(s_r v) > \ell(v)$, then, since $\ell(wv) < \ell(w) + \ell(v)$, there exists some integer $j \in \mathbb{Z}$ such that $1 \leq j \leq r-1$

$$\ell(s_{j+1} \cdots s_r v) = r - (j+1) + q \quad \text{and} \quad \ell(s_j s_{j+1} \cdots s_r v) < \ell(s_{j+1} \cdots s_r v),$$

and thus, by the relations in (3.1.1) and (3.1.2), we have

$$\varepsilon_w \varepsilon_v = \varepsilon_{s_1 \cdots s_{j-1}} \varepsilon_{s_j} \varepsilon_{s_{j+1} \cdots s_r} \varepsilon_v = \varepsilon_{s_1 \cdots s_{j-1}} \varepsilon_{s_j} \varepsilon_{s_{j+1} \cdots s_r v} = \varepsilon_{u'} (a_{s_j} \varepsilon_{v'} + b_{s_j} \varepsilon_{s_j v'}),$$

$v' = s_{j+1} \cdots s_r v$ and $u' := s_1 \cdots s_{j-1}$, and thus

$$\flat(\varepsilon_w \varepsilon_v) = a_{s_j} \flat(\varepsilon_{u'} \varepsilon_{v'}) + b_{s_j} \flat(\varepsilon_{u'} \varepsilon_{s_j v'}). \tag{3.1.24}$$

Then, by the induction hypothesis, since

$$\ell(u') + \ell(s_j v') < \ell(u') + \ell(v') = j - 1 + r - (j+1) + q = r + q - 2 < \ell(w) + \ell(v),$$

the equality in (3.1.24) becomes

$$\flat(\varepsilon_w \varepsilon_v) = a_{s_j} \varepsilon_{v'^{-1}} \varepsilon_{u'^{-1}} + b_{s_{s_j}} \varepsilon_{v'^{-1} s_j} \varepsilon_{u'^{-1}} = (a_{s_j} \varepsilon_{v'^{-1}} + b_{s_{s_j}} \varepsilon_{v'^{-1} s_j}) \varepsilon_{u'^{-1}}. \tag{3.1.25}$$

But

$$\ell(v'^{-1} s_j) = \ell(s_j v') < \ell(v') = \ell(v'^{-1}),$$

so by the second relation in (3.1.2), we have

$$a_{s_j} \varepsilon_{v'^{-1}} + b_{s_{s_j}} \varepsilon_{v'^{-1} s_j} = \varepsilon_{v'^{-1}} \varepsilon_{s_j},$$

and thus, (3.1.25) becomes

$$\flat(\varepsilon_w \varepsilon_v) = \varepsilon_{v'^{-1}} \varepsilon_{s_j} \varepsilon_{u'^{-1}} = \varepsilon_{v^{-1} s_r \cdots s_{j+1}} \varepsilon_{s_j} \varepsilon_{s_1 \cdots s_{j-1}} = \varepsilon_{v^{-1}} \varepsilon_{s_r \cdots s_{j+1}} \varepsilon_{s_j} \varepsilon_{s_{j-1} \cdots s_1} = \flat(\varepsilon_v) \cdot \flat(\varepsilon_w),$$

an the proof is now complete. □

Proposition 3.1.2. *The family of generators $(\varepsilon_s)_{s \in S}$ and the relations*

$$\begin{aligned} \varepsilon_s^2 &= a_s \varepsilon_s + b_s \varepsilon_1 && \text{for } s \in S \\ (\varepsilon_s \varepsilon_{s'})^m &= (\varepsilon_{s'} \varepsilon_s)^m && \text{for } s, s' \in S \text{ such that } ss' \text{ is finite of order } 2m \\ (\varepsilon_s \varepsilon_{s'})^m \varepsilon_s &= (\varepsilon_{s'} \varepsilon_s)^m \varepsilon_{s'} && \text{for } s, s' \in S \text{ such that } ss' \text{ is finite of order } 2m+1 \end{aligned}$$

form a presentation of the generic algebra $\mathcal{E}_R(a_s, b_s)$.

3.2 Iwahori-Hecke algebras

Let $\mathscr{A} := \mathbb{Z}\left[q^{1/2}, q^{-1/2}\right]$ denote the ring of Laurent polynomials in the indeterminate $q^{1/2}$. In this section we will define an algebra $\mathcal{H}$, called the *Iwahori-Hecke algebra of* (W, S). Our first aim is to show that $\mathcal{H}$ is a free $\mathscr{A}$-module with a canonical basis indexed by the elements $w \in W$ using the results in Section 3.1. In particular, we will use the generic algebra $\mathcal{E}_{\mathscr{A}}(q-1, q)$ of Theorem 3.1.1 with $R = \mathscr{A}$ and parameters $a_s = q - 1$ and $b_s = q$ for any $s \in S$. Note that in this case, the values of a_s are the same for all $s \in S$, and the same is true for the values of b_s.

Definition 3.2.1. Let $\mathcal{H}$ be the $\mathscr{A}$-algebra defined by the generators T_s $(s \in S)$ and the relations

$$(T_s + 1)(T_s - q) = 0 \quad \text{for } s \in S \tag{3.2.1}$$

$$\mathsf{prod}(m_{ss'}; T_s, T_{s'}) = \mathsf{prod}(m_{ss'}; T_{s'}, T_s) \quad \text{for any } (s, s') \in S_F \text{ with } s \neq s'. \tag{3.2.2}$$

For w in W, define $T_w \in \mathcal{H}$ by

$$T_w := T_{s_1} \cdots T_{s_r} \tag{3.2.3}$$

where $w = s_1 \cdot s_r$ is a reduced expression of w.

Proposition 3.2.1. *For each $w \in W$, the element $T_w \in \mathcal{H}$ given in* (3.2.3) *is well-defined and independent of the choice of reduced expression. In particular, we have $T_1 = 1_{\mathcal{H}}$.*

Proof. Consider the set M given by

$$M := \{T_{s_1} \cdots T_{s_r} \mid s_1, \ldots, s_r \in S \text{ for some integer } r \geq 0\},$$

which is a monoid with operation given by multiplication, and let ψ be the map from S to M defined by

$$\begin{aligned} \psi : S &\to M \\ s &\mapsto T_s. \end{aligned}$$

By the relation in (3.2.2), we have that for any $s, s' \in S$, the map $\psi : S \to M$ satisfies

$$\mathsf{prod}\left(m_{ss'}; \psi(s), \psi(s')\right) = \mathsf{prod}\left(m_{ss'}; \psi(s'), \psi(s)\right),$$

and thus, by Proposition 2.3.2, there exists a map $\tilde{\psi} : W \to M$ from W to M such that

$$\tilde{\psi}(w) = \psi(s_1) \cdots \psi(s_r) = T_{s_1} \cdots T_{s_r}$$

for all $w \in W$. Hence the element T_w is well-defined and independent of the choice of reduced expression of w. □

Proposition 3.2.2. *For any $s \in S$ and $w \in W$, we have*

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (q-1)\, T_w + q T_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases} \tag{3.2.4}$$

Proof. Take any $s \in S$ and any $w \in W$, and let $w = s_1 \cdots s_r$ be a reduced expression of w with $s_i \in S$. By Proposition 3.2.1,

$$T_w = T_{s_1} \cdots T_{s_r}.$$

If $\ell(sw) > \ell(w)$, then $sw = ss_1 \cdots s_r$ is a reduced expression os sw, and hence, again by Proposition 3.2.1, we obtain

$$T_{sw} = T_{ss_1 \cdots s_r} = T_s T_{s_1} \cdots T_{s_r} = T_s T_w,$$

which proves the first relation in (3.2.4). If on the contrary $\ell(sw) < \ell(w)$, then we may assume that our reduced expression of w is chosen so that $s_1 = s$. Then using the defining relations (3.2.1) and (3.2.2) for $\mathcal{H}$, we obtain

$$T_s T_w = T_{s_1} T_{s_1} \cdots T_{s_r} = ((q-1) T_{s_1} + q T_1)\, T_{s_2} \cdots T_{s_r}. \tag{3.2.5}$$

But $sw = s_2 \cdots s_r$ is a reduced expression of sw, and so, by Proposition 3.2.1, the equality in (3.2.5) becomes

$$T_s T_w = (q-1) T_w + q T_{sw},$$

which proves the second relation in (3.2.4) and completes the proof. □

Proposition 3.2.3. *The $\mathscr{A}$-algebra $\mathcal{H}$ is generated by $(T_w)_{w \in W}$ as an $\mathscr{A}$-module.*

Proof. Let $\mathcal{H}' \subseteq \mathcal{H}$ be the $\mathscr{A}$-submodule of $\mathcal{H}$ generated by $(T_s)_{w \in W}$. First, note that by the relations (3.2.4) in Proposition 3.2.2, we see that $T_s T_w \in \mathcal{H}'$ for any $s \in S$ and any $w \in W$, which shows that the $\mathscr{A}$-submodule $\mathcal{H}'$ of $\mathcal{H}$ is stable under left multiplication by all the T_s such that $s \in S$, which shows that $\mathcal{H}'$ is a left ideal in $\mathcal{H}$. Moreover, the identity element $1_{\mathcal{H}} = T_1$ of $\mathcal{H}$ belongs to the collection $(T_w)_{w \in W}$, and thus belongs to $\mathcal{H}'$. Therefore, since we have shown that $1_{\mathcal{H}} = T_1 \in \mathcal{H}'$ and $\mathcal{H}'$ is a left ideal in $\mathcal{H}$, we conclude that $\mathcal{H}' = \mathcal{H}$, and thus $\mathcal{H}$ is generated by $(T_w)_{w \in W}$ as an $\mathscr{A}$-module. □

Proposition 3.2.4. *$(T_w)_{w\in W}$ is a $\mathscr{A}$-basis of $\mathcal{H}$.*

Proof. Take any $\cdot w \in W$. In the uniqueness part of the proof of Theorem 3.1.1, we showed that $\varepsilon_w = \varepsilon_{s_1}\cdots\varepsilon_{s_r}$ whenever $w = s_1\cdots s_r$ is a reduced expression of w. Moreover, by Proposition 3.1.2, $\mathcal{E}_{\mathscr{A}}(q-1,q)$ has a presentation as an $\mathscr{A}$-algebra given by the generators $(e_s)_{s\in S}$ and the relations

$$\varepsilon_s^2 = (q-1)\varepsilon_s + q\varepsilon_1 \quad \text{for } s\in S \tag{3.2.6}$$

$$\mathsf{prod}(m_{ss'};\varepsilon_s,\varepsilon_{s'}) = \mathsf{prod}(m_{ss'};\varepsilon_{s'},\varepsilon_s) \quad \text{for any } (s,s')\in S_F \text{ with } s\neq s'. \tag{3.2.7}$$

Note that the relation in (3.2.6) can be rewritten as

$$(\varepsilon_s+1)(\varepsilon_s-q)=0.$$

We therefore see that there is a unique algebra homomorphism $\phi : \mathcal{H}\to\mathcal{E}$ between the $\mathscr{A}$-modules $\mathcal{H}$ and $\mathcal{E}$ such that

$$\begin{aligned}\phi : \mathcal{H} &\to \mathcal{E}\\ T_1 &\mapsto \varepsilon_1\\ T_s &\mapsto \varepsilon_s,\end{aligned}$$

and in particular, it maps $T_w\mapsto\varepsilon_w$ for any $w\in W$. Now, if $c_w\in\mathscr{A}$ are such that $c_w=0$ for all but finitely many $w\in W$ and

$$\sum_{w\in W} c_w T_w = 0 \tag{3.2.8}$$

in $\mathcal{H}$, then, applying the algebra homomorphism $\phi:\mathcal{H}\to\mathcal{E}$ on both sides of (3.2.8), gives

$$\sum_{w\in W} c_w\varepsilon_w = \sum_{w\in W} c_w\phi(T_w) = \phi(0) = 0.$$

But then, since $\mathcal{E}$ is a free $\mathscr{A}$-module with canonical basis $(\varepsilon_w)_{w\in W}$, it follows, by the linear independence of the ε_w ($w\in W$), that $c_w=0$ for all $w\in W$, and thus the T_w are linearly independent generators of the $\mathscr{A}$-module $\mathcal{H}$, which shows that $(T_w)_{w\in W}$ is an $\mathscr{A}$-basis of $\mathcal{H}$. □

3.3 Hecke algebras and the Bruhat ordering

From now on, $\mathcal{H}$ will be the Hecke algebra over $\mathscr{A}=\mathbb{Z}[q^{-1/2},q^{1/2}]$ of Section 3.2, which is a free $\mathscr{A}$-module with basis $(T_w)_{w\in W}$ and the multiplication is defined by (3.2.4), or equivalently, by

$$T_wT_{w'} = T_{ww'} \quad \text{if } \ell(ww') = \ell(w)+\ell(w) \tag{3.3.1}$$

$$(T_s+1)(T_s-q)=0 \qquad \text{if } s\in S. \tag{3.3.2}$$

Also, let $\mathscr{A}^+ := \mathbb{Z}\left[q^{1/2}\right]$. In this section we consider the invertibility of the elements T_w for $w\in W$. Once we have established the invertibility of such elements of $\mathcal{H}$, our aim is to express the inverse T_w^{-1} of the element T_w for any w in W as a linear combination of the canonical basis elements T_x ($x\in W$). In order to achieve this, we introduce a family of polynomials, called the 'R-polynomials' which enable us to express T_w^{-1}, for any $w\in W$, as a combination of those basis elements T_w such that $x\leqslant w$ in the Bruhat ordering, described in Section 2.10. We then give some basic properties of the R-polynomials. The main reference for this section is [21].

Proposition 3.3.1. *For each $s\in S$, the generator $T_s\in\mathcal{H}$ is invertible, with inverse given by*

$$T_s^{-1} = q^{-1}T_s - (1-q^{-1})T_1. \tag{3.3.3}$$

Proof. Note that

$$T_s\left(q^{-1}T_s - (1-q^{-1})T_1\right) = q^{-1}T_s^2 - (1-q^{-1})T_s. \tag{3.3.4}$$

Hence, substituting into (3.3.4) the value of T_s^2 given by the defining relation (3.2.1) of $\mathcal{H}$, yields

$$T_s\left(q^{-1}T_s - (1-q^{-1})T_1\right) = q^{-1}(q-1)T_s + q^{-1}qT_1 - (1-q^{-1})T_s = T_1,$$

which proves the proposition. □

Corollary 3.3.1. *For each $w \in W$, the basis element T_w of $\mathcal{H}$ is invertible, with inverse given by*

$$T_w^{-1} = T_{s_r}^{-1} \cdots T_{s_1}^{-1} \tag{3.3.5}$$

whenever $w = s_1 \cdots s_r$ is a reduced expression of w.

Proof. Let $w = s_1 \cdots s_r$ be any reduced expression of w. By Proposition 3.2.1,

$$T_w = T_{s_r} \cdots T_{s_1},$$

and so, by Proposition 3.3.1, we see that

$$T_w T_{s_r}^{-1} \cdots T_{s_1}^{-1} = T_{s_1} \cdots T_{s_r} T_{s_r}^{-1} \cdots T_{s_1}^{-1} = T_1,$$

and the proof is complete. □

Remark 3.3.1. As $\ell(w)$ increases, it becomes increasingly difficult to compute the inverse explicitly as a linear combination of the canonical basis $(T_w)_{w\in W}$ of $\mathcal{H}$. To ease the notation, we define $\varepsilon_w := (-1)^{\ell(w)}$ and $q_w := q^{\ell(w)}$ and we write F instead of $F(q)$ whenever $F(q)$ is a polynomial in $\mathbb{Z}[q]$ or a polynomial in $\mathscr{A}$.

Proposition 3.3.2. *For all $w \in W$,*

$$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \sum_{\substack{x\in W\\ x\leqslant w}} \varepsilon_x R_{x,w} T_x, \tag{3.3.6}$$

where $R_{x,w} \in \mathbb{Z}[q]$ is a polynomial of degree $\ell(w) - \ell(x)$ in q, and where $R_{w,w} = 1$.

Proof. We proceed by induction on the length $\ell(w)$ of w. If $\ell(w) = 0$, then $w = 1$ and we have

$$T_{1^{-1}} T_{1^{-1}} = T_1 T_1 = T_1, \tag{3.3.7}$$

so (3.3.6) trivially holds. If $\ell(w) = 1$, then $w = s$ for some $s \in S$, and thus, in view of (3.3.3), if we set

$$R_{1,s} := q - 1,$$

and noting that

$$\varepsilon_1 = 1, \quad \varepsilon_s = -1, \text{and} \quad q_s = q,$$

we obtain

$$T_{s^{-1}}^{-1} = T_s^{-1} = \varepsilon_s q_s^{-1} \varepsilon_s R_{s,s} T_s + \varepsilon_s q_s^{-1} \varepsilon_1 R_{1,s} T_1.$$

So let $w \in W$ be such that $\ell(w) > 1$ and assume that (3.3.6) holds for any $w' \in W$ such that the length of w' is less than the length of w. For convenience, set $R_{x,w} = 0$ whenever $x \in W$ is such that $x \not\leqslant w$. Note that since $\ell(w) > 1$, there exists some $s \in S$ such that $w = sv$ for some $v \in W$ with

$$\ell(v^{-1}) = \ell(v) = \ell(w) - 1 < \ell(w) = \ell(w^{-1}) = \ell(v^{-1}s).$$

We therefore have

$$\varepsilon_w = (-1)^{\ell(w)} = (-1)^{\ell(v)+1} = -\varepsilon_v \quad \text{and} \quad q_w = q^{\ell(w)} = q^{\ell(v)+1} = qq_v, \tag{3.3.8}$$

so using Proposition 3.1.1 and the relation (3.3.1) defining multiplication in $\mathcal{H}$, we obtain

$$T_{w^{-1}}^{-1} = (\flat\,(T_w))^{-1} = (\flat\,(T_sT_v))^{-1} = (\flat\,(T_v)\cdot\flat\,(T_s))^{-1} = T_s^{-1}T_{v^{-1}}^{-1},$$

and thus, substituting into (3.3.3), we obtain

$$T_{w^{-1}}^{-1} = \left(q^{-1}T_s - (1-q^{-1})T_1\right)T_{v^{-1}}^{-1} = q^{-1}\left(T_s - (q-1)T_1\right)T_{v^{-1}}^{-1}. \tag{3.3.9}$$

Moreover, by the induction hypothesis, we know that

$$T_{v^{-1}}^{-1} = \varepsilon_v q_v^{-1} \sum_{\substack{x\in W\\ x\leqslant v}} \varepsilon_x R_{x,v} T_x,$$

where $R_{x,v} \in \mathbb{Z}[q]$ is a polynomial of degree $\ell(w) - \ell(x)$ in q and $R_{v,v} = 1$, and so substituting this into (3.3.9) yields

$$T_{w^{-1}}^{-1} = \varepsilon_v q_v^{-1} q^{-1} \left(\sum_{\substack{x\in W\\ x\leqslant v}} \varepsilon_x R_{x,v} T_s T_x - (q-1) \sum_{\substack{x\in W\\ x\leqslant v}} \varepsilon_x R_{x,v} T_x \right). \tag{3.3.10}$$

In view of (3.3.8), we can further simplify (3.3.10) to obtain

$$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \left(-\sum_{\substack{x\in W\\ x\leqslant v}} \varepsilon_x R_{x,v} T_s T_x + (q-1) \sum_{\substack{x\in W\\ x\leqslant v}} \varepsilon_x R_{x,v} T_x \right). \tag{3.3.11}$$

Now, if $x \in W$ is involved in the first sum in (3.3.11), we have two possibilities: either $sx < x$ or $x < sx$. In the first case, by the first relation in (3.2.4), the term corresponding to x is equal to

$$\varepsilon_x R_{x,v} T_s T_x = (q-1)\varepsilon_x R_{x,v} T_x + q\varepsilon_x R_{x,v} T_{sx}, \tag{3.3.12}$$

and in the latter case, the term corresponding to x is equal to

$$\varepsilon_x R_{x,v} T_s T_x = \varepsilon_x R_{x,v} T_{sx}. \tag{3.3.13}$$

We therefore see that if $sx < x$, the first term in (3.3.12) cancels a term in the second sum in (3.3.11), which can thus be rewritten as

$$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \left(-q \sum_{\substack{x\in W\\ x\leqslant v\\ sx<x}} \varepsilon_x R_{x,v} T_{sx} - \sum_{\substack{x\in W\\ x\leqslant v\\ x<sx}} \varepsilon_x R_{x,v} T_{sx} + (q-1) \sum_{\substack{x\in W\\ x\leqslant v\\ x<sx}} \varepsilon_x R_{x,v} T_x \right). \tag{3.3.14}$$

Also note that if $x \leqslant v$ then, by the transitivity of " $\leqslant$ ", we have $x < w$, and thus, by Lemma 2.10.2, we also have $sx \leqslant w$. Also notice that if $y \in W$ is such that $y \leq w$, then, by Theorem 2.10.1, y occurs either as an x with $x \leqslant v$ or as an sx with $x \leq v$ or both, so we just need to check the coefficient of T_y for each $y \in W$ with $y \leqslant w$. So let $y \in W$ be any element such that $y \leqslant w$.

If $sy < y$, then T_y appears only in the second sum in (3.3.14) with $y = sx$ and coefficient

$$-\varepsilon_x R_{x,v} = -\varepsilon_{sy} R_{sy,v} = \varepsilon_y R_{sy,sw},$$

with

$$\deg_q(\varepsilon_y R_{sy,sw}) = \ell(sw) - \ell(sy) = \ell(w) - 1 - \ell(y) + 1 = \ell(w) - \ell(y).$$

Note that if $y = w$, then $x = v$ and we have $R_{v,v} = 1$, by the induction hypothesis. Hence the polynomial $R_{y,w}$ can be defined, in this case, as

$$R_{y,w} := R_{sy,sw}, \tag{3.3.15}$$

which we have shown satisfies the requirements in the proposition.

If on the contrary $y < sy$, then we automatically have $y < w$, and we have two possibilities: either $sy < v$ or $sy \not\leqslant v$. In the first case T_y appears in the first sum in (3.3.14) with $y = sx$ and coefficient

$$-q\varepsilon_x R_{x,v} = -q\varepsilon_{sy} R_{sy,v} = -q\varepsilon_{sy} R_{sy,sw},$$

with

$$\deg_q(-q\varepsilon_{sy} R_{sy,sw}) = \ell(sw) - \ell(sy) + 1 = \ell(w) - 1 - \ell(y) + 1 + 1 = \ell(w) - \ell(y) + 1,$$

as well as in the third sum (3.3.14) with $y = x$ and coefficient

$$(q-1)\varepsilon_x R_{x,v} = (q-1)\varepsilon_y R_{y,sw}$$

with

$$\deg_q((q-1)\varepsilon_y R_{y,sw}) = \ell(sw) - \ell(y) + 1 = \ell(w) - 1 - \ell(y) + 1 = \ell(w) - \ell(y).$$

Therefore, the coefficient in (3.3.14) of T_y in this case is equal to

$$-q\varepsilon_{sy} R_{sy,sw} + (q-1)\varepsilon_y R_{y,sw}$$

and

$$\deg_q\big(-q\varepsilon_{sy} R_{sy,sw} + (q-1)\varepsilon_y R_{y,sw}\big) = \ell(w) - \ell(y).$$

We can thus define, in this case, the polynomial $R_{y,w}$ as

$$R_{y,w} := qR_{sy,sw} + (q-1)R_{y,sw}, \tag{3.3.16}$$

which satisfies the requirements in the proposition. In the latter case T_y appears only in the third sum in (3.3.14) with $y = x$ and coefficient

$$(q-1)\varepsilon_x R_{x,v} = (q-1)\varepsilon_y R_{y,sw}$$

with

$$\deg_q\big((q-1)\varepsilon_y R_{y,sw}\big) = \ell(w) - \ell(y).$$

Hence, since by convention $R_{sy,v} = 0$ as $sy \not\leqslant v$, the polynomial $R_{y,w}$ can be defined, in this case, as in (3.3.16), which we have shown satisfies the requirements in the proposition. The proof is now complete. □

Remark 3.3.2. Even though we defined, in Section 2.10, the Bruhat ordering for (W, S) without reference to the Hecke algebra $\mathcal{H}$ of (W, S), Proposition 3.3.2 shows that $R_{y,w} \neq 0$ if and only if $y \leqslant w$, which shows that the Bruhat ordering of (W, S) is actually forced on us by the way inversion works in $\mathcal{H}$.

Remark 3.3.3. Note that the proof of Proposition 3.3.2 actually produces an algorithm for the computation of R-polynomials. More explicitly, given the fact that

$$R_{y,w} = \begin{cases} 0 & \text{if } y \nleqslant w \\ 1 & \text{if } y = w \end{cases} \tag{3.3.17}$$

for any $y, w \in W$, we use induction on $\ell(w) \geq 1$ to compute $R_{y,w}$ assuming that the $R_{y,x}$ are known for all $x \in W$ with $\ell(x) < \ell(w)$. In particular, if we fix any $s \in S$ such that $sw < w$, then we can compute $R_{y,w}$ according to

$$R_{y,w} = \begin{cases} R_{sy,sw} & \text{if } y < w \text{ and } sy < y \\ qR_{sy,sw} + (q-1)R_{y,sw} & \text{if } y < w \text{ and } y < sy < sw \\ (q-1)R_{y,sw} & \text{if } y < w \text{ and } y < sy \nleqslant sw. \end{cases} \tag{3.3.18}$$

Moreover, the formula in (3.3.6) can be rewritten as

$$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \left(\sum_{\substack{y \in W \\ y \leqslant w \\ sy < y}} \varepsilon_y R_{sy,sw} T_y + \sum_{\substack{y \in W \\ y < w \\ y < sy}} \left(\varepsilon_y q R_{sy,sw} + (q-1) R_{y,sw}\right) T_y \right), \tag{3.3.19}$$

with the convention that $R_{z,x} = 0$ whenever $z \nleqslant x$.

Proposition 3.3.3. (Exercise 1 in Chapter 7.5 in [21]) *If $x, w \in W$ are elements such that $x \leqslant w$, then $R_{x,w}$ is a monic polynomial with constant term $\varepsilon_w \varepsilon_x$.*

Proof. We proceed by induction on the length $\ell(w)$ of $w \in W$. First note that if $\ell(w) = 0$, then there is nothing to prove. So suppose that $\ell(w) \geq 1$, and assume that $R_{x,y}$ is a monic polynomial with constant term $\epsilon_y \epsilon_x$ whenever $y \in W$ such that $\ell(y) < \ell(w)$. Now choose any $s \in S$ such that $sw < w$. If $sx < x$, then, by (3.3.18), we have $R_{x,w} = R_{sx,sw}$. Since in this case we have

$$\ell(w) - \ell(x) = \ell(sw) + 1 - \ell(sx) - 1 = \ell(sw) - \ell(sx),$$

the claim holds for w in this case. If on the contrary $x < sx$, then, by (3.3.18) again, we have that

$$R_{x,w} = qR_{sx,sw} + (q-1)R_{x,sw}.$$

Since in this case we have that

$$\ell(sw) - \ell(sx) = \ell(w) - 1 - \ell(x) - 1 = \ell(w) - \ell(x) - 2,$$

and

$$\ell(sw) - \ell(x) = \ell(w) - 1 - \ell(x) = \ell(w) - \ell(x) - 1,$$

by the induction hypothesis, $R_{sx,sw}$ and $R_{x,sw}$ are both monic polynomials of constant term $\epsilon_{sx}\epsilon_{sw}$ and $\epsilon_x \epsilon_{sw}$, respectively. Note that even if $sx \nleqslant sw$, in which case $R_{sx,sw} = 0$, the result in the lemma would still hold in this case, and the proof is now complete. □

Proposition 3.3.4. *If $x, w \in W$ are elements such that $x \leq w$ and $\ell(w) - \ell(x) \leq 2$, then*

$$R_{x,w} = (q-1)^{\ell(w)-\ell(x)}.$$

Proof. Let $x, w \in W$ be such that $x \leq w$ and $\ell(w) - \ell(x) \leq 2$. If $\ell(w) - \ell(x) = 0$, then the result trivially holds and there is nothing to prove. So suppose that $\ell(w) - \ell(x) = 1$, and let $w = s_1 \cdots s_r$ be a reduced expression of w. Since the length difference between x and w is 1, we can obtain x from w be removing a single s_i in the reduced expression $w = s_1 \cdots s_r$ of w. Repeatedly using the relations in (3.3.18), we are reduced to the case when $x = 1$ and $w = s_i$. As we saw in the proof of Proposition 3.3.2, we obtain R_{1,s_i} directly from $T_{s_i}^{-1}$, and in particular, from (3.3.3), we obtain $R_{1,s_{s_i}} = q - 1$.

