SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Quadratic differential forms applied to stability and dissipativeness in the behavioural framework

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2021 - No M9

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Yishao Zhou

2021

Abstract

In dynamical systems it is sometimes not just the system variables that are of interest, but also functionals of these variables. For linear systems these functionals are often quadratic differential forms. We take a behavioural approach to linear systems and quadratic differential forms, focusing on how these can be described using polynomial matrices. Two areas of application are considered, the first being stability where quadratic differential forms are used as Lyapunov functions. The second is dissipative systems and its close connection to LQ-control problems.

Acknowledgements

I want to thank my supervisor Yishao Zhou for all her help, in particular with finding useful material and many productive discussions.

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1 Introduction

In dynamical systems theory we study how a system evolves over time, described using a function of time. Sometimes it is not just the system variables themselves that are of interest, but also functionals of these variables. For linear systems these functionals are often quadratic in the system variables and their derivatives, we call these quadratic differential forms (QDFs for short). A classic example is mechanical systems. If the system variables describe for example the position of something that is moving, then the mechanical energy can be written as a quadratic differential form of the position.

In this text we will look at systems that are described using linear differential equations. The strength of using QDFs is that we can sometimes use them to derive properties of a system without having to solve the differential equations. An example is stability theory, where QDFs appear in the form of Lyapunov functions and can be used to derive several types of stability properties.

In chapter 2 we study linear differential systems, that is solution sets to linear differential equations. We primarily work in the behavioural framework introduced by Jan C Willems, as opposed to the state space framework that is more standard in the field. In the behavioural framework much emphasis is put on how linear systems are represented using polynomial matrices in one indeterminate, and how algebraic properties of these matrices describe the systems they represent.

The main focus is on solutions to differential equations that are also smooth (infinitely many times differentiable). We also look at more general solutions that may not be differentiable everywhere. To do this we introduce weak solutions of differential equations, which solve the equations in the sense of distributions.

The most important representations of behaviours is the kernel representation, including input/output representations. Introducing latent variables also gives us some important representations, the two most important versions being the image representation and state space representations. Both are particularly simple versions of the latent variable representation but in different ways. We also cover two important concepts in systems theory, controllability and observability.

In chapter 3 we introduce quadratic differential forms, which are nicely described using polynomial matrices in two indeterminates. We cover how the algebraic properties of these matrices affect the properties of the QDFs they describe. We also look at how their interplay with the polynomial matrices representing systems affect how the QDFs interplay with systems we apply them to. In particular we look at positivity (and negativity) of QDFs and how this relates to factorizations of the matrices representing the QDFs.

In stability theory we are concerned with the asymptotic behaviour of systems, what happens with them after a long period of time. In chapter 4 we explore the stability of linear systems. We look at four forms of stability; Lyapunov stability, semistability, asymptotic stability and bounded input-bounded output stability. We put particular focus on how QDFs can be used as Lyapunov functions to determine stability properties of linear systems.

Chapter 5 deals with the area of dissipative systems. In these systems a supply (of for example energy) is lost over time through dissipation. This dissipation is suitably described by QDFs. We look at how this can be related to one of the most important type of problems in control theory, LQ-control problems.

2 Linear differential systems

In dynamical systems theory we study how something changes over time in some predictable way. In applications these systems can often be described as solutions to some equations, for example differential equations. There are several ways to present such systems and we will use the behavioural framework, introduced by Jan C Willems. This is in contrast to the state space framework that can be considered the standard approach for the field.

There are several differences between the behavioural framework and the state space framework, some of them purely mathematical in nature while others are of a more philosophical character. In the state space framework one works with variables referred to as state, input and output variables respectively, each with specific properties. For example we have a notion of causality, the input influences the output but not the other way around. In many systems however there is no preferred direction of causality, and our mathematical models should be able to reflect this.

In the behavioural framework we therefore study sets of functions called behaviours (for example solutions to some equations) without any predetermined roles. These ideas are covered in more detail in [6], as well as how this relates to mathematical modelling of real phenomena. We will elaborate on the difference between the different frameworks once we introduce inputs, outputs and state variables.

The theory of this section is in large part based on [6] and [14]. We begin by introducing dynamical systems as they are defined in the behavioural framework.

Definition 2.1. A dynamical system is a triple $\Sigma = (\mathcal{T}, \mathcal{W}, \mathcal{B})$, where $\mathcal{T} \subseteq \mathbb{R}$ is called the time axis, $\mathcal{W}$ is a set called the signal space and $\mathcal{B} \subseteq \mathcal{W}^{\mathcal{T}}$ is called the behaviour. Functions $w \in \mathcal{B}$ are called external signals or external variables.

In this text the time axis $\mathcal{T}$ will always be an interval in $\mathbb{R}$, often $\mathbb{R}$ itself. Such systems are called continuous-time systems, in contrast with discrete-time systems where we choose an interval in $\mathbb{Z}$ instead. We will for the most part use the signal space $\mathcal{W} = \mathbb{R}^d$ for some $d \in \mathbb{N}$ so the external signals are real valued functions. In some cases we will also use complex valued functions, that is $\mathcal{W} = \mathbb{C}^d$.

The elements of $\mathcal{B}$ are usually characterized as the solutions to some differential equations. In this text we will look at solutions to equations of the form

$$R\left(\frac{d}{dt}\right) w = 0 \tag{1}$$

where R is a polynomial matrix with coefficients independent of t. Since w is a map from $\mathbb{R}$ to $\mathbb{R}^d$, R must have d columns. The number of rows in R can vary however. We let $\mathbb{R}^{e \times d}[\xi]$ denote the set of polynomial matrices of size $e \times d$ in the indeterminate ξ with real coefficients , so that $R \in \mathbb{R}^{e \times d}[\xi]$ for some e.

The next consideration to make is what smoothness criterion to put on w. The easiest for proving results is of course to assume that w is smooth. This may seem a bit restrictive, indeed there are applications where functions with for example (a finite number of) discontinuities are of interest. If one wants to study this in full generality, one would need to work with distributions, which is beyond the scope of this text.

A middle ground that is often good enough is to study functions w that are locally integrable. The differential equations are then interpreted to hold in the weak sense (that is, in the sense of distributions). This turns out to be a particularly suitable class of functions to study, as the smooth functions are dense in the locally integrable functions in a certain sense. We will elaborate on the locally integrable functions, their properties and relation to smooth functions in section 2.1.

The behaviours we will study are of the form

$$\mathcal{B} = \left\{ w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d) \middle| R\left(\frac{d}{dt}\right) w = 0 \right\} \tag{2}$$

for some $R \in \mathbb{R}^{e\times d}[\xi]$. We let $\mathfrak{L}^d$ denote the set of all sets that are of the form (2) for some $R \in \mathbb{R}^{e\times d}[\xi]$ (here d is held fixed, but e is not). While $R \in \mathbb{R}^{e\times d}[\xi]$ uniquely determines a behaviour in $\mathfrak{L}^d$ via equation (1), behaviours do not correspond to a unique polynomial matrix. Hence we say that R gives a representation of the behaviour if (1) holds. We will explore the relationship between different representations of the same behaviour, as well as several different types of representations.

The behaviours in $\mathfrak{L}^d$ have two very important properties. They are linear and time invariant. By linear we mean that $\mathcal{B} \in \mathfrak{L}^d$ is a (real) linear vector space. This follows simply from the fact that differentiation and matrix multiplication are linear operators. Note that this extends also to complex valued solutions, the set of complex solutions to some linear equation is a complex linear vector space.

Time invariance is the property that if $w(t) \in \mathcal{B}$, then also $w(t + t_0) \in \mathcal{B}$ for any $t_0 \in \mathbb{R}$. Since t does not appear by itself anywhere in the differential equations studied, we have

$$R\left(\frac{d}{dt}\right)(w(t + t_0)) = \left(R\left(\frac{d}{dt}\right) w\right)(t + t_0).$$

which shows that all behaviours in $\mathfrak{L}^d$ are time invariant. In ODE theory such systems are usually called autonomous, however in systems theory the term autonomous is often used to refer to something else (see Definition 2.12).

The set $\mathfrak{L}^d$ does not include all behaviours in $(\mathbb{R}^d)^{\mathbb{R}}$ that are linear and time invariant however. An example of a linear and time invariant behaviour not on this form is the linear span of all functions of the form e^{nt} where $n \in \mathbb{Z}$. It does not belong to $\mathfrak{L}^1$. The key here is that this behaviour has infinite dimension, and we will see in Proposition 2.13 that the only behaviour in $\mathfrak{L}^1$ that has infinite dimension is the trivial case $\mathcal{B} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$.

2.1 Locally integrable functions and weak solutions

In many applications it is necessary to consider not only smooth function. For example we could have a system where at some time, say $t = 0$, we turn on the power, and so in an instant the power changes from zero to some positive value. To handle such situations we need a broader definition of what it means to solve a differential equation that models the system, one that works even for functions that are not differentiable everywhere.

Take for example the Heaviside step function defined as

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0. \end{cases}$$

Due to the point of discontinuity at $t = 0$, it does not have a derivative in the classical sense. Hence it would not make sense as a solution to any differential equation. By introducing distributions it is possible to give H a derivative, and so we can make sense of it as a solution to a differential equation.

