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Adic spaces and etale cohomology

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Part I
Introduction

Adic spaces were introduced by Roland Huber in [Hub93a], [Hub93b] and [Hub94] with the aim to provide a natural framework to work with in non-archimedean geometry, generalizing rigid analytic varieties and formal schemes. Despite the simplicity and strength of the theory, it only gained popularity after Peter Scholze who used adic spaces in [Sch11] to introduce perfectoid spaces. Ever since, adic spaces have become crucial in the development of p-adic geometry and p-adic Hodge theory. The primary aim of this thesis is to give an introduction to the theory of adic spaces. A secondary aim is to study its relation with classical rigid and algebraic geometry. To justify this relation further, we introduce etale cohomology of adic spaces and present two comparison theorems. Although there are many other recent texts that serve as an introduction to adic spaces, most notably [Wed19] and [Mor19], the author hopes that what is discussed in this thesis will give new insights to the reader. The author believes that the direction taken here is different ,in many ways, than other such texts and that many notes and remarks have not been made elsewhere. On the other hand, this is still (mostly) an introduction to the basic theory and so i) the material intersects to a great extent with the material of other similar texts and ii) this thesis contains only a fraction of the material contained in other similar texts. Moreover, no result is originally due to the author.

Outline

The thesis is structured as follows

Section 1. This is the core of the thesis. Here we introduce the commutative algebra behind the theory by defining valuations, Huber rings, Huber pairs and we prove basic results. Moreover, we introduce the adic spectrum associated to a Huber pair, topologize it and define a presheaf of topological rings on it. We define adic spaces and among other things, we prove Tate's acyclicity theorem in this context. Lastly we define morphisms of certain type between adic spaces that are to be used later in the thesis.

Section 2. In this section we describe a functor from schemes locally of finite type over a non-archimidean field K to adic spaces over K and a functor from rigid analytic varieties over a non-archimidean field K to adic spaces over K. This leads to a commutative diagram

$$\begin{array}{ccc} \text{(schemes lft over } K\text{)} & \xrightarrow{\quad ad \quad} & \text{(adic spaces over } K\text{)} \\ & \searrow^{rig} \qquad \nearrow_{r} & \\ & \text{(rigid analytic varieties over } K\text{)} & \end{array}$$

where "rig" refers to the rigid analytification.

Section 3. Here we introduce the etale site of stable adic spaces. Moreover we discuss about differentials and prove basic results on etale morphisms of adic spaces, ultimately leading to the theorems of section 4.

Section 4. Here we present two comparison theorems in etale cohomology related to the diagram of section 2. In particular we prove that the etale cohomology of a rigid analytic variety is equivalent to the etale cohomology of its associated adic space.

Part 4. All of the material discussed here is essential for the main part of the thesis. In particular we use the material of A, B and D in section 1 and the material of C in sections 2 and 4.

As a general motivation for the subject and for the outline of this thesis, the reader should keep in mind the following: adic spaces provide a framework to work with in rigid geometry without the need of Grothendieck topologies. It also provides a framework to work with in etale cohomology of rigid varieties (again without all the difficulties that come with working with classical rigid analytic varieties). All the above will be explained in this thesis (and a big part of this thesis is devoted to explaining the above).

References, Notation

Every result in the main part of the thesis can be found in [Hub93b], [Hub94] and [Hub96] and those are the main references for this text. We should note that notation in Huber's original papers differs from the modern one. In particular the term "f-adic ring" has been replaced by "Huber ring" and the term "affinoid ring", denoted by $A = (A^{\triangleright}, A^{+})$ has been replaced by "Huber pair", denoted by (A, A^{+}). We do our best to follow the notation that is mostly used (for example the notation in [SW20]). We should also say that in this thesis every ring is commutative and every group is abelian.

Part II
Basics

1 Basics of Adic Spaces

In this section we introduce the category of adic spaces. Recall that the category of affine schemes is the opposite category of rings. We aim to do something similar for adic spaces. Namely, given a Huber pair (A, A^+), we want to have an adic space associated to it so that the opposite category of Huber pairs is the category of "affine" (later to be called affinoid) adic spaces. This is not exactly the picture we end up with, but it is close! There are two problems with the latter. First of all, not every Huber pair (A, A^+) gives a locally topologically ringed space and a priori to the topological space associated to (A, A^+) we can only define a presheaf. Secondly, Huber pairs give the same space up to completion. So we may say for now that the category of "affine" adic spaces will be the opposite category of "nice" complete Huber pairs. Adic spaces will be defined roughly in the following way: consider the set of continuous valuations (1.1) on a Huber pair (A, A^+), topologize it and define a presheaf of topological rings on it (1.3). The latter will be called the adic spectrum of (A, A^+) and will be denoted $Spa(A, A^+)$. Lastly, glue adic spectra to get adic spaces (1.4). Moreover, in 1.4 we present some cases when the presheaf on $Spa(A, A^+)$ is a sheaf. In particular we prove Tate's acyclicity theorem. The references for 1.1-1.4 are [Hub93a],[Hub93b] and [Hub94]. For a pure introduction to adic spaces, the results on spectrality in 1.3 could be omitted, but we will use them later in an essential way. In 1.5 we define morphisms of "finite type" between adic spaces and prove that under some (favorable) conditions, fiber products of adic spaces exist. Lastly in 1.6 we define "quasi-finite" morphisms of adic spaces which will later be used to understand etale morphisms of adic spaces. The main references for 1.5 and 1.6 are [Wed19] and [Hub96]. All over this section, we use the results on topological rings, spectral spaces and Cech Cohomology, discussed in the appendix.

1.1 Valuations

Definition 1.1. *A totally ordered abelian group $(\Gamma, \times)$ is a group with a total order $\leq$ on its underlying set such that if $a \leq b$ then for all $\gamma \in \Gamma$, $a \cdot \gamma \leq b \cdot \gamma$.*

A morphism $\phi : \Gamma_1 \to \Gamma_2$ of totally ordered abelian groups is a morphism of groups with $a \leq_1 b$ if and only if $\phi(a) \leq_2 \phi(b)$ for all $a, b \in \Gamma_1$, where $\leq_1$ and $\leq_2$ are the total orders of Γ_1 and Γ_2 respectively.

Definition 1.2. *Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is called convex if for every $a, b, c \in \Gamma$ with $a \leq c \leq b$, $a, b \in \Delta$ implies that $c \in \Delta$.*

Let $(\Gamma, \times)$ be a totally ordered abelian group. We add an element 0 in Γ and extend the group law by $0 \cdot \gamma = \gamma \cdot 0 = 0$ for all $\gamma \in \Gamma$ and the order by $0 \leq \gamma$ for all $\gamma \in \Gamma$.

Definition 1.3. *Let A be a ring and $(\Gamma, \times)$ a totally ordered abelian group. A valuation on A is a map $|\cdot| : A \to \Gamma \cup \{0\}$ such that*
i) $|0| = 0, |1| = 1$
ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in A$
iii) $|a + b| \leq max\{|a|, |b|\}$ *for all* $a, b \in A$

Definition 1.4. *Let $x : A \to \Gamma \cup \{0\}$ be a valuation on a ring A.*
i) The set $\{a \in A : x(a) = 0\}$ is called the support of x and is denoted by $supp(x)$.
ii) The subgroup of Γ generated by $im(x) \setminus \{0\}$ is called the value group of x and is denoted by Γ_x.
iii) The convex subgroup of Γ generated by $\{x(a) : a \in A, x(a) \geq 1\}$ is called the characteristic group of x and is denoted by $c\Gamma_x$.
iv) The rank of x is the height of Γ_x, i.e the ordinal of the set of the convex subgroups of Γ_x.

Note. In the above definition, we wrote x instead of $|\cdot|$ to denote a valuation. We will eventually stick with this notation, as points of the topological spaces that we will study are valuations.

Definition 1.5. *Two valuations $|\cdot|_1, |\cdot|_2$ on a ring A are equivalent if there exists an isomorphism of ordered abelian groups $\phi : \Gamma_1 \to \Gamma_2$,where Γ_1 and Γ_2 are the value groups of $|\cdot|_1$ and $|\cdot|_2$ respectively, such that $\phi \circ |\cdot|_1 = |\cdot|_2$.*

Lemma 1.6. *Two valuations $|\cdot|_1, |\cdot|_2$ on a ring A are equivalent if and only if for every $a, b \in A$, $|a|_1 \leq |b|_1$ if and only if $|a|_2 \leq |b|_2$.*

Proof. We claim first that we may replace A by a field. This is because in any case, $supp(|\cdot|_1) = supp(|\cdot|_2) = p$ a prime ideal and since $|a|_1$ and $|a|_2$ only depend on the image of a in A/p, we may replace A by A/p. Furthermore, A/p is an integral domain and $supp(|\cdot|_1) = supp(|\cdot|_2) = \{0\}$ which means that we may extend $|\cdot|_1$ and $|\cdot|_2$ to $Frac(A/p)$ (by mapping $\frac{a}{b}$ to $|a||b|^{-1}$).
Let us now resume the actual proof. We write K instead of A. Obviously if $|\cdot|_1$ and $|\cdot|_2$ are equivalent, then the condition holds. Assume now that $|a|_1 \leq |b|_1$ if and only if $|a|_2 \leq |b|_2$ for all $a, b \in K$. Then, $\{a \in K : |a|_1 \geq 1\} = \{a \in K : |a|_2 \geq 1\}$. Moreover $|\cdot|_1 : K^\times \to \Gamma_{|\cdot|_1}$ and $|\cdot|_2 : K^\times \to \Gamma_{|\cdot|_2}$ are surjective homomorphisms that have the same kernel. Therefore there exists a unique group homomorphism $f : \Gamma_{|\cdot|_1} \to \Gamma_{|\cdot|_2}$ such that $f \circ |\cdot|_1 = |\cdot|_2$. Note then that f maps elements ≤ 1 to elements ≤ 1 and so is a homomorphism of totally ordered abelian groups since if $a \leq b$ then $ab^{-1} \leq 1$ and so $\phi(ab^{-1}) \leq 1$ which implies that $\phi(a) \leq \phi(b)$. □

Definition 1.7. *Let A be a topological ring. A valuation $|\cdot| : A \to \Gamma \cup \{0\}$ is called continuous if for every $\gamma \in \Gamma$, the set $\{a \in A : |a| < \gamma\}$ is open in R.*

1.2 Huber rings and pairs

Definition 1.8. *A Huber ring is a topological ring A for which there exists an open subring A_0 of A that is adic with respect to a finitely generated ideal of definition I.*

For the definition of an adic ring, see A.10. For a Huber ring A and A_0, I as in the above definition, we call (A_0, I) a couple of definition. Morphisms of Huber rings are just morphisms of topological rings. If a Huber ring A contains a topologically nilpotent unit ϖ, then A is called Tate.

Example 1.9. *Every discrete ring A is a Huber ring with $A_0 = A$ and $I = 0$*

Example 1.10. *Any non-archimedean field K is a Huber ring with ring of definition K^o and ideal of definition (ϖ) where ϖ is any element of K with $0 < |\varpi| < 1$. Also the $K-$algebra $K\langle T_1, \ldots, T_n\rangle$ (see C.3) is a Huber ring with ring of definition the subring $K^o\langle T_1, \ldots, T_n\rangle$ and ideal of definition (ϖ).*

Example 1.11. *A case of interest of latter example is the field of p-adic numbers $\mathbb{Q}_p$ is a Huber ring with ring of definition $\mathbb{Z}_p$ and ideal of definition (p).*

Definition 1.12. *A morphism of Huber rings $\phi : A \to B$ is called adic if there exist rings of definition A_0 and B_0 of A and B respectively and an ideal of definition I of A such that $\phi(A_0) \subset B_0$ and $\phi(I)B_0$ is an ideal of definition of B_0.*

Definition 1.13. *A subset B of a Huber ring A is called bounded if for every neighborhood U of 0 there exists a neighborhood V of 0 such that $vb \in U$ for all $v \in V$ and $b \in B$.*

Definition 1.14. *An element x of a Huber ring A is called power-bounded if the set $\{x^n : n \geq 0\}$ is bounded. We denote A^o the set of power-bounded elements of A. A subset T of A is called power-bounded if the set $\bigcup_{n\in\mathbb{N}} T(n)$ is bounded, where $T(n) = \{t_1 \cdots t_n : t_i \in T\}$.*

Note. A subset of a bounded subset is bounded, by definition.

Definition 1.15. *An element x of a Huber ring A is called topologically nilpotent if the sequence x^n converges to 0. We denote A^{oo} the set of topologically nilpotent elements of A.*

Proposition 1.16. *A subring A_0 of a Huber ring A is a ring of definition if and only if it is open and bounded.*

Proof. Assume first that $A_0 \subset A$ is a ring of definition and let I be a finitely generated ideal of A_0. By definition, A_0 is open. To show that it is bounded, it is enough to show that for every n there exists a neighborhood V of 0 in A such that $V \cdot A_0 \subset I^n$. For that, it is enough to choose $V = I^n$.
Assume now that A_0 is an open and bounded subring of A. Let U be a subset of A such that $(U^n)_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0 in A and let T be a finite subset of U such that $T \cdot U = U^2 \subset U$ (take for example U to be equal to some ideal of definition of A and T a finite system of generators of this ideal). Since $(U^n)_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0 and $T \subset U$, there exists k such that $T(k) \subset A_0$. Let $I = T(k) \cdot A_0$. We will show that $(I^n)_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0. Let l be such that $U^l \subset A_0$. Then $I^n = T(nk)\cdot A_0 \supset T(nk)\cdot U^l = U^{l+nk}$. Therefore since U^{l+nk} is a neighborhood of 0, it follows that I^n is a neighborhood of 0. Now let V be a neighborhood of 0. Since A_0 is assumed to be bounded, there exists a neighborhood Y of 0 with $Y \cdot A_0 \subset V$. Let m be such that $U^m \subset V$. Then $U^m \cdot A_0 \subset V$ and so $I^m \subset V$. Now since I is a finitely generated of A_0, the result follows. □

Corollary 1.17. *Let A be a Huber ring. Then A^o is the union of all rings of definition of A. In particular, A^o is open.*

Proof. Let A_0 be a ring of definition of A. Since A_0 is bounded, for every $x \in A_0$ the set $\{x^n : n \geq 0\}$ is bounded as a subset of A_0. Therefore $x \in A^o$ and so $A_0 \subset A^o$. Let now $x \in A^o$. We need to show that x belongs in some ring of definition of A. We have that $\{x^n : n \geq 0\}$ is bounded. Let A_0 be some ring of definition of A. Then $A_0 \cdot \{x^n : n \geq 0\}$ is open and bounded and thus a ring of definition that contains x. □

Proposition 1.18. *Let A be a Huber ring. Then A^{oo} is the union of all ideals of definition. In particular A^{oo} is open.*

Proof. We need to show that $f \in A$ is topologically nilpotent if and only if it belongs in some ideal of definition. We may assume that A is adic (this will be visible by the end of the proof). If $f \in I$ for some ideal of definition I, then since I^n form a fundamental system of neighborhoods of 0, we have that $f \in A^{oo}$. Conversely assume that $f \in A^{oo}$. Let I be an ideal of definition. There exists n such that $x^n \in I$. Let $J = I + xA$. Then J is an open ideal of A and $I \subset J$, $J^n \subset I$ and so J is an ideal of definition of A that contains x, as we wished. □

Proposition 1.19. *Let A be a Huber ring with couple of definition (A_0, I). Consider $\varprojlim A/I^n$ as a group with the inverse limit topology. There is a ring structure on $\varprojlim A/I^n$ making the natural map $i : A \to \varprojlim A/I^n$ a ring homomorphism.*

To prove the above, we first note the following

Note. The group homomorphisms $A_0/I^n \to A/I^n$ give rise to an injective group homomorphism $j : \widehat{A_0} \to \varprojlim A/I^n$. It is easy to see that an element $x = (x_1 + I, x_2 + I^2, \dots) \in A/I^n$ is an image of j if and only if $x_1 \in A_0$. Thus the quotient of $(A_0 + I, A + I^2, A + I^3, \dots)$ in $\varprojlim A/I^n$ is equal to $\widehat{A_0}$ and is also open. As a final note, we see that every element x of $\widehat{A}$ can be written as $x = y + i(a)$ for some $y \in \widehat{A_0}$ and some $a \in A$. Indeed, if $x = (x_1 + I, x_2 + I^2, \dots)$, then it is enough to take $y = (0, x_2 - x_1 + I^2, x_3 - x_1 + I^3, \dots)$ and $a = x_1$.

Proof. For every $a \in A$ there exists $c > 0$ such that $a\cdot I^c \subset A_0$. Therefore for all $n \geq 1$, $a\cdot I^{n+c} \subset I^n$. Let $x = (x_1+I, x_2+I^2, \dots)$ be an element of $\varprojlim A/I^n$. We consider $y = (ax_{c+1}+I, ax_{c+2}+I^2, \dots)$ which is an element of $\varprojlim A/I^n$ from the above relation. We define $x\cdot i(a)$ to be equal to y. Now let x_1 and x_2 be two elements of $\varprojlim A/I^n$. From the note, we can write x_1 and x_2 as $y_1+i(a_1)$ and $y_2+i(a_2)$ respectively, for some $y_1, y_2 \in \widehat{A_0}$ and some $a_1, a_2 \in A$. We define $x_1 \cdot x_2 = y_1 \cdot y_2 + y_1 \cdot i(a_2) + y_2 \cdot i(a_1) + i(a_1) \cdot i(a_2)$. It follows immediately that i is a ring homomorphism. □

We denote $\widehat{A}$ the topological group $\varprojlim A/I^n$ together with the ring structure of the above proposition and call it the completion of A.

Note. In the proof we defined the product $x \cdot i(a)$ by making a choice on the presentation of x as a sum of some element of $\widehat{A_0}$ and some element in the image of i. Moreover we defined $x_1 \cdot x_2$ by making a choice on the presentation of x_1 and x_2 as sums of some elements of $\widehat{A_0}$ and some elements of the image of i. Strictly speaking one should check that the definitions are independent of the choices we make. This is standard (and tedious in this case).

Note. From the way we defined multiplication in $\varprojlim A/I^n$ it follows immediately that $j : \widehat{A_0} \to \widehat{A}$ is a ring homomorphism.

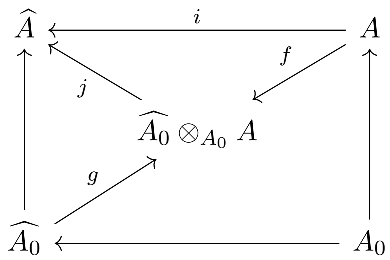
Note. For a Huber ring A, $\varprojlim A/I^n$ is the completion of A viewed as a topological group. Therefore since we defined a multiplication in $\varprojlim A/I^n$ that respects the topology and the morphism $i : A \to \widehat{A}$, the completion of A is the completion in the sense of topological rings.

Proposition 1.20. *The completion of a Huber ring A is a Huber ring.*

Proof. As we have shown already, $\widehat{A_0}$ is an open subring of $\widehat{A}$. Moreover $\widehat{A_0}$ is adic with ideal of definition $I\widehat{A_0}$ which is finitely generated. Thus $\widehat{A}$ is a Huber ring with couple of definition $(\widehat{A_0}, I\widehat{A_0})$. □

Proposition 1.21. *Let A be a Huber ring with couple of definition (A_0, I). Then $\widehat{A} = \widehat{A_0} \otimes_{A_0} A$.*

Proof. Consider the following diagram



Given $\hat{a} \in \widehat{A}$, we may write it as a sum $i(a) + a_0$ for some $a \in A$ and $a_0 \in \widehat{A_0}$. Then we define $h : \widehat{A} \to \widehat{A_0} \otimes_{A_0} A$ given by $h(i(a) + a_0) = f(a) + g(a_0)$. h is easily seen to be well defined (i.e independent of the presentation we choose for $\hat{a}$). We note from the definition that h has the following properties:
i) h is additive
ii) $f = h \circ i$
iii) $g = h|_{\widehat{A_0}}$
From this and the way we defined h, it follows that $j \circ h = id_{\widehat{A}}$ and $h \circ j(x) = x$ for all $x \in f(A) \cup g(\widehat{A_0})$. Therefore if we show that h is a ring homomorphism then it will follow that $h \circ j$ is the identity on $\widehat{A_0} \otimes_{A_0} A$ and so j will be an isomoprhism. For this, it is enough to find a topology on $\widehat{A_0} \otimes_{A_0} A$ that makes it into a topological ring such that f and h are continuous. This is done by considering the topology on $\widehat{A_0} \otimes_{A_0} A$ for which g is a topological embedding. For the remaining details see [Hub93b]. □

Definition 1.22. *A subring $A^+ \subset A$ is called a ring of integral elements if it is open in A, integrally closed in A and contained in A^o.*

In the following theorem, the "usual bijection" refers to the bijection between the open subgroups of A and the open subgroups of $\widehat{A}$ (see A.8).

Proposition 1.23. *Let $i : A \to \widehat{A}$ be the completion of a Huber ring A. Then*
i) $\widehat{A^o} = \widehat{A}^o$
ii) if G and H are open subgroups of A and $\widehat{A}$ respectively that correspond to each other under the

usual bijection, then G is a ring of definition of A of and only if H is a ring of definition of $\widehat{A}$.
iii) If G and H are open subgroups of A and $\widehat{A}$ respectively that correspond to each other under the usual bijection, then G is a ring of integral elements of A if and only if H is a ring of integral elements of H.

Proof. i) Since open subgroups of A correspond bijectively to open subgroups of $\widehat{A}$ and since the topologies on A and $\widehat{A}$ are generated by open subgroups, $E \subset A$ is bounded if and only if $i(E)$ is bounded. Therefore, $i^{-1}(\widehat{A}^o) \subset A^o$ and so $\widehat{A}^o \subset \widehat{A^o}$. On the other hand we have $i(A^o) \subset \widehat{A}^o \subset \widehat{A^o}$. Since $i(A^o)$ is dense in $\widehat{A^o}$, it follows that $\widehat{A}^o$ is dense in $\widehat{A^o}$. But $\widehat{A}^o$ is open in $\widehat{A}$ and so it is closed and therefore $\widehat{A}^o = \widehat{A^o}$.
ii) First of all $i(G)$ is dense in H and therefore since $i(A)$ is a subring of $\widehat{A}$, G is a subring of A if and only if H is a subring of $\widehat{A}$. It is enough to show that G is bounded if and only if H is bounded. If $i(G)$ is bounded, then so is G and so if H is bounded then so is G. Assume now that G is bounded. Then $i(G)$ is bounded and so we can find n such that $I^n \subset H$ where I is an ideal of definition of $\widehat{A}$. Then $H = i(G) + I^n$ since $i(G)$ is dense in H and therefore H is bounded.
iii) As in ii), G is a subring of A if and only if H is a subring of $\widehat{A}$. Moreover, $G \subset A^o$ if and only if $H \subset \widehat{A}^o$. Therefore, we need to show that G is an integrally closed open subring of A if and only if H is an integrally closed open subring of $\widehat{A}$. Assume first that G is an integrally closed open subring of A. let $x \in \widehat{A}$ be integral over H. Then $x^n + a_1x + \cdots + a_n = 0$ for some $a_i \in H$. Since H is an open neighborhood of $\widehat{A}$, there exist $x' \in A$ and $a_1', \ldots, a_n' \in G$ such that $x - i(x') \in H$ and $(x^n + a_1x^{n-1} + \cdots + a_n) - (i(x')^n + a_1'i(x')^{n-1} + \cdots + i(a_n')) \in H$. Hence it follows that $x'^n + a_1'x'^{n-1} + \cdots + a_n' \in G$ which implies that $x' \in G$ since G is integrally closed in A. Therefore $i(x') \in H$ and so $x = (x - i(x')) + i(x') \in H$ which means that H is integrally closed in $\widehat{A}$. If on the other hand H is integrally closed in $\widehat{A}$ and $x \in A$ is integral over G, then $i(x)$ is integral over H and so $i(x) \in H$ which implies that $x \in G$ which means that G is integrally closed in A. □

Definition 1.24. *A pair (A, A^+) where A is a Huber ring and A^+ a ring of definition is called a Huber pair. A morphism $(A, A^+) \to (B, B^+)$ of Huber pairs is a morphism of topological rings $\phi : A \to B$ with $\phi(A^+) \subset B^+$. Moreover ϕ is said to be adic if $\phi : A \to B$ is adic.*

The following follows directly from proposition 1.23.

Corollary 1.25. *For a Huber pair (A, A^+), the ring $\widehat{A^+}$ is a ring of integral elements of $\widehat{A}$ and thus $(\widehat{A}, \widehat{A^+})$ is a Huber pair. We call $(\widehat{A}, \widehat{A^+})$ the completion of (A, A^+).*

Remark. There is a natural morphism of Huber rings $i : (A, A^+) \to (\widehat{A}, \widehat{A^+})$ induced by the morphism $i : A \to \widehat{A}$, which is adic.

Definition 1.26. *Let (A, A^+) be a Huber pair and I an ideal of A. Then we define $(A, A^+)/I$ to be the Huber pair $(A/I, A^+/(A^+ \cap I)^{int})$ where A/I is endowed with the quotient topology.*

1.3 Adic spectrum and Rational Subsets

Let x be an equivalence class of valuations on a ring A and let $f \in A$. We write $|f(x)|$ for $|f|$ where $|\cdot|$ is some valuation in the class x. We only care about the inequalities valuations impose and not about the actual values, so any two choices in the same class will not affect the definitions. We will not always use this notation (especially in proofs where this would make things more confusing) and in this case we will write $x(f)$ instead. The latter point about working with equivalence classes of valuations is already visible in the following definition.

Definition 1.27. *Let A be a commutative ring. We consider the set $Spv(A)$ of equivalence classes of valuations on A with the topology generated by the sets of the form*

$$U(\tfrac{f_1,\ldots,f_n}{g}) = \{x \in Spv(A) : |f_i(x)| \leq |g(x)| \neq 0 \text{ for all } i\}$$

and call it the valuation spectrum of A.

Example 1.28. *Ostrowski's theorem states that on $\mathbb{Q}$, every absolute value is equivalent to the real absolute value or some p-adic absolute value (and those are valuations). Using that, one can prove that $Spv(\mathbb{Q})$ and $Spec(\mathbb{Z})$ are isomorphic topological spaces.*

Proposition 1.29. *$Spv(A)$ is spectral and its boolean algebra of constructible subsets is generated by the sets of the form $\{x \in Spv(A) : |a(x)| \leq |b(x)|\}$ for the different $a, b \in A$.*

Proof. Let u be a valuation on A. Then u defines a binary relation $|_u$ on A by $a|_u b$ if and only if $u(a) \geq u(a)$. Note that two valuations on A are equivalent if and only if $|_{u_1} = |_{u_2}$. Therefore we can define a map $\phi : Spv(A) \to P(A \times A)$ given by $x \mapsto |_u$ where u is any valuation in the class x. Note that this is injective. We consider $P(A \times A)$ as the topological space $\{0,1\}^{A \times A}$ where $\{0,1\}$ is endowed with the discrete topology. It is standard to check that the image of ϕ in $P(A \times A)$ is the set of all relations $|$ such that for all $a, b \in A$
1) $a|b$ or $b|a$
2) $a|b$ and $b|c$ implies that $a|c$
3) $a|b$ and $a|c$ implies that $a|b+c$
4) $a|b$ implies that $ac|bc$ for all $c \in A$
5) if $ac|bc$ and $0 \not| c$ implies that $a|b$
6) $0 \not| 1$
Therefore it follows that the sets $U(\frac{f_1,\ldots,f_n}{g})$ are open and closed with respect to the topology induced by ϕ. Therefore we get that X is spectral and that the sets of the form $U(\frac{f_1,\ldots,f_n}{g})$ form a basis of quasi-compact open subsets of $Spv(A)$. □

Definition 1.30. *Let A be a commutative ring. We consider the set $Cont(A)$ of equivalence classes of continuous valuations on A with the topology generated by the sets of the form*

$$U_{cont}(\tfrac{f_1,\ldots,f_n}{g}) = \{x \in Cont(A) : |f_i(x)| \leq |g(x)| \neq 0 \textit{ for all } i\}$$

and call it the continuous spectrum of A.

Note. $Cont(A)$ can be viewed as a subspace of $Spv(A)$.

We have the following important theorem

Theorem 1.31. *For every Huber ring A, $Cont(A)$ is spectral and the subsets of the form*

$$U_{cont}(\tfrac{f_1,\ldots,f_n}{g}) = \{x \in Cont(A) : |f_i(x)| \leq |g(x)| \neq 0 \textit{ for all } i\}$$

with $A^{oo} \subset \sqrt{(f_1, \ldots, f_n)}$ form a basis of quasi-compact subsets of $Cont(A)$.

Proof. For every ideal I of A such that $Spec(A) - V(I)$ is quasi-compact (this is equivalent to saying that there exists a finitely generated ideal J of A such that $\sqrt{I} = \sqrt{J}$) and every valuation $x \in Spv(A)$, we define $c\Gamma_x(I)$ to be equal to $c\Gamma_x$ if $x(I) \cap c\Gamma_x \neq \emptyset$. Otherwise, we define $c\Gamma_x(I)$ to be equal to the greatest convex subgroup of Γ_x such that $x(a)$ is cofinal for all $a \in I$ (meaning that for all $h \in H$ there is $n \in \mathbb{N}$ such that $x(a)^n < h$). Note that groups as the latter exist, so we may consider the greatest one: for example, if T is a finite set of generators of I, we consider the convex subgroup of Γ_x generated by $max\{x(t) : t \in T\}$. We define

$$Spv(A, I) = \{x \in Spv(A) : c\Gamma_x(I) = \Gamma_x\}$$

and endow it with the subspace topology induced by $Spv(A)$. We have

i) $Spv(A, J)$ is spectral and a basis of quasi-compact subsets is given by the sets of the form

$$U_J(\tfrac{f_1,\ldots,f_n}{g}) = \{x \in Spv(A, J) : |f_i(x)| \leq |g(x)| \neq 0 \text{ for all } i\}$$

for which $J \subset \sqrt{(f_1, \ldots, f_n)}$.

We note that $x(I) \cap H \neq \emptyset$ if and only if $x(\sqrt{I}) \cap H \neq \emptyset$ for every subgroup H of Γ_x. That is because $x(I) \cap H \subset x(\sqrt{I}) \cap H$ and if $x(\sqrt{I}) \cap H \neq \emptyset$, then there is $a \in A$ such that $x(a) \in H$. But then $a^n \in I$ for some n and so $x(a)^n \in x(I) \cap H$.
So $Spa(A, I)$ only depends on $\sqrt{I}$ and so since there exists a finitely generated ideal J such that $\sqrt{I} = \sqrt{J}$, we may assume that I is finitely generated. The sets of the form $U_J(\frac{f_1,\ldots,f_n}{g})$ are obviously open as

$$U_J(\tfrac{f_1,\ldots,f_n}{g}) = Spa(A, J) \cap U(\tfrac{f_1,\ldots,f_n}{g})$$

and moreover $U_J(\frac{f_1,\ldots,f_n}{g}) = U_J(\frac{f_1,\ldots,f_n,g}{g})$. Let now T, T' be finite subsets of A with $J \subset \sqrt{T}$ and $I \subset \sqrt{T'}$ and let $g, g' \in A$. Then $J \subset \sqrt{TT'}$. Moreover we may assume that $g \in T$ and $g' \in T'$ and then

$$U_J(\tfrac{T}{g}) \cap U_J(\tfrac{T'}{g'}) = U_J(\tfrac{TT'}{gg'})$$

We now want to show that the sets of the form $U_J(\frac{f_1,\ldots,f_n}{g})$ with $J \subset \sqrt{(f_1, \ldots, f_n)}$ form a basis for the topology on $Spv(A, J)$. Of course this would be obvious if there was no restriction on $f_1, \ldots, f_n$. For this, it is enough to show that for every $x \in Spv(A, J)$ and every open U of $Spv(A)$ with $x \in U$, there exist $f_1, \ldots, f_n \in A$ with $J \subset \sqrt{(f_1, \ldots, f_n)}$ such that $U_j(\frac{f_1,\ldots,f_n}{g}) \subset U$. Let U be such an open subset. Let $f_1, \ldots, f_n, g \in A$ such that $x \in U(\frac{f_1,\ldots,f_n}{g}) \subset U$.
If $\Gamma_x = c\Gamma_x$, there exists $a \in A$ such that $x(a) \geq x(g)^{-1}$ (this follows directly from the definition of $c\Gamma_x$), i.e $x(ag) \geq 1$. Then

$$x \in U_J(\tfrac{af_1,\ldots,af_n,1}{ag}) \subset U(\tfrac{f_1,\ldots,f_n}{g}) \subset U$$

If on the other hand $\Gamma_x \neq c\Gamma_x$ and $a_1, \ldots, a_m$ are generators of J, then there exists $k \in \mathbb{N}$ such that $x(a_i)^k < x(g)$ for all $i \in \{1, \ldots, m\}$ (this is because $x \in Spa(A, J)$ is equivalent to $\Gamma_x = x\Gamma_x$ or $x(a)$ is cofinal for every $a \in J$, which follows from the definitions). Then

$$x \in U_J(\tfrac{f_1,\ldots,f_n,a_1^k,\ldots,a_m^k}{g}) \subset U(\tfrac{f_1,\ldots,f_n}{g}) \subset U$$

We now consider the map $r : Spv(A) \to Spv(A, J)$ given by $x \mapsto x|_{c\Gamma_x(J)}$. Let $T \subset A$ be finite such that $J \subset J$ and $g \in A$. We will show that $r^{-1}(U_J(\frac{T}{g})) = U(\frac{T}{g})$. From the way we defined r, it follows that $r^{-1}(U_J(\frac{T}{g})) \subset U(\frac{T}{g})$. Let now $x \in U(\frac{T}{g})$. We will show that $r(x) \in U_J(\frac{T}{g})$. If $r(J) = \{0\}$, then $r(x) = x$. If $r(J) \neq \{0\}$ then $r(x)(t) \leq r(x)(g)$ for all $t \in T$ and so we only need to show that $r(x)(g) \neq 0$. Assume that $r(x)(g) = 0$. Then $r(x)(t) = 0$ for all $t \in T$ and so $r(x)(j) = 0$ for all $j \in J$ since $J \subset \sqrt{T}$. But $x(J) \cap c\Gamma_x(J) \neq \emptyset$ and so there is $j \in J$ with $r(x)(j) \neq 0$ which contradicts the assumption.

Let us now combine the above to finally prove i). Consider $SpvA$ with its constructible topology. Moreover consider the boolean algebra of the subsets of $Spv(A, J)$ of the form $U_J(\frac{T}{g})$ with $J \subset \sqrt{(T)}$ and let X be $Spv(A, J)$ equipped with the topology generated by the latter algebra. Then $r : (SpvA)_{cons} \to X$ is continuous and since $(SpvA)_{cons}$ is quasi-compact and r surjective, X is quasi-compact. Moreover we conclude that $Spv(A, J)$ is spectral and all the subsets of the form $U_J(\frac{T}{g})$ (with the known assumptions on T) are constructible. Moreover r is spectral.

ii) $Cont(A) = \{x \in Spv(A, A^{oo} \cdot A) : \text{for all } a \in A^{oo}, |a(x)| < 1\}$

Let $x \in Cont(A)$. Then for $x \in A^{oo}$ and every $\gamma \in \Gamma_x$ there is n such that $x(a)^n < \gamma$, i.e $x(a)$ is cofinal in Γ_x. This implies that $x(a) < 1$ and $\Gamma_x = c\Gamma_x(A^{oo} \cdot A)$.
Let now $x \in Spv(A, A^{oo} \cdot A)$: for all $a \in A^{oo}$, $x(a) < 1\}$. We first prove that $x(a)$ is cofinal in Γ_x for

all $a \in A^{oo}$. If $\Gamma_x \neq c\Gamma_x$, then we already know it. If $\Gamma_x = c\Gamma_x$, fix $a \in A$ and $\gamma \in \Gamma_x$. There is t such that $x(t) \neq 0$ and $x(t)^{-1} \leq \gamma$. Moreover there is n such that $ta^n \in A^{oo}$ and then $x(ta^n) < 1$ which implies that $x(a)^n < \gamma$. Let U be an open subset of A such that $(U^n)_{n\in\mathbb{N}}$ forms a fundamental system of neighborhoods of 0 in A and T a finite subset of U such that $T \cdot U = U^2 \subset U$. Since $U \subset A^{oo}$, we have $x(u) < 1$ for all $u \in U$. For a given $\gamma \in \Gamma_x$, there exists n such that $(max\{x(t), t \in T\})^n < \gamma$ and so for every $a \in T^n \cdot U = U^{n+1}$ we have $x(a) < \gamma$. This implies that x is continuous.

iii) Put $J = A^{oo} \cdot A$. Then

$$Cont(A) = Spv(A, J) - \bigcup_{g \in A^{oo}} U_J(\tfrac{1}{g})$$

and so $Cont(A)$ is a closed subset of $Spa(A, J)$

By combining all the above the result follows. Note that for an ideal J of A we have $A^{oo} \subset \sqrt{J}$ if and only if J is open. That is because if J is open, then for $a \in A^{oo}$,$a^n \in J$ for n large enough and so $a \in \sqrt{J}$. If on the other hand $A^{oo} \subset \sqrt{J}$ and (A_0, I) is a couple of definition of A, then $I \subset A^{oo}$ and so $I \subset \sqrt{J}$. Hence, if $I = (a_1, \ldots, a_k)$ for some $a_i \in A_0$, there is N such that $a_1^N, \ldots, a_k^N \in J$ and so $I^{kN} \subset J$. Therefore J is open. Using this, we see that the requirement on $f_1, \ldots, f_n$ is that they generate an open ideal. □

Let (A, A^+) be a Huber pair.

Definition 1.32. *We define $Spa(A, A^+)$ to be the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$ with the topology generated by the sets of the form*

$$\{x \in Spa(A, A^+) : |f(x)| \leq |g(x)| \neq 0 \text{ for all } i\}$$

for the different elements $g, f \in A$.

Definition 1.33. *Let $(A, A+)$ be a Huber pair and consider $X = Spa(A, A^+)$. Let $s \in A$ and $t_1, \ldots, t_n \in A$ be such that they generate an open ideal of A. We define the set*

$$X(\tfrac{t_1,\ldots,t_n}{s}) = \{x \in X : max|t_i(x)| \leq |s(x)| \neq 0 \text{ for all } i\}$$

and call such sets rational subsets of X.

Note. The intersection of any two rational subsets is rational: it is enough to show this for two rational subsets and this comes down to showing that for finite subsets $\{t_1, \ldots, t_n\}, \{t'_1, \ldots, t'_m\} \subset A$ such that $(t_1, \ldots, t_n), (t'_1, \ldots, t'_m)$ are open, the ideal generated by the elements $t_i t'_j$ is open. But this is true since $I^k \subset (t_1, \ldots, t_n)$ and $I^l \subset (t'_1, \ldots, t'_m)$ for some k, l and therefore $I^{k+l} \subset (t_i t'_j)$.

Note. A morphism $\phi : (A, A^+) \to (B, B^+)$ of Huber pairs gives rise to a morphism of topological spaces $Spa(\phi) : Spa(B, B^+) \to Spa(A, A^+)$ which maps x to $x \circ \phi$. Although this is continuous, $Spa(\phi)^{-1}(U)$ might not be rational where U is some rational subset of $Spa(B, B^+)$. This is because of the condition that $t_1, \ldots, t_n$ generate an open ideal. If ϕ is adic, then this property holds.

Theorem 1.34. *$Spa(A, A^+)$ is spectral and the rational subsets form a basis of quasi-compact subsets of $Spa(A, A^+)$.*

Proof. Write X for $Spa(A, A^+)$. For $a \in A^+$ we have

$$\{x \in Cont(A) : |a(x)| \leq 1 \text{ for all } a \in A\} = U_{cont}(\tfrac{1,a}{1})$$

which is a quasi-compact subset of $Cont(A)$ and so is constructible. Hence

$$Spa(A, A^+) = \bigcap_{a \in A^+} U_{cont}(\tfrac{1,a}{1})$$

which is pro-constructible. Therefore since $Cont(A)$ is spectral, it follows that $Spa(A, A^+)$ is spectral and the inclusion $Spa(A, A^+) \to Cont(A)$ is spectral. Now since

$$X(\frac{f_1,\dots,f_n}{g}) = X \bigcap U_{cont}(\frac{f_1,\dots,f_n}{g})$$

the result follows. □

Lemma 1.35. *Let A be a Huber ring and T a subset of A such that the elements of T generate an open ideal of A. Then for every $n \in \mathbb{N}$ and every neighborhood V of 0 in A, the set $T^n \cdot V$ is open.*

Proof. Since $T \cdot A$ is open it follows that $T^n \cdot A = (T \cdot A)^n$ is open. Consider a finite subset U of A and a finite subset S of of U such that $(U^m)_{m\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0 in A, $S \cdot U = U^2 \subset U$ and $U \subset T^n \cdot A$. We have seen earlier that such U, S exist. Let R be a finite subset of A such that $S \subset T^n \cdot R$ and $k \in \mathbb{N}$ such that $R \cdot U^k \subset V$. From the above relations it follows that $U^{k+1} \subset T^n \cdot V$ and so the result follows. □

Proposition 1.36. *The natural morphism $i : (A, A^+) \to (\widehat{A}, \widehat{A}^+)$ induces an isomorphism of topological spaces $Spa(\widehat{A}, \widehat{A}^+) \to Spa(A, A^+)$ and $U \subset Spa(A, A^+)$ is rational if and only if $Spa(i)^{-1}(U)$ is rational.*

The following 3 lemmata are the key components to the proof of the latter. Those are mostly valuation-theoretic, but of great independent interest (especially the last 2) and so we proceed to prove them.

Lemma 1.37. *Let A be a Huber ring and $i : A \to \widehat{A}$ its completion. The induced map $Cont(\widehat{A}) \to Cont(A)$ is bijective.*

Proof. Let $\mathcal{F}$ be a Cuachy filter on A and let $|\cdot| : A \to \Gamma \cup \{0\}$ be a valuation on A. We claim that one of the following is true:
i) for every $\gamma \in \Gamma$ there is $F \in \mathcal{F}$ such that $|a| < \gamma$ for all $a \in F$
ii) there is $F \in \mathcal{F}$ such that $|\cdot|$ is constant on F.
Indeed, if i) does not hold, then there exists $\gamma_0 \in \Gamma$ such that for every $F \in \mathcal{F}$, there is $a \in F$ with $|a| \geq \gamma_0$. The set $\{a \in A : |a| < \gamma_0\}$ is an open neighborhood of 0 and since $\mathcal{F}$ is Cauchy, there exists $F \in \mathcal{F}$ such that $|a - b| < \gamma_0$ for all $a, b \in F$. Let $a_0 \in F$ be such that $|a_0| \geq \gamma_0$. Then for all $a \in F$ we have $|a - a_0| < g \leq |a_0|$ and so from the strong triangle inequality it follows that $|a| = |a_0|$, so ii) follows.
We resume the proof. Since $i(A)$ is dense in $\widehat{A}$, every continuous valuation on $\widehat{A}$ is completely determined by its restriction to $i(A)$ and so $Cont(\widehat{A}) \to Cont(A)$ is injective.
Let now $|\cdot| : A \to \Gamma \cup \{0\}$ be a continuous valuation on A and let $a \in ker i$ and $\mathcal{F}$ be the filter of neighborhoods of a in A ,which is Cauchy. If $|\cdot|$ satisfies the condition i) as above, then $|a| < \gamma$ for every $\gamma \in \Gamma$ and so $|a| = 0$. If it satisfies condition ii) then there is $F \in \mathcal{F}$ such that $|\cdot|$ is constant on F. But $a \in F$ and $0 \in F$ and so $|a| = |0| = 0$. In any case we have that $|a| = 0$ and so it follows that $|\cdot|$ factors through $i(A)$ and so we may assume that i is injective.
Let $a \in \widehat{A}$ and consider a Cauchy filter $\mathcal{F}$ on A that converges to a. If $\mathcal{F}$ satisfies condition i), then we set $|a|' = 0$. If it satisfies condition ii) then consider $F \in \mathcal{F}$ such that $|\cdot|$ is constant on F and set $|a|' = |b|$ for any $b \in F$. It is straightforward to check the resulting map $|\cdot|' : \widehat{A} \to \Gamma \cup \{0\}$ is a valuation and is independent of the choices made. In fact $|\cdot|'$ is continuous as for every $\gamma \in \Gamma$, $\{a \in A : |a| < \gamma\}$ is an open subgroup of A and by definition of $|\cdot|'$, its closure in $\widehat{A}$ is contained in $\{a \in \widehat{A} : |a|' < \gamma\}$ which implies that the latter is open. Therefore $Cont(\widehat{A}) \to Cont(A)$ is surjective and the result follows. □

Lemma 1.38. *Let A be a complete Huber ring and $t_1, \dots, t_n, s \in A$ such that $(t_1, \dots, t_n)$ is an open ideal of A. There exists a neighborhood U of 0 in A such that for all $s' \in s + U, t'_1 \in t_1 + U, \dots, t'_n \in t_n + U$, $(t'_1, \dots, t'_n)$ is an open ideal of A and $X(\frac{t_1,\dots,t_n}{s}) = X(\frac{t'_1,\dots,t'_n}{s'})$*

Proof. We write I for the ideal generated by $t_1, \ldots, t_n$. Let A_0 be a ring of definition of A. Let $r_1, \ldots, r_m \in A_0 \cap I$ such that $J = (r_1, \ldots, r_m)$ is an open ideal of A_0. By [Bou72] III 2.10 cor. 3, there is an open neighborhood V of 0 in A_0 such that for all $r'_1 \in r_1 + V, \ldots, r'_m \in r_m + V$, $J = (r'_1, \ldots, r'_m)$. Therefore there exists an open neighborhood U' of 0 in A such that $(t'_1, \ldots, t'_n)$ is open in A for all $t'_1 \in t_1 + U', \ldots, t'_n \in t_n + U'$.
For every $i \in \{0, 1, \ldots, n\}$ we consider the rational subset $R_i = X(\frac{s,t_1,\ldots,t_n}{t_i})$ where by convention $t_0 = s$. Every R_i is quasi-compact and for every $x \in R_i$, $x(t_i) \neq 0$. Therefore by lemma 1.39, there exists a neighborhood U'' of 0 in A such that $x(u) < x(t_i)$ for every $u \in U''$,every i and every $x \in R_i$. We consider $U = U' \cap U'' \cap A^{oo}$ and we will show that it has the desired property. Let $t'_1 \in t_1 + U, \ldots, t'_n \in t_n + U, s' \in s + U$. Let $x \in X(\frac{t_1,\ldots,t_n}{s})$. Since $t'_i - t_i \in U''$, it follows that $x(t'_i - t_i) < x(s)$ and so

$$x(t'_i) = x(t_i + (t'_i - t_i)) \leq max\{x(t_i), x(t'_i - t_i)\} \leq x(s) = x(s')$$

Therefore $X(\frac{t_1,\ldots,t_n}{s}) \subset X(\frac{t'_1,\ldots,t'_n}{s'})$.
Let now $x \in Spa(A, A^+)$ such that $x \notin X(\frac{t_1,\ldots,t_n}{s})$. We will show that $x \notin X(\frac{t'_1,\ldots,t'_n}{s'})$. If $x(t_i) = 0$ for all i, then $supp(x)$ is open, as it contains I which is open. Therefore $s' - s \in supp(x)$, as $s' - s \in A^{oo}$ and so $s' \in supp(x)$. Therefore $s' \in X(\frac{t'_1,\ldots,t'_n}{s'})$. If now $x(t_i) \neq 0$ for some i, then we may choose t_j such that $x(t_j)$ is maximal among $x(s), x(t_n), \ldots, x(t_n)$. Then $x(t_0) < x(t_j)$ since otherwise $x \in X(\frac{t_1,\ldots,t_n}{s})$. Moreover note that $x \in R_j$ and since $t'_i - t_i \in U''$, it follows that $x(t'_i - t_i) < x(t_j)$ for all i. Therefore it follows that

$$x(t'_0) = x(t_0 + t'_0 - t_0) \leq max\{x(t_0), x(t'_0 - t_0)\} < x(t_j) = x(t'_j)$$

Therefore $x \notin X(\frac{t'_1,\ldots,t'_n}{s'})$. □

Lemma 1.39. *Le (A, A^+) be a Huber pair, Y a quasi-compact subset of $Spa(A, A^+)$ and $s \in A$ such that $x(s) \neq 0$ for all $x \in Y$. There exists a neighborhood U of 0 in A such that $x(u) < x(s)$ for every $u \in U$ and $x \in Y$.*

Proof. Let T be a finite subset of A^{oo} such that $T \cdot A^{oo}$ is open. For every $n \in \mathbb{N}$ we consider $Y_n = \{x \in Spa(A, A^+) : x(t) \leq x(s) \neq 0 \text{ for all } t \in T^n\}$ that is open in $Spa(A, A^+)$. Moreover, as $T \subset A^{oo}$, it follows that for every $x \in Y$, $x \in Y_n$ for some n large enough, so $Y \subset \bigcup_m Y_m$. Therefore as Y is quasi-compact, $Y \subset Y_m$ for some m. Put $U = T^m \cdot A^{oo}$ which satisfies the property we wish. □

Proof. (of proposition 1.36) We use lemmata 1.37,1.38,1.39. First of all $i : A \to \widehat{A}$ gives rise to a bijection $Cont(\widehat{A}) \to Cont(A)$. Therefore by the definition of the adic spectrum, it follows directly that $Spa(i) : Spa(\widehat{A}, \widehat{A}^+) \to Spa(A, A^+)$ is bijective. Let $X(\frac{t_1,\ldots,t_n}{s})$ be a rational subset of $Spa(A, A^+)$. The ideal generated by $i(t_1), \ldots, i(t_n)$ is open and we have

$$Spa(i)^{-1}(U(\tfrac{t_1,\ldots,t_n}{s})) = X(\tfrac{i(t_1),\ldots,i(t_n)}{i(s)})$$

Therefore the inverse image of a rational subset is rational and so $Spa(i)$ is continuous. Hence $Spa(i)$ is an isomorphism of topological spaces. We shall now prove that the image of a rational subset of $Spa(\widehat{A}, \widehat{A^+})$ is a rational subset of $Spa(A, A^+)$. Since $i(A)$ is dense in $\widehat{A}$, it follows that every rational subset of $Spa(\widehat{A}, \widehat{A^+})$ is of the form

$$X(\tfrac{i(t_1),\ldots,i(t_n)}{i(s)})$$

for some $t_1, \ldots, t_n, s \in A$ such that $i(t_1), \ldots, i(t_n)$ generated an open ideal. Since this is quasi-compact and $|i(s)(x)| \neq 0$ for all $x \in U$, it follows that there is an open neighborhood G of 0 in A such that $|i(g)(x)| < |i(s)(x)|$ for all $g \in G$ and $x \in U$. Therefore for any $t'_1, \ldots, t'_m \in G$ such that $(t'_1, \ldots, t'_m)$ is open, we have that

$$X(\frac{t_1,\dots,t_n,t'_1,\dots,t'_m}{s})$$

is a rational subset and it is equal to $Spa(i)(U)$. □

Proposition 1.40. *For a complete Huber pair* (A, A^+)
i) $Spa(A, A^+) = \emptyset$ *if and only if* $A/\overline{\{0\}} = 0$.
ii) $A^+ = \{f \in A : |f(x)| \leq 1$ *for all* $x \in Spa(A, A^+)\}$
iii) $f \in A$ *if invertible if and only if* $|f(x)| \neq 0$ *for all* $x \in Spa(A, A^+)$.

Proof. i) Let us assume that $Spa(A, A^+) = \emptyset$. The main idea is proving that $\overline{\{0\}}$ is an open ideal of $Spa(A, A^+)$. This is done, by first proving an analogous theorem for $Spa(A, A^+)_{an}$ which is the set of analytic points of A, i.e the set of points x for which $supp(x)$ is not open in A. Namely, one can prove that $Spa(A, A^+) = \emptyset$ if and only if $A/\overline{\{0\}}$ is discrete. For a proof of this fact, see [Hub93b] Lemma 3.6 ii. Let us use that $\overline{\{0\}}$ is open as a fact. For every prime ideal p of A, consider the valuation $|\cdot|_p$ on A given by $|a|_p = |amodp|_{triv}$ where $|\cdot|_{triv}$ is the trivial valuation on A/p. The valuations $|\cdot|_p$ for which $\overline{\{0\}} \subset supp(|\cdot|_p)$ are elements of $Spa(A, A^+)$. By the assumption, there should be no such p, i.e there is no prime ideal p of A with $\overline{\{0\}} \subset supp(|\cdot|_p) = p$. This implies that $A = \overline{\{0\}}$.
ii) Let $\mathcal{F}_A$ denote the set of pro-constructible subsets of $Cont(A)$. Such sets are intersections of the form

$$\bigcap_{a\in A'}\{x \in Cont(A) : x(a) \leq 1\}$$

where A' can be any subset of A. Moreover we consider the set $\mathcal{G}_A$ that consists of the subrings of A that are open and integrally closed. There is a bijection between $\mathcal{F}_A$ and $\mathcal{G}_A$ mapping $G \in \mathcal{G}_A$ to $\{x \in Cont(A) : x(g) \leq 1$ for all $a \in G\}$ with inverse mapping $F \in \mathcal{F}_A$ to $\{a \in A : x(a) \leq 1$ for all $x \in F\}$. This fact uses heavily techniques that we used in theorem 1.31. For a proof see [Hub93b] Lemma 3.3. This implies directly our proposition.
iii) We will first show that every maximal ideal m of A is closed and that there exists $x \in Spa(A, A^+)$ such that $m = supp(x)$. Consider A^{oo} which is an open subring of A. Then $1 + A^{oo}$ is open in A and as A is complete, $1 + A^{oo}$ is an open subring of the group of units of A. Therefore $A^\times$ is open in A. Assume that m is not closed. Then its closure is A, which contradicts that $A \setminus A^\times$ is closed. Hence m is closed. Therefore A/m is Hausdorff and

$$\{x \in Spa(A, A^+) : supp(x) \supseteq m\} = \{x \in Spa(A, A^+) : supp(x) = m\} = Spa((A, A^+)/m)$$

which proves our claim. Therefore since $f \in A$ is invertible if and only if it does not belong in any maximal ideal of A, the result follows.

□

Note. Part ii) of the above proposition is often called "adic nullstellensatz". The assumptions on A^+ that it should be open and integrally closed in A are just to ensure that we can recover A^+ from A and $Spa(A, A^+)$ while the fact that $A^+ \subset A^o$ is the crucial condition for the theory to work.

Theorem 1.41. *Let* $U \subset X$ *be a rational subset. There exists a complete Huber pair* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *with a morphism of Huber pairs* $\phi : (A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *such that the induced morphism* $Spa(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to Spa(A, A^+)$ *factors over* U *and for every morphism* $f : (A, A^+) \to (B, B^+)$ *where* (B, B^+) *is complete such that the induced morphism* $Spa(\phi) : Spa(B, B^+) \to Spa(A, A^+)$ *factors over* U*, there exists a unique morphism* $\psi : (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (B, B^+)$ *such that* $f = \psi \circ \phi$.

To prove this, we need the following two lemmata.

Lemma 1.42. *Let* A *be a Huber ring with couple of definition* (A_0, I)*,* $s \in A$ *and* T *be a finite subset of* A *such that the elements of* T *generate an open ideal of* A*. We consider a group topology on the localization* $A[\frac{1}{s}]$ *such that* $(I^n \cdot B)_{n\in\mathbb{N}}$ *is a fundamental system of neighborhoods of* 0*, where* $B = A_0[\frac{t}{s}, t \in T]$*. The latter is a topological ring which we denote* $A(\frac{T}{s})$ *and is independent of the choice of the couple of definition* (A_0, I)*.*

Proof. We first show that it is a topological ring. By lemma 1.35, $T \cdot A_0$ is open in A. Therefore $I^n \subset T \cdot A_0$ for some n and so $\frac{t}{s} \cdot I^n \subset A_0[\frac{t}{s}, t \in T]$ which implies that for every $x \in A[\frac{1}{s}]$ and every neighborhood U of 0 in $A(\frac{T}{s})$ there exists a neighborhood V of 0 with $x \cdot V \subset U$, so indeed $A(\frac{T}{s})$ is a topological ring. With the same argument it follows easily that two couples of definition $(A_1, I_1), (A_2, I_2)$ on A give the same topology. □

Let (A, A^+) be a Huber pair. Let $s \in A$ and $T \subset A$ be a finite such that the elements of T generate an open ideal. Let $A(\frac{T}{s})^+$ be the integral closure of $A^+[\frac{T}{s}]$ in $A[\frac{1}{s}]$. Then $A(\frac{T}{s})^+$ is a ring of integral elements of $A(\frac{T}{s})$. We will sometimes denote the Huber pair $(A(\frac{T}{s}), A(\frac{T}{s})^+)$ as $(A, A^+)(\frac{T}{s})$.

Note. From the definition it follows that the natural morphism $(A, A^+) \to (A(\frac{T}{s}), A(\frac{T}{s})^+)$ is adic.

Lemma 1.43. *Let $s \in A$ and $\{t_1, \ldots, t_n\} \subset A$ be such that $(t_1, \ldots, t_n)$ is open. There exists a complete Huber pair $(A\langle\frac{t_1,\ldots,t_n}{s}\rangle, A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+)$ with a morphism of Huber pairs $\phi : (A, A^+) \to (A\langle\frac{t_1,\ldots,t_n}{s}\rangle, A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+)$ such that $\phi(s)$ is invertible in $A\langle\frac{t_1,\ldots,t_n}{s}\rangle$ and $\phi(\frac{t_i}{s}) \in A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+$ for all i and is universal for all such Huber pairs, i.e for any complete Huber pair (B, B^+) with a morphism $f : (A, A^+) \to (B, B^+)$ such that $f(s)$ in invertible in B and $f(\frac{t_i}{s}) \in B^+$ for all i, there exists a unique morphism $g : (A\langle\frac{t_1,\ldots,t_n}{s}\rangle, A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+) \to (B, B^+)$ such that $f = g \circ \phi$.*

Proof. The Huber pair $(A(\frac{t_1,\ldots,t_n}{s}), A(\frac{t_1,\ldots,t_n}{s})^+)$ satisfies the same property without assuming that (B, B^+) is complete. By the universal property of completions, it follows that the completion of $(A(\frac{t_1,\ldots,t_n}{s}), A(\frac{t_1,\ldots,t_n}{s})^+)$ satisfies the desired properties. □

Proof. (of theorem 1.41) We set $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (A\langle\frac{t_1,\ldots,t_n}{s}\rangle, A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+)$. Let (B, B^+) be a complete Huber pair with a morphism $f : (A, A^+) \to (B, B^+)$ such that the induced morphism $Spa(B, B^+) \to Spa(A, A^+)$ factors over U. First we note that $|f(s)(x)| \neq 0$ for all $x \in Spa(B, B^+)$ and thus it follows from proposition 1.40 that $f(s)$ in invertible in B. Moreover, $|(\frac{f(t_i)}{f(s)})(x)| \leq 1$ for all i and for all $x \in Spa(A, A^+)$ and thus it follows from proposition 1.40 that $\frac{f(t_i)}{f(s)} \in B^+$ for all i. The result follows from the universal property of $(A\langle\frac{t_1,\ldots,t_n}{s}\rangle, A\langle\frac{t_1,\ldots,t_n}{s}\rangle^+)$. □

In addition to theorem 1.41, we have the following

Proposition 1.44. *Let (A, A^+) be a Huber pair and U a rational subset of $Spa(A, A^+)$.*
i) If $V \subset U$ is a rational subset of $Spa(A, A^+)$, there exits a morphism of Huber pairs $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (\mathcal{O}_X(V), \mathcal{O}_X^+(V))$
ii) The morphism $Spa(\mathcal{O}_X(U), \mathcal{O}_x^+(U)) \to Spa(A, A^+)$ is a homeomorphism onto U and it induces a bijection between the rational subsets of $Spa(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ and the rational subsets of $Spa(A, A^+)$ that are contained in U.

Proof. i) follows from the universal property of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$
ii) Assume that $U = X(\frac{T}{s})$ where $T \subset A$ is a finite subset of A such that its elements generate an open ideal. Factor the morphism $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ as

$$(A, A^+) \xrightarrow{f} (A(\tfrac{T}{s}), A(\tfrac{T}{s})^+) \xrightarrow{i} (A\langle\tfrac{T}{s}\rangle, A\langle\tfrac{T}{s}\rangle^+)$$

As f and i are adic, it follows that $i \circ f$ is adic and therefore the inverse image of any rational subset of $Spa(A, A^+)$ is a rational subset of $Spa(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$. Since i induces an isomorphim $Spa(A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+) \to Spa(A(\frac{T}{s}), A(\frac{T}{s})^+)$ mapping rational subsets to rational subsets, it is enough to prove that $Spa(f) : Spa(A(\frac{T}{s}), A(\frac{T}{s})^+) \to Spa(A, A^+)$ is a homeomorphism onto U mapping rational subsets to rational subsets. The first claim follows from the definition of $A(\frac{T}{s})$. We write Y for $Spa(A(\frac{T}{s}), A(\frac{T}{s})^+)$ and let $V = Y(\frac{l_1,\ldots,l_n}{g})$ be a rational subset of Y, for some $l_1, \ldots, l_n, g \in A(\frac{T}{s}) = A[\frac{1}{s}]$. By multiplying $l, g_1, \ldots, g_n$ by a suitable power of s, we may assume that there exist $d, c_1, \ldots, c_n \in A$ with $l = f(d)$ and $g_i = f(c_i)$ for all i. Since V is quasi-compact, there exists a neighborhood E of 0 in A such that $x(f(e)) \leq x(l)$ for every $e \in E$ and $x \in V$. Let

D be a finite subset of E such that $D \cdot A$ is open in A. Then $X(\frac{\{c_1,\dots,c_n\}\cup D}{d})$ is a rational subset of $Spa(A,A^+)$ and $Spa(f)(V) = U \bigcap X(\frac{\{c_1,\dots,c_n\}\cup D}{d})$ which is rational in $Spa(A,A^+)$

□

Definition 1.45. *We define a presheaf on $X = Spa(A,A^+)$ in the following way: If $U \subset X$ is a rational subset, then $\mathcal{O}_X(U)$ is an in theorem 1.42. For a general open $W \subset X$, we define*

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W, rational} \mathcal{O}_X(U)$$

which we equip with the inverse limit topology. This is a presheaf of complete topological rings, for obvious reasons. In the same way we define a presheaf $\mathcal{O}_X^+$.

Note. For an open subset U of X, one can prove that $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1$ for all $x \in U\}$. This follows almost directly from the adic nullstellensatz in case U is a rational subset and then follows the general case. In particular, if $\mathcal{O}_X$ is a sheaf, then so is $\mathcal{O}_X^+$.

Note. $\mathcal{O}_X$ is not always a sheaf as the following example shows. We consider $A = \mathbb{Z}[X_1, X_2, X_3]_{X_1 X_2}$ and let B be the subring of A generated by $X_2, X_1X_2, X_1^{-1}X_2, X_1^n X_2^{-n} X_3, X_1^{-n} X_2^{-n} X_3$ for all $n \in \mathbb{N}$. We endow A with a topology such that $\{X_2^n B\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of 0. Let A^+ be any ring of integral elements of A. We claim that $\mathcal{O}_{Spa(A,A^+)}$ is not a sheaf. We consider the sets $U = \{x \in Spa(A,A^+) : x(X_1) \leq 1\}$ and $V = \{x \in Spa(A,A^+) : x(X_1) \geq 1\}$. Let X be the element of $\mathcal{O}_{Spa(A,A^+)}(Spa(A,A^+))$ given by X_3. One can see that $X|_U = X|_V = 0$ while $X \neq 0$. This was the first example of such a Huber pair in the literature. In [Mih16] and [BV15] further such examples are given, notably, in the case of Tate Huber pairs.

Definition 1.46. *A Huber pair (A,A^+) is called sheafy if $\mathcal{O}_{Spa(A,A^+)}$ is a sheaf. A Huber ring A is called sheafy if for every ring of integral elements A^+ of $\widehat{A}$, $\mathcal{O}_{Spa(\widehat{A},A^+)}$ is a sheaf.*

Definition 1.47. *For $x \in X$ we define the ring $\mathcal{O}_{X,x}$ which we call the stalk of X at the point x to be $\varinjlim_{x\in U, open} \mathcal{O}_X(U)$*

Note. In the above definition, we can replace the open subsets with rational subsets since they form a basis for the topology on X. Therefore we have $\mathcal{O}_{X,x} = \varinjlim_{x\in V, rational} \mathcal{O}_X(V)$

For a rational subset U of X and a valuation $x : A \to \Gamma_x \cup \{0\}$ we can uniquely extend x to a continuous valuation $|\cdot|_U : \mathcal{O}_X(U) \to \Gamma_x \cup \{0\}$ (this follows from proposition 1.44). Therefore we can define, in the only natural way, a valuation $|\cdot|_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{0\}$.

Proposition 1.48. *For every $x \in X$ the stalk is a local ring with maximal ideal the support of the valuation $|\cdot|_x$.*

Proof. Let $U \subset Spa(A,A^+)$ be open with $x \in U$ and consider any $s \in \mathcal{O}_X(U)$ with $|s|_x \neq 0$. For a rational subset $W \subset U$ of $Spa(A,A^+)$, the valuation $|\cdot|_x$ can be restricted to a valuation on W that we denote $|\cdot|_W$. As $|s|_x \neq 0$, it follows that $|s|_w \neq 0$ and thus there is a finite subset T of $\mathcal{O}_X(W)$ such that $T \cdot \mathcal{O}_X(W)$ is open and $|t|_W \leq |s|_W$ for all $t \in T$. Hence we have a rational subset $Y(\frac{T}{s})$ of $Y = Spa(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ that corresponds to some rational subset S of $Spa(A,A^+)$ contained in W. Since $|\cdot|_W \in Y(\frac{T}{s})$ and $s \in \mathcal{O}_Y(Y(\frac{T}{s}))^\times$, it follows that s is a unit in $\mathcal{O}_X(S)$ and so it is a unit in $\mathcal{O}_{X,x}$. The result follows. □

1.4 Definition of adic spaces

Definition 1.49. *We define a category (V) with objects $(X, \mathcal{O}_X, (v_x)_{x\in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings and v_x is a class of continuous valuations on $\mathcal{O}_{X,x}$ for every element x of X. Moreover we define a morphism $(X, \mathcal{O}_X, (v_x)_{x\in X}) \to (Y, \mathcal{O}_Y, (v_y)_{y\in Y})$ between two objects of (V) to be a morphism of the underlying topologically ringed spaces such that for every $x \in X$ the following diagram commutes up to equivalence for all $x \in X$*

$$\begin{array}{ccc} \mathcal{O}_{X,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

Definition 1.50. *An adic space is an object $(X, \mathcal{O}_X, (v_x)_{x\in X})$ in the category (V) that admits a covering by topological spaces U_i such that for all i the object $(U_i, \mathcal{O}_{X|U_i}, (v_x)_{x\in U_i})$ is isomorphic to $Spa(A_i, A_i^+)$ for some sheafy Huber pair (A_1, A_i^+).*

Note. Let us explain why adic spaces are defined in the category (V). First of all, it is clear that an adic space should be a locally topologically ringed space, so this note is made to justify the importance of the data of valuations on stalks. Note that we can not just work in the category of locally topologically ringed spaces. That is because given a locally topologically ringed space X that is locally isomorphic to adic spectra, there is no way to recover $\mathcal{O}_X^+$ from X and $\mathcal{O}_X$. But how can we do that in (V)? Let $(X, \mathcal{O}_X, (v_x)_{x\in X})$ be an adic space. Define $\mathcal{O}_X^+$ by

$$\mathcal{O}_X^+(U) = \{f \in .\mathcal{O}_X(U) : v_x(f) \le 1 \text{ for all } x \in U\}$$

This agrees with the definition of $\mathcal{O}_X^+$ when X is affinoid, by the adic Nullstellensatz. That said, an adic space is roughly a locally topologically ringed space but with two sheaves that is locally isomorphic to adic spectra. This can actually be made formal and we can define adic spaces in this way. This was clear to the author who later realised that this idea is described in [Stu17]. So let us explain this by using the notation in [Stu17]. Define a category (V') with objects triples $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ where X is a topological space and $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X^+)$ are locally topologically ringed spaces and $\mathcal{O}_X^+ \subset \mathcal{O}_X$. Morphisms in (V') are defined in the obvious way. Then an adic space is an object in (V') that is locally isomorphic to adic spectra.

Note. Adic spaces glue. Namely, given adic spaces X_i, open subspaces $X_{ij} \subset X_i$ and isomorphisms $X_{ij} \xrightarrow{\phi_{ij}} X_{ji}$ with the usual properties (as in [Sta18] 26.14), then we can glue them to get an adic space X with morphisms $X_i \xrightarrow{\psi_i} X$ that are isomorphisms onto open subspaces of X and $X = \bigcup \psi_i(X_i)$ (and other usual properties). This is done exactly as with schemes, with the only difference being that for every $x \in X$ we have an equivalence class of valuations on $\mathcal{O}_{X,x}$. This is inherited locally from the adic spaces X_i in the obvious way.

Theorem 1.51. *Let $\phi : (A, A^+) \to (B.B^+)$ be a morphism of Huber pairs. The induced morphism $Spa(\phi) : Spa(B, B^+) \to Spa(A, A^+)$ of topological spaces is a morphism of adic spaces. Moreover if (B, B^+) is complete, then*

$$Hom((A, A^+), (B, B^+)) \to Hom(Spa(B, B^+), Spa(A, A^+))$$

is bijective.

Proof. Write X and Y for $Spa(A, A^+)$ and $Spa(B, B^+)$ respectively. Let U be a rational subset of Y and V a rational subset of X such that $Spa(\phi)(U) \subset V$. From the universal property of the Huber pairs $(\mathcal{O}_X(V), \mathcal{O}_X^+(V)), (\mathcal{O}_Y(U), \mathcal{O}_Y^+(U))$ it follows that ϕ gives rise to a commutative diagram

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \downarrow \\ (\mathcal{O}_X(V), \mathcal{O}_X^+(V)) & \longrightarrow & (\mathcal{O}_Y(U), \mathcal{O}_Y^+(U)) \end{array}$$

and the morphisms $(\mathcal{O}_X(V), \mathcal{O}_X^+(V)) \to (\mathcal{O}_Y(U), \mathcal{O}_Y^+(U))$ induce morphisms of sheaves of topological rings $\mathcal{O}_X \to Spa(\phi)_*\mathcal{O}_Y$ and the induced morphism of rings $\mathcal{O}_{X,Spa(\phi)(x)} \to \mathcal{O}_{Y,x}$ is compatible with the valuations on the stalks. From now on we write $Spa(\phi)$ to denote the induced morphism of adic spaces and not simply topological rings.
Assume now that (B, B^+) is complete. Let $\phi_1, \phi_2 : (A, A^+) \to (B, B^+)$ be morphisms of Huber rings with $Spa(\phi_1) = Spa(\phi_2)$. We get commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_Y(Y) \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\phi_2} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_Y(Y) \end{array}$$

and since B is complete, $B \to \mathcal{O}_Y(Y)$ is an isomorphism which implies that $\phi_1 = \phi_2$. Therefore $Hom((A, A^+), (B, B^+)) \to Hom(Spa(B, B^+), Spa(A, A^+))$ is injective.
Let now $r : Y \to X$ be a morphism of adic spaces. We will show that there exists a morphsim $\phi : (A, A^+) \to (B, B^+)$ of Huber pairs such that $r = Spa(\phi)$. We consider the induced morphism of Huber pairs $(\mathcal{O}_X(X), \mathcal{O}_X^+(X)) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$. Since $(B, B^+) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ is an isomorphism, there exists a morphism of Huber pairs $f : (A, A^+) \to (B.B^+)$ such that the following diagram commutes

$$\begin{array}{ccc} (A, A^+) & \xrightarrow{\quad f \quad} & (B, B^+) \\ \downarrow & & \downarrow \\ (\mathcal{O}_X(X), \mathcal{O}_X^+(X)) & \longrightarrow & (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \end{array}$$

For every $y \in Y$ the induced ring homomorphism $\mathcal{O}_{X,r(y)} \to \mathcal{O}_{Y,y}$ is compatible with the valuations on the stalks and thus from the commutativity of the above diagram it follows that $r(y) = Spa(f)(y)$ for every $y \in Y$. Let $U = U(\frac{t_1,\ldots,t_n}{s})$ be a rational subset of X. Then $V = r^{-1}(U) = Spa(f)^{-1}(U)$ is a rational subset of Y. Consider the canonical ring homomorphisms $A \to \mathcal{O}_X(U)$ and $B \to \mathcal{O}_Y(V)$ and let $a : \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ and $b : \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ be the morphisms of topological rings induced by r and $Spa(f)$ respectively. The following diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_X(U) & \xrightarrow{a} & \mathcal{O}_Y(V) \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_X(U) & \xrightarrow{b} & \mathcal{O}_Y(V) \end{array}$$

Since s is invertible in $\mathcal{O}_X(U)$ and $A[\frac{1}{d}]$ is dense in $\mathcal{O}_X(U)$, it follows that $a = b$. Therefore $r = Spa(f)$. Hence $Hom((A, A^+), (B, B^+)) \to Hom(Spa(B, B^+), Spa(A, A^+))$ is surjective and the result follows. □

Corollary 1.52. *Let X be an adic space and $Spa(A, A^+)$ an affonoid adic space. There is a natural bijection between the morphisms of adic spaces $X \to Spa(A, A^+)$ and the morphisms of Huber pairs $(A, A^+) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$.*

Note. The morphism $Spa(i) : Spa(\widehat{A}, \widehat{A^+}) \to Spa(A, A^+)$ induced by $i : (A, A^+) \to (\widehat{A}, \widehat{A^+})$ is an isomorphism in (V)

We give a few basic examples.

Example 1.53. *(The Final object)*
We consider $Spa(\mathbb{Z}, \mathbb{Z})$ where $\mathbb{Z}$ is discrete. This is the final object in the category of adic spaces as $(\mathbb{Z}, \mathbb{Z})$ with $\mathbb{Z}$ discrete is the initial object in the category of Huber pairs. This is because, given a Huber pair (A, A^+), consider the ring homomorphism $\phi : Z \to A$ and since A^+ is a subring of A, it

follows that $\phi(\mathbb{Z}) \subset A^+$. Moreover as $\mathbb{Z}$ is discrete, ϕ is continuous and thus there is a (necessarily unique) morphism of Huber pairs $(\mathbb{Z},\mathbb{Z}) \to (A, A^+)$.

Example 1.54. *(The Adic closed unit disk over $\mathbb{Z}$)*
We consider $Spa(\mathbb{Z}[T],\mathbb{Z}[T])$ where $Z[T]$ is discrete. A morphism $X \to Spa(\mathbb{Z}[T],\mathbb{Z}[T])$ of adic spaces is induced by a morphism $(\mathbb{Z}[T],\mathbb{Z}[T]) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ and thus by a morphism $\mathbb{Z}[T] \to \mathcal{O}_X^+(X)$, which is completely determined by the image of T (and vice versa). Therefore

$$Hom(X, Spa(\mathbb{Z}[T],\mathbb{Z}[T]) = \mathcal{O}_X^+(X)$$

which motivates why we think of it as the closed unit disk over $\mathbb{Z}$.

Example 1.55. *(The Adic affine line over $\mathbb{Z}$) We consider $Spa(\mathbb{Z}[T],\mathbb{Z})$ where $\mathbb{Z}[T]$ is discrete. A morphism $X \to Spa(\mathbb{Z}[T],\mathbb{Z})$ is induced by a morphism $(\mathcal{O}_X(X), \mathcal{O}_X^+(X)) \to (\mathbb{Z}[T],\mathbb{Z})$ and every such morphism is completely determined by the image of T in $\mathcal{O}_X(X)$ (and vice versa). Therefore*

$$Hom(X, Spa(\mathbb{Z}[T],\mathbb{Z}) = \mathcal{O}_X(X)$$

which motivates why we think of it as the affine line over $\mathbb{Z}$

Example 1.56. *(The Adic point) Let K be a non-archimedean field. We consider $X = Spa(K, K^o)$ that is the adic point and its underlying set is the valuation on K.*

Definition 1.57. *Let K be a non-archimedean field. An adic space over K is an adic space X for which there is a morphism $X \to Spa(K, K^o)$.*

Example 1.58. *(Adic closed unit disk over K) Let K be a non-archimedean field. We consider the adic space $X = Spa(K\langle T\rangle, K^o\langle T\rangle)$. This deserves to be called the closed unit disk over K as $Hom(X, Spa(K\langle T\rangle, K^o\langle T\rangle)) = \mathcal{O}_X^+(X)$*

Example 1.59. *(Perfectoid Spaces) A perfectoid ring is a Huber ring that is Tate, uniform (meaning that A^o is bounded) and contains a topologically nilpotent unit $\overline{\omega}$ such that $\overline{\omega}|p$ in A^o and such that the p-th power map $A^o/\overline{\omega} \to A^o/\overline{\omega}^p$ is an isomorphism. A perfectoid space is an adic space that is covered by affinoids $Spa(A_i, A_i^+)$ where A_i is perfectoid for all i.*

We say that a Huber Tate ring A is strongly Noetherian Tate if

$$A\langle X_1, \ldots, X_n\rangle = \{\textstyle\sum_\nu a_\nu X^\nu \in A[[X]] : a_\nu \to 0 \text{ as } |\nu| \to \infty\}$$

is Noetherian for all n. For example every non-archimedean field is strongly Noetherian. Moreover, we say that a Tate Huber pair (A, A^+) is stably uniform if for every rational subset U of $X = Spa(A, A^+)$, $\mathcal{O}_X(U)$ is uniform. We have the following main theorem.

Theorem 1.60. *A Huber pair (A, A^+) is sheafy in the following cases*
i) $\widehat{A}$ is discrete
ii) A is strongly Noetherian Tater ring
iii) $\widehat{A}$ has a Noetherian ring of definition
iv) A is perfectoid
v) (A, A^+) is stably uniform.

Proof. We only prove i) and ii), as those are the cases of interest in this thesis. For iii) see [Hub94], for iv) see [Sch11] and for v) see [BV15]. One can also use v) to prove iv).

i) As $Spa(A, A^+)$ is isomorphic to $Spa(\widehat{A}, \widehat{A^+})$, we may assume that A is discrete. We consider the map $\psi : SpecA \to Spa(A, A^+)$ given by $p \mapsto (A \to A/p \to \{0,1\})$, where $A/p \to \{0,1\}$ is the trivial valuation (i.e nonzero elements map to 1 and 0 maps to 0). Note that for a rational subset

$U(\frac{t_1,\ldots,t_n}{s})$ of $Spa(A, A^+)$ we have

$$\psi^{-1}(U(\tfrac{t_1,\ldots,t_n}{s})) = \{p \in SpecA : s \notin p\} = D(s)$$

Hence it follows that $\mathcal{O}_{Spa(A,A^+)} = \psi_* \mathcal{O}_{Spec(A)}$ and thus $\mathcal{O}_{Spa(A,A^+)}$ is a sheaf.

ii) For this proof we follow [Wed19]. We proceed in several steps. Write X for $Spa(A, A^+)$.

1. Let $(V_j)_{j\in J}$ be a covering of $Spa(A, A^+)$ by open subsets. There exists a refinement of $(V_j)_{j\in J}$ of the form $(X(\frac{T}{t}))_{t\in T}$ for some finite subset T of A such the ideal generated by the elements of T is A.

proof: Let $x \in Spa(A, A^+)$ and consider the characteristic group $c\Gamma_x$ of x. Moreover, consider the valuation y on A given by $y(a) = x(a)$ if $x(a) \in c\Gamma_x$ and $y(a) = 0$ otherwise. Then $\Gamma_y = c\Gamma_y$. Let y belong in some V_j. There exists $s \in A$ and a finite subset T of A with $1 \in T$ such that $y \in X(\frac{T}{s}) \subset V_j$. We may do that for all $x \in Spa(A, A^+)$ and conclude (using the fact that $Spa(A, A^+)$ is quasi-compact) that there exist $s_1, \ldots, s_n \in A$ and $T_1, \ldots, T_n \subset A$ finite subsets of A with $1 \in T_i$ such that $(X(\frac{T_i}{s_i}))_{1\le i\le n}$ is a refinement of $(V_j)_{j\in J}$. We may assume that $s_i \in T_i$ and consider $T = \{t_1 \cdots t_n : t_i \in T_i\}$ and $S = \{t_1 \cdots t_n : t_i \in T_1$ and $t_i = s_i$ for at least one $i\}$. Then for every $(t_1, \ldots, t_n) \in T_1 \times \cdots T_n$ we have

$$X(\tfrac{T}{t}) = \bigcap X(\tfrac{T_i}{t_i})$$

where $t = t_1 \cdots t_n$. Moreover for every $1 \le i \le n$ we have

$$Spa(A, A^+) = \textstyle\bigcup_{t\in T_i} X(\frac{T_i}{t})$$

and so

$$Spa(A, A^+) = \textstyle\bigcup_{s\in S} X(\frac{T}{s})$$

Hence from the above it follows that $X(\frac{T}{s}) = X(\frac{S}{s})$ for every $s \in S$ and $X(\frac{T}{s})$ is contained in some $X(\frac{T_i}{s_i})$ for every $s \in S$. Therefore for every $s \in S$, $X(\frac{S}{s})$ is contained in some V_j. We are left to show that $(X(\frac{S}{s}))_{s\in S}$ is a covering of $Spa(A, A^+)$. We first note that the condition that T generates A is equivalent to saying that for every $x \in Spa(A, A^+)$ there is $t \in T$ with $|t(x)| \ne 0$. Indeed if the ideal generated by T is contained in some maximal ideal, then there exists $x \in Spa(A, A^+)$ with $|t(x)| = 0$ for all $t \in T$. Conversely, if T generates A, then there exist $a_1, \ldots, a_n \in A$ such that $\sum a_i t_i = 1$ and so if $x \in Spa(A, A^+)$ is such that $|t_i(x)| = 0$ for all i, it will follow that $1 \le max(|t_i||a_i|) = 0$. Therefore ,back in our case, given $x \in Spa(A, A^+)$, consider $s_x \in S$ such that $|s_x(x)| = max\{|s(x)|, x \in X\}$. Then $x \in X(\frac{S}{s_x})$

We note at this point, that from 1 and the results in the appendix on Cech cohomology, it is enough to show that covers of $Spa(A, A^+)$ of the form $(X(\frac{T}{t}))_{t\in T}$ (where T is finite and generates A) are $\mathcal{O}_X$-acyclic.

2. Let A be a complete Noetherian Tate ring and M a finitely generated A-module. There exists a unique topology on M that makes it a Hausdorff complete topological A-module having a countable fundamental system of neighborhoods of 0. Moreover, if $f : M \to N$ is an A-linear map where M, N are finitely generated modules endowed with the latter topology, then f is continuous and open onto its image. A proof of this is very similar to the case of Banach spaces. See [Hen14].

From now on A denotes a strongly Noetherian Tate ring. Moreover every finitely generated A-module M is endowed with the topology as in 2, which we call the natural topology on M. Let M a finitely generated A-module endowed with its natural topology. We denote by $M\langle X\rangle$ the $A\langle X\rangle$-submodule of $M[[X]]$ of elements $f = \sum_{\nu \geq 0} m_\nu X^\nu$ such that for every neighborhood U of 0 in M we have $m_\nu \in U$ for almost all ν.

3. For every finitely generated A-module M with its natural topology, the morphism

$$M \otimes_A A\langle X\rangle \to M\langle X\rangle$$

given by $m \otimes a \mapsto ma$ is an isomorphism of $A\langle X\rangle$-modules.

4. For every $f \in A$, $A\langle X\rangle/(f - X)$ and $A\langle X\rangle/(1 - fX)$ are flat over A.

proof: First we note that $A\langle X\rangle$ is flat over A. This is because for any injective homomorphism $i : M \to N$ of finitely generated A-modules, the induced homomorphism

$$i \otimes id_{A\langle X\rangle} : M \otimes_A A\langle X\rangle \to N \otimes_A A\langle X\rangle$$

is injective (from 3). Let $g \in A\langle X\rangle$ be such that for every finitely generated A-module M, the multiplication by g morphism $M\langle X\rangle \to M\langle X\rangle$ is injective. We will show that $A\langle X\rangle/(g)$ is flat over A. Consider the sequence

$$0 \to A\langle X\rangle \xrightarrow{u_g} A\langle X\rangle \xrightarrow{p} A\langle X\rangle/(g) \to 0$$

where u_g is the multiplication by g morphism. This is exact. Write B for $A\langle X\rangle/(g)$. To show that this is flat over A, it is enough to show that $Tor_1^A(M, B) = 0$ for every finitely generated A-module M. The above exact sequence gives rise to a long exact sequence

$$\cdots Tor_2^A(M, A\langle X\rangle) \longrightarrow Tor_2^A(M, A\langle X\rangle) \longrightarrow Tor_2^A(M, B) \longrightarrow$$
$$\longrightarrow Tor_1^A(M, A\langle X\rangle) \longrightarrow Tor_1^A(M, A\langle X\rangle) \longrightarrow Tor_1^A(M, B) \longrightarrow$$
$$\longrightarrow A\langle X\rangle \otimes_A M \xrightarrow{w_g} A\langle X\rangle \otimes_A M \longrightarrow A \otimes_A M \longrightarrow 0$$

where $w_g = id_M \otimes_A u_g$. As $Tor_1^A(M, A\langle X\rangle) = 0$, it is enough to show that w_g is injective. But by using what we proved for $M\langle X\rangle$, we see that w_g is the homomorphism $M\langle X\rangle \to M\langle X\rangle$ given by multiplication by g, which proves our claim. Now for $g = f - X$ or $g = 1 - fX$, it is easy to see the multiplication by g morphism $M\langle X\rangle \to M\langle X\rangle$ is injective and so the result follows.

5. Let $U \subseteq V \subseteq X$ be two rational subsets. Then $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is flat.

proof: From the construction of the presheaf $\mathcal{O}_X$ it can be seen that $\mathcal{O}_X(V)$ is a strongly Noetherian Tate ring. Therefore we may assume that $X = V$ and that A is complete. We may also assume that U is of the form $\{x \in X : |f(x)| \leq 1\}$ or $\{x \in X : |f(x)| \geq 1\}$. To prove the latter claim, let $U = X(\frac{f_1,\ldots,f_n}{g})$ for some $f_1, \ldots, f_n, g \in A$. Since $|g(x)| \neq 0$ for all $x \in U$, there exists an open neighborhood U' of 0 in A such that for all $u' \in U'$, $|u'(x)| < |g(x)|$. As A is assumed to be Tate, we may find a unit $u \in U'$. Consider the set

$$X_0 = \{x \in Spa(A, A^+) : |\tfrac{g}{u}(x)| \geq 1\}$$

and define inductively

$$X_i = \{x \in X_{i-1} : |\tfrac{f_i}{g}(x)| \leq 1\}$$

and obtain a sequence of rational subsets $Spa(A, A^+) \supseteq X_0 \supseteq \cdots \supseteq X_n = U$.
Therefore as we may assume U to be either equal to $X(\frac{1,f}{1})$ or $X(\frac{1}{f})$, it is enough to show that $\mathcal{O}_X(X(\frac{1,f}{1}))$ and $\mathcal{O}_X(X(\frac{1}{f}))$ are flat over $\mathcal{O}_X(X) = A$. A simple computation using the definition shows that the latter are equal to $A\langle X\rangle/(1 - fX)$ and $A\langle X\rangle/(f - X)$, so by 4 the result follows.

6. Let $f \in A$ and let $U_1 = \{x \in X : |f(x)| \leq 1\}$ and $U_2 = \{x \in X : |f(x)| \geq 1\}$. Moreover let $\psi : \mathcal{O}_X(X) \to \mathcal{O}_X(U_1)$ and $\phi : \mathcal{O}_X(X) \to \mathcal{O}_X(U_2)$ be the restriction morphisms. Then

$$0 \to \mathcal{O}_X(X) \xrightarrow{\psi\oplus\phi} \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) \xrightarrow{\psi-\phi} \mathcal{O}_X(U_1 \cap U_2) \to 0$$

is exact.

proof: For simplicity we write ϵ for $\psi\oplus\phi$ and δ for $\psi-\phi$. As in 5, we have $\mathcal{O}_X(U_1) = A\langle\zeta\rangle/(f-\zeta)$, $\mathcal{O}_X(U_2) = A\langle\eta\rangle/(1-f\eta)$, $\mathcal{O}_X(U_1 \cap U_2) = A\langle\zeta,\zeta^{-1}\rangle/(f-\zeta)$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (f-\zeta)A\langle\zeta\rangle \oplus (1-f\eta)A\langle\eta\rangle & \xrightarrow{\lambda'} & (f-\zeta)A\langle\zeta,\zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i'_1} & & \\
0 & \longrightarrow & A & \xrightarrow{\mu} & A\langle\zeta\rangle \oplus A\langle\eta\rangle & \xrightarrow{\lambda} & A\langle\zeta,\zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i'_2} & & \\
0 & \longrightarrow & A & \xrightarrow{\epsilon} & \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) & \xrightarrow{\delta} & \mathcal{O}_X(U_1 \cap U_2) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where i is the natural morphism, λ is given by $(g(\zeta), h(\zeta)) \mapsto g(\zeta) - h(\zeta^{-1})$ and λ' is induced by λ. The columns are exact. Moreover, ϵ is faithfully flat and so is injective. By diagram chasing, if the first two rows are exact then so is the third. We have that

$$A\langle\zeta,\zeta^{-1}\rangle = A\langle\zeta\rangle + \zeta^{-1}A\langle\zeta^{-1}\rangle$$
$$(f-\zeta)A\langle\zeta,\zeta^{-1}\rangle = (f-\zeta)A\langle\zeta\rangle + (1-f\zeta^{-1})A\langle\zeta^{-1}\rangle$$

which imply that λ and λ' are surjective. Moreover

$$\textstyle\sum_{k\geq 0} a_k\zeta^k - \sum_{k\geq 0} b_k\eta^k = \lambda(\sum_{k\geq 0} a_k\zeta^k, \sum_{k\geq 0} b_k\eta^k) = 0$$

if and only if $a_k, b_k = 0$ for all $k \neq 0$ and $a_0 = b_0$. This happens exactly when

$$\textstyle(\sum_{k\geq 0} a_k\zeta^k, \sum_{k\geq 0} b_k\eta^k) \in im\mu$$

and so $ker\lambda = im\mu$ and the result follows.

We are finally ready to prove the last step. So far, we have managed to prove that the covering of X by $X(\frac{1,f}{1}), X(\frac{1}{f})$ is $\mathcal{O}_X$-acyclic. Moreover, as we noted early in the proof, it is enough to prove that every covering of the form $(X(\frac{T}{t}))_{t\in T}$ where $T \subset A$ is finite that generates A is $\mathcal{O}_X$-acyclic. The idea is to prove the latter, by reducing it to the case of $X(\frac{1,f}{1}), X(\frac{1}{f})$. Once one knows the properties of Cech cohomology that we discuss in the appendix, this becomes simple. We prove it below

7. Let $\mathcal{U}$ be a cover of X consisting of rational subsets of the form $X(\frac{T}{t})$ where T is a finite subset of A that generates A and $t \in T$. Then $\mathcal{U}$ is $\mathcal{O}_X$-acyclic.

proof: Let us write $\mathcal{U}_f$ for the covering of X by $X(\frac{1,f}{1})$ and $X(\frac{1}{f})$. Moreover, let us call a cover of the form $(X(\frac{T}{t}))_{t\in T}$ a rational cover generated by T. With induction, it follows that covers of the form $\mathcal{V} = \mathcal{U}_{f_1} \times \cdots \times \mathcal{U}_{f_n}$ are $\mathcal{O}_X$-acyclic. Such a cover is called a Laurent cover generated by $f_1, \ldots, f_n$. Note that it is also a rational cover generated by $\{\prod_{j\in J} f_j, J \subset \{1, \ldots, n\}\}$. If $U = X(\frac{T}{s})$ is a rational subset of X, then $\mathcal{V}|_U$ is a Laurent cover of X generated by the $f_{1|U}, \ldots, f_{n|U}$ where $f_{i|U}$ is the image of f_i under the homomorphism $A \to A(\frac{T}{s})$. Let now $\mathcal{U}$ be a rational cover generated by $T = \{f_1, \ldots, f_n\}$. Then for all $x \in X$ there is f_i such that $|f_i(x)| \neq 0$ and so there is a unit $s \in S^\times$ such that $x(s) < x(f_i)$ for all i. Consider the Laurent cover generated by $s^{-1}f_1, \ldots, s^{-1}f_n$. The latter has the property that it is a Laurent cover $(V_j)_{j\in J}$ of X such that $\mathcal{U}_{|V_j}$ is a rational cover generated by the units of $\mathcal{O}_X(V_j)$ for all $j \in J$. Now every rational cover $\mathcal{U}$ of X which is generated by units $f_1, \ldots, f_n$ has a refinement by a Laurent cover, as we may simply consider the Laurent cover generated by $f_i f_j^{-1}$. Moreover by the above, it follows that restrictions of rational covers generated by units to rational subsets are $\mathcal{O}_X$-acyclic. By combining the above, the result follows.

□

Remark. For every adic space X as in the above theorem, we have $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$. In the case of strongly Noetherian Tate rings, this is easy to see by the results in Cech cohomology discussed in the appendix and the above proof. In [KL15] it is proved that if (A, A^+) is a complete analytic (meaning that $Spa(A, A^+)$ is analytic) sheafy Huber pair, then $H^i(Spa(A, A^+), \mathcal{O}_{Spa(A,A^+)}) = 0$ for all $i \geq 1$.

Note. The case ii) of theorem 1.60 is essentially Tate's acyclicity theorem. This will become more apparent in section 2.2.

1.5 Morphisms of finite type and Fiber Products

Definition 1.61. *Let A be a Huber ring and let $M_1, \ldots, M_n$ be finite subsets of A such that $M_i \cdot A$ is open for all i. We define*

$$A\langle X\rangle_M = A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} =$$

$$\{\textstyle\sum_\nu a_\nu X^\nu \in \widehat{A}[[X_1, \ldots, X_n]] : \text{for every neighborhood } U \text{ of } 0 \text{ in } \widehat{A},\ a_\nu \notin M_1^{\nu_1} \cdots M_n^{\nu_n} \cdot U \text{ for only finitely many } \nu\}$$

We equip $A\langle X\rangle_M$ with a topology such that for the neighborhoods U of 0 in $\widehat{A}$, the sets $U_{\langle X\rangle} = \{\sum_\nu a_\nu X^\nu \in A\langle X\rangle_M : a_\nu \in M_1^{\nu_1} \cdots M_n^{\nu_n} \cdot U$ for every $\nu\}$ form a fundamental system of neighborhoods of 0 in $A\langle X\rangle_M$. This gives a topological ring which is complete.

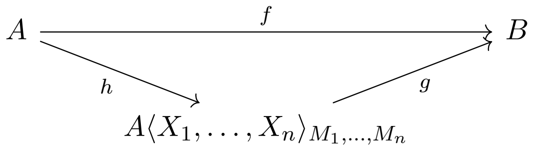
Proposition 1.62. *$A\langle X\rangle_M$ is a Huber ring with ring of definition*

$$\{\textstyle\sum_\nu a_\nu X^\nu \in A\langle X\rangle_M : a_\nu \in M_1^{\nu_1} \cdots M_n^{\nu_n} \cdot A_0 \text{ for every } \nu\}$$

and ideal of definition

$$\{\textstyle\sum_\nu a_\nu X^\nu \in A\langle X\rangle_M : a_\nu \in M_1^{\nu_1} \cdots M_n^{\nu_n} \cdot I \text{ for every } \nu\}$$

and the natural ring homomorphism $h : A \to A\langle X\rangle_M$ is continuous. Moreover, $h(m)X_i$ is power-bounded for all $m \in M_i$ and all i and every morphism of topological rings $A \to B$ where B is a complete non-archimedean topological ring such that $f(m)b_i$ is power bounded for all $m \in M_i$ and all i for some elements $b_1, \dots, b_n \in B$, factors in the following way



for some morphism of topological rings $g : A\langle X\rangle_M \to B$ with $g(X_i) = b_i$ for all i.

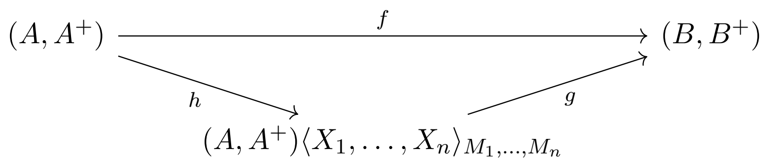
Definition 1.63. *A morphism $f : A \to B$ between Huber rings where B is complete is called topologically of finite type if there exist finite subsets $M_1, \dots, M_n$ of A where $M_i \cdot A$ is open for all i such that there exists a surjective and open morphism of topological rings $g : A\langle X\rangle_M \to B$ with $f = g \circ h$. In case $M_1, \dots, M_n = \{1\}$, we say that ϕ is strictly topologically of finite type.*

Let now (A, A^+) be a Huber pair and $M_1, \dots, M_n$ be finite subsets of A such that $M_i \cdot A$ is open for all i. Consider

$$B = \{\sum a_\nu X^\nu \in A\langle X\rangle_M : a_\nu \in M^\nu \cdot \widehat{A^+} \text{ for every } \nu\}$$

This is a subring of $A\langle X\rangle_M$ and its integral closure in $A\langle X\rangle_M$,that we denote by C, is a ring of integral elements of $A\langle X\rangle_M$. We denote the Huber pair $(A\langle X\rangle_M, C)$ by $(A, A^+)\langle X\rangle_M$. It satisfies the following universal property

Proposition 1.64. *Let $h : (A, A^+) \to (A, A^+)\langle X\rangle_M$ be the natural morphism of Huber pairs. Then $h(m)X_i \in C$ for every $m \in M_i$ and every i. Moreover if $f : (A, A^+) \to (B, B^+)$ is a morphism of Huber pairs where B is complete, such that $f(m)b_i \in B^+$ for every $m \in M_i$ and every i, for some $b_1, \dots, b_n \in B$, then f factors in the following way*



for some morphism of Huber pairs $g : (A, A^+)\langle X\rangle_M \to (B, B^+)$ with $g(X_1) = b_i$.

We may replace A be any non-archimedean topological ring and define $A\langle X\rangle_M$ in the same way. This will have the exact same properties as in the above propositions. Also, given a non-archimedean ring A and finite subsets $M_1, \dots, M_n$ as above, we can consider $\widehat{A}[X_1, \dots, X_n]$ with a topology such that for the neighborhoods U of 0 in $\widehat{A}$, the sets

$$U_{[X]} = \{\textstyle\sum_\nu a_\nu X^\nu \in \widehat{A}[X_1, \dots, X_n] : a_\nu \in M_1^{\nu_1} \cdots M_n^{\nu_n} \cdot U\}$$

form a fundamental system of neighborhoods of 0 in $\widehat{A}[X]$. This gives a topological ring which we denote by $A[X]_M$. In the same way, one can define $(A, A^+)[X]_M$. Moreover, if A is a Huber ring, then $A[X]_M$ is a Huber ring with ring of definition

$$\{\textstyle\sum_\nu a_\nu X^\nu \in A[X]_M : a_\nu \in M^\nu \cdot A_0\}$$

and ideal of definition

$$\{\textstyle\sum_\nu a_\nu X^\nu \in A[X]_M : a_\nu \in M^\nu \cdot I\}$$

where (A_0, I) is a pair of definition of A. To prove propositions 1.62 and 1.64 one can proceed in the following way: First prove the exact analogous statements for $A[X]_M$ and $(A, A^+)[X]_M$, without the completeness assumptions in the universal property. Then, prove that $A\langle X\rangle_M$ is the completion of $A[X]_M$. By using the universal property of completions, everything will follow.

Definition 1.65. *A morphism $\phi : (B, B^+) \to (C, C^+)$ of Huber pairs is called a quotient mapping if $\phi : B \to C$ is surjective, open and C^+ is the integral closure of $\phi(B^+)$ in C.*

Definition 1.66. *A morphism $\phi : (A, A^+) \to (B, B^+)$ of Huber pairs where (B, B^+) is complete is called topologically of finite type if there exist finite subsets $M_1, \ldots, M_n$ of A such that $M_i \cdot A$ is open for all i and a quotient mapping $g : (A, A^+)\langle X\rangle_M \to (B, B^+)$ such that the following diagram commutes*

$$\begin{array}{ccc} (A, A^+) & \xrightarrow{\phi} & (B, B^+) \\ {\scriptstyle h}\downarrow & \nearrow_{g} & \\ (A, A^+)\langle X\rangle_M & & \end{array}$$

If $M_1, \ldots, M_n = \{1\}$, we say that ϕ is strictly topologically of finite type.

We shall first present the following propositions. For a proof of 1.67, see [Lud20] 2.2. The hard part is to show that *ii)* implies *iii)* which uses proposition 1.21. The proof of 1.68 is similar.

Proposition 1.67. *Let A be a Huber ring, B a complete Huber ring and $\phi : A \to B$ a morphism of Huber rings. The following are equivalent*
i) ϕ is topologically of finite type.
ii) The exist rings of definition $A_0 \subset A$ and $B_0 \subset B$ such that B_0 is an A_0-algebra strictly topologically of finite type and B is finitely generated over $A \cdot B_0$.
iii) Given an open subring $A_0 \subset A$, there exists an open subring $B_0 \subset B$ with $\phi(A_0) \subset B_0$ such that B_0 is an A_0-algebra strictly topologically of finite type and B is finitely generated over $A \cdot B_0$.

Proposition 1.68. *Let A be a Huber ring, B a complete Huber ring and $\phi : A \to B$ a morphism of Huber rings. The following are equivalent*
i) ϕ is strictly topologically of finite type
ii) There exist ring of definitions $A_0 \subset A$ and $B_0 \subset B$ with $\phi(A_0) \subset B_0$ such that B_0 is an A_0-algebra strictly topologically of finite type and $B = A \cdot B_0$
iii) For every open subring $A_0 \subset A$ there is an open subring $B_0 \subset B$ such that $\phi(A_0) \subset B_0$, B_0 is an A_0-algebra strictly topologically of finite type and $B = A \cdot B_0$

Proposition 1.69. *Let A be a Tate ring and B a complete Huber ring. Then $\phi : A \to B$ is topologically of finite type if and only if it is strongly topologically of finite type.*

Proof. We first prove the following

1. Let A be a Tate ring and A_0 a ring of definition. We claim that there is a topologically nilpotent unit in A_0. Moreover, we claim that for such an element $\varpi \in A_0$, $A = A_{0\varpi}$ and ϖA_0 is an ideal of definition of A. Indeed, let ϖ be a topologically nilpotent element of A. For n large enough, $\varpi^n \in A_0$. Moreover for every $a \in A$, there is n such that $a\varpi^n \in A_0$. Therefore $A = A_{0\varpi}$. As multiplication by ϖ^n gives an isomorphism $A \to A$, $\varpi^n A_0$ is open.

2. Let $\phi : A \to B$ be a morphism of Huber rings. Assume that A is Tate. Then B is Tate, ϕ is adic and for every ring of definition B_0 of B we have $\phi(A) \cdot B_0 = B$ Indeed, let ϖ be a topologically nilpontent unit in A. Then $\phi(\varpi)$ is a topologically nilpotent element of B and so B is Tate. By 1, there is a ring of definition B_0 of B such that $f(\varpi) \cdot B_0$ is an ideal of definition of B. Hence ϕ is adic. Let B_0 be a ring of definition of B. We may assume that $\varpi \in B_0$ (as otherwise, we may replace ϖ be some power of it). Then $A = A_{0\varpi}$ and $B = B_{0\phi(\varpi)}$ so $\phi(A) \cdot B_0 = B$.

By combining 2 with propositions 1.67 and 1.68 the result follows.

□

Proposition 1.70. *The composition of morphisms of Huber rings (resp. of Huber pairs) that are topologically of finite type is topologically of finite type.*

Proof. Note that given morphisms of Huber pairs $\phi : (A, A^+) \to (B, B^+)$ and $\psi : (B, B^+) \to (C, C^+)$ such that B^+ is the integral closure of $\phi(A^+)$ and C^+ is the integral closure of $\psi(B^+)$, then C^+ is the integral closure of $\psi(\phi(A^+))$. Thus it is enough to prove the proposition only for Huber rings, which follows directly from proposition 1.67iii). □

Definition 1.71. *A morphism $f : X \to Y$ of adic space is called adic if for every open affinoid subspaces U and V of X and Y respectively with $f(U) \subset V$, the induced morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic.*

Definition 1.72. *A morphism $f : X \to Y$ of adic spaces is called*
i) locally of weakly finite type if for every $x \in X$ there are open affinoid subsaces U, V of X and Y respectively with $x \in U$ and $f(U) \subset V$ such that the induced morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is topologically of finite type.
ii) locally of finite type if for every $x \in X$ there are open affonoid subspaces U, V of X and Y respectively with $x \in U$ and $f(U) \subset V$ such that the induced morphism $\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is topologically of finite type.
iii) locally of finite presentation if for every $x \in X$ there are open affinoid subspaces U, V of X and Y respectively with $x \in U$ and $f(U) \subset V$ such that the induced morphism $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is topologically of finite type and in case $\mathcal{O}_Y(V)$ is discrete, $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

Note. {morphisms locally of finite presentation} ⊂ {morphisms locally of finite type} ⊂ { morphisms of weakly finite type} ⊂ {adic morphisms}. The first two inclusions follow from the definitions while the last follows from 1.67.

From proposition 1.70 and the definitions, we have the following

Proposition 1.73. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of adic spaces that are weakly of finite type (resp. locally of finite type, resp. locally of finite presentation). Then $g \circ f$ is weakly of finite type (resp. locally of finite type, resp. locally of finite presentation).*

Definition 1.74. *A Huber ring A is called stably sheafy if every $\widehat{A}$-algebra topologically of finite type is sheafy.*

Proposition 1.75. *A is stably sheafy in the following cases:*
i) $\widehat{A}$ is discrete
ii) A is a strongly Noetherian Tate ring
iii) $\widehat{A}$ has a Noetherian ring of definition

Proof. i) follows from the definitions.
ii) We first note that $A\langle X_1, \ldots, X_n\rangle$ is Noetherian if and only if every Tate ring topologically of finite type over A is Noetherian. Indeed this follows directly from the fact that given a Tate ring A, a morphism $\phi : A \to B$ is topologically of finite type if and only if ϕ is strictly topologically of finite type. Therefore given a strongly Noetherian Tate ring A and a morphism $\phi : A \to B$ topologically of finite type, consider the natural morphism $B \to B\langle X_1, \ldots, X_n\rangle$ which is topologically of finite type and the composition $A \to B\langle X_1, \ldots, X_n\rangle$ which is topologically of finite type. Then $B\langle X_1, \ldots, X_n\rangle$ is Noetherian.
iii) We may find a Noetherian ring of definition of $A[X]_M$, since if A_0 is a Noetherian ring of definition of A, then $\widehat{A_0}$ is a Noetherian ring of definition of $\widehat{A}$ (as it is adic) and so $\{\sum_\nu a_\nu X^\nu \in A[X]_M : a_\nu \in M^\nu \cdot \widehat{A_0}\}$ is Noetherian, as it is finitely generated. So by completing such a ring of definition of $A[X]_M$ (and since it is adic), we get a Noetherian ring of definition of $A\langle X\rangle_M$. □

We can now prove the existence of fiber products of adic spaces in some special (but very general!) cases. Before we do that, let us note the following

Note. Fiber products of adic spaces do not exist in general. For example, in [SW20] it is explained that given any non-archimidean field K and $\mathbb{Z}$, $\mathbb{Z}[X_1, X_2, \dots]$ with the discrete topology, then the fiber product

$$Spa(\mathbb{Z}[X_1, X_2, \dots], \mathbb{Z}) \times_{Spa(\mathbb{Z},\mathbb{Z})} Spa(K, K^o)$$

does not exist in the category of adic spaces.

Definition 1.76. *An adic space X is called stable if it admits an open covering by affinoid subspaces $Spa(A_i, A_i^+)$ where A_i are stably sheafy for every i.*

Proposition 1.77. *Assume that $f : X \to Z$ and $g : Y \to Z$ are morphisms of adic spaces with Y stable. The fiber product $X \times_Z Y$ exists (in the category of adic spaces) in the following cases*
i) f is locally of finite type.
ii) f is locally of weakly finite type and g is adic.

Proof. With standard arguments, the existence of the fiber product of f and g follows if we can find open covers by affinoid subspaces $Z = \bigcup Z_i$, $f^{-1}(Z_i) = \bigcup X_{ij}$ and $g^{-1}(Z_i) = \bigcup Y_{ik}$ such that all products $X_{ij} \times_{Z_i} Y_{ik}$ exist. Therefore we may assume that X, Y, Z are affinoid. Since Y is assumed to be stable, we may assume that $Y = Spa(B, B^+)$ where B is stably sheafy.

i) We write $(A, A^+) = (\mathcal{O}_Z(Z), \mathcal{O}_Z^+(Z)), (B, B^+) = (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $(C, C^+) = (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$. Let $\phi : (A, A^+) \to (B, B^+)$ and $\psi : (A, A^+) \to (C, C^+)$ be the morphisms of Huber pairs induced by g and f respectively. By assumption we may take ψ to be topologically of finite type. Therefore we can write $(C, C^+) = (A, A^+)\langle X_1, \dots, X_n\rangle_{M_1,\dots,M_n}/I$ for some finite subsets $M_1, \dots, M_n$ of B such that $M_i \cdot B$ is open in B and for some ideal I of $A\langle X\rangle_M$. Let $L \subset B^{oo}$ be finite such that $(\phi(M_i) \cup L^m) \cdot B$ is open in B for all $i \in \{1, \dots, n\}$ and all $m \in \mathbb{N}$. Such a subset exists. For example we may take L to be a finite set of generators of an ideal of definition of B. We consider the Huber pair

$$(B, B^+)\langle X\rangle_{\phi(M)\cup L^m} := (B, B^+)\langle X_1, \dots, X_n\rangle_{\phi(M_1)\cup L^m,\dots,\phi(M_n)\cup L^m}$$

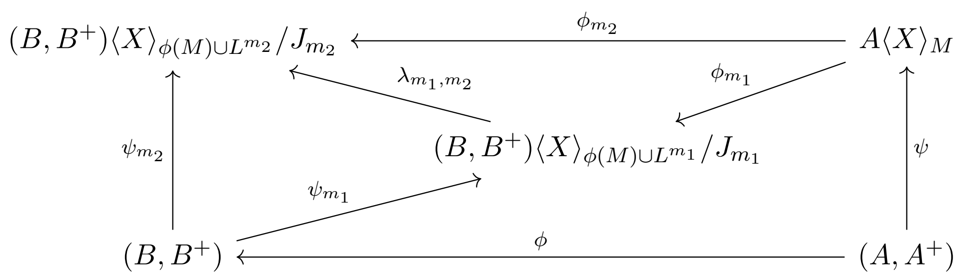
and the natural morphisms $\psi' : (A, A^+) \to (A, A^+)\langle X\rangle_M$ and $\psi'_m : (B, B^+) \to (B, B^+)\langle X\rangle_{\phi(M)\cup L^m}$. For every $m \in M_i$ and every i, the element $\psi'_m(\phi(m))X_i$ belongs in the rings of integral elements of the Huber pair $(B, B^+)\langle X\rangle_{\phi(M)\cup L^m}$ and so there exists a morphism of Huber pairs $\phi'_m : A\langle X\rangle_M \to (B, B^+)\langle X\rangle_{\phi(M)\cup L^m}$ such that the following diagram commutes

$$\begin{array}{ccc} (B, B^+)\langle X\rangle_{\phi(M)\cup L^m} & \xleftarrow{\ \phi'_m\ } & A\langle X\rangle_M \\ \Big\uparrow{\scriptstyle \psi'_m} & & \Big\uparrow{\scriptstyle \psi'} \\ (B, B^+) & \xleftarrow{\ \phi\ } & (A, A^+) \end{array}$$

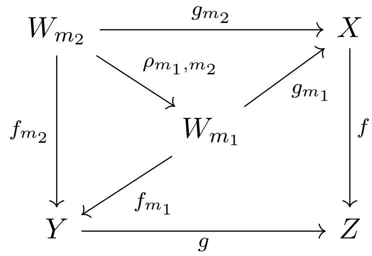
Now for $m_1, m_2 \in \mathbb{N}$ with $m_1 \geq m_2$, there exists a unique morphism of Huber pairs

$$\lambda'_{m_1,m_2} : (B, B^+)\langle X\rangle_{\phi(M)\cup L^{m_1}} \to (B, B^+)\langle X\rangle_{\phi(M)\cup L^{m_2}}$$

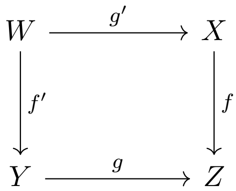
such that $\lambda'_{m_1,m_2}(X_i) = X_i$ and $\psi'_{m_2} = \lambda'_{m_1,m_2} \circ \psi'_{m_1}$. It also follows that $\phi'_{m_2} = \lambda'_{m_1,m_2} \circ \phi'_{m_1}$. We consider the closure of the ideal of $B\langle X\rangle_{\phi(M)\cup L^m}$ generated by $\phi'_m(I)$ and denote it by J_m. Now $\phi'_m, \psi'_m, \lambda'_{m_1,m_2}$ induce morphisms of Huber pairs as in the commutative diagram below



For every $m \in \mathbb{N}$ consider the adic space $W_m = Spa((B,B^+)\langle X\rangle_{\phi(M)\cup L^m}/J_m)$. The latter diagram gives rise to a commutative diagram of adic spaces



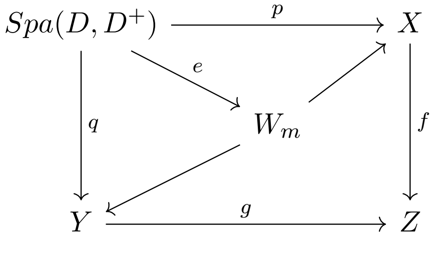
It is easy to check that ρ_{m_1,m_2} is an isomorphism onto $\{x \in W_{m_1} : x(lT_i) \leq 1$ for all $i \in \{1,\dots,n\}$ and $l \in L^{m_2}\}$ which is a rational subset of W_{m_1} and so ρ_{m_1,m_2} is an open embedding. Therefore we may consider the limit W of the system of the adic spaces $W_m, m \in \mathbb{N}$ with the morphisms ρ_{m_1,m_2} for all $m_1 \geq m_2$. The morphisms f_m and g_m induce morphisms $f' : W \to Y$ and $g' : W \to X$ respectively such that the following diagram commutes



All is left to do, is prove that the latter is cartesian in the category of adic spaces. Let $Spa(D,D^+)$ be an adic space with morphisms $p : Spa(D,D^+) \to X$ and $q : Spa(D,D^+) \to Y$ such that $f \circ p = g \circ q$. Moreover let $\sigma : (C,C^+) \to (D,D^+)$ and $\tau : (B,B^+) \to (D,D^+)$ be the morphisms of Huber pairs that correspond to p and q. Then $\sigma \circ \psi = \tau \circ \phi$. For every i let $\overline{X_i}$ be the image of X in $A\langle X\rangle_M/I$. For $a \in M_i$ we have

$$\tau(\phi(a)) \cdot \sigma(\overline{X_i}) = \sigma(\psi(a)) \cdot \sigma(\overline{X_i}) = \sigma(\psi(a))\overline{X_i}) \in D^+$$

Moreover since L is a finite subset of B^{oo} for m large enough and every $l \in L^m$ we have $\tau(l) \cdot \sigma(\overline{X_i}) \in D^+$. Therefore there exists a unique morphism $\epsilon' : (B,B^+)\langle X\rangle_{\phi(M)\cup L^m} \to (D,D^+)$ such that $\tau = \epsilon' \circ \psi'_m$ and $\epsilon'(X_i) = \sigma(\overline{X_i})$ and so we also have $\epsilon' \circ \phi'_m = \sigma \circ \delta$ where δ is the natural morphism $(A,A^+)\langle X\rangle_M \to (A,A^+)\langle X\rangle_M/I$. Therefore $\epsilon'(J_m) = 0$ and so ϵ' factors through a morphism of Huber pairs $\epsilon : (B,B^+)\langle X\rangle_{\phi(M)\cup L^m}/J_m \to (D,D^+)$. Hence there is m such that the following diagram commutes



where e is the morphism of adic spaces induced by ϵ. This completes the proof.

ii) We write $(A, A^+) = (\mathcal{O}_Z(Z), \mathcal{O}_Z^+(Z)), (B, B^+) = (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $(C, C^+) = (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$. Let $\phi : (A, A^+) \to (B, B^+)$ and $\psi : (A, A^+) \to (C, C^+)$ be the morphisms of Huber pairs that correspond to g and f respectively. By assumption ϕ is adic and ψ is topologically of weakly finite type. Let A_0, B_0, C_0 be rings of definition of A, B, C respectively such that $\phi(A_0) \subset B_0$ and $\psi(A_0) \subset C_0$. Moreover let I be an ideal of definition of A_0. We consider $D = C \otimes_A B$ and let D_0 be the image of $C_0 \otimes_{A_0} B_0$ in D. Moreover let D^+ be the integral closure of $C^+ \otimes_{A^+} B^+$ in D. We topologize D such that $\{I^n \cdot D_0 : n \in \mathbb{N}\}$ forms a fundamental system of neighborhoods of 0 in D. The following diagram

$$\begin{array}{ccc} (D, D^+) & \longleftarrow & (C, C^+) \\ \uparrow & & \uparrow \psi \\ (B, B+) & \xleftarrow[\phi]{} & (A, A^+) \end{array}$$

is cocartesian in the category of Huber pairs: for a Huber pair (D', D'^+) such that the following diagram commutes

$$\begin{array}{ccc} (D', D'^+) & \longleftarrow & (C, C^+) \\ \uparrow & & \uparrow \psi \\ (B, B+) & \xleftarrow[\phi]{} & (A, A^+) \end{array}$$

we get a morphism of rings $f : D \to D'$. by the construction of the topology on D one can see that this is continuous and $f(D^+) \subset D'^+$. Therefore $X \times_Z Y = Spa(D, D^+)$. □

Remark. The fiber product of affinoid adic spaces might not be affinoid, as seen in case i) of the latter proposition.

Example 1.78. *Fiber products exists in the category of perfectoid spaces: Given perfectoid spaces $X = Spa(A, A^+), Y = Spa(B, B^+)$ and $Z = Spa(C, C^+)$ and morphisms $X \to Z$ and $Y \to Z$, then those morphisms are adic (this is the case in general for Tate rings) and so the fiber product $X \times_Z Y = Spa(D, D^+)$ exists in the category of adic spaces. In [Sch11] it is proved that D is perfectoid and so $X \times_Z Y$ is a perfectoid space.*

Corollary 1.79. *Let*

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

be a cartesian digram in the category of adic spaces. Then
i) If f is locally of finite presentation (resp. locally of finite type), then so is f'
ii) If f is locally of weakly finite type and g is adic then
a) f' is locally of weakly finite type and g' is adic
b) If f (resp. g) is quasi-compact (resp. quasi-separated) then so is f' (resp. g')
c) for every $x \in X$ and $y \in Y$ with $f(x) = g(y)$ there is $w \in W$ with $y = f'(w)$ and $x = g'(w)$

Proof. Everything follows from the proof of 1.25 except for ii)c), for which we refer the reader to [Hub94]3.9i. □

1.6 Quasi-finite Morphisms

In this section we introduce quasi-finite and locally quasi-finite morphisms. A detailed analysis of such morphisms would require many results about proper and finite morphisms which would distract our attention away from the purpose of this thesis. So we give the definitions and prove a basic characterization, but we cite results when needed.

Definition 1.80. *Let X be an adic space and consider a point x of X. We define $k(x)$ to be the residue class of $\mathcal{O}_{X,x}$ and $k(x)^+$ to be the valiation subring of $k(x)$ that corresponds to the valuation on $\mathcal{O}_{X,x}$ (i.e if $|\cdot|$ is the valuation on the stalk, then $k(x)^+$ is the set of elements f of $k(x)$ with $|f| \leq 1$). We topologize $k(x)$ in the following way: if x is analytic, then we endow $k(x)$ with the topology induced by $k(x)^+$. Otherwise, we endow $k(x)$ with the discrete topology.*

Definition 1.81. *A morphism $\phi : (A, A^+) \to (B, B^+)$ of complete Huber pairs is called finite if it is topologically of finite type and the ring homomorphisms $A \to B$ and $A^+ \to B^+$ are integral.*

Definition 1.82. *A morphism $f : X \to Y$ of adic spaces is called finite if for every $y \in Y$ there exists an open affinoid neighborhood V of Y that contains y such that $f^{-1}(V)$ is affinoid and the induced morphism of Huber pairs $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite.*

Definition 1.83. *Let $f : X \to Y$ be a morphism of adic spaces.*
i) f is called locally quasi-finite if f is locally of weakly finite type and for all $y \in Y$, $f^{-1}(y)$ is discrete.
ii) f is called quasi-finite if it is quasi-compact and locally quasi-finite.

Let us recall at this point that given a topological space X and points $x, y \in X$, x is called a generalization of y if x is contained in every neighborhood of y. In this case, we write $x \succ y$ and call x maximal if it is maximal with respect to $\succ$. For an analytic adic space X and points x, y of X with $x \succ y$, the ring homomorphism $\mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$ is flat. That follows directly from the fact that given rational subsets U, V with $U \subset V$, $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is flat. Moreover, for a morphism $f : X \to Y$ of analytic adic spaces and a maximal point $x \in X$, $f(x)$ is a maximal point of Y. This is because we can characterize the maximal points of X as the points whose valuation on the stalk has rank 1. We will usually refer to the following as the "basic characterization" of quasi-finite morphisms.

Proposition 1.84. *Let $f : X \to Y$ be a morphism of analytic adic spaces that is weakly of finite type. The following are equivalent*
i) f is quasi-finite
ii) For every $y \in Y$, $f^{-1}(y)$ is finite
iii) For every $y \in Y$ maximal, $f^{-1}(y)$ is finite
iv) For every $x \in X$, $\widehat{k(x)}$ is finite over $\widehat{k(f(x))}$
v) For every maximal $x \in X$, $\widehat{k(x)}$ is finite over $\widehat{k(f(x))}$
vi) For every maximal point $y \in Y$, the morphism $X \times_Y Spa(k(y), k(y)^o) \to Spa(k(y), k(y)^o)$ is finite

Proof. We may assume that $X = Spa(A, A^+)$ and $Y = Spa(K, K^+)$ for some complete Huber pairs (A, A^+) and (K, K^+) where K is a field and A is of topologically of finite type over K. Indeed, choose a point $y \in Y$ and consider the natural morphism $Spa(K(y), K(y)^o) \to Y$ which is adic and so we can consider the fiber product $X \times_Y Spa(K(y), K(y)^o)$. From the following diagram and what we saw in last section about fiber products

$$\begin{array}{ccc} X \times_Y Spa(K(y), K(y)^o) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Spa(K(y), K(y)^o) & \longrightarrow & Y \end{array}$$

to prove the above for f, it is enough to prove it for the laft vertical morphism of the above diagram. If we recall the proof of the existence of fiber products in the above case, we see that indeed $X \times_Y Spa(K(y), K(y)^o)$ is affinoid.
Now from Tate algerbas we know that there exists $d \in \mathbb{N}$ and an injective finite K-algebra homomorphsim

$$K\langle X_1, \dots, X_d\rangle \to A$$

Moreover A^o is integral over $T_d(K)^o$. Therefore the induced morphism of adic spaces $h : Spa(A, A^o) \to Spa(T_d(K), T_d(K)^o)$ is finite and surjective. Consider the following commutative diagram

$$\begin{array}{ccccc} Spa(T_d(K), T_d(K)^o) & \xleftarrow{\;h\;} & Spa(A, A^o) & \xrightarrow{\;i\;} & Spa(A, A^+) \\ & \searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & Spa(K, K^o) & \longrightarrow & Spa(K, K^+) \end{array}$$

The conditions $ii), iii), iv)$ and $v)$ are easily seen to be equivalent to the condition $d = 0$. Moreover $i)$ and $ii)$ are equivalent and $iii)$ and $vi)$ are equivalent. □

2 A relation of adic spaces with schemes and rigid analytic varieties

2.1 Adic Analytification

Fix a non-archimedean field K. We want to describe a functor from the category of schemes locally of finite type over K to the category of adic spaces over K. We use [Hub94]. A logical choice for a functor from schemes to adic spaces would be $Spec(R) \mapsto Spa(R, R)$ where on the right hand side R is discrete. This indeed glues to give a functor but it is not what we want. That is because given a scheme $Spec(R)$ locally of finite type over K, then we do not expect $Spa(R, R)$ to be over K as this would require a morphism of Huber pairs $(K, K^o) \to (R, R)$ and as R is discrete, it is unlikely that we can have that in most cases. Let us first make clear what properties we wish our functor to have. This is done in the next proposition, in a much more general setting, but we will focus on our case right after.

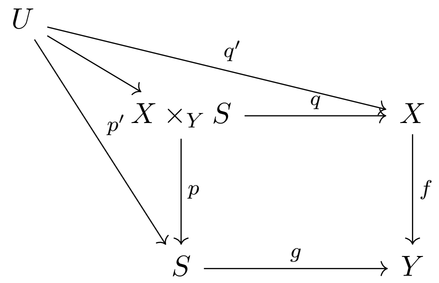
Proposition 2.1. *Let X, Y be schemes, S a stable adic space, $f : X \to Y$ a morphism of schemes locally of finite type and $g : S \to Y$ a morphism of locally ringed spaces. There exists an adic space that we denote $X \times_Y S$ together with a morphism of adic spaces $p : X \times_Y S \to S$ and a morphism of locally ringed spaces $q : X \times_Y S \to X$ such that the following diagram commutes (where every morphism is viewed as a morphism of locally ringed spaces)*

$$\begin{array}{ccc} X \times_Y S & \xrightarrow{\;q\;} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\;g\;} & Y \end{array}$$

and such that for every adic space U together with a morphism of locally ringed spaces $q' : U \to X$ and a morphism of adic spaces $p' : U \to S$ such that the following diagram commutes (where every morphism is viewed as a morphism of locally ringed spaces)

$$\begin{array}{ccc} U & \xrightarrow{\;q'\;} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\;g\;} & Y \end{array}$$

there exists a unique morphism of adic spaces $U \to X \times_Y S$ such that the following diagram commutes (where every morphism is viewed as a morphism of locally ringed spaces)



In the special case when $S = Spa(K, K^o)$ and $Y = SpecK$, we get an adic space over K associated to the scheme X which we call the adic analytification of X and denote it by X^{ad}. We only prove proposition 2.1 in this very special case, by suggesting a proof that follows closely the analogous proof of rigid analytification. We first need the following

Proposition 2.2. *Let $Spa(A, A^+)$ be an affonoid adic space where A is Tate. There exists a natural morphism $\pi : Spa(A, A^+) \to SpecA$ of locally ringed spaces such that any morphism $Spa(A, A^+) \to SpecR$ of locally ringed spaces factors as a composition $Spa(A, A^+) \xrightarrow{\pi} SpecA \to SpecR$ for some morphism $SpecA \to SpecR$. In particular we have*

$$Hom_{LRS}(Spa(A, A^+), Spec(R)) = Hom(R, A)$$

Proof. We define $\pi : Spa(A, A^+) \to SpecA$ given by $|\cdot| \mapsto supp(|\cdot|) = \{f \in A : |f| = 0\}$. We verify that it has the properties of morphisms of locally ringed spaces. Consider an element ϖ of A that is topologically nilpotent. Write X for $Spa(A, A^+)$.
i) π is continuous: Let $f \in A$. Then $\pi^{-1}(D_f) = \{x \in Spa(A, A^+) : x(f) \neq 0\} = \cup_{n \geq 0} X(\frac{\varpi^n}{f})$ since if $x \in Spa(A, A^+)$ is such that $x(f) \neq 0$, then there is n such that $x(\varpi^n) \leq x(f)$ and thus $x \in X(\frac{\varpi^n}{f})$ and vice versa. The sets $X(\frac{\varpi^n}{f})$ are open and so the claim follows.
ii) π induces a morphism $\mathcal{O}_{SpecA} \to \pi_* \mathcal{O}_X$: From i), it follows that $\mathcal{O}_X(\pi^{-1}(D_f)) = \varprojlim A\langle \frac{\varpi^n}{f} \rangle$, since $\mathcal{O}_X(X(\frac{\varpi^n}{f})) = A\langle \frac{\varpi^n}{f} \rangle$. Therefore there is a morphism of rings $\mathcal{O}_{SpecA}(D_f) \to \mathcal{O}_X(\pi^{-1}(D_f))$ since there are morphisms of rings $A_f \to A\langle \frac{\varpi^n}{f} \rangle$. Moreover, those morphisms are easily seen to commute with restrictions.
iii) The homomorphism $\mathcal{O}_{SpecA, \pi(x)} \to \mathcal{O}_{X,x}$ is local: this follows from the way we defined valuations on stalks of $Spa(A, A^+)$. □

Moreover, in the case of affine schemes, we have the following

Lemma 2.3. *Let $X = Spec(K[X, \dots, X_n]/I)$ and r be an element of K with $0 < |r| < 1$. Then $\bigcup_{i \geq 0} Spa(K\langle r^i X_1, \dots, r^i X_n \rangle/(I), (K\langle r^i X_1, \dots, r^i X_n \rangle/(I))^o)$ is the adic analytification of X.*

Proof. We follow closely the proof of the analogous statement in the case of rigid analytic varieties (see appendix). First of all we note that there is a sequence of morphisms of rings

$$K[X_1, \dots, X_n]/I \to \cdots \to K\langle r^i X_1, \dots, r^i X_n \rangle/(I) \to \cdots \to K\langle X_1, \dots, X_n \rangle/(I)$$

and thus from the latter proposition we get a morphism of locally ringed spaces

$$\bigcup_{i \geq 0} Spa(K\langle r^i X_1, \dots, r^i X_n \rangle/(I), (K\langle r^i X_1, \dots, r^i X_n \rangle/(I))^o) \to X$$

Let Y be an adic space over K such that the following diagram commutes

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spa(K, K^o) & \longrightarrow & SpecK \end{array}$$

We may assume that $Y = Spa(B, B^+)$ is affinoid. By proposition 2.2, the morphism $Y \to X$ is induced by a morphism of rings $K[X_1, \ldots, X_n]/I \to B$ which is in fact a morphism of K-algebras, since the above diagram is commutative. Therefore from the proof of rigid analytification, we can find i large enough so that the following diagram commutes

$$\begin{array}{ccccc} K[X_1, \ldots, X_n]/I & & \longrightarrow & & B \\ \downarrow & & \searrow & & \nearrow \\ K\langle r^j X_1, \ldots, r^j X_n\rangle/(I) & \longrightarrow & K\langle r^i X_1, \ldots, r^i X_n\rangle/(I) & & \end{array}$$

for all $j \geq i$. In addition to that, for i large enough, $K\langle r^i X_1, \ldots, r^i X_n\rangle/(I) \to B$ maps $(K\langle r^i X_1, \ldots, r^i X_n\rangle/(I))^o$ to a subset of B^+, as $K\langle r^i X_1, \ldots, r^i X_n\rangle/(I) \to B$ emerges from extending $K \to B$, which in this case maps K^o to a subset of B^+. Therefore there is i large enough so that there is a morphism of Huber pairs

$$K\langle r^i X_1, \ldots, r^i X_n\rangle/(I), (K\langle r^i X_1, \ldots, r^i X_n\rangle/(I))^o \to (B, B^+)$$

which induces a morphism of adic spaces $Y \to \bigcup_{i \geq 0} Spa(K\langle r^i X_1, \ldots, r^i X_n\rangle/(I), (K\langle r^i X_1, \ldots, r^i X_n\rangle/(I))^o)$ with the desired properties. □

Proof. (of the special case of proposition 2.1) Let X be a scheme locally of finite type over K. Assume that X is covered by affine open subschemes X_i. As we showed in lemma 2.3, X_i admit adic analytifications $f_i : X_i^{ad} \to X_i$. From the universal property it follows that for all i, j, $f_i(X_i \cap X_j)$ is the adic analytification of $X_i \cap X_j$ and therefore we can glue X_i^{ad} and f_i to an adic space X^{ad} with a morphism $f : X^{ad} \to X$ which is the adic analytification of X. □

Remark. Let $f : X \to Y$ be a morphism of schemes locally of finite type over K. Moreover, let X^{ad} be the adic analytification of X and $i : X^{ad} \to X$ the corresponding morphism of locally ringed spaces. Then $f \circ i$ as a morphism of locally ringed spaces $X^{ad} \to Y$ that makes the diagram of proposition 2.1 commute. Therefore by the universal property, there exists a morphism of adic spaces $X^{ad} \to Y^{ad}$ that we denote by f^{ad}. By using the universal property one can easily prove that for morphisms $f : X \to Y$ and $g : Y \to Z$ of schemes locally of finite type over K, $(gf)^{ad} = g^{ad} f^{ad}$.

Corollary 2.4. *From Theorem 2.1. and the latter remark, we conclude that mapping a scheme X to its adic analytification X^{ad} and a morphism of schemes f to f^{ad} gives a functor from the category of schemes that are locally of finite type over K to the category of adic spaces over K.*

Example 2.5. *The adic analytification of $Spec(K)$ is $Spa(K, K^o)$*

Example 2.6. *The adic analytification of $Spec(K[T])$ is $\bigcup_{i \geq 0} Spa(K\langle r^i T\rangle, K^o\langle r^i T\rangle)$.*

2.2 Rigid Analytic Varieties as Adic Spaces

Fix a non-archimedean field K. We want to describe a functor from the category of rigid analytic varieties over K to the category of adic spaces over K. For that, we use [Hub93b] and [Sta15]. We write $\mathcal{R}_K$ for the category of rigid analytic varieties over K and $\mathcal{A}_K$ for the category of adic spaces over K. Recall that given an affinoid algebra A over K and $x \in Max(A)$, then A/x if finite over K and so we can extend the valuation on K to get a valuation $|\cdot|_x$ of A/x. As x is closed in A, this easily shows that $|\cdot|_x$ is continuous. Moreover $|\cdot|_x \in Spa(A, A^o)$. This gives an injective mapping $SpaA \to Spa(A, A^o)$ given by $x \mapsto |\cdot|_x$. From now on, we treat SpA as a subset of $Spa(A, A^o)$ under the above injection. One of the key propositions in understanding the relation between rigid analytic varieties and adic spaces is the following.

Proposition 2.7. *Let A be an affonoid K-algebra and consider $X = Spa(A, A^o)$ and $X_0 = Sp(A)$. There is a bijection between the constructible subsets of X and the constructible subsets of X_0 given by $C \mapsto C \cap X_0$.*

Proof. We shall first show that if C is non-empty, then $C \cap X_0$ is non-empty. Let C be a constructible subset of X. Then $C = \bigcup_{i=1}^{n} U_i \cap (X - V_i)$ for some quasi-compact open $U_i, V_i \subset X$. In order to show that $C \cap X$ is non-empty, we may assume that C is of the form $U \bigcap (X - V)$ for some quasi-compact open $U, V \subset X$. As U, V are finite unions of rational domains, we may also assume that U is rational and write $V = \bigcup_{i=1}^{n} V_i$. Therefore we may assume that $U = X$, since otherwise we simply consider V_i to be rational in U. Therefore we are left to show that $(X - \bigcup_{i=1}^{n} V_i) \cap Sp(A)$ is non-empty. Note that the latter is empty if and only if $Sp(A) = \bigcup V_i$ and in this case, we will show that $\bigcup_{i=1}^{n} V_i = X$. So assume that indeed V_i cover $Sp(A)$. Then $V_i \cap Sp(A)$ cover $Sp(A)$ and thus we can refine V_i so that $V_i \cap Sp(A)$ becomes a rational covering. Hence we may write $V_i = X(\frac{f_1,\ldots,f_n}{f_i})$ for some f_i that generate the unit ideal. Now every $x \in X$ is easily seen to belong in some V_i. We proved so far that if C is non-empty then $C \cap Sp(A)$ is non-empty. Let C_1 and C_2 be constructible subsets of X with $C_1 \cap Sp(A) = C_2 \cap Sp(A)$. Then $C' = (C_1 - C_2) \bigcup (C_2 - C_1)$ is constructible with $C' \cap Sp(A) = \emptyset$ and so it must be $C_1 = C_2$. □

Now we present the main result:

Proposition 2.8. *There exists a fully faithful functor $r_K : \mathcal{R}_K \to \mathcal{A}_K$*

Proof. The idea is that $SpaA \to Spa(A, A^o)$ can be extended to give such a functor. Once one knows the Gerritzen-Grauert theorem and proposition 2.7, this becomes easy. We follow the presentation found in [Sta15]. We present the main ideas.

1. We claim that if X is affinoid, then given open immersions $U \to X$ and $V \to X$ where U, V are affinoid, the morphisms $r(U) \to r(X)$ and $r(V) \to r(X)$ are open immersions and $r(U \cap V) = r(U) \cap r(V)$. Indeed, by the Gerritzen-Grauert theorem, U is covered by finitely many rational domains $U_i \subset X$ with $U_i \cap U_j$ rational domains of X (thus also of U_i and U_j). Therefore $r(U) = \bigcup r(U_i)$ and by proposition 2.7, the rational subsets of $r(U)$, $r(U_i \cap U_j)$ and $r(U_i) \cap r(U_i)$ coincide. But $r(U_i)$ are rational subsets of $r(X)$ and so $r(U_i \cap U_j) = r(U_i) \cap r(U_j)$ in $r(X)$. We do the same for V and if, say, V is covered by finitely many rational domains $\{V_k\}$, then we have

$$r(U) \cap r(V) = \bigcup (r(U_i) \cap r(V_k)) = \bigcup r(U_i \cap V_k) = r(U \cap V)$$

which proves 1.

2. Gluing in the separated case: Let X be a separated rigid analytic variety. Given affinoid open subspaces $U, V \subset X$, $U \cap V$ is affinoid open as X is separated. Thus $r(U \cap V) \to r(U)$ and $r(U \cap V) \to r(V)$ are open immersions by step 1. Moreover, given $W \subset X$ an open subspace, we have $r(U \cap W) \cap r(V \cap W) = r(U \cap V \cap W)$, by applying step 1 to W. Therefore, we may consider all the affinoid open subspaces $U \subset X$ and define $r(X)$ to be the gluing of all $r(U)$. Given admissible open subspaces $U, V \subset X$ and open immersions $U \to X$ and $V \to X$, then there are unique maps $r(U) \to r(X)$ and $r(V) \to r(X)$ which are open immersions and $r(U \cap V) = r(U) \cap r(V)$.

3. Let X be a separated rigid analytic variety over K. A collection $\{U_i\}$ of admissible open subspaces of X is an admissible covering if and only if $r(X) = \bigcup r(U_i)$.

4. Gluing for general rigid analytic varieties: Let X be a rigid analytic variety over K. We define $r(X)$ as we did in step 2, by replacing affinoid open subspaces by separated open subspaces and everything works as in step 2. Moreover, the proposition of step 3 still works. This proves the existence of the functor.

5. r_K is fully faithful. This is reduced to the affinoid case where we have that a morphism of affinoid K-algebras $B \to A$ induces a morphism of Huber pairs $(B, B^o) \to (A, A^o)$ and any morphism of Huber pairs $(B, B^o) \to (A, A^o)$ by definition corresponds to a morphism of affinoid K-algebras $B \to A$.

□

From the above, the following is easily deduced

Proposition 2.9. *i) r induces an equivalence between the category of quasi-separated rigid analytic varieties over X, where X is a rigid analytic variety, to the category of adic spaces over $r(X)$ with quasi-separated and locally of finite type morphisms.*
ii) r commutes with fiber products.
iii) $r(X)$ is affinoid if and only if X is affinoid.

Remark. Let X be a scheme locally of finite type over K. Let X^{rig} be the rigid analytification of X and X^{ad} the adic analytification of X. Then $X^{ad} = r(X^{rig})$. This leads to the diagram that we presented in the introduction.

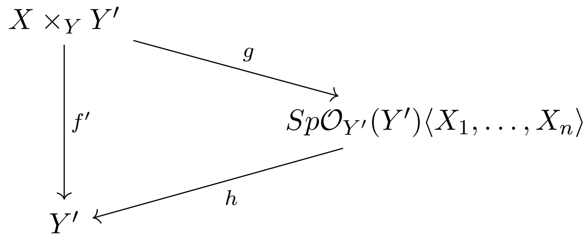
We also note the following

Note. From proposition 2.8 it follows that one can do rigid geometry without the need of Grothendieck topologies.

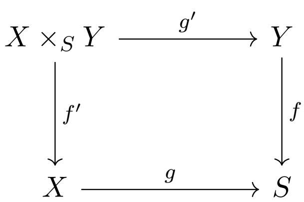
We conclude this section with the following proposition, which will later be used to understand the relation between etale morphisms of rigid analytic varieties and etale morphisms of adic spaces.

Proposition 2.10. *Let $f : X \to Y$ be a morphism of rigid analytic varieties and $r(f) : r(X) \to r(Y)$ the associated morphism of adic spces. Then f is locally quasi-finite (meaning that $dim f^{-1}(y) \leq 0$ for all $y \in Y$) if and only if $r(f)$ is locally quasi-finite.*

Proof. We may assume that X, Y are affinoid. Let y be a maximal point of $r(Y)$ such that $r(f)^{-1}(y) \neq \emptyset$. By [Hub96] 1.5.8, there is a locally closed affinoid rigid analytic subvariety Y' of Y and an integer $n \geq 0$ such that $y \in r(Y') \subset r(Y)$ and such that the morphism $f' : X \times_Y Y' \to Y'$ factors as



where h is the natural morphism and g is finite surjective. Moreover, the following general result holds, which can be proved by using the basic characterization of quasi-finite morphisms (see cor. 1.53 [Hub96]): Consider the following commutative diagram



where f is locally quasi-finite and locally of finite type. Then f' is locally quasi-finite. Moreover, for every $x \in X$ and $y \in Y$ with $f(y) = g(x)$ there is $a \in X \times_S Y$ with $x = f'(a)$ and $a = g'(a)$. Summing up the above, we get a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times_Y Y' & & \\ \Big\downarrow f & & \Big\downarrow f' & \overset{g}{\searrow} & \\ & & & & Sp\mathcal{O}_{Y'}(Y')\langle X_1, \dots, X_n \rangle \\ & & & \overset{h}{\swarrow} & \\ Y & \longleftarrow & Y' & & \end{array}$$

which gives a commutative diagram

$$\begin{array}{ccccc} r(X) & \longleftarrow & r(X) \times_{r(Y)} r(Y') & & \\ \Big\downarrow r(f) & & \Big\downarrow r(f') & \overset{r(g)}{\searrow} & \\ & & & & r(Sp\mathcal{O}_{Y'}(Y')\langle X_1, \dots, X_n \rangle) \\ & & & \overset{r(h)}{\swarrow} & \\ r(Y) & \longleftarrow & r(Y') & & \end{array}$$

By combining the above with the basic characterization of quasi-finite morphisms, the result follows. □

Note. What we need this for, basically, is because we will have to prove that an etale morphism $f : X \to Y$ of rigid analytic varieties gives an etale morphism of adic spaces $r(f) : r(X) \to r(Y)$ and this is done by noticing in particular that f is locally quasi-finite and so is $r(f)$. Then for a locally quasi-finite morphism of adic spaces we will have good criteria to determine when it is etale.

Part III
Etale Cohomology

3 Etale morphisms, Etale site and Etale cohomology

In order to define the etale site of an adic space, we only consider stable adic spaces. Let us explain why. Say we want to study the etale cohomology of some class of adic spaces and let X belong in this class. Then the adic spaces that are etale (which is still to be defined, but it is in particular topologically of finite type) over X have to belong in this class. By considering stable adic spaces, we definitely achieve that. Also stable adic spaces contain all the adic spaces that are of interest (so far), so this condition does not seem to restrict us much.

Note. In [Hub96], a smaller class of adic spaces is considered. Namely, all the adic spaces that are considered are locally adic spectra $Spa(A, A^+)$ where A has one of the following properties:
i) $\widehat{A}$ has a Noetherian ring of definition over which $\widehat{A}$ is finitely generated
ii) A is strongly Noetherian Tate
iii) $\widehat{A}$ is discrete
Related to the above, see this answer [hs] to a question of the author.

Beyond this slight generalization, in general, we follow [Hub96]. In particular, in 3.2 and 3.3 we follow the exposition of [Hub96] to a great extent.

3.1 The etale site

Definition 3.1. *A morphism $f : X \to Y$ of adic spaces is called etale (resp. smooth, resp. unramified) if it is locally of finite presentation (resp. locally of finite presentation, resp. locally of finite type) and for any Huber pair (A, A^+), any ideal I of A with $I^2 = \{0\}$ and any morphism $Spa(A, A^+) \to Y$, the induced map $Hom_Y(Spa(A, A^+), X) \to Hom_Y(Spa((A, A^+)/I), X)$ is bijective (resp. surjective, resp. injective)*

Using this definition, we can define etaleness at a point in the following way. A morphism $f : X \to Y$ is etale at a point $x \in X$ if there is an open neighborhood U of X with $x \in U$ and such that $f(U) \subset V$ for some open subspace $V \subset Y$ and $f : U \to V$ is etale. In the same way one can define what it means for f to be smooth or unramified at x. Moreover, just like with schemes, it can be proved that f is etale (resp. smooth, resp. unramified) if and only if it is etale (resp. smooth, resp. unramified) at every point $x \in X$. The following elementary propositions follow from the definitions, but let us spell out the arguments.

Proposition 3.2. *The composition of etale morphisms of adic spaces is etale.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be etale morphisms of adic spaces. Of course $g \circ f$ is locally of finite presentation, so we need to show that $Hom_Z(Spa((A, A^+)/I), X) \to Hom_Z(Spa(A, A^+), X)$ is bijective. Let s be an element of $Hom_Z(Spa((A, A^+)/I), X)$ and consider $f \circ s$. Since g is etale, there is $s' : Spa(A, A^+) \to Y$ with $f \circ s = s' \circ h$. Therefore, since f is etale, there is $s'' : Spa(A, A^+) \to X$ with $s = s'' \circ h$ which proves that $Hom_Z(Spa((A, A^+)/I), X) \to Hom_Z(Spa(A, A^+), X)$ is surjective. Now let t_1, t_2 be elements of $Hom_Z(Spa(A, A^+), X)$ with $t_1 \circ h = t_2 \circ h$. Then, $f \circ t_1 \circ h = f \circ t_2 \circ h$ and so since g is etale there is $t : Spa(A, A^+) \to Y$ with $f \circ t_1 \circ h = f \circ t_2 \circ h = t \circ h$. Therefore since f is etale, it follows that $t_1 = t_2$. The result follows. □

Proposition 3.3. *Let $f : X \to Y$ be an etale morphism of adic spaces and $g : Y' \to Y$ any morphism of adic spaces. Then $X \times_Y Y' \to Y'$ is etale.*

Proof. We know that $X \times_Y Y' \to Y'$ is locally of finite presentation. Therefore, we need to show that for any Huber pair (A, A^+), any ideal I of A with $I^2 = \{0\}$ and any morphism $Spa(A, A^+) \to Y'$, the induced mapping $Hom_{Y'}(Spa(A, A^+), X \times_Y Y') \to Hom_{Y'}(Spa((A, A^+)/I), X \times_Y Y')$ is bijective.

We first show that it is surjective. Write g' for the morphism $X \times_Y Y' \to X$ and f' for the morphism $X \times_Y Y' \to Y'$. Let $t : Spa((A, A^+)/I) \to A \times_Y Y'$ be a morphism of adic spaces and consider $g' \circ t$. Since f is etale, there is exactly one morphism $t' : Spa(A, A^+) \to X$ such that $g' \circ t = t' \circ h$ where h is the natural morphism $Spa((A, A^+)/I) \to Spa(A, A^+)$. By the universal property of fiber products, there is a morphism $s : Spa(A, A^+) \to X \times_Y Y'$ with $t' = g' \circ s$. We have $g' \circ t = g' \circ s \circ h$ and $f' \circ t = f' \circ s \circ h$ and this is enough to conclude that $t = s \circ h$. Therefore $Hom_{Y'}(Spa(A, A^+), X \times_Y Y') \to Hom_{Y'}(Spa((A, A^+)/I), X \times_Y Y')$ is surjective. Let now $s_1, s_2 : Spa(A, A^+) \to X \times_Y Y'$ be two morphisms of adic spaces and assume that $s_1 \circ h = s_2 \circ h$ where h is the natural morphism $Spa((A, A^+)/I) \to Spa(A, A^+)$. It follows that $g' \circ s_1 \circ h = g' \circ s_2 \circ h$. But since f is etale, there exists exactly one morphism $s : Spa(A, A^+) \to X$ with $s \circ h = g' \circ s_1 \circ h = g' \circ s_2 \circ h$. It follows that $s = g' \circ s_1 = g' \circ s_2$ and we also have $f' \circ s_1 = f' \circ s_2$ and this is enough to conclude that $s_1 = s_2$. □

Remark. Similarly to the above we have that the composition of smooth morphisms are smooth and the composition of unramified morphisms is unramified.

Definition 3.4. *Let X be an adic space. We consider the category Et/X of adic spaces that are etale over X. We consider a Grothendieck topology on Et/X such that $(U_i \xrightarrow{f_i} U)$ is a covering if and only if $U = \bigcup f_i(U_i)$.*

Perhaps the only part that is not obvious about the above being a site, is stability under base change which follows from corollary 1.79ii)c.

Note. Since perfectoid spaces are undoubtedly an important class of adic spaces and since we defined them in example 1.56, we should make this important remark: in [Sch11], etale morphisms of perfectoid spaces are not defined in the same way as in definition 2.1. In the next few lines we will explain why is that and why this is not a problem for our theory. Let R be a perfectoid ring. Then we can define addition on the multiplicative monoid

$$R^\flat = \varprojlim_{x \mapsto x^p} R$$

so that $R^\flat$ becomes a perfectoid ring of charateristic p. This extends to give a functor $X \mapsto X^\flat$ from perfectoid spaces to perfectoid spaces "of characteristic p". In [Sch11], etale morphisms of perfectoid spaces are defined so that $f : X \to Y$ is etale if and only if $f^\flat : X^\flat \to Y^\flat$ is etale. This ultimately leads to an equivalence between X_{et} and $X^\flat_{et}$ (where the etale site is defined by considering surjective families of etale morphisms in the category of perfectoid spaces over X). Although this is a different definition (and it is actually different in many cases), it agrees with the definition of etale morphisms between analytic adic spaces. In particular this definition agrees with the definition of etale morphisms between adic spaces coming from rigid analytic varieties. In any case, an adic space over a perfectoid space is analytic. This justifies at least that our definition works. The latter will of course be visible later.

Now given a stable adic space X and an abelian sheaf $\mathcal{F}$ on X_{et}, the etale cohomology groups of X with respect to $\mathcal{F}$ are just the sheaf cohomology groups of X_{et} with respect to $\mathcal{F}$ (just like with schemes). This is explained in the following definition.

Definition 3.5. *Let X be a stable adic space and consider the category $Ab(X_{et})$ of abelian sheaves on X_{et}. Then $Ab(X_{et})$ has enough injectives and so given $\mathcal{F} \in Ab(X_{et})$, we can consider the right derived functors of $\mathcal{F}$. Consider the global sections functor $\Gamma(X, -) : Ab(X_{et}) \to \boldsymbol{Ab}$ and define the n-th etale cohomology group of X with respect to $\mathcal{F}$ by*

$$H^n(X_{et}, \mathcal{F}) = R^n\Gamma(X, \mathcal{F}).$$

We note that we give this definition just for the sake of it and we cannot really prove much without more machinery from category theory and homological algebra (for example Leray spectral sequences) that we don't cover in this thesis. That said, although we will present two corollaries of the comparison theorems about etale cohomology groups, we will not do anything more in this direction.

3.2 Differentials

Here we define universal derivations in our context. This is very similar to the case of schemes with the only difference being that we should endow $\Omega_{B|A}$ with a suitable (”natural”) topology.

Definition 3.6. *Let A, B be complete Huber rings and $A \to B$ a morphism of Huber rings topologically of finite type. A universal A-derivation of B is a continuous A-derivation $d_{B|A} : B \to \Omega_{B|A}$ where $\Omega_{B|A}$ is a complete topological B-module such that for any continuous A-derivation $d : A \to M$ where B is a complete topological B-module, there exists a unique continuous B-linear map $f : \Omega_{B|A} \to M$ such that the following diagram commutes*

$$\begin{array}{ccc} B & \xrightarrow{d_{B|A}} & \Omega_{B|A} \\ & \searrow^{d} & \downarrow f \\ & & M \end{array}$$

Let A, B, C be complete Huber rings and $A \to B$ and $B \to C$ be morphisms that are topologically of finite type. We have the following properties:

1) By assumption, there are finite subsets $M_1, \ldots, M_n$ of A with $M_i \cdot A$ open for all i and an ideal I of $C := A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$ such that $B = C/I$. Let Ω be the free C-module of rank n generated by $dX_1, \ldots, dX_n$ and let $d : C \to \Omega$ be given by

$$d(s) = \textstyle\sum_{i=1}^{n} \frac{\partial s}{\partial X_i} dX_i$$

Moreover, let $\Omega_{B|A}$ be the quotient of Ω by the C-submodule generated by $P \cdot \Omega \cup \{d(p) : p \in P\}$. There exists a mapping $d_{B|A} : B \to \Omega_{B|A}$ such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{d} & \Omega \\ \downarrow & & \downarrow \\ B & \xrightarrow{d_{B|A}} & \Omega_{B|A} \end{array}$$

and $(\Omega_{B|A}, d_{B|A})$ is a universal A-derivation of B.

2) Let I be the kernel of the ring homomorphism $B\widehat{\otimes}_A B \to B$ given by $b\widehat{\otimes}b' \mapsto bb'$. Then the universal A-derivation of B is $\Omega_{B|A} := I/I^2$ with $d_{B|A} : B \to \Omega_{B|A}$ the mapping induced by $b \mapsto 1\widehat{\otimes}b - b\widehat{\otimes}1$.

3) If B is finite over A, then $d_{B|A} : B \to \Omega_{B|A}$ is a universal A-derivation of the ring B.

4) There is an exact seuquence of finitely generated C-modules

$$\Omega_{B|A} \otimes_B C \to \Omega_{C|A} \to \Omega_{C|B} \to 0$$

5) If $B \to C$ is surjective and open with kernel I, then there exists an exact sequence of finitely generated C-modules

$$I/I^2 \to \Omega_{B|A} \otimes_B C \to \Omega_{C|A} \to 0$$

6) Let A' be a complete Huber ring and $A \to A'$ an adic morphism. Consider $B' := C\widehat{\otimes}_A A'$. Then

$$\Omega_{B'|A'} = \Omega_{B|A} \otimes_B B'$$

Definition 3.7. *Let $f : X \to Y$ be a morphism of adic spaces weakly of finite type. Consider the diagonal morphism $\Delta : X \to X \times_Y X$ and let $\mathcal{I}$ be $ker(\mathcal{O}_{X\times_Y X} \to \Delta_* \mathcal{O}_X)$. We define the $\mathcal{O}_X$-module*

$$\Omega_{X|Y} := \mathcal{I} \otimes_{\mathcal{O}_{X\times_Y X}} \mathcal{O}_X$$

and call it the sheaf of differentials of X over Y.

Note. $\Omega_{X|Y}$ should be thought as a "global" construction of $\Omega_{A|B}$ and we can deduce results about it by working locally (and by using the properties above). This is justified by the following: If U and V are open affinoid subspaces of X and Y respectively with $f(U) \subset V$ and $\mathcal{O}_X(U)$ is topologically of finite type over $\mathcal{O}_Y(V)$, then $\Omega_{X|Y}|_U$ is the $\mathcal{O}_X|_U$-module associated with the finite $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)|\mathcal{O}_Y(V)}$.

By using the definitions of unramified and smooth morphisms, the properties of universal derivations and the above note on the "global" character of $\Omega_{X|Y}$ one can prove the following propositions

Proposition 3.8. *Let $f : X \to Y$ be a morphism of adic spaces that is locally of finite type. The following are equivalent*
i) f is unramified
ii) $\Omega_{X|Y} = 0$
iii) the diagonal morphism $X \to X \times_Y X$ is an open embedding
iv) The restriction of f to any open affinoid subspace U of X such that $f(U)$ is contained in an open affinoid subspace of Y factors as

$$U \xrightarrow{g} X \xrightarrow{h} Y$$

where Z is an affinoid adic space, g is a closed embedding and h is etale.

Proposition 3.9. *Let $f : X \to Y$ be a smooth morphism of adic spaces. Then*
i) $\Omega_{X|Y}$ is a locally free $\mathcal{O}_X$-module
ii) If Z is an adic space locally of finite type over Y and $f : X \to Z$ is a Y-morphism, then f is etale at $x \in X$ if and only if $d : (\Omega_{Z|Y} \otimes_{\mathcal{O}_Z} \mathcal{O}_X)_x \to (\Omega_{X|Y})_x$ is bijective.

3.3 Properties of etale morphisms

In this section we study properties of etale morphisms. We start with the following fundamental characterization.

Proposition 3.10. *Let $g : X \to Y = Spa(A, A^+)$ be a morphism of affinoid adic spaces. The following are equivalent*
i) g is etale
ii) There exist finite subsets $M_1, \dots, M_n$ of A with $M_i{\cdot}A$ open for all i and $f_1, \dots, f_n \in A\langle X_1, \dots, X_n\rangle_{M_1,\dots,M_n}$ such that if we put $(B, B^+) = (A, A^+)\langle X_1, \dots, M_n\rangle_{M_1,\dots,M_n}$ and if I is the ideal of B generated by $f_1, \dots, f_n$, then the image of $det(\frac{\partial f_i}{\partial X_i})_{i,j=1,\dots,n} \in B$ in B/I is a unit of B/I and X is Y-isomorphic to $Spa(B, B^+)/I$.
iii) There exist finite subsets $M_1, \dots, M_n$ of A with $M_i \cdot A$ open for all i and $f_1, \dots, f_n \in A[X_1, \dots, X_n]$ such that if we put $(B, B^+) = (A, A^+)\langle X_1, \dots, X_n\rangle_{M_1,\dots,M_n}$ and if I is the ideal of B generated by $f_1, \dots, f_n$, then the image of $det(\frac{\partial f_i}{\partial X_i})_{i,j=1,\dots,n} \in A[X_1, \dots, X_n]$ in B/I is a unit of B/I and X is Y-isomorphic to $Spa(B, B^+)/I$. In case A is Tate, we can choose $M_1, \dots, M_n = \{1\}$.

Proof. A proof of this fact is long and technical (thus omitted). See [Hub96] proposition 1.7.1 and corollary 1.7.2. ☐

Corollary 3.11. *Let (A, A^+) be a Huber pair, D a commutative ring and $\phi : D \to A$ a ring homomorphism. Then ϕ induces a morphism of locally ringed spaces $Spa(A, A^+) \to Spec(D)$ given by the composition*

$$Spa(A, A^+) \to Spec(A) \to Spec(D)$$

where $Spa(A, A^+)$ maps x to $supp(x)$ and $Spec(A) \to Spec(D)$ is the usual morphism of schemes induced by ϕ. If X is locally of finite type over $Spec(D)$, them we may consider $X \times_{Spec(D)} Spa(A, A^+)$. We have that if $X \to Spec(D)$ is etale, then $X \times_{Spec(D)} Spa(A, A^+) \to Spa(A, A^+)$ is etale.

Proof. i) We may assume that X is affine. Since $X \to SpecD$ is etale, there exist $f_1, \dots, f_n \in A[X_1, \dots, X_n]$ such that X is $Spec(A)$-isomorphic to $Spec(A[X_1, \dots, X_n]/(f_1, \dots, f_n))$ and the image of $det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\dots,n}$ in $A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ is a unit. Let x be a point of $X \times_{SpecD} Spa(A, A^+)$. By the construction of $X \times_{SpecD} Spa(A, A^+)$, there is an open neighborhood U of x and finite subsets $M_1, \dots, M_n$ of A with $M_i \cdot A$ open for all i such that U is $Spa(A, A^+)$-isomorphic to $Spa(B, B^+)/I$ where $(B, B^+) = (A, A^+)\langle X_1, \dots, X_n\rangle$ and I is the ideal of B generated by $f_1, \dots, f_n$. It follows directly from proposition 3.10 that $X \times_{SpecD} Spa(A, A^+) \to Spa(A, A^+)$ is etale at x. $\square$

Before we prove more corollaries, let us give a definition that we will use in what follows.

Definition 3.12. *We say that a morphism $f : A \to B$ of Huber rings is of algebraically finite type if it is adic and there are rings of definition A_0 and B_0 of A and B respectively such that $A \to B$ and $A_0 \to B_0$ are of finite type.*

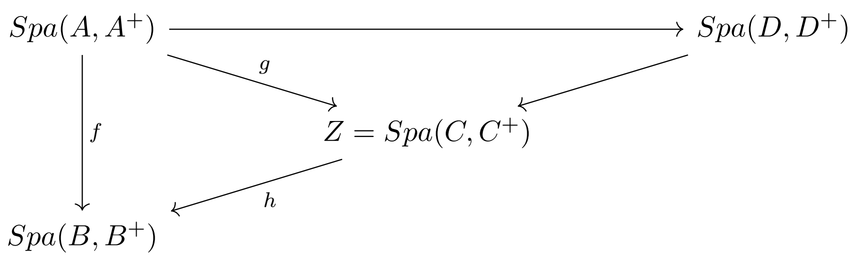
Definition 3.13. *We say that a morphism $f : (A, A^+) \to (B, B^+)$ of Huber pairs is of algebraically finite type if $A \to B$ is of algebraically finite type and there exists an open subring D of B such that B^+ is the integral closure of D in B and $A^+ \to D$ is of algebraically finite type.*

Corollary 3.14. *Let $f : Y \to Spa(A, A^+)$ be an etale morphism of affinoid adic spaces. There is a Huber pair (B, B^+), a morphism $(A, A^+) \to (B, B^+)$ of algebraically finite type such that $A \to B$ is etale and Y is $Spa(A, A^+)$-isomorphic to $Spa(B, B^+)$.*

Proof. See [Hub96] 1.7.3iii). $\square$

Corollary 3.15. *Unramified morphisms of adic spaces are locally quasi-finite.*

Proof. We may work locally and prove this for affinoid adic spaces. Let $f : Spa(A, A^+) \to Spa(B, B^+)$ be an unramified morphism of adic spaces. Then by combining proposition 3.8 and corollary 3.14 we get the following commutative diagram



where h is a closed embedding, g is etale, $Spa(A, A^+) \to Spa(D, D^+)$ is an isomorphism, (D, D^+) is of algebraically finite type over (C, C^+) and D is etale over C. Then $Spa(A, A^+) \to Spa(D, D^+)$, $Spa(D, D^+) \to Spa(C, C^+)$ and h are locally quasi-finite and so is f. $\square$

Corollary 3.16. *Let $f : X \to Y$ be an etale morphism of affinoid adic spaces with $\mathcal{O}_Y(Y)$ being a strongly Noetherian Tate ring. Then the induced morphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is flat.*

Proof. Write (A, A^+) for $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$. Then by proposition 3.10, there is a natural number n and $f_1, \dots, f_n \in A[X_1, \dots, X_n]$ such that if we put $B = A\langle X_1, \dots, X_n\rangle$ and I be the ideal of B generated by $f_1, \dots, f_n$, then the image of $\Delta = det(\frac{\partial f_i}{\partial X_j})_{i,j=1,\dots,n}$ in B/I is a unit of B/I and $\mathcal{O}_X(X) = B/I$. If we show that $A\langle X_1, \dots, X_n\rangle$ is flat over $A[X_1, \dots, X_n]$, then the result will follow. Indeed, in this case, B/I is flat over $C = A[X_1, \dots, X_n]/(f_1, \dots, f_n)$. If we write C_Δ for the localization of C by Δ, then $C \to B/I$ factors through $C_\Delta \to B/I$ (since the image of Δ in B/I is a unit) which is flat. Moreover C_Δ is flat over A and so B/I is flat over A. Therefore let us show that $A\langle X_1, \dots, X_n\rangle$ is flat over $A[X_1, \dots, X_n]$. We may assume that $n = 1$. As in the proof of sheafyness of strongly Noetherian Tate rings, $A\langle X\rangle$ is flat over A and since $A[X]$ is flat over A, it is enough to show that for every residue field K of A, $A[X] \otimes_A K \to A\langle X\rangle \otimes_A K$ is flat (see [DG67] III 10.2.5). Let p be a prime ideal of A. Then the ideal $p \cdot A\langle X\rangle$ of $A\langle X\rangle$ is closed and so it is equal to $\{\sum a_i X^i \in A\langle X\rangle : a_i \in p \text{ for all } i\}$, so it is a prime ideal of $A\langle X\rangle$. Then $A\langle X\rangle/p \cdot A\langle X\rangle$ is a torsion free $A[X]/p \cdot A[X]$-module. Write M for $A\langle X\rangle/p \cdot A\langle X\rangle$, C for $A[X]/p \cdot A[X]$ and S for $A - p$. Then M_S is a torsion free C_S-module and since C_S is a Dedekind domain, M_S is flat over C_S. □

Note. The latter holds true in the cases considered in [Hub96] but it is probably not true for all stable adic spaces (see [hs]).

Let us for simplicity call "strongly Noetherian Tate" the adic spaces that are glued by adic spectra of strongly Noetherian Tate Huber pairs. We have the following characterization of etale morphisms of such spaces.

Proposition 3.17. *Let $f : X \to Y$ be a morphism of strongly Noetherian Tate adic spaces that is locally of finite type. The following are equivalent.*
i) f is etale
ii) f is flat and unramified
iii) For every maximal point $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}/m_{f(x)}\mathcal{O}_{X,x}$ is a finite seperable extension of $k(f(x))$.

Note. This is still true for analytic adic spaces considered in [Hub96] but for us this does not make any difference.

We first prove the following

Lemma 3.18. *Let $f : X \to Y$ be a morphism between strongly Noetherian Tate adic spaces that is locally quasi-finite. Then*
i) For every maximal point $x \in X$, $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$ and the $\mathcal{O}_{Y,f(x)}$-derivation $d : \mathcal{O}_{X,x} \to (\Omega_{X|Y})_x$ induced by $d_{X|Y} : \mathcal{O}_X \to \Omega_{X|Y}$ is a universal $\mathcal{O}_{Y,f(x)}$-derivation of $\mathcal{O}_{X,x}$.
ii) If f is flat, then for every morphism of strongly Noetherian Tate adic spaces $Y' \to Y$, the projection $f' : X \times_Y Y' \to Y'$ is flat.

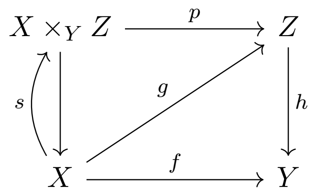
Proof. i) The second claim follows from the first and from property 3 of universal derivations. We may assume that X and Y are affinoid. The idea for the first claim is the following (this is just a sketch, for a complete proof see [Hub96] 1.5.4): Write (A, A^+) and (B, B^+) for $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ respectively. Let x be a maximal point of X and write y for $f(x)$ which is a maximal point of Y. If (B, B^+) is topologically of finite type over (A, A^+), then we can find an affinoid neighborhood V of y such that $f^{-1}(V) \to V$ is finite. Let $f^{-1}(y) = \{x_0, \dots, x_n\}$. Then

$$\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in W} \mathcal{O}_X(f^{-1}(W)) = \prod_{i=0}^n \mathcal{O}_{X,x_i}$$

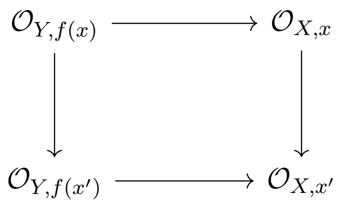
so $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$. In general we only have that B is topologically of finite type over A but it can be reduced to the above case in the following way. Let C be a ring of integral elements of B such that (B, C) is topologically of finite type over (A, A^+). Moreover define $Z = Spa(B, B^o)$, $X' = Spa(B, C)$ and let $i : Z \to X$ and $j : Z \to X$ be the natural morphisms. Then Z is quasi-finite over Y as j is a topological embedding. Moreover by the fundamental characterization of quasi-finite morphisms, X' is quasi-finite over Y. If x is a maximal point of X, then let z be a maximal point of

Z with $x = i(z)$. Moreover, let $x' = i(z)$. Then $\mathcal{O}_{X',x'}$ is finite over $\mathcal{O}_{Y,y}$ by the case we treat above. Moreover, we have isomorphisms $\mathcal{O}_{X',x'} \to \mathcal{O}_{Z,z}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ so it follows that $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,y}$.
ii) Write X' for $X \times_Y Y'$ and let x' be a maximal point of X'. Moreover, let $y' \in Y', x \in X$ and $y \in Y$ be the images of x under the morphisms $X' \to Y'$, $X' \to X$ and $X' \to Y$. By the argument in i), we have that $\mathcal{O}_{X',x'}$ is a direct factor of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}$ so $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{Y',y'}$. it follows that f' is flat (this is the same argument as in the subsequent proof). $\square$

Proof. (of proposition 3.17)
i) $\implies$ *ii)* Every etale morphism is by definition unramified. By the latter lemma f is also flat.
ii) $\implies$ *i)* Consider the following commutative diagram



where Z is an affinoid adic space, g is a closed embedding and h is etale (see proposition 3.8). To show that f is etale, it is enough to show that g is etale, for which it is enough to show that g is an open embedding. This is because open embeddings are easily seen to be etale. But g is a closed embedding and so it is enough to show that it is flat. But to show that g is flat, it is enough to show that s and p are flat. The fact that p is flat follows from the latter lemma (as f is locally quasi-finite, since it is unramified). The fact that s is flat follows from the fact that it is an open embedding as it is it is the pullback of $\Delta : Z \to Z \times_Y Z$ by $g \times id : X \times_Y Z \to Z \times_Y Z$ and Δ is an open embedding (as f is unramified, by 3.8).
iii) $\implies$ *ii)* Let $x \in X$ and let $x' \in X$ be maximal with $x' \succ x$. We have a commutative diagram



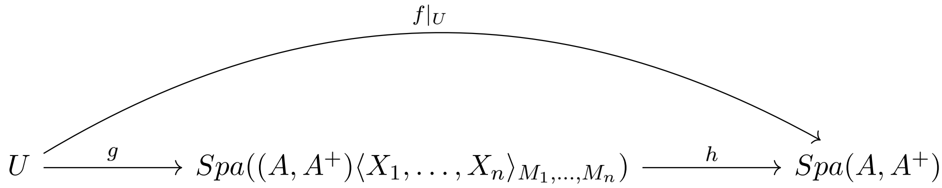
from which it follows that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat (as everything else is flat). Therefore f is flat. From the basic characterization of locally quasi-finite morphisms, it follows that f is locally quasi-finite, by the assumption. By the latter lemma it follows that $(\Omega_{X|Y})_x = 0$ for every maximal point x and so $(\Omega_{X|Y})_x = 0$ for every $x \in X$. It follows that $\Omega_{X|Y} = 0$, so f is unramified by proposition 3.8.
ii) $\implies$ *iii)* Since f is unramified, it is locally quasi-finite. Write A for $\mathcal{O}_{X,x}/m_{f(x)}\mathcal{O}_{X,x}$. By the latter lemma we have that A is finite over $k(f(x))$ and since $\Omega_{X|Y} = 0$ (as f is unramified) we can deduce that $\Omega_{A|k(f(x))} = 0$. From property 3 of universal derivations, $\Omega_{A|k(f(x))}$ is the universal derivation in the sense of rings. Therefore A is a finite separable extension of $k(f(x))$.

$\square$

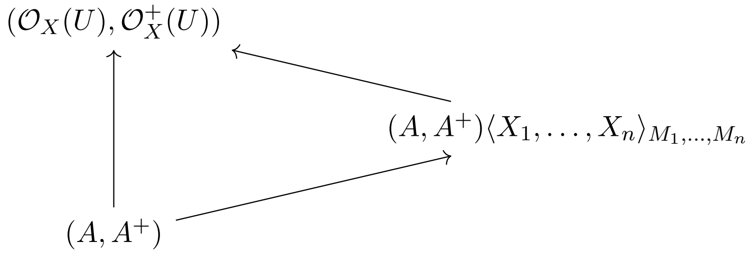
Proposition 3.19. *Etale morphisms are open.*

Proof. In fact we prove that smooth morphisms are open. We proceed in several steps.

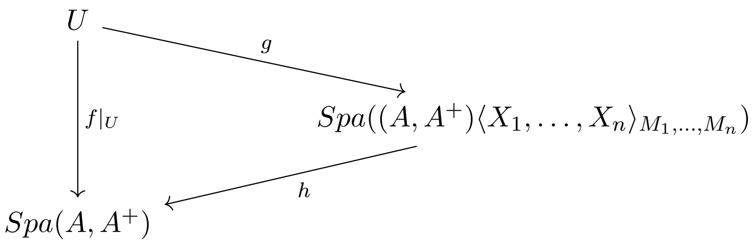
1 A morphism $f : X \to Spa(A, A^+)$ of adic spaces is smooth if and only if for every $x \in X$ there is an open neighborhood U of x in X, finite subsets $M_1, \ldots, M_n \subset A$ with $M_i \cdot A$ open for all i and an etale morphism $g : U \to Spa((A, A^+)\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n})$ such that the following diagram commutes



where h is the natural morphism. Indeed, h is smooth, so if g is etale, then $h \circ g$ is smooth, so f is smooth at x and so f is smooth. Assume now that f is smooth and write Y for $Spa(A, A^+)$. Then, $\Omega_{X|Y}$ is a locally free $\mathcal{O}_X$-module, i.e there is an open affinoid neighborhood U of x and $s_1, \ldots, s_n \in \mathcal{O}_X(U)$ such that $\Omega_{X|Y}|_U$ is a free $\mathcal{O}_X|_U$-module with basis $d_{X|Y}(s_1), \ldots, d_{X|Y}(s_n) \in \Omega_{X|Y}(U)$. Consider finite subsets $M_1, \ldots, M_n$ of A with $M_i \cdot A$ open for all i and $M_i \cdot s_i \in \mathcal{O}_X^+(U)$. Moreover consider the following commutative diagram



where $(A, A^+)\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ maps X_i to s_i. This give rise to the following commutative diagram



and all we are left to do, is show that g is etale. Write Z for $Spa((A, A^+)\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$. Then g is etale at a point $x \in U$ if and only if the $\mathcal{O}_{X,x}$-linear mapping

$$(\Omega_{Z|Y} \otimes_{\mathcal{O}_Z} \mathcal{O}_U)_x \to (\Omega_{X|Y}|_U)_x$$

is bijective, which follows from the above and property 1 of universal derivations.

2 By combining step 1 with corollary 3.15 we get the following: Let $Y \to Spa(A, A^+)$ be a smooth morphism of adic spaces. Then for every $y \in Y$ there is an open affinoid neighborhood of y, a Huber pair (B, B^+) and a morphism of Huber pairs $(A, A^+) \to (B, B^+)$ that is of algebraically finite type such that U is $Spa(A, A^+)$-isomorphic to $Spa(B, B^+)$ and $SpecB \to SpecA$ is smooth.

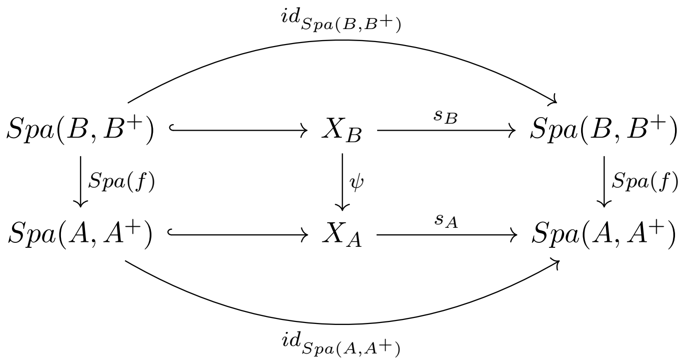
3 Let $f : (A, A^+) \to (B, B^+)$ be an adic morphism of Huber pairs with $f : A \to B$ flat and finitely presented. Moreover, let L be a finite subset of B^+ such that B^+ is the integral closure of $A^+[L]$ in B. Then $Spa(f) : Spa(B, B^+) \to Spa(A, A^+)$ is open. To show this, it is enough to show that the image of $Spa(B, B^+)$ is open is $Spa(A, A^+)$ as it is enough to show that for every rational subset U of $Spa(B, B^+)$, $Spa(f)(U)$ is open, but we may replace B by $\mathcal{O}_{Spa(B,B^+)}(U)$. Consider

$$X_A = \{x \in SpvA : x(a) \leq 1 \text{ for every } a \in A^+ \text{ and } x(a) < 1 \text{ for every } a \in A^{oo}\}$$

and

$$X_B = \{x \in SpvB : x(b) \leq 1 \text{ for every } b \in B^+ \text{ and } x(b) < 1 \text{ for every } b \in B^{oo}\}$$

f gives rise to a morphism $Spv(f) : SpvB \to SpvA$ which restricts to $\psi : X_B \to X_A$. We have retractions $r_A : SpvA \to Spv(A, A^{oo} \cdot A)$ and $r_B : SpvB \to Spv(B, B^{oo} \cdot B)$ that restrict to retractions $s_A : X_A \to Spa(A, A^+)$ and $s_B : X_B \to Spa(B, B^+)$. We have the following commutative diagram



Here comes the crucial part. In [KH94] it is proved that if $f : A \to B$ is flat and finitely presented, then $Spv(f)$ is open. Therefore, since X_B is open in $Spv(f)^{-1}(X_B)$, it follows that $im(\psi)$ is open in X_A. Now $im(Spa(f)) = im(\psi) \cap Spa(A, A^+)$ and so the result follows.

To show that a smooth morphism $f : X \to Y$ of adic spaces is open, we may assume that X and Y are affinoid and by factoring f as in 2, we are left to show that a morphism $g : Spa(B, B^+) \to Spa(A, A^+)$ of adic spaces that is induced by a morphism of Huber pairs $(A, A^+) \to (B, B^+)$ that is of algebraically finite type such that $Spec(B) \to Spec(A)$ is smooth is open. But in such a case one can apply 3 (see [Hub96] remark 1.26).

□

4 Comparison theorems

Huber initially wanted to develop a theory in which etale cohomology of rigid analytic varieties is accessible. The subsection 4.1 is concerned with that. Moreover, in 4.2 we present a comparison theorem in etale cohomology for schemes and adic spaces. This presentation is largely based on the exposition of [Hub96].

4.1 A comparison theorem for rigid analytic varieties and adic spaces

Let X be a rigid analytic variety over K and $r(X)$ its associated adic space. We will prove that the etale topoi associated to X and $r(X)$ are equivalent.

Proposition 4.1. *Let $f : X \to Y$ be a morphism of rigid analytic varieties. Then f is etale if and only if $r(f)$ is etale.*

Proof. We think of X and Y as subsets of $r(X)$ and $r(Y)$ respectively. Assume first that $r(f)$ is etale. Then, as $r(X)$ and $r(Y)$ are analytic, it follows that for every maximal point x of $r(X)$, $\mathcal{O}_{r(X),x}$ is flat over $\mathcal{O}_{r(Y),r(f)(x)}$ and $\mathcal{O}_{r(X),x}/m_{r(f)(x)}\mathcal{O}_{r(X),x}$ is flat over $k(r(f)(x))$. Now since the points of $r(X)$ coming from X are maximal (as by definition those are valuations of rank 1), it follows that $f : X \to Y$ is flat and unramified, i.e etale. Assume now that f is etale. Let U and V be admissible open subsets of X and Y respectively with $f(U) \subset V$. Then, $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_Y(V)$. Therefore, for every $x \in r(X)$, we have that $\mathcal{O}_{r(X),x}$ is flat over $\mathcal{O}_{r(Y),r(f)(x)}$. As f is locally quasi-finite, it follows that $r(f)$ is locally quasi-finite. So, by lemma 3.18 it follows that $(\Omega_{r(X)|r(Y)})_x = 0$ for every $x \in X$.Now every maximal point of $r(X)$ comes from a point in X and so $\Omega_{r(X)|r(Y)} = 0$ so $r(f)$ is unramified. The result follows. □

Proposition 4.2. *Let $f : X \to Y$ be an etale morphism of rigid analytic varieties. Then f is strongly surjective if and only if $r(f)$ is surjective.*

Proof. We think of X and Y as subsets of $r(X)$ and $r(Y)$ respectively. Let us assume first that $f : X \to Y$ is strongly surjective. Then it is enough to show that $r(V) \subset im(r(f))$ for every quasi-compact quasi-separated admissible open subset of Y (as we can cover Y by such subsets). Since f is strongly surjective, there exist quasi-compact admissible open subsets $U_1, \ldots, U_n$ of X with $V = \bigcup_{i=1}^n f(U_i)$. So we have $U_i \subset f^{-1}(V)$ and so $r(U_i) \subset r(f^{-1}(V)) = r(f)^{-1}(r(V))$. Therefore the morphism $r(f) : r(X) \to r(Y)$ restricts to a mapping $f_i : r(U_i) \to r(V)$. Note that f_i is spectral and $r(U_i)$ is quasi-compact, which implies that $im(f_i)$ is a pro-constructible subset of $r(V)$. Therefore $\bigcup_{i=1}^n im(f_i)$ is a pro-constructible subset of $r(V)$ which contains V. Recall that every constructible subset of $r(V)$ is uniquely determined by its intersection with V and so $\bigcup_{i=1}^n im(f_i) = r(V)$. Assume now that $r(f)$ is surjective and let V be a quasi-compact admissible open subset of Y. Then $r(V)$ is a quasi-compact open subset of $r(Y)$. Since f is etale, it follows that $r(f)$ is etale and so it is open. Therefore, there exists a quasi-compact open subset W of $r(X)$ with $V = r(f)(W)$. Since the sets of the form $r(U)$ where U is any quasi-compact admissible open subset of X form a basis of Y, it follows that there are $U_1, \ldots, U_n$ quasi-compact admissible open subsets of X such that $W = \bigcup_{i=1}^n r(U_i)$. We will show that $V = \bigcup_{i=1}^n f(U_i)$. First of all, we note from the above that $f(U_i) \subset V$. So, the morphism $f : X \to Y$ restricts to a mapping $f : U_i \to V$. Given $v \in V$ and since $r(V) = r(f)(W)$ and $W = \bigcup_{i=1}^n r(U_i)$, there is j with $v \in r(f_i)(r(U_j)) = r(f_j)(r(U_j))$. Its follows that $f_j^{-1}(v) \neq \emptyset$ as r commutes with fiber products and therefore $v \in \bigcup_{i=1}^n f(U_i)$.

□

Definition 4.3. *Let $u : B \to C$ be a functor of categories where C is a site. The topology induced on B is the finest one such that u is continuous.*

The following proposition is known, in topos theory, as the comparison lemma.

Proposition 4.4. *Let B be a small category, C a site and $u : B \to C$ a fully faithful functor. Consider B as a site with the topology induced by u. If every object $x \in C$ has a covering $(u(x_a) \to x)$ by objects x_a of b, then u induces an equivalence $B^\sim \to C^\sim$.*

Proof. See [Awo05] C2.2. □

We now prove the main theorem of this section.

Theorem 4.5. *Let X be a rigid analytic variety over a non-archimedean field K and let $r(X)$ be the adic space associated to X. Then the etale topos $X_{et}^\sim$ is equivalent to the etale topos $r(X)_{et}^\sim$.*

Proof. Recall that for a rigid analytic variety X, the etale site X_{et} is the category Et/X equipped with a Grothendieck topology such that $(U_i \xrightarrow{f_i} U)$ is a covering if and only if $\bigsqcup f_i : \bigsqcup U_i \to U$ is strongly surjective. Moreover, for a morphism $f : X \to Y$ of rigid analytic varieties, we get a morphism of sites $f : X_{et} \to Y_{et}$ induced by the functor $Et/Y \to Et/X$ given by $U \mapsto U \times_Y X$. Moreover for an adic space X, the etale site X_{et} is the category Et/X equipped with a Grothendieck topology such that $(U_i \xrightarrow{f_i} U)$ is a covering if and only if $U = \bigcup f_i(U_i)$. Moreover for a morphism $f : X \to Y$ of adic spaces, we get a morphism of sites $f : X_{et} \to Y_{et}$ (exactly as in the case of rigid analytic varieties). Let now X be a rigid analytic variety. We get a functor

$$\rho_X^{-1} : Et/X \to Et/r(X)$$

given by $U \mapsto r(U)$. Since ρ_X^{-1} commutes with fiber products and sends coverings to coverings, we get a morphism of sites

$$\rho_X : r(X)_{et} \to X_{et}$$

Let $f : X \to Y$ be a morphism of rigid analytic varieties. We get a commutative diagram

$$\begin{array}{ccc} r(X)_{et} & \xrightarrow{\rho_X} & X_{et} \\ \downarrow{\scriptstyle r(f)} & & \downarrow{\scriptstyle f} \\ r(Y)_{et} & \xrightarrow{\rho_Y} & Y_{et} \end{array}$$

Moreover recall that the functor r induces an equivalence between the category of rigid analytic varieties over X with quasi-separated structure morphisms and the category of adic spaces over $r(X)$ with quasi-separated structured morphisms.

We write Et/X' for the category of rigid analytic varieties that are etale and quasi-separated over X. Moreover we write $Et/r(X)'$ for the category of adic spaces that are etale and quasi-separated over $r(X)$. Finally, we write X'_{et} and $r(X)'_{et}$ for the sites with underlying categories Et/X' and $Et/r(X)'$ and with topologies induced by the topologies on X_{et} and $r(X)_{et}$ respectively. By the above, ρ_X induces an equivalence

$$\rho'_X : r(X)'_{et} \to X'_{et}$$

Consider the natural morphisms $a : X_{et} \to X'_{et}$ and $b : r(X)_{et} \to r(X)'_{et}$. Since we can cover every object of Et/X be objects of Et/X', by the comparison lemma, $a^{\sim}$ is an equivalence. In the same way $b^{\sim}$ is an equivalence. We have a commutative diagram

$$\begin{array}{ccc} r(X)_{et} & \xrightarrow{\rho_X} & X_{et} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} \\ r(X)'_{et} & \xrightarrow{\rho'_X} & X'_{et} \end{array}$$

Since $a^{\sim}, b^{\sim}, \rho'^{\sim}_X$ are equivalences, so is $\rho^{\sim}_X$.

□

One can prove the following

Corollary 4.6. *Let X be a rigid analytic variety over K and let $r(X)$ be its associated adic space. Moreover let $\mathcal{F}$ be a sheaf of abelian groups on X_{et}. Then for every $q \geq 0$, there is an isomorpism*

$$H^q(X_{et}, \mathcal{F}) \to H^q(r(X)_{et}, \mathcal{F})$$

4.2 A comparison theorem for schemes and adic spaces

In this section we present a comparison theorem between the etale cohomology of adic spaces and the etale cohomology of schemes. This is motivated by the usual comparison theorem for schemes locally of finite type over $\mathbb{C}$ (see [Mil80]). Recall that for a scheme X, a sheaf of abelian groups on X_{et} is called constructible if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes U_i such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.

Fix an algebraically closed non-archimedean field K. Let $f : X \to Y$ be a morphism in the category of schemes that are locally of finite type over K. Consider the natural morphisms of sites $\phi_X : (X^{ad})_{et} \to X_{et}$ and $\phi_Y : (Y^{ad})_{et} \to Y_{et}$ which give a commutative diagram

$$\begin{array}{ccc} (X^{ad})_{et} & \xrightarrow{\phi_X} & X_{et} \\ \downarrow{\scriptstyle f^{ad}} & & \downarrow{\scriptstyle f} \\ (Y^{ad})_{et} & \xrightarrow{\phi_Y} & Y_{et} \end{array}$$

We have the following theorem

Theorem 4.7. *Consider the above situation and let n be a natural number that is prime to the char(K) (in case that is positive). Then for every constructible $\mathbb{Z}/n\mathbb{Z}$-module $\mathcal{F}$ on X_{et} and every positive integer q, the base change morphism*

$$\phi_Y^* R^q f_* \mathcal{F} \to R^q (f^{ad})_* \phi_X^* \mathcal{F}$$

is bijective.

A proof of this theorem is far beyond the scope of this thesis. We can only remark what is mentioned in [Hub96]: that the proof in the case of 0 characteristic is the same as in the usual comparison theorem in [AGV71], while in positive characteristic, the proof is the same as in the comparison theorem in [Ber93]. One can prove the following

Corollary 4.8. *Let X be a scheme locally of finite type over K and let $\mathcal{F}$ be a constructible $\mathbb{Z}/n\mathbb{Z}$-module on X_{et} where n is prime to char(K). Then there is an isomorphism*

$$H^q(X_{et}, \mathcal{F}) \to H^q((X_{et})^{ad}, \mathcal{F})$$

Note. Without the assumption that n is prime to $char(K)$, this fails in general. For example, in [JdJ95], it is proved that for an algebraically closed field K of characteristic p and $\mathbb{A}^1$ the affine line over K, the map

$$H^1(\mathbb{A}^1_{et}, \mathbb{Z}/p\mathbb{Z}) \to H^1(\mathbb{A}^{1,ad}_{et}, \mathbb{Z}/p\mathbb{Z})$$

is injective but not surjective.

Part IV
Appendix

A Topological groups and rings

In this section we present some basic results on topological groups and rings. Most of the results can be found in [Bou98], but there are many other great texts that one can learn this material from, available online, for example [Mur06] and [Mur05].

Definition A.1. *A topological group A is a group with a topology such that $A \times A \to A$, $(a,b) \mapsto ab$ and $A \to A$, $a \mapsto a^{-1}$ are continuous.*

Note. Translations $A \to A$, $a \mapsto ab$ and $a \mapsto ba$ are homeomorphisms.

Proposition A.2. *Let A be a topological group. The set N of open neighborhoods of 0 in A satisfies:*
i) For every $U \in N$ and $c \in U$ there exists $V \in N$ such that $c + V \subset U$
ii) For every $U \in N$ there is $V \in N$ such that $V + V \subset U$
iii) For every $U \in N$, $-U \in N$.
Moreover if G is an abelian group that in addition to $i), ii), iii)$ satisfies (a) every element of N contains 0 and (b) for every $U, V \in N$ there is $W \in N$ such that $W \subset U \cap V$, then there is a unique topology on G making it a topological group with N a fundamental system of neighborhoods of 0.

Definition A.3. *A topological ring A is a ring with a topology such that $(A, +)$ is a topological group and $A \times A \to R$, $(a,b) \mapsto ab$ is continuous.*

Proposition A.4. *The set N of open neighborhoods of 0 in a topological ring A satisfies properties $i), ii), iii)$ of proposition A.1 and*
iv) For every $c \in A$ and every $U \in N$ there is $V \in N$ such that $cV \subset U$ and $Vc \subset U$
v) For every $U \in N$ there is $V \in N$ such that $V \cdot V \subset U$
Moreover if N is a set of subsets of a ring A that in addition to i),ii),iii),iv),v) satisfies (a) every elements of N contains 0 and (b) for every $U, V \in N$ there is $W \in N$ such that $W \subset U \cap V$, then there is a unique topology on A making it a topological ring with fundamental system of neighborhoods of 0 the set N.

Note. The topology of a topological group (or a topological ring) is completely determined by a fundamental system of neighborhoods of 0.

Definition A.5. *Let A be an abelian topological group. We say that a filter F on A is Cauchy if for every neighborhood U of 0 there exists an element E of the filter such that $x - y \in U$ for all $x, y \in E$.*

Definition A.6. *An abelian topological group A is complete if it is Hausdorff and every Cauchy filter on A has a limit. A topological ring is complete if its underlying topological group is complete.*

Note. This is the same as saying that every Cauchy sequence on A converges.

Theorem A.7. *Let A be an abelian topological group (resp. topological ring). Then there exists a complete abelian topological group (resp. topological ring) $\widehat{A}$, called the completion of A, together with a morphism of topological groups (resp. topological rings) $i : A \to \widehat{A}$ that is universal for such topological groups (resp. topological rings).*

Proposition A.8. *Let A be an abelian topological group and $i : A \to \widehat{A}$ its completion. The set of open subgroups of A corresponds bijectively to the set of open subgroups of $\widehat{A}$ under the map that sends an open subgroup H of A to $\overline{i(H)} = \widehat{H}$.*

Proposition A.9. *Let A be an abelian topological group with a sequence of subgroups $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ forming a fundamental system of neighborhoods of 0. Then $\widehat{A} \cong \varprojlim A/A_n$.*

Definition A.10. *A topoogical ring is called adic if there exists an ideal I of A such that $\{I^n\}_{n \in \mathbb{N}}$ forms a fundamental system of neighborhoods of 0 in A. We call I an ideal of definition of A.*

B Spectral spaces

In this section we define spectral spaces and present basic results, which are used in section 1 in order to prove the spectrality results on the valuation and adic spectra. Those are results of [Hoc69] and [Hoc67]. The reader can have a look at the "Spectral Spaces" chapter of [Wed19], but basically all we need is remark 2.1 of [Hub93b], which is presented below.

Definition B.1. *A topological space T is called spectral if one of the following equivalent holds:*
i) $T = SpecR$ for some commutative ring R.
ii) $T = \varprojlim T_i$ for some inverse system $\{T_i\}$ of some finite T_0 spaces.
iii) T is quasi-compact, there exists a basis of quasi-compact open subsets of T that is stable under finite intersections (i.e T is quasi-separated) and T is Sober (meaning that every irreducible closed subset has a unique generic point).

A topological space is called locally spectral if it admits a covering by spectral spaces.

Definition B.2. *A continuous map of locally spectral spaces $f : X \to Y$ is called spectral if for every open spectral subspace V of Y and every open spectral subspace U of $f^{-1}(U)$, the induced map $U \to V$ is quasi-compact.*

Definition B.3. *Let X be a quasi-compact and quasi-separated topological space and $Y \subset X$ a subset of X.*
i) Y is called constructible if it is a finite union of subsets of the form $U \cap (X - V)$ where U, V are quasi-compact open subsets of X
ii) Y is called pro-constructible if it is the intersection of constructible subsets of X.

For a spectral space X we denote X_{cons} the set X together with the topology generated by the constructible open subsets of the topological space X. Alternatively, a map $f : X \to Y$ is spectral if and only if $X \to Y$ and $X_{cons} \to Y_{cons}$ are continuous. The following is part of remark 2.1 in [Hub93b].

Proposition B.4. *Let X be a spectral space and T a pro-constructible subset of X. Then*
i) T is quasi-compact in X and in X_{cons}.
ii) T is constructible if and only if $X \setminus T$ is pro-constructible.
iii) T is spectral (with the subspace topology) and a subset U of T is (pro-)constructible if and only if $U = V \cap T$ for some (pro-)constructible subset V of X.
iv) Let Z be a set and $\mathcal{T}$ a quasi-compact topology on Z. Moreover, let $\mathcal{Z}$ be the set of open and closed subsets of $(Z, \mathcal{T})$ and let $\mathcal{T}'$ be the topology on Z generated by a subset of $\mathcal{Z}$. Then $(Z, \mathcal{T}')$ is spectral and $\mathcal{Z}$ is the set of constructible subsets of $(Z, \mathcal{T}')$.

C Rigid Analytic varieties

In this section we treat the basic theory of rigid analytic varieties. This is far from being a detailed or complete introduction to the theory, for which the reader should consult [Bos14]. We also refer the reader to [Con08] for a short but very motivated introduction to the topic and [Fre04] for an introduction to etale cohomology of rigid analytic varieties.

C.1 Algebraic Preliminaries

Definition C.1. *A non-archimedean field K is a field together with a valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that K is complete with respect to the topology induced by $|\cdot|$*

Throughout this section, K denotes a non-archimedean field.

Proposition C.2. *The valuation on K admits a unique extension to $\overline{K}$ and each finite subextension of $\overline{K}/K$ is complete with respect to the topology induced by the extension of $|\cdot|$.*

Consider the closed unit disk in $\overline{K}^n$

$$\mathbb{B}^n(\overline{K}) = \{(x_1, \ldots, x_n) \in \overline{K}^n : |x_i| \leq 1\}$$

Definition C.3. *We define the n-th Tate algebra $T_n(K) = K\langle X_1, \ldots, X_n\rangle$ that consists of the formal power series of $K[[X_1, \ldots, X_n]]$ that converge on $\mathbb{B}^n(\overline{K})$, i.e*

$$T_n(K) = \{\textstyle\sum_\nu a_\nu X^\nu \in K[[X_1, \ldots, X_n]] : a_\nu \to 0 \text{ as } |v| \to \infty\}$$

We define the so-called Gauss norm on $T_n(K)$ given by

$$||\textstyle\sum_\nu a_\nu X^\nu|| = max_\nu |a_\nu|$$

which has the following property

Proposition C.4. *$T_n(K)$ endowed with the Gauss norm is a Banach K-algebra, i.e a K-algebra that is complete.*

Proposition C.5. *(Maximum Modulus Principle) For every $f \in T_n(K)$ we have*

$$||f|| = max_{x \in \mathbb{B}^n(\overline{K})} |f(x)|$$

Proposition C.6. *Let m be a maximal ideal of $T_n(K)$. Then $T_n(K)/m$ is a finite extension of K.*

Remark. The latter is really important: We will be able to extend the valuation on K to a valuation on $T_n(K)/m$!

Corollary C.7. *There is a bijection between the maximal ideals of $T_n(K)$ and the orbits of $\mathbb{B}^n(\overline{K})$ under the action of $Gal(\overline{K}/K)$.*

Proof. As a full proof of this fact is usually omitted in literature, we proceed to prove it. Consider a map

$$\mathbb{B}^n(\overline{K}) \to MaxT_n(K)$$

given by $x \mapsto m_x = \{f \in T_n(K) : f(x) = 0\}$. We will show that this is surjective. First of all we see that this is well defined, as m_x is the kernel of the evaluation morphism $T_n(K) \to K(x)$ given by $f \mapsto f(x)$ which is surjective. Let now m be a maximal ideal of $T_n(K)$ and consider the embedding $T_n(K)/m \to \overline{K}$. Consider the induced morphism $\phi : T_n(K) \to \overline{K}$ with kernel m and let $x = (\phi(X_1), \ldots, \phi(X_n))$. Then ϕ coincides with the evaluation map we defined above and so $m = m_x$. We will now show that $m_x = m_y$ if and only if x and y belong in the same conjugacy class under the action of $Gal(\overline{K}/K)$ on $\mathbb{B}^n(\overline{K})$, which will complete the proof. Given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{B}^n(\overline{K})$, we have that $m_x = m_y$ if and only if there is a K-isomorphism $K(x) \to K(y)$ that sends x_i to y_i. But such an isomorphism extends to an automorphism of $\overline{K}$, so there is $\gamma \in Gal(\overline{K}/K)$ such that $\gamma(x) = y$. □

Using the above, we are able to describe the underlying set of a rigid analytic variety. Note that if K is algebraically closed, then there is a bijection between $MaxT_n(K)$ and $\mathbb{B}^n(K)$. A few more properties of $T_n(K)$ are the following

Proposition C.8. *$T_n(K)$ is Noetherian, Jacobson and its Krull dimension is n.*

Proposition C.9. *Every ideal of $T_n(K)$ is closed.*

Definition C.10. *An affinoid algebra A is a K-algebra isomorphic to $T_n(K)/a$ for some ideal a of $T_n(K)$.*

As with Tate algebras, we have the following

Proposition C.11. *Let A be an affinoid K-algebra. Then A is Noetherian and Jacobson.*

Consider a K-algebra $A = T_n(K)/a$. For $f \in A$, we define

$$|f|_{sup} = max_{x \in MaxA}|f(x)|$$

where $f(x)$ denotes the residue class of f in A/x. This is called the supremum norm on A. Again, we have a maximum modulus principle

Theorem C.12. *(Maximum Modulus Principle) Let A be an affinoid algebra. For every $f \in A$ we have*

$$|f|_{sup} = max_{x \in Max(A)}|f(x)|$$

For an affinoid K-algebra A, the elements of A can be thought as functions of $MaxA$ in the way we have described. From now on, we will write SpA to denote $MaxA$ together with its K-algebra of functions on A and call it the affinoid K-space associated to A.

A morphism $\sigma : A \to B$ of affinoid K-algebras induces a map

$$\sigma : SpA \to SpB$$

given by $m \mapsto \sigma^{-1}(m)$. Indeed $\sigma^{-1}(m)$ is a maximal ideal of B as we have injections $K \to B/\sigma^{-1}(m) \to A/m$ and A/m is finite over K. In general, a morphism of affinoid K-spaces $SpA \to SpB$ is a morphism induced by a morphism of affinoid K-algebras $B \to A$ in the way we described. That said, the category of affinoid K-spaces can be thought as the opposite of the category of affinoid K-algebras.

Definition C.13. *Consider an affinoid K-space $X = SpA$. The topology on X generated by the sets of the form*

$$X(f; \epsilon) = \{x \in X : |f(x)| \leq \epsilon\}$$

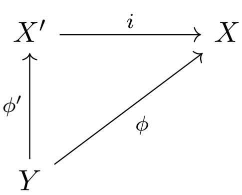
with $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$ is called the canonical topology on X.

Definition C.14. *Let $X = SpA$ be an affinoid K-space and let $f_1, \ldots, f_n, g \in A$ generate A. Then the set*

$$X(\tfrac{f_1,\ldots,f_n}{g}) = \{x \in X : |f_i(x)| \leq |g(x)|\}$$

is called a rational domain of X.

Definition C.15. *Let $X = SpA$ be an affinoid K-space. A subset U of X is called affinoid subdomain of X if there exists a morphism of affinoid K-spaces $i : X' \to X$ such that $i(X') \subset U$ and for every morphism of affinoid K-spaces $\phi : Y \to X$ with $\phi(Y) \subset U$, there exists a unique morphism $\phi' : Y \to X'$ such that the following diagram commutes*



The following theorem is fundamental in rigid geometry

Theorem C.16. *(Gerritzen-Grauert) Let X be an affinoid K-space and $U \subset X$ an affinoid subdomain. Then U is a finite union of rational subdomains of X.*

Let X be an affinoid K-space. For an affinoid subdomain U of X we define $\mathcal{O}_X(U)$ to be the K-algebra of "functions" corresponding to U. This is a presheaf of K-algerbas on the category of affinoid subdomains of X. For $x \in X$ we define

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$$

where U runs in the set of affinoid subdomains that contain x.

Proposition C.17. *$\mathcal{O}_{X,x}$ is a local ring with maximal ideal $m_x \mathcal{O}_{X,x}$.*

Theorem C.18. *(Tate's Acyclicity) Let X be an affinoid K-space. The presheaf $\mathcal{O}_X$ is $\mathcal{U}$-acyclic for all finite coverings $\mathcal{U} = (U_i)$ of X by affinoid subdomains.*

C.2 Rigid Analytic Varieties

In order to define rigid analytic varieties, we restrict our attention to a special kind of Grothendieck topologies. in particular, given a set X, we consider a Grothendieck topology on a category of subsets of X (with inclusions) and with coverings $(U_i \to U)_{i \in I}$ being set-theoretic coverings, i.e $U = \bigcup U_i$. We call X with the latter Grothendieck topology a G-topological space.

Definition C.19. *Let X be an affinoid K-space. Consider the category of affinoid subdomains of X with inclusions and endow it with a Grothendieck topology such that $(U_i \to U)_{i \in I}$ is a covering whenever I is finite and $U = \bigcup_{i \in I} U_i$. This is called the weak topology on X.*

Definition C.20. *Let X be an affinoid K-space.*
i) A subset $U \subset X$ is called admissible open if there exists a covering $U = \bigcup_{i \in I} U_i$ by affinoid subdomains U_i such that for every morphism of affinoid K-spaces $\phi : Z \to X$ satisfying $\phi(Z) \subset U$, the cover $(\phi^{-1}(U_i))$ of Z admits a finite refinement by affinoid subdomains.
ii) A covering $V = \bigcup_{j \in J} V_j$ of an admissible open subset $V \subset X$ by admissible open subsets $V_j \subset X$ is called admissible if for every morphism $\phi : Z \to X$ of affinoid K-spaces satisfying $\phi(Z) \subset V$, the cover $(\phi^{-1}(V_j))_{j \in J}$ of Z admits a finite refinement by affinoid subdomains.
The Grothendieck topology on X with admissible open subsets and admissible coverings is called the strong topology on X.

Proposition C.21. *The strong Grothendieck topology on an affinoid K-space X has the following properties*
G_0) $\emptyset$ and X are admissible open
G_1) If $(U_i)_{i \in I}$ is an admissible cover of an admissible open subset U of X and V is a subset of U such that $U_i \cap V$ is admissible open for all i, then V is admissible open.
G_2) If $(U_i)_{i \in I}$ is a cover of an admissible open subset U by admissible open subsets $U_i \subset X$ such that it admits an admissible cover of U as refinement ,then $(U_i)_{i \in I}$ is itself admissible.

Proposition C.22. *Every morphism $\phi : X \to Y$ of affinoid K-spaces is continuous with respect to the strong Grothendieck topologies on X and Y.*

Proposition C.23. *Let X be an affinoid K-space and $\mathcal{F}$ a sheaf with respect to the weak Grothendieck topology on X. Then $\mathcal{F}$ admits a unique extension with repsect to the strong topology on X. In particular, by Tate's acyclicity theorem we may extend the structure sheaf $\mathcal{O}_X$ to get a sheaf with respect to the strong Grothendieck topology on X.*

Definition C.24. *A G-ringed K-space is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of algebras on it. If in addition to the above, the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$, then $(X, \mathcal{O}_X)$ is called a locally G-ringed K-space.*

Definition C.25. *A morphism of G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (ϕ, ϕ^*) where $\phi : X \to Y$ is a morphism of G-topological spaces and ϕ^* is a system of K-homomorphisms $\phi_V^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(\phi^{-1}(V))$,for every admissible open subset V, that are compatible with restrictions, i.e the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{O}_Y(V) & \xrightarrow{\phi_V^*} & \mathcal{O}_X(\phi^{-1}(V)) \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(V') & \xrightarrow{\phi_{V'}^*} & \mathcal{O}_X(\phi^{-1}(V'))
\end{array}$$

Furthermore if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally G-ringes K-spaces, then (ϕ, ϕ^*) is a morphism between them if in addition to the above, the ring homomorphisms

$$\phi_x^* : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$$

are local for all $x \in X$.

Now every affinoid K-space can be viewed naturally as a locally G-ringed space $(X, \mathcal{O}_X)$, where X is endowed with the strong Grothendieck topology and O_X is its structure sheaf. Moreover, if $\phi : X \to Y$ is a morphism of affinoid K-spaces, then it induces a morphism of locally G-ringed K-spaces. In fact we have the following

Proposition C.26. *Let X, Y be affinoid K-spaces. The map which sends a morphism $X \to Y$ of affinoid K-spaces to the induced morphism of locally G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a bijection.*

Definition C.27. *A rigid analytic variety over K is a locally G-ringed K-space $(X, \mathcal{O}_X)$ such that*
i) X satisfies conditions G_0, G_1, G_2 and
ii) X admits an admissible covering $(X_i)_{i \in I}$ where $(X_i, \mathcal{O}_{X|X_i})$ is an affinoid K-space for all $i \in I$.

A morphism of rigid analytic varieties over K is a morphism in the sense of locally G-ringed K-spaces. For an admissible open subset $U \subset X$, we call $(U, \mathcal{O}_{X|U})$ an open subspace of $(X, \mathcal{O}_X)$. We can glue rigid analytic varieties in the following way

Proposition C.28. *Consider the following data*
i) rigid analytic varieties X_i, $i \in I$
ii) open subspaces $X_{ij} \subset X_i$ and isomoprhisms $\phi_{ij} : X_{ij} \to X_{ji}$ for all $i, j \in I$
such that
a) $\phi_{ij} \circ \phi_{ij} = id$, $X_{ii} = X_i$ and $\phi_{ii} = id$ for all $i, j \in I$
b) ϕ_{ij} induces isomorphisms $\phi_{ijk} : X_{ij} \cap X_{ik} \to X_{ji} \cap X_{jk}$ that satisfy $\phi_{ijk} = \phi_{kji} \circ \phi_{ikj}$ for all $i, j, k \in I$.
Then there exists a unique (up to isomorphism) rigid analytic variety X over K with an admissible covering $(X'_i)_{i \in I}$ and isomorphisms $\psi_i : X_i \to X'_i$ that restrict to isomorphisms $\phi_{ij} : X_{ij} \to X'_i \cap X'_j$ such that the following diagram commutes

$$\begin{array}{ccc}
X_{ij} & \xrightarrow{\psi_{ij}} & X'_i \cap X'_j \\
\downarrow{\scriptstyle \phi_{ij}} & & \downarrow{\scriptstyle id} \\
X_{ji} & \xrightarrow{\psi_{ji}} & X'_j \cap X'_i
\end{array}$$

We can also glue morphisms of rigid analytic varieties in the following way

Proposition C.29. *Let X, Y be rigid analytic varieties over K and $(X_i)_{i \in I}$ an admissible cover of X. Let $\phi_i : X_i \to X$ be morphisms of rigid analytic varieties such that $\phi_i|_{X_i \cap X_j} = \phi_j|_{X_i \cap X_j}$ for all $i, j \in I$. There is a unique morphism of rigid analytic varieties $\phi : X \to Y$ such that $\phi|_{X_i} = \phi_i$.*

Note. This is of course exactly the same as gluing locally ringed spaces (say schemes), except that now we have to deal with Grothendieck topologies, which doesn't make it much different.

Proposition C.30. *Let X be a rigid analytic variety over K and Y an affonoid K-space. Then $Hom(X,Y) \to Hom(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ given by $\phi \mapsto \phi_Y^*$ is bijective.*

Let us now give a few examples.

Example C.31. *In rigid geometry the "affine" line is realized as*

$$\varinjlim_{n\geq 1} SpK\langle \overline{\omega}^n X\rangle$$

where $\overline{\omega} \in K$ is such that $|\overline{\omega}| < 1$. The underlying set is

$$K = \bigcup_{n\geq 1} \mathbb{D}(0, |\overline{\omega}|^{-n})$$

where $\mathbb{D}(0,r) = \{x \in K : |x| \leq r\}$. In fact (as we will see) this is the rigid analytification of $SpecK[X]$ (the affine line over K).

Example C.32. *Let $\overline{\omega} \in K$ be such that $|\overline{\omega}| < 1$. We considee the rigid analytic variety*

$$\bigcup_{n\in\mathbb{Z}} SpK\langle \tfrac{\overline{\omega}^{n+1}}{X}, \tfrac{X}{\overline{\omega}^n}\rangle$$

which has underlying set

$$\bigcup_{n\in\mathbb{Z}}\{x \in K : |\overline{\omega}|^{n+1} \leq |x| \leq |\overline{\omega}|^n\} = \bigcup_{n\geq 1}\{x \in K : |\overline{\omega}|^n \leq |x| \leq |\overline{\omega}|^{-n}\} = K^\times$$

This is the rigid analytification of G_m.

Example C.33. *Write $G_{m,K}^{rig}$ for the rigid analytic variety of the last example.For $q \in K$ with $|q| < 1$, the Tate curve is given by $G_{m,K}^{rig}/q^{\mathbb{Z}}$. Let us describe what this means. Note that $G_{m,K}^{rig}$ remains the same regardless of the choice of $\overline{\omega}$. This we may choose $\overline{\omega}$ such that $|\overline{\omega}|^k = |q|$ for some $k \geq 2$. We may cover $G_{m,K}^{rig}$ by A_n, $n \in \mathbb{Z}$ where $A_n = SpK\langle \frac{\overline{\omega}^{n+1}}{X}, \frac{X}{\overline{\omega}^n}\rangle$. Note that multiplication by q induces an isomorphism from A_n to A_{n+k} and so the Tate curve can be realized as the union of k copies of rigid varieties of the form A_n glued along their boundary in the way multiplication by q forces them to.*

Definition C.34. *Let X and Y be rigid analytic varieties over K.*
i) X is called quasi-compact if it admits a finite admissible affinoid cover. A morphism $f : X \to Y$ is called quasi-compact if for every quasi-compact open subspace $Y' \subset Y$, $f^{-1}(Y')$ is quasi-compact.
ii) A morphism $f : X \to Y$ is called separated (resp. quasi-separated) if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion (resp. quasi-compact).
iii) X is called separated (resp. quasi-separated) if the structural morphism $X \to SpK$ is separated (resp. quasi-separated).

C.3 Fiber Products of Rigid Analytic Varieties

Definition C.35. *Let A be a Banach K-algebra and E, F two Banach A-modules. Consider the tensor product $E \otimes_A F$. For $x \in E \otimes_A F$ we define*

$$||x|| = inf_{x=\sum e_i\otimes f_i} max||e_i|| \cdot ||f_i||$$

where the sum runs over all such representations of x. The latter is easily seen to be a seminorm. We define $E\widehat{\otimes}_A F$ to be the completion of $E \otimes_A F$, which is an A-module and call it and completed tensor product of E and F. We have natural maps $e : E \to E\widehat{\otimes}_A F$ and $f : F \to E\widehat{\otimes}_A F$. The completed tensor product with the above maps is universal in the sense that for any Banach A-module M and any continuous A-linear maps $\phi : E \to M$ and $\psi : F \to M$ there exists a unique continuous A-linear map $\phi\widehat{\otimes}\psi : E\widehat{\otimes}_A F \to M$ such that $(\phi\widehat{\otimes}\psi) \circ e = \phi$ and $(\phi\widehat{\otimes}\psi) \circ f = \psi$. By the universal property of tensor products we see that there is a map $\phi \otimes \psi : E \otimes_A F \to M$ and by the universal property of completions we then get $\phi\widehat{\otimes}\psi$.

Proposition C.36. *Let $Sp(B) \to Sp(A)$ and $Sp(C) \to Sp(A)$ be morphisms of affinoid spaces. Then $B \otimes_A C$ is an affinoid algebra and $Sp(B\widehat{\otimes}_A C)$ is the fiber product of the morphisms above.*

Proof. First of all the morphisms $Sp(B) \to Sp(A)$ and $Sp(C) \to Sp(A)$ are induced by morphisms of K-algebras $A \to B$ and $A \to C$. Hence we write $A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_k)$ and $A\langle X_1, \dots, X_m\rangle/(g_1, \dots, g_l)$ for B and C respectively, for some $f_1, \dots, f_k, g_1, \dots, g_l$. It follows that $A \otimes_A C =$

$$A\langle X_1, \otimes 1, \dots, X_n \otimes 1, 1 \otimes X_1, \dots, 1 \otimes X_m\rangle/(f_i \otimes 1, 1 \otimes g_j, 1 \leq i \leq k, 1 \leq j \leq l)$$

and since A is an affinoid K- algebra, so is the latter. It follows directly from the universal property of completed tensor product that $Sp(B\widehat{\otimes}_A C)$ is the fuber product. □

Proposition C.37. *Fiber products exist in the category of rigid analytic varieties over K*

Proof. Assume that $Y \to X$ and $Z \to X$ are morphisms of rigid analytic varieties over K. If X, Y, Z are affinoid, then $Sp(\mathcal{O}_Y(Y)\widehat{\otimes}_{\mathcal{O}_X(X)}\mathcal{O}_Z(Z))$ is their fiber product. The general case follows by gluing. □

C.4 Rigid Analytification

Fix a non-archimedean field K. We will construct a functor from the category of schemes locally of finite type over K to the category of rigid analytic varieties over K. In particular we prove the following

Theorem C.38. *Let X be a scheme locally of finite type over K. There exists a rigid analytic variety X^{rig} over K with a morphism of locally G-ringed spaces $i : X^{rig} \to X$ such that for any rigid analytic variety Y over K, any morphism of locally G-ringed spaces $Y \to X$ factors uniquely through $X^{rig} \to X$ for some morphism of rigid analytic varieties $Y \to X^{rig}$. We call X^{rig} the rigid analytification of X.*

We need the following

Lemma C.39. *Let X be an affine scheme over K and Y a rigid analytic vatiery over K. The set of morphisms of locally G-ringed spaces $Y \to X$ corresponds bijectively to the set of K-algebra homomorphisms $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$.*

Proposition C.40. *Let $X = Spec(K[X_1, \dots, X_n]/I)$ and r be an element of K with $0 < |r| < 1$. Then $X^{rig} = \bigcup_{i\geq 0} Sp(K\langle r^i X_1, \dots, r^i X_n\rangle/(I))$ is the rigid analytification of X.*

Proof. We consider the sequence of natural morphisms of K-algebras

$$K[X_1, \dots, X_n]/I \to \cdots \to K\langle r^i X_1, \dots, r^i X_n\rangle/(I) \to \cdots \to K\langle X_1, \dots, X_n\rangle/(I)$$

that gives rise to inclusions

$$SpK\langle X_1, \dots, X_n\rangle/(I) \hookrightarrow \cdots \hookrightarrow SpK\langle r^i X_1, \dots, r^i X_n\rangle/(I) \hookrightarrow \dots$$

which explains why we can glue them to make $\bigcup_{i\geq 0} Sp(K\langle r^i X_1, \dots, r^i X_n\rangle/(I))$. There are morphisms of locally G-ringed spaces $SpK\langle r^i X_1, \dots, r^i X_n\rangle/(I) \to X$. The latter morphisms are easily seen to agree on intersections and so there exists a morphism of locally G-ringed spaces $i : \bigcup_{i\geq 0} Sp(K\langle r^i X_1, \dots, r^i X_n\rangle/(I)) \to X$. We now verify the universal property. Let Y be a rigid analytic variety over K with a morphism of locally G-ringed spaces $f : Y \to X$. Without loss of generality we may assume that $Y = SpB$ for some Tate algebra B. The morphism $Y \to X$ is induced by a $K-$algebra homomorphism $K[X_1, \dots, X_n]/I \to B$. It is enough to show that there exists i such that the latter fctors as

$$K[X_1, \dots, X_n]/I \to K\langle r^i X_1, \dots, r^i X_n\rangle/(I) \to B$$

for some morphism of topological rings $K\langle r^iX_1,\dots,r^iX_n\rangle/(I) \to B$. We consider the composition

$$K[X_1,\dots,X_n]/I \to K[X_1,\dots,X_n]/I \to B$$

and let y_i be the image in B of X_i under the above morphism. From the maximum modulus principle there exists an i such that $||y_k||_{sup} \leq |r|^{-i}$ for all k. Therefore we can extend the composition morphism to a morphism $K\langle r^iX_1,\dots,r^iX_n\rangle \to B$ which factors through a morphism $K\langle r^iX_1,\dots,r^iX_n\rangle/(I) \to B$ which is the desired one. □

Proof. (of theorem C.38.) Let X be covered by affine open subschemes X_i. Those admit analytifications $f_i : X_i^{rig} \to X_i$. Moreover, from the universal property it follows that for all i,j, $f_i^{-1}(X_i \cap X_j)$ is the analytification of $X_i \cap X_j$. Therefore gluing, we can construct a rigid analytic variety X^{rig} over K with a morphism of locally G-ringed spaces $X^{rig} \to X$ which is the rigid analytification of X. □

Remark. Let $f : X \to Y$ be a morphism of affine schemes that are locally of finite type over K. Consider the analytifications X^{rig} and Y^{rig} of X and Y respectively. There is a morphism of locally G-ringed spaces $i_X : X^{rig} \to X$ and thus if we compose the latter with f (viewed as a morphism of the underlying locally ringed spaces of X and Y), we get a morphism of locally G-ringed spaces $X^{rig} \to Y$. Therefore from the universal property of analytifications, there should be a morphism $X^{rig} \to Y^{rig}$ of rigid analytic varieties over K. We denote the latter morphism by f^{rig}. It is easy to see that for two morphisms $f : X \to Y$ and $g : Y \to Z$ of schemes locally of finite type over K, $(gf)^{rig} = g^{rig}f^{rig}$

Corollary C.41. *We conclude that mapping a scheme X to its analytification X^{rig} and a morphism f of schemes to f^{rig} gives a functor from the category of schemes that are locally of finite type over K to the category of rigid analytic varieties over K.*

Example C.42. *The rigid analytification of $X = SpecK$ is the rigid variety $X^{rig} = SpK$.*

Example C.43. *The rigid analytification of $X = SpecK[T]$ is the rigid variety $X^{rig} = \cup_{i\geq 0} SpK\langle r^iT\rangle$. Note that since K is algebraically closed, $SpK\langle r^iT\rangle$ can be thought as the closed disk of radius r^{-i} and so the analytification of the affine line is a union of closed disks of infinitely increasing radius, which geometrically makes sense.*

C.5 The etale site of a rigid analytic variety

Definition C.44. *A morphism $f : X \to Y$ of rigid analytic varieties over K is called etale if for every $x \in X$ the induced morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat and unramified.*

Definition C.45. *An etale morphism $f : X \to Y$ of rigid analytic varieties is called strognly surjective if for every quasi-compact admissible open subset V of Y there exist finitely many quasi-compact admissible open subsets U_i of X such that $V = \bigcup f(U_i)$.*

Definition C.46. *The etale site X_{et} if a rigid analytic variety is the category Et/X of the rigid analytic varieties that are etale over X equipped with a Grothendieck topology such that $(U_i \xrightarrow{f_i} U)$ is a covering if and only if $\bigsqcup f_i : \bigsqcup U_i \to U$ is strongly surjective.*

D Cech Cohomology

In this section we introduce Cech cohomology, by following closely the exposition of [Wed19]. We need propositions D.5 and D.6 to prove Tate's acyclicity theorem. Fix a topological space X. Moreover, let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be open covers of X. for $(i_0,\dots,i_p) \in I^{p+1}$ we define

$$U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$$

Moreover for a subspace Y of X we define

$$\mathcal{U}|_Y = (U_i \cap Y)_{i\in I}$$

and

$$\mathcal{U} \times \mathcal{V} = (U_i \cap V_j)_{i\in I, j\in J}$$

and we set $\check{C}^p(\mathcal{U},\mathcal{F}) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\ldots i_p})$ and define $d : \check{C}^n(\mathcal{U},\mathcal{F}) \to \check{C}^{n+1}(\mathcal{U},\mathcal{F})$ given by

$$d(f)_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k f_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}$$

It is easy to see that $d^2 = 0$ and so we obtain a complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ called the Cech complex associated to $\mathcal{F}$ and the covering $\mathcal{U}$.

Definition D.1. *The cohomology groups $H^i(\check{C}^\bullet(\mathcal{U},\mathcal{F}))$ are called the Cech cohomology groups associated to $\mathcal{F}$ and the covering $\mathcal{U}$ and are denoted by $\check{H}^i(\mathcal{U},\mathcal{F})$.*

Definition D.2. *We say that $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $j \in J$ there exists $i_j \in I$ such that $V_i \subset U_{i_j}$, i.e there is a map $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$.*

Let $\mathcal{V}$ be a refinement of $\mathcal{U}$ and choose a map $\tau : J \to I$ as in the above definition. Then τ induces homomorphisms of complexes $\tau^\bullet : \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet(\mathcal{V},\mathcal{F})$ and the induced homomorphism

$$\check{H}^i(\tau^\bullet) : \check{H}^i(\mathcal{U},\mathcal{F}) \to \check{H}^i(\mathcal{V},\mathcal{F})$$

is independent of the choice of τ. This gives the following definition.

Definition D.3. *We define the Cech cohomology on X with values in $\mathcal{F}$ given by*

$$\check{H}^i(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U},\mathcal{F})$$

where $\mathcal{U}$ run over the set of open covers of X preordered by refinement.

Definition D.4. *$\mathcal{U}$ is called $\mathcal{F}$-acyclic if the homomorphism $\epsilon : \mathcal{F}(X) \to \check{C}^\bullet(\mathcal{U},\mathcal{F})$ induces an isomorphism $\mathcal{F}(X) \to \check{H}^0(\mathcal{U},\mathcal{F})$ and $\check{H}^i(\mathcal{U},\mathcal{F}) = 0$ for all $i \geq 1$.*

We have the following two propositions

Proposition D.5. *Let $\mathcal{F}$ be a presheaf of abelian groups and let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be coverings of X such that $\mathcal{V}|_{U_{i_0\cdots i_n}}$ is $\mathcal{F}$-acyclic for all $(i_0,\ldots,i_n) \in I^{n+1}$ and all $n \geq 0$. Then we have*
i) If $\mathcal{U}_{V_{i_0\cdots i_n}}$ is $\mathcal{F}$-acyclic for all $(i_0,\ldots,i_n) \in I^{n+1}$ and all $n \geq 0$, then $\mathcal{U}$ is $\mathcal{F}$-acyclic if $\mathcal{V}$ is $\mathcal{F}$-acyclic.
ii) If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\mathcal{U}$ is $\mathcal{F}$-acyclic if and only if $\mathcal{V}$ is $\mathcal{F}$-acyclic.
iii) $\mathcal{U} \times \mathcal{V}$ is $\mathcal{F}$-acyclic if and only if $\mathcal{V}$ is $\mathcal{F}$-acyclic.

Proposition D.6. *Let $\mathcal{B}$ be a basis of the topological space X that is stable under finite intersections. Let $\mathcal{F}'$ be a presheaf of abelian groups on $\mathcal{B}$ and $\mathcal{F}$ a presheaf of abelian groups on X given by*

$$\mathcal{F}(V) = \varprojlim_{U\subset V, U\in\mathcal{B}} \mathcal{F}'(U)$$

Assume that for every $U \in \mathcal{B}$ and every open cover $\mathcal{U}$ of U, the presheaf $\mathcal{F}$ is $\mathcal{U}$-acyclic. Then $\mathcal{F}$ is a sheaf and for every open subset U of X and every $i \geq 1$, the groups $H^i(U,\mathcal{F})$ and $\check{H}^i(U,\mathcal{F})$ are isomorphic.

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