Now, suppose that $\ell(w) - \ell(x) = 2$, and let $w = s_1 \cdots s_r$ be a reduced expression of w. Since the length difference is two, we can obtain x from w be removing exactly two s_i, s_j in the reduced expression $w = s_1 \cdots s_r$ of w. Assume, without loss of generality that $i < j$. Again, repeatedly using the relations in (3.3.18), we are reduced to the case $w = s_i \cdots s_j$ and $x = s_{i+1} \cdots s_{j-1}$. If we then take $s = s_i$, we have $sw < w$ and $x < sx$, and thus the second relation in (3.3.18) gives

$$R_{x,w} = qR_{sx,sw} + (q-1)R_{x,sw}.$$

Since $\ell(sw) - \ell(x) = 1$, we know from the previous case that $R_{x,sw} = q - 1$. Moreover, $\ell(sx) = \ell(w)$ but $sxsw$, and thus $R_{sx,sw}$. We thus obtain $R_{x,w} = (q-1)^2$, as required. □

Remark 3.3.4. Carrying out the explicit calculations for $R_{x,w}$ whenever $x, w \in W$ are such that $x \leqslant w$ and $\ell(w) - \ell(x) \geq 3$ becomes increasingly difficult due to the existence of a wider range of subexpressions whenever we omit some s_i in some reduced expression $s_1 \cdots s_r$ of w.

3.4 The bar involution

In this section we introduce a ring automorphism of $\mathcal{H}$, called the *the bar involution of* $\mathcal{H}$. Such an involution has an important role in the definition of the Kazhdan-Lusztig polynomials in Section 3.5. Our first aim is to prove the existence of such involution of $\mathcal{H}$. We then study some of its properties by analysing the role it plays in the proofs of further properties of the R-polynomials and its relations to other morphisms of the algebra $\mathcal{H}$. The main reference for this section is [24].

Lemma 3.4.1. *Let* $\bar{\ } : \mathscr{A} \to \mathscr{A}$ *be the ring involution of* $\mathscr{A}$ *mapping* $q^{1/2}$ *to* $q^{-1/2}$*, that is,*

$$\overline{q^{1/2}} = q^{-1/2}. \tag{3.4.1}$$

There exists a unique ring homomorphism $\bar{\ } : \mathcal{H} \to \mathcal{H}$ *which is* $\mathscr{A}$*-semilinear with respect to* $\bar{\ } : \mathscr{A} \to \mathscr{A}$ *and satisfies*

$$\overline{T}_s = T_s^{-1}. \tag{3.4.2}$$

Moreover, such ring homomorphsim is an involution of $\mathcal{H}$ *and takes* T_w *to* $T_{w^{-1}}^{-1}$*, that is,*

$$\overline{T}_w = T_{w^{-1}}^{-1}. \tag{3.4.3}$$

Proof. First, let $\bar{\ } : \mathcal{H} \to \mathcal{H}$ be an $\mathscr{A}$-semilinear map such that $\overline{T}_s = T_s^{-1}$. Since it must be an $\mathscr{A}$-semilinear map, by definition $\bar{\ } : \mathcal{H} \to \mathcal{H}$ must satisfy

$$\begin{aligned} \overline{h + h'} &= \overline{h} + \overline{h'} \quad \text{for any } h, h' \in \mathcal{H} \\ \overline{ch} &= \overline{c}\overline{h} \qquad \text{for any } c \in \mathscr{A} \text{ and any } h \in \mathcal{H}. \end{aligned}$$

Given such a semilinear map, note that for any $s \in S$, we have

$$\overline{(T_s + 1)}\,\overline{(T_s - q)} = \left(T_s^{-1} + 1\right)\left(T_s^{-1} - q^{-1}\right),$$

and so substituting (3.3.3) into the above yields

$$\overline{(T_s + 1)}\,\overline{(T_s - q)} = \left(q^{-1}T_s - (1 - q^{-1}) + 1\right)\left(q^{-1}T_s - (1 - q^{-1}) - q^{-1}\right),$$

which then simplifies to

$$\overline{(T_s+1)}\,\overline{(T_s-q)} = q^{-1}\left(T_s+1\right))\, q^{-1}\left(T_s-q\right) = q^{-2}\left(T_s+1\right)\left(T_s-q\right).$$

Hence, using the defining relation (3.2.1) of $\mathcal{H}$, the above becomes

$$\overline{(T_s+1)}\,\overline{(T_s-q)} = 0.$$

Also, for any $s, s' \in S$ such that $s \neq s'$ and $(s, s') \in S_F$, we have

$$\mathsf{prod}(m_{ss'}; \overline{T}_s, \overline{T}_{s'}) = \mathsf{prod}\left(m_{ss'}; T_s^{-1}, T_{s'}^{-1}\right) = \mathsf{prod}\left(m_{ss'}; T_{s'}, T_s\right)^{-1}.$$

Hence, using the relation defining relation (3.2.2) of $\mathcal{H}$, the above gives

$$\mathsf{prod}(m_{ss'}; \overline{T}_s, \overline{T}_{s'}) = \mathsf{prod}\left(m_{ss'}; T_s, T_{s'}\right)^{-1} = \mathsf{prod}\left(m_{ss'}; T_{s'}^{-1}, T_s^{-1}\right) = \mathsf{prod}(m_{ss'}; \overline{T}_{s'}, \overline{T}_s).$$

Since we have shown that the $\mathscr{A}$-semilinear map $^{-} : \mathcal{H} \to \mathcal{H}$ preserves the relations (3.2.1) and (3.2.2) defining $\mathcal{H}$ as an $\mathscr{A}$-algebra, it follows that the map has a unique extension to a ring homomorphism $^{-} : \mathcal{H} \to \mathcal{H}$ which is $\mathscr{A}$-semilinear with respect to a $^{-} : \mathscr{A} \to \mathscr{A}$.

Now, let $s \in S$. By (3.4.2) and (3.3.3), we have that

$$\overline{\overline{T}_s} = \overline{T_s^{-1}} = \overline{q^{-1}T_s - (1-q^{-1})T_1} = qT_s^{-1} - (1-q)T_1.$$

Substituting (3.3.3) again into the right-hand side of the above equation, we obtain

$$\overline{\overline{T}_s} = qq^{-1}T_s - q(1-q^{-1})T_1 - (1-q)T_1 = T_s + (1-q)T_1 - (1-q)T_1 = T_s,$$

which shows, since $(T_s)_{s\in S}$ generates $\mathcal{H}$ as an $\mathscr{A}$-algebra that, that $^{-} : \mathcal{H} \to \mathcal{H}$ is an involution of $\mathcal{H}$.

Finally, let $w \in W$ be any element of W. By Proposition 3.2.1, we know that

$$T_w = T_{s_1} \cdots T_{s_r} \quad \text{and} \quad T_{w^{-1}} = T_{s_r} \cdots T_{s_1}$$

for any reduced expression $w = s_1 \cdots s_r$ of W and T_w and $T_{w^{-1}}$ are independent of the choice of reduced expression of w. We then have that

$$\overline{T}_w = \overline{T_{s_1} \cdots T_{s_r}} = T_{s_1}^{-1} \cdots T_{s_r}^{-1} = (T_{s_r} \cdots T_{s_1})^{-1} = T_{w^{-1}}^{-1},$$

and the proof is now complete. □

Remark 3.4.1. Since $\mathcal{H}$ is generated as an $\mathscr{A}$-module by $(T_w)_{w\in W}$ the unique $\mathscr{A}$-semilinear ring homomorphism $^{-} : \mathcal{H} \to \mathcal{H}$ with respect to $^{-} : \mathscr{A} \to \mathscr{A}$ from Lemma 3.4.1 is given by

$$\overline{\sum_{w\in W} c_w T_w} = \sum_{w\in W} \overline{c}_w T_{w^{-1}}^{-1} \quad \text{for any } c_w \in \mathscr{A}. \tag{3.4.4}$$

Proposition 3.4.1. *Let $y, w \in W$ be any elements of W.*

(a) We have

$$\overline{R}_{y,w} = \varepsilon_y \varepsilon_w q_y q_w^{-1} R_{y,w}.$$

(b) Let $\delta_{,w}$ denote the Kronecker delta function. We have

$$\sum_{\substack{x\in W\\ y\leqslant x\leqslant w}} \varepsilon_y \varepsilon_x R_{y,x} R_{x,w} = \delta_{y,w}.$$

Proof. (a) We prove the equality inductively using the rules in (3.3.17) and (3.3.18). First, note that if $y = w$ or $y \not\leqslant w$, the equality trivially holds since we have $R_{y,w} = \overline{R}_{y,w}$ in both cases. So suppose now that we are in the case $y < w$, $sy < y$ and $sw < w$, so that $R_{y,w} = R_{sy,sw}$ and $\overline{R}_{y,w} = \overline{R}_{sy,sw}$. By induction, we have

$$\overline{R}_{sy,sw} = \varepsilon_{sy}\varepsilon_{sw}q_{sy}q_{sw}^{-1}R_{sy,sw} = (-\varepsilon_y)(-\varepsilon_w)q^{-1}q_y\left(q^{-1}q_w\right)^{-1}R_{sy,sw} = \varepsilon_y\varepsilon_w q_y q_w^{-1}R_{y,w}, \quad (3.4.5)$$

as required. Suppose on the contrary that we are in the case$y < w$, $y < sy$ and $sw < w$, so that

$$R_{y,w} = (q-1)R_{y,sw} + qR_{sy,sw}$$

which, after applying $^{-} : \mathcal{H} \to \mathcal{H}$ yields

$$\overline{R}_{y,w} = (q^{-1}-1)\overline{R}_{y,sw} + q^{-1}\overline{R}_{sy,sw} = -q^{-1}(q-1)\overline{R}_{y,sw} + q^{-1}\overline{R}_{sy,sw}. \quad (3.4.6)$$

By induction, we have

$$\overline{R}_{y,sw} = \varepsilon_y\varepsilon_{sw}q_y q_{sw}^{-1}R_{y,sw} = -\varepsilon_y\varepsilon_w q_y q_w^{-1}qR_{y,sw}$$

and

$$\overline{R}_{sy,sw} = \varepsilon_{sy}\varepsilon_{sw}q_{sy}q_{sw}^{-1}R_{sy,sw} = \varepsilon_y\varepsilon_w qq_y q_w^{-1}qR_{y,sw} = \varepsilon_y\varepsilon_w q_y q_w^{-1}q^2R_{sy,sw}.$$

Substituting these two equalities into (3.4.6) then yields

$$\overline{R}_{y,w} = q^{-1}(q-1)\varepsilon_y\varepsilon_w q_y q_w^{-1}qR_{y,sw} + q^{-1}\varepsilon_y\varepsilon_w q_y q_w^{-1}q^2R_{sy,sw}.$$

Hence, cancelling the corresponding terms and substituting in (3.4.5), we obtain

$$\overline{R}_{y,w} = \varepsilon_y\varepsilon_w q_y q_w^{-1}\left((q-1)R_{y,sw} + qR_{sy,sw}\right) = \varepsilon_y\varepsilon_w q_y q_w^{-1}R_{y,w},$$

as required. This completes the proof of (a).

(b) We first apply the bar involution $^{-} : \mathcal{H} \to \mathcal{H}$ to both sides of the equality (3.3.6) in Proposition 3.3.2 to obtain

$$T_w = \varepsilon_w q_w \sum_{\substack{x\in W\\ x\leqslant w}} \varepsilon_x \overline{R}_{x,w}T_{x^{-1}}^{-1}. \quad (3.4.7)$$

Substituting the result we have just proven for $\overline{R}_{y,w}$ in (a) into (3.4.7) with x instead of y gives

$$T_w = \varepsilon_w q_w \sum_{\substack{x\in W\\ x\leqslant w}} \varepsilon_x\varepsilon_x\varepsilon_w q_x q_w^{-1}R_{x,w}T_{x^{-1}}^{-1} = \sum_{\substack{x\in W\\ x\leqslant w}} q_x R_{x,w}T_{x^{-1}}^{-1}. \quad (3.4.8)$$

Then, substituting into (3.4.8) the expression in (3.3.6) for $T_{x^{-1}}^{-1}$, yields

$$T_w = \sum_{\substack{x\in W\\ x\leqslant w}} q_x R_{x,w}\varepsilon_x q_x^{-1}\sum_{\substack{y\in W\\ y\leqslant x}}\varepsilon_y R_{y,x}T_y = \sum_{\substack{x\in W\\ x\leqslant w}} R_{x,w}\varepsilon_x \sum_{\substack{y\in W\\ y\leqslant x}}\varepsilon_y R_{y,x}T_y. \quad (3.4.9)$$

Note that the coefficient of T_y on the left-hand side of (3.4.9) is 0 if $y \neq w$ and 1 if $y = w$, that is, $\delta_{y,w}$. Moreover, the coefficient of T_y on the right-hand side of (3.4.9) is

$$\sum_{\substack{x\in W\\ y\leqslant x\leqslant w}} \varepsilon_x\varepsilon_y R_{y,x}R_{x,w}.$$

Equating these two coefficients gives the result in (b), and the proof is now complete. □

Corollary 3.4.1. *For any element $w \in W$ we have*

$$T_{w^{-1}}^{-1} = \sum_{\substack{y \in W \\ y \leqslant w}} q_y^{-1} \overline{R}_{y,w} T_y. \tag{3.4.10}$$

Proof. If we substitute the result in (a) of Proposition 3.4.1 into the right-hand side of the equality (3.3.6) in Proposition 3.3.2 with y instead of x we obtain

$$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \sum_{\substack{y \in W \\ y \leqslant w}} \varepsilon_y \varepsilon_y^{-1} \varepsilon_w^{-1} q_y^{-1} q_w \overline{R}_{y,w} T_y = \sum_{\substack{y \in W \\ y \leqslant w}} q_y^{-1} \overline{R}_{y,w} T_y,$$

as required. □

Remark 3.4.2. The proof of (b) of Proposition 3.4.1 provides a method for computing the inverse of the matrix of R-polynomials. In particular, in view of the equalities in (3.4.8) and (3.4.10), we conclude that the matrices $(R_{x,w} q_x)$ and $\left(\overline{R}_{x,w} q_x^{-1}\right)$ are inverses of each other and, in view of the equality in (a) of Proposition 3.4.1, we deduce that the matrices $(R_{y,w} q_y)$ and $\left(\varepsilon_y \varepsilon_w R_{x,w} q_w^{-1}\right)$ are inverses of each other too.

Proposition 3.4.2. *Let $\flat : \mathcal{H} \to \mathcal{H}$ be the anti-automorphism of the algebra $\mathcal{H}$ in Proposition 3.1.1, that is,*

$$\begin{aligned} \flat : \mathcal{H} &\to \mathcal{H} \\ T_w &\mapsto T_{w^{-1}} \quad (w \in W). \end{aligned}$$

The bar involution $^{-} : \mathcal{H} \to \mathcal{H}$ of $\mathcal{H}$ and $\flat : \mathcal{H} \to \mathcal{H}$ commute.

Proof. First note that

$$\overline{\flat\left(T_w\right)} = \overline{T}_{w^{-1}} = T_w^{-1} = \flat\left(T_{w^{-1}}^{-1}\right) = \flat\left(\overline{T}_w\right) \tag{3.4.11}$$

for any element $w \in W$. Therefore, since for any element

$$\sum_{w \in W} c_w T_w \in \mathcal{H} \quad \text{where } c_w \in \mathscr{A} \text{ for all } w \in W,$$

we have, by the $\mathscr{A}$-linearity of $\flat : \mathcal{H} \to \mathcal{H}$, the $\mathscr{A}$-semilinearity of $^{-} : \mathcal{H} \to \mathcal{H}$ and the addition preserving property of both, that

$$\overline{\flat\left(\sum_{w \in W} c_w T_w\right)} = \sum_{w \in W} \overline{c_w \flat\left(T_w\right)} = \sum_{w \in W} \overline{c}_w\, \overline{\flat\left(T_w\right)},$$

and

$$\flat\left(\overline{\sum_{w \in W} c_w T_w}\right) = \flat\left(\sum_{w \in W} \overline{c}_w \overline{T}_w\right) = \sum_{w \in W} \overline{c}_w \flat\left(\overline{T}_w\right),$$

it follows, by (3.4.11), that

$$\overline{\flat\left(\sum_{w \in W} c_w T_w\right)} = \flat\left(\overline{\sum_{w \in W} c_w T_w}\right),$$

which shows the commutativity of $^{-} : \mathcal{H} \to \mathcal{H}$ of $\mathcal{H}$ and $\flat : \mathcal{H} \to \mathcal{H}$. □

3.5 Kazhdan-Lusztig basis

In this section we present a new basis $(C_w)_{w\in W}$ of the $\mathscr{A}$-module $\mathcal{H}$, called the *Kazhdan-Lusztig basis of* $\mathcal{H}$, which was introduced for the first time by Kazhdan and Lusztig in [24]. An important feature of such basis is the fact that each basis element is fixed under the bar involution $^{-} : \mathcal{H} \to \mathcal{H}$, and this property is one of the properties that uniquely determine the Kazhdan-Lusztig polynomials. First we state the basic theorem of Kazhdan and Lusztig [24] about the existence a uniqueness of the elements $(C_w)_{w\in W}$ fixed under the bar involution of $\mathcal{H}$. Then we make some remarks about the statement of the theorem and give a proof for it. We also prove the fact that $(C_w)_{w\in W}$ forms a basis of $\mathcal{H}$ as an $\mathscr{A}$-module. Finally, we conclude this section by proving some key properties of the Kazhdan-Lusztig polynomials and giving some examples.

Theorem 3.5.1. *For any $w \in W$, there is a unique element $C_w \in \mathcal{H}$ such that*

$$\overline{C}_w = C_w \tag{3.5.1}$$

$$C_w = \varepsilon_w q_w^{1/2} \sum_{\substack{y\in W \\ y\leqslant w}} \varepsilon_y q_y^{-1} \overline{P}_{y,w} T_y \tag{3.5.2}$$

where $P_{y,w} \in \mathbb{Z}[q]$ is a polynomial in q satisfying $P_{w,w} = 1$ and

$$\deg_q P_{y,w} \leq \frac{1}{2} \left(\ell(w) - \ell(y) - 1\right) \ \text{for } y < w. \tag{3.5.3}$$

The polynomials $P_{y,w}$ are called the Kazdan-Lusztig polynomials. For short, we refer to them as KL-polynomials.

Remark 3.5.1. First, the image of T_1 under the bar involution of $\mathcal{H}$ is T_1, so if we define $C_1 := T_1$, we see that $\overline{C}_1 = C_1$ and (3.5.2) trivially hold. Moreover, if we consider the image of the element $(T_s - qT_1) \in \mathcal{H}$ under the bar involution, then, using (3.3.3), we get

$$\overline{T_s - qT_1} = T_s^{-1} - q^{-1}T_1 = q^{-1} \left(T_s - qT_1\right).$$

Therefore, if we define

$$C_s := q^{-1/2} \left(T_s - qT_1\right), \tag{3.5.4}$$

we see that $\overline{C}_s = C_s$, and C_s can be written in the form

$$C_s = -q^{1/2} \left(-q^{-1}T_s + T_1\right)$$

as in (3.5.2) with $P_{s,s} = 1 = P_{1,s}$, so that (3.5.3) also holds.

Remark 3.5.2. Given the way canonical basis $(T_w)_{w\in W}$ of $\mathcal{H}$ is constructed from $(T_s)_{s\in S}$, we could consider building up the new basis $(C_w)_{w\in S}$ consisting of elements fixed under the bar involution from $(C_s)_{s\in S}$ in the same way, where C_s is as in (3.5.4). So let $s, s' \in S$ be distinct. Then, using our definition for C_s and $C_{s'}$ in (3.5.4) we compute

$$C_{s'}C_s = q^{-1} \left(T_{s'} - qT_1\right) \left(T_s - qT_1\right) = q^{-1}(T_{s'}T_s - qT_{s'} - qT_s + q^2T_1). \tag{3.5.5}$$

But recall that, since $s \neq s'$, we have $T_{s'}T_s = T_{s's}$, so (3.5.4) becomes

$$C_{s'}C_s = q^{-1}(T_{s's} - qT_{s'} - qT_s + q^2T_1),$$

and we could label this $C_{s's}$. Note that the only other possible length two reduced expression consisting of s and s' is ss', so if $ss' = s's$, then we would have obtained the same element from the product $C_sC_{s'}$. Unfortunately, for reduced expressions consisting of several s and s' of length greater that two, such construction does not work as well. For example, suppose $\ell(ss's) = 3$. Proceeding as before we would have

$$C_sC_{s'}C_s = q^{-3/2}\left(T_s - qT_1\right)\left(T_{s's} - qT_{s'} - qT_s + q^2T_1\right),$$

which, after expanding out the brackets, substituting the relation (3.3.2) for T_s^2 and collecting equal terms, becomes

$$C_sC_{s'}C_s = q^{-3/2}\left(T_{ss's} - qT_{s's} - qT_{ss'} + q^2T_{s'} + q^2(1+q^{-1})T_s - q^3(1+q^{-1})T_1\right).$$

Now note that if we want to label the element in (3.5.2) as $C_{ss's}$, we would face a problem in the case when $ss's = s'ss'$, as

$$C_{s'}C_sC_{s'} = q^{-3/2}\left(T_{s'ss'} - qT_{ss'} - qT_{s's} + q^2T_s + q^2(1+q^{-1})T_{s'} - q^3(1+q^{-1})T_1\right),$$

and thus $C_sC_{s'}C_s \neq C_{s'}C_sC_{s'}$. But,

$$C_sC_{s'}C_s - C_s = q^{-3/2}\left(T_{ss's} - qT_{s's} - qT_{ss'} + q^2T_{s'} + q^2T_s - q^3T_1\right)$$

is another element of $\mathcal{H}$ that is fixed under the bar involution, the coefficients of the T_x for $x \leqslant ss's$ are simpler than those of the T_x for $x \leqslant ss's$ in the expression for $C_sC_{s'}C_s$, and in the case when $ss's = s'ss'$ we have

$$C_sC_{s'}C_s - C_s = C_{s'}C_sC_{s'} - C_{s'},$$

eliminating our earlier problem. We thus see that building up the new basis in the same way the canonical basis $(T_w)_{w\in W}$ is constructed from the T_s ($s \in S$) is not entirely a good idea, but this can be solved by introducing some correction term. In particular, note that the only correction term needed in our example was C_s, which is the element C_x corresponding to the unique $x \in W$ such that $x \leqslant ss's$ and

$$\deg_q\left(q_{ss's}^{1/2}q_x^{-1}Q_{x,ss's}\right) > \frac{1}{2}\left(\ell(ss's) - \ell(x) - 1\right),$$

where $Q_{x,ss's}$ denotes the coefficient of T_x in the expression for $C_sC_{s'}C_s$. Moreover, note that such x is the unique $x \in W$ such that $x \leqslant ss's$ and $x \leqslant s's$ and $sx < x$. It thus make sense to define the following, which will be explored in the proof of Theorem 3.5.1.

Definition 3.5.1. Given $x, w \in W$ we say that $x \prec w$ if $x < w$, $\varepsilon_x = -\varepsilon_w$, and

$$\deg_q P_{x,w} = \frac{1}{2}\left(\ell(w) - \ell(x) - 1\right).$$

In this case, the non-zero integer coefficient of the highest power of q in $P_{x,w}$ is denoted by $\mu(x,w)$. If $w \prec x$ we set $\mu(w,x) = \mu(x,w)$.

Remark 3.5.3. As for R-polynomials, it is useful to make the convention that $P_{x,w} = 0$ whenever $x \not\leqslant w$. Nevertheless, the statement of Theorem 3.5.1 already shows a way in which the KL-polynomials are significantly more subtle than the R-polynomials as it shows that the precise degrees of the KL-polynomials are not predictable, whereas the precise degree for the R-polynomials are known, by Proposition 3.3.2.

Proof of Theorem 3.5.1. We first assume that, for each $w \in W$, the element

$$C_w = \sum_{\substack{x\in W\\ x\leqslant w}} \jmath(x,w)\overline{P}_{x,w}T_x \quad \text{where } \jmath(x,w) := \varepsilon_w\varepsilon_x q_w^{1/2}q_x^{-1} \tag{3.5.6}$$

satisfying properties (3.5.1) and (3.5.3) exists, and prove its uniqueness. Now, note that given (3.5.6), the equality $\overline{C}_w = C_w$ can be written as

$$\sum_{\substack{y \in W \\ y \leqslant w}} \varepsilon_w \varepsilon_y q_w^{-1/2} q_y P_{y,w} T_{y^{-1}}^{-1} = \sum_{\substack{x \in W \\ x \leqslant w}} \jmath(x,w) \overline{P}_{x,w} T_x. \tag{3.5.7}$$

Then substituting for $T_{y^{-1}}^{-1}$ the expression in (3.3.6) with y in place of w, the equality in (3.5.7) can also be written as

$$\sum_{\substack{y \in W \\ y \leqslant w}} \varepsilon_w \varepsilon_y q_w^{-1/2} q_y P_{y,w} \sum_{\substack{x \in W \\ x \leqslant y}} \varepsilon_y q_y^{-1} \varepsilon_x R_{x,y} T_x = \sum_{\substack{x \in W \\ x \leqslant w}} \jmath(x,w) \overline{P}_{x,w} T_x,$$

which, after the appropriate cancellations becomes

$$\varepsilon_w q_w^{-1/2} \sum_{\substack{y \in W \\ y \leqslant w}} \sum_{\substack{x \in W \\ x \leqslant y}} \varepsilon_x T_x P_{y,w} R_{x,y} = \sum_{\substack{x \in W \\ x \leqslant w}} \jmath(x,w) \overline{P}_{x,w} T_x. \tag{3.5.8}$$

Hence, proving the uniqueness of the element C_w amounts to showing that the polynomials $P_{x,w}$ can be chosen in exactly one way for all $x \in W$. We show this by induction on $\ell(w)-\ell(x)$, starting with the condition $P_{w,w} = 1$ for the case the lenght difference is zero. So let $\ell(w) - \ell(x) \geq 1$ and assume that $P_{y,w}$ is uniquely determined for each $y \in W$ such that $x < y \leqslant w$. Comparing coefficients on both sides of the equality in (3.5.8) of a fixed T_x for each $x \leqslant w$, gives

$$\varepsilon_w q_w^{-1/2} \sum_{\substack{y \in W \\ x \leqslant y \leqslant w}} \varepsilon_x P_{y,w} R_{x,y} = \jmath(x,w) \overline{P}_{x,w} \quad \text{for all } x < w, \tag{3.5.9}$$

which is equivalent to

$$q_w^{-1/2} \sum_{\substack{y \in W \\ x < y \leqslant w}} P_{x,w} R_{x,y} = q_w^{1/2} q_x^{-1} \overline{P}_{x,w} - q_w^{-1/2} P_{x,w} R_{x,x} \quad \text{for all } x < w.$$

Multiplying both sides by $q_x^{1/2}$ and using the fact that $R_{x,x} = 1$, the above becomes

$$q_w^{-1/2} q_x^{1/2} \sum_{\substack{y \in W \\ x < y \leqslant w}} P_{y,w} R_{x,y} = q_w^{1/2} q_x^{-1/2} \overline{P}_{x,w} - q_w^{-1/2} q_x^{1/2} P_{x,w} \quad \text{for all } x < w. \tag{3.5.10}$$

Now, by the degree assumption on (b) of the theorem, we know that

$$\ell(x) - \ell(w) \leq \deg_{q^{1/2}} \left(q_w^{-1/2} q_x^{1/2} P_{x,w} \right) \leq -1,$$

so since $x < w$, it follows that $q_w^{-1/2} q_x^{1/2} P_{x,w}$ is a polynomial in $q^{-1/2}$ without constant term. Similarly, we know that

$$1 \leq \deg_{q^{1/2}} \left(q_w^{1/2} q_x^{-1/2} \overline{P}_{x,w} \right) \leq \ell(w) - \ell(x),$$

so since $x < w$, it follows that $q_w^{1/2} q_x^{-1/2} \overline{P}_{x,w}$ is a polynomial in $q^{1/2}$ without constant term. Hence, there cannot be any cancellations between these two terms, and there is at most one choice for $P_{x,w}$ satisfying (3.5.10). The uniqueness part of the proof has now been dealt with.

We now deal with the existence of the elements C_w ($w \in W$) satisfying the properties in the theorem. We proceed by induction en the length of $\ell(w)$ of W, and the base case is dealt with in

Remark 3.5.1. So suppose $\ell(w) \geq 1$ and assume that the existence of $C_{w'}$ satisfying (3.5.1) and (3.5.2) has already been proved for $w' \in W$ such that $\ell(w') < \ell(w)$. Now, since $\ell(w) \geq 1$, there is some $s \in S$ such that $w = sv$ and $\ell(v) = \ell(w) - 1$, and so by the induction hypothesis, C_v has already been constructed and we may thus apply Definition 3.5.1 to C_v so that $z \prec v$ and the corresponding integer $\mu(z, v)$ have a meaning. Now define

$$C_w := C_s C_v - \sum_{\substack{z \in W \\ z \prec v \\ sz < z}} \mu(z, v) C_z. \tag{3.5.11}$$

Note that C_w is fixed by the bar involution since from Remark 3.5.1 we know that $\overline{C}_s = C_s$ and by the induction hypothesis we know that $\overline{C}_v = C_v$ and $\overline{C}_z = C_z$ for all $z \in W$ such that $z \prec v$. Moreover, substituting (3.5.4) for C_s into (3.5.11) gives

$$C_w = q^{-1/2} \left(T_s - qT_1\right) C_v - \sum_{\substack{z \in W \\ z \prec v \\ sz < z}} \mu(z, v) C_z, \tag{3.5.12}$$

so we see that C_w is an $\mathscr{A}$-linear combination of element T_x such that $x \leqslant w$. At this point, we are only left to prove that C_w as defined in (3.5.11) can be written as in (3.5.2) with the $P_{y,w}$ satisfying the degree property in (3.5.3). We prove this by analyzing the coefficient of T_y for each fixed y in (3.5.12). First, if $y = w$, then since $v < sv = w$, the element T_w can only occur in the product $T_s C_v$, and it will appear with coefficient

$$q^{-1/2} \jmath(v, v) \overline{P}_{v,v} = q^{-1/2} q_v^{1/2} q_v^{-1} = q^{(-1/2)(\ell(v)+1)} = q_w^{-1/2},$$

which agrees with (3.5.2), with $P_{w,w} = 1$. Next, take any $y < w$. In this case T_y can occur in $C_s C_v$ either directly in C_v or indirectly in $T_s C_v$ when T_s is multiplied by T_{sx}. The former occurs if $y \leqslant v$ and the latter occurs if $sy \leqslant v$, and we should distinguish between the two possibilities $y < sy$ and $sy < y$.

If $y < sy$, then $s \cdot sy = y < sy \leqslant v$ and so, by the second relation in (3.2.4) of Proposition 3.2.2, we have

$$T_s T_{sy} = (q - 1) T_{sy} + q T_y.$$

Hence T_y appears in $q^{-1/2} T_s C_v$ with coefficient

$$q^{-1/2} q \jmath(sy, v) \overline{P}_{sy,v} = -q^{-1/2} \jmath(y, v) \overline{P}_{sy,v} = q^{-1} \jmath(y, w) \overline{P}_{sy,v}. \tag{3.5.13}$$

In this case T_y also appears in $q^{1/2} T_1 C_v$ with coefficient

$$-\, q^{1/2} \jmath(y, v) \overline{P}_{y,v} = \jmath(y, w) \overline{P}_{y,v}. \tag{3.5.14}$$

Combining (3.5.13) and (3.5.14), we see that the coefficient of T_y in $C_s C_v$ is

$$q^{-1} \jmath(y, w) \overline{P}_{sy,v} + \jmath(y, w) \overline{P}_{y,v}.$$

If on the contrary $sy < y$, then $sy < y = s \cdot sy$ and so, by the relations in (3.2.4) of Proposition 3.2.2, we have

$$T_s T_{sy} = T_y \quad \text{and} \quad T_s T_y = (q - 1) T_y + q T_1.$$

Hence T_y appears twice in $-q^{-1/2} T_s C_v$, each of the times with coefficient

$$q^{-1/2} \jmath(sy, v) \overline{P}_{sy,v} = -q^{1/2} \jmath(y, v) \overline{P}_{sy,v} = \jmath(y, w) \overline{P}_{sy,v}, \tag{3.5.15}$$

and

$$q^{-1/2} (q - 1) \jmath(y, v) \overline{P}_{y,v} = (q^{-1} - 1) \jmath(y, w) \overline{P}_{y,v} \tag{3.5.16}$$

In this case T_y also appears in $-q^{1/2}T_1C_v$ with coefficient

$$-\,q^{1/2}\jmath(y,v)\overline{P}_{y,v} = \jmath(y,w)\overline{P}_{y,v}. \tag{3.5.17}$$

Combining (3.5.15), (3.5.16) and (3.5.17), we see that the coefficient of T_y in C_sC_v is

$$\jmath(y,w)\overline{P}_{sy,v} + q^{-1}\jmath(y,w)\overline{P}_{y,v}.$$

Finally, since $\varepsilon_z\varepsilon_w = 1$ whenever $z \in W$ is such that $z \prec v$, it follows that the coefficient of T_y in the last sum in (3.5.12) is always of the form

$$-\sum_{\substack{z\in W\\ z\prec v\\ sz<z}} \mu(z,v)\jmath(y,z)\overline{P}_{y,z} = -\sum_{\substack{z\in W\\ z\prec v\\ sz<z}} \mu(z,v)q_w^{-1/2}q_z^{1/2}\jmath(y,w)\overline{P}_{y,z}.$$

We now combine these calculations to express C_w as in (3.5.2), where

$$P_{y,w} := q^{1-c}P_{sy,v} + q^cP_{y,v} - \sum_{\substack{z\in W\\ z\prec v\\ sz<z}} \mu(z,v)q_w^{1/2}q_z^{-1/2}P_{y,z} \tag{3.5.18}$$

and

$$c = \begin{cases} 0 & \text{if } y < sy \\ 1 & \text{if } sy < y, \end{cases}$$

with the convention that $P_{y,z} = 0$ whenever $y \nleq z$. Now, in the case when $y < sy$ and thus $c = 0$, we obtain, by the induction hypothesis, that

$$\deg_q P_{y,w} \leq 1 + \frac{1}{2}\left(\ell(v) - \ell(sy) - 1\right) = \frac{1}{2}\left(\ell(w) - \ell(y) - 1\right).$$

On the other hand, in the case when $sy < y$ and thus $c = 1$, we obtain, by the induction hypothesis, that

$$\deg_q P_{sy,v} \leq \frac{1}{2}\left(\ell(v) - \ell(sy) - 1\right) = \frac{1}{2}\left(\ell(w) - \ell(y) - 1\right),$$

$$\max_{\substack{z\in W\\ z\prec v\\ sz<z}} \deg_q\left(\mu(z,v)q_w^{-1/2}q_z^{1/2}P_{y,z}\right) \leq \max_{\substack{z\in W\\ z\prec v\\ sz<z}} \frac{1}{2}\left(\ell(w) - \ell(y) - 1\right) = \frac{1}{2}\left(\ell(w) - \ell(y) - 1\right).$$

Moreover, we have

$$\deg_q\left(qP_{y,v}\right) \leq 1 + \frac{1}{2}\left(\ell(v) - \ell(y) - 1\right) = \frac{1}{2}\left(\ell(w) - \ell(y)\right). \tag{3.5.19}$$

But note that if the bound in (3.5.19) is tight, then, since in this case we have $y \prec z$ as $sy < y$, there is a term for $z = y$ in the last sum in (3.5.18), namely

$$\mu(y,v)q_w^{1/2}q_y^{-1/2}P_{y,y} = \mu(y,v)q_w^{1/2}q_y^{-1/2},$$

which is equal to the highest degree term of $qP_{y,v}$, and thus cancels it. Since the the last sum in (3.5.18) involves $P_{y,y}$ only there, there are no further powers of q equal to the bound in (3.5.19) terms in the sum, it follows that

$$\deg_q P_{y,w} \leq \frac{1}{2}\left(\ell(w) - \ell(y) - 1\right)$$

as required. The proof is now complete. □

Remark 3.5.4. The the proof of Theorem 3.5.1 gives very explicit information about the KL-polynomials and the formula in (3.5.18) as well as that in (3.5.9) can be used to compute them. Applying such formula repeatedly by hand for rank greater than three can become unmanageable rapidly. However, such formula is well suited for computers, and the only limit on computing KL-polynomials is that for large rank, the number of polynomials becomes too large, exceeding the storage capacity of computers.

Remark 3.5.5. The proof of Theorem 3.5.1 also gives explicit information about how multiplication of the elements C_w ($w \in W$) works. We say more of this in Section 3.6.

Remark 3.5.6. The proof of Theorem 3.5.1 we gave follows the original proof of Kazhdan and Lusztig in [24], making some of the steps there more explicit. However, Lusztig outlines a more elegant proof later in [28]. The idea is to reverse the steps of the uniqueness part of the proof of Theorem 3.5.1 by showing, inductively on $\ell(w) - \ell(x)$, that the equation in (3.5.10) can be solved for $P_{x,w}$. Since applying the bar involution $^{-} : \mathcal{H} \to \mathcal{H}$ of $\mathcal{H}$ to the right-hand side of the equation in (3.5.10) just changes the sign, the key of the proof lies in showing that the same is true for the left-hand side. He applies the bar involution to the left-hand side, then, by induction, substitutes for $\overline{P}_{y,w}$ the already known formula of the type in (3.5.9), and finishes by using the inversion formula for R-polynomials in (*b*) of Proposition 3.4.1. Once this is shown, the bound of the degree of $P_{x,w}$ follows and the elements C_w fixed under the bar involution can be defined as in (3.5.6). In his proof in [28], Lusztig actually uses a different basis $(\widetilde{T}_w)_{w\in W}$ which he conveniently introduces at the beginning of his paper as

$$\widetilde{T}_w := q_w^{-1/2} T_w \quad \text{for each } w \in W,$$

for which the conditions in (3.3.1) and (3.3.2) give

$$\widetilde{T}_w \widetilde{T}_{w'} = \widetilde{T}_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w') \tag{3.5.20}$$

$$\left(\widetilde{T}_s + q_w^{-1/2}\right)\left(\widetilde{T}_s - q_w^{1/2}\right) = 0 \qquad \text{if } s \in S \tag{3.5.21}$$

The steps of the are essentially the same which ever basis is used, as every formula used in terms of $(T_w)_{w\in W}$ can be easily written in terms of $(\widetilde{T}_w)_{w\in W}$. Moreover, the elements of $(\widetilde{T}_w)_{w\in W}$ are also invertible, with

$$\widetilde{T}_s^{-1} = \widetilde{T}_s + \left(q^{-1/2} - q^{1/2}\right), \tag{3.5.22}$$

and such that

$$\overline{\widetilde{T}}_w = \overline{q_w^{-1/2} T_w} = q_w^{1/2} T_w^{-1} = \widetilde{T}_{w^{-1}}^{-1}.$$

With respect to this basis, we can write the elements C_w in Theorem 3.5.1 using (3.5.2) as

$$C_w = \varepsilon_w q_w^{1/2} \sum_{\substack{y\in W\\ y\leqslant w}} \varepsilon_y q_y^{-1/2} \overline{P}_{y,w} \widetilde{T}_y = \widetilde{T}_w + \varepsilon_w q_w^{1/2} \sum_{\substack{y\in W\\ y< w}} \varepsilon_y q_y^{-1/2} \overline{P}_{y,w} \widetilde{T}_y, \tag{3.5.23}$$

and thus, using the degree property in (3.5.3) of Theorem 3.5.1, we see that

$$C_w \in \widetilde{T}_w + q^{1/2} \sum_{\substack{y\in W\\ y< w}} \mathscr{A}^{+} \cdot \widetilde{T}_y. \tag{3.5.24}$$

Proposition 3.5.1. *The collection $(C_w)_{w\in W}$ forms a basis of $\mathcal{H}$ as an $\mathscr{A}$-module, called the KL-basis of $\mathcal{H}$.*

Proof. They collection $(C_w)_{w\in W}$ is a generating set as each C_w is a $\mathscr{A}$-linear combination of the standard basis elements T_w. The linear independence also follows from this. □

Proposition 3.5.2. (Exercise 7.11 in [21]) *If $x, w \in W$ are elements such that $x \leqslant w$, then $P_{x,w}(0) = 1$.*

Proof. To prove this we use the formula in (3.5.18) together with induction on $\ell(w)$. First, if $\ell(w) = 0$ then $w = 1$ and thus $x \leqslant 1$ implies that $x = w = 1$. In this case $P_{x,w}(0) = P_{1,1}(0)$ trivially holds. If $\ell(w) = 1$, the $w = s'$ for some $s' \in S$ and thus, the unique $s \in S$ such that $ss' < s'$ is $s = s'$ itself. In this case we have $x \leqslant w = s$ implies that $x = 1$ or $x = s$. If $x = s$, then $P_{x,w}(0) = P_{s,s}(0) = 1$ trivially holds, and if $x = 1$, then $1 = x < s'x$ implies that $c = 0$ in (3.5.18) and we have

$$P_{x,w}(0) = P_{1,s'}(0) = P_{1,s'w}(0) - \sum_{\substack{z \in W \\ z \prec s'w \\ s'z<z}} \mu(z, s'w) q_{s'}^{1/2} q_z^{-1/2} P_{1,z}(0). \tag{3.5.25}$$

But $s'w = 1$, so there is no $z \in W$ such that $z \prec s'w$, the last sum in (3.5.25) is zero and $P_{1,1}(0) = 1$, and thus $P_{x,w}(0) = 1$ also holds in this case.

Now, let $x, w \in W$ be such that $\ell(w) > 1$ and $x \leq w$, and assume that if $y, w' \in W$ are such that $y \leqslant w'$ then $P_{y,w'}(0) = 1$ holds for all $y, w' \in W$ with $\ell(w') < \ell(w)$. Since $\ell(w) > 1$, there is some $s \in S$ such that $w = sv$ where $v \in W$ such that $\ell(v) = \ell(w) - 1$. Then, according to (3.5.18), we have

$$P_{x,w}(0) = 0^{1-c} P_{sx,v} + 0^c P_{x,v} - \sum_{\substack{z \in W \\ z \prec v \\ sz<z}} \mu(z, v) 0^{(1/2)\ell(w)} 0^{(-1/2)\ell(z)} P_{x,z}(0), \tag{3.5.26}$$

where $c = 0$ is $x < sx$ and $c = 1$ if $sx < x$. First note that since $\ell(w) - \ell(z) \geq 1$ for any $z \in W$ such that $z \prec v$, we have that

$$0^{(1/2)\ell(w)} 0^{(-1/2)\ell(z)} = 0,$$

and moreover, since $\ell(z) < \ell(w)$, it follows, by induction, that $P_{x,z}(0) = 1$ for all $z \in W$ such that $z \prec v$, $sz < z$ and $x \leqslant z$. We thus see that the last sum in (3.5.26) is zero, and thus the equality becomes

$$P_{x,w}(0) = 0^{1-c} P_{sx,v}(0) + 0^c P_{x,v}(0). \tag{3.5.27}$$

Now, if $sx < x$ and thus $c = 0$, the equality in (3.5.27) becomes $P_{x,w}(0) = P_{x,v}(0)$. Since $\ell(v) < \ell(w)$, it follows, by induction, that $P_{x,w}(0) = 1$, as required. If on the contrary $x < sx$ and thus $c = 1$, the equality in (3.5.27) becomes $P_{x,w}(0) = P_{sx,v}(0)$. Again, since $\ell(v) < \ell(w)$, it follows, by induction, that $P_{x,w}(0) = 1$, as required. This completes the proof. □

Remark 3.5.7. Proposition 3.5.2 shows that the constant term of the KL-polynomial $P_{x,w}$ is precisely 1 whenever $x, w \in W$ are such that $x \leqslant w$. The proof of Proposition 3.5.2 serves as another example of a way in which the KL-polynomials are significantly more subtle than the R-polynomials. In particular, in the proof of Proposition 3.3.3 we were able to compute in a straightforward way the constant term for R-polynomials, but the proof of Proposition 3.5.2 required more involved tools, such as the recursive formula in (3.5.18).

Corollary 3.5.1. (Exercise 7.11 in [21]) *If $x, w \in W$ are elements such that $x \leqslant w$ and $l(w) - l(x) \leq 2$, then $P_{x,w}(q) = 1$.*

Proof. By Theorem 3.5.1, we know that

$$\deg_q P_{x,w} \leq \frac{1}{2} (\ell(w) - \ell(x) - 1) \leq \frac{1}{2},$$

where the last inequality follows from the hypothesis $\ell(w) - \ell(x) \leq 2$. Therefore, since $P_{x,y} \in \mathbb{Z}[q]$, it follows that $P_{x,y}$ must be a constant polynomial. Hence since by Proposition 3.5.2 the constant term of any KL-polynomial is 1, we deduce that $P_{x,w} = 1$, as required. □

Example 3.5.1. Suppose W is a dihedral group with $S = \{s, s'\}$. We show, by induction on $\ell(w) \geq 0$, that $P_{x,w} = 1$ for all $x, w \in W$ such that $x \leqslant w$. We also use Example 2.10.1 where we showed that $x < w$ if and only if $\ell(x) < \ell(w)$. First, if $\ell(w) = 0$ this is trivial since $w = 1$ and thus if $x \in W$ is such that $x \leqslant w = 1$ then $x = 1$. Next, if $\ell(w) = 1$, then $w = s$ or $w = s'$. In either case, if $x \in W$ is such that $x \leqslant w$ then $x \in \{1, w\}$, and thus $\ell(w) - \ell(x) \leq 1$. Hence, by Corollary 3.5.1, it follows that $P_{x,w} = 0$, as required.

So suppose $x, w \in W$ are such that $r = \ell(w) > 1$ and $x \leqslant w$, and assume that $P_{y,w'} = 1$ holds for any elements $y, w' \in W$ such that $\ell(w') < \ell(w)$ and $y \leqslant w'$. If $x = w$, we already have $P_{x,w} = 1$, so suppose that $x < w$. Now, since any reduced expression of w is of the form

$$w = \mathsf{prod}(r; s, s') \quad \text{or} \quad w = \mathsf{prod}(r; s', s),$$

we may assume, without loss of generality, that $w = \mathsf{prod}(r; s, s')$, and let $s'' \in S$ be an element of S such that

$$w = s'' \cdot \mathsf{prod}(r - 1; s, s'),$$

with $\ell(s''w) = r - 1$, and let $u := \mathsf{prod}(r - 1; s, s')$. Note that if W is infinite, then such element $s'' \in S$ would be unique, and if W is finite, such element $s'' \in S$ would also be unique unless $r = m_{ss'}$, in which case we would have $w = w_\circ$ is the longest element of W. We therefore let s'' the be the leftmost element of the reduced expression $w = \mathsf{prod}(r; s, s')$ of w. The formula in (3.5.18) then gives

$$P_{x,w} = q^{1-c} P_{s''x,u} + q^c P_{x,u} - \sum_{\substack{z \in W \\ z \prec u \\ s''z < z}} \mu(z, u) q^{r/2} q_z^{-1/2} P_{x,z}, \tag{3.5.28}$$

with $c = 0$ is $x < s''x$ and $c = 1$ if $s''x < x$. Now, by Definition 3.5.1, we have that $s \in W$ is such that $z \prec v$ if $z < u$, $\varepsilon_x = -\varepsilon_u$ and

$$\deg_q P_{z,u} = \frac{1}{2} (\ell(u) - \ell(z) - 1) = \frac{1}{2} (r - 1 - \ell(z) - 1) = \frac{1}{2} (r - \ell(z) - 2).$$

But since $\ell(u) < \ell(w)$, the induction hypothesis tells us that $P_{z,u} = 1$ for all $z \in W$ such that $z < u$ and thus $z \prec u$ only if $\ell(z) = r - 2 = \ell(u) - 1$. In particular we have $\mu(z, u) = 1$. Therefore, since the are exactly two elements of W of length $r - 2$, one whose unique reduced expression starts with s and the other whose unique reduced expression starts with s', it follows that there is a unique $z \in W$ such that $\ell(z) = r - 2$ and $s''z < z$, namely $z = \mathsf{prod}(r - 2; s, s')$, and the formula in (3.5.28) becomes

$$P_{x,w} = q^{1-c} P_{s''x,u} + q^c P_{x,u} - q^{(1/2)(r-r+2)} P_{x,\mathsf{prod}(r-2;s,s')}. \tag{3.5.29}$$

Now, if $s''x < x$ so that $c = 1$ in (3.5.29), then $\ell(s''x) < \ell(x) < \ell(w)$, and thus

$$\ell(s''x) < \ell(x) \leq \ell(u),$$

which implies that $s''x < x \leqslant u$. But $u < s''u$ so $x \neq u$ and we actually have $s''x < x < u$ and thus $x \leqslant z$. Hence, by the induction hypothesis, the formula in (3.5.29) gives

$$P_{x,w} = 1 + q - q = 1,$$

as required.

If on the contrary $x < s''x$ so that $c = 0$ in (3.5.29), then $\ell(x) < \ell(s''x) \leq \ell(w)$, and we have two possibilities: either $\ell(s''x) = \ell(w)$ or $\ell(s''x) < \ell(w)$. If $\ell(s''x) = \ell(w)$, then

$< r-2 < \ell(u) = \ell(x) < \ell(w)$, which implies, since also $u < s''u$, that $\mathsf{prod}(r-2; s, s') < u = x < s''x$. Hence, by convention, we have

$$P_{s''x,u} = 0 = P_{x,\mathsf{prod}(r-2;s,s')},$$

so the formula in (3.5.29) becomes $P_{x,w} = 1$. On the other hand, if $\ell(s''x) < \ell(w)$, then $\ell(x) < \ell(s''x) \leq \ell(u)$, which implies that $x < s''x \leqslant u$. But $u < s''u$ and $s'' \cdot s''x = x < s''x$ so $s''x \neq u$ and we actually have $x < s''x < u$ and thus $\ell(x) \leq r-3$. Hence, by the induction hypothesis, since $r-2 < \ell(u) < \ell(w)$, the formula in (3.5.29) gives

$$P_{x,w} = q + 1 - q = 1,$$

as required. This completes the proof.

Remark 3.5.8. Example 3.5.1 shows, in particular, that for every rank 2 Coxeter group the KL-polynomials are either equal to 1 or equal to 0.

Example 3.5.2. In rank 3, the Coxeter group $\mathcal{S}_4$ does give KL-polynomials different from 1. Recall that in this case, if we let $S = \{s_1, s_2, s_3\}$, with $s_1 = (12)$, $s_2 = (23)$ and $s_3 = (34)$, then $\mathcal{S}_4$ has a presentation given by

$$\mathcal{S}_4 = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_1)^2 = 1 \right\rangle.$$

These are

$$P_{s_2,s_2s_1s_3s_2} = q + 1 = P_{s_1s_3,s_1s_3s_2s_3s_1}.$$

Note that in both cases we have $x \prec w$ but $\ell(w) - \ell(x) > 1$.

Remark 3.5.9. Kazhdan and Lusztig conjectured in [24] that the KL-polynomials have non-negative coefficients. The conjecture was proved in [25] in the case when the underlying Coxeter group is a Weyl group or an affine Weyl group. It was proved in the general case in [11].

3.6 Action of the canonical basis on the Kazhdan-Lusztig basis

In this section we use the results obtained in Section 3.5 to study how the elements T_s $(s \in S)$ act on the KL-basis of $\mathcal{H}$. From this, we then derive a crucial property of the KL-polynomials. We proceed as in [24].

Proposition 3.6.1. *Let $s \in S$ and $w \in W$. Then*

$$T_sC_w = \begin{cases} -C_w & \text{if } sw < w, \quad (3.6.1) \\ qC_w + q^{1/2}C_{sw} + q^{1/2} \sum_{\substack{z \in W \\ z \prec w \\ sz < z}} \mu(z,w)C_z & \text{if } w < sw. \quad (3.6.2) \end{cases}$$

Proof. First, we prove (3.6.2). Note that if $s, w \in W$ are such that $w < sw$, then rearranging (3.5.11) with w in place of v and sw in place of w, we obtain

$$q^{-1/2}T_sC_w = C_{sw} + q^{1/2}C_w + \sum_{\substack{z \in W \\ z \prec w \\ sz < z}} \mu(z,w)C_z,$$

so multiplying both sides by $q^{1/2}$ gives

$$T_sC_w = q^{1/2}C_{sw} + qC_w + q^{1/2} \sum_{\substack{z \in W \\ z \prec w \\ sz < z}} \mu(z,w)C_z,$$

which is exactly (3.6.2).

We now prove (3.6.1). In this case, since $sw < w$, we know that $\ell(w) \geq 1$, and we proceed by induction on $\ell(w)$. If $\ell(w) = 1$, then $w = s$ since the only element of S whose length is reduced after left multiplication by s is s itself. We can then directly compute T_sC_s using the definition of C_s in (3.5.4) to get

$$T_sC_s = q^{-1/2}T_s^2 - q^{1/2}T_s.$$

Then, substituting the relation in (3.3.2) for T_s^2 and collecting equal terms we get

$$T_sC_s = q^{-1/2}\,(q-1)\,T_s + q^{1/2}T_1 - q^{1/2}T_s = -q^{-1/2}\,(T_s - qT_1) = -C_s,$$

as required.

Now, suppose that $\ell(w) > 1$ and assume that (3.6.1) holds for any $s', w' \in W$ such that $s'w' < w'$ and $\ell(w') < \ell(w)$. Since $sw < w = s \cdot sw$, we are in the case just proven and (3.6.2) applies with s and sw. We therefore have

$$T_sC_{sw} = qC_{sw} + q^{1/2}C_w + q^{1/2}\sum_{\substack{z\in W\\ z\prec sw\\ sz<z}} \mu(z, sw)C_z,$$

and we can rearrange and multiply both sides by $q^{-1/2}T_s$ to get

$$T_sC_w = T_s\left(q^{-1/2}T_sC_{sw} - q^{1/2}C_{sw} - \sum_{\substack{z\in W\\ z\prec sw\\ sz<z}} \mu(z, sw)C_z\right). \tag{3.6.3}$$

Then substituting into (3.6.3) the relation in (3.3.2) for T_s^2 and collecting equal terms we get

$$T_sC_w = -q^{-1/2}T_sC_{sw} + q^{1/2}C_{sw} - \sum_{\substack{z\in W\\ z\prec sw\\ sz<z}} \mu(z, sw)T_sC_z. \tag{3.6.4}$$

Since $sz < z < w$, then, by the induction hypothesis, we have that $T_sC_z = -C_z$ for all $z \in W$ contributing a term in the last sum in (3.6.4), and thus

$$T_sC_w = -\left(q^{-1/2}T_sC_{sw} - q^{1/2}C_{sw} - \sum_{\substack{z\in W\\ z\prec sw\\ sz<z}} \mu(z, sw)T_sC_z\right) = -C_w,$$

as required. This completes the proof. □

Remark 3.6.1. Since $(T_s)_{s\in S}$ generates $\mathcal{H}$ as an $\mathscr{A}$-algebra, the formulas in Proposition 3.6.1 describe the action of $\mathcal{H}$ on itself in the left regular representation of $\mathcal{H}$, relative to the KL-basis. We say more about this in Section 4.1.