We will not go into the details of the theory of distributions here, but only present some of the most important ideas needed to explain weak solutions. For a more thorough exposition and proofs of claims made here we refer to [5]. We begin by defining the space $\mathscr{D}(\mathbb{R}, \mathbb{R}^d) \subset \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ as the set of smooth functions that have compact support. In the context of distributions the elements of $\mathscr{D}(\mathbb{R}, \mathbb{R}^d)$ are usually called test functions. The space $\mathscr{D}(\mathbb{R}, \mathbb{R}^d)$ can then be endowed with a specific topology. The space of distributions, denoted $\mathscr{D}'(\mathbb{R}, \mathbb{R}^d)$, is then defined as the space of continuous linear functionals on $\mathscr{D}(\mathbb{R}, \mathbb{R}^d)$.

For example, any $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ defines a distribution by the linear functional

$$\varphi \mapsto \int_{\mathbb{R}} w^T \varphi dt. \tag{3}$$

Since the elements of $\mathscr{D}'(\mathbb{R}, \mathbb{R}^d)$ are not actually functions from $\mathbb{R}$ to $\mathbb{R}^d$, a smooth function w is not itself a distribution. We instead use this integral formula to characterize functions in $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ as distributions.

For $T \in \mathscr{D}'(\mathbb{R}, \mathbb{R}^d)$ it is customary to write $\langle T, \varphi \rangle$ for $T(\varphi)$ when $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. Thus the integral in equation (3) is written $\langle w, \varphi \rangle$.

The integral (3) is well defined even for more general w, for example the Heaviside step function H above. We say that $w : \mathbb{R} \to \mathbb{R}^d$ is locally integrable if

$$\int_K |w| dt < \infty$$

for every compact $K \subset \mathbb{R}$. We define a Lebesgue space, denoted $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$, of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^d$, though to be precise the elements of $L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ are equivalence classes of functions, two functions being equivalent if they are equal almost everywhere (with respect to the Lebesgue measure). The space L^1_{loc} is in many ways similar to L^p-spaces.

It is not difficult to see that the map (3) is a well defined map from $\mathscr{D}(\mathbb{R}, \mathbb{R}^d)$ into $\mathbb{R}$ for all $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$. It is in fact a continuous and linear functional on $\mathscr{D}(\mathbb{R}, \mathbb{R}^d)$, and hence such w also define distributions. Less obvious is that for any distribution that is given by a function $w : \mathbb{R} \to \mathbb{R}^d$ and the formula (3), we have $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ (for a proof see [11]). Not all distributions are on this form however. An example of a distribution that is not, is the Dirac delta δ_0 defined by

$$\langle \delta_0, \varphi \rangle = \varphi(0).$$

Sometimes the Dirac delta is written as

$$\delta_0(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0, \end{cases}$$

though this presentation does not make it entirely clear how the distribution behaves. The idea is that δ_0 represent an instantaneous impulse at $t = 0$. It turns out that δ_0 is exactly what we need to make sense of a derivative of the Heaviside function. Away from the origin H is constant so the derivative should be zero, but at zero H makes a jump and in this instant H makes a big leap in value, hence the "infinite" derivative.

To give a precise definition of the distributional derivative we first go back to the smooth case. For any $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ we have by the integration by parts formula

$$\int_{\mathbb{R}} \dot{w}^T \varphi dt = -\int_{\mathbb{R}} w^T \dot{\varphi} dt$$

which we can also write as $\langle \dot{w}, \varphi \rangle = -\langle w, \dot{\varphi} \rangle$. We now define that a distribution u is the weak derivative, or distributional derivative, of w if $\langle u, \varphi \rangle = -\langle w, \dot{\varphi} \rangle$ for every $\varphi \in \mathscr{D}$. This is a generalization of the classical derivative in the sense that if w is differentiable, then its classical derivative satisfies this criterion by the integration by parts formula. It is possible to show that all distributions in fact have a weak derivative in this sense.

It is a more general notion, and we illustrate this with the Heaviside function and δ_0. For any $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R})$ we have

$$-\langle H, \dot{\varphi} \rangle = -\int_{\mathbb{R}} H \dot{\varphi} dt = -\int_0^\infty \dot{\varphi} dt = \varphi(0) = \langle \delta_0, \varphi \rangle$$

so δ_0 really is the weak derivative of H.

The weak derivative allows us to define weak solutions to a differential equation as essentially a function whose weak derivatives satisfy the equation. To be more precise, we generalize weak derivatives to higher order by using integration by parts repeatedly to get

$$\int_{\mathbb{R}} \varphi^T \frac{d^k w}{dt^k} dt = (-1)^k \int_{\mathbb{R}} w^T \frac{d^k \varphi}{dt^k} dt$$

for $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d), \varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. Motivated by this we say that u is the kth weak derivative of w if $\langle u, \varphi \rangle = (-1)^k \langle w, \frac{d^k \varphi}{dt^k} \rangle$. We can now generalize this to linear differential operators $R(\frac{d}{dt})$, where $R \in \mathbb{R}^{e\times d}[\xi]$ by the following. If w is smooth, then

$$\int_{\mathbb{R}} \varphi^T R\left(\frac{d}{dt}\right) w dt = \int_{\mathbb{R}} w^T R^T \left(-\frac{d}{dt}\right) \varphi dt$$

for all $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. We then say that $w \in \mathscr{D}'(\mathbb{R}, \mathbb{R}^d)$ is a weak solution of the equation $R(\frac{d}{dt})\, w = 0$ if it satisfies that $\langle w, R^T\left(-\frac{d}{dt}\right)\varphi\rangle = 0$ for all $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. In particular $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ is a weak solution if it satisfies

$$\int_{\mathbb{R}} w^T R^T \left(-\frac{d}{dt}\right) \varphi dt = 0$$

for all $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$.

Allowing our functions to be in L^1_{loc} is quite generous, as the space includes most functions that we may want to work with. All continuous functions are included in L^1_{loc}, since on any compact set a continuous function is bounded and hence the integral of the function is also bounded. Also all L^p functions, for $1 \le p \le \infty$, are also in L^1_{loc}. For $p = 1$ this is easy to see. For $p > 1$, let q be its Hölder conjugate, that is, q is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = \infty$, take $q = 1$). Then by the Hölder inequality we have, for any $K \subset \mathbb{R}$ compact and $w \in L^p(\mathbb{R}, \mathbb{R}^d)$,

$$\int_K |w| dt = \int_{\mathbb{R}} |w| \chi_K dt \le \left(\int_{\mathbb{R}} |w|^p dt\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \chi_K^p dt\right)^{\frac{1}{q}}$$

where χ_K denotes the characteristic function of K. The first factor in the right hand side is finite by assumption, and the second factor is equal to the Lebesgue measure of K to the $\frac{1}{q}$ power, also finite. Hence we have $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$.

As noted earlier, all functions in L^1_{loc} define distributions and hence have weak derivatives. The weak derivative need not itself be L^1_{loc} however. For example the Heaviside function is L^1_{loc} but its derivative δ_0 is not.

Another reason that the space L^1_{loc} is nice to work with is that it can be seen as a sort of closure for $\mathscr{C}^\infty$. As we will see, this notion of convergence is also well behaved with regard to behaviours in $\mathfrak{L}^d$.

Definition 2.2. A sequence $w_k \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ converges to $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ in the sense of L^1_{loc} if for every compact set $K \subset \mathbb{R}$

$$\lim_{k\to\infty} \int_K |w - w_k| dt = 0.$$

This definition and most results about weak solutions that we will derive using it are from [6]. However, in [6] weak solutions are defined in a slightly different way (not using distributions), so the proofs have been adjusted to reflect this. We will seldom work with this definition directly, but rather use the following two results.

Proposition 2.3. *Suppose $\{w_k\} \subset L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ is a sequence of weak solutions to the equation $R\left(\frac{d}{dt}\right) w = 0$ for some $R \in \mathbb{R}^{e\times d}[\xi]$. If w_k converges to w in the sense of L^1_{loc}, then w is a weak solution of $R\left(\frac{d}{dt}\right) w = 0$.*

Note in particular that we can apply this to a sequence $\{w_k\} \subset \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ of strong solutions. The limit, which may or may not itself be smooth, is at least a weak solution.

Proof. Take any $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$, and let $K = \operatorname{supp} \varphi$. Since w_k are weak solutions, we have for every k that $\int_{\mathbb{R}} w_k^T R^T \left(-\frac{d}{dt}\right) \varphi dt = 0$. We get

$$\begin{aligned}
&\left|\int_{\mathbb{R}} w^T R^T \left(-\frac{d}{dt}\right) \varphi dt\right| = \left|\int_{\mathbb{R}} w^T R^T \left(-\frac{d}{dt}\right) \varphi dt - \int_{\mathbb{R}} w_k^T R^T \left(-\frac{d}{dt}\right) \varphi dt\right| \le \\
&\le \int_{\mathbb{R}} \left|(w - w_k)^T R^T \left(-\frac{d}{dt}\right) \varphi\right| dt = \int_K \left|(w - w_k)^T R^T \left(-\frac{d}{dt}\right) \varphi\right| dt \le \\
&\le \int_K |w - w_k| \left|R^T \left(-\frac{d}{dt}\right) \varphi\right| dt
\end{aligned}$$

Since $\left|R^T\left(-\frac{d}{dt}\right)\varphi\right|$ is continuous on the compact set K, it attains some maximum M. We then have

$$\left|\int_{\mathbb{R}} w^T R^T \left(-\frac{d}{dt}\right) \varphi dt\right| \le M \int_K |w - w_k| \, dt.$$

The right hand side converges to zero as $k \to \infty$ by assumption, and as the left hand side does not depend on k it must be zero. Hence w is a weak solution to the equation. ☐

Proposition 2.4. *For every $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ there is a sequence $\{w_k\} \subset \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ such that w_k converges to w in the sense of L^1_{loc}. Furthermore, if w satisfies $R\left(\frac{d}{dt}\right) w = 0$ weakly, then the w_k can be chosen to satisfy $R\left(\frac{d}{dt}\right) w = 0$ (strongly).*

The main idea here is to convolve w with a sequence of mollifiers to get the sequence w_k. To do this we need one additional property of Lebesgue spaces, that for any $w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$ and any compact $K \subset \mathbb{R}$ we have

$$\lim_{\tau\to 0} \int_K |w(t) - w(t - \tau)| dt = 0.$$

We can also express this as $w(t + \tau)$ converging to $w(t)$ in the L^1_{loc}-sense. A proof of this fact as well as the following proof can be found in [10].

Proof. The proof is by explicitly constructing such a sequence w_k. To start with we define

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, |t| < 1 \\ 0, |t| \ge 1. \end{cases}$$

A straight forward calculation shows that this function is smooth and it clearly has compact support, so $f \in \mathscr{D}(\mathbb{R}, \mathbb{R})$, and then so is the normalized version

$$\psi(t) = \frac{f(t)}{\int_{\mathbb{R}} f(\tau) d\tau}.$$

Now let $\psi_k = k\psi(kt)$, which are then clearly also smooth. The support of ψ_k is contained in the support of ψ, so we have $\psi_k \in \mathscr{D}(\mathbb{R}, \mathbb{R})$. Finally let

$$w_k(t) = \int_{\mathbb{R}} w(t-\tau)\psi_k(\tau) d\tau$$

be the convolution product. By a change of integration variable this can be written

$$w_k(t) = \int_{\mathbb{R}} w(\tau)\psi_k(t-\tau) d\tau$$

Since ψ_k is smooth, and passing derivatives under the integral sign, w_k is also smooth. We get, for any compact $K \subset \mathbb{R}$

$$\begin{aligned}\int_K |w - w_k| dt &= \int_K \left| w \int_{\mathbb{R}} \psi_k(\tau) d\tau - \int_{\mathbb{R}} w(t-\tau)\psi_k(\tau) d\tau \right| dt \leq \\ &\leq \int_K \int_{\mathbb{R}} \psi_k(\tau) |w(t) - w(t-\tau)| d\tau dt = \\ &= \int_{\mathbb{R}} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau.\end{aligned}$$

We now use that $\lim_{\tau \to 0} \int_K |w(t) - w(t-\tau)| dt = 0$, i.e. for any $\epsilon > 0$ there is $\delta > 0$ such that $|\tau| < \delta$ implies $\int_K |w(t) - w(t-\tau)| dt < \epsilon$. Then we have

$$\begin{aligned}&\int_{\mathbb{R}} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau = \int_{|\tau|<\delta} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau + \\ &+ \int_{|\tau|\geq\delta} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau \leq \\ &\leq \int_{|\tau|<\delta} \psi_k(\tau)\epsilon d\tau + \int_{|\tau|\geq\delta} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau \leq \\ &\leq \epsilon + \int_{|\tau|\geq\delta} \psi_k(\tau) \int_K |w(t) - w(t-\tau)| dt d\tau.\end{aligned}$$

The support of ψ_k is $[-\frac{1}{k}, \frac{1}{k}]$ and so for every $\delta > 0$ there is some k such that $[-\frac{1}{k}, \frac{1}{k}] \subset [-\delta, \delta]$ and hence $\psi_k(\tau) = 0$ for all $|\tau| \geq \delta$. It follows that for every $\epsilon > 0$ we have

$$\int_K |w - w_k| dt \leq \epsilon$$

for all k large enough. Hence w_k converge to w in the L^1_{loc} sense, proving the first part of the statement.

If furthermore w satisfies $R\left(\frac{d}{dt}\right) w = 0$ weakly, then the w_k as we have defined them satisfy, for any $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$

$$\int_{\mathbb{R}} w_k^T R^T \left(-\frac{d}{dt}\right) \varphi dt = \int_{\mathbb{R}} \int_{\mathbb{R}} (w(t-\tau)\psi_k(\tau))^T d\tau R^T \left(-\frac{d}{dt}\right) \varphi dt =$$
$$= \int_{\mathbb{R}} \psi_k \int_{\mathbb{R}} w^T(t-\tau) R^T \left(-\frac{d}{dt}\right) \varphi dt d\tau = 0.$$

Here we use in the last step that if $w(t)$ is a solution, then so is $w(t-\tau)$, i.e. time invarience. Hence all w_k are weak solutions, but since they are smooth functions, they are in fact strong solutions. $\square$

Combining these two results, we see that all limits (in the L^1_{loc} sense) of elements of a behaviour $\mathcal{B} \in \mathfrak{L}^d$ are weak solutions to the corresponding equation, and all weak L^1_{loc}-solutions can be approximated by smooth (and hence strong) solutions. In this sense the set of (weak) L^1_{loc}-solutions can be seen as a closure of the behaviour. We will hence write $\overline{\mathcal{B}}$ for the set of weak solutions.

Another consequence of these results is that the set of weak solutions is completely determined by the set of strong solutions. For this reason we can in many theorems focus on proving them for the smooth solutions, the case for weak solutions following simply by taking limits. Of course not all properties are preserved when taking limits, and we will comment on which results hold for weak solutions and which do not throughout the text.

2.2 Kernel representations of $\mathcal{B} \in \mathfrak{L}^d$

As stated earlier, the behaviours of $\mathfrak{L}^d$ are solutions sets to differential equations of the form $R\left(\frac{d}{dt}\right) w = 0$. We can view $R\left(\frac{d}{dt}\right)$ as a map from $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ into $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^e)$ and then the behaviour $\mathcal{B}$ is the kernel of this map. Hence we call this a kernel representation of $\mathcal{B}$, and write $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$. It should be noted that while a polynomial matrix $R \in \mathbb{R}^{e\times d}[\xi]$ completely determines a behaviour $\mathcal{B}$, the converse is not true. For example, we could have two polynomial matrices with different number of rows giving the same behaviour. To see how, take a matrix $R \in \mathbb{R}^{e\times d}[\xi]$ and make a new matrix $\tilde{R} \in \mathbb{R}^{(e+1)\times d}[\xi]$ by setting $\tilde{R} = \begin{pmatrix} R \\ r \end{pmatrix}$, where r is any of the rows of R multiplied by a real polynomial, then R and $\tilde{R}$ will define the same behaviour. This is because the new row added to $\tilde{R}$ makes no new requirements of the solutions. These kind of vacuous rows can in fact be removed, and we will define a notion of matrix rank to handle this.

However even polynomial matrices of the same size can define the same behaviour without being equal. To describe these relations we need to define a class of very useful polynomial matrices.

Definition 2.5. A polynomial matrix $U \in \mathbb{R}^{e\times e}[\xi]$ is unimodular if there is a polynomial matrix $U^{-1} \in \mathbb{R}^{e\times e}[\xi]$ such that $U(\xi)U^{-1}(\xi) = I$.

Unimodular matrices can be characterized by the fact that their determinants are non-zero constants. Indeed, if U is unimodular, then the determinants of both U and U^{-1} are polynomials and $\det(U^{-1}) = \frac{1}{\det U}$. Hence $\det U$ must be a non-zero constant polynomial. Conversely, if $\det U(\xi)$ is a non-zero constant, then the inverse of U is given by $U^{-1}(\xi) = (\det U)^{-1} \operatorname{adj} U$, where $\operatorname{adj} U$ is the adjugate of U. U^{-1} is a polynomial matrix since $\operatorname{adj} U$ is polynomial and $\det U$ is constant.

Examples of unimodular matrices include invertible constant matrices and upper (or lower) triangular matrices with non-zero constants on the main diagonal since their determinants are non-zero constants. A particular example that we will make liberal use of are permutation matrices. As we will soon show, unimodular matrices are very useful to transition between different matrices that represent the same behaviour.

Proposition 2.6. *Suppose $R_1, R_2 \in \mathbb{R}^{e\times d}[\xi]$ and that there is a unimodular matrix $U \in \mathbb{R}^{e\times e}[\xi]$ such that $R_1 = UR_2$. Then* $\ker R_1\left(\frac{d}{dt}\right) = \ker R_2\left(\frac{d}{dt}\right)$.

Proof. Whenever $w \in \ker R_2\left(\frac{d}{dt}\right)$, we have $R_1\left(\frac{d}{dt}\right) w = U\left(\frac{d}{dt}\right) R_2\left(\frac{d}{dt}\right) w = U\left(\frac{d}{dt}\right) 0 = 0$, so $\ker R_2\left(\frac{d}{dt}\right) \subseteq \ker R_1\left(\frac{d}{dt}\right)$. Switching the roles of R_1 and R_2 and using that $R_2 = U^{-1}R_1$, shows the opposite inclusion. □

The converse of this statement is also true, but we will have to postpone the proof somewhat (see Theorem 2.19). If we instead multiply from the right the behaviour does change, but in a predictable way. Suppose $R_1, R_2 \in \mathbb{R}^{e\times d}[\xi]$ and $U \in \mathbb{R}^{d\times d}[\xi]$ unimodular such that $R_1 = R_2U$. If $w \in \ker R_1\left(\frac{d}{dt}\right)$, then

$$R_2\left(\frac{d}{dt}\right)\left(U\left(\frac{d}{dt}\right) w\right) = 0$$

so we see that $U\left(\frac{d}{dt}\right) w \in \ker R_2\left(\frac{d}{dt}\right)$. So we can view $U\left(\frac{d}{dt}\right)$ as a map

$$\begin{aligned} U\left(\frac{d}{dt}\right) : \ker R_1\left(\frac{d}{dt}\right) &\to \ker R_2\left(\frac{d}{dt}\right) \\ w &\mapsto U\left(\frac{d}{dt}\right) w. \end{aligned}$$

This map is linear and hence a homomorphism, and in fact it is an isomorphism, its inverse given simply by multiplying by $U^{-1}\left(\frac{d}{dt}\right)$. The most important use of this is that we can permute the components of w, and simultaneously the columns of R. We are then technically changing to a different behaviour, but they are isomorphic so the change is not so significant.

When looking at weak solutions the situation is a bit more complicated. For multiplication with unimodular matrices from the left it does not matter if the external signals w are smooth or just locally integrable, the argument in the proposition above works either way. When multiplying from the right however, we run into a problem. If w is locally integrable, $U\left(\frac{d}{dt}\right) w$ may not be locally integrable (the derivatives here are taken in the weak sense). Hence $U\left(\frac{d}{dt}\right)$ does not give an isomorphism between the sets of weak solutions of $R_1\left(\frac{d}{dt}\right) w = 0$ and $R_2\left(\frac{d}{dt}\right) w = 0$, even if $R_1 = R_2 U$.

We can remedy this problem using the fact that the weak solutions are completely determined by the strong solutions. If $\mathcal{B}_1 = \ker R_1\left(\frac{d}{dt}\right), \mathcal{B}_2 = \ker R_2\left(\frac{d}{dt}\right)$ denote the sets of strong solutions, and $R_1 = R_2 U$, then while

$$U\left(\frac{d}{dt}\right)\overline{\mathcal{B}}_1 \neq \overline{\mathcal{B}}_2$$

in general, we do have

$$\overline{U\left(\frac{d}{dt}\right)\mathcal{B}_1} = \overline{\mathcal{B}_2}.$$

So while we cannot extend the isomorphism to weak solutions, we can get around the problem quite easily. Next we prove a technical lemma that is useful when we want to multiply with unimodular matrices. The proof is from [6].

Lemma 2.7. *Let $p_1, \ldots, p_d \in \mathbb{R}[\xi]$ be polynomials with no common factor and let p be the row vector $p = (p_1, \ldots, p_d)$. Then there is a unimodular matrix $U \in \mathbb{R}^{d\times d}[\xi]$ such that the last row of U is p.*

Since permutation of rows can be done by multiplying with a unimodular matrix we can actually get p as any row of U we want. Also, by transposing U we can get a unimodular matrix with a column specified by p^T instead.

Proof. Pick p_j with minimal degree. We now perform division with remainder on the other p_i:s, that is we find $q_i, r_i \in \mathbb{R}[\xi]$ such that

$$p_i = q_i p_j + r_i$$

and deg(r_i) <deg(p_j) for all $i \neq j$. If we set $r_j = p_j$ and $r = (r_1, \ldots, r_d)$ then this means that we have the equality $pV_1 = r$ where

$$V_1 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ -q_1 & \ldots & -q_{j-1} & 1 & -q_{j+1} & \ldots & -q_d \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

where the q_i:s are on the j:th row. Note that $\det V_1 = 1$ and hence V_1 is unimodular. Also, the minimal degree of the r_i:s is strictly smaller than that of the p_i:s. Also the maximal degree of the r_i:s is the minimal degree of the p_i:s.

We can now repeat the process, giving us a sequence of unimodular matrices $V_1, V_2, \ldots, V_k$. Since the degrees of the remainders decrease in every step, after a finite number, say k steps we have

$$pV_1 \cdots V_k = (0, \ldots, 0, \tilde{r}, 0, \ldots, 0)$$

for some $\tilde{r} \in \mathbb{R}[\xi]$. Since permutation matrices are unimodular we can choose V_{k+1} such that

$$pV_1 \cdots V_{k+1} = (0, \ldots, 0, \tilde{r}).$$

If we now set $\tilde{U} = (V_1 \cdots V_{k+1})^{-1}$, then we have $p = (0, \ldots, 0, \tilde{r})\tilde{U}$ and so we see that $\tilde{r}$ divides all of the p_i:s. The p_i:s have no common factor by assumption, so $\tilde{r}$ must be constant. Setting $U = \frac{1}{\tilde{r}}\tilde{U}$, we get

$$p = (0, \ldots, 0, 1)U,$$

which means that the last row of U is p. □

For $R \in \mathbb{R}^{e \times d}[\xi]$ we define its rank, denoted $\mathrm{rk}(R)$, as its rank as a matrix over the field of real rational functions. Hence the rows r_k of R are considered linearly dependent if there are real rational functions a_k, not all zero, such that

$$\sum_k a_k(\xi) r_k(\xi) = 0.$$

Note that by multiplying this equation by all denominators of the a_k we get some polynomials b_k such that

$$\sum_k b_k(\xi) r_k(\xi) = 0.$$

Further we could cancel all common factors of the b_k, getting some new polynomials c_k not all zero and with no common factors such that

$$\sum_k c_k(\xi) r_k(\xi) = 0.$$

Going forward, we will use this to characterize linear dependence of polynomial vectors.

For any $\lambda \in \mathbb{C}$, we let $\mathrm{rk}(R(\lambda))$ denote the rank of $R(\lambda)$ as a matrix over $\mathbb{C}$. The two notions of matrix rank for $R \in \mathbb{R}^{e \times d}[\xi]$ are related by the following proposition.

Proposition 2.8. *For any* $R \in \mathbb{R}^{e \times d}[\xi]$ *we have* $\mathrm{rk}(R) = \max_{\lambda \in \mathbb{C}} \mathrm{rk}(R(\lambda))$.

Proof. Let $r_k(\xi)$ denote the k:th row of $R(\xi)$. If some of the rows of $R(\xi)$ are linearly dependent, then there are $a_k \in \mathbb{R}[\xi]$ with no common factors such that

$$\sum_k a_k(\xi) r_k(\xi) = 0$$

For any $\lambda \in \mathbb{C}$ we thus have

$$\sum_k a_k(\lambda) r_k(\lambda) = 0$$

and since the a_k have no common roots, not all $a_k(\lambda)$ are zero. Hence the corresponding rows of $R(\lambda)$ are also linearly dependent, and so $\mathrm{rk}(R) \geq \mathrm{rk}(R(\lambda))$ for every $\lambda \in \mathbb{C}$.

Next we note that for any linear combination $\sum_k a_k(\xi) r_k(\xi)$ of some rows of $R(\xi)$, with none of the polynomials a_k the zero polynomial, we can find some $\lambda \in \mathbb{C}$ that is not the root of any of the polynomials a_k nor the root of any component of $R(\xi)$. For such λ we can have

$$\sum_k a_k(\lambda) r_k(\lambda) = 0$$

only if

$$\sum_k a_k(\xi) r_k(\xi) = 0.$$

Hence if some rows of $R(\lambda)$ are linearly dependent for all $\lambda \in \mathbb{C}$, then the corresponding rows of $R(\xi)$ are also linearly dependent. Thus there is some $\lambda \in \mathbb{C}$ such that $\mathrm{rk}(R) = \mathrm{rk}(R(\lambda))$, and so $\mathrm{rk}(R) = \max_{\lambda \in \mathbb{C}} \mathrm{rk}(R(\lambda))$. □

Going forward, if we just say "rank" of a polynomial matrix we are referring to its rank as a matrix of real rational functions. When we want to refer to rank over $\mathbb{C}$ we will always write out the dependence on the complex variable λ.

Example 2.9. Consider the matrix

$$R(\xi) = \begin{pmatrix} (\xi-3)(\xi+1) & 0 & \xi^2 \\ 0 & \xi+1 & \xi+3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Looking at the polynomials on the main diagonal, we find that for $\lambda \in \mathbb{C}$

$$\mathrm{rk}(R(\lambda)) = \begin{cases} 1 & \text{if } \lambda = -1 \\ 2 & \text{if } \lambda = 3 \\ 3 & \text{otherwise} \end{cases}.$$

It then follows from Proposition 2.8 that $\mathrm{rk}(R) = 3$ in this case.

From Proposition 2.8 it is easy to deduce that multiplication by unimodular matrices leaves rank unchanged. Indeed if U and V are unimodular, then for every $\lambda \in \mathbb{C}$

$$\mathrm{rk}(U(\lambda)R(\lambda)V(\lambda)) = \mathrm{rk}(R(\lambda))$$

since $U(\lambda)$ and $V(\lambda)$ are always invertible matrices over $\mathbb{C}$. Hence the maximum of both sides over all $\lambda \in \mathbb{C}$ is also the same, so the rank of R is preserved. Another thing we can do with a polynomial matrix that leaves its rank unchanged is to remove any row with just zeros. By the following result from [6], multiplying by unimodular matrices allows us to find a full row rank representation from any kernel representation.

Proposition 2.10. *For any $R \in \mathbb{R}^{e\times d}[\xi]$, there is a unimodular $U \in \mathbb{R}^{e\times e}[\xi]$ and $\tilde{R} \in \mathbb{R}^{\mathrm{rk}(R)\times d}[\xi]$ of full row rank such that*

$$UR = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix},$$

where in the right hand side we have $e - \mathrm{rk}(R)$ zero-rows. Furthermore we have $\ker R\left(\frac{d}{dt}\right) = \ker \tilde{R}\left(\frac{d}{dt}\right)$

Proof. If $\mathrm{rk}(R) = e$ then there is nothing to prove. If $\mathrm{rk}(R) < e$, then there are polynomials a_k, not all zero and with no common factors, such that

$$\sum_{k=1}^{e} a_k(\xi) r_k(\xi) = 0,$$

where r_k denotes the kth row of R. Now by Lemma 2.7 there is a unimodular U_0 such that its last row is $(a_1, \dots, a_e)$. Then

$$U_0 R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

for some $R_1 \in \mathbb{R}^{(e-1)\times d}[\xi]$. Note that $\mathrm{rk}(R) = \mathrm{rk}\begin{pmatrix} R_1 \\ 0 \end{pmatrix} = \mathrm{rk}(R_1)$ since deleting a zero-row does not change the rank of a matrix. Repeating the process for R_1 gives us another unimodular matrix $U_1 \in \mathbb{R}^{(e-1)\times(e-1)}[\xi]$ and matrix $R_2 \in \mathbb{R}^{(e-2)\times d}[\xi]$ such that

$$U_1 R_1 = \begin{pmatrix} R_2 \\ 0 \end{pmatrix}$$

and $\mathrm{rk}(R_1) = \mathrm{rk}(R_2)$. Note that then

$$\begin{pmatrix} U_1 & \\ & 1 \end{pmatrix} U_0 R = \begin{pmatrix} R_2 \\ 0 \\ 0 \end{pmatrix},$$

and $\begin{pmatrix} U_1 & \\ & 1 \end{pmatrix} U_0$ is unimodular. After repeating the process $e - \mathrm{rk}(R)$ times we can take $\tilde{R} = R_{e-\mathrm{rk}(R)}$ which then has $\mathrm{rk}(R)$ rows and $\mathrm{rk}\left(\tilde{R}\right) = \mathrm{rk}(R)$. In each step we only multiply from the left by a unimodular matrix, hence $\tilde{R}$ is on the desired form.

That $\ker R = \ker \tilde{R}$ now follows from Proposition 2.6 and the fact that deleting zero-rows does not change the set of solutions. □

The rank of $R \in \mathbb{R}^{e\times d}[\xi]$ is important as it describes the minimum number of equations needed to describe the corresponding behaviour. We will soon show that this is really a property of the behaviour, that is, the rank is always the same for matrices that define the same behaviour.

The next result (also from [6]) will be very important going forward. It tells us that using unimodular matrices we can get a triangular or diagonal form. This allows us to reduce many problems to the one dimensional case and this is a standard method used in many proofs when working in the behavioural framework.

Proposition 2.11. *Let $R \in \mathbb{R}^{e\times d}[\xi]$. Then the following holds*

(i) *There is a unimodular $U \in \mathbb{R}^{e\times e}[\xi]$ such that UR is an upper triangular polynomial matrix.*

(ii) *There are unimodular $U \in \mathbb{R}^{e\times e}[\xi], V \in \mathbb{R}^{d\times d}[\xi]$ such that URV is a diagonal polynomial matrix.*

Proof. The proof is based on using division with remainder, similar to the proof of Lemma 2.7. In that proof we saw that choosing an element of a row and performing division with remainder on the other elements of the row constitutes multiplying by a unimodular matrix from the right. Note also that performing division with remainder on the column of a given element is instead given by multiplication by a unimodular matrix from the left.

(i) The proof is by induction on d. Consider first the case $d = 1$. If R only has zeroes then there is nothing to prove. Suppose instead that there are some non-zero elements of R. Find the element of lowest non-zero degree in R and perform division with remainder on the other elements. Repeat this until there is only one non-zero element left. Since the lowest non-zero degree of elements decreases in every step this will take only a finite number of steps. Finally we permute the rows so that the non-zero element is in the first row. Since all these manipulations are done by multiplication with unimodular matrices from the left this proves the case $d = 1$.

Next we suppose the statement holds for all matrices with less than d columns and show that it then also holds for matrices with d columns. For $R \in \mathbb{R}^{e\times d}[\xi]$, apply the same algorithm we used for the $d = 1$ case on the first column of R, but multiplying all of R by the unimodular matrices in the process. Hence there is unimodular $U_1 \in \mathbb{R}^{e\times e}[\xi]$ such that the first column of U_1R is non-zero only in the first row. Now let $\tilde{R}$ be the submatrix of U_1R where we have deleted the first row and first column. By the induction hypothesis there is unimodular $U_2 \in \mathbb{R}^{(e-1)\times(e-1)}[\xi]$ such that $U_2\tilde{R}$ is upper triangular. Setting

$$U = \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} U_1$$

we have that UR is upper triangular, and we are done.

(ii) We begin with the case when R is square and has full rank. Find a non-zero element of minimal degree in R and perform division with remainder on the corresponding row and column. We now repeat this step. Since the minimal degree of non-zero elements decreases in each step we will, after a finite number of steps, find that some element is the only non-zero element in its row and its column. We then continue the algorithm on the rest of the rows and columns. After a finite number of steps we hence arrive at a matrix wherein every row and every column has exactly one non-zero element (since R has full rank no row or column can be all zeroes). We can then permute the rows and columns to get a diagonal matrix. Each of these steps are given by multiplication by unimodular matrices, either from the right or from the left, and hence the statement is proven for the square and full rank case.

Now consider the general case $R \in \mathbb{R}^{e\times d}[\xi]$. By Proposition 2.10 there is a unimodular matrix $U_1 \in \mathbb{R}^{e\times e}[\xi]$ such that

$$U_1 R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

where R_1 has $\mathrm{rk}(R)$ rows. Now apply Proposition 2.10 to $(U_1R)^T$, giving us V_1 such that

$$V_1(U_1R)^T = \begin{pmatrix} R_2^T & 0 \\ 0 & 0 \end{pmatrix}$$

where R_2^T has $\mathrm{rk}(R)$ rows and columns and is of full rank. Transposing again we have

$$U_1RV_1^T = \begin{pmatrix} R_2 & 0 \\ 0 & 0 \end{pmatrix}$$

Since R_2 is square and of full rank there are unimodular matrices $U_2, V_2 \in \mathbb{R}^{\mathrm{rk}(R)\times\mathrm{rk}(R)}[\xi]$ such that $U_2R_2V_2$ is diagonal. Then

$$\begin{pmatrix} U_2 & 0 \\ 0 & I \end{pmatrix} U_1RV_1^T \begin{pmatrix} V_2 & 0 \\ 0 & I \end{pmatrix},$$

where I denotes identity matrices of appropriate sizes, is diagonal and so we are done. □

We call elements w of a behaviour an external signal. The name suggests that w describes some sort of interaction that a system has with the outside. The idea is usually that some of the components of w describe input imposed on the system, and the other components describe some output from the system that we can observe. How this split can or should be done is often guided by the physical interpretation of the model. A more precise definition is as follows. For $\mathcal{B} \in \mathfrak{L}^d$, partition $w \in \mathcal{B}$ as $w = \begin{pmatrix} u \\ y \end{pmatrix}$, possibly rearranging w first. Here $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^p), y \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and $p + q = d$. Such a decomposition is an input/output (i/o) decomposition, with u the input (or control), y the output, if the following holds:

(i) u is free; that is, for every $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ there is $y \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ such that $\begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{B}$

(ii) u is maximally free; that is, given $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ none of the components of y can be chosen freely and still have $\begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{B}$.

If $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$, then there is a one to one correspondence of partitioning w as (u, y) and partioning the columns of R as $R(\xi) = \begin{pmatrix} -Q(\xi) & P(\xi) \end{pmatrix}$ so that the equation

$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u \tag{4}$$

holds. Equation (4) is then called an i/o representation of $\mathcal{B}$. In fact whether a decomposition of w is an i/o decomposition can be characterized by properties of the matrices P and Q in the decomposition of R, see Theorem 2.18 below. We will speak of i/o decompositions in terms of w and in terms of the columns of R somewhat interchangeably.

Decompositions into input/output are in general not unique. A simple example to illustrate this is the behaviour given by the equation $w_1 = w_2$. Here either w_1 or w_2 could be chosen as input, but by doing so the other is completely determined. Hence one variable must be input and one output, but either choice is fine.

The output can not be chosen freely, meaning it is affected by the choice of input. The input does not in general completely determine the output as the output also depends on the past of the signal (usually described using initial conditions). There is still a direction of the relation between input and output, from the former to the latter.

This causal direction is sometimes wanted, and sometimes not. Consider for example a simple circuit with a resistor with some resistance R. The voltage V and current I through the circuit are related by Ohm's law:

$$V = IR.$$

Either one of V, I could be input and the other output, and both choices imply a causal direction. This holds both mathematically in terms of the equations, and in the sense that we could control either voltage or current in a circuit. This demonstrates the usefulness of foregoing the labels input and output in the behavioural framework. Ohm's law does not impose any kind of causality, and so our model should not need to either.

Note that we can have either p or q equal to zero. If $q = 0$, then all of the components of w can be chosen freely, i.e. $\mathcal{B} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^p)$. This case is of course not very interesting by itself, but once we introduce latent variables in the next section it can be of interest. In the case $p = 0$, there is no input, no outside influence on the system. The system, or equivalently the behaviour, is then said to be autonomous. Autonomous behaviours can be given the following behavioural definition. We will soon show that this implies that we must have $p = 0$ in the case of $\mathfrak{L}^d$-systems.

Definition 2.12. A behaviour $\mathcal{B}$ is autonomous if for every $w_1, w_2 \in \mathcal{B}$

$$\big(\forall t \leq 0\, w_1(t) = w_2(t)\big) \implies w_1 = w_2.$$

In an autonomous behaviour the future of the behaviour is completely determined by its past. If $\mathcal{B}$ is linear, and setting $w = w_1 - w_2$ this can be formulated as

$$\big(\forall t \leq 0\, w(t) = 0\big) \implies w = 0.$$

In physical systems this can be interpreted as the notion that without any external forces (inputs) a system at rest will remain at rest.

It turns out that if we have a square polynomial matrix with full rank, then its kernel can be completely described using the determinant of the matrix.

Proposition 2.13. *Suppose $P \in \mathbb{R}^{q\times q}[\xi]$ has full rank and*

$$\det P(\xi) = c \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$$

for some $c \in \mathbb{R} \setminus \{0\}$. Then every complex solution of $P\big(\frac{d}{dt}\big)\, w = 0$ is of the form

$$w(t) = \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} b_{k,l} t^l e^{\lambda_k t} \tag{5}$$

for some $b_{k,l} \in \mathbb{C}^d$ satisfying

$$\sum_{l=m}^{n_k-1} \binom{l}{m} P^{(l-m)}(\lambda_k) b_{k,l} = 0 \tag{6}$$

for every $k = 1, \ldots, N$ and every $m = 0, \ldots, n_k - 1$. Furthermore the dimension of $\ker P\big(\frac{d}{dt}\big)$ is $n := \deg \det P(\xi)$.

Here the λ_k:s are the roots of $\det P(\xi)$, and the n_k:s their multiplicities. The number N is the number of unique roots of $\det P(\xi)$, and $P^{(i)}(\xi)$ denotes the ith derivative of P with respect to ξ. Note also that since P is square, that it has full rank is equivalent to $\det P(\xi)$ being non-zero. This ensures that the determinant can be written on the desired form with non-zero c. Since the roots λ_k will in general be complex numbers, when working with autonomous systems it is often convenient to work with complex valued solutions. To recover the real valued solutions we need only take real parts of the complex ones.

The scalar case of this statement is a classic result from ODE theory, and can be found in most textbooks on the subject, for example [1]. Note that in this case we have $P^{(l-m)}(\lambda_k) = 0$ for all k and all $l - m < n_k$, and so equation (6) is satisfied automatically. In other words the coefficients $b_{k,l}$ can be chosen arbitrarily in the scalar case. The proof of the general case is from [6].

Proof. Let $\mathcal{B}$ be the complex solution set to $P\left(\frac{d}{dt}\right) w = 0$. The first step of the proof is to show that the dimension of $\mathcal{B}$ over $\mathbb{C}$ is equal to n. We have already covered the case $q = 1$, let us now look at the case $q \geq 2$.

By Proposition 2.11 there are unimodular matrices $U, V \in \mathbb{R}^{q\times q}[\xi]$ such that $UPV = \operatorname{diag}(d_1(\xi), \dots, d_q(\xi)) =: D$. Let $\mathcal{B}' = V^{-1}\left(\frac{d}{dt}\right) \mathcal{B}$, and note that the solution set of the i:th component of $w \in \mathcal{B}'$ has dimension equal to $\deg d_i(\xi)$. It follows that the dimension of $\mathcal{B}'$ is equal to $\sum_{i=1}^{q} \deg d_i(\xi) = \deg \det D(\xi)$. Since $\det P(\xi)$ and $\det D(\xi)$ differ only by a constant factor given by the determinants of U and V, they have the same degree. Furthermore $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic and hence have the same dimension, and so the dimension of $\mathcal{B}$ is equal to n.

The next step is to note that the elements of $\mathcal{B}'$ must be of the form

$$\sum_{k=1}^{N} \sum_{l=0}^{n_k-1} \tilde{b}_{k,l} t^l e^{\lambda_k t}$$

where $\tilde{b}_{k,l} \in \mathbb{C}^d$. This is because since D is diagonal, the equation $D\left(\frac{d}{dt}\right) w = 0$ splits into q separated scalar equations, so we can refer back to the scalar case. Hence every $w \in \mathcal{B}$ is of the form

$$V\left(\frac{d}{dt}\right) \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} \tilde{b}_{k,l} t^l e^{\lambda_k t} = \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} b_{k,l} t^l e^{\lambda_k t}.$$

Finally we show that the solutions must satisfy (6). The key here is that for any integer $l \geq 0$ we have $t^l e^{\lambda_k t} = \frac{d^l}{d\lambda_k^l} e^{\lambda_k t}$ and the fact that differentiation with respect to λ_k commutes with differentiation with respect to t. For any w of the form (5) we have

$$\begin{aligned}
P\left(\frac{d}{dt}\right) w &= \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} P\left(\frac{d}{dt}\right) b_{k,l} t^l e^{\lambda_k t} = \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} P\left(\frac{d}{dt}\right) b_{k,l} \frac{d^l}{d\lambda_k^l} (e^{\lambda_k t}) = \\
&= \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} \frac{d^l}{d\lambda_k^l} \left(P\left(\frac{d}{dt}\right) b_{k,l} e^{\lambda_k t} \right) = \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} \frac{d^l}{d\lambda_k^l} \left(P(\lambda_k) b_{k,l} e^{\lambda_k t} \right) = \\
&= \sum_{k=1}^{N} \sum_{l=0}^{n_k-1} \sum_{m=0}^{l} \binom{l}{m} P^{(l-m)}(\lambda_k) b_{k,l} t^m e^{\lambda_k t} = \\
&= \sum_{k=1}^{N} \sum_{m=0}^{n_k-1} \left(\sum_{l=m}^{n_k-1} \binom{l}{m} P^{(l-m)}(\lambda_k) b_{k,l} \right) t^m e^{\lambda_k t}.
\end{aligned}$$

Since $t^m e^{\lambda_k t}$ are linearly independent for different m and k, $P\left(\frac{d}{dt}\right) w = 0$ if and only if (6) holds for every $k = 1, \dots, N$ and every $m = 0, \dots, n_k - 1$. $\square$

Corollary 2.14. *If $P \in \mathbb{R}^{q\times q}[\xi]$ has full rank, then $\mathcal{B} = \ker P\left(\frac{d}{dt}\right)$ is autonomous.*

Proof. Since $\mathcal{B}$ is linear we need to show that if for $w \in \ker P\big(\frac{d}{dt}\big)$, $w(t) = 0$ for $t \leq 0$, then $w = 0$. By Proposition 2.13 any $w \in \mathcal{B}$ is of the form (5). Note that $t^l e^{\lambda_k t}$ are linearly independent elements of $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$. Hence if $w(t) = 0$ for $t \leq 0$, then all coefficient vectors $b_{k,l}$ must be equal to zero, and so $w = 0$. $\square$

Remark 2.15. A kind of converse of Corollary 2.14 also holds. If $\mathcal{B} \in \mathfrak{L}^q$ is autonomous, then $\mathcal{B} = \ker P\big(\frac{d}{dt}\big)$ for some $P \in \mathbb{R}^{q\times q}[\xi]$ with full rank. In particular the elements of autonomous behaviours are always on the form (5). We have already seen that every behaviour has a kernel representation that has full row rank. If such a representation is not square (has fewer than q rows) then at least one component of the external signal can be chosen as input. This will be proven in Theorem 2.18 below. Since inputs are chosen freely they are not determined by the past, so such systems can not be autonomous.

Remark 2.16. For any solution of the form (5), if we single out the terms corresponding to a specific root λ_k then this is also a solution. In fact there are always n_k linearly independent solutions involving only the root λ_k, for each k. In the scalar case these must involve all the elements of $S = \{e^{\lambda_k t}, te^{\lambda_k t}, \ldots, t^{h_k-1}e^{\lambda_k t}\}$.

In the more general case it matters which of the polynomials d_i that λ_k is a root of. If λ_k is only the root of one of the d_i:s, then to get n_k linearly independent solutions we again need all the terms of S. If λ_k has at most multiplicity one in each of the d_i:s, then we can find n_k linearly independent choices for $b_{k,0}$. Hence all solutions involving just λ_k are in this case on the form $b_{k,0}e^{\lambda_k t}$.

It is worth noting that for autonomous systems there are virtually no weak solutions that are not also strong solutions. As we saw in the proof above, autonomous behaviours are finite dimensional, and hence closed. Since in L^1_{loc} we identify functions that are equal almost everywhere we can find weak solutions that are equal to a smooth solution almost everywhere, but not necessarily everywhere. We can however not find any other weak solutions.

Let us now return to a matrix $R \in \mathbb{R}^{q\times d}[\xi]$ of full row rank. The rank must then necessarily be q satisfying $q \leq d$. The idea is now choosing q linearly independent columns to form P (possibly rearranging the columns first), and the corresponding components of w are the output. The rest of the columns form $-Q$ and the corresponding components of w will be the free variables, i.e. the input.

For the smooth case it is enough to choose P such that $\det P(\xi)$ is non zero, however if we want to generalize this to the L^1_{loc}-case we need another assumption, namely that $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions. A rational function is proper if the degree of its numerator does not exceed the degree of the denominator. The matrix $P^{-1}(\xi)Q(\xi)$ is called the transfer function and is important in frequency space considerations.

The case when $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions is still important to us as we will use it to prove the more general case. The key here is for proper rational functions we can use a partial fraction decomposition to construct an output for any given input. The properness property ensures that in the decomposition ξ is only present in the denominators.

Lemma 2.17. *Suppose $P \in \mathbb{R}^{q\times q}[\xi], Q \in \mathbb{R}^{q\times p}[\xi]$ such that $\det P(\xi) \neq 0$ and $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions. If the partial fraction decomposition of $P^{-1}(\xi)Q(\xi)$ is*

$$P^{-1}(\xi)Q(\xi) = A_0 + \sum_{k=1}^{N}\sum_{l=0}^{n_k-1} \frac{A_{k,l}}{(\xi-\lambda_k)^l},$$

then any pair $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ and

$$y_{\mathrm{p}}(t) := A_0 u(t) + \sum_{k=1}^{N}\sum_{l=0}^{n_k-1} A_{k,l}\int_0^t \frac{(t-\tau)^{l-1}}{(l-1)!}e^{\lambda_k(t-\tau)}u(\tau)d\tau$$

satisfy $P\big(\frac{d}{dt}\big)\, y_{\mathrm{p}} = Q\big(\frac{d}{dt}\big)\, u$.

Like before, the λ_k:s are the roots of $\det P(\xi)$, n_k:s their multiplicities, and N is the number of unique roots. Since the roots may be non-real, the coefficient matrices $A_0, A_{k,l}$ are complex valued $q\times p$ matrices. Because of this the solution y_{p} we find may not be real valued, but taking the real (or imaginary) part, we get a real i/o decomposition solving the same equation. For the proof we refer to [6].

The subscript p in y_{p} denotes a particular solution. The idea here is that if (u, y_{p}) satisfies $P\big(\frac{d}{dt}\big)\, y_{\mathrm{p}} = Q\big(\frac{d}{dt}\big)\, u$ and y is another output with the same input u, i.e. they satisfy equation (4), then $y_{\mathrm{h}} := y - y_{\mathrm{p}}$ satisfies $P\big(\frac{d}{dt}\big)\, y_{\mathrm{h}} = 0$. We can view y_{h} as a homogeneous solution, in the sense that in the equation $P\big(\frac{d}{dt}\big)\, y_{\mathrm{h}} = 0$ every term is a constant times either y_{h} or one of its derivatives. Hence any (u, y) that satisfy equation (4) can be decomposed into a homogenous part and a particular part that depends on u,

$$(u, y) = (0, y_{\mathrm{h}}) + (u, y_{\mathrm{p}}).$$

This decomposition can be done for all of $\mathcal{B}$, so $\mathcal{B}$ can be decomposed as a direct sum. Since P is square and of full rank, $\mathcal{B}_{\mathrm{aut}} := \ker P\big(\frac{d}{dt}\big)$ is autonomous. The other part we denote $\mathcal{B}_{\mathrm{cont}}$, the controllable part. So $\mathcal{B} = \mathcal{B}_{\mathrm{aut}} + \mathcal{B}_{\mathrm{cont}}$. We will return to this decomposition when discussing controllability.

For any polynomial matrix R with full row rank we can always choose $\mathrm{rk}(R)$ columns to form P, so every $\mathfrak{L}^d$-system admits an i/o representation, and in fact all i/o representations are on this form.

Theorem 2.18. *Suppose $R \in \mathbb{R}^{e\times d}[\xi]$ is of full rank and decompose R (possibly rearranging the columns) as $R(\xi) = \begin{pmatrix} -Q(\xi) & P(\xi)\end{pmatrix}$ and make the corresponding decomposition of w as $w = \begin{pmatrix} u \\ y \end{pmatrix}$. Then this is an i/o decomposition if and only if P is square and $\det P(\xi) \neq 0$. In particular, the output y must have $\mathrm{rk}(R)$ components.*

The "if" part of the following proof is from [6].

Proof. We begin with the "if" part. If $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions then by Lemma 2.17 there is for every u a y satisfying (4), so u is free in this case. If $P^{-1}(\xi)Q(\xi)$ is not a matrix of proper rational functions, then there is $k \in \mathbb{N}$ such that $\xi^{-k}P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions. Setting $\tilde{P}(\xi) = \xi^k P(\xi)$, there is then, by applying Lemma 2.17 to the pair $\tilde{P}, Q$, for every u, a $\tilde{y}_{\mathrm{p}}$ such that $\tilde{P}\left(\frac{d}{dt}\right)\tilde{y}_{\mathrm{p}} = Q\left(\frac{d}{dt}\right)u$. But then $y_{\mathrm{p}} := \frac{d^k}{dt^k}\tilde{y}_{\mathrm{p}}$ satisfies $P\left(\frac{d}{dt}\right)y_{\mathrm{p}} = Q\left(\frac{d}{dt}\right)u$, so u can be chosen freely in this case as well.

To see that u is maximally free, suppose that (u, y_1) and (u, y_2) are two solutions to (4). By linearity $(0, y_1 - y_2)$ is then also a solution to (4), which in this case reduces to $P\left(\frac{d}{dt}\right)(y_1 - y_2) = 0$. Since $\det P(\xi) \neq 0$, the system $\ker P\left(\frac{d}{dt}\right)$ is autonomous, and hence the possible solutions $y_1 - y_2$ are completely determined by the roots of P. In particular no components of $y_1 - y_2$ can be chosen freely.

For the "only if" part we assume that $R(\xi) = \begin{pmatrix} -Q(\xi) & P(\xi) \end{pmatrix}$ is an i/o decomposition and consider a few different cases regarding the columns of P.

(i) Consider first the case when $\mathrm{rk}(P) \geq \mathrm{rk}(R)$ and P has more than $\mathrm{rk}(R)$ columns total. If we then make a new choice of $\mathrm{rk}(R)$ linearly independent columns among those in P, then by the if part of the theorem, this new choice gives us an i/o representation. However, when making this change some outputs are turned into inputs. As outputs these could not be free, but as inputs they must be free. Since the behaviour remains the same, we have a contradiction.

(ii) Next we consider the case when $\mathrm{rk}(P) < \mathrm{rk}(R)$ and P has more than $\mathrm{rk}(P)$ columns total, and consider i/o pairs of the form $(0, y)$. They must satisfy $P\left(\frac{d}{dt}\right)y = 0$. By selecting $\mathrm{rk}(P)$ linearly independent columns we can split P and y as $P(\xi) = \begin{pmatrix} -\tilde{Q}(\xi) & \tilde{P}(\xi) \end{pmatrix}$ and $y = \begin{pmatrix} \tilde{u} \\ \tilde{y} \end{pmatrix}$ and by the if part of the theorem this split gives $\tilde{u}, \tilde{y}$ as an i/o representation of $\ker P\left(\frac{d}{dt}\right)$. This means that $\tilde{u}$, some of the components of y can be chosen freely, which is a contradiction since y is the output of $\ker R\left(\frac{d}{dt}\right)$.

(iii) Finally consider the case when $\mathrm{rk}(P) < \mathrm{rk}(R)$ and P has full column rank. We then form $\tilde{P}$ by taking the columns of P and switching some columns of Q to $\tilde{P}$ in such a way that $\tilde{P}$ has $\mathrm{rk}(R)$ linearly independent columns. Then we get a new i/o pair $\tilde{u}, \tilde{y}$ but some of the components of $\tilde{y}$ were part of the original output u so they should be free. But then $\tilde{u}$ is not maximally free, a contradiction.

The only possibility left is that $\mathrm{rk}(P) = \mathrm{rk}(R)$ and P has full column rank, but that means precisely that P is square and $\det P(\xi) \neq 0$, and so the proof is complete. □

With these results we are finally ready to prove the converse of Proposition 2.6, also from [6].

Theorem 2.19. *Suppose $R_1, R_2 \in \mathbb{R}^{e\times d}[\xi]$ and let $\mathcal{B}_1 = \ker R_1\left(\frac{d}{dt}\right)$ and $\mathcal{B}_2 = \ker R_2\left(\frac{d}{dt}\right)$. Then $\mathcal{B}_1 = \mathcal{B}_2$ if and only if there is a unimodular matrix $U \in \mathbb{R}^{e\times e}[\xi]$ such that $R_1 = UR_2$.*

Proof. The if part was proven in Proposition 2.6. For the only if part, we will divide the proof in two parts.

(i) Consider first the case when R_1, R_2 are square and R_1 of full rank. We then proceed by induction of the size d.

For $d = 1$, the system given by $R_1\left(\frac{d}{dt}\right) w = 0$ is autonomous, and hence the solutions w are given explicitly by the roots of $R_1(\xi)$. Since R_2 defines the same system, it must have the same roots, i.e. there is a constant $U \in \mathbb{R}$ such that $R_1 = UR_2$, proving the statement for $d = 1$.

Next we assume that the statement holds for matrices of size d and show that it then also holds for size $d + 1$. The first step is to note that by Proposition 2.11, by multiplying by a unimodular matrix from the left, R_i, $i \in \{1, 2\}$, can be assumed to be upper triangular, so

$$R_i = \begin{pmatrix} R_{11}^{(i)} & R_{12}^{(i)} \\ 0 & R_{22}^{(i)} \end{pmatrix}$$

where $R_{11}^{(i)} \in \mathbb{R}^{d\times d}[\xi], R_{12}^{(i)} \in \mathbb{R}^{d\times 1}[\xi], R_{22}^{(i)} \in \mathbb{R}[\xi]$.

Now we also decompose w as $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ so that w_2 is scalar valued. Then for any $\begin{pmatrix} w_1 \\ 0 \end{pmatrix} \in \mathcal{B}$ we have

$$R_{11}^{(1)}\left(\frac{d}{dt}\right) w_1 = R_{11}^{(2)}\left(\frac{d}{dt}\right) w_1 = 0$$

and so by the induction hypothesis there is unimodular U_{11} such that $R_{11}^{(1)} = U_{11}R_{11}^{(2)}$. Also, for any $\begin{pmatrix} 0 \\ w_2 \end{pmatrix} \in \mathcal{B}$ we get

$$R_{22}^{(1)}\left(\frac{d}{dt}\right) w_2 = R_{22}^{(2)}\left(\frac{d}{dt}\right) w_2 = 0$$

so by the $d = 1$ case there is a non-zero constant U_{22} such that $R_{22}^{(1)} = U_{22}R_{22}^{(2)}$. So far we have concluded that

$$R_1 = \begin{pmatrix} U_{11}R_{11}^{(2)} & R_{12}^{(1)} \\ 0 & U_{22}R_{22}^{(2)} \end{pmatrix}.$$

Looking only at the first d rows of R_1 and noting that $\det U_{11}R_{11}^{(2)} \neq 0$ there is by Theorem 2.18 for every w_2 a w_1 so that

$$U_{11}R_{11}^{(2)}\left(\frac{d}{dt}\right) w_1 + R_{12}^{(1)}\left(\frac{d}{dt}\right) w_2 = 0$$

(w_2 is the free input, w_1 the output). Hence for any $w_2 \in \ker R_{22}^{(2)}$ there is w_1 so that $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{B}$. For such w_1, w_2 we have

$$\begin{aligned} U_{11} R_{11}^{(2)} \left(\frac{d}{dt}\right) w_1 + R_{12}^{(1)} \left(\frac{d}{dt}\right) w_2 &= 0 \\ R_{11}^{(2)} \left(\frac{d}{dt}\right) w_1 + R_{12}^{(2)} \left(\frac{d}{dt}\right) w_2 &= 0. \end{aligned}$$

Taking the first of these equations and subtracting U_{11} times the second we get

$$\left(R_{12}^{(1)} - U_{11} R_{12}^{(2)}\right) \left(\frac{d}{dt}\right) w_2 = 0. \tag{7}$$

In other words $R_{22}^{(2)} \left(\frac{d}{dt}\right) w_2 = 0$ implies that (7) holds. This means that $R_{22}^{(2)}$ divides every element of $R_{12}^{(1)} - U_{11} R_{12}^{(2)}$, so there is a polynomial column vector U_{12} such that $U_{12} R_{22}^{(2)} = R_{12}^{(1)} - U_{11} R_{12}^{(2)}$. But then we have

$$R_1 = \underbrace{\begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}}_{U} R_2$$

and since U_{11} and U_{22} are unimodular, so is U. The statement is then proven for the $(d+1) \times (d+1)$ case, and by induction it holds for every d.

(ii) Now we look at the general case. By Proposition 2.10, by multiplication by unimodular matrix R_1, R_2 can be assumed to be of the form

$$R_i = \begin{pmatrix} \tilde{R}_i \\ 0 \end{pmatrix}$$

with $\tilde{R}_i$ full row rank. Now decompose $\tilde{R}_1$ as $\tilde{R}_1 = \begin{pmatrix} -\tilde{Q}_1 & \tilde{P}_1 \end{pmatrix}$ so that $\tilde{P}_1$ is square and of full rank (by possibly rearranging the columns) and make the corresponding decompositions $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $\tilde{R}_2 = \begin{pmatrix} -\tilde{Q}_2 & \tilde{P}_2 \end{pmatrix}$. Note that if $w_2 \in \ker \tilde{P}_1$, then $\begin{pmatrix} 0 \\ w_2 \end{pmatrix} \in \mathcal{B}$ and hence also $w_2 \in \ker \tilde{P}_2$ and vice versa. It then follows from part (i) of the proof that there is a unimodular matrix U such that $\tilde{P}_1 = U \tilde{P}_2$.

By Theorem 2.18 this decomposition is an i/o decomposition for $\tilde{R}_1$ so there is for every w_1 a w_2 so that $w \in \mathcal{B}$. Hence for every w_1 we can find w_2 so that

$$\tilde{P}_1 \left(\frac{d}{dt}\right) w_2 = \tilde{Q}_1 \left(\frac{d}{dt}\right) w_1, \quad \tilde{P}_2 \left(\frac{d}{dt}\right) w_2 = \tilde{Q}_2 \left(\frac{d}{dt}\right) w_1.$$

Taking the first equation minus U times the second gives

$$\left(\tilde{Q}_1 - U\tilde{Q}_2\right)\left(\frac{d}{dt}\right) w_1 = 0.$$

Since w_1 was arbitrary, this implies that $\tilde{Q}_1 = U\tilde{Q}_2$, and so $\tilde{R}_1 = U\tilde{R}_2$. But then

$$R_1 = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} R_2,$$

where I is an identity matrix of appropriate size, and the statement is proven. □

With this theorem we can fully explain when two polynomial matrices define the same behaviour.

Corollary 2.20. *Let $R_1 \in \mathbb{R}^{e_1 \times d}[\xi]$ and $R_2 \in \mathbb{R}^{e_2 \times d}[\xi]$ where $e_1 \leq e_2$. Then R_1 defines the same behaviour as R_2 if and only if R_2 can be recovered from R_1 by first possibly adding some zero rows to R_1 and then multiplying R_1 from the left by a unimodular matrix.*

Proof. The if direction follows from the fact that adding zero rows or multiplying from the left by a unimodular matrix has no impact on the resulting behaviour. For the only if direction, note that we can always add $e_2 - e_1$ zero rows to R_1. A suitable unimodular matrix to multiply with to get R_2 then exists by Theorem 2.19. □

As we alluded to earlier, it also follows that different matrices that represent the same behaviour always have the same rank, since adding zero rows and multiplication by unimodular matrix does not change the rank. The rank is therefore in a sense a property of the behaviour, describing the smallest number of equations that can describe the behaviour.

2.3 Latent variable representations of $\mathcal{B} \in \mathfrak{L}^d$

We will now look at another representation, the latent variable representation. The idea here is that it may be difficult to formulate differential equations using only the external signal w, say from known laws of physics. It can then help to define some new variables, latent variables, with which the formulation of equations becomes easier, even if the latent variables themselves are of little to no interest. If $w \in \mathcal{B}$ if and only if there is $l \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ such that

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) l$$

for some some $R \in \mathbb{R}^{e \times d}[\xi], M \in \mathbb{R}^{e \times n}[\xi]$, then we say that this is a latent variable representation of $\mathcal{B}$ with l as the latent variable. Every kernel representation can of course be seen as a latent variable representation with $M = 0$. However the latent variable representation is not actually more general because

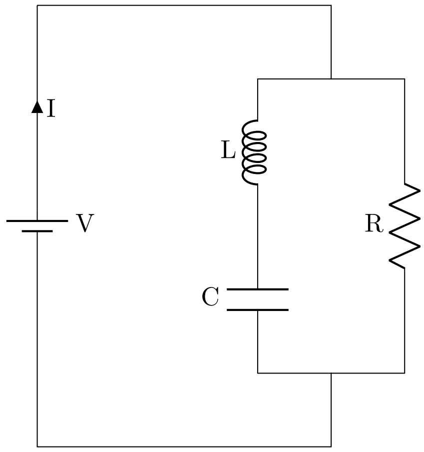

Figure 1: RLC circuit.

we can essentially eliminate the latent variable to recover a kernel representation for $\mathcal{B}$. To eliminate l we first use Proposition 2.10 to find a unimodular matrix U to multiply with so that we get UM on the form

$$UM = \begin{pmatrix} \tilde{M} \\ 0 \end{pmatrix}$$

with $\tilde{M}$ having full row rank. By a suitable decomposition of $UR = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ we can write this as

$$R_1\left(\frac{d}{dt}\right) w = \tilde{M}\left(\frac{d}{dt}\right) l \tag{8}$$

$$R_2\left(\frac{d}{dt}\right) w = 0. \tag{9}$$

Equation (9) of course puts some constraints on w, but equation (8) does not in fact restrict w in any way. To see this note that by choosing $\mathrm{rk}\left(\tilde{M}\right)$ columns of $\tilde{M}$ and a corresponding choice of components of l, these can be viewed as outputs in the sense of Theorem 2.18. The rest of the components of l as well as w are then the inputs and can hence be chosen freely and still satisfy equation (9). Therefore R_2 gives us a kernel representation for $\mathcal{B}$. Let us demonstrate the procedure with an example.

Example 2.21. Consider the RLC circuit shown in Figure 1. We are interested in how the voltage $V(t)$ across the external port and the current $I(t)$ through the external port vary over time. In the circuit we have a resistor with resistance R, an inductor with inductance L and a capacitor with capacitance C. The values of R, L and C are assumed to be positive real constants.

To derive the relation between V and I we introduce some latent variables. We let V_R, V_L and V_C denote the (time varying) voltages across the resistor, inductor and capacitor respectively, and similarly I_R, I_L and I_C denote the currents through the respective components. These quantities must satisfy the constitutive equations for the different components and Kirchhoffs current and voltage laws:

$$\begin{cases} V_R = RI_R \\ L\frac{d}{dt}(I_L) = V_L \\ C\frac{d}{dt}(V_C) = I_C, \end{cases} \qquad \begