SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Local Model Structures on Presheaves

av

Isak Ohlsson Ångnell

2021 - No M6

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Local Model Structures on Presheaves

Isak Ohlsson Ångnell

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Dan Petersen

2021

Acknowledgements

I am very thankful to have had Dan Petersen as supervisor for this thesis, for his support and guidance and for suggesting such an interesting field of study. I also extend my gratitude to Kathryn Hess and Nima Rasekh for introducing me to homotopy theory. This thesis is dedicated to my friends, family and my partner Mollie.

Abstract

We give an exposé of the current understanding of local model structures on presheaves and suggest possible future results. We give an introduction to combinatorial model categories, left Bousfield localizations and the injective and projective model structures on categories of presheaves. After this, we present Jardine's local model structure on simplicial presheaves and describe Dugger, Hollander and Isaksen's way of obtaining said model structure using hyperdescent. This is followed by descriptions of other local model structures constructed by Ayoub, Drew and Meadows.

Contents

Chapter 1

Introduction

This Master's thesis concerns presheaves taking values in specific kinds of model categories. In a first course in algebraic geometry, the reader may encounter sheaves of abelian groups, rings or modules. In homotopical algebra, we are interested in sheaves taking values in some *higher category* commonly said to be *presented* by a *model category*. The latter is an ordinary category equipped with a *model structure*.

The study of model categories began in 1967 with Daniel Quillen's *Homotopical Algebra* [Q]. Model categories were modeled after the category of topological spaces and the category of chain complexes, inspired by the homotopy theory being used there. Since then, the purpose of studying them have shifted and model categories are now primarily used to study the aforementioned higher categories. Higher categories are categories which do not only have objects and arrows, but also arrows between the arrows, and arrows between those arrows, *ad infinitum*. Working in such a setting requires some extra care. We have attached a section in the appendix, Section 8.4, explaining briefly how a reader may obtain an ∞-category[1], a sort of higher category, from a model category via a process called *simplicial localization*. Understanding this transition is not necessary to read this report, although possible implications should become more apparent.

We are in this paper particularly interested in categories of presheaves. However, we do not consider presheaves to necessarily associate an abelian group or ring to each open subset of a topological space; instead we are content with regarding presheaves as contravariant functors between two categories. We will in this paper often consider source categories endowed with a *Grothendieck Topology*, defined in Section 4.1, and we will let the target category be a model category.

There are a number of different conditions we may impose on this model category, of which requiring it to be *combinatorial* is the most important for this paper. Historically,

[1]Technically, what we mean by ∞-categories are $(\infty, 1)$-categories. These are the only sort of higher category we are concerned with however and we will stick to calling these ∞-categories for short.

combinatorial model categories were introduced by Jeff Smith in seminars, although to the author's knowledge, no printed source has yet been published. Instead numerous results inspired by Smith have been produced by a number of people, see for example [BAR] and [D1]. Oftentimes combinatorial model structures allow us to construct other model structures. If the target category is a combinatorial model category, we get induced model structures on the category of presheaves, from which we may construct new model structures by performing a *left Bousfield localization*.

A prominent result utilizing a left Bousfield localization is [DHI, Thm. 1.2 & Thm. 1.3]. The authors show in said paper that interesting model structures on simplicial presheaves, i.e. presheaves taking values in the category of simplicial sets, may be obtained via left Bousfield localizations with respect to a class of morphisms which they call hypercovers. These model structures was first described by [J] and [BL] and are *local* in the sense that the weak equivalences may be checked stalkwise if the site has enough points.

Since the work done by Jardine [J] and Dugger, Hollander and Isaksen [DHI], there have been a number of similar results where the presheaf model category varies. The original goal of this thesis was to prove the existence of a local model structure on left proper simplicial combinatorial model categories, but alas, no definitive results are found thus far. We do however make some remarks in the last chapter regarding interesting and potentially fruitful approaches to finding such structures.

The structure of this paper is as follows: We begin with a chapter which mainly concerns combinatorial model categories, although we also introduce some necessary constructions such as homotopy (co)limits and (co)ends. This is followed up by a chapter introducing left Bousfield localizations which as mentioned above will be used to, from an induced model structure on a presheaf category, construct a local model structure on the same presheaf category.

The last three chapters specializes to talking about categories of presheaves. Chapter four introduces presheaves in a category theoretic language and describes two natural model structures, the injective and projective model structures, which we may equip a presheaf category with, under "nice" conditions. We also introduce *descent conditions* which one may impose on a presheaf to obtain some kind of local-to-global property. The reader may be familiar with the difference between a sheaf and a presheaf, which we may formulate as "a sheaf is a presheaf which satisfies Čech descent". We will in particular be interested in a condition called hyperdescent, where we require even stronger properties of the presheaf. To obtain the local model structures we will combine hyperdescent with a left Bousfield localization.

Constructing these local model structures is the content of chapter five. We give an overview over known such structures, with some emphasis put on the local model structures on simplicial presheaves first described by Jardine and Blander. The last chapter concerns circumstances under which the author conjectures it to be possible to construct local model structures, although as of yet no such result exists to the author's knowledge.

1.1 Knowledge Expected of the Reader

The reader is expected to have taken a course in homotopical algebra and to be familiar with model categories and homotopy categories. It does not hurt to know some things about quasi-categories either - the authors favorite way to think about ∞-categories. Classic references regarding model categoriesinclude [H], [HOV], and a more modern source is [R3].

As we are interested in presheaves it might be helpful to know some algebraic geometry as well. A standard reference in this field is Hartshorne's book *Algebraic Geometry* [HAR], although the reader wishes to include Richard Borcherds' video lectures for good intuition [BO1] [BO2].

1.2 Notation

Here we introduce common notation. A lot of the definitions appear in the text, but compiling common ones here ensures that they are easy to find when needed.

1. We write $(\mathcal{M}, \mathrm{WE}, \mathrm{Cof}, \mathrm{Fib})$ for a model category with WE being the weak equivalences, Cof being the cofibrations and Fib being the fibrations. Often we will only use $\mathcal{M}$ to reference the model category.
2. We take $\bullet \twoheadrightarrow \bullet$ to mean a fibration, $\bullet \rightarrowtail \bullet$ to mean a cofibration and $\bullet \xrightarrow{\sim} \bullet$ to mean a weak equivalence.
3. We take $\cong$ to mean isomorphism and $\simeq$ to mean weakly equivalent.
4. Set is the category of sets.
5. sSet is the category of simplicial sets. When referenced as a model category without further specified structure, the classical Kan-Quillen model structure is intended.
6. Let S, T be classes of morphisms such that S is the morphisms with the left lifting property (LLP) with respect to T and such that T is the class of morphisms with the right lifting property (RLP) with respect to S. Then we write $S \boxslash T$.
7. Letting S be a class of morphisms in a category. Then $S^{\boxslash}$ is the class of morphisms which have the RLP with respect to S and ${}^{\boxslash}S$ is the class of morphisms which have the LLP with respect to S.
8. Subscripts on categories often indicate which model structure is intended, e.g. $\mathrm{sSet}_{\mathrm{Joyal}}$ is the category of simplicial sets equipped with the Joyal model structure.
9. With ∞-categories we will always mean $(\infty, 1)$-categories. At times these will be used interchangeably with quasi-categories.
10. Let $\mathcal{C}$ be a category and let X be an object in $\mathcal{C}$. We write rX for the representable Set-valued presheaf $\hom_{\mathcal{C}}(-, X)$.

    If $\mathcal{M}$ is a symmetric monoidal model category with unit 1, we will by $y_{\mathcal{M}}X$ mean the $\mathcal{M}$-valued presheaf on $\mathcal{C}$ given by $y_{\mathcal{M}}X = 1 \cdot rX$. If $\mathcal{M}$ is the category of simplicial sets with the classical Kan-Quillen model structure we will often by abuse of notation write rX instead of $y_{\mathrm{sSet}_{\text{Kan-Quillen}}}X$.
11. Suppose that $\mathcal{C}$ is enriched over a symmetric monoidal model category $\mathcal{D}$. We write $\hom_{\mathcal{C}}^{\mathcal{D}}(-, -)$ for the hom-functor which takes objects in $\mathcal{C}$ to an object in $\mathcal{D}$. Sometimes the superscript if dropped if no confusion may be had.

    If $\underline{\hom}$ is underlined, this means the internal hom in a category.

    For a simplicial category, we instead of hom use $\mathrm{map}(-, -)$ for the hom functor, unless the simplicial category is sSet or the category of simplicial presheaves in which case we use $\mathrm{Map}(-,-)$ with capital M.

12. For a functor $F : \mathcal{M} \to \mathcal{C}$ with $\mathcal{M}$ a model category and $\mathcal{C}$ any category, we write $\mathbb{L}F$ for the left derived functor of F and $\mathbb{R}F$ for the right derived functor of F.

13. We will interchangeably use both $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ and $\mathcal{D}\,\mathrm{pre}(\mathcal{C})$ for the presheaf category with source category $\mathcal{C}$ and target category $\mathcal{D}$.

14. For a bifunctor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ we will use $\int_{c:\mathcal{C}} F(c,c)$ to denote the end of F and $\int^{c:\mathcal{C}} F(c,c)$ to denote the coend of F.

15. For a model category $\mathcal{M}$ and a class of morphisms H for which the left Bousfield localization of $\mathcal{M}$ exists, we write $\mathcal{L}_H\mathcal{M}$ for the model category obtained by said localization.

    If $\mathcal{M}$ is a model category enriched over a model category $\mathcal{V}$ and H is a class of morphisms for which the enriched left Bousfield localization of $\mathcal{M}$ exists, we write $\mathcal{L}_{H/\mathcal{V}}\mathcal{M}$ for the obtained model category.

Chapter 2

Prerequisites

This paper concerns functor categories of a special kind. Namely, we wish to work with presheaves taking values in some fixed left proper combinatorial model category enriched over $\mathrm{sSet}_{\text{Kan-Quillen}}$. Here we introduce and explore such a target category, as well as introduce some general constructions constructions, e.g. homotopy (co)limits and (co)ends. The concepts from model category theory are mostly standard and may be found in either [H] or [HOV], both great sources. We begin by introducing the notion of weak factorization systems as means of gaining control over the morphisms in a given category.

Notation 1. We shall divert somewhat from classical notation and use the suggestive symbol $\boxslash$, as far as we know introduced by Emily Riehl, to signal certain lifting properties. Namely, if $\mathcal{D}$ is a collection of morphisms in a category $\mathcal{C}$, we denote by $\mathcal{D}^{\boxslash}$ the collection of morphisms in $\mathcal{C}$ which have the *right lifting property* with respect to $\mathcal{D}$. Likewise, ${}^{\boxslash}\mathcal{D}$ are the morphisms in $\mathcal{C}$ with the left lifting property with respect to $\mathcal{D}$.

Definition 1. Let $\mathcal{C}$ be a category, and let $(\mathcal{L}, \mathcal{R})$ be two classes of morphisms in $\mathcal{C}$. We say that $(\mathcal{L}, \mathcal{R})$ is a *weak factorisation system* on $\mathcal{C}$ if

1. any morphism in $\mathrm{Mor}\,\mathcal{C}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$, and
2. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$, i.e. $\mathcal{L}$ are all morphisms with the left lifting property with respect to all the morphisms in $\mathcal{R}$, and $\mathcal{R}$ are all morphisms with the right lifting property with respect to all morphisms in $\mathcal{L}$.

The reader may note that any model category comes equipped with two weak factorisation systems - $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{WE})$ and $(\mathrm{Cof} \cap \mathrm{WE}, \mathrm{Fib})$. In fact, a concise definition of a model category is in manners of weak factorization systems, as described in [R2].

In showing the existence of a particular model structure it is therefore useful to know when we have a weak factorization system on our hands. The perhaps most common way of doing so is using Quillen's small object argument. This ultimately leads us to describe cofibrantly generated model categories. Later on in the paper, all model categories considered will in fact be cofibrantly generated.

2.1 Cofibrantly Generated Model Categories

It can be quite difficult showing that a particular choice of subcategories constitute a model structure. One of the most common ways of constructing model structures utilizes the *the small object argument*, described already in Quillen's book on homotopy theory in 1967 [Q, Lemma 3 & the following remark.]. An important consequence of the small object argument is the ability to construct cofibrantly generated model categories - model categories which have a small set of generating cofibrations and acyclic cofibrations in the sense that many properties need only be checked on the generators rather than the whole class of cofibrations and acyclic cofibrations.

Example 1. A classic example of a cofibrantly generated model category is the category of simplicial sets equipped with the classical Quillen model structure. Here the generating cofibrations are the boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$, and the generating acyclic cofibrations are the horn inclusions $\Lambda[n]_i \hookrightarrow \Delta[n]$.

2.1.1 The Small Object Argument

The small object argument is used to construct and detect weak factorization systems. The argument is built on what is called *transfinite induction* - a generalization of the induction the reader may be used to seeing. In standard induction, one considers a sequence often indexed over $\mathbb{N}$. Then the reader may infer that an object in the sequence has a specific property given that all previous objects in the sequence has said property. The natural extension to this is working with sequences indexed by another ordinal. This allows us to formalize the notion of smallness which as the name suggests is essential to the small object argument.

Definition 2. An *ordinal* is an equivalence class of well-ordered sets, where the equivalence relation is given by isomorphism.

Definition 3. Let λ be an ordinal. We say that the *successor ordinal of* λ is the smallest ordinal larger than λ.

Definition 4. Let λ be an ordinal. We say that λ is a *successor ordinal* if it is the successor ordinal of some ordinal. We say that λ is a *limit ordinal* if λ is neither 0 nor a successor ordinal.

The most common examples are the finite ordinals, often thought of being the numbers from 0 up to some natural number $n-1$, with the ordinary $\leq$-order. The first infinite ordinal may be represented by the natural numbers and is usually denoted $\aleph_0$.

Remark 1. There is a canonical way in which to regard a partially ordered set as a category where the objects are the elements in the set and there is an arrow $A \to B$ if and only if $A \leq B$. In particular, a totally ordered set has for any pair of objects A and B either a unique arrow with direction given by the ordering, as well as the identities. A consequence is that ordinal diagrams are of the shape

$$X_0 \to X_1 \to \dots$$

Definition 5. Let λ be an ordinal and let $\mathcal{C}$ be a cocomplete category, i.e. a category with all small colimits. A *λ-sequence* is a functor $X : \lambda \to \mathcal{C}$, commonly written

$$X_0 \to X_1 \to \dots \to X_\alpha \to \dots .$$

We say, by slight abuse of notation as the colimit is unique only up to unique isomorphism, that the induced morphism $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$ is the *composition* of X.

If $\mathcal{D}$ is a class of morphisms in $\mathcal{C}$ such that each morphism in the λ-sequence is in $\mathcal{D}$, we say that $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$ is a *transfinite composition* of morphisms in $\mathcal{D}$.

Notation 2. Let A be a set. Then we denote by $|A|$ the smallest ordinal to which there is a bijection $A \to |A|$. A *cardinal* is an ordinal κ such that $|\kappa| = \kappa$.

Definition 6. Let κ be a cardinal. We say that an ordinal λ is *κ-filtered* if

1. λ is a limit ordinal, and
2. if $A \subseteq \lambda$ and $|A| < \kappa$, then $\sup A < \lambda$.

Here $\sup A$ is the usual notion of supremum of the elements in A.

We are now ready to present a notion of smallness. The intuition one should have is that an object A of a category is small if every morphism from A to a large enough composition $\operatorname{colim} X_\bullet$ factors through one of the factors X_β.

Definition 7. Let $\mathcal{C}$ be a category and A an object in $\mathcal{C}$. Let κ be a cardinal and let $\mathcal{D}$ be a class of morphisms in $\mathcal{C}$. We say that A is *κ-small relative to $\mathcal{D}$* if for every κ-sequence

$$X_0 \to \dots \to X_\beta \to \dots$$

with every morphism $X_\beta \to X_{\beta+1}$, $\beta + 1 < \kappa$, being in $\mathcal{D}$, any morphism from A to the composition of X factors as $A \to X_\beta \to \operatorname{colim}_{\alpha<\kappa} X_\alpha$, for some $\beta < \kappa$. In other words, we have an isomorphism

$$\operatorname{colim}_{\alpha<\kappa} \hom_{\mathcal{C}}(A, X_\alpha) \cong \hom_{\mathcal{C}}(A, \operatorname{colim}_{\alpha<\kappa} X_\alpha).$$

We say that A is *small relative to $\mathcal{D}$* if A is κ-small with respect to $\mathcal{D}$ for some κ and we say that A is small if it is small relative to $\mathcal{D} = \mathcal{C}$.

Definition 8. Let I be a collection of morphisms in a category $\mathcal{C}$.

1. We say that a map f is *I-injective* or *injective with respect to I* if $f \in I^{\boxslash}$.
2. We say that a map f is *I-projective* or *projective with respect to I* if $f \in {}^{\boxslash}I$.
3. We say that a map f is an *I-cofibration* if $f \in {}^{\boxslash}(I^{\boxslash})$. We call the class of morphisms ${}^{\boxslash}(I^{\boxslash})$ the *I-cofibrations*.
4. We say that a map f is an *I-fibration* if $f \in ({}^{\boxslash}I)^{\boxslash}$. We call the class of morphisms $({}^{\boxslash}I)^{\boxslash}$ the *I-fibrations*.

With these notions we approach the I-cell complexes which will then be used to prove the small object argument.

Definition 9. Let I be a collection of morphisms in a category $\mathcal{C}$ containing all small colimits. We say that a morphism f in $\mathcal{C}$ is a *relative I-cell complex* if f is a transfinite composition of pushouts of morphisms in I. We denote the collection of relative I-cell complexes by *I-cell*. We say that an object $C \in \mathcal{C}$ is an *I-cell complex* if the unique morphism from the initial object to C is a relative I-cell complex.

We expand upon this definition. Let

$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots$$

be a λ-sequence for some ordinal λ, where each morphism $X_\beta \to X_{\beta+1}$ is a pushout of a morphism in I, for all $\beta + 1 < \lambda$. That is, there is a pushout square

$$\begin{array}{ccc} A & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\downarrow & & \downarrow \\ B & \longrightarrow & X_{\beta+1} \end{array}$$

where g_β is in I. Then the transfinite composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ is a relative I-cell complex.

Remark 2. Ensuring that all small colimits exist implies that the initial object exists as the initial object is the colimit of the empty diagram. Dually, if $\mathcal{C}$ contains all small limits, $\mathcal{C}$ has a terminal object.

We now present the small object argument which is a convenient way of finding model structures which in many instances will carry good properties. One such property is that the small object argument often produces *cofibrantly generated* model structures, described in Section 2.1.2.

Theorem 1 (The small object argument, [HOV, Thm.2.14]). *Suppose $\mathcal{C}$ is a category containing all small colimits and let I be a set of morphisms in $\mathcal{C}$. If the domains of the morphisms in I are small relative to I-cell, then there exists a functorial factorization (γ, δ) such that*

1. *For any morphism $f \in \mathcal{C}$, $\gamma(f)$ is a relative I-cell complex.*
2. *For any morphism $f \in \mathcal{C}$, $\delta(f)$ is in $I^{\boxslash}$.*
3. $f = \delta(f) \circ \gamma(f)$.

Proof. We use transfinite induction. By letting κ be a cardinal large enough to ensure that domains of I are κ-small relative to I-cell, we may choose an ordinal λ which is κ-filtered.

Consider for a morphism $f : X \to Y$ the constant λ-sequence $Y \to Y \to \dots$ which we by abuse of notation will call Y. Our aim is to construct another λ-sequence with a natural transformation to Y. By taking the composition of this λ-sequence we get an induced morphism to Y which will be $\delta(f)$. We construct a λ-sequence Z and a natural transformation ρ inductively:

Let $Z_0 = X$ and $\rho_0 = f$. Let β be a limit ordinal and suppose that Z_α and ρ_α is defined for all $\alpha < \beta$. Then we let $Z_\beta = \mathrm{colim}_{\alpha<\beta}$ and ρ_β be the morphism from Z_β to Y induced by the universal property of the colimit.

Now we wish to define $Z_{\beta+1}$ for any successor ordinal $\beta + 1$. Let S_β be the set of commutative squares of the form

$$\begin{array}{ccc} A & \longrightarrow & Z_\beta \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & Y \end{array}$$

with $g \in I$. For a fixed commutative square $s \in S$, let g_s be the corresponding morphism in I. We may define $Z_{\beta+1}$ to be the pushout

$$\begin{array}{ccc} \coprod_s A_s & \longrightarrow & Z_\beta \\ {\scriptstyle \coprod_s g_s}\big\downarrow & & \big\downarrow \\ \coprod_s B_s & \longrightarrow & Z_{\beta+1} \end{array}$$

and let $\rho_{\beta+1}$ be the corresponding induced morphism.

The composition of Z so defined is by [HOV, Lemma 2.1.13.] in I-cell and we set this to be $\gamma(f)$. We get an induced morphism from colim Z_β to Y which we wish to choose as $\delta(f)$. To do this we need to show that $\delta(f) \in I^{\boxslash}$. Consider a commutative square

$$\begin{array}{ccc} A & \longrightarrow & \operatorname{colim} Z_\beta \\ \Big\downarrow & & \Big\downarrow \\ B & \longrightarrow & Y \end{array}$$

in which $A \to B$ is in I. The information left that is not used is that the domains of the morphisms in I are small relative to I-cell. With this assumption, we may factor $A \to \operatorname{colim} Z_\beta$ into $A \to Z_\beta \to \operatorname{colim} Z_\beta$, for some $\beta < \lambda$. But by construction there is a morphism $B \to Z_{\beta+1}$ such that the composite $B \to Z_{\beta+1} \to \operatorname{colim} Z_\beta$ works as a lift. ■

Hence we have at this point a way of knowing when weak factorization systems exist. Using the definition of a model category found in [R2], a large part of finding model structures would be lying in finding two sets of morphisms which allow the small object argument.

2.1.2 Cofibrantly Generated Model Categories

Definition 10. We say that a model category $\mathcal{M}$ is *cofibrantly generated* if there exists two sets of morphisms I and J satisfying the following.

1. The domains of the morphisms of I are small relative to I-cell.
2. The domains of the morphisms of J are small relative to J-cell.
3. The class of fibrations are the morphisms which have the right lifting property with respect to J, i.e. $\text{Fib} = J^{\boxslash}$.
4. The class of trivial fibrations are the morphisms which have the right lifting property with respect to I, i.e. $\text{Fib} \cap \text{WE} = I^{\boxslash}$.

We will call I the *generating cofibrations* and J the *generating acyclic cofibrations*.

This sort of model structure is especially nice to work with since we by definition have a small set (as opposed to a proper class) of morphisms against which we may detect fibrations and acyclic fibrations. In fact, more can be deduced. A succinct compilation of important properties is Proposition 2.1.18 in [HOV]:

Theorem 2. *Let $\mathcal{M}$ be a cofibrantly generated model category, with generating cofibrations I and generating acyclic cofibrations J.*

1. *The cofibrations form the class ${}^{\boxslash}(I^{\boxslash})$.*
2. *The acyclic cofibrations form the class ${}^{\boxslash}(J^{\boxslash})$.*
3. *Every cofibrations is a retract of a relative I-cell complex.*
4. *Every acyclic cofibration is a retract of a relative J-cell complex.*

5. *The domains of the morphisms of I are small relative to the cofibrations.*

6. *The domains of the morphisms of J are small relative to the acyclic cofibrations.*

One may as written above use the small object argument to show the following, which is a standard result included for example as Theorem 2.1.19 in [HOV].

Theorem 3. *Let $\mathcal{M}$ be a bicomplete category. Let $\mathcal{W}$ be a subcategory of $\mathcal{M}$ and let I and J be sets of morphisms of $\mathcal{M}$. Then there is a cofibrantly generated model structure on $\mathcal{M}$ with $\mathcal{W}$ being the weak equivalences, I being the generating cofibrations and J being the generating acyclic cofibrations if and only if:*

1. *The subcategory $\mathcal{W}$ satisfies the 2-out-of-3 property and is closed under retracts.*

2. *The domains of I are small relative to I-cell.*

3. *The domains of J are small relative to J-cell.*

4. $J - \text{cell} \subseteq {}^{\boxslash}(I^{\boxslash}) \cap \mathcal{W}$.

5. $I^{\boxslash} \subseteq J^{\boxslash}$.

6. *Either* $({}^{\boxslash}(I^{\boxslash})) \cap \mathcal{W} \subseteq {}^{\boxslash}(J^{\boxslash})$ *or* $J^{\boxslash} \cap \mathcal{W} \subseteq I^{\boxslash}$.

In other words, we may get by without checking the factorization axioms, and we get Theorem 2 for free, should we be able to show that the conditions in the above theorem are satisfied.

Recall at this point that we are interested in combinatorial model categories. These are model categories which are both cofibrantly generated and locally presentable. We are interested in such model structures since it turns out that from them, we may often construct other model structures which in turn inherit a lot of structure. An example is Theorem 10 which states that if $\mathcal{M}$ is a combinatorial left proper model category, then the left Bousfield localization described in Chapter 3 exists in a large number of cases. Another example is Theorem 11 which states that combinatorial model categories induce combinatorial model structures on presheaf categories under sufficiently nice conditions.

2.2 Locally Presentable Categories

The second condition for a model category being combinatorial is that the underlying category has to be locally presentable. I refer the reader to [AR] for a thorough investigation.

Definition 11. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *locally presentable* if

(i) $\mathcal{C}$ is cocomplete,

(ii) $\mathcal{C}$ is locally small, and

(iii) There is a regular cardinal λ and a λ-small set S such that every object of $\mathcal{C}$ may be written as a colimit of a λ-filtered sequence of objects in S.

If γ is a cardinal such that condition 3 is met, we say that $\mathcal{C}$ is locally γ-presentable.

Definition 12. We say that a model category $\mathcal{M}$ is combinatorial if it is

1. cofibrantly generated, and
2. the underlying category is locally presentable.

Although not explicitly used in any arguments in this text, local presentability is an assumption which is often necessary in category theory. As an example, the ever so necessary theorem that the localization of a left proper combinatorial model category exists (Theorem 10) uses implicitly that the underlying category is locally presentable. See for example [BAR, Prop. 2.2] for a result regarding locally presentable categories used to show said theorem.

2.3 Enriched Model Structures

Recalling that in a 1-categorical setting, we have (if the category is locally small) sets of morphisms between the objects, it is a natural extension to work in a setting in which the morphisms between objects form not only sets, but also carry additional structure. In higher category theory we do not only have arrows between objects but instead n-arrows relating the $n-1$-arrows for any natural number $n \geq 2$, whence we instead obtain spaces, i.e. simplicial sets, of morphisms between objects. This is a special case of an enriched category, where the category in question is enriched over the category of simplicial sets. We may analogously consider enrichment over a another category $\mathcal{C}$, where the morphisms between objects may be represented by an object in $\mathcal{C}$. We must however require that $\mathcal{C}$ be *symmetric monoidal* for enrichment to make sense.

We have written an exposé of symmetric monoidal categories for the interested reader, but there is unfortunately a morass of machinery needed to capture what is ultimately quite an intuitive idea. Hence that section is moved to the appendix, see Section 8.3. For this paper it is probably sufficient and easier for the reader to think of a symmetric monoidal model category as a category $\mathcal{S}$ with a composition operation $- \otimes - : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ and a unit object $1 \in \mathcal{S}$ satisfying that $1 \otimes X \cong X \cong X \otimes 1$ for any object $X \in \mathcal{S}$.

Suppose that $\mathcal{C}$ is a category enriched over a symmetric monoidal category $\mathcal{S}$. This means that for any two objects $A, B \in \mathcal{C}$, there is a hom-*object*, denoted by $\hom_{\mathcal{C}}^{\mathcal{S}}(A, B) \in \mathcal{S}$, which models the morphisms from A to B.

Example 2. The category of sets with the cartesian product is symmetric monoidal where we may choose any singleton to be the unit.

Example 3. Any locally small category has Set-valued hom-objects.

Definition 13. Let $\mathcal{S}$ be a symmetric monoidal category and let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *enriched over* $\mathcal{S}$ if

1. For any pair of objects $X, Y \in \mathcal{C}$ there is an object $\hom^{\mathcal{S}}_{\mathcal{C}}(X, Y)$ in $\mathcal{S}$ called the *hom-object* from X to Y.

2. For any triple of object $X, Y, X \in \mathcal{C}$, there is a *composition morphism*

$$\hom^{\mathcal{S}}_{\mathcal{C}}(Y, Z) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) \to \hom^{\mathcal{S}}_{\mathcal{C}}(X, Z).$$

3. For each object $X \in \mathcal{C}$ there is a morphism $1 \to \hom^{\mathcal{S}}_{\mathcal{C}}(X, X)$.

4. The hom-objects are compatible with the symmetric monoidal structure of $\mathcal{S}$ in the sense that for any quadruple of objects X, Y, Z, W in $\mathcal{C}$, the diagrams

$$\begin{array}{ccc}
 & & \hom^{\mathcal{S}}_{\mathcal{C}}(Y, W) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) \\
 & \nearrow & \downarrow \\
(\hom^{\mathcal{S}}_{\mathcal{C}}(Z, W) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(Y, Z)) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) & & \\
\downarrow & & \hom^{\mathcal{S}}_{\mathcal{C}}(X, W) \\
\hom^{\mathcal{S}}_{\mathcal{C}}(Z, W) \otimes (\hom^{\mathcal{S}}_{\mathcal{C}}(Y, Z) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y)) & & \uparrow \\
 & \searrow & \\
 & & \hom^{\mathcal{S}}_{\mathcal{C}}(Z, W) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Z)
\end{array}$$

and

$$\begin{array}{ccccc}
\hom^{\mathcal{S}}_{\mathcal{C}}(X, X) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) & \longrightarrow & \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) & \longleftarrow & \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(Y, Y) \\
\uparrow & \nearrow & & \nwarrow & \uparrow \\
1 \otimes \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) & & & & \hom^{\mathcal{S}}_{\mathcal{C}}(X, Y) \otimes 1
\end{array}$$

commute.

Notation 3. If no confusion may be had, we will drop the superscripts and subscripts of the hom-object.

We now wish for an analogue of enrichment for model categories. In order to enrich a model category over another model category, some extra structure is necessary. The model category which we enrich over needs to be a *symmetric monoidal model category* (see Def. 73).

This is a symmetric monoidal category $\mathcal{S}$ which

1. admits an *internal* hom-*functor* $\underline{\hom}(-,-) : \mathcal{S}^{\mathrm{op}} \times \mathcal{S}$ such that for any object b in $\mathcal{S}$, $- \otimes b : \mathcal{S} \leftrightarrows \mathcal{S} : \underline{\hom}(b,-)$ forms an adjunction, and
2. which we may equip with a model structure in a compatible way.

We also require of the enriched model category what are called *tensor products* and *cotensor products*, which are ways in which to combine objects in the enriched category with objects in the category enriched over. Tensoring and cotensoring are not hingent on the involved categories having model structures and analogously to the above, we may see the tensoring as a left adjoint to the hom-functor.

Definition 14. Let $\mathcal{C}$ be a category which is enriched over $\mathcal{S}$. We say that $\mathcal{C}$ is *tensored* and *cotensored* over $\mathcal{S}$ if there exist functors

1. *(Tensor)*
$(-) \otimes_{\mathcal{C}} (-) : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$, and
2. *(Cotensor or power)*
$(-)^{(-)} : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$,

such that for any objects $X, Y \in \mathcal{C}$ and $S \in \mathcal{S}$ we have natural isomorphisms

$$\hom_{\mathcal{C}}^{\mathcal{S}}(X, Y^S) \cong \underline{\hom}_{\mathcal{S}}(S, \hom_{\mathcal{C}}^{\mathcal{S}}(X,Y)) \cong \hom_{\mathcal{C}}^{\mathcal{S}}(X \otimes_{\mathcal{C}} S, Y).$$

Remark 3. Note that we have two adjunctions $(-) \otimes_{\mathcal{C}} S \dashv (-)^S$ and $X \otimes_{\mathcal{C}} (-) \dashv \hom_{\mathcal{C}}^{\mathcal{S}}(X,-)$.

Notation 4. If no confusion may be had, the subscripts on the hom-objects and the tensoring are often dropped.

In order to finally make sense of what it means to enrich one model category over another we just need to require that the tensor and cotensor products are compatible with the model structure.

Definition 15. Let $\mathcal{S}$ be a symmetric monoidal model category and let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is an *$\mathcal{S}$-enriched model category* or *enriched over $\mathcal{S}$* if

1. As categories, $\mathcal{M}$ is enriched, tensored and cotensored over S.
2. For every cofibration $A \to B$ and any fibration $X \to Y$, the induced morphism on the pullback

$$\hom_{\mathcal{M}}^{\mathcal{S}}(B,X) \to \hom_{\mathcal{M}}^{\mathcal{S}}(A,X) \times_{\hom_{\mathcal{M}}^{\mathcal{S}}(A,Y)} \hom_{\mathcal{M}}^{\mathcal{S}}(B,Y)$$

is a fibration which is an acyclic fibration whenever any of $A \to B$ or $X \to Y$ is a weak equivalence.

Notation 5. An important special case is when a category $\mathcal{C}$ is enriched over sSet, in which case we call $\mathcal{C}$ a *simplicial category*. If $\mathcal{M}$ is a model category, we say that $\mathcal{M}$ is a *simplicial model category* if $\mathcal{M}$ is an $\mathrm{sSet}_{\text{Kan-Quillen}}$-enriched model category. If $\mathcal{M}$ is a simplicial model category, it is customary to write $\mathrm{map}_{\mathcal{M}}(X, Y)$ for the hom-object from X to Y. We refer to the simplicial set $\mathrm{map}_{\mathcal{M}}(X, Y)$ as the *simplicial mapping space* from X to Y. If no confusion may be had we will drop the subscript.

The simplicial mapping space acts as a *homotopy function complex* (see Section 8.1) between cofibrant and fibrant objects.

Theorem 4 ([H, Ex. 17.1.4]). *Let $\mathcal{M}$ be a simplicial model category and suppose that X is a cofibrant object in $\mathcal{M}$ and that Y is a fibrant object in $\mathcal{M}$. Then* $\mathrm{map}(X, Y)$ *is a homotopy function complex from X to Y.*

This will be of importance when describing local objects and local equivalences in Section 3.1.

2.4 Homotopy (Co)Limits and (Co)Ends

In homotopy theory, it turns out that limits and colimits are often ill-equipped to cooperate with model structures. One major limitation is that even though we may have a morphism of diagrams in $\mathcal{M}$ which is an objectwise weak equivalence, the induced morphism between the limits or indeed the colimits may not be a weak equivalence. Hence we introduce the homotopy limit and the homotopy colimit. These remedy the situation somewhat. An overwiew of the basic theory may be seen in [H, Ch. 18-19].

As an intuitive start, the reader is encouraged to think of the homotopy limit as a construction which preserves fibrancy, and if we have a homotopy limit of a fibrant diagram, then it coincides with the limit. Dually, if we have a homotopy colimit of a cofibrant diagram, the result is a cofibrant object which is simultaneously the colimit of said diagram. Note here the subtlety that we talk about fibrant diagrams and cofibrant diagrams, which require that we equip the functor category with some sort of model structure. There are a number of ways of doing so, with the most natural structures being the injective and projective model structures of Section 4.1.1.

Recalling that we may, for any category $\mathcal{C}$ and any small category $\mathcal{D}$, consider the colimit and limit of the $\mathcal{D}$-diagrams in $\mathcal{C}$ as the left and right adjoints of the constant functor from $\mathcal{C}$ to the diagram category $[\mathcal{D}, \mathcal{C}]$, we have a natural way to extend this to a homotopy theoretic setting.

Definition 16. Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a small category. Make the diagram category $[\mathcal{C}, \mathcal{M}]$ into a category with weak equivalences by taking the morphisms which are objectwise weak equivalences in $\mathcal{M}$ to be weak equivalences in $[\mathcal{C}, \mathcal{M}]$. The *homotopy limit* is defined as the right derived functor of the limit functor. Dually, the *homotopy colimit* is defined to be the left derived functor of the limit functor.

From this it is clear that homotopy limits of fibrant diagrams are ordinary limits, and the dual result for cofibrant diagrams also follows. The following result is very important and acts as motivation for the necessity of homotopy (co)limits as well.

Theorem 5 ([H, Thm. 19.4.2])**.** *Let $\mathcal{M}$ be a framed model category and let $\mathcal{C}$ be a small category.*

1. *If f is a morphism between $\mathcal{C}$-diagrams in $\mathcal{M}$ and f is an objectwise weak equivalence between fibrant objects, then f induces a weak equivalence between the fibrant homotopy limits.*
2. *If f is a morphism between $\mathcal{C}$-diagrams in $\mathcal{M}$ and f is an objectwise weak equivalence between cofibrant objects, then f induces a weak equivalence between the cofibrant homotopy colimits.*

We have the following result regarding homotopy function complexes (see Section 8.1).

Theorem 6 ([H, Thm. 19.4.4])**.** *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a small category.*

1. *If $\mathbf{X}$ is a cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$ and $Y \in \mathcal{M}$ is fibrant, we have a natural weak equivalence*
$$\operatorname{map}(\operatorname{hocolim} \mathbf{X}, Y) \simeq \operatorname{holim} \operatorname{map}(\mathbf{X}, Y).$$
2. *If $X \in \mathcal{M}$ is cofibrant and $\mathbf{Y}$ is a fibrant $\mathcal{C}$-diagram in $\mathcal{M}$, then there is a natural weak equivalence*
$$\operatorname{map}(X, \operatorname{holim} \mathbf{Y}) \simeq \operatorname{holim} \operatorname{map}(X, \mathbf{Y}).$$
where map *denotes a homotopy function complex.*

2.4.1 Ends and Coends

The homotopy limit and homotopy colimit is part of more general structures known as ends and coends, described in chapters 18 and 19 of [H]. Although this generalization is interesting, perhaps more important for this text is that the end and coend will be necessary for us to look at inherited enrichment in presheaf categories.

We may think of the end as a special kind of limit of a bifunctor, and dually we may think of the coend as a special kind of colimit of a bifunctor. To formalize this we introduce the wedge.

Definition 17. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a bifunctor. A *wedge* with respect to F is an object $w \in \mathcal{D}$ and a collection of morphisms $\pi_c : w \to F(c, c)$ for all $c \in \mathcal{C}$ such that the following family of diagrams commute:

$$\begin{array}{ccc} w & \longrightarrow & F(c,c) \\ \downarrow & & \downarrow \\ F(c',c') & \longrightarrow & F(c,c') \end{array}$$

Definition 18. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a bifunctor. A *cowedge* with respect to F is an object $w \in \mathcal{D}$ and a family of morphisms $\omega_c : F(c,c) \to w$ such that the following family of diagrams commute:

$$\begin{array}{ccc} F(c',c) & \longrightarrow & F(c,c) \\ \big\downarrow & & \big\downarrow \\ F(c',c') & \longrightarrow & w \end{array}$$

Definition 19. Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a category. Suppose that $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a bifunctor.

1. We define the *end* $\int_{c:\mathcal{C}} F(c,c)$ of F to be the universal wedge in the sense that any other wedge factors uniquely through $\int_{c:\mathcal{C}} F(c,c)$.

2. We define the *coend* $\int^{c:\mathcal{C}} F(c,c)$ of F to be the universal cowedge such that for any other cowedge w, the morphisms $F(c,c) \to \int^{c:\mathcal{C}} F(c,c)$ factor uniquely through w.

Remark 4. The convention to use the integration symbol for ends and coends is due to there being an analogue of Fubini's theorem for ends and coends. We will not use it in this paper but the interested reader is encouraged to read for example [K].

It is not at all clear how to see the homotopy limit and homotopy colimit as ends and coends but it turns out that if one chooses a specific kind of functor, both holim and hocolim are part of this construction. We will not cover this as this requires an alternative approach to the homotopy limit and colimit but the interested reader is referred to [H, Ex. 19.2.8]. In Section 4.1.2 we will also use the theory of ends and coends to enrich presheaf categories.

Chapter 3

Bousfield Localizations

Recall that a localization of a category with respect to a class of morphisms may be viewed as formally inverting said morphisms, keeping the objects constant. This hopefully reminds the reader with a background in commutative algebra of the localization of a ring or module. The fundamental example in homotopy theory is the homotopy category of a model category. Namely, for any model category $\mathcal{M}$, its homotopy category $\mathrm{Ho}(\mathcal{M})$ may be viewed as the localization of $\mathcal{M}$ with respect to the weak equivalences. In general there is no guarantee that a given localization exists, though there are some prominent cases where the existence is known.

The natural extension of the concept of a localization of a category to the setting of model categories is to not require that the morphisms we localize with respect to, to become inverted, but instead to become invertible *up to homotopy*. In other words, we wish to expand the class of weak equivalences to include some chosen class of morphisms.

This paper focuses on two natural ways of constructing such localizations, known as *left and right Bousfield localizations*.

Definition 20. Let $\mathcal{M}$ be a model category. A *left (right) Bousfield localization of* $\mathcal{M}$ is a new model category $\mathcal{LM}$ on the same underlying category such that

- The class of cofibrations (fibrations) of $\mathcal{LM}$ is the same as the class of cofibrations (fibrations) of $\mathcal{M}$, and
- the weak equivalences in $\mathcal{LM}$ includes the weak equivalences in $\mathcal{M}$.

Note that for a left (right) Bousfield localization, the identity functor on the underlying category $\mathrm{Id} : \mathcal{M} \to \mathcal{LM}$ gives a left (right) Quillen functor.

It is convenient to ask of such this localization, should it exist, to have some tractable properties. One of these properties is that we hope for a homotopy theoretic analogue of a *reflective localization*.

Definition 21. We say that a localization of a category is *reflective* if the localization functor admits a fully faithful right adjoint.

The correct homotopy theoretic analogue is the following.

Definition 22. Let $\mathcal{M} \leftrightarrows \mathcal{N}$ be an adjunction between two model categories. We say that this adjunction is a *Quillen reflection* if the derived adjunction counit is an objectwise weak equivalence.

Theorem 7. *A left Bousfield Localization of a model category $\mathcal{M}$ gives rise to a Quillen reflection:*

$$\mathrm{Id} : \mathcal{M} \rightleftarrows \mathcal{L}\mathcal{M} : \mathrm{Id}\,.$$

Proof. The counit is given by the identity functor, whence the derived counit is given by an objectwise cofibrant approximation in $\mathcal{M}$. Let $f : X^c \overset{\sim}{\twoheadrightarrow} X$ be a cofibrant approximation. We need to see that f is a weak equivalence in $\mathcal{L}\mathcal{M}$. But the cofibrations coincide in both $\mathcal{M}$ and $\mathcal{L}\mathcal{M}$, whence the acyclic fibrations coincide. Hence f is still an acyclic fibration. ■

To see that the Quillen reflection is indeed a natural extension of the ordinary reflective localization we have the following.

Theorem 8. *An Quillen pair between two model categories is a Quillen reflection if and only if the induced adjunction between the homotopy categories is a reflective localization.*

Proof. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen reflection. Then the counit ε is an objectwise weak equivalence by definition. Hence composing with the homotopy category functor Ho we obtain an objectwise isomorphism as the counit of $\mathbb{L}F \dashv \mathbb{R}G$. ■

Although the above definition of a left Bousfield localization is compact, it hides a lot of information with respect to what maps are inverted. What one wishes for is a Bousfield localization *with respect to certain morphisms*. In order to do this formally we introduce local objects and morphisms.

3.1 Local Objects and Local Equivalences

Fix in this section a model category $\mathcal{M}$. Is there a way of measuring how close a morphism is to being a weak equivalence? This is where the homotopy function complexes from Section 8.1 are useful. With them we may introduce local objects and local equivalences. The local objects of a class of morphisms are the objects which make the morphisms "look" like weak equivalences. Hence we may expect that if the class of morphisms is the weak equivalences, all objects are local. Dually, the local equivalences are the morphisms which "look" like weak equivalences at all local objects.

Definition 23. Let $\mathcal{M}$ be a model category and let I be a collection of morphisms in $\mathcal{M}$. An *I-local object* $M \in \mathcal{M}$ is a fibrant object such that for any morphism $f : A \to B$, the induced morphism of homotopy function complexes

$$f^* : \mathrm{map}(B, M) \to \mathrm{map}(A, M)$$

is a weak equivalence.

If I consists of a single map g, then M will be called *g-local*.

Dually, we have the notion of a local equivalence.

Definition 24. Let $\mathcal{M}$ be a model category and let I be a collection of morphisms in $\mathcal{M}$. An *I-local equivalence* is a morphism $f : A \to B$ in $\mathcal{C}$ such that for every I-local object M, the induced morphism of homotopy function complexes

$$f^* : \mathrm{map}(B, M) \to \mathrm{map}(A, M)$$

is a weak equivalence.

The first result regarding local equivalences is one we should wish for.

Theorem 9. *All weak equivalences are I-local equivalences for any class of morphisms I in $\mathcal{M}$.*

In other words, all weak equivalences are local equivalences. The converse is certainly not true. This shortcoming is in fact a strength. We should think of the I-local equivalences as the minimal class of morphisms which must become weak equivalences in order to expand the weak equivalences to include I. An important property of the I-local equivalences are that they satisfy the 2-out-of-3 property.

In general, the function complexes may be hard to control. Especially good control may however be obtained under the added assumption that we are working in a simplicial model category (see section 8.4). Here we may use Theorem 4 to conclude that the simplicial mapping space $\mathrm{map}(X, Y)$, where X is cofibrant and Y is fibrant, acts as a homotopy function complex from X to Y. An analogue which works when working in any enriched model category is described in [BAR, Ch. 4], and which we describe in Section 3.2.2.

3.2 Bousfield Localizations with Respect to a Class of Morphisms

As mentioned in the beginning of the chapter, we often want to choose what morphisms to perform a left or right Bousfield localization at.

Definition 25. Let $\mathcal{M}$ be a model category with WE being its class of weak equivalences. Further fix a class of morphisms H in $\mathcal{M}$. A *left (right) Bousfield localization* of $\mathcal{M}$ with respect to H is a new model category $\mathcal{L}_H\mathcal{M}$ with the following properties:

1. The underlying category of $\mathcal{M}$ is the same as that of $\mathcal{L}_H\mathcal{M}$.

2. The class of cofibrations (fibrations) of $\mathcal{M}$ is the same as the class of cofibrations (fibrations) in $\mathcal{L}_H\mathcal{M}$.

3. The class of weak equivalences in $\mathcal{L}_H\mathcal{M}$ are the H-local equivalences of $\mathcal{M}$.

In other words, a left Bousfield localization with respect to a class of morphisms I endows the underlying category with a new model structure. This model structure has the minimal amount of weak equivalences such that H becomes weak equivalences, while keeping the cofibrations fixed.

Although the concepts of left and right Bousfield localizations are dual, it is in general much easier to work with left Bousfield localizations (Barwick even names his chapter on the right variant "The dreaded right Bousfield localization" [BAR, Ch. 5]). By certain slackness we will by "localization" mean the left Bousfield localization.

3.2.1 Existence and Properties of a Left Bousfield Localization

In general it may be hard to show that a given model structure exists, but the following theorem is a major reason why we wish to work with combinatorial model categories.

Theorem 10 ([BAR], Thm. 4.7)**.** *Let $\mathcal{M}$ be a left proper combinatorial model category and let S be a set of equivalence classes in* $\operatorname{Ho}\mathcal{M}$*. Then the left Bousfield localization $\mathcal{L}_S\mathcal{M}$ exists and is left proper and combinatorial.*

Remark 5. Beware of the subtlety that we require S to be a set. The statement is not true in general when S is a proper class.

3.2.2 Enriched Left Bousfield Localization

As mentioned in Section 3.1 we may generalize the usage of simplicial mapping spaces to detect local equivalences whenever we are working in an enriched setting. Let $\mathcal{S}$ be a symmetric monoidal model category and let $\mathcal{M}$ be a model $\mathcal{S}$-category, i.e. a model category $\mathcal{M}$ enriched $\mathcal{S}$.

Definition 26. Suppose H is a set of morphisms in $\mathcal{M}$. A *left Bousfield localization of $\mathcal{M}$ with respect to H, enriched over $\mathcal{S}$* is a new model $\mathcal{S}$-category, $\mathcal{L}_{H/\mathcal{S}}\mathcal{M}$ with a left Quillen $\mathcal{S}$-enriched functor $\mathcal{M} \to \mathcal{L}_{H/\mathcal{S}}\mathcal{M}$ which is initial among all left Quillen $\mathcal{S}$-enriched functors which turn H into weak equivalences.

Using the $\mathcal{S}$-valued hom-objects one may obtain a more intuitive notion of local equivalences and local objects.

Definition 27. An object X in $\mathcal{M}$ is called $(H/\mathcal{S})$*-local* if for all $f : A \to B$ in H, the induced morphism on the right derived hom-objects

$$\mathbb{R}\hom^{\mathcal{S}}_{\mathcal{M}}(B, X) \to \mathbb{R}\hom^{\mathcal{S}}_{\mathcal{M}}(A, X)$$

is an isomorphism in the homotopy category $\operatorname{Ho}(\mathcal{S})$.

Definition 28. A morphism $f : A \to B$ in $\mathcal{M}$ is called $(H/\mathcal{S})$-*local* if for all $(H/\mathcal{S})$-local objects X, the induced morphism on the right derived hom-objects

$$\mathbb{R}\hom^{\mathcal{S}}_{\mathcal{M}}(B, X) \to \mathbb{R}\hom^{\mathcal{S}}_{\mathcal{M}}(A, X)$$

is a weak equivalence.

Definition 29. We say that a combinatorial model category is *tractable* if we may choose a set of generating cofibrations where all domains are cofibrant and a set of generating acyclic cofibrations where all domains are cofibrant.

Proposition 1 (Thm. 4.46, [BAR]). *Suppose that $\mathcal{M}$ is a left proper, tractable combinatorial model category enriched over a tractable symmetric monoidal model category $\mathcal{S}$. Further suppose that H is a small set of morphisms in $\mathcal{M}$.*

Then the enriched left Bousfield localization of $\mathcal{M}$ with respect to $(H/\mathcal{S})$ exists and is a new model structure $\mathcal{L}_{H/\mathcal{S}}\mathcal{M}$ in which

1. *$\mathcal{L}_{\mathcal{S}}\mathcal{M}$ is tractable and left proper,*
2. *the underlying category is the same as that of $\mathcal{M}$,*
3. *the cofibrations are the same as in $\mathcal{M}$,*
4. *the fibrant objects are the fibrant $(H/\mathcal{S})$-local objects of $\mathcal{M}$, and*
5. *the weak equivalences are the $(H/\mathcal{S})$-local equivalences.*

Note that we may in this enriched version use the right derived hom-objects instead of the homotopy function complexes which gives us extra structure to work with.

Chapter 4

Model Structures on Presheaves

4.1 Presheaves on a Site

For the reader with a background in algebraic geometry, the notion of a presheaf is hopefully familiar. Here we first translate the classic presheaf into a categorical setting and go on to generalize both the target category and the source category. A good source regarding presheaf and sheaf categories is [BOR], although to transfer presheaf categories to a homotopy theoretic setting the reader is encouraged to consult [L]. Recall first the definition presheaves of abelian groups as defined by Hartshorne.

Definition 30 ([HAR, First def. Ch. 2])**.** Let X be a topological space. A *presheaf* F of abelian groups on X consists of the data

a) for every open subset $U \subseteq X$, an abelian group $F(U)$, and

b) for every inclusing $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : F(U) \to F(V)$,

subject to the conditions

1. $F(\varnothing) = 0$, where $\varnothing$ is the empty set,
2. ρ_{UU} is the identity map $F(U) \to F(U)$, and
3. if $W \subseteq V \subseteq U$ are open subsets of X, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 6. The condition $F(\varnothing) = 0$ is to the author's knowledge not standard outside of [HAR], and when changing the source category we may not have any initial object on which to evaluate the presheaf. Hence we will ignore this condition.

We wish to rephrase this in a more compact and categorical way. We may for any topological space X consider its category of open subsets $\mathrm{Op}(X)$ where the morphisms are given by inclusion. Then we may say that a presheaf of abelian groups on X is a contravariant functor $F : \mathrm{Op}(X)^{\mathrm{op}} \to \mathfrak{Ab}$. The reader may of course replace the target category with their favorite category, such as $\mathfrak{Grp}$ or $\mathfrak{Ring}$.

Example 4. The reader is encouraged to think about some topological concepts and to try to formulate them in category theoretic terms. Here we describe the intersection of two open subsets U and V.

Since the arrows signify inclusions, we know that the intersection $U \cap V$ has arrows to both U and V. It is therefore intuitive to think of $U \cap V$ as the pullback of U and V over some object. As all objects have inclusions into the whole topological space, it is natural to assume that $U \cap V$ is the pullback $U \times_X V$ and a short calculation may convince the reader that this is indeed the case. The choice of $U \cap V$ being the pullback of U and V over X is somewhat arbitrary however, as X may be substituted for any open subset containing both U and V.

We may at this point define a category of presheaves in a broad sense.

Definition 31. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be a category. The *$\mathcal{D}$-valued presheaves on $\mathcal{C}$* is the functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ which we shall often denote $\mathcal{D}\,\mathrm{Pre}(\mathcal{C})$. If $\mathcal{D} = \mathrm{sSet}$ we will call a simplicial set-valued presheaf a simplicial presheaf and we will write $\mathrm{sPre}(\mathcal{C})$ for the simplicial presheaf category on $\mathcal{C}$.

4.1.1 The Injective and Projective Model Structures on Presheaves

There are two natural model structures of which to equip a presheaf category with.

Definition 32. Let $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ be the category of $\mathcal{M}$-valued presheaves on a site $\mathcal{C}$. We say that a morphism of $\mathcal{M}$-valued presheaves $f : F \to G$ is an *injective cofibration* (resp. *injective weak equivalence*) if f is an objectwise cofibration (resp. objectwise weak equivalence). If the injective cofibrations and injective weak equivalences constitute a model structure, we refer to said structure as *the injective model structure on* $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ and write it as $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{inj}}$.

Definition 33. Let $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ be the category of $\mathcal{M}$-valued presheaves on a site $\mathcal{C}$. We say that a morphism of $\mathcal{M}$-valued presheaves $f : F \to G$ is a *projective fibration* (resp. *projective weak equivalence*) if f is an objectwise fibration (resp. objectwise weak equivalence). If the projective fibrations and projective weak equivalences constitute a model structure, we refer to said structure as *the projective model structure on* $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ and write it as $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{proj}}$.

Remark 7. One may consider these model structures for any functor category. In particular one may consider the injective and projective model structures on the functor categories discussed in section 2.4.

We have the convenient existence theorem stated below.

Theorem 11 ([BAR, Thm. 2.14 & Thm. 2.16]). *Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{C}$ be a site. Then both the projective and injective model structure on $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ exists, both of which are again combinatorial.*

We also know that the presheaf categories preserve properness.

Theorem 12 ([BAR, Prop. 2.18]). *Suppose that $\mathcal{M}$ is left (right) proper and combinatorial, then $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{inj}}$ and $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{proj}}$ are both left (right) proper.*

Note in particular that this means that if $\mathcal{M}$ is both combinatorial and left proper, then both $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{inj}}$ and $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})_{\mathrm{proj}}$ may be localized at any small set of morphisms by Theorem 10.

4.1.2 Enriched Presheaves

Recalling Section 3.2.2, it is natural to consider whether or not the injective and projective model structures on presheaves inherits enriched properties. Here we may under sufficiently nice conditions use the theory of ends and coends of Section 2.4.1 to get induced enrichment on the category of presheaves. Many of these theorems are found in [BAR].

Theorem 13. *[[BAR, Not. 3.35]] Suppose $\mathcal{S}$ is a symmetric monoidal category and that $\mathcal{C}$ is a small site. Let further S be an object in $\mathcal{S}$ and suppose F and G are objects in $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$. Then $\mathcal{S}\,\mathrm{Pre}$ is enriched, tensored and cotensored over $\mathcal{S}$ where the hom-objects are given by*

$$\hom^{\mathcal{M}}_{\mathcal{M}\,\mathrm{Pre}(\mathcal{C})}(F,G) = \int_{c:\mathcal{C}} \underline{\hom}(F(c), G(c)).$$

the tensor product is given by

$$(M \otimes_{\mathcal{M}\,\mathrm{Pre}(\mathcal{C})} F)(c) := M \otimes_{\mathcal{M}} F(c)$$

and the cotensoring is given by

$$\hom_{\mathcal{M}\,\mathrm{Pre}(\mathcal{C})}(M,F)(c) := \underline{\hom}_{\mathcal{M}}(M, F(c)).$$

It is also a general fact that if $\mathcal{M}$ is enriched over a model category, then $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ with both the injective and projective model structure is similarly enriched over the same model category (if the structures exist). It is an unfortunate thing that the only source the author has found so far for this general statement is the website Model Structure on Functors - nLab.

Specifically for this paper we are however only interested in simplicial combinatorial model categories. Then the presheaf category equipped with either the injective or projective model structure is simplicial by [MO, Thm. 5.4 & Rmk. 5.5].

4.2 Sheaves on a Site

We are often interested in some sort of local-to-global property, i.e. some kind of sheaf condition, on a presheaf taking values in a model category. There are a number of ways of doing this but what we would want is a definition which has a homotopy theoretic analogue and which may generalize to other similar local-to-global properties. We are particularly interested in a definition which leads naturally to the definition of a hypersheaf, which we will work with in Section 5.1.1.

Recall first the conditions imposed for a presheaf to be a sheaf in the classical setting:

Definition 34 ([HAR, Second def, Ch.2.])**.** A presheaf F of abelian groups on X is a *sheaf* if it satisfies the following supplementary conditions:

4. if U is an open set, if $\{V_i\}$ is an open covering of U and if $s \in F(U)$ is an element such that $s\mid_{V_i}= 0$ for all i, then $s = 0$;

5. if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in F(V_i)$ for each i, with the property that for each i, j, $s_i \mid_{V_i \cap V_j} = s_j \mid_{V_i \cap V_j}$, then there is an element $s \in F(U)$ such that $s \mid_{V_i} = s_i$ for all i.

Note that suddenly the notion of a *cover* or *covering* is very important. When changing the source category to something other than the open subsets of a topological space we must define some new sort of cover. Hence we introduce the *Grothendieck topology*.

Definition 35. Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$. A *sieve* over c is a collection of morphisms with c as codomain which is closed under precomposition, i.e. if $d \to c$ is in s, then for any morphism $e \to d$, the composition $e \to d \to c$ is in S.

Definition 36 (Grothendieck Topology)**.** Let $\mathcal{C}$ be a category. A *Grothendieck Topology* τ on $\mathcal{C}$ is an assignment $c \mapsto \tau(c)$, for every object $c \in \mathcal{C}$, where $\tau(c)$ is a collection of sieves over c. These sieves satisfy the following:

1. If S is a sieve covering c and $g : d \to c$ is a morphism in $\mathcal{C}$, then the pullback sieve $g^*S := \bigcup_{e \in \mathcal{C}} \{f : e \to d \mid gf \in S\}$ is a sieve covering d.

2. If F is a sieve over c and $\bigcup_{e \in \mathcal{C}} \{g : e \to c \mid g^*F \text{ covers } e\}$ contains a sieve which covers c, then F covers c.

3. $\hom(-, c)$ covers c.

The set of sieves assigned to c are said to *cover* c.

Definition 37 (Site)**.** A tuple $(\mathcal{C}, \tau)$ where $\mathcal{C}$ is a category and τ is a Grothendieck topology on $\mathcal{C}$ is called a *site*.

Example 5. Consider again the classical setting where the source category is $\text{Op}(X)$ for a topological space X. We describe the covering sieves for an open subset U. Let S be a sieve over U, i.e. a collection of morphisms with U as codomain. We may consider evaluating S on another subset V to obtain all morphisms in S from V to U. Note that $S(V)$ in this case is either empty or consists of one morphism. We may then say that S is a covering sieve if the union over all V such that $S(V) \neq \varnothing$ is U.

There are a number of equivalent ways of stating Hartshorne's sheaf condition, Definition 34, in category theory. But recall that we want a definition which generalizes to other local-to-global properties. We first introduce an intuitive way of thinking of sheaves but which is not ideal for later purposes.

Definition 38. Let $\mathcal{C}$ be a site and let $\mathcal{D}$ be a category. We say that a $\mathcal{D}$-valued presheaf $F \in \mathcal{D}\,\mathrm{Pre}(\mathcal{C})$ is a *$\mathcal{D}$-valued sheaf* if for any object $X \in \mathcal{C}$ and any covering sieve R over X, we have an isomorphism

$$F(X) \cong \lim_{Y \in R/\mathcal{C}} F(Y).$$

This definition of a sheaf is easy to state and it has a natural homotopy theoretic analogue where we switch the limit for the homotopy limit, and instead of $F(X)$ being isomorphic to $\mathrm{holim}_{Y \in R/\mathcal{C}} F(Y)$ we instead require this to hold only by weak equivalence.

Definition 39 ([BAR, Def. 4.54])**.** Let $\mathcal{M}$ be a symmetric monoidal model category. Then an $\mathcal{M}$-valued presheaf on a small site $(\mathcal{C}, \tau)$ is called an *$\mathcal{M}$-valued sheaf* if for any X and any covering sieve $R \in \tau(X)$, the morphism

$$F(X) \to \mathrm{holim}_{Y \in R/\mathcal{C}} F(Y)$$

is a weak equivalence.

It is not too hard to show the existence of a model structure which presents $\mathcal{M}$-valued sheaves as the fibrant objects when $\mathcal{M}$ is left proper, combinatorial and symmetric monoidal.

Proposition 2 (Prop 4.56, [BAR])**.** *Suppose $\mathcal{M}$ has a cofibrant unit 1 and equip $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ with the projective (resp. injective) model structure. Then we may perform a left Bousfield localization to obtain a model structure in which the projective (resp. injective) fibrant $\mathcal{M}$-valued sheaves are the fibrant objects.*

We introduce some results regarding symmetric monoidal model categories in order to show this.

Definition 40 (Enriched Yoneda Embedding)**.** Let $\mathcal{M}$ be a symmetric monoidal model category with unit 1. For any $X \in \mathcal{C}$ we define the *$\mathcal{M}$-enriched representable presheaf* to be

$$y_{\mathcal{M}}X := 1 \cdot rX.$$

In the special case where $\mathcal{M} = \mathrm{sSet}_{\text{Kan-Quillen}}$ we will sometimes by abuse of notation write rX for $y_{\mathrm{sSet}_{\text{Kan-Quillen}}}X$ when no confusion may be had. We refer to such presheaves as *simplicial representables*.

Remark 8. Note that the simplicial representables may be obtained by viewing the ordinary representable functor as a discrete simplicial presheaf.

Notation 6. "The $\mathcal{M}$-enriched representable presheaf" seems a bit tedious to write, so if no confusion may be had, we will simply refer to the $\mathcal{M}$-enriched representable presheaves as representables.

Lemma 1 (Enriched Yoneda Lemma, Thm 6.3.5 [BOR]). *The two objects in $\mathcal{M}$, $\hom(y_{\mathcal{M}}X, F)$ and $F(X)$, are isomorphic.*

Proof. We show this when $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ has the projective model structure. Since $\mathcal{M}$ is assumed to be symmetric monoidal, we may regard it as canonically as enriched over itself. Using the characterization of $(H/\mathcal{M})$-local equivalences found in 3.2.2, we wish to find a suitable set of morphism on which to localize. Consider for a covering sieve R over X a presheaf $\mathbb{1}_R$ defined by

$$\mathbb{1}_R(Y) = \begin{cases} 1 \cdot y_{\mathcal{M}}Y, & \text{if } Y \text{ is a domain of a morphism in } R \text{, and} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Thus we by postcomposition obtain for every sieve an induced morphism $\mathbb{1}_R \to y_{\mathcal{M}}X$. Denote the set of all such morphisms H. Then we have that $\mathcal{L}_{H/\mathcal{S}}\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ exists by Theorem 10, and any fibrant object F in $\mathcal{L}_{H/\mathcal{S}}\mathcal{M}$ is fibrant in $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$ and satisfies that

$$\hom(y_{\mathcal{M}}X, F) \xrightarrow{\sim} \hom(\mathbb{1}_R, F).$$

By the Lemma 1, $\hom(y_{\mathcal{M}}X, F) = F(X)$. One may easily verify that $\mathbb{1}_R = \mathrm{colim}_{Y \in R/\mathcal{C}}\, y_{\mathcal{M}}Y$. Since the unit is cofibrant, we hence obtain a colimit of cofibrant objects, i.e. a homotopy colimit. Hence we may compute

$$\hom(\mathrm{hocolim}\, y_{\mathcal{M}}Y, F) = \mathrm{holim}\, \hom(y_{\mathcal{M}}Y, F) = \mathrm{holim}\, F(Y)$$

whence the proposition is a restatement of Definition 39 of F being an $\mathcal{M}$-valued sheaf. ■

A consequence of the above theorem is that the weak equivalences between the fibrant objects in the described structure are objectwise weak equivalences.

4.2.1 Hypercovers and Descent Conditions

Although Definition 39 is perhaps the most natural, it does not generalize to other local-to-global notions easily. If we want to formulate some stronger local-to-global property we must instead use another definition. What we can do is rephrase Definition 39 into what is known as *Čech descent*, which then naturally extends to *hyper descent*.

Definition 41. Let $\mathcal{C}$ be a site and let $U \to X$ be a morphism in $\mathcal{C}$. The *Čech complex* associated to a covering $\{U_i\}$ of an object X in $\mathcal{C}$, is the simplicial object in $\mathcal{C}$ which in level n is given by $\coprod_{a_0,\ldots a_n} U_{a_0} \times_X \ldots \times_X U_{a_n}$.

Definition 42. We say that an $\mathcal{M}$-valued presheaf F on a site $\mathcal{C}$ satisfies *Čech descent* if for all objects $X \in \mathcal{C}$ and all coverings $\{U_a\}$ of X, $F(X)$ is the same as the inverse limit of the cosimplicial diagram

$$\prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_a \times_X U_b) \rrightarrow \dots$$

Remark 9. We may state the definition more compactly as follows: A presheaf F satisfies Čech descent if $F(X) \cong \lim_n F(\check{C}U_n)$ where $\check{C}U$ is the Čech complex associated to the covering $\{U_i\}$ of X. Also note that a presheaf satisfying Čech descent is the same as a sheaf.

These sort of descent properties generalize naturally to a homotopy theoretic setting by substituting limits by homotopy limits and isomorphisms by weak equivalences, precisely as in Definition 39.

Definition 43. We say that an $\mathcal{M}$-valued presheaf satisfies Čech descent if

$$F(X) \simeq \operatorname{holim}\left(\prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_a \times_X U_b) \rrightarrow \dots\right)$$

for all objects $X \in \mathcal{C}$ and all coverings $\{U_a\}$ of $\mathcal{C}$.

What the Čech descent property also allows us to do is vary the *complex* for which a presheaf may satisfy descent. Thus we may obtain what are known as *hypersheaves*. The general idea is to construct a simplicial object in $\mathcal{C}$ which in level n is a cover of the $n+1$-fold intersections.

Remark 10. If $\mathcal{S}$ is a symmetric monoidal model category, we may lift the complexes into $\mathcal{S}\,\mathrm{pre}(\mathcal{C})$ by using the enriched Yoneda embedding.

It is in fact using the above remark that one major result regarding local model structures was shown. Lifting hypercovers into the category of simplicial presheaves, Dugger, Hollander and Isaksen showed the existence of a local model structure obtained by performing a left Bousfield localization with respect to them. See Section 5.1.1, or indeed the original source [DHI].

Chapter 5

Local Model Structures on Presheaves

In this chapter we introduce another kind of local model structure on the category of $\mathcal{M}$-valued presheaves, where the weak equivalences may be checked stalkwise. Of course objectwise weak equivalences satisfy this condition but there are also stalkwise weak equivalences which are not objectwise. Although a number of such model structures have been described now, the first such structure was shown to exist by Jardine [J].

5.1 Jardine's Local Model Structure

In 1987, Jardine [J] showed the existence of a model structure on presheaves taking values in the category of simplicial sets which was local in nature. A lot of fruitful research has in some way or another utilized this structure. A prominent case which comes to mind is the paper "$\mathbb{A}^1$-Homotopy Theory of Schemes" by Morel and Voevodsky [MV]. Here, the authors developed a new homotopic scheme theory which was modelled after Jardine's model structure. Their work laid ground for a whole field of study and the reader is encouraged to read about the consequences.

The model structure was constructed by performing a left Bousfield localization on the injective model structure on $\mathrm{sPre}(\mathcal{C})$. Concretely, we may for any simplicial set X consider its geometric realization $|X|$ on which we may compute the homotopy groups, or indeed take the path components in the zeroth level. In such a way we obtain a presheaf of groups from a simplicial presheaf. Thus, a morphism $f : F \to G$ of simplicial presheaves induce morphisms of sheaves of homotopy groups.

Definition 44. Let $f : F \to G$ be a morphism of simplicial presheaves on a site $\mathcal{C}$. Let $\pi_0(X)$ be the set of path components of a topological space X and let $\pi_n(X, x_0)$ be its n:th homotopy group. Further, for a presheaf A, let A^+ be its associated sheaf. We say that f is a *local simplicial equivalence* if

1. $\pi_0^+(F) \to \pi_0^+(G)$ is an isomorphism in $\mathrm{Sh}(\mathcal{C})$.
2. $\pi_n^+(F|_U, x) \to \pi_n^+(G|_U, f(X))$ is an isomorphism for all $U \in \mathcal{C}$, and $x \in F_0(U)$. Here $F|_U$ denotes the pullback to the over category $\mathcal{C}/U$ and $F_0(U)$ are the zero-simplicies of $F(U)$.

Remark 11. The quite technical definition above reduces to a morphism f being a local simplicial equivalence if and only if it induces simplicial equivalences on all stalks when the site $\mathcal{C}$ has enough points.

Jardine shows the existence of a model structure on the category of simplicial presheaves where the weak equivalences are the local equivalences.

Theorem 14. *There exists a model structure on the category of simplicial presheaves in which the cofibrations are the monomorphisms and where the weak equivalences are the local simplicial equivalences. We refer to this structure as Jardine's local model structure.*

The dual result, that we may obtain a local model structure in the projective case was shown by Blander [BL].

Theorem 15 ([BL, Thm. 1.6])**.** *Let $\mathcal{C}$ be an essentially small site. Then* $\mathrm{sPre}(\mathcal{C})$ *admits a proper simplicial model structure if we define weak equivalences to be the local simplicial equivalences, cofibrations to be the projective cofibrations, and fibrations to be the morphisms which has the RLP with respect to the acyclic cofibrations.*

Later on, Dugger, Hollander and Isaksen [DHI] showed that we may obtain both Jardine's local model structure and the local projective model structure via a left Bousfield localization with respect to a class of morphisms called *hypercovers* (see Definition 47).

Remark 12. Just as in Remark 5, we need to be careful. The above local model structures are not obtained via an application of Theorem 10, as we are not certain that the local simplicial equivalences constitute a small set rather than a proper class.

5.1.1 The Left Bousfield Localization at the Hypercovers

In a paper published in 2002, Dugger, Hollander and Isaksen [DHI] showed that we may construct both Jardine's and Blander's local model structures by (left Bousfield) localizing at a class of morphisms which are the aforementioned hypercovers, introduced in Section 4.2.1 as one way of generalizing Čech complexes. In order to define the simplicial hypercovers properly, we introduce some related concepts.

Definition 45. Let $f : F \to G$ be a morphism of simplicial presheaves. We say that f is a *local fibration* if for every diagram

$$\begin{array}{ccc} \Lambda[n]_i \otimes rX & \longrightarrow & F \\ \downarrow & & \downarrow \\ \Delta[n] \otimes rX & \longrightarrow & G \end{array}$$

we may find a covering sieve R such that for all $U \to X$ in R, the diagram obtained by restricting to U

$$\begin{array}{ccc} \Lambda[n]_i \otimes rU & \longrightarrow & F \\ \downarrow & & \downarrow \\ \Delta[n] \otimes rU & \longrightarrow & G \end{array}$$

has a lifting.

Remark 13. Any objectwise fibration is a local fibration since the set of all morphisms of the form $\Lambda[n]_i \otimes rX \hookrightarrow \Delta[n] \otimes rX$ generates the projective acyclic cofibrations.

Definition 46. A *local acyclic fibration* is a morphism which is both a local weak equivalence and a local fibration.

We now arrive at the definition of a simplicial hypercover.

Definition 47. Let U be a simplical presheaf with an augmentation $U \to rX$. We say that $U \to rX$ is a *simplicial hypercover* if U in each level is a coproduct of representables and $U \to rX$ is a local acyclic fibration.

Although not obvious from this compact definition of a simplicial hypercover, this is indeed what Remark 10 referred to. The category of simplicial presheaves is symmetric monoidal with the unit being the constant simplicial presheaf which to any object assigns the terminal object. To see the equivalence we introduce *generalized covers* and reformulate the latter condition of the definition.

Definition 48. Let $f : F \to G$ be a morphism of set-valued presheaves. We say that f is a *generalized cover* if given any map $rX \to G$ there is a covering of X, R, such that for every morphism $U \to X \in R$ the composite $rU \to rX \to G$ lifts through f.

Definition 49. For a simplicial presheaf F, we denote by $\tilde{M}_n F$ the Set-valued presheaf which is given by

$$X \mapsto \{f : \partial\Delta[n] \to F(x) \mid f \in \operatorname{Mor} \mathrm{sSet}\}.$$

Remark 14. It may in fact be shown [DHI, Prop. 3.2] that a local acyclic fibration $f : F \to G$

is precisely a morphism satisfying that

$$F_n \to \tilde{M}_n F \times_{\tilde{M}_n G} G$$

are all generalized covers.

Using these observations, we may define a hypercover precisely as a simplicial presheaf U with an augmentation $U \to X$ such that $U_n \to \tilde{M}_n U, \forall n \geq 1$ are all generalized covers. A Čech complex is precisely a hypercover for which all these morphisms are isomorphisms.

Theorem 16 (Theorem 1.2, [DHI]). *The local projective (injective) model structure on* $\mathrm{sPre}(\mathcal{C})$ *may be obtained by performing a left Bousfield localization at the collection of hypercovers in the projective (injective) model structure on* $\mathrm{sPre}(\mathcal{C})$.

As the reader may have become accustomed to at this point, the standard warning regarding localizing at proper classes applies. Localizing at the class of hypercovers does not guarantee a model structure. Instead the authors show the important Theorem 17.

Definition 50. A *refinement* of a hypercover $U \to X$ is another hypercover $V \to X$ which factors through $U \to X$.

Definition 51. We say that a set S of hypercovers is *dense* if every hypercover may be refined through a hypercover in S.

Theorem 17. *The class of hypercovers contains a dense (small) set of hypercovers.*

We are thus allowed to localize at such a dense set of hypercovers and it is this localization which is shown to coincide with both the localization at all hypercovers as well as Jardine's local model structure. An added benefit of this approach is that the authors were able to classify the fibrations in Jardine's local structures on simplicial presheaves.

Notation 7. For a simplicial set X, we write cX for the constant cosimplicial simplicial set.

Definition 52. We say that a morphism of simplicial presheaves $F \to G$ *satisfies descent* for a hypercover $U \to X$ if the induced morphism

$$F(X) \to \mathrm{holim}_{\Delta} \left[cG(X) \times_{G(U)} F(U) \right]$$

is a weak equivalence.

Theorem 18. *The fibrations in the injective (projective) local model structure on the category of simplicial presheaves are the injective (projective) fibrations satisfying descent for all hypercovers.*

Perhaps more importantly, they characterize a set of generating acyclic cofibrations in the local model structures. Often when using a left Bousfield localization, one loses control over the generating acyclic cofibrations.

For any hypercover $U \to X$, let $A(U) := \mathrm{hocolim}_n U_n$ and factor the morphism $A(U) \to X$ into

$$A(U) \rightarrowtail B(U) \xrightarrow{\sim} X$$

Theorem 19. *Let $J_{\mathcal{C}}$ consist of all morphisms of the form $\Lambda[n]_i \otimes rX \hookrightarrow \Delta[n] \otimes rX$ for all $X \in \mathcal{C}$, and all morphisms of the form*

$$[A(U) \otimes \Delta[n]] \coprod_{A(U)\otimes\partial\Delta[n]} \to B(U) \otimes \Delta[n]$$

for all hypercovers $U \to X$ in some dense set of hypercovers S. Then $J_{\mathcal{C}}$ is a set of generating acyclic cofibrations for $\mathrm{sPre}(\mathcal{C})_{\mathrm{proj,loc}}$.

5.2 General Results

Apart from the results on simplicial presheaves with the classical model structure there have been numerous results published. In the doctoral thesis of Meadows [M2], supervised by Jardine, he constructs local model structures on three model categories: the category of simplicial sets equipped with the Joyal model structure, the category of simplicial sets equipped with the model structure for complete Segal spaces and lastly for the Bergner model structure on the category of simplicial categories. However, Meadows uses a different sort of localization - that of Boolean localization which is beyond the scope of this thesis. A quick description of the local Joyal model structure on simplicial presheaves may however be found in Section 8.2. We will instead focus on works by Ayoub [AY] and Drew [DR] in this section.

5.2.1 Coefficient Model Categories

Ayoub presents in his paper [AY] a local structure on presheaves taking values in a special kind of combinatorial model categories. The precise conditions on the model category are very technical and the author has so far not been able to decipher where the assumptions are necessary. We will establish the language of Ayoub in order to state the theorem. All introduced definitions before Theorem 20 may be found in [AY, Section 4.4].

Definition 53. We let $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C}, \mathrm{Spc})$ be the ∞-categorical Yoneda embedding described for example in [L, §5.1.3].

Definition 54. We say that a small category $\mathcal{C}$ is *β-small* for some cardinal β if the cardinality of $\mathcal{C}$ is smaller than or equal to β.

Definition 55. Let $\mathcal{C}$ be a category and let X be an object in $\mathcal{C}$. Let α be a cardinal. We say that X is *α-accessible* if for any functor $\mathbf{Y} : \mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ a totally ordered set such that $|\mathcal{I}| \geq \alpha$, the map

$$\mathrm{colim}_{i\in\mathcal{I}} \hom_{\mathcal{C}}(\mathbf{Y}(i), X) \to \hom_{\mathcal{C}}(\mathrm{colim}_{i\in\mathcal{I}} \mathbf{Y}(i), X)$$

is an isomorphism.

Definition 56 ([AY], Def.4.2.16)**.** Let $\mathcal{C}$ be a category and let α be a cardinal. We say that $\mathcal{C}$ is *α-presentable* if the following conditions are satisfied where β is any cardinal bigger than or equal to α:

(i) The small limits and small colimits are representables in $\mathcal{C}$. The filtered colimits commute with finite limits. The β-filtered colimits commutes with limits indexed by β-small categories.

(ii) The monomorphisms in $\mathcal{C}$ are universal.

(iii) All objects in $\mathcal{C}$ are accessible. A subobject of a β-accessible object is still β-accessible.

(iv) Any object in $\mathcal{C}$ is the β-filtered colimit of its β-accessible subobjects.

(v) The subcategory $\mathcal{C}_\beta$ formed by its β-accessible objects is essentially small.

We say that $\mathcal{C}$ is finitely presentable if α is finite and greater than or equal to 5. We say that $\mathcal{C}$ is presentable if the conditions hold for some cardinal α.

Definition 57. We say that a model category $\mathcal{M}$ is *α-presentable through cofibrations* if the following conditions are satisfied:

(i) The underlying category of $\mathcal{M}$ is α-presentable.

(ii) The cofibrations are the monomorphisms.

(iii) Let Cof_α be the class of cofibrations with α-accessible targets. Then $\mathrm{Fib} = (\mathrm{Cof}_\alpha \cap \mathrm{WE})^{\boxslash}$ and $\mathrm{Fib} \cap \mathrm{WE} = (\mathrm{Cof}_\alpha)^{\boxslash}$.

Next we need some kind of compactness-condition on the objects in a stable model category. For reference, and since the notion of compactness is important, not only in general but also for the results of Drew, we include the classical definition of a compact object.

Definition 58. Let $\mathcal{C}$ be a locally small category with filtered colimits. An object X of $\mathcal{C}$ is said to *κ-compact* for a cardinal κ if $\hom(X, -)$ commutes with κ-filtered colimits. X is said to be *compact* if this holds for some unspecified cardinal κ.

Ayoub uses a more specified version of compactness.

Definition 59. Let $\mathcal{M}$ be a stable model category. An object $A \in \mathcal{M}$ is said to be *homotopically compact* if for all $n \in \mathbb{Z}$, the functor $\hom(A, -[n]) : \mathcal{M} \to \mathrm{Set}$ commutes with small filtered colimits.

Remark 15. It is not at all clear at first glance what $M[n]$ would mean for an object $M \in \mathrm{Ho}\,\mathcal{M}$. It turns out however that the homotopy category of a stable model category is *triangulated* whence objects naturally have such a structure. A good source for information on triangulated categories is for example [NE].

With this notation we may define a coefficient category.

Definition 60 ([AY, Def. 4.4.23])**.** Let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is a *coefficient model category* if the following conditions are satisfied:

(i) $\mathcal{M}$ is left proper, presentable and stable.

(ii) The weak equivalences are stable under finite coproducts.

(iii) There exists a set $\mathcal{E}$ of homotopically compact objects in $\mathcal{M}$ which generate $\operatorname{Ho}\mathcal{M}$ with infinite sums.

We take the set $\mathcal{E}$ to be part of the data of a coefficient category.

Next we construct what may be seen as related to the sheaves of homotopy groups found in the definition of a local simplicial equivalence, or indeed in Theorem 21 to come. Since $\mathcal{M}$ is not assumed to be simplicial in Ayoub's work, it is not immediately clear what construction should be used.

Definition 61. Let M be an object in $\mathcal{M}$ and let H be a set. We define $H\underline{\otimes}M$ to be the colimit of the diagram

$$H \to * \xrightarrow{M} \mathcal{M}$$

where we view H as a discrete category. We may view $H \otimes M$ as a coproduct of $|H|$ copies of M.

Definition 62. Let F be a Set-valued presheaf and let G be an $\mathcal{M}$-valued presheaf. We denote by $F\underline{\otimes}G$ the $\mathcal{M}$-valued presheaf given by

$$U \mapsto F(U)\underline{\otimes}G(U).$$

Definition 63. Let H and K be $\mathcal{M}$-valued presheaves. We denote by $\underline{\prod}_0(H,K)$ the Set-valued presheaf defined by

$$U \mapsto \hom_{\operatorname{Ho}\mathcal{M}\operatorname{Pre}(\mathcal{C})(\mathcal{C})}(U \overset{L}{\underline{\otimes}} H, K),$$

where the derived functor $U \overset{L}{\otimes} -$ is taken relative to the injective model structure on $\mathcal{M}\operatorname{Pre}(\mathcal{C})$. We denote by

$$\underline{\prod}_0^{\text{top}}(H,K)$$

the sheaf associated to $\underline{\prod}_0(H,K)$.

Remark 16. In the case where $H = A_{\text{cst}}$ is constant, we have

$$\underline{\prod}_0^{\text{top}}(H,K)(U) = \hom_{\operatorname{Ho}\mathcal{M}}(A, K(U))$$

and we will for simplicity denote $\underline{\prod}_0^{\text{top}}(H,K)$ by $\prod_0^{\text{top}}(A,K)$

Definition 64. Suppose that $(\mathcal{M}, \mathcal{E})$ is a coefficient model category and let $f : F \to G$ be a morphism of $\mathcal{M}$-valued presheaves. We say that f is *top-local equivalence* if for all $n \in \mathbb{Z}$, and all $M \in \mathcal{E}$, the induced morphism on the Set-valued sheaves

$$\prod_0^{\text{top}}(A, F[n]) \to \prod_0^{\text{top}}(A, G[n])$$

are isomorphisms. We denote the class of all top-local equivalences by $\mathcal{L}_{\text{top}}$.

We here arrive at the model structure shown by Ayoub:

Theorem 20. *Let $\mathcal{M}$ be a coefficient model category and let $\mathcal{C}$ be a small site. The left Bousfield localization at the class of* top*-local equivalences exists for both the projective and the injective model structures on $\mathcal{M}\operatorname{Pre}(\mathcal{C})$. The model structures thus obtained will be called the projective* top*-local model structure and the injective* top*-local model structure.*

Proof. This is [AY, Prop. 4.4.31 & Def. 4.4.33]. ∎

5.2.2 Locally Presentable ∞-categories

Drew on the other hand works in an ∞-categorical setting. There are a number of equivalent ways of thinking about ∞-categories, one of which is to think of an ∞-category as a quasi-category. This is not an obvious observation, as the quasi-category is by definition a simplicial set such that any inner horn has fillers, i.e. that for any $n \geq 2$ and any $0 < i < n$, we may find a lifts in any diagram of the form

$$\begin{array}{ccc} \Lambda[n]_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

For a good introduction to how we may view such simplicial sets as ∞-categories I refer the reader to [L, §1.1]. The intuition one should have is that the n-faces act as n-morphisms and that the filler condition may be viewed as ways to compose morphisms.

Example 6. Let S be a quasi-category and let $f : x \to y$ and $g : y \to z$ be elements in S_1. Then we may construct a morphism from $\Lambda[2]_1$ into S, constructing the diagram:

$$\begin{array}{ccc} & & z \\ & \nearrow & \big\uparrow \\ x & \longrightarrow & y \end{array}$$

where the dotted line is obtained by extending $\Lambda[2]_1 \to S$ to $\Delta[2] \to S$. We may view the dotted arrow as a composition of f and g.

Hopefully the reader now has at least an inclination as to why quasi-categories are a good model for ∞-categories. There are however other equivalent ways to model ∞-categories.

Remark 17. In the following paragraphs regarding Drew's theorem, Theorem 21, it is implied that we may handle a locally presentable quasi-category $\mathcal{M}$ as a simplicial category. By Dugger's theorem, Theorem 22, any locally presentable quasi-category arises as the bifibrant objects in a simplicial combinatorial model category however, whence this makes sense. Another way to see this is to use the adjunction described in Section 8.4.

Now we move on to describe Drew's model structure before we compare it to the top-local structures. In the discussion in Section 2 of [DR], Drew points out the existence of a local model structure on any category of presheaves which take values in a locally presentable quasi-category. The following theorem characterizes the weak equivalences in this local model structure given that the quasi-category is locally $\aleph_0$-presentable.

Theorem 21 ([DR], Prop. 2.5)**.** *Let $\mathcal{M}$ be a locally $\aleph_0$-presentable quasi-category, $f : F \to G$ a morphism of $\mathcal{M}$-valued presheaves and $*$ the terminal object in $\mathcal{C}$. The following are equivalent:*

1. *For each $n \in \mathbb{N}_{>0}$, each 0-simplex x of $\hom_{\mathcal{M}}(M, F(*))$ and each $M \in \mathcal{M}_{\aleph_0}$, the τ-sheafifications of the morphisms of presheaves of sets*

$$\pi_0(\hom_{\mathcal{M}}(M, F(-))) \to \pi_0(\hom_{\mathcal{M}}(M, G(-)))$$

   and

$$\pi_n(\hom_{\mathcal{M}}(M, F(-)), x) \to \pi_n(\hom_{\mathcal{M}}(M, G(-)), f(x))$$

   are all isomorphisms of sheaves.

2. *For each $M \in \mathcal{M}_{\aleph_0}$, the morphism*

$$\mathfrak{l}_\tau \hom_{\mathcal{M}}(M, F(-)) \to \mathfrak{l}_\tau \hom_{\mathcal{M}}(M, F(-))$$

   is an equivalence in the full sub-quasi-category consisting of the hypersheaves in $\mathcal{M}\,\mathrm{Pre}(\mathcal{C})$.

3. *The morphism $f : F \to F'$ is a τ-local equivalence.*

Drew's result shows that for any simplicial combinatorial model category whose underlying ∞-category is locally $\aleph_0$-presentable may be equipped with a local model structure with weak equivalences as described in Theorem 21. In particular we have a corollaries comparing Drew's structure structure with the top-local structure of Theorem 20.

Corollary 1. *Let $\mathcal{M}$ be a combinatorial model category which is also a coefficient category (Definition 60) and whose underlying quasi-category is locally $\aleph_0$-presentable. Then the τ-local model structure on the $\mathcal{M}$-valued presheaves exists and is canonically equivalent to its underlying quasi-category. The weak equivalences are those described in Theorem 21.*

Corollary 2. *If $\mathcal{M}$ is a coefficient model category, then the* top*-local model structures of Theorem 20 coincides with the structure described by Drew in Corollary 1.*

Proof. This is [DR, Rmk. 2.6]. ■

There is a subtlety here in which we require $\mathcal{M}$ to be locally $\aleph_0$-presentable rather than just locally presentable. In the proof presented by Drew, he shows an equivalence between the left exact functors

$$\mathrm{Fun}^{\mathrm{lex}}\left((\mathcal{V}_{\aleph_0})^{\mathrm{op}}, \widehat{\mathrm{Sh}}_\tau(\mathcal{C}, \mathrm{Spc})\right) \simeq \widehat{\mathrm{Sh}}_\tau(\mathcal{C}, \mathcal{V})$$

which only holds in this particular setting.

Remark 18. Since a site with enough points by definition means that isomorphisms of set-valued sheaves may be checked stalkwise, the first condition guarantees that we have stalkwise isomorphisms on the set-valued sheaves

$$\widetilde{\pi}_0(\mathrm{hom}_{\mathcal{M}}(M, F(-)) \to \widetilde{\pi}_0(M, G(-))$$

and

$$\widetilde{\pi}_n(\mathrm{hom}_{\mathcal{M}}(M, F(-), x) \to \widetilde{\pi}_n(M, G(-), f(x))$$

if and only if the morphism $F \to G$ is τ-local.

Lemma 2. *Let $(\mathcal{C}, \tau)$ be a site with enough points and let $\mathcal{M}$ be a simplicial combinatorial model category with a locally $\aleph_0$-presentable underlying quasi-category. A morphism of $\mathcal{M}$-valued presheaves $F \to G$ is a τ-local equivalence if and only if the morphisms $\mathrm{map}_{\mathcal{M}}(M, F) \to \mathrm{map}_{\mathcal{M}}(M, G)$ are stalkwise weak equivalences for all $M \in \mathcal{M}_{\aleph_0}$.*

Proof. We have that $\mathrm{map}(M, F(-))$ and $\mathrm{map}(M, G(-))$ are simplicial presheaves. Hence [DHI, Thm. 1.1 & Thm. 1.2] show that Condition 3 in Theorem 21 is true if and only if the induced morphisms on the stalks are weak equivalences. ■

Corollary 3. *Let $\mathcal{C}$ have enough points. For a morphism of simplicial presheaves $F \to G$, the induced morphisms on the stalks $F_p \to G_p$ as p ranges over the points on the site are weak equivalences if and only if $\mathrm{map}(M, F(-)) \to \mathrm{map}(M, G(-))$ is a local weak equivalence for every $M \in \mathrm{sSet}_{\aleph_0}$.*

Proof. Combine [DHI] and 2. ■

Remark 19. The alert reader may have noted a difference in the used sheaves of homotopy groups. [J] uses $\pi_n(F(-), x)$ as the homotopy groups whereas [DR] uses the homotopy groups of the function complex $\pi_n(\mathrm{map}(M, F(-)), x)$. The latter includes the former in that we may choose M to be the tensor unit. Hence the corollary above is mainly interesting for the other direction.

What one would wish for is a similar statement for an arbitrary simplicial model category, i.e. that τ-local equivalences may be checked stalkwise in an arbitrary model category presenting a quasi-category such as in Theorem 21. The author was not able to provide such a result in the allotted time frame and the partial findings are presented in the next chapter.

Chapter 6

The Case for Left Proper Simplicial Combinatorial Model Categories

We begin with the important Dugger's theorem after Daniel Dugger.

Theorem 22 ([D1, Thm.1.2]). *Any $(\infty, 1)$-category may be presented by a left proper, simplicial combinatorial model category in which every object is cofibrant.*

It is natural to ask whether we may translate Drew's $(\infty, 1)$-categorical findings into concrete model categorical settings. By the above theorem, left proper simplicial combinatorial model categories are the primary suspects on which to find a hypercompletion. Unless otherwise stated, $\mathcal{M}$ will in this section be such a model category, and M will be an object in $\mathcal{M}$.

Thinking of Drew's results, one might expect there to exist a model structure when we localize at morphisms of the form $M \otimes U \to M \otimes X$ where $U \to X$ is a simplicial hypercover. Indeed, by adjointness we have isomorphisms:

$$\hom(M \otimes U, F) \cong \hom(U, \mathrm{map}(M, F)).$$

and hence the following lemma says that an injective-fibrant F being local with respect to $M \otimes U \to M \otimes X$ is the same as $\mathrm{map}(M, F)$ satisfying descent for the hypercover $U \to X$.

Lemma 3 ([DHI, Lemma 4.4]).

1. *A simplicial presheaf F satisfies descent for a hypercover $U \to X$ if and only if*

$$\mathrm{Map}(X, \tilde{F}) \to \mathrm{Map}(U, \tilde{F})$$

is a weak equivalence of simplicial sets, where $\tilde{F}$ is an injective-fibrant replacement for F.

2. *A simplicial presheaf F satisfies descent for a hypercover $U \to X$ if and only if*

$$\mathrm{Map}(X, \hat{F}) \to \mathrm{Map}(U', \hat{F})$$

is a weak equivalence of simplicial sets, where U' is a projective-cofibrant approximation for U and $\hat{F}$ is an objectiwise fibrant approximation for F.

Of course we have some cardinality issues in that the cofibrant objects are not necessarily a small set. We do however have some small inclination that this is the correct way of proceeding.

Remark 20. As we wish for our result to coincide with Dugger, Hollander and Isaksens when $\mathcal{M} = \mathrm{sSet}$, one would wish for that a presheaf satisfies descent for the hypercover $U \to X$ if and only if it is local with respect to all morphisms $K \otimes U \to K \otimes X$ where K is any simplicial set. In other words if $\mathrm{Map}(M \otimes U', F) \to \mathrm{Map}(M \otimes X, F)$ being weak equivalences is indeed equivalent to $\mathrm{Map}(U', F) \to \mathrm{Map}(X, F)$ being weak equivalences.

Letting $M = *$, it is clear that the former imply the latter. Suppose now that $\mathrm{Map}(U', F) \to \mathrm{Map}(X, F)$ is a weak equivalence and let M be a cofibrant simplicial set, i.e. any simplicial set. We have

$$\begin{aligned}
\mathrm{Map}(\mathrm{hocolim} \prod M \otimes U_n, F) &= \mathrm{holim} \prod \mathrm{Map}(M \otimes U_n^a, F) \\
&= \mathrm{holim}\, \mathrm{Map}(U_n^a, \mathrm{Map}(M, F(-))) \\
&= \mathrm{holim} \prod \mathrm{Map}(M, F(U_n^a)) \\
&= \mathrm{Map}(M, \mathrm{holim} \prod F(U_n^a))
\end{aligned}$$

and

$$\mathrm{Map}(M \otimes X, F(-)) = \mathrm{Map}(M, F(X)).$$

Since M is cofibrant and $\mathrm{holim} \prod F(U_n^a) \to F(X)$ is a weak equivalence of fibrant objects, $\mathrm{Map}(M \otimes X, \mathrm{holim} \prod F(U_n^a)) \to \mathrm{Map}(M, F(X))$ is a weak equivalence of fibrant objects. Hence the local objects with respect to the morphisms $M \otimes U \to M \otimes X$ are precisely the hypersheaves when $\mathcal{M} = \mathrm{sSet}_{\text{Kan-Quillen}}$.

Now we have an intuitive way to lift hypercovers to our model category. By considering for any simplicial hypercover $U \to X$ the family of morphisms $M \otimes U \to M \otimes X$ in $\mathcal{M}$, we obtain a reasonable class of morphisms to localize at. In order to get around the cardinality issues, one might consider imposing the condition that $\mathcal{M}$ be monoidal with cofibrant unit and only consider morphisms of the form $1 \otimes U \to 1 \otimes X$. One may also consider using the generating cofibrations in a smart way. However, the author has as of yet not been able to proceed in deriving properties of, or showing existence of, such a localization. In any case, we hope that the reader has found the field interesting and should they feel inclined to do so, they are encouraged to continue where this paper ends.

Chapter 7

Bibliography

[AM] Artin, M. & Mazur, B. (1969). Étale Homotopy. *Lecture Notes in Mathematics*, **100**. Springer-Verlag, Berlin.
ISBN: 978-3-540-36142-8

[AR] Adámek, J. & Rosicky, J. (1994). Locally Presentable and Accessible Categories. *London Mathematical Society Lecture Note Series*. Cambridge University Press.
doi:10.1017/CBO9780511600579

[AY] Ayoub, J. (2007). Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II.

[BAR] Barwick, C. (2010). On Left and Right Model Categories and Left and Right Bousfield Localizations. *Homology, Homotopy and Applications*, **12**(2), 245–320, International Press of Boston.
https://www.intlpress.com/site/pub/pages/journals/items/hha/content/vols/0012/0002/a009/index.php

[BL] Blander, B. (2001). Local Projective Model Structures on Simplicial Presheaves. *K-theory*, **24**(3).
DOI:10.1023/A:1013302313123

[BOR] Borceaux, F. (1994). Handbook of Categorical Algebra 2. *Cambridge University Press*.
ISBN: 9780521441797

[BO1] Borcherds, R. *Algebraic Geometry I*. (2020). Youtube Playlist retrieved 2021-05-09: [Link].

[BO2] Borcherds, R. *Algebraic Geometry II*. (2020). Youtube Playlist retrieved 2021-05-09: [Link].

[DR] Drew, B. (2018). Motivic Hodge Modules.

https://arxiv.org/abs/1801.10129

[D1] Dugger, D. (2000). Combinatorial Model Categories have Presentations. *Advances in Mathematics*, **164**(1), 177-201.
https://doi.org/10.1006/aima.2001.2015

[D2] Dugger, D. (2001). Universal Homotopy Theories. *Advances in Mathematics*, **164**(1), 144-176,
https://doi.org/10.1006/aima.2001.2014.

[DHI] Dugger, D., Hollander, S., & Isaksen, D. (2004). Hypercovers and simplicial presheaves. *Mathematical Proceedings of the Cambridge Philosophical Society*, **136**(1), 9-51.
doi:10.1017/S0305004103007175

[DI1] Dugger, D. & Isaksen, D. C. (2000). Topological Hypercovers and $\mathbb{A}^1$-realizations. *Mathematische Zeitschrift*, published 2004.
DOI: 10.1007/s00209-003-0607-y

[DI2] Dugger, D. & Isaksen, D. C. (2002). Weak equivalences of simplicial presheaves, *Proc. Conf. in Algebraic Topology*
arXiv:math/0205025

[DS]

[H] Hirschhorn, P. (2003). Model Categories and Their Localizations. *American Mathematical Society, Mathematical Surveys and Monographs*, **99**.
https://doi.org/http://dx.doi.org/10.1090/surv/099

[HAR] Hartshorne, R. (1977). Algebraic Geometry. *Springer-Verlag*, New York.
doi:10.1007/978-1-4757-3849-0

[HOV] Hovey, M. (1991). Model Categories. *Mathematics Subject Classification*, **55**.

[J] Jardine, J.F. (1987). Simplicial Presheaves. *Journal of Pure and Applied Algebra*, **47**(1), 35-87.
[PDF]

[K] Kelly, G.M. (1982), reprint (2005). Basic Concepts of Enriched Category Theory. *Reprints in Theory and Applications of Categories*, **10**.
ISBN: 978-0521287029

[L] Lurie, J. (2008). Higher Topos Theory.
arXiv:math/060804v4

[M1] Meadows, N. (2016). The Local Joyal Model Structure. *Theory and Applications of Categories* **31**.

[M2] Meadows, N. (2018). Local Higher Category Theory. *Electronic Thesis and Dissertation Repository. 5483.*

https://ir.lib.uwo.ca/etd/5483

[MO] Moser, L. (2017). Injective and Projective Model Structures on Enriched Diagrams. *Homology, Homotopy and Appliciations*, **21**(2). DOI:10.4310/HHA.2019.v21.n2.a15

[MV] Morel, F. & Voevodsky, V. (1999). $\mathbb{A}^1$-Homotopy Theory of Schemes. *Publications mathématiques de l'I.H.É.S.*, **90**, 45-143.

[NE] Neeman, A. (2001). Triangulated Categories. *Annals of Mathematics Studies*, **148**. Princeton University Press. ISBN: 9780691086866

[nLab] Quillen Reflection in nLab. [Link]

[Q] Quillen, D. G. (1967). Homotopical Algebra. *Lecture Notes in Mathematics*, **43**, Springer-Verlag, Berlin.

[R1] Riehl, E. (2010). On the structure of simplicial categories associated to quasi-categories. *Math. Proc. Camb. Phil. Soc.*, **150**, 489 doi:10.1017/S0305004111000053

[R2] Riehl, E. (2009). A Concise Definition of a Model Category. [PDF]

[R3] Riehl, E. (2014). Categorical Homotopy Theory. (New Mathematical Monographs), *Cambridge University Press*. doi:10.1017/CBO9781107261457

[SGA4(2)] Artin, M., Grothendieck, A. & Verdier, J.-L. (1963-1964). Théorie des Topos et Cohomologie Étale des Schémas, Tome 2.

Chapter 8

Appendix

In this chapter we include some material on homotopy function complexes and symmetric monoidal categories and the interconnection between simplicial categories and infinity categories. We expand somewhat on the local Joyal model structure on simplicial presheaves and finally give an elegant way of thinking about local equivalences and local objects in terms of an antitone Galois connection.

8.1 Homotopy Function Complexes

The homotopy function complex is a simplicial set which models the morphisms between two objects in a model category. In a non-obvious way, this lets us measure for which objects in a model category certain morphisms behave as weak equivalences. For such objects it is then valuable to know which morphisms in $\mathcal{M}$ which share this property.

In order to formalize homotopy function complexes, we need to introduce cosimplicial and simplicial resolutions. For these resolutions to make sense, the reader needs to know of a model structure which some functor categories may be endowed with, called the *Reedy model structure*. This model structure is quite technical to define and not particularly illuminating in the author's mind. The interested reader is referred to [H, Ch. 15]. For a thorough investigation about simplicial and cosimplicial resolutions, the reader may consult [H, Ch. 16].

As usual, let $\mathcal{M}$ be a model category.

Definition 65. Let M be an object in $\mathcal{M}$. Then we define $\mathrm{cc}_* M$ to be constant cosimplicial object in $\mathcal{M}$ and we define $\mathrm{cs}_* M$ to be the constant simplicial object in $\mathcal{M}$.

Definition 66. Let M be an object in $\mathcal{M}$. A *cosimplicial resolution* of M is defined to be a cofibrant approximation of $\mathrm{cc}_* M$ in the Reedy model structure on $[\Delta, \mathcal{M}]$.

A *simplicial resolution* of M is defined to be a fibrant approximation of $\mathrm{cs}_* M$ in the Reedy model structure on $[\Delta^{\mathrm{op}}, \mathcal{M}]$.

Definition 67. Let X and Y be objects in $\mathcal{M}$. A *left homotopy function complex* from X to Y is a triple $(\widetilde{\mathbf{X}}, Y^f, \mathcal{M}(\widetilde{\mathbf{X}}, Y^f))$ where $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X, Y^f is a fibrant approximation of Y and $\mathcal{M}(\widetilde{\mathbf{X}}, Y^f)$ is the simplicial set given by

$$\mathcal{M}(\widetilde{\mathbf{X}}, Y^f)_n = \mathcal{M}(\widetilde{\mathbf{X}}^n, Y^f).$$

As the name suggests, there is a right analogue of the homotopy function complex as well.

Definition 68. Let X and Y be objects in $\mathcal{M}$. A *right homotopy function complex* from X to Y is a triple $(X^c, \widehat{\mathbf{Y}}, \mathcal{M}(X^c, \widehat{\mathbf{Y}}))$, where X^c is a cofibrant approximation of X, $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y and $\mathcal{M}(X^c, \widehat{\mathbf{Y}})$ is the simplicial set given by

$$\mathcal{M}(X^c, \widehat{\mathbf{Y}})_n = \mathcal{M}(X^c, \widehat{\mathbf{Y}}_n).$$

There is in this case a third analoguous construction.

Definition 69. Let X and Y be objects in $\mathcal{M}$. A *two-sided homotopy function complex* from X to Y is a triple $(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}))$, where $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X, $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y and $\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$ is the simplicial set given by

$$\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})_n = \mathcal{M}(\widetilde{\mathbf{X}}^n, \widehat{\mathbf{Y}}_n).$$

Notation 8. Often we will not care which construction is used, hence the notation $\mathrm{map}(X, Y)$ will be taken to mean either a left, right or two-sided homotopy function complex from X to Y. By abuse of notation we will often write "the" homotopy function complex although it of course depends on the choices of resolutions. One may however show [H, 17.4.14] that any two homotopy function complexes from X to Y are connected via an essentially unique zig-zag of weak equivalences.

It may also be shown that $\pi_0 \mathrm{map}(X, Y) \cong \mathrm{Ho}\,\mathcal{M}(X, Y)$. Hence when constructing the *derived mapping space from X to Y* - the induced simplicial set of $\mathrm{Ho}\,\mathcal{M}(X, Y)$, any two choices of homotopy function complexes give derived mapping spaces connected via an essentially unique isomorphism.

8.2 The Local Joyal Model Structure

As the simplicial hypercovers introduced in Definition 47 are intrinsic to the category of simplicial presheaves, the most natural generalization is to see what happens when changing the model structure. There are a number of model structures on the category of simplicial sets which are important in the field of homotopy theory. Apart from the classical

Kan-Quillen structure, perhaps the most important is the Joyal model structure. The Joyal model structure serves as a presentation of the infinity categories as the fibrant objects are the quasi-categories.

Quite recently there has been work done by Meadows, showing amongst other results the existence of a local model structure on the Joyal simplicial presheaves [M1].

Definition 70. We say that a morphism is a *quasi-injective fibration* if it has the right lifting property with respect to all morphisms which are both monomorphisms and local Joyal equivalences.

Theorem 23 (The Local Joyal Model Structure [M2, Thm. 2.3.3])**.** *There exists a model structure on simplicial presheaves such that*

1. *the cofibrations are the monomorphisms,*
2. *the weak equivalences on a site with enough points are stalkwise Joyal equivalences, and*
3. *The fibrations are the quasi-injective fibrations.*

As mentioned before, the method used by Meadows is not a Bousfield localization. Instead he uses another form of localization called Boolean localization which exceeds the scope of this thesis.

8.3 (Symmetric) Monoidal Model Categories

We describe in this section symmetric monoidal categories. There is a surprising amount of machinery necessary to capture the quite intuitive idea behind such a category. If the reader is somewhat comfortable with these structures she is encouraged to skip this section.

Definition 71. We say that a category $\mathcal{C}$ is *symmetric monoidal* if there is a functor called the *tensor product*

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

together with a *unit object* (or simply "unit"), $1 \in \mathcal{C}$ such that there are two natural isomorphisms called the *left unitor*

$$\lambda_c : 1 \otimes c \longrightarrow c$$

and *right unitor*,

$$\rho_c : c \otimes 1 \longrightarrow c.$$

We further require the existence of a natural isomorphism called the *associator*,

$$\alpha_{c_1,c_2,c_3} : (c_1 \otimes c_2) \otimes c_3 \longrightarrow c_1 \otimes (c_2 \otimes c_3).$$

and another natural isomorphism called the *braiding*,

$$B_{c_1,c_2} : c_1 \otimes c_2 \to c_2 \otimes c_1.$$

The associator and unitors are required to satisfy the **triangle identity**, i.e. that diagrams of the following form

$$\begin{tikzcd} (c_1 \otimes 1) \otimes c_2 \arrow[rr, "\alpha_{c_1,1,c_2}"] \arrow[dr, "\rho_{c_1} \otimes \mathrm{Id}_{c_2}"'] & & c_1 \otimes (1 \otimes c_2) \arrow[dl, "\mathrm{Id}_{c_1} \otimes \lambda_{c_2}"] \\ & c_1 \otimes c_2 & \end{tikzcd}$$

commute, and we further require from the associator that the **pentagon identity** holds, i.e. that all diagrams of the form

$$\begin{tikzcd} & (c_1 \otimes c_2) \otimes (c_3 \otimes c_4) \arrow[dr, "\alpha_{c_1,c_2,(c_3 \otimes c_4)}"] & \\ ((c_1 \otimes c_2) \otimes c_3) \otimes c_4 \arrow[ur, "\alpha_{(c_1 \otimes c_2),c_3,c_4}"] \arrow[d, "\alpha_{c_1,c_2,c_3} \otimes \mathrm{Id}_{c_4}"'] & & c_1 \otimes (c_2 \otimes (c_3 \otimes c_4)) \\ (c_1 \otimes (c_2 \otimes c_3)) \otimes c_4 \arrow[rr, "\alpha_{c_1,(c_2 \otimes c_3),c_4}"'] & & c_1 \otimes ((c_2 \otimes c_3) \otimes c_4) \arrow[u, "\mathrm{Id}_{c_1} \otimes \alpha_{c_2,c_3,c_4}"'] \end{tikzcd}$$

commute.

The associator and the braiding must satisfy the **first hexagon identity**, i.e. that all diagrams of the form

$$\begin{tikzcd} (c_1 \otimes c_2) \otimes c_3 \arrow[r] \arrow[d] & c_1 \otimes (c_2 \otimes c_3) \arrow[r] & (c_2 \otimes c_3) \otimes c_1 \arrow[d] \\ (c_2 \otimes c_1) \otimes c_3 \arrow[r] & c_2 \otimes (c_1 \otimes c_3) \arrow[r] & c_2 \otimes (c_3 \otimes c_1) \end{tikzcd}$$

commute and finally we require of the braiding that

$$B_{c_2,c_1} \circ B_{c_1,c_2} = 1_{c_1 \otimes c_2}.$$

As the reader may have noted, the braiding acts as a sort of symmetry whence the "symmetric" monoidal name.

There are a lot of categories which are symmetric monoidal. The most prominent in category theory is perhaps Set, although in homotopy theory our favorite category sSet also satisfies being symmetric monoidal. In Set we may take the unit to be any singleton and the product to be the cartesian product. In sSet we may similarly take the unit to be a singleton in each level, i.e. the terminal object, and the tensor product to be the level-wise cartesian product.

As one would expect there is a notion of a monoidal *model* category as well. The first condition we need is that of a *closed* monoidal category.

Definition 72. A *closed (symmetric) monoidal category* is a symmetric monoidal category in which the tensor product $- \otimes b$ admits a left adjoint $\underline{\text{hom}}(-, b)$ called the *internal hom-functor*, for all objects $b \in \mathcal{C}$, i.e. we have natural isomorphisms

$$\text{hom}(a \otimes b, c) \cong \text{hom}(a, \underline{\text{hom}}(b, c))$$

for all objects $a, b, c \in \mathcal{C}$.

To incorporate the symmetric monoidal structures into a homotopy theoretic setting we need extra conditions.

Definition 73. A *symmetric monoidal model category* $\mathcal{M}$ is a closed symmetric monoidal category equipped with a model structure satisfying

1. For every pair of cofibrations in $\mathcal{M}$, $I \to J$ and $K \to L$, the induced morphism

$$I \otimes L \coprod_{I \otimes K} J \otimes K \to J \otimes L$$

is a cofibration which is also a weak equivalence if any of $I \to J$ and $J \to K$ is.

2. For any cofibrant object $M \in \mathcal{M}$ and any cofibrant approximation of the unit $\tilde{1}$, the morphism

$$\tilde{1} \otimes M \to 1 \otimes M \cong M$$

is a weak equivalence.

The next section handles simplicial model categories which is a particularly important kind of enriched model category.

8.4 Simplicial Model Categories and Simplicial Sets

In this section we first recall briefly what a simplicial model category is and go on to describe an important adjunction between the category of simplicial categories and the category of simplicial sets. This adjunction is important because it entails why we may see a model category as a presentation of an ∞-category. Namely we may for any model category associate to it a simplicial category via *simplicial localization*, then use this adjunction to associate a quasi-category to the model category. It is perhaps not important to understand the results in this text, but nonetheless it is one of the cornerstones of homotopy theory.

Definition 74. We say that a model category $\mathcal{M}$ is a *simplicial model category* if it is enriched over $\text{sSet}_{\text{Kan-Quillen}}$.

There are numerous results regarding simplicial model categories. The interested reader is encouraged to consult [H, Ch. 9] for a thorough investigation. One thing that is recurring is however that simplicial mapping spaces are well-behaved only when it is from a cofibrant object into a fibrant one.

An important property used in Section 5.2 is that there is an adjunction between the category of simplicial categories, Cat_Δ, and the category of simplicial sets, sSet.

Theorem 24. *There is an adjunction*

$$\mathfrak{C} : \text{Cat}_\Delta \leftrightarrows \text{sSet} : \mathbb{N}.$$

We call the right adjoint $\mathbb{N}$ *the* homotopy coherent nerve.

We shall describe at least part of this adjunction. At least enough to uniquely determine the whole adjunction, although a description for how the adjunction works on every object is hard to achieve. We may define as a preliminary measure

$$(\mathbb{N}\mathcal{C})_n := \text{Cat}_\Delta(\mathfrak{C}\Delta[n], \mathcal{C})$$

which only moves our attention to how $\mathfrak{C}$ behaves. To understand the nerve, we will however only need to know how $\mathfrak{C}$ acts on the simplices. For a detailed outline, we refer the reader to a paper by Riehl [R1]. The objects of $\mathfrak{C}\Delta[n]$ are the same as those of $[n]$, i.e. $\{0, \ldots, n\}$. The morphisms are as follows:

$$\mathfrak{C}\Delta[n](i,j) = \begin{cases} \varnothing, & \text{if } i > j, \\ \Delta[0], & \text{if } i = j, \textit{and} \\ \Delta[1]^{j-i-1}, & \text{if } j > i. \end{cases}$$

These may be derived by considering $P_{i,j}$ to be the partially ordered set of subsets of the interval $[i, j]$ containing both endpoints (with $P_{i,j}$ understood to be empty if $j < i$). We may then think of $\mathfrak{C}\Delta[n](i, j)$ as the (ordinary) nerve of $P_{i,j}$.

One may show using Kan extensions that this description uniquely determines $\mathfrak{C}X$ for an arbitrary simplicial set X, although in a convoluted way. Again the reader is referred to [R1].

Remark 21. The adjunction descibed in 24 is a Quillen equivalence when Cat_Δ is equipped with the Bergner model structure and sSet is equipped with the Joyal model structure.

An important consequence is that we may view any simplicial category as a simplicial set and vice versa. Viewing quasi-categories as ∞-categories, we may see a deep connection between simplicial categories and ∞-categories. In fact, the Bergner model structure may be used to present ∞-categories in an equivalent way to the quasi-categories.

8.5 The Galois Connection Between Local Objects and Local Equivalences

An elegant way of thinking about local objects and local equivalences is that of a (antitone) Galois connection. In this section we define what such a connection is and show that we may find a Galois connection between classes of objects on a site, and morphisms in the target category.

Definition 75. Let $(P, \leq)$ and $(P', \leq)$ be two partially ordered classes. We say that $F : P \to P'$ and $G : P' \to P$ constitute an *antitone Galois connection* if the following holds: $b \leq F(a)$ if and only if $a \leq G(b)$.

Proposition 3. *Let $\mathcal{M}$ be a model category. Let $\mathcal{P}(\mathrm{Ob}\,\mathcal{M})$ be the partially ordered set of subsets of objects in $\mathcal{M}$, with $T \leq T'$ if T is contained in T', and let $\mathcal{P}(\mathrm{Mor}\,\mathcal{M})$ be the partially ordered set of subsets of morphisms in $\mathcal{C}$, again with $S \leq S'$ if S is contained in S'.*

Let $F : \mathcal{P}(\mathrm{Ob}\,\mathcal{M}) \to \mathcal{P}(\mathrm{Mor}\,\mathcal{M})$ be the function which sends a class of objects S to the S-local equivalences, and let $G : \mathcal{P}(\mathrm{Mor}\,\mathcal{M}) \to \mathcal{P}(\mathrm{Ob}\,\mathcal{M})$. Then F and G constitute an antitone Galois connection between $\mathcal{P}(\mathrm{Ob}\,\mathcal{M})$ and $\mathcal{P}(\mathrm{Mor}\,\mathcal{M})$.

Proof. Note first that for $S, S' \in \mathcal{P}(\mathrm{Ob}\,\mathcal{M})$, $S \leq S' \implies F(S') \leq F(S)$. Similarly, for $T, T' \in \mathcal{P}(\mathrm{Mor}\,\mathcal{M})$, $T \leq T' \implies G(T') \leq G(T)$. In other words, both F and G are order reversing.

Suppose that S is a class of objects in $\mathcal{M}$ and T is some class of morphisms T in $\mathcal{M}$. We wish to show that $T \leq F(S)$ if an only if $S \leq G(T)$.

Now suppose first that $T \leq F(S)$. By the observation above we have that

$$T \leq F(S) \implies GF(S) \leq G(T)$$

but S is contained in $GF(S)$ and hence $S \leq G(T)$ whence one implication is clear.

Next suppose that $S \leq G(T)$. Then by the above observation we have that

$$FG(T) \leq F(S).$$

Hence $T \leq F(S)$ and we are done. ∎