Remark 3.6.2. An identical proof interchanging left an right shows that

$$C_wT_s = \begin{cases} -C_w & \text{if } ws < w, \quad (3.6.5)\\ qC_w + q^{1/2}C_{ws} + q^{1/2}\sum_{\substack{z\in W\\ z\prec w\\ zs<z}} \mu(z, w)C_z & \text{if } w < ws. \quad (3.6.6)\end{cases}$$

As in Remark 3.6.1, the formulas in (3.6.5) and (3.6.6) describe the action of $\mathcal{H}$ on itself in the right regular representation of $\mathcal{H}$, relative to the KL-basis.

Remark 3.6.3. Following Remark 3.5.5, we are now able to fully describe multiplication between the KL-basis elements. In particular, (3.5.11) tells us that if $w \in W$ and $s \in S$ such that $w < sw$, then

$$C_s C_w = C_{sw} + \sum_{\substack{z \in W \\ z \prec w \\ sz < z}} \mu(z,w) C_z. \tag{3.6.7}$$

On the other hand, if $w \in W$ and $s \in S$ such that $sw < w$, then we can directly use the definition of C_s in (3.5.4) and (3.6.1) in Proposition 3.6.1 to obtain

$$C_s C_w = q^{-1/2}\left(T_s - qT_1\right) C_w = q^{-1/2} T_s C_w - q^{1/2} C_w = -\left(q^{-1/2} + q^{1/2}\right) C_w. \tag{3.6.8}$$

Similarly, from the formulas Remark 3.6.3 corresponding to the 'right-hand' version the formulas in Proposition 3.6.1, we also obtain

$$C_w C_s = \begin{cases} -\left(q^{-1/2} + q^{1/2}\right) C_w & \text{if } ws < w, \quad (3.6.9) \\ C_{ws} + \sum\limits_{\substack{z \in W \\ z \prec w \\ zs < z}} \mu(z,w) C_z & \text{if } w < ws. \quad (3.6.10) \end{cases}$$

Corollary 3.6.1. *If $x, w \in W$ are elements of W such that $sw < w$ and $x < sx$ for some $s \in S$, then $P_{x,w} = P_{sx,w}$.*

Proof. Let $x, w \in W$ be elements of W such that $sw < w$ and $x < sx$ for some $s \in S$. By (3.6.1) in Proposition 3.6.1 we have $T_s C_w = -C_w$. Using the expression in (3.5.2) of Theorem 3.5.1 for C_w, this equality can be written as

$$\varepsilon_w q_w^{1/2} \sum_{\substack{y \in W \\ x \leqslant w}} \varepsilon_y q_y^{-1} \overline{P}_{y,w} T_s T_y = -\varepsilon_w q_w^{1/2} \sum_{\substack{z \in W \\ z \leqslant w}} \varepsilon_z q_z^{-1} \overline{P}_{z,w} T_z,$$

which, after multiplying both sides by $\varepsilon_w q_w^{-1/2}$, becomes

$$\sum_{\substack{y \in W \\ y \leqslant w}} \varepsilon_y q_y^{-1} \overline{P}_{y,w} T_s T_y = -\sum_{\substack{z \in W \\ z \leqslant w}} \varepsilon_z q_z^{-1} \overline{P}_{z,w} T_z. \tag{3.6.11}$$

Note that, by Lemma 2.10.2, we have that $sx \leqslant w$, so T_{sx} appears on the right-hand side of (3.6.11), with coefficient

$$-\varepsilon_{sx} q_{sx}^{-1} \overline{P}_{sx,w} = \varepsilon_x q^{-1} q_x^{-1} \overline{P}_{sx,w},$$

where we have used the facts that $\varepsilon_{sx} = -\varepsilon_x$ and $q_{sx} = qq_x$. Since $x < sx$, by the relation in (3.2.4) in Proposition 3.2.2, the element T_{sx} appears on the left-hand side of (3.6.11) directly in the product $T_s T_x$, with coefficient $\varepsilon_x q_x^{-1} \overline{P}_{x,w}$, and indirectly in the product $T_s T_{sx}$, with coefficient $(q-1)\varepsilon_{sx} q_{sx}^{-1} \overline{P}_{sx,w}$. Therefore, the coefficient of T_{sx} on the left-hand side of (3.6.11) is equal to

$$\varepsilon_x q_x^{-1} \overline{P}_{x,w} + (q-1)\varepsilon_{sx} q_{sx}^{-1} \overline{P}_{sx,w} = \varepsilon_x q_x^{-1} \overline{P}_{x,w} - \varepsilon_x q_x^{-1}\left(1 - q^{-1}\right) \overline{P}_{sx,w},$$

where we have again used the facts that $\varepsilon_{sx} = -\varepsilon_x$ and $q_{sx} = qq_x$. Hence equating the coefficients of T_{sx} on both sides of (3.6.11) yields

$$\varepsilon_x q_x^{-1} \overline{P}_{x,w} - \varepsilon_x q_x^{-1}\left(1 - q^{-1}\right) \overline{P}_{sx,w} = \varepsilon_x q_x^{-1} q^{-1} \overline{P}_{sx,w},$$

which then implies that

$$\overline{P}_{x,w} = q^{-1} \overline{P}_{sx,w} + \left(1 - q^{-1}\right) \overline{P}_{sx,w} = \overline{P}_{sx,w},$$

which is equivalent to $P_{x,w} = P_{sx,w}$, as required. □

Remark 3.6.4. As in Remark 3.6.2, an identical proof interchanging left and write shows the 'right-hand' version of Corollary 3.6.1, that is, if $x, w \in W$ are elements of W such that $ws < w$ and $x < xs$ for some $s \in S$, then $P_{x,w} = P_{xs,w}$.

3.7 Inversion

In this section we introduce 'inverse KL-polynomials' $Q_{x,w}$ following [25], or [29] where Lusztig defined them only for Weyl groups, and prove two properties of these that are mentioned in both [25] and [30] without proof. We also try to compute the polynomials $Q_{1,w}$ for any $w \in W$. We then use these polynomials $Q_{x,q}$ to define a 'basis' $(D_w)_{w \in W}$ dual to the KL-basis following [30] and introduce a -linear map $\tau : \mathcal{H} \to \mathscr{A}$ together with some of its properties. These three tools will be of great importance in Section **??**. The main references for this section are [25], [29] and [30].

Definition 3.7.1. For each $y, w \in W$ with $y \leq w$, define a polynomial $Q_{y,w} \in \mathbb{Z}[q]$ by the identities

$$\sum_{\substack{z \in W \\ y \leqslant z \leqslant w}} \varepsilon_y \varepsilon_z Q_{y,z} P_{z,w} = \delta_{y,w}. \tag{3.7.1}$$

Lemma 3.7.1. *Let $y, w \in W$ be elements such that $y \leqslant w$.*

(a) If $y = w$, then $Q_{y,w} = 1$.

(b) If $y < w$, then $Q_{y,w} \in \mathbb{Z}[q]$ is a polynomial in q with

$$\deg_q Q_{y,w} \leq (\ell(w) - \ell(y) - 1)\,. \tag{3.7.2}$$

Proof. First, choosing $y = w$ in (3.7.1), we see that $Q_{w,w} = 1$ an (a) is proved.

To prove (b), we proceed by induction on $\ell(w) - \ell(y) \geq 1$. If $\ell(w) - \ell(y) = 1$, then $y \leq w$ implies that $sy = w$ for some $s \in S$ and thus, using the facts that $\varepsilon_y = -\varepsilon_w$ and $\varepsilon_y^2 = 1$, the identity in (3.7.1) reads

$$Q_{y,y} P_{y,w} - Q_{y,w} P_{w,w} = 0. \tag{3.7.3}$$

Since $\ell(w) - \ell(y) = 1$, we know, by Corollary 3.5.1, that $P_{y,w} = 1$. Moreover $P_{w,w} = 1$. Substituting these and the result in (a) into (3.7.3), we obtain $Q_{y,w} = P_{y,w}$. Hence, by Theorem 3.5.1, the degree bound in (3.7.2) of (b) follows in this case.

Now, suppose that $\ell(w) - \ell(y) > 1$, and assume that for any $x, x' \in W$ such that $x \leqslant x'$ and $0 < \ell(x') - \ell(x) < \ell(w) - \ell(y)$ the degree bound in (3.7.2) of (b) holds. Now note that since $P_{w,w} = 1$, identity in (3.7.1) can be written as

$$Q_{y,w} = -\sum_{\substack{z \in W \\ y \leq z < w}} \varepsilon_y \varepsilon_z Q_{y,z} P_{z,w}.$$

Since for every $z \in W$ with $y \leqslant z < w$ we have that $0 < \ell(z) - \ell(y) < \ell(w) - \ell(y)$, it follows, by the induction hypothesis and the degree bound of the KL-polynomials in Theorem 3.5.1, that

$$\deg_q Q_{y,z} P_{z,w} \leq \frac{1}{2}(\ell(z) - \ell(y) - 1) + \frac{1}{2}(\ell(w) - \ell(z) - 1) = \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

for each $z \in W$ contributing a term to the right-hand side of the equality in (3.7), proving the degree bound in (3.7.2) in (b). □

Example 3.7.1. Let $w \in W$ be any non-identity element, and take $y = 1$ in (3.7.1) of Definition 3.7.1, which gives

$$\sum_{\substack{z \in W \\ 1 \leqslant z \leqslant w}} \varepsilon_z Q_{1,z} P_{z,w} = 0. \tag{3.7.4}$$

Note that if $\ell(w) = 1$, then $w = s$ for some $s \in S$ and (3.7.4) yields

$$0 = \varepsilon_1 Q_{1,1} P_{1,s} + \varepsilon_s Q_{1,s} P_{s,s},$$

so by (a) of Lemma 3.7.1, Corollary 3.5.1 and the fact that $P_{s,s} = 1$, to obtain $Q_{1,s} = 1$. Now, suppose that $\ell(w) > 1$ and pick $s \in S$ such that $sw < w$. By induction, $Q_{1,z} = 1$ for all $z \in W$ such that $1 \leqslant z < w$, so by definition,

$$\varepsilon_w Q_{1,w} = -\sum_{\substack{z\in W\\ 1<z<w\\ sz<z}} \varepsilon_z P_{z,w} - \sum_{\substack{z\in W\\ 1\leqslant z<w\\ z<sz}} \varepsilon_z P_{z,w}. \tag{3.7.5}$$

But, by definition, $\varepsilon_{sz} = -\varepsilon_z$ and, by Corollary 3.6.1, $P_{z,w} = P_{sz,w}$ for every $z \in W$ in the second sum in (3.7.5), which becomes

$$\varepsilon_w Q_{1,w} = -\sum_{\substack{z\in W\\ 1<z<w\\ sz<z}} \varepsilon_z P_{z,w} + \sum_{\substack{z\in W\\ 1\leqslant z<w\\ z<sz}} \varepsilon_{sz} P_{sz,w}. \tag{3.7.6}$$

Moreover, since for every $z \in W$ such that $z < sz$, we have that $s \cdot sz = z < sz$, setting $y = sz$ for such $z \in W$, the equality in (3.7.6) becomes

$$\varepsilon_w Q_{1,w} = -\sum_{\substack{z\in W\\ 1<z<w\\ sz<z}} \varepsilon_z P_{z,w} + \sum_{\substack{y\in W\\ 1<y\leqslant w\\ sy<y}} \varepsilon_y P_{y,w}. \tag{3.7.7}$$

Every term in the second sum in the right-hand side of (3.7.7) except $\varepsilon_w P_{w,w} = \varepsilon_w$ is cancelled by a term in the first sum in the right-hand side of (3.7.7), which gives

$$\varepsilon_w Q_{1,w} = \varepsilon_w,$$

which in turn implies, since $\ell(w) > 1$, that $Q_{1,w} = 1$. Since $w \in W$ was arbitrary, it follows that $Q_{1,w} = 1$ for all $w \in W$.

Definition 3.7.2. For any $y \in W$, define

$$D_y := \sum_{\substack{x\in W\\ y\leqslant x}} \overline{Q}_{y,x} q_y^{-1/2} T_x = \sum_{\substack{x\in W\\ y\leqslant x}} \overline{Q}_{y,x} q_x^{1/2} q_y^{-1/2} \widetilde{T}_x. \tag{3.7.8}$$

Remark 3.7.1. The object D_y in Definition 3.7.2 is in fact an element of the set $\hat{\mathcal{H}}$ of formal sums

$$\sum_{w\in W} \alpha_w \widetilde{T}_w$$

with coefficients $\alpha_w \in \mathscr{A}$ for each $w \in W$. In particular, using (a) and (b) in Lemma 3.7.1, we see that

$$D_y = \widetilde{T}_y + \sum_{\substack{x\in W\\ y<x}} \overline{Q}_{y,x} q_x^{1/2} q_y^{-1/2} \widetilde{T}_x \in \widetilde{T}_y + q^{1/2} \sum_{\substack{x\in W\\ y<x}} \mathscr{A}^+ \cdot \widetilde{T}_x, \tag{3.7.9}$$

where the sum could be infinite. Since $\mathcal{H}$ is the set of formal sums such that $\alpha_w = 0$ for all but finitely many $w \in W$, we have that $\mathcal{H} \subseteq \hat{\mathcal{H}}$ and the left $\mathcal{H}$-module structure on $\mathcal{H}$ extends naturally to a left $\mathcal{H}$-module structure on $\hat{\mathcal{H}}$. For example, for any $s \in S$ and any $\alpha_w \in \mathscr{A}$ for each $w \in W$, we have

$$\widetilde{T}_s \left(\sum_{w\in W} \alpha_w \widetilde{T}_w \right) = \sum_{\substack{w\in W\\ w<sw}} \alpha_w \widetilde{T}_{sw} + \sum_{\substack{w\in W\\ sw<w}} \alpha_w \left(\widetilde{T}_{sw} + (q^{1/2} - q^{-1/2}) \widetilde{T}_w \right), \tag{3.7.10}$$

where we have used the relations (3.5.20) and (3.5.21) defining multiplication with respect to the basis $\big(\widetilde{T}_w\big)_{w\in W}$. Since a sum of terms $\alpha_w\widetilde{T}_{sw}$ for $w < sw$ plus a sum of terms $\alpha_w\widetilde{T}_{sw}$ for $sw < w$ with w exhausting W is equal to a sum of terms $\alpha_{sw}\widetilde{T}_w$ for $sw < w$ plus a sum of terms $\alpha_{sw}\widetilde{T}_w$ for $w < sw$ with w exhausting W, the equality in (3.7.10) can be written as

$$\widetilde{T}_s\left(\sum_{w\in W}\alpha_w\widetilde{T}_w\right) = \sum_{\substack{w\in W\\ w<sw}}\alpha_{sw}\widetilde{T}_w + \sum_{\substack{w\in W\\ sw<w}}\Big(\alpha_{sw} + (q^{1/2} - q^{-1/2})\alpha_w\Big)\widetilde{T}_w. \tag{3.7.11}$$

Similarly, $\hat{\mathcal{H}}$ has a right $\mathcal{H}$-module structure.

Definition 3.7.3. Define an $\mathscr{A}$-linear map $\tau : \hat{\mathcal{H}} \to \mathscr{A}$ by

$$\tau\left(\alpha_w\sum_{w\in W}\widetilde{T}_w\right) = \alpha_1. \tag{3.7.12}$$

Lemma 3.7.2. *The $\mathscr{A}$-linear map $\tau : \hat{\mathcal{H}} \to \mathscr{A}$ in Definition 3.7.3 has the following properties:*

(a) For any $y, w \in W$, we have $\tau(\widetilde{T}_y\widetilde{T}_w) = \delta_{yw,1}$.

(b) For any $h \in \mathcal{H}$ and $\hat{h} \in \mathcal{H}$, we have $\tau(h\hat{h}) = \tau(\hat{h}h)$.

Remark 3.7.2. The statement in *(a)* Lemma 3.7.2 can be directly checked by applying the definition of τ and the multiplication relations in (3.5.20) and (3.5.21). A similar technique is used in the following corollary. The statement in *(b)* follows from the fact that the left and right $\mathcal{H}$-module structures on $\hat{\mathcal{H}}$ commute with each other.

Corollary 3.7.1. *For any $y, w \in W$, we have $\tau(C_wD_y) = \tau\big(D_yC_w\big) = \delta_{wy,1}$.*

Proof. For any $y, w \in W$, we know, by (3.5.24) and (3.7.9) that

$$\begin{aligned}\tau(D_yC_w) = \tau(\widetilde{T}_y\widetilde{T}_w) &+ \varepsilon_w q_w^{1/2}\sum_{\substack{z\in W\\ z<w}}\varepsilon_z q_z^{-1/2}\overline{P}_{z,w}\tau(\widetilde{T}_y\widetilde{T}_z) + q_y^{-1/2}\sum_{\substack{x\in W\\ y<x}}q_x^{1/2}\overline{Q}_{y,x}\tau(\widetilde{T}_x\widetilde{T}_w)\\ &+ \varepsilon_w q_w^{1/2}q_y^{-1/2}\sum_{\substack{x\in W\\ y<x}}\sum_{\substack{z\in W\\ z<w}}q_x^{1/2}\varepsilon_z q_z^{-1/2}\overline{Q}_{y,x}\overline{P}_{z,w}\tau(\widetilde{T}_x\widetilde{T}_z).\end{aligned} \tag{3.7.13}$$

Now, if $yw = 1$, we have, by (a) in Lemma 3.7.2, that $\tau(\widetilde{T}_y\widetilde{T}_w) = 1$ and

$$\tau(\widetilde{T}_y\widetilde{T}_z) = \tau(\widetilde{T}_x\widetilde{T}_w) = \tau(\widetilde{T}_x\widetilde{T}_z) = 0$$

for all $x, z \in W$ such that $z < w$ and $y < x$, which shows that $\tau(D_yC_w) = 1$. On the other hand, if $yw \neq 1$, then $\tau(\widetilde{T}_y\widetilde{T}_w) = 0$ and either $y < w^{-1}$ or $w^{-1} < y$. In the latter case, we have, by Proposition 2.10.2, that $w < y^{-1}$, so $\tau(\widetilde{T}_y\widetilde{T}_z) = 0$ for all $z \in W$ such that $z < w$, and similarly, the fact that $w^{-1} < y$ implies $\tau(\widetilde{T}_x\widetilde{T}_w) = 0$ for all $x \in W$ such that $y < x$. Moreover, for any fixed $x \in W$ such that $y < x$ in the third sum in (3.7.13), we have that if $z \in W$ is going to contribute a term for such a given x, then $z = x^{-1}$. But $w < y^{-1} < x^{-1} = z$, so the last sum in (3.7.13) is actually zero too. Hence $\tau(D_yC_w) = 0$ in this case, as required. Now, in the first case, by Proposition 2.10.2 again, we have that $y^{-1} < w$, and thus if $x \in W$ such that $y < x$, then $x = w^{-1}$ is the only element contributing a term two the second sum in (3.7.13) since otherwise $\tau(\widetilde{T}_x\widetilde{T}_w) = 0$, and similarly, if $z \in W$ such that $z < w$, then $z = y^{-1}$ is the only element contributing a term two the first sum in (3.7.13) since otherwise $\tau(\widetilde{T}_y\widetilde{T}_z) = 0$. Moreover, for any given $x \in W$ such that $y < x$ in the third sum in (3.7.13), we have that if $z \in$ W is an element that is going to contribute a term for such a given x, then $z = x^{-1}$ and $z < w$, which is compatible with $y < x < w^{-1}$. The equality in (3.7.13) thus becomes

$$\tau(D_y C_w) = q_w^{1/2} q_y^{-1/2} \varepsilon_w \varepsilon_y \left(\overline{P}_{y^{-1},w} + \varepsilon_w \varepsilon_y \overline{Q}_{y,w^{-1}} + \sum_{\substack{x \in W \\ y < x < w^{-1}}} \varepsilon_y \varepsilon_x \overline{Q}_{y,x} \overline{P}_{x,w^{-1}} \right), \tag{3.7.14}$$

where we have used the facts that $\varepsilon_v = \varepsilon_{v^{-1}}$ and $q_v = q_{v^{-1}}$ for all $v \in W$. But (3.7.1) with w^{-1} instead of w gives

$$\overline{P}_{y,w^{-1}} + \varepsilon_y \varepsilon_w \overline{Q}_{y,w^{-1}} + \sum_{y < x < w^{-1}} \varepsilon_y \varepsilon_x \overline{Q}_{y,x} \overline{P}_{x,w^{-1}} = 0, \tag{3.7.15}$$

and thus, combining (3.7.14) and (3.7.15) and using the fact that $\overline{P}_{y^{-1},w} = \overline{P}_{y,w^{-1}}$, we obtain $\tau(D_y C_w) = 0$, as required. Finally, the fact that $\tau(D_y C_w) = \tau(C_w D_y)$ follows by (b) of Lemma 3.7.2. □

4 Kazhdan-Lusztig Cells

4.1 Left, right and two-sided KL-cells

The notion of *Kazhdan-Lusztig-cells (KL-cells)* first appeared in [24] as a result of the desire of Kazhdan and Lusztig to explicitly construct representations of the Hecke algebra $\mathcal{H}$. In this section we start with some definitions, together with a combinatorial description and some general properties of KL-cells. These turn out to partition W. We consider the case when W is dihedral. We then see how KL-cells relate to the left regular representation of $\mathcal{H}$ relative to the KL-basis and present a result in [30] relating the products $C_w D_y$ and $D_y C_w$ with the preorder "$\leqslant_L$" constructed in this section and used to define KL-cells. We also give some examples of KL-cells. For the rest of Chapter 4, whenever we give examples in rank 2 and ranks 3, we describe explicitly the KL-cell decomposition of W viewing W as a set of chambers/alcoves in a Euclidean plane, where each alcove is coloured according to the left/two-cell KL-cell it belongs to. We end this section by presenting a series of conjectures by Lusztig about the partition of W into KL-cells as well as the relations between these. The main references for this section are [24] and [28], and the conjectures can be found, either explicitly or implicitly in [32].

Definition 4.1.1. We say that y, w are *joined* if we have $y \prec w$ or $w \prec y$ and we denote this by $y — w$; we then set

$$\hat{\mu}(y,w) = \begin{cases} \mu(y,w) & \text{if } y \prec w \\ \mu(w,y) & \text{if } w \prec y. \end{cases}$$

Definition 4.1.2. For any $w \in W$, set

$$\mathscr{L}(w) := \{s \in S \mid sw < w\} \quad \text{and} \quad \mathscr{R}(w) := \{s \in S \mid ws < w\}.$$

Then $\mathscr{L}(w)$ is called the *left descent set of* w and $\mathscr{R}$ the *right descent set of* w.

Example 4.1.1. For any Coxeter system (W, S) we have $\mathscr{L}(1) = \emptyset = \mathscr{R}(1)$.

Example 4.1.2. For any finite Coxeter system (W, S) we have $\mathscr{L}(w_\circ) = S = \mathscr{R}(w_\circ)$.

Remark 4.1.1. Note that if we call elements $w, w' \in W$ equivalent if $\mathscr{L}(w) = \mathscr{L}(w')$, we get a partition of W. The same is true if we use right descent sets.

Definition 4.1.3. Given elements $w, w' \in W$, we say that $w \leqslant_L w'$ if there exists a sequence $w = w_0, w_1, \ldots, w_n = w'$ of elements of W such that for each $i = 1, \ldots, n$, we have $w_{i-1} — w_i$ and $\mathscr{L}(w_{i-1}) \nsubseteq \mathscr{L}(w_i)$. We say that $w \leqslant_{LR} w'$ if there exists a sequence $w = w_0, w_1, \ldots, w_n = x'$ of elements of W such that for each $i = 1, \ldots, n$, we either have $w_{i-1} \leqslant_L w_i$ or $w_{i-1}^{-1} \leqslant_L w_i^{-1}$.

Remark 4.1.2. Note that the relations "$\leqslant_L$" and "$\leqslant_{LR}$" are transitive and reflexive, so they are preorders. However, these preorders are not anti-symmetric, so they are not order relations.

Definition 4.1.4. Let "$\sim_L$" be the equivalence relation associated to the preorder "$\leqslant_L$"; thus $w \sim_L w'$ means that $w \leqslant_L w'$ and $w' \leqslant_L w$. The corresponding equivalence classes are called *left cells of* W. A *right cell of* W is a set of the form

$$\{w \in W \mid w^{-1} \in \mathbf{\Gamma},\ \mathbf{\Gamma} \text{ is a left cell}\}.$$

Let "$\sim_{LR}$" be the equivalence relation associated to the preorder "$\leqslant_{LR}$"; thus $w \sim_{LR} w'$ means that $w \leqslant_{LR} w'$ and $w' \leqslant_{LR} w$. The corresponding equivalence classes are called *two-sided cells.*

Remark 4.1.3. From Definition 4.1.3 and Definition 4.1.4, it follows that each two-sided cell is a union of left cells, respectively, right cells.

Remark 4.1.4. If $\mathcal{H}$ is commutative, then the relations " $\leqslant_L$" and " $\leqslant_{LR}$" coincide. Hence the relations " $\sim_L$" and " $\sim_{LR}$" coincide, and thus the sets of left cells, right cells and two-sided cells all coincide too.

Proposition 4.1.1. *Let (W, S) be a Coxeter system. Then:*

(a) The identity element of W lies in a left cell by itself. Hence lies in a unique two-sided cell.

(b) If W is finite, then the unique longest element $w_\circ$ of W lies in a left cell by itself.

Proof. Assume that $w \in W$ is any non-identity element belonging to the same left cell as the identity 1. Then, be definition, there must exist a sequence of elements

$$1 = w_0 \text{ --- } w_1 \text{ --- } \cdots \text{ --- } w_n = w \tag{4.1.1}$$

such that $\mathscr{L}(w_{i-1}) \nsubseteq \mathscr{L}(w_i)$ for each $i = 1, \cdots n$. But $\mathscr{L}(1) = \emptyset$, which is contained in $\mathscr{L}(x)$ for any $x \in W$. This contradicts the existence of a sequence as in (4.1.1). We hence conclude that there is no non-identity element of W belonging to the same left cell as 1, and part (a) follows.

Now, assume that W is finite and let $w' \in W$ be any element such that $w \neq w_\circ$ and w' lies in the same left cell as w_0. Then, be definition, there must exist a sequence of elements

$$w' = w'_0 \text{ --- } w'_1 \text{ --- } \cdots \text{ --- } w'_n = w_\circ \tag{4.1.2}$$

such that $\mathscr{L}(w'_{i-1}) \nsubseteq \mathscr{L}(w'_i)$ for each $i = 1, \cdots n$. But $\mathscr{L}(w_\circ) = S$, which contains $\mathscr{L}(x)$ for any $x \in W$. This contradicts the existence of a sequence as in (4.1.2). We hence conclude that there is no element element of W different from $w_\circ$ belonging to the same left cell as $w_\circ$, and part (b) follows. ☐

Example 4.1.3. Let $W = \mathcal{D}_m$ be a dihedral group of order $m \leq \infty$, with generating set $S = \{s, s'\} \in W$. From Example 3.5.1 we know that $P_{x,w} = 1$ whenever $x \leqslant w$, which we know, by Example 2.10.1, is the case if and only if $\ell(x) \leq \ell(w)$. All this together implies that $x \prec w$ if and only if $\ell(w) - \ell(x) = 1$, and thus $x \text{ --- } w$ if and only if the $\ell(w) - \ell(x) = 1$. Now, recall that every element of W, except the longest element $w_\circ$ if $m < \infty$, has a unique reduced expression, and thus $\mathscr{L}(w) = \{s\}$ if such expression begins with s and $\mathscr{L}(w) = \{s'\}$ if such expression begins with s'. Hence, for non-identity elements x, w, the condition $\mathscr{L}(x) \nsubseteq \mathscr{L}(w)$ is equivalent to the condition that the unique reduced expressions of x and w have a different leftmost element. We can therefore construct two different chains

$$\begin{gathered} s \text{ --- } s's \text{ --- } ss's \text{ --- } s'ss's \text{ --- } ss'ss's \text{ --- } \cdots \\ s' \text{ --- } ss' \text{ --- } s'ss' \text{ --- } ss'ss' \text{ --- } s'ss'ss' \text{ --- } \cdots \end{gathered}$$

and this will exhaust all elements of W. This shows that $x \leqslant_L w$ if $x, w \in W$ such that $\ell(x) < \ell(w)$ and the rightmost element in their reduced expressions is the same for both. If $x \leqslant_L w$ with chain

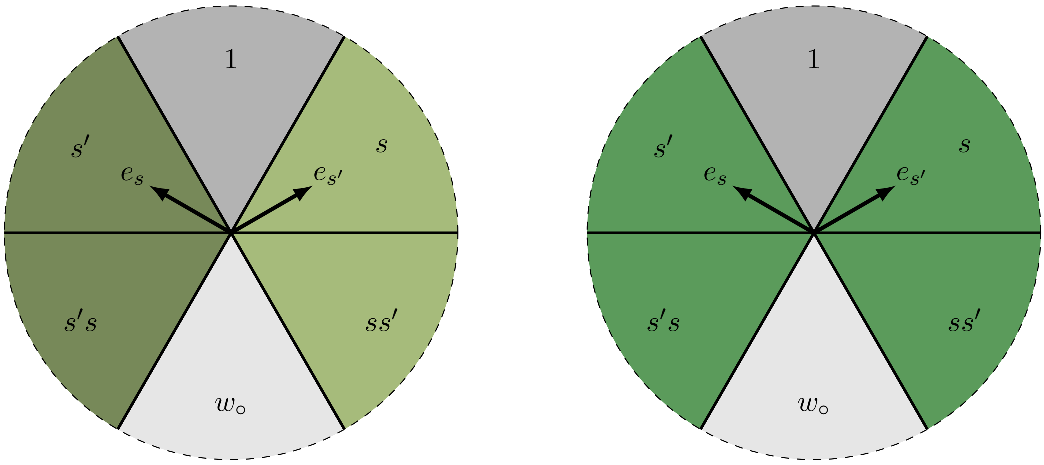
$$x = w_0 \text{ --- } w_1 \text{ --- } \cdots \text{ --- } w_{n-1} \text{ --- } w_n = w,$$

then the reverse chain also meets the required conditions for $w \leqslant_L x$. Hence, by definition $x \sim_L w$. Conversely, if two elements $x', w' \in W$ have reduced expressions whose rightmost element is different, then any chain between these two elements has to contain only elements sharing their leftmost element in their reduced expression. In this case we have $\mathscr{L}(w'_{i-1}) \subseteq \mathscr{L}(w'_i)$ for each pair of adjacent elements w'_{i-1}, w'_i in the chain, and thus $x' \neq w'$. This shows that if $m = \infty$, then W has three left cells, and if $m < \infty$, then W has four left cells. Using the same argument, but this time working with elements sharing their leftmost element in their reduced expression, we can get the right cells of W. In then, follows by definition that every element of W, except the identity and $w_\circ$ in the case where W is finite, lie in the same two-sided cell of W. For example, if W is of type A_2, with presentation as in Example 2.9.1, then the above discussion shows that the the partition of W into left KL-cells is given by

$$\{1\}, \{s, s's\}, \{s', ss'\}, \{w_\circ\},$$

and depicted in Figure 4.1a, and the partition of W into two-sided KL-cells is given by

$$\{1\}, \{s, s's, s', ss'\}, \{w_\circ\}.$$



(a) Left KL-cells (b) Two-sided KL-cells

Figure 4.1: KL-cells when (W, S) is of type A_2

Proposition 4.1.2. *If $x, w \in W$ are elements such that $x \leqslant_L w$, then $\mathscr{R}(w) \subseteq \mathscr{R}(x)$.*

Proof. Let $x, w \in W$ be elements such that $x \leqslant_L w$. First note that, by the transitivity of the preorder "$\leqslant_L$", it is enough to prove this for the case when x — w with $\mathscr{L}(x) \nsubseteq \mathscr{L}(w)$, and recall that we have two possibilities: either $x \prec w$ or $w \prec x$.

Consider the case $w \prec x$ and let $s \in \mathscr{L}(x) \setminus \mathscr{L}(w)$. In particular we have $w < sw$ and $sx < x$ and so, by Corollary 3.6.1, we obtain $P_{w,x} = P_{sw,x}$. This implies that $x = sw$, since otherwise

$$\deg_q P_{w,x} = \deg_q P_{sw,x} \leq \frac{1}{2} \left(\ell(x) - \ell(sw) - 1\right) < \frac{1}{2} \left(\ell(x) - \ell(w) - 1\right),$$

which contradicts the fact that $w \prec x$. Then, in view of the facts that $x = sw$ and $w < x$, we deduce that $\mathscr{R}(w) \subseteq \mathscr{R}(x)$.

Now, consider the case $x \prec w$ and assume, for a contradiction, that there exists some $s' \in \mathscr{R}(w) \setminus \mathscr{R}(x)$. Then $ws' < w$ and $x < xs'$, so by the 'right-hand' version of Corollary 3.6.1 (see Remark 3.6.4), we obtain $P_{x,w} = P_{xs',w}$. Note that this implies that $w = xs'$, since otherwise

$$\deg_q P_{x,w} = \deg_q P_{xs',w} \leq \frac{1}{2} \left(\ell(w) - \ell(xs') - 1\right) < \frac{1}{2} \left(\ell(w) - \ell(x) - 1\right),$$

which contradicts the the fact that $x \prec w$. Then, in view of the facts that $w = xs'$ and $x < w$, we deduce that $\mathscr{L}(x) \subseteq \mathscr{L}(w)$, which contradicts our initial assumption that $\mathscr{L}(x) \nsubseteq \mathscr{L}(w)$. The proof is now complete. □

Corollary 4.1.1. *If $x, w \in W$ are elements such that $x \sim_L w$, then $\mathscr{R}(w) = \mathscr{R}(x)$.*

Proof. If $x, w \in W$ are elements such that $x \sim_L w$, then, by definition of "$\sim_L$", we have that $x \leqslant_L w$ and $w \leqslant_L x$. Hence, by Proposition 4.1.2, the equality $\mathscr{R}(w) = \mathscr{R}(x)$ follows. □

Remark 4.1.5. Corollary 4.1.1 shows that decomposition of W into left KL-cells refines the decomposition into sets with a common right descent set in Remark 4.1.1.

Remark 4.1.6. Having described KL-cells combinatorially, we return to the formulas for the action of the canonical basis on the KL-basis in Proposition 3.6.1 and consider what these tell us about the left regular representation of $\mathcal{H}$. First, if $w \in W$ is an element such that $w < sw$, then $P_{w,sw} \neq 0$ and

$$\deg_q \leq \frac{1}{2}\left(\ell(sw) - \ell(w) - 1\right) = 0,$$

which shows that $w \prec sw$, and in particular $sw — w$. Moreover, $s \in \mathscr{L}(sw)$ but $s \notin \mathscr{L}(w)$, so $\mathscr{L}(sw) \nsubseteq \mathscr{L}(w)$. It follows, by definition of "$\leqslant_L$", that $sw \leqslant_L w$. Also, for the given s such that $w < sw$, any element $z \prec w$ in the final sum of (3.6.2) satisfies $sz < z$, and thus $s \in \mathscr{L}(z)$ but $s \notin \mathscr{L}(w)$, which shows that $\mathscr{L}(z) \nsubseteq \mathscr{L}(w)$. This shows that $z \leqslant_L w$. We therefore see that for any $w \in W$, left multiplication by T_s takes C_w into an $\mathscr{A}$-linear combination of itself and some C_x for which $x \leqslant_L w$, and thus

$$\mathcal{H} \cdot C_w \subseteq \sum_{\substack{x \in W \\ x \leqslant_L w}} \mathscr{A} \cdot C_x. \tag{4.1.3}$$

We can do the same thing for the formulas in Remark 3.6.2 and consider what these tell us about the right regular representation of $\mathcal{H}$. If $w \in W$ is an element such that $w < ws$, then, by Proposition 2.10.2, we have $w^{-1} < sw^{-1}$, so the same argument as above shows that $sw^{-1} \leqslant_L w^{-1}$. Similarly, for the given s such that $w < ws$, any element $z \prec w$ in the final sum of (3.6.6) satisfies $zs < z$, and so it satisfies $sz^{-1} < z^{-1}$. The same argument as above then gives $z^{-1} \leqslant_L w^{-1}$. We therefore see that for any $w \in W$, right multiplication by T_s takes C_w into an $\mathscr{A}$-linear combination of itself and come C_x for which $x^{-1} \leqslant_L w^{-1}$, and thus

$$C_w \cdot \mathcal{H} \subseteq \sum_{\substack{x \in W \\ x^{-1} \leqslant_L w^{-1}}} \mathscr{A} \cdot C_x. \tag{4.1.4}$$

Hence, combining (4.1.3) and (4.1.4) and using the definition of "$\leqslant_{LR}$", it follows that

$$\mathcal{H} \cdot C_w \cdot \mathcal{H} \subseteq \sum_{\substack{x \in W \\ x \leqslant_{LR} w}} \mathscr{A} \cdot C_x. \tag{4.1.5}$$

Remark 4.1.7. Each left KL-cell gives rise to a representation of $\mathcal{H}$. Let $\mathbf{\Gamma} \subseteq W$ be a left cell and define

$$\mathcal{I}_{\mathbf{\Gamma}} := \operatorname{span}_{\mathscr{A}}\left(C_x \mid x \in \mathbf{\Gamma} \text{ or } x \leqslant_L w \text{ for some } w \in \mathbf{\Gamma}\right),$$

that is, $\mathcal{I}_{\mathbf{\Gamma}}$ is the $\mathscr{A}$-span of all C_w such that $w \in \mathbf{\Gamma}$ and all C_x such that $x \leqslant_L w$ for some $w \in \mathbf{\Gamma}$. Also define

$$\mathcal{J}_{\mathbf{\Gamma}} := \operatorname{span}_{\mathscr{A}}\left(C_x \mid x \leqslant_L w \text{ for some } w \in \mathbf{\Gamma} \text{ and } x \notin \mathbf{\Gamma}\right),$$

that is, $\mathcal{J}_{\mathbf{\Gamma}}$ is the $\mathscr{A}$-span of all C_x such that $x \leqslant_L w$ for some $w \in \mathbf{\Gamma}$ but $x \notin \Gamma$. From (4.1.3) in Remark 4.1.6, it follows that $\mathcal{I}_{\mathbf{\Gamma}}$ is a left ideal of $\mathcal{H}$. Moreover, the transitivity of the preorder "$\leqslant_L$" together with the definition of a left cell, it follows that $\mathcal{J}_{\mathbf{\Gamma}}$ is also a left ideal of $\mathcal{H}$. Hence the quotient $\mathcal{I}_{\mathbf{\Gamma}}/\mathcal{J}_{\mathbf{\Gamma}}$ is an $\mathcal{H}$-module. In particular, it is a free $\mathscr{A}$-module with basis $(C_w + \mathcal{J}_{\mathbf{\Gamma}})_{w \in \mathbf{\Gamma}}$, and a left $\mathcal{H}$-module satisfying

$$T_s(C_w + \mathcal{J}_{\mathbf{\Gamma}}) = \sum_{x \in \mathbf{\Gamma} \setminus \{w\}} q^{1/2} \mu_{s,w,x} C_x + \mathcal{J}_{\mathbf{\Gamma}},$$

where

$$\mu_{s,w,x} = \begin{cases} -1 & \text{if } sw < w = x \\ q^{1/2} & \text{if } x = w < sw \\ \hat{\mu}(x,w) & \text{if } sx < x < w < sw. \end{cases}$$

Such module is called a *left cell module of* $\mathcal{H}$ and has rank equal to the to the number of elements in the left cell. The representation afforded by the left cell module is then a *cell representation of* $\mathcal{H}$. Similarly, we can define right ideals giving rise to right $\mathcal{H}$-modules associated with the different right KL-cells of W, as well as two-sided ideals giving rise to right $\mathcal{H}$-modules associated with the different two-sided KL-cells of W.

Example 4.1.4. In [24] showed that if W is a Weyl group of type A_n, then the left cell representations are irreducible. Furthermore, any irreducible representation can be realized as a left cell representation.

Remark 4.1.8. The reader interested in representations associated with KL-cells should turn to the discussions in [41], [18], [15], [19] or [26] about the topic.

Lemma 4.1.1. *Let $w, y \in W$. If $C_w D_y \neq 0$, then $y^{-1} \leqslant_L w$. If $D_y C_w \neq 0$, then $y \leqslant_L w^{-1}$.*

Proof. First, assume that $C_w D_y \neq 0$. Using the definitions of D_y and C_w in (3.7.9) and (3.5.23), respectively, as an $\mathscr{A}$-linear combinations of the elements in the basis $\big(\widetilde{T}_w\big)_{w\in W}$, we see that the product $C_w D_y$ can be written as a $\mathscr{A}$-linear combination of elements in the basis $\big(\widetilde{T}_w\big)_{w\in W}$, and so in particular, as

$$C_w D_y = \sum_{z\in W} \alpha_z D_z \quad \text{with } \alpha_z \in \mathscr{A}.$$

Moreover, the fact $C_w D_y \neq 0$ implies that $\alpha_z \neq 0$ for at least one $z \in W$. For such an element z, say $z = x$, we have

$$\tau\big(C_{x^{-1}} C_w D_y\big) = \tau\left(C_{x^{-1}} \sum_{z\in W} \alpha_z D_z\right) = \sum_{z\in W} \alpha_z \tau\big(C_{x^{-1}} D_z\big) = \alpha_x \neq 0.$$

Recall that we may also write the product $C_{x^{-1}} C_w$ as

$$C_{x^{-1}} C_w = \sum_{z\in W} h_{x^{-1},w,z} C_z, \tag{4.1.6}$$

and thus, using the result in Corollary 3.7.1, we also get another expression for $\tau\big(C_{x^{-1}} C_w D_y\big)$ as

$$\tau\big(C_{x^{-1}} C_w D_y\big) = \tau\left(\sum_{z\in W} h_{x^{-1},w,z} C_z D_y\right) = \sum_{z\in W} h_{x^{-1},w,z} \tau\big(C_z D_y\big) = h_{x^{-1},w,y^{-1}},$$

which shows that the coefficient $h_{x^{-1},w,y^{-1}}$ of $C_{y^{-1}}$ in the expansion of $C_{x^{-1}} C_w$ in (4.1.6) is equal to

$$h_{x^{-1},w,y^{-1}} = \alpha_x \neq 0.$$

Hence, using (4.1.3), we deduce that $y^{-1} \leqslant_L w$, as required. The other assertion of the lemma is proved in exactly the same way but interchanging left and right. □

Example 4.1.5. (Type $\widetilde{\mathrm{A}}_2$) Let (W, S) is an affine Weyl group of type $\widetilde{\mathrm{A}}_2$ so that $S = \{s_1, s_2, s_3\}$ and W has a presentation

$$W = \big\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_s^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^3 = 1 \big\rangle.$$

as in Example 2.9.3. Now, let W_{ij} denote the standard parabolic subgroup generated by s_i and s_j for each pair $1 \leq i, j \leq 3$, and define, for each pair $1 \leq i, j \leq 3$, the set

$$W^{ij} := \{w \in W \mid \mathscr{R}(w) = \{s_i, s_j\}\}.$$

The sets W^{12}, W^{13} and W^{23} correspond to the green, pink and brown shaded areas in Figure 4.2. Similarly, for each $1 \leq i \leq 3$, define the set

$$W^{i} := \{w \in W \mid \mathscr{R}(w) = \{s_i\}\}.$$

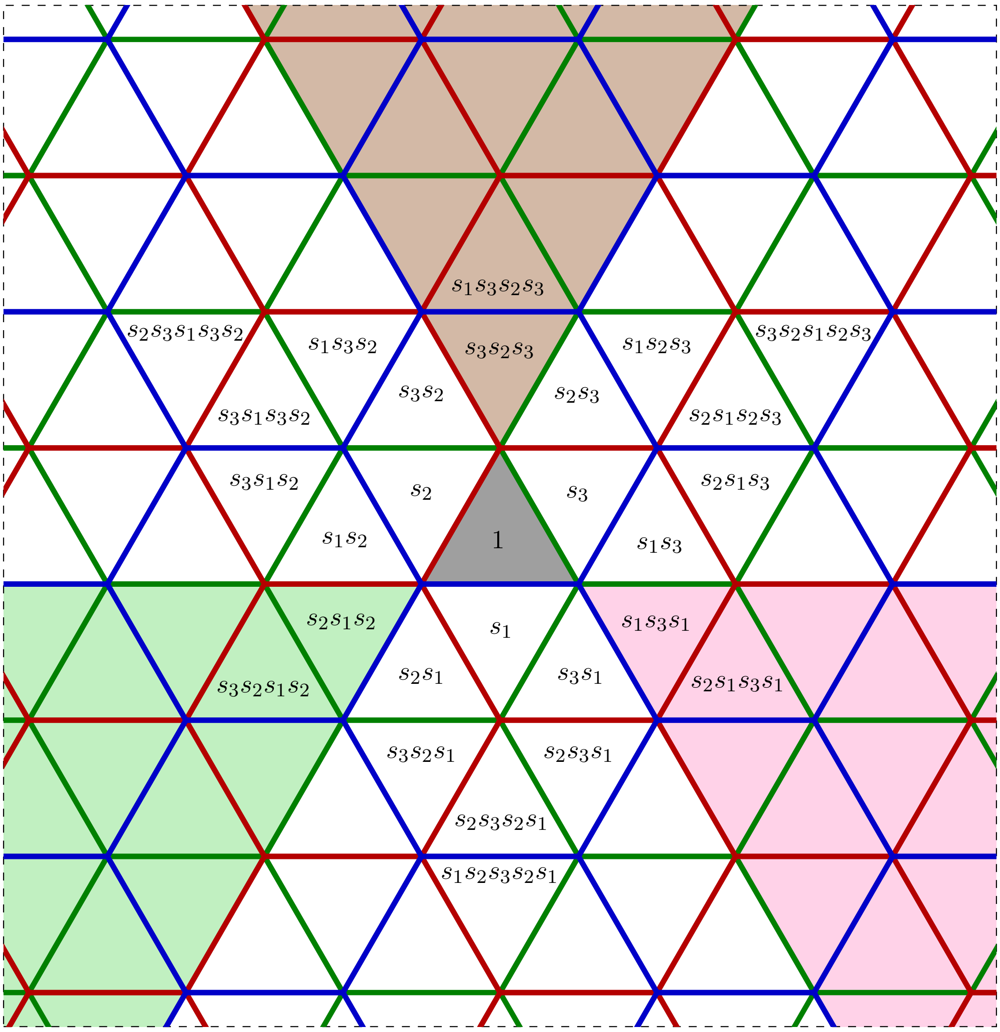


Figure 4.2: Sets W^{12}, W^{13} and W^{23} for W affine Weyl group of type $\widetilde{\mathrm{A}}_2$

First, by Corollary 4.1.1, the map $w \mapsto \mathscr{R}(w)$ is constant on the left KL-cells. Since the sets W^{12}, W^{13} and W^{23} are three different particular fibres of this map, it follows that each of the sets W^{12}, W^{13} and W^{23} is a union of left cells. Now, notice that

$$s_3s_2s_1s_2 \longrightarrow s_2s_1s_2 \quad \text{and} \quad \mathscr{L}(s_3s_2s_1s_2) = \{s_3\} \nsubseteq \{s_1, s_2\} = \mathscr{L}(s_2s_1s_2),$$

so $s_3s_2s_1s_2 \leqslant_L s_2s_1s_2$, and the reverse chain gives $s_2s_1s_2 \leqslant_L s_3s_2s_1s_2$. Now, if $w \in W^{12}$ with $\ell(w) = \ell(s_3s_2s_1s_2) + 1$ has either $\mathscr{L}(w) = \{s_2, s_3\}$ or $\mathscr{L}(w) = \mathscr{L}(s_1)$. In either case we have

$$w \longrightarrow s_3s_2s_1s_2 \longrightarrow s_2s_1s_2 \quad \text{and} \quad \mathscr{L}(w) \nsubseteq \mathscr{L}(s_3s_2s_1s_2) = \{s_3\} \nsubseteq \{s_1, s_2\} = \mathscr{L}(s_2s_1s_2),$$

so $w \leqslant_L s_3 s_2 s_1 s_2 \leqslant_L s_2 s_1 s_2$. Moreover, if $\mathscr{L}(w) = \{s_1\}$, then the reverse chain gives the relations $s_2 s_1 s_2 \leqslant_L s_3 s_2 s_1 s_2 \leqslant_L w$, and if $\mathscr{L}(w) = \{s_2, s_3\}$, then the chain $s_2 s_1 s_2 — w — s_3 s_2 s_1 s_2$ satisfies

$$\mathscr{L}(s_2 s_1 s_2) = \{s_1, s_2\} \not\subseteq \mathscr{L}(w) \not\subseteq \{s_3\} = \mathscr{L}(s_2 s_1 s_2),$$

which gives the relations $s_2 s_1 s_2 \leqslant_L w \leqslant_L s_3 s_2 s_1 s_2$. It thus follows that $w \sim_L s_3 s_2 s_1 s_2 \sim_L s_2 s_1 s_2$. Since for every element in W^{12} we can proceed in this way, we see that W^{12} is in fact a single left cell. By the symmetry between the three sets W^{12}, W^{13} and W^{23}, the same is also true for the sets W^{13} and W^{23}.

No take the set $W^{12} s_3$. This is the yellow set in Figure 4.3. Note that the shortest element of this set is the element $s_2 s_1 s_2 s_3$, and if $w \in W$ is any element of $W^{12} s_3$, then w is of the form $w = x s_2 s_1 s_2 s_3$ for some $x \in W$ with $x s_2 s_1 s_2 \in W^{12}$ and $\ell(w) = \ell(x) + \ell(s_2 s_1 s_2)$. In particular note that as we know that $x s_2 s_1 s_2 \sim_L w s_3$

$$w — s_2 s_1 s_2 s_3 \quad \text{and} \quad \mathscr{L}(w) \not\subseteq \mathscr{L}(s_2 s_1 s_2 s_3),$$

which gives $w \leqslant_L s_2 s_1 s_2 s_3$, and the reverse sequence gives $s_2 s_1 s_2 s_3 \leqslant_L w$, and thus $w \sim_L s_2 s_1 s_2 s_3$. Hence $W^{12} s_3$ is a left cell too. Again, by the symmetry with the sets $W^{13} s_2$ and $W^{23} s_1$ are left KL-cells too and these correspond to the regions shaded purple and orange, respectively, in Figure 4.3.

Finally consider the set $W^3 \setminus W^{12} s_3$. This corresponds to the dark blue region in Figure 4.3. Again, by Corollary 4.1.1, since the map $w \mapsto \mathscr{R}(w)$ is constant on the left KL-cells and the set $W^3 \setminus W^{12} s_3$ is a particular fibre of this map, it follows that it is a union of left cells. But note that the elements of $W^3 \setminus W^{12} s_3$ are exactly

$$s_3, s_2 s_3, s_1 s_2 s_3, s_3 s_1 s_2 s_3, s_2 s_3 s_1 s_2 s_3, s_1 s_2 s_3 s_1 s_2 s_3, s_3 s_1 s_2 s_3 s_1 s_2 s_3, \ldots,$$
$$s_1 s_3, s_2 s_1 s_3, s_3 s_2 s_1 s_3, s_1 s_3 s_2 s_1 s_3, s_2 s_1 s_3 s_2 s_1 s_3, s_3 s_2 s_1 s_3 s_2 s_1 s_3, \ldots,$$

all which have singled-element left descent sets, and we can choose a chain containing all the elements of elements of the set such that each pair of adjacent elements in the chain is joined and the chain has alternating left descent sets, and the same is true for the reverse sequence. We therefore see that $W^3 \setminus W^{12} s_3$ is another left cell of W, and by the symmetry, so are the sets $W^1 \setminus W^{23} s_1$ and $W^2 \setminus W^{12} s_2$, which correspond to the regions shaded light blue and red, respectively, in Figure 4.3. Since we have exhausted all the elements of W, we have shown that W is partitioned into 10 left cells and the partition is given by:

$$\begin{array}{llll} \mathbf{A}_{13} := W^{13}, & \mathbf{A}_2 := \mathbf{A}_{13} s_2 & \mathbf{B}_1 := W^1 \setminus \mathbf{A}_1, & \mathbf{C}_\emptyset = W^\emptyset \\ \mathbf{A}_{12} := W^{12}, & \mathbf{A}_3 := \mathbf{A}_{12} s_3, & \mathbf{B}_2 := W^2 \setminus \mathbf{A}_2, & \\ \mathbf{A}_{23} := W^{23}, & \mathbf{A}_1 := \mathbf{A}_{23} s_1 & \mathbf{B}_3 := W^3 \setminus \mathbf{A}_3 & \end{array}$$

The left KL-cell decomposition of W is described in Figure 4.3, where the different cells have been given different colours.

Now, consider the union of all the left KL-cells of W whose name contains a fixed capital letter, so that we obtain the following partition of W:

$$\begin{aligned} &\mathbf{A} = \mathbf{A}_{13} \cup \mathbf{A}_{12} \cup \mathbf{A}_{23} \cup \mathbf{A}_2 \mathbf{A}_3 \cup \mathbf{A}_1 \\ &\mathbf{B} := \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3, \\ &\mathbf{C} := \mathbf{C}_\emptyset. \end{aligned}$$

Notice that under the automorphism $w \mapsto w^{-1}$, we have

$$\mathbf{A} \mapsto \mathbf{A}, \quad \mathbf{B} \mapsto \mathbf{B} \quad \text{and} \quad \mathbf{C} \mapsto \mathbf{C}_\emptyset,$$

which implies that each of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is contained in a two-sided cell of W. Since $\mathbf{C}$ is a two-sided KL-cell, by (a) in Proposition 4.1.1. It remains for us check whether or not $\mathbf{A}$ and $\mathbf{B}$

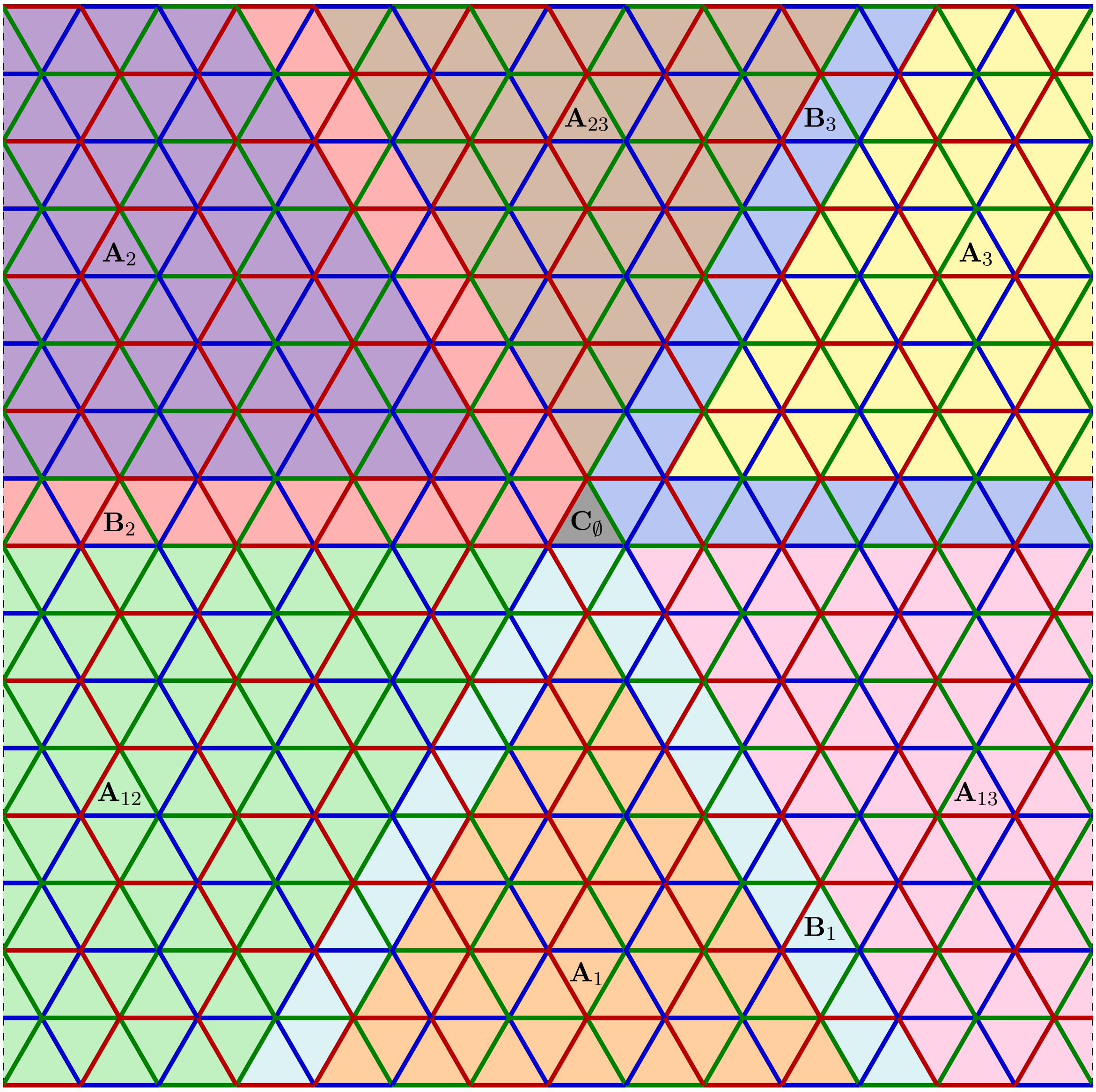


Figure 4.3: Left KL-cells of W of type $\widetilde{A}_2$

belong to the same two-sided KL-cell of W. The tool we introduce in the next section will help us solve this.

Remark 4.1.9. In Example 4.1.5, we have used a slightly different approach to the one Lusztig presents in [30], where he also presents explicit decompositions of KL-cells for the affine Weyl groups of type $\widetilde{\mathrm{B}}_2$ and $\widetilde{\mathrm{G}}_2$.

Definition 4.1.5. A subset $X \subseteq W$ is said to be *left-connected* if, for all $x, y \in X$, there exists a sequence $s_1, s_2, \ldots s_r$ of elements of S such that $y = s_r \cdots s_2 s_1 x$ and $s_i \cdots s_2 s_1 x \in X$ for all integers $i \in \mathbb{Z}$ such that $1 \leq i \leq r$. A subset X is said to be *right-connected* if X^{-1} is left-connected. The maximal left-connected subsets of X are called the *left-connected components of* X.

Remark 4.1.10. Note that every subset of W is the disjoint union of its left-connected components.

Conjectures 4.1.1. ***(Lusztig)***

(L1) *Every left KL-cell contains an involution.*

(L2) *If $w \in W$, then $w \sim_{LR} w^{-1}$.*

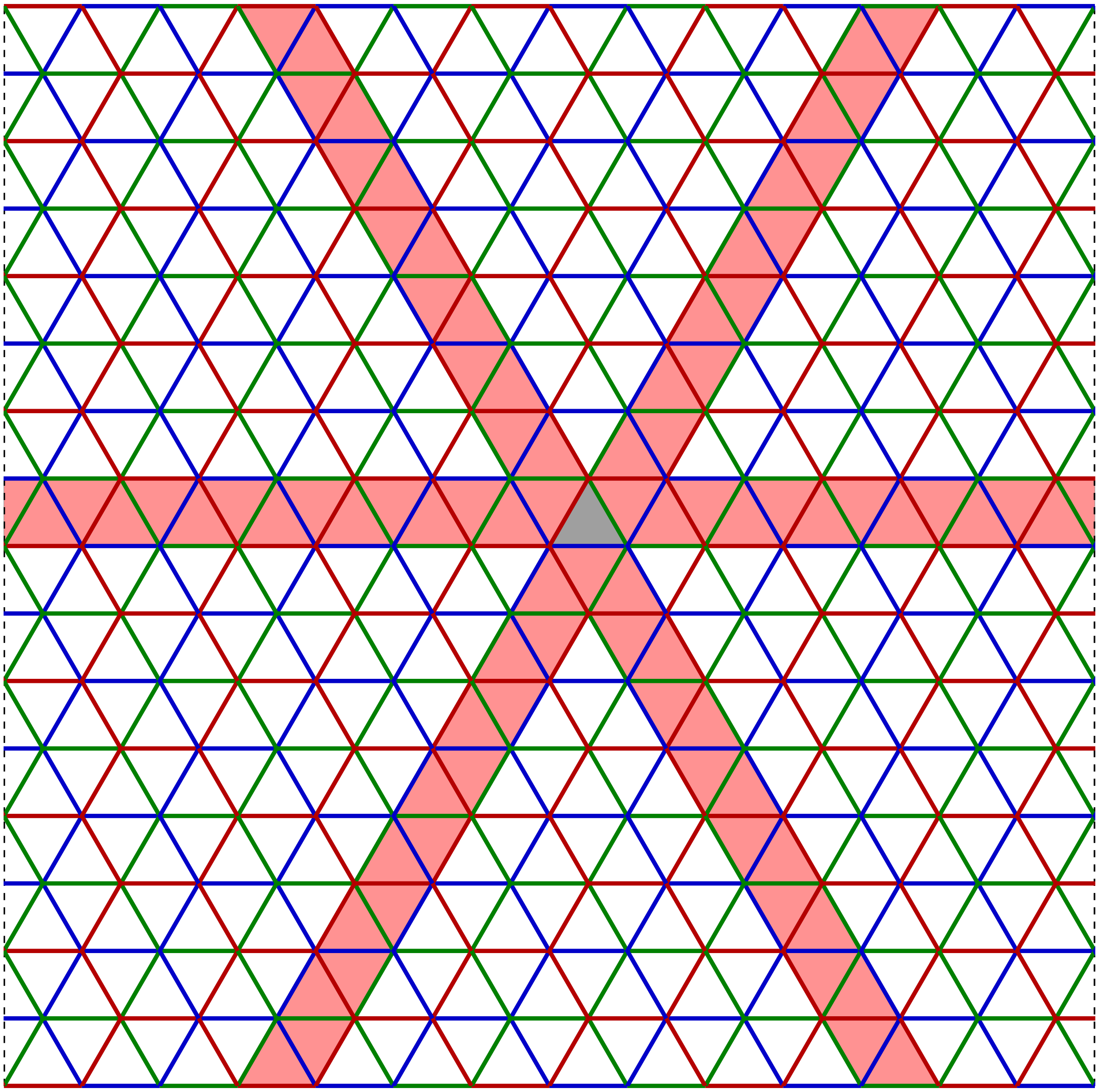


Figure 4.4: Two-sided cells of W of type $\widetilde{A}_2$

(L3) *The equivalence relation* "$\sim_{LR}$" *is generated by* "$\sim_L$" *and* "$\sim_R$".

(L4) *If* $x, y \in W$ *are such that* $x \leqslant_L y$ *and* $x \sim_{LR} y$, *then* $x \sim_L y$.

(L5) *Every left KL-cell is left-connected.*

(L6) *If* $\mathbf{\Gamma}$ *is a left KL-cell, then the left-connected components of* Γ *are the left cells of* W *contained in* $\mathbf{\Gamma}$.

(L7) *Every two-sided KL-cell meets a finite standard parabolic subgroup of* W.

(L8) *If* S *is finite, then the number of two-sided cells is finite.*

Remark 4.1.11. Conjectures (L7) and (L8) can be found in [32] and [20]. The remaining are implied by a series of conjectures in [32].

Remark 4.1.12. (L7) has been proved by Geck in [17] whenever W is finite.

Proposition 4.1.3. *The conjectures* (L1)-(L8) *are related in the following way:*

(a) (L1) *implies* (L2).

(b) (L7) *implies* (L8).

Proof. First, assume that (L1) holds. Let $\mathbf{\Gamma}$ be a left KL-cell of W and let $w \in \mathbf{\Gamma}$. Then, there exists an involution $d \in W$ such that $w \sim_L d$. But, then, by definition of a right KL-cell, we also have $w^{-1} \sim_R d^{-1} = d$, and so in particular, have $w^{-1} \leqslant_R d$ and $d \leqslant_R w^{-1}$. So we have a sequence w, d, w^{-1} such that $w \leqslant_L d$ and $d \leqslant_L w^{-1}$, which implies that $w \leqslant_{LR} w^{-1}$, and we have a sequence w^{-1}, d, w, such that $\left(w^{-1}\right)^{-1} \leqslant_L d$ and $d \leqslant_L w$, which implies that $w^{-1} \leqslant_{LR} w$. Hence $w \sim_{LR} w^{-1}$ and (L2) holds, which proves (a).

Now, assume that (L7) holds, and suppose that S is finite. Since S is finite, the number of elements of W contained in a standard parabolic subgroup is finite. Hence, since by (L7), the number of distinct two-sided KL-cells is bounded above by the number of distinct elements in the standard parabolic subgroups of W, it follows by the fact that such number is finite, that the number of two-sided KL-cells is finite and (L8) holds, which proves (b). □

Remark 4.1.13. We see from the collection (L1)-(L8) in Conjectures 4.1.1 that involutions play an important role in the number of KL-cells as well as that the particular choice for the set S of simple reflections.

4.2 The a-function

Lusztig's **a**-*function* is the most subtle invariant of cells that has been introduced so far. It was first introduced in [30] for Weyl groups and affine Weyl groups, where Lusztig defined $\mathbf{a}(w)$ as the order of the worst pole of the coefficient of C_w in a product $\widetilde{T}_x\widetilde{T}_y$ of two basis elements $\widetilde{T}_x, \widetilde{T}_y$ of the Hecke Algebra $\mathcal{H}$. The definition of the **a**-function is Lusztig's main contribution in [30], where he develops various techniques for computing the left and two-sided cells of Weyl groups or affine Weyl groups. It was later extended to arbitrary Coxeter groups in [16]. In this section we formally define Lusztig's **a** and study its connections with the decomposition of W into left, right, and two-sided KL-cells. We present some properties of the **a**-function that apply to general Coxeter systems (W, S), including a lower bound. The main references for this section are [25] and [30].

Definition 4.2.1. Given $w \in W$, consider the set

$$\mathscr{I}_w := \left\{ i \in \mathbb{N}_0 \mid q^{i/2}\tau(\widetilde{T}_x\widetilde{T}_y D_w) \in \mathscr{A}^+ \text{for all } x, y \in W \right\}. \tag{4.2.1}$$

If $\mathscr{I}_w$ is non-empty, we denote by $\mathbf{a}(w)$ the smallest integer in $\mathscr{I}_w$. If $\mathscr{I}_w$ is empty, we set $\mathbf{a}(w) = \infty$. This defines a function

$$\mathbf{a} : W \to \mathbb{N} \cup \{\infty\},$$

called the **a**-function.

Remark 4.2.1. Recall that $C_1 = T_1 = \widetilde{T}_1$. Moreover, for any $s \in S$, we see, from the definitions of C_s and $\widetilde{T}_s$, that

$$C_s = \widetilde{T}_s - q^{1/2}\widetilde{T}_1,$$

which shows that

$$\widetilde{T}_s = C_s + q^{1/2}C_1.$$

Now suppose that $w \in W$ is an element such that $\ell(w) > 1$, and assume that for any $x \in W$ with $\ell(x) < \ell(w)$, the following holds:

$$\widetilde{T}_x \in C_x + q^{1/2} \sum_{z<x} \mathscr{A}^+ \cdot C_z.$$

Now, the relation in (3.5.23) can be written as

$$\widetilde{T}_w = C_w - \varepsilon_w q_w^{1/2} \sum_{\substack{y\in W \\ y<w}} \varepsilon_y q_y^{-1/2} \overline{P}_{y,w} \widetilde{T}_y \in C_w + q^{1/2} \sum_{\substack{y\in W \\ y<w}} \mathscr{A}^+ \cdot \widetilde{T}_y,$$

where the containment as an element follows from the degree property (3.5.3) of the KL-polynomials. Since for any $y \in W$ such that $y < w$ we have that $\ell(y) < \ell(w)$, it follows, by the induction hypothesis, that

$$\widetilde{T}_w \in C_w + q^{1/2} \sum_{\substack{y\in W \\ y<w}} \mathscr{A}^+ \cdot C_y. \tag{4.2.2}$$

From the relations (3.5.24) and (4.2.2), it then follows that the set $\mathscr{I}_w$ is actually equal to

$$\mathscr{I}_w = \left\{ i \in \mathbb{N}_0 \mid q^{i/2} \tau(C_x C_y D_w) \in \mathscr{A}^+ \text{for all } x, y \in W \right\}. \tag{4.2.3}$$

Remark 4.2.2. In the introduction of this section, we described the **a**-function in a seemingly different way, but in fact the two definitions are equivalent. Let

$$\widetilde{T}_x \widetilde{T}_y = \sum_{z\in W} \alpha_{x,y,z} C_z \quad \text{with } \alpha_{x,y,z} \in \mathscr{A} \text{ for all } x, y, z \in W. \tag{4.2.4}$$

Consider the coefficient $\alpha_{x,y,w^{-1}}$ of $C_{w^{-1}}$ when $\widetilde{T}_x \widetilde{T}_y$ is written as an $\mathscr{A}$-combination of the KL-basis as in (4.2.4) and consider the order of the pole at 0 of such coefficient. Then, as $x, y \in W$ vary, the order of such pole may be bounded, and in this case $\mathbf{a}(w)$ is the largest such order, or it may be unbounded, in which case $\mathbf{a}(w) = \infty$.

Proposition 4.2.1. *Let (W, S) be any Coxeter system. We have $\mathbf{a}(w) = \mathbf{a}(w^{-1})$ for any $w \in W$.*

Proof. Recall the anti-homormorphism $\flat : \mathcal{H} \to \mathcal{H}$ of $\mathcal{H}$ in 3.1.2. Applying it to the equality in (4.2.4), we get

$$\widetilde{T}_{y^{-1}} \widetilde{T}_{x^{-1}} = \sum_{z\in W} \alpha_{x,y,z} C_{z^{-1}},$$

which shows that for any $x, y, z \in W$, we have $\alpha_{y^{-1},x^{-1},z^{-1}} = \alpha_{x,y,z}$. By the definition of $\mathbf{a} : W \to \mathbb{N}_0$, $\mathbf{a}(w) = \mathbf{a}(w^{-1})$ follows. □

Proposition 4.2.2. *Let (W, S) be any Coxeter system. Then $\mathbf{a}(w) = 0$ if and only if $w = 1$.*

Proof. By definition of D_1 and by the result in Example 3.7.1, it follows that

$$D_1 = \sum_{w\in W} q_w^{1/2} \widetilde{T}_w. \tag{4.2.5}$$

We no show, by induction on $\ell(x) \geq 1$ that $\widetilde{T}_x D_1 = q_x^{1/2} D_1$. First, for any $s \in S$, we have, by (3.7.11)

$$\widetilde{T}_s D_1 = \sum_{\substack{w\in W \\ w<sw}} q^{1/2} q_w^{1/2} \widetilde{T}_w + \sum_{\substack{w\in W \\ sw<w}} q^{-1/2} q_w^{1/2} \left(1 + (q-1)\right) \widetilde{T}_w = q^{1/2} D_1.$$

Now suppose $x \in W$ is an element such that $\ell(x) > 1$ and assume that $\widetilde{T}_y D_1 = q_y^{1/2} D_1$ for any $y \in W$ such that $\ell(y) < \ell(x)$. Pick $s' \in S$ such that $x = sy$ with $\ell(y) < \ell(x)$, so that $\widetilde{T}_x = \widetilde{T}_s \widetilde{T}_y$. Then, by the induction hypothesis, it follows that

$$\widetilde{T}_x D_1 = \widetilde{T}_s \widetilde{T}_y D_1 = \widetilde{T}_s q_y^{1/2} D_1 q^{1/2} D_1 = q_{sy}^{1/2} D_1 = q_x^{1/2} D_1,$$

as required. Hence

$$\tau(\widetilde{T}_x\widetilde{T}_yD_1) = q_y^{1/2}\tau(\widetilde{T}_xD_1) = q_y^{1/2}q_x^{1/2}\tau(D_1),$$

and since, by (3.7.9) and by definition of τ, we have

$$\tau(D_1) = \tau\Big(\sum_{w\in W} q_w^{1/2}\widetilde{T}_w\Big) = 1,$$

it follows that

$$\tau(\widetilde{T}_x\widetilde{T}_yD_1) = q_y^{1/2}q_x^{1/2} \in \mathscr{A}^+$$

for all $x, y \in W$. It follows that $\mathbf{a}(1) = 0$.

On the other hand, assume that $w \in W$ is a non-identity element an let $s \in S$ be such that $sw < w$. Then, using the relation for multiplication with respect to to the basis $\big(\widetilde{T}\big)_{w\in W}$ in (3.5.20) and (3.5.21), we get

$$\widetilde{T}_s\widetilde{T}_w = \widetilde{T}_{sw} + \big(q^{1/2} - q^{-1/2}\big)\widetilde{T}_w,$$

which by (3.5.24) we know is of the form

$$\widetilde{T}_s\widetilde{T}_w = \big(q^{1/2} - q^{-1/2}\big)C_w + \mathscr{A}\text{-linear combination of elements } C_{w'} \text{ for } w' < w.$$

Since $\tau\big(C_{w'}D_{w^{-1}}\big) = 0$ for any $w' < w$, we have that

$$\tau\big(\widetilde{T}_s\widetilde{T}_wD_{w^{-1}}\big) = \tau\Big(\big(q^{1/2} - q^{-1/2}\big)C_wD_{w^{-1}}\Big) = q^{1/2} - q^{-1/2},$$

which implies that $0 \in \mathscr{I}_{w^{-1}}$. This implies that $\mathbf{a}(w^{-1}) \geq 1$, and thus, by Proposition 4.2.1, we conclude that $\mathbf{a}(w) \geq 1$. The proof is now complete. □

Proposition 4.2.3. *Let J be a subset of S which generates a finite subgroup of W, and let w_J be the longest element in this subgroup. Let $w, w', w'' \in W$ such that $w = w'w_Jw''$ and $l(w) = l(w') + l(w_J) + l(w'')$. Then $\mathbf{a}(w) \geq l(w_J)$.*

Proof. First note that since every time we multiply w_J by an element $s \in J$ we get $sw_J < w_J$, so the coefficient of $\widetilde{T}_{w_J}$ in the product $\widetilde{T}_{T_{w_J}}\widetilde{T}_{w_J}$ will be of the form

$$\big(q^{1/2} - q^{-1/2}\big)^{\ell(w_J)} + \text{smaller powers of } (q^{1/2} - q^{-1/2}),$$

and thus

$$\widetilde{T}_{w_J}\widetilde{T}_{w_J} = \big(\varepsilon_{w_J}q_{w_J}^{-1/2} + \text{higher powers of } q^{1/2}\big)\ \widetilde{T}_{w_J} + \sum_{\substack{y\in W\\ y<w_J}} f_{w_J,w_J,y}\widetilde{T}_y,$$

where $\alpha_{w_J,w_J,y} \in \mathscr{A}$ for all $y \in W$ such that $y < w_J$. Then, since

$$\widetilde{T}_{w'w_J}\widetilde{T}_{w_Jw''} = \widetilde{T}_{w'}\widetilde{T}_{w_J}\widetilde{T}_{w_J}\widetilde{T}_{w''},$$

it follows that

$$\widetilde{T}_{w'w_J}\widetilde{T}_{w_Jw''} = \big(\varepsilon_{w_J}q_{w_J}^{-1/2} + \text{higher powers of } q^{1/2}\big)\widetilde{T}_{w'}\widetilde{T}_{w_J}\widetilde{T}_{w''} + \sum_{\substack{y\in W\\ y<w_J}} f_{w_J,w_J,y}\widetilde{T}_{w'}\widetilde{T}_y\widetilde{T}_{w''}.$$

But since $l(w) = l(w') + l(w_J) + l(w'')$, the multiplication formula in (3.5.20) tells us that $\widetilde{T}_{w'}\widetilde{T}_{w_J}\widetilde{T}_{w''} = \widetilde{T}_w$, so the above becomes

$$\widetilde{T}_{w'w_J}\widetilde{T}_{w_Jw''} = \big(\varepsilon_{w_J}q_{w_J}^{-1/2} + \text{higher powers of } q^{1/2}\big)\widetilde{T}_w + \sum_{\substack{z\in W\\ z<w}} f_{w_J,w_J,z}\widetilde{T}_z.$$

Then, using (4.2.2), we obtain

$$\widetilde{T}_{w'w_J}\widetilde{T}_{w_Jw''} = \big(\varepsilon_{w_J} q_{w_J}^{-1/2} + \text{higher powers of } q^{1/2}\big)C_w + \sum_{\substack{z\in W\\ z<w}} \alpha_{w_J,w_J,z} Cz. \tag{4.2.6}$$

Since $\tau\big(C_z D_{w^{-1}}\big) = 0$ for all $z \in W$ such that $z < w$ and $\tau\big(C_w D_{w^{-1}}\big) = 1$, multiplying the equality in (4.2.6) by $D_{w^{-1}}$ then applying τ to it gives

$$\tau\big(\widetilde{T}_{w'w_J}\widetilde{T}_{w_Jw''}D_{w^{-1}}\big) = \big(\varepsilon_{w_J} q_{w_J}^{-1/2} + \text{higher powers of } q^{1/2}\big),$$

which shows that $\mathbf{a}(w^{-1}) \geq \ell(w_J)$, and thus, by Proposition 4.2.1, $\mathbf{a}(w) \geq \ell(w_J)$. □

Remark 4.2.3. Proposition 4.2.3 gives a lower bound for $\mathbf{a}(w)$, where $w \in W$ and W is any Coxeter group. At the moment, there is no known description of a general upper bound for the $\mathbf{a}$-function. The closest we are to a bound is given by the following:

Lemma 4.2.1. *Let (W, S) be any Coxeter system and for any $x, y, z \in W$, define polynomials $f_{x,y,z} \in \mathscr{A}$ so that*

$$\widetilde{T}_x\widetilde{T}_y = \sum_{z\in W} f_{x,y,z}\widetilde{T}_z. \tag{4.2.7}$$

Then, for any $x, y, z \in W$ we have that $f_{x,y,z}$ is a polynomial in $\xi := q^{1/2} - q^{-1/2}$ of degree bounded by the smallest of $\ell(x), \ell(y)$ and $\ell(z)$.

Proof. See Lemma 7.4 in [30]. □

Remark 4.2.4. Note that the bound in Lemma 4.2.1 does not necessarily mean that $\deg_\xi f_{x,y,z}$ is actually finite, since it may well be the case that x, y and z are all of infinite length. However, Lusztig conjectures the following in [32]:

Conjecture 4.2.1. *There exists an integer $N \geq 0$ such that $q^{N/2} f_{x,y,z} \in \mathscr{A}^+$ for all $x, y, z \in W$. In particular*

$$N = \max_{\substack{J\subseteq S\\ |W_J|<\infty}} \big\{\ell(w_J)\big\},$$

where $|W_J|$ denotes the order of the the subgroup W_J of W generated by the elements in $J \subseteq S$.

Of course, Conjecture 4.2.1 is known to hold whenever W is finite, by Lemma 4.2.1, since in this case the lengths of all elements of W are finite. In [30], Lusztig also shows that it holds if W is an affine crystallographic Coxeter group. In particular he first proves the following slightly stronger version of Lemma 4.2.1:

Lemma 4.2.2. *Let (W, S) be an affine crystallographic Coxeter system. Then for any $x, y, z \in W$, $f_{x,y,z^{-1}}$ is a polynomial in $\xi := q^{1/2} - q^{-1/2}$ with integral, non-negative coefficients, whose degree satisfies*

$$\deg_\xi f_{x,y,z^{-1}} \leq \min\{\ell(x), \ell(y), \ell(z)\}. \tag{4.2.8}$$

Proof. See Lemma 7.4 in [30]. The idea is to first use induction on $\ell(x)$ to show that $f_{x,y,z^{-1}}$ in a polynomial in ξ with integral non-negative coefficients of degree at most $\ell(x)$. Then, do a induction on $\ell(y)$ to conclude that $f_{x,y,z^{-1}}$ has degree as a polynomial in ξ at most $\ell(y)$. Then use these results together with the equalities

$$f_{x,y,z^{-1}} = \tau\big(\widetilde{T}_x\widetilde{T}_y\widetilde{T}_z\big) = \tau\big(\widetilde{T}_y\widetilde{T}_z\widetilde{T}_x\big) = f_{y,z^{-1},x}$$

to deduce that $\deg_\xi f_{y,z^{-1},x} \leq \ell(z)$, hence $\deg_\xi f_{x,y,z^{-1}} \leq \ell(z)$. □

He then uses Lemma 4.2.2 and the realization of W in terms of alcoves in Section 2.9 together with the free-module $\mathcal{M}$ with basis corresponding to the alcoves, to prove the following:

Theorem 4.2.1. *For any $x, y, z \in W$ the polynomial $f_{x,y,z} \in \mathscr{A}$ is a polynomial in ξ with integral, non-negative coefficients, of degree*

$$\deg_{\xi} f_{x,y,z} \leq \operatorname{card} \Phi^{+},$$

where $\operatorname{card} \Phi^{+}$ *denotes the number of positive roots in the corresponding root system of* (W, S).

Corollary 4.2.1. *For any $w \in W$, we have $\mathbf{a}(w) \leq \operatorname{card} \Phi^{+}$.*

Proof. From Theorem 4.2.1, we see that

$$q^{(1/2)\cdot \operatorname{card} \Phi^{+}} f_{x,y,z} \in \mathscr{A}^{+}$$

for all $x, y, z \in W$. On the other hand, we know, by the relation in (4.2.2) that

$$\widetilde{T}_{z^{-1}} \in \sum_{u \in W} \mathscr{A}^{+} \cdot C_u,$$

so it follows that

$$q^{(1/2)\cdot \operatorname{card} \Phi^{+}} \widetilde{T}_x \widetilde{T}_y \in \sum_{u \in W} \mathscr{A}^{+} \cdot C_u,$$

and the corollary follows, by definition of the **a**-function. □

Example 4.2.1. (Type $\widetilde{\mathrm{A}}_2$) Let (W, S) is an affine Weyl group of type $\widetilde{\mathrm{A}}_2$ for which we explicitly describe its decomposition into left KL-cells in Example 4.1.5. We now want to compute the values of the **a**-function on the elements of W. Since the number of positive roots in the root system corresponding to W is 3, then we know that for any $w \in W$, we have $\mathbf{a}(w) \leq 3$. Now, since every element w in the left KL-cell $\mathbf{A}_{12}$ is of the form $w = w' s_2 s_1 s_2$ where $w' \in W$ such that $\ell(w) = \ell(w') + \ell(s_1 s_2 s_1)$ and $s_1 s_2 s_1$ is the longest element in the subgroup W_{12} of W generated by s_1 and s_2, it follows, by Proposition 4.2.3, that $\mathbf{a}(w) \geq 3$, and thus $\mathbf{a}(w) = 2$ for any $w \in \mathbf{A}_{12}$. Since the left KL-cell $\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are all sets with the same property as $\mathbf{A}_{12}$ with their corresponding longest element each, which is either $s_1 s_2 s_1$, $s_2 s_3 s_2$ or $s_1 s_3 s_1$, it follows that $\mathbf{a}(w) = 3$ for every element in one of these left KL-cells. Using similar arguments, the $\mathscr{A}$-linear map τ and the multiplication relations of the KL-basis, we compute

$$\mathbf{a}(w) = \begin{cases} 0 & w \in \mathbf{C} \\ 1 & w \in \mathbf{B} \\ 3 & w \in \mathbf{A}. \end{cases} \tag{4.2.9}$$

A different method is used in Section 4.4, once we have established further results. We thus see that the **a**-function is constant on the set **B** with a value different to the constant value on the set **A**. We later see that the **a** is constant on the two-sided cells, allowing to deduce that each of **A**, **B** and **B** is indeed a two-sided cell of W, and the partition represented in Figure 4.4 is the correct partition into two-sided KL-cells. Also, since each of the parabolic subgroups is a Coxeter system of type $\widetilde{\mathrm{A}}_2$, we see that in this case, for any $I \subset S$, if $w \in W$ is such that $w \in W_I$, we have that $\mathbf{a}_I(w) = \mathbf{a}(w)$.

Remark 4.2.5. Conjecture 4.2.1 also holds, for instance, if the Coxeter graph is complete, proved by Xi in [42], or if the rank is 3, proved by Zhou in [43].

Remark 4.2.6. In the case of affine Weyl group, given the result in Theorem 4.2.1, the study of the set

$$\{w \in W \mid \mathbf{a}(w) = \operatorname{card} \Phi^{+}\}$$

becomes relevant for affine Weyl groups. It turns out to be a two-sided cell of W. This has been studied in [7] and [35] and [36]. In particular, the last two articles study the left KL-cells contained in such set as well as the exact number of such left KL-cells.

Remark 4.2.7. In [30], Lusztig proves that in the case of finite crystalographic Coxeter groups $\mathbf{a}(w) \leq \ell(w)$ for any $w \in W$ using the fact that since W is finite, then $\mathcal{H} = \hat{\mathcal{H}}$ and so the products $D_x \widetilde{T}_y D_z$ and $\widetilde{T}_y D_z D_x$ are actually well-defined, but he claims that he does not see how to carry the proof for infinite crystallographic Coxeter groups. Later in [31], Lusztig generalizes his result using a proof shown to him by Springer based on a "positivity property" of the coefficient of the structures constants with respect to the KL-basis for crystallographic Coxeter groups. It has not been until recently that the inequality $\mathbf{a}(w) \leq \ell(w)$ has been shown to hold for arbitrary Coxeter groups. This has been possible thanks to the proof of Conjecture 3.5.9 about the non-negativity of the coefficients of $P_{y,w}$ by Ellias and Williamson in [11]. Using an adaptation of Lusztig's proof suggested by Springer's, we can prove the following for general Coxeter groups:

Proposition 4.2.4. *Let (W, S) be an arbitrary Coxeter system. Then, for any $w \in W$, we have $\mathbf{a}(w) \leq \ell(w)$.*

Proof. For any $x, y, w \in W$, define polynomials $h_{x,y,w} \in \mathscr{A}^+$ so that

$$C_x C_y = \sum_{w \in W} h_{x,y,w} C_w. \tag{4.2.10}$$

If we then apply the $\mathscr{A}$-linear map $\tau : \hat{\mathcal{H}} \to \mathscr{A}$ to both sides of the equality in (3.5.24), we get

$$\tau(C_w) = \varepsilon_w q_w^{1/2} \sum_{\substack{z \in W \\ z < w}} \varepsilon_z q_z^{-1/2} \overline{P}_{z,w} \tau\big(\widetilde{T}_z\big) = \varepsilon_w q_w^{1/2} \varepsilon_1 q_1^{-1/2} \overline{P}_{1,w} = \varepsilon_w q_w^{1/2} \overline{P}_{1,w},$$

Therefore, applying $\mathscr{A}$-linear map $\tau : \hat{\mathcal{H}} \to \mathscr{A}$ to both sides of the equality in (4.2.10), gives

$$\tau(C_x C_y) = \sum_{w \in W} h_{x,y,w} \varepsilon_w q_w^{1/2} \overline{P}_{1,w}. \tag{4.2.11}$$

But note that, writing C_x and C_y as an $\mathscr{A}$-linear combination of the $\widetilde{T}_w$ $(w \in W)$ as in (3.5.24), we get

$$C_x C_y = \sum_{\substack{x' \in W \\ x' < x}} \sum_{\substack{y' \in W \\ y' < y}} \varepsilon_x q_x^{1/2} \varepsilon_y q_y^{1/2} \varepsilon_{x'} q_{x'}^{-1/2} \overline{P}_{x',x} \varepsilon_{y'} q_{y'}^{-1/2} \overline{P}_{y',y} \widetilde{T}_{x'} \widetilde{T}_{y'},$$

and so, by (a) in Lemma 3.7.2, we see that

$$\tau(C_x C_y) = \varepsilon_x q_x^{1/2} \varepsilon_y q_y^{1/2} \sum_{\substack{x' \in W \\ x' < x \\ x'^{-1} < y}} q_{x'}^{-1} \overline{P}_{x',x} \overline{P}_{x'^{-1},y}, \tag{4.2.12}$$

if there is some $x' < x$ such that $x'^{-1} = y' < y$, and $\tau(C_x C_y) = 0$ otherwise. Therefore, since by the degree property in (3.5.3) of Theorem 3.5.1, we have, using the equality $\ell(x') = \ell(x'^{-1})$, that

$$\deg_q q_{x'} P_{x',x} P_{x'^{-1},y} \leq \ell(x') + \frac{1}{2}\Big(\ell(x) - \ell(x') - 1 + \ell(y) - \ell(x'^{-1}) - 1\Big) = \frac{1}{2}\left(\ell(x) + \ell(y) - 2\right),$$

and thus, in the first case, we have that the lowest power of q in $\tau(C_x C_y)$ is

$$\frac{1}{2}\left(\ell(x) + \ell(y)\right) - \max_{\substack{x' \in W \\ x' < x \\ x'^{-1} < y}} \deg_q P_{x',x} P_{x'^{-1},y} \geq 1.$$

It thus follows that in either case we have $\tau(C_x C_y) \in \mathscr{A}^+$, and from (4.2.11), we recover

$$\sum_{w \in W} h_{x,y,w} \varepsilon_w q_w^{1/2} \overline{P}_{1,w} = \tau(C_x C_y) \in \mathscr{A}^+. \tag{4.2.13}$$

Now, by the "positivity property" of the coefficients of the KL-polynomials, we have that the the coefficient of $\overline{P}_{1,w}$ is non-negative for every $w \in W$ in the sum, and moreover, given the multiplication formulas in (3.6.7)-(3.6.10), the same is true for $h_{x,y,w}$ as a polynomial in $-q^{1/2}$ and $-q^{-1/2}$. Hence the sum in (4.2.13) has no cancellations, and we have

$$h_{x,y,w}\varepsilon_w q_w^{1/2}\overline{P}_{1,w} \in \mathscr{A}^+ \quad \text{for all } x, y, w \in W. \tag{4.2.14}$$

Since $P_{1,w} \neq 0$, and so $\overline{P}_{1,w} \neq 0$, it follows that $h_{x,y,w}\varepsilon_w q_w^{1/2} \in \mathscr{A}^+$, which shows that $\mathbf{a}(w) \leq \ell(w)$. □

Remark 4.2.8. Note that the proof of Proposition 4.2.4 not only tells us that $h_{x,y,w}\varepsilon_w q_w^{1/2} \in \mathscr{A}^+$, but also, from (4.2.14), we see that $h_{x,y,w}\varepsilon_w q_w^{1/2} q^{-2\delta(w)} \in \mathscr{A}^+$, where $\delta(w) := \deg_q P_{1,w}$, and so it follows that

$$\mathbf{a}(w) \leq \ell(w) - 2\delta(w) \quad \text{for all } w \in W.$$

4.3 Distinguished involutions

In [23], Joseph shows that for each left KL-cell of a Weyl group, the function $x \mapsto \ell(w) - 2\delta(w)$, reaches its minimum at a unique element of that left KL-cell, which he calls "Duflo involution". Inspired by this, and in view of Remark 4.2.8, Lusztig shows in [31] that such minimum value is in fact $\mathbf{a}(w)$ for each w in the left KL-cell. Of course in [31], given that the "positivity property" for general Coxeter groups has only been recently proved in [11], Lusztig only proves this for Weyl and affine Weyl groups using, as starting point, the result in Remark 4.2.8, for which he had, at the time, a proof for in these cases. For this, he assumes that (W, S) is such that $\mathbf{a}(w) < \infty$, and defines a set

$$\mathscr{D} := \{w \in W \mid \mathbf{a}(w) = \ell(w) - 2\delta(w)\},$$

which enables him to prove several properties of the $\mathbf{a}$-function that were proved in [30] only for Weyl groups, this time for a larger class of Coxeter groups, including the affine Weyl groups. Today, the "positivity property" for general Coxeter groups allows us to define the set $\mathscr{D}$ whithout the boundedess assumption, following Remark 4.2.8. In this section, we construct the set $\mathscr{D}$ following [31] but updating the results taking into account the recently proved positivity property for general Coxeter groups.

Definition 4.3.1. For any $x, y, w \in W$ define $\gamma_{x,y,w} \in \mathbb{Z}$ by

$$(-1)^{\mathbf{a}(w)} q^{\mathbf{a}(w)/2} h_{x,y,w^{-1}} - \gamma_{x,y,w} \in -q^{1/2}\mathscr{A}^+.$$

Remark 4.3.1. Note that applying the unique algebra anti-automorphism $\flat : \mathcal{H} \to \mathcal{H}$ mapping $T_w \mapsto T_{w^{-1}}$ for any $w \in W$ in Definition 3.1.2 to the equality in (4.2.7), we deduce that

$$f_{x,y,w} = f_{y^{-1},x^{-1},z^{-1}} \quad \text{for any } x, y, z \in W. \tag{4.3.1}$$

Similarly, since $\flat(C_w) = C_{w^{-1}}$ for any $w \in W$, we deduce from (4.2.10), that

$$h_{x,y,w} = h_{y^{-1},x^{-1},w^{-1}} \quad \text{for any } x, y, w \in W. \tag{4.3.2}$$

From this together with the definition of $\gamma_{x,y,w}$, we deduce

$$\gamma_{x,y,w} = \gamma_{y^{-1},x^{-1},w^{-1}} \quad \text{for any } x, y, w \in W. \tag{4.3.3}$$

From (4.3.2) we immediately see that $\mathbf{a}(w) = \mathbf{a}(w^{-1})$ for any $w \in W$.

Proposition 4.3.1. *Let (W, S) be any Coxeter system.*

(a) Let $d \in W$. Then $d \in \mathscr{D}$ if and only if $d^{-1} \in \mathscr{D}$.

(b) *Let* $d \in \mathscr{D}$. *Then if* $x, y \in W$ *are such that* $\gamma_{x,y,d} \neq 0$, *then* $x = y^{-1}$ *and* $\gamma_{y^{-1},y,d} = 1$.

(c) *For any* $y \in W$, *there is a unique* $d \in \mathscr{D}$ *such that* $\gamma_{y^{-1},y,d} \neq 0$.

(d) *If* $d \in \mathscr{D}$, *then* $d^2 = 1$.

Proof. From (4.3.1) and (4.3.2), we immediately see that $\mathbf{a}(w) = \mathbf{a}(w^{-1})$ for any $w \in W$. Therefore, by definition of $\mathscr{D}$ and the fact that $P_{1,w} = P_{1,w^{-1}}$, we have that $d \in \mathscr{D}$ if and only if

$$\mathbf{a}(d^{-1}) = \mathbf{a}(d) = \ell(d) - 2\delta(d) = \ell(d^{-1}) - 2\delta(d^{-1}),$$

which is the case if and only if $d^{-1} \in \mathscr{D}$. This proves (a).

Now, let $x, y \in W$ and $d \in \mathscr{D}$ be as in (b) and consider the inclusion in (4.2.14) with $w = d^{-1}$. The left-hand side has constant term equal to $\gamma_{x,y,d} n_d \in$, where $n_d \in \mathbb{Z}$ such that

$$P_{1,d} = P_{1,d^{-1}} = n_d q^{\delta(d)} + \text{lower powers of } q.$$

As seen in the proof of Proposition 4.2.4, each term $h_{x,y,w}\varepsilon_w q_w^{1/2}\overline{P}_{1,w}$ in the sum in (4.2.13) has non-negative constant term. Since the sum of these constant terms is the constant term of $\tau(C_x C_y)$, this shows that we have that the constant term of $\tau(C_x C_y)$ is at least $\gamma_{x,y,d} n_d > 0$. But recall, from the proof of Proposition 4.2.4, that $\tau(C_x C_y) = 0$ if $x \neq y^{-1}$, and if $x = y^{-1}$, then $\tau(C_x C_y)$ is given by (4.2.12) and so from the bounds of the degree of the polynomials $P_{x',x}$ and $P_{x',x^{-1}}$, we see immediately that the constant term of $\tau(C_x C_y)$ is exactly 1 in this case. Combining this with the fact that constant term of $\tau(C_x C_y)$ is at least $\gamma_{x,y,d} n_d > 0$, it must be the case that $x = y^{-1}$, so that $\gamma_{y^{-1},y,d} n_d = 1$, and since $\gamma_{y^{-1},y,d}, n_z \in \mathbb{Z}$, it follows that

$$\gamma_{y^{-1},y,d} = n_d = 1,$$

which proves (b).

Now, let $y \in W$. For this, recall the discussion in the previous paragraph about the constant term of $\tau(C_x C_y)$, which gives that the constant term of the expression in (4.2.13) for $x = y^{-1}$ is equal to 1. Since each term in the sum in the left-hand side of the equality in (4.2.13) has non-negative constant term, it follows that $h_{y^{-1},y,w}\varepsilon_w q_w^{1/2}\overline{P}_{1,w}$ has constant term 1 for a unique w, say w', and has constant term equal to 0 for all $w \in W$ with $w \neq w'$. Also notice that we may write

$$h_{y^{-1},y,w}\varepsilon_w q_w^{1/2}\overline{P}_{1,w} = (-1)^{\mathbf{a}(w)} q_w^{\mathbf{a}(w)/2} h_{y^{-1},y,w} \cdot q_w^{\delta(w)}\overline{P}_{1,w} \cdot (-q^{1/2})^{\ell(w)-\mathbf{a}(w)-2\delta(w)},$$

with

$$(-1)^{\mathbf{a}(w)} q_w^{\mathbf{a}(w)/2} h_{y^{-1}} \in \mathscr{A}^+, \quad q_w^{\delta(w)}\overline{P}_{1,w} \in \mathscr{A}^+, \quad \text{and} \quad \ell(w) - \mathbf{a}(w) - 2\delta(w) \geq 0.$$

Then, for $w = w'$, we have $\gamma_{y^{-1},y,w^{-1}} \neq 0$ and $\ell(w) - \mathbf{a}(w) - 2\delta(w) \geq 0$, which shows that $w' \in \mathscr{D}$, and for $w \neq w'$, we have $\gamma_{y^{-1},y,w} = 0$. By the result in (a), the result in (c) follows.

Finally, take any $d \in \mathscr{D}$, and find $x, y \in W$ such that $\gamma_{x,y,d} \neq 0$. By the result in (a), we have $x = y^{-1}$ so that $\gamma_{x^{-1},x,d} \neq 0$. From the equality in(4.3.3), we obtain

$$\gamma_{y^{-1},y,d^{-1}} = \gamma_{y^{-1},y,d^{-1}} \neq 0,$$

and thus, by the uniqueness in the the result in (c), it follows that $d = d^{-1}$. This proves (d) and completes the proof. □

Remark 4.3.2. In view of (b) in Proposition 4.3.1, we call the elements $d \in \mathscr{D}$ *distinguished involutions of* W, and the set $\mathscr{D}$ is called *the set of distinguished involutions of* W.

Example 4.3.1. Let (W, S) be any Coxeter system. By Proposition 4.2.2, we have $\mathbf{a}(1) = 0$. Therefore, since $\ell(w) = 0$ and $\delta(1) = \deg_q P_{1,1} = 0$, we see that

$$\mathbf{a}(1) = \ell(1) - \delta(1),$$

and thus $1 \in \mathscr{D}$.

Example 4.3.2. Let (W, S) be any Coxeter system and let $s \in S$ be an arbitrary simple reflection. By Proposition 4.2.4, we have $\mathbf{a}(s) \leq \ell(s) = 1$, and thus, by Proposition 4.2.2, it follows that $\mathbf{a}(s) = 1$. Moreover, since $\ell(s) - \ell(1) = 1 \leq 2$, we know by Corollary 3.5.1, that $P_{1,s} = 1$, and thus $\delta(s) = 0$. We therefore see that

$$\mathbf{a}(s) = 1 = \ell(s) = \ell(s) - 2\delta(s),$$

which shows that $s \in \mathscr{D}$. Since $s \in S$ is chosen arbitrarily, it follows that $S \subseteq \mathscr{D}$.

Lemma 4.3.1. *Let (W, S) be any Coxeter system. If $y, w \in W$ are elements such that $y \leqslant_{LR} w$, then $\mathbf{a}(y) \geq \mathbf{a}(w)$. Hence, if $y \sim_{LR} w$, then $\mathbf{a}(y) = \mathbf{a}(w)$.*

Proof. Take any elements $y', w' \in W$ and any $s \in S$ such that $y's < y$ and $w' < w's$ and such that $h_{w',s,y'} \neq 0$, and let $x, z \in W$ be any elements such that $\gamma_{x,z,w'} \neq 0$. Then we have $(-q)^{\mathbf{a}(w')/2} h_{x,z,'w^{-1}} \neq 0$, so there exists $x' \in W$ such that $(-q)^{\mathbf{a}(w')/2} h_{x',z,y'^{-1}}$ has a non-zero constant term. In particular, we have $\mathbf{a}(w') \leq \mathbf{a}(y')$.

Now, if $y, w \in W$ are such that $y \leqslant_{LR} w$, then we may assume, by the transitivity of "$\leqslant_L$" that, $y \leqslant_L w$ or $y^{-1} \leqslant_L w^{-1}$. In fact we may assume that y — w and $\mathscr{L}(y) \nsubseteq \mathscr{L}(w)$ or that y^{-1} — w^{-1} and $\mathscr{L}(y^{-1}) \nsubseteq \mathscr{L}(w^{-1})$. In the second case we are directly in the case described in the previous paragraph with $y' = y$ and $w' = w$ as $\mathscr{R}(y) \nsubseteq \mathscr{R}(w)$, and thus $\mathbf{a}(w) \leq \mathbf{a}(y)$ follows. In the first case we have $\mathscr{R}(y^{-1}) \nsubseteq \mathscr{R}(w^{-1})$ so if we take $y' = y^{-1}$ and $w' = w^{-1}$ in the previous paragraph, it follows that $\mathbf{a}(w^{-1}) \leq \mathbf{a}(y^{-1})$, and thus, by Proposition 4.2.1, we have $\mathbf{a}(w) \leq \mathbf{a}(y)$, as required. □

Lemma 4.3.2. *Let (W, S) be any Coxeter system and pick any $x, y, z \in W$ and any $d \in \mathscr{D}$ such that $\gamma_{x,y,z} \neq 0$, such that $\gamma_{z^{-1},z,d} = 1$ and such that $\mathbf{a}(d) = \mathbf{a}(z) =: a$. Then $\gamma_{x,y,z} = \gamma_{y,z,x}$.*

Proof. Take $x, y, z \in W$ and any $d \in \mathscr{D}$ as in the statement of Lemma 4.3.2. Since $\gamma_{x,y,z} \neq 0$, we deduce that $h_{x,y,z^{-1}} \neq 0$, and thus, by (4.1.3), we see that $z^{-1} \leqslant_L x$ and so in particular, we have $z^{-1} \leqslant_{LR} x$. Hence, by Lemma 4.3.1, it follows that $\mathbf{a}(x) \leq \mathbf{a}(z^{-1}) = a$. By the associativity of $\mathcal{H}$, if we let $h_{x,y,z,d}$ denote the coefficient of C_d in the expansion of the product $C_x C_y C_z$ with respect to the KL-basis, we can compute $h_{x,y,z,d}$ in two ways, and get

$$h_{x,y,z,d} = \sum_{u \in W} h_{x,y,u} h_{u,z,d} = \sum_{v \in W} h_{x,v,d} h_{y,z,v}. \tag{4.3.4}$$

Now, since $h_{u,z,d} \neq 0$ implies $d \leqslant_{LR} u$ and $h_{x,v,d} \neq 0$ implies $d \leqslant_{LR} v$, then, by Lemma 4.3.1, we have $\mathbf{a}(u) \leq \mathbf{a}(d) = a$ and $\mathbf{a}(v) \leq \mathbf{a}(d) = 0$, and thus the equalities in (4.3.4) can be written as

$$h_{x,y,z,d} = \sum_{\substack{u \in W \\ \mathbf{a}(u) \leq n}} h_{x,y,u} h_{u,z,d} = \sum_{\substack{v \in W \\ \mathbf{a}(v) \leq n}} h_{x,v,d} h_{y,z,v}. \tag{4.3.5}$$

Since $h_{u,z,d} \neq 0$ and $h_{x,v,d} \neq 0$ imply that $(-q)^{a/2} \gamma_{u,z,d^{-1}} \neq 0$ and $(-q)^{a/2} \gamma_{x,v,d^{-1}} \neq 0$, it follows, by (b) in Proposition 4.3.1, that $u = z^{-1}$ and $v = x^{-1}$, so, using (d) in Proposition 4.3.1 and the hypothesis $\gamma_{z^{-1},z,d} = 1$, the left-hand side of (4.3.5) is

$$\gamma_{x,y,z}(-q)^a + \text{strictly smaller powers of } -q^{1/2},$$

and similarly, the right-hand side is

$$\gamma_{x,x^{-1},d} \cdot \mathsf{coeff}_a(h_{y,z,x^{-1}}) \cdot (-q)^a + \text{strictly smaller powers of } -q^{1/2},$$

where $\mathsf{coeff}_a(h_{y,z,x^{-1}})$ denoted the coefficient of the $(-q^{1/2})^a$ in $h_{y,z,x^{-1}}$. Hence, comparing coefficients on both sides of (4.3.5) and the hypothesis $\gamma_{x,y,z} \neq 0$, we get

$$0 \neq \gamma_{x,y,z} = \gamma_{x,x^{-1},d} \cdot \mathsf{coeff}_a(h_{y,z,x^{-1}}), \tag{4.3.6}$$

which implies that

$$\gamma_{x,x^{-1},d} \neq 0 \quad \text{and} \quad \mathsf{coeff}_a(h_{y,z,x^{-1}}) \neq 0.$$

Now, since $\gamma_{x,x^{-1},d} \neq 0$, we have, by (*b*) in Proposition 4.3.1, that $\gamma_{x,x^{-1},d} = 1$, so from (4.3.6), we recover

$$\gamma_{x,y,z} = \mathsf{coeff}_a(h_{y,z,x^{-1}}), \tag{4.3.7}$$

which implies that $\mathbf{a}(x^{-1}) \geq a$. But recall that $\mathbf{a}(w) \leq a$, so we have $\mathbf{a}(x) = \mathbf{a}(x^{-1})$, and by definition, $\mathsf{coeff}_a(h_{y,z,x^{-1}}) = \gamma_{y,z,x}$. This together with (4.3.7) gives $\gamma_{x,y,z} = \gamma_{y,z,x}$, as required. □

Lemma 4.3.3. *Let (W,S) be any Coxeter system such that $\ell(w) < \infty$ for all $w \in W$, and let $y \in W$ and $d \in \mathscr{D}$ be elements such that $\gamma_{y-1,y,d} \neq 0$. Then $\mathbf{a}(z) = \mathbf{a}(d)$.*

Proof. We proof this by descending induction on $\mathbf{a}(y)$ since $\mathbf{a}(y)$ is bounded above by $\ell(y)$, and by assumption, $\ell(y)$ is finite for all $z \in W$. Therefore, we assume that the if $y \in W$ and $d \in W$ are elements such that $\gamma_{y^{-1},y,d} \neq 0$ and $\mathbf{a}(y) \geq N_0$ for a given integer $N_0 \geq 0$, then $\mathbf{a}(y) = \mathbf{a}(d)$, and we shall deduce that that it is also true when $\mathbf{a}(y) = N_0$.

So let $y \in W$ and $d \in \mathscr{D}$ be as in the statement of the lemma with $\mathbf{a}(y) = N_0 \geq 0$. Since $\gamma_{y^{-1},y,d} \neq 0$, it follows that $h_{y^{-1},z,d} = h_{y^{-1},z,d^{-1}} \neq 0$, and hence, by (4.1.3), we see that $d \leqslant_L y^{-1}$, and so in particular, $d \leqslant_{LR} y^{-1}$. By Lemma 4.3.1, it follows that $\mathbf{a}(y) = \mathbf{a}(y^{-1}) \leq \mathbf{a}(d)$. So assume, for a contradiction, that $\mathbf{a}(d) > \mathbf{a}(y) = N_0$, and let $d' \in \mathscr{D}$ be such that $\gamma_{d^{-1},d,d'} \neq 0$, which exists and is unique by (*c*) in Proposition 4.3.1. Now, by the induction hypothesis applied to d, d' instead of y, d, we have $\mathbf{a}(d) = \mathbf{a}(d')$. Therefore, since we also have $\gamma_{y^{-1},y,d} \neq 0$ and, $\gamma_{d^{-1},d,d'} \neq 0$, Lemma 4.3.2 applies, and we get

$$\gamma_{y,d,y^{-1}} = \gamma_{y^{-1},y,d'} \neq 0.$$

It follows that $h_{y,d,y} \neq 0$ and hence, again by (4.1.3), we have $y \leqslant_L d$, and hence, by Lemma 4.3.1, we obtain $\mathbf{a}(d) \leq \mathbf{a}(y)$. But this contradicts the assumption $\mathbf{a}(d) > \mathbf{a}(y) = N_0$. Hence we must have $\mathbf{a}(y) = \mathbf{a}(d)$, and the proof is complete. □

Theorem 4.3.1. *Let (W,S) be any Coxeter system such that $\ell(w) < \infty$ for all $w \in W$. For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.*

Proof. Let $x, y, z \in W$ be arbitrary elements. First assume that $\gamma_{x,y,z} \neq 0$. By (*c*) in Proposition 4.3.1, there exists a unique $d \in \mathscr{D}$ such that $\gamma_{z^{-1},z,d} \neq 0$, and by Lemma 4.3.3, we have $\mathbf{a}(d) = \mathbf{a}(z)$. Since we also have $\gamma_{x,y,z} \neq 0$ and $\gamma_{z^{-1},z,d} \neq 0$, Lemma 4.3.2 applies, and gives

$$\gamma_{x,y,z} = \gamma_{y,z,x},$$

as required.

On the contrary, assume that $\gamma_{x,y,z} = 0$, and assume, for a contradiction, that $\gamma_{y,z,x} \neq 0$. As in the previous paragraph, $\gamma_{y,z,x} \neq 0$ implies that

$$\gamma_{z,x,y} = \gamma_{y,z,x} \neq 0,$$

which in turn implies that

$$\gamma_{x,y,z} = \gamma_{z,x,y} \neq 0,$$

which contradicts the assumption that $\gamma_{x,y,z} = 0$. The proof is now complete. □

Remark 4.3.3. Notice that the proofs of Proposition 4.3.1, Lemma 4.3.1 and Lemma 4.3.2 did not require any boundedness assumption on the length of the elements of the (W, S), so they apply generally to any Coxeter group. However the proof of Lemma 4.3.3 did require that the length of the elements of the (W, S) was finite for all elements of W in order to bound the **a**-function. Therefore Lemma 4.3.3 actually works for any Coxeter system (W, S) such that $\mathbf{a}(w) < \infty$ for all $w \in W$ (see Remark 4.2.3 to see when this applies). We say more about this in Section 4.4.

4.4 Lusztig's conjectures

As we have seen so far, the **a**-function turns out to very difficult to compute. However, in view of Lusztig's Conjecture 4.2.1 and the fact that the only requirements needed to prove the results in [31] were the boundedness of the **a**-function and the "possitivity property" resulting from the fact that (W, S) was assumed to be crystallographic, Lusztig presents in [32] a series of conjectures (P1)-(P15) which seem to govern the behaviour of the **a**-function. As one can imagine from the results in Section 4.3, these are closely related to the set $\mathscr{D}$ and the integers $\gamma_{x,y,z}$ for $x, y, z \in W$. In this section we present the original set of conjectures (P1)-(P15) in [32] and we make some remarks about these now that the "positivity property" is known to hold for general Coxeter groups, by [11]. Next, we study the relation between these conjectures as well as their implications for the partition of W into KL-cells. We also finish the example on W of type A_2 to illustrate the results. The main reference for this section is [32], and its updated version [33], which includes the more recent results in [11].

Conjectures 4.4.1. (Lusztig) *Let (W, S) be any Coxeter system. The following properties hold.*

(P1) *For any $w \in W$ we have $\mathbf{a}(w) \leq l(w) - 2\delta(w)$.*

(P2) *If $d \in \mathscr{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*

(P3) *If $y \in W$, there exists a unique $d \in \mathscr{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.*

(P4) *If $z', z \in W$ with $z' \leqslant_{LR} z$, then $\mathbf{a}(z') \geq \mathbf{a}(z)$. Hence if $z' \sim_{LR} z$, then $\mathbf{a}(z') = \mathbf{a}(z)$.*

(P5) *If $d \in \mathscr{D}$, $y \in W$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*

(P6) *If $d \in \mathscr{D}$, then $d^2 = 1$.*

(P7) *For any $x, y, z \in W$ we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.*

(P8) *Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$, $z \sim_L x^{-1}$.*

(P9) *If $z, z' \in W$ with $z' \leqslant_L z$ and $a(z') = a(z)$, then $z' \sim_L z$.*

(P10) *If $z, z' \in W$ with $z' \leqslant_R z$ and $a(z') = a(z)$, then $z' \sim_R z$.*

(P11) *If $z, z' \in W$ with $z' \leqslant_{LR} z$ and $a(z') = a(z)$, then $z' \sim_{LR} z$.*

(P12) *Let $I \subseteq S$. If $y \in W_I$, then $\mathbf{a}(y)$ computed in terms of W_I is equal to $\mathbf{a}(y)$ computed in terms of W.*

(P13) *Any left KL-cell γ of W contains a unique element $d \in \mathscr{D}$. We have $\gamma_{x^{-1},x,d} \neq 0$ for all $x \in \Gamma$.*

(P14) *For any $z \in W$, we have $z \sim_{LR} z^{-1}$.*

(P15) *Let v be a second indeterminate and let $h' \in \mathbb{Z}[v, v^{-1}]$ be obtained from $h_{x,y,z}$ by the* *substitution* $q^{1/2} \mapsto v$*. If* $x, x', y, w \in W$ *satisfy* $\mathbf{a}(w) = \mathbf{a}(y)$*, then*

$$\sum_{y'} h'_{w,x',y'} h_{x,y',y} = \sum h_{x,w,y'} h_{y',x',y}.$$

Remark 4.4.1. Notice that (P1) is proved in Remark 4.2.8, (P2), (P3), (P5) and (P6) are exactly the statements in $(b)-(d)$ in Proposition 4.3.1, and (P4) is exactly Lemma 4.3.1. Therefore, since the proofs of these in Section 4.3 take into account the "positivity property" in [11] for general Coxeter groups with no further assumptions, (P1)-(P6) are thus known to hold for general Coxeter groups and are no longer conjectures. In particular, notice from the proofs of $(b)-(d)$ in Proposition 4.3.1, that (P1) and (P3) imply (P5), and that (P2) and (P3) imply (P6). Also notice that (P7) is the statement in Theorem 4.3.1 without the assumption that $\ell(w) < \infty$ for all $w \in W$, and its proof in Section 4.3 makes use of (P3), and Lemmas 4.3.3 and 4.3.2. Now, the proof of Lemma 4.3.3 makes use of (P3) and Lemmas 4.3.2 and 4.3.1, and the proof of Lemma 4.3.2 makes use of (P2), (P5), (P6) and Lemma 4.3.1, which in turn uses (P4). We therefore see that (P7) is implied by (P2)-(P5), given that Conjecture 4.2.1 holds.

Remark 4.4.2. In order to study the the remaining implications between Conjectures (P1)-(P15), denote by (P0) the following:

(P0) Let $x, y, z, w \in W$ be such that $\gamma_{x,y,z^{-1}} \neq 0$ and $w \,—\, z$ with $\mathscr{L}(w) \not\subseteq \mathscr{L}(z)$. Then there exists $x' \in W$ such that $\mathsf{coeff}_{\mathbf{a}(z)}(h_{x',y,w}) \neq 0$. In particular, $\mathbf{a}(w) \geq \mathbf{a}(z)$.

Notice that (P0) was actually proved for general Coxeter groups in the first paragraph of the proof of Lemma 4.3.1, and was used to prove such Lemma. Hence, from the discussion in Remark 4.4.1 and the proof of Lemma 4.3.1 we see that (P0) actually implies (P4).

Proposition 4.4.1. *Assume that Conjecture 4.2.1 holds for any Coxeter system* (W, S)*. We also have the following implications between Lusztig's Conjectures (P1)-(P14) and the Property (P0) of general Coxeter groups:*

(a) (P7) *implies* (P8).

(b) (P0), (P4) *and* (P8) *imply* (P9).

(c) (P9) *implies* (P10).

(d) (P4), (P9) *and* (P10) *imply* (P11).

(e) (P3), (P4) *and* (P8) *for* W *and* W_I *imply* (P12).

(f) (P0), (P2), (P3) *and* (P7) *imply* (P13).

(g) (P6) *and* (P13) *imply* (P14)

Proof of (a) in Proposition 4.4.1. (a) Assume that (P7) holds and let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $h_{x,y,z^{-1}} \neq 0$, and hence from (4.1.3) and (4.1.4), we deduce that $z^{-1} \leqslant_L y$ and $z \leqslant_L x^{-1}$. Now, by (P7), we also have $\gamma_{y,z,x} \neq 0$, hence by the same argument $x^{-1} \leqslant_L z$ and $x \leqslant_L y^{-1}$, and $\gamma_{z,x,y} \neq 0$, hence by the same argument $y^{-1} \leqslant_L x$ and $y \leqslant_L z^{-1}$. The relations $y^{-1} \leqslant_L x$ and $x \leqslant_L y^{-1}$ then give $x \sim_L y^{-1}$, the relations $z^{-1} \leqslant_L y$ and $y \leqslant_L z^{-1}$ then give $y \sim_L z^{-1}$, and the relations $x^{-1} \leqslant_L z$ and $z \leqslant_L x^{-1}$ finally give $z \sim_L x^{-1}$, as required. □

Proof of (b)-(e) in Proposition 4.4.1. See [33] for the proofs. □

Proof of (f) in Proposition 4.4.1. Assume that (P0), (P2), (P3), and (P7) hold and let $\mathbf{\Gamma}$ be any left KL-cell of W and $x \in W$ be any element such that $x \in \mathbf{\Gamma}$. Note that, by (a) in Proposition 4.4.1, this implies that (P8) also holds, and moreover, by Remark 4.4.1, this implies that (P4) and (P6) also hold. Now, by (P3), there exists a unique $d \in \mathscr{D}$ such that $\gamma_{x^{-1},x,d} \neq 0$, so by (P8), we have $x \sim_L d^{-1}$, which shows that $d^{-1} \in \mathbf{\Gamma}$. Hence, by (P6), we have $d \in \mathbf{\Gamma}$.

To show that uniqueness of d in the previous paragraph, let $d' \in \mathscr{D}$ such that $d' \in \mathbf{\Gamma}$, and find $y, z \in W$ such that $\gamma_{y,z,d'} \neq 0$. By (P2), we have $y = z^{-1}$ and by (P8) we have $z \sim_L d'$ and thus $z \in \mathbf{\Gamma}$. By definition of a left KL-cell, since we also have $x \sim_L d$ with $x, d \in \mathbf{\Gamma}$, there exists a sequence $x = w_0, w_1, \ldots, w_n = z$ such that $w_{i-1} \mathrel{—} w_i$ and $\mathscr{L}(w_{i-1}) \nsubseteq \mathscr{L}(w_i)$ for integer i such that $1 \leq i \leq n$. Moreover, since $x \sim_L z$, we have $w_i \in \mathbf{\Gamma}$ for all integers i such that $0 \leq i \leq n$, so for each w_i with $1 \leq i \leq n-1$, let $d_i \in \mathscr{D}$ be such that $\gamma_{w_i^{-1},w_i,d_i} \neq 0$. Then, as in the beginning of the proof of (f), we have $d_i \in \mathbf{\Gamma}$ for each $1 \leq i \leq n-1$. Now, since $w_{i-1} \mathrel{—} w_i$ and $\mathscr{L}(w_{i-1}) \nsubseteq \mathscr{L}(w_i)$, applying (P0) to w_i, d_i, w_i, w_{i-1} instead of x, y, z, w in the statement of (P0), we get that there exists $u \in W$ such that $\mathsf{coeff}_{\mathbf{a}(w_i)}(h_{u,d_i,w_{i-1}}) \neq 0$. Also, since $w_{i-1} \sim_L w_i$, we have in particular, that $w_{i-1} \sim_{LR} w_i$, so by (P4), we get that $\mathbf{a}(w_{i-1}) = \mathbf{a}(w_i)$, and hence

$$\gamma_{u,d_i,w_{i-1}^{-1}} = \mathsf{coeff}_{\mathbf{a}(w_i)}(h_{u,d_i,w_{i-1}}) \neq 0,$$

so by (P7), we obtain

$$\gamma_{w_{i-1}^{-1},u,d_i} = \gamma_{u,d_i,w_{i-1}^{-1}} \neq 0.$$

But then, by (P2), we have $u = w_{i-1}$ and $\gamma_{w_{i-1}^{-1},w_{i-1},d_i} \neq 0$. Since we also have $\gamma_{w_{i-1}^{-1},w_{i-1},d_{i-1}} \neq 0$, it follows by the uniqueness in (P3), that $d_{i-1} = d_i$. Hence $d = d'$, as required. $\square$

Proof of (g) in Proposition 4.4.1. Assume that (P6) and (P13) and take $z \in W$. By (P13), there exists a unique $d \in \mathscr{D}$ such that $z \sim_L d$ and since $d = d^{-1}$ by (P6), it follows that $z \sim_L d^{-1}$. Hence, by definition of "$\sim_L$", we have

$$z \leqslant_L d, \quad d \leqslant_L z, \quad z \leqslant_L d^{-1} \quad \text{and} \quad d^{-1} \leqslant_L z.$$

Therefore, we see that the sequence z, d, z^{-1} satisfies $z \leqslant_L d$ and $d^{-1} \leqslant_L (z^{-1})^{-1} = z$, and thus, by definition of "$\leqslant_{LR}$", it follows that $z \leq z^{-1}$. Similarly, the sequence z^{-1}, d, z satisfies $(z^{-1})^{-1} = z \leqslant_L d^{-1}$ and $d \leqslant_L z$, and thus $z^{-1} \leqslant_{LR} z$. Hence, by definition of "$\sim_{LR}$", it follows that $z \sim_{LR} z^{-1}$, as required. $\square$

Remark 4.4.3. Note that Remark 4.4.1 and Proposition 4.4.1 shows that (P7)-(P14) hold for (W, S) given the $\mathbf{a}$-function is bounded. Therefore, if Conjecture 4.2.1 holds, then (P7)-(P14) hold all Coxeter systems (W, S).

Example 4.4.1. (Type $\widetilde{\mathrm{A}}_2$) Let (W, S) be of type $\widetilde{\mathrm{A}}_2$ as in Examples 2.9.3, 4.1.5, and 4.2.1. From Examples 4.3.1 and 4.3.2, we know that $1, s_1, s_2, s_3 \in \mathscr{D}$.

Now, consider the element $s_2s_1s_2 \in W$. Since

$$s_2 \cdot s_2s_1s_2 = s_1s_2 < s_2s_1s_2 \quad \text{and} \quad 1 < s_2 \cdot 1,$$

we know, by Corollary 3.6.1, that $P_{1,s_2s_1s_2} = P_{s_2,s_2s_1s_2}$, and thus, since $\ell(s_2s_1s_2) - \ell(s_2) = 3 - 1 = 2$, it follows, by Corollary 3.5.1, that

$$P_{1,s_2s_1s_2} = P_{s_2,s_2s_1s_2} = 1,$$

and thus $\delta(s_2s_1s_2) = 0$. We can see in Figure 4.2 that $s_2s_1s_2 \in W^{12} = \mathbf{A}_{12} \subseteq \mathbf{A}$, and thus, by (4.2.9) in Example 4.2.1, we know that $\mathbf{a}(s_2s_1s_2) = 3$. Therefore, we have

$$\mathbf{a}(s_2s_1s_2) = 3 = \ell(s_2s_1s_2) = \ell(s_2s_1s_2) - 2\delta(s_2s_1s_2),$$

which shows that $s_2s_1s_2 \in \mathscr{D}$. Similarly, since the element $s_{13}s_1 \in W$ satisfies

$$s_1 \cdot s_1s_3s_1 = s_3s_1 < s_1s_3s_1, \quad 1 < s_1 \cdot 1 \quad \text{and} \quad \ell(s_1s_3s_1) - \ell(s_1) = 2,$$

and the element $s_{32}s_3 \in W$ satisfies

$$s_3 \cdot s_3s_2s_3 = s_2s_3 < s_3s_2s_3, \quad 1 < s_3 \cdot 1 \quad \text{and} \quad \ell(s_3s_2s_3) - \ell(s_3) = 2,$$

exactly the same argument shows that

$$P_{1,s_1s_3s_1} = P_{s_1,s_1s_3s_1} = 1 \quad \text{and} \quad P_{1,s_3s_2s_3} = P_{s_3,s_3s_2s_3} = 1.$$

We can then see in Figure 4.2 that $s_1s_3s_1 \in W^{13} = \mathbf{A}_{13} \subseteq \mathbf{A}$ and $s_3s_2s_3 \in W^{23} = \mathbf{A}_{23} \subseteq \mathbf{A}$, and thus, by (4.2.9) in Example 4.2.1, we know that $\mathbf{a}(s_1s_3s_1) = 3 = \mathbf{a}(s_3s_2s_3)$. Therefore, we have

$$\begin{aligned}\mathbf{a}(s_1s_3s_1) = 3 = \ell(s_1s_3s_1) = \ell(s_1s_3s_1) - 2\delta(s_1s_3s_1),\\ \mathbf{a}(s_3s_2s_3) = 3 = \ell(s_3s_2s_3) = \ell(s_3s_2s_3) - 2\delta(s_3s_2s_3),\end{aligned}$$

which shows that $s_1s_3s_1, s_3s_2s_3 \in \mathscr{D}$.

Now, consider the element $s_1s_2s_3s_2s_1 \in W$. Since

$$\mathscr{L}(s_1s_2s_3s_2s_1) = \{s_1\} \quad \text{and} \quad s_1 \notin \mathscr{L}(1),$$

we know by Corollary 3.6.1, that

$$P_{1,s_1s_2s_3s_2s_1} = P_{s_1,s_1s_2s_3s_2s_1}, \tag{4.4.1}$$

and we have no further reductions. By the inductive formula to compute KL-polynomials in (3.5.18) with $c = 1$ since $s_1 \cdot s_1 = 1 < s_1$, we have

$$P_{s_1,s_1s_2s_3s_2s_1} = P_{1,s_2s_3s_2s_1} + qP_{s_1,s_2s_3s_2s_1} - q^{5/2} \sum_{\substack{z\in W\\ z\prec s_2s_3s_2s_1\\ s_1z<z}} \mu(z, s_2s_3s_2s_1)q_z^{-1/2}P_{s_1,z}. \tag{4.4.2}$$

Again, since $s_3, s_2 \in \mathscr{L}(s_2s_3s_2s_1)$ and $s_2 \notin \mathscr{L}(1)$, $s_3 \notin \mathscr{L}(s_2)$ and $s_2 \notin \mathscr{L}(s_1)$, we have, by Corollary 3.6.1, that

$$P_{1,s_2s_3s_2s_1} = P_{s_2,s_2s_3s_2s_1} = P_{s_3s_2,s_2s_3s_2s_1} \quad \text{and} \quad P_{s_1,s_2s_3s_2s_1} = P_{s_2s_1,s_2s_3s_2s_1},$$

and thus, since

$$\ell(s_2s_3s_2s_1) - \ell(s_3s_2) = 4 - 2 = 2 \quad \text{and} \quad \ell(s_2s_3s_2s_1) - \ell(s_2s_1) = 4 - 2 = 2,$$

it follows by Corollary 3.5.1, that

$$P_{1,s_2s_3s_2s_1} = 1 = P_{s_1,s_2s_3s_2s_1}. \tag{4.4.3}$$

We now need to find the $z \in W$ that contribute a term to the sum in (4.4.2). But note that the only subexpression of $s_2s_3s_2s_1$ having s_1 in its descent set is s_1 itself, so by Theorem 2.10.1, this is the unique element satisfying $z < s_2s_3s_2s_1$ and $s_1z < z$. Since we have already computed $P_{s_1,s_2s_3s_2s_1} = 1$ in (4.4.3), we see that

$$\deg_q P_{s_1,s_2s_3s_2s_1} = 0 < \frac{1}{2}(4-1-1) = \frac{1}{2}\left(\ell(s_2s_3s_2s_1) - \ell(s_1) - 1\right),$$

and thus $s_1 \not\prec s_2s_3s_2s_1$ and the last sum in (4.4.2) is 0. This together with (4.4.1), (4.4.2) and (4.4.3), gives

$$P_{1,s_1s_2s_3s_2s_1} = 1 + q,$$

and thus $\delta(s_1s_2s_3s_2s_1) = 1$. We can see from Figure 4.2 and Figure 4.3 that $s_1s_2s_3s_2s_1 \in \mathbf{A}_1 \subseteq \mathbf{A}$, and thus, by (4.2.9) in Example 4.2.1, we know that $\mathbf{a}(s_1s_2s_3s_2s_1) = 3$. Therefore, we have

$$\mathbf{a}(s_1s_2s_3s_2s_1) = 3 = 5 - 2 = \ell(s_1s_2s_3s_2s_1) - 2\delta(s_1s_2s_3s_2s_1),$$

which shows that $s_1s_2s_3s_2s_1 \in \mathscr{D}$. Similarly, if we consider the elements $s_2s_3s_1s_3s_2 \in \mathbf{A}_2$ and $s_3s_2s_1s_2s_3 \in \mathbf{A}_3$, by the symmetry, applying exactly the same argument but interchanging s_1 with s_2 and s_3, respectively in each case, we see that $s_2s_3s_1s_3s_2, s_3s_2s_1s_2s_3 \in \mathscr{D}$. Note that we have already found 10 elements of W belonging to the set $\mathscr{D}$, and by Example 4.1.5, we know that this is exactly the number of left KL-cells partitioning W. Since, by Example 4.2.1, we know that the **a**-function is bounded in this case, Remark 4.4.3 tells us that (P13) holds, and thus we have found every distinguished involution in W. More precisely,

$$\mathscr{D} = \{1, s_1, s_2, s_3, s_2s_1s_2, s_1s_3s_1, s_3s_2s_3, s_1s_2s_3s_2s_1, s_2s_3s_1s_3s_2, s_3s_2s_1s_2s_3\},$$

and each alcove corresponding to one of the distinguished involutions appears with a black dot in Figure 4.5.

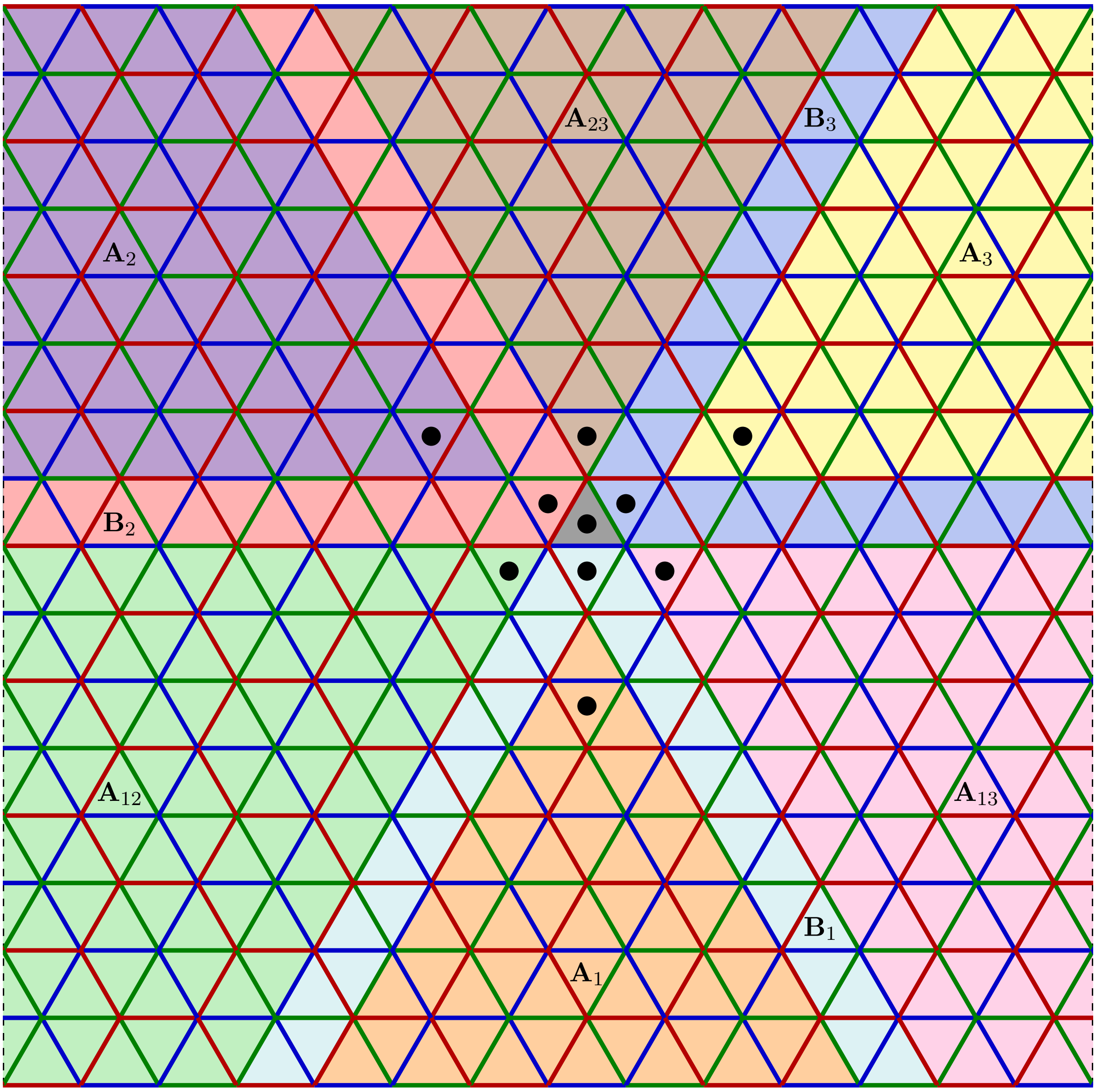


Figure 4.5: Distinguished involutions and left KL-cells of W of type $\widetilde{A}_2$

It is easy to see in Figure 4.5 that each left KL-cell contains a unique distinguished involution of W, which agrees with the fact that (P13) holds for W of type $\widetilde{\mathrm{A}}_2$. Moreover, we can easily see in Figure 4.5 the relation between the of the symmetry of the relations in the presentation of W and the symmetry of the decomposition of W into KL-cell and the symmetry of the set $\mathscr{D}$.

Remark 4.4.4. From the computations in Example 4.4.1, we can already see that that explicitly computing the set $\mathscr{D}$ becomes difficult very quickly simply even if we stay in rank 3, as increasing the order of $m_{ss'}$ of the product ss' of a single pair of generator, already makes the computations a lot harder. Of course, increasing the rank also increases the difficulty rapidly, as for each given length value, there are more possible elements and thus more elements that need to be checked as candidates for distinguished involutions.

4.5 Bibliographical Remarks

This Chapter focuses on exploring the concept of Kazhdan-Lusztig cells and the corresponding partition of the Coxeter group, as well as some of tools and methods that have played and continue to play a huge paper in the development of the Kazhdan-Lusztig theory of cells. Since we cannot give a complete account of every result in which the partition into cells is known, we give a brief account, definitely incomplete, together with references of the cases in which the partition into KL-cells is known.

The cells of finite Coxeter groups of type A was dealt with by Kazhdan and Lusztig's original paper [24], we presented the case of finite dihedral groups in this thesis (see Example 4.1.3), types B and D have been obtained by Barbasch and Vogan in [3] and by Garfinkle in [12], [13] and [14] and Types H_3 and H_4 have been dealt with by Alvis [2].

The cells of affine Weyl groups have been described in [30] for type $\widetilde{\mathrm{A}}_2, \widetilde{\mathrm{B}}_2$ and $\widetilde{\mathrm{G}}_2$. Those of type $\widetilde{\mathrm{C}}_3$, have been studied by Bédard in [6] and Du [22], and by Du in [10] for type $\widetilde{\mathrm{B}}_3$. The KL-cells of type $\widetilde{\mathrm{A}}_{n-1}$ have been studied by Shi in [34] and Lawton in [27]

The KL-cells of some hyperbolic Coxeter groups of rank 3 have also been studied by Bédard in [7] and [8].

5 Concluding remarks

This thesis focuses on exploring the concept of Kazhdan-Lusztig cells and the corresponding partition of the Coxeter group, as well as some of tools and methods that have played and continue to play a huge paper in the development of the Kazhdan-Lusztig theory of cells. The definition of KL-cells was generalized to the case where the simple reflections are given different weights, giving rise to Hecke algebras with unequal parameters. We would have liked to explore further this topic, since the knowledge on whether certain results that hold in the equal parameter case extend to the general case is still very scarce. Since many of the established results in the equal parameter case rely on the "positivity property" established in 1001[11], they break in the unequal parameter case, where the "positivity property" no longer holds. We would have therefore liked to explored different tools in the equal parameter case that avoid the use of the "positivity property" and do not require some kind of geometric interpretation. Some of these methods would include studying KL-cells and the behavoiur of the KL-basis in the parabolic subgroups, as well some induction process such as the Guilhot Induction process that enable us to extend results from more well-known, or manageable cases that are easier to study.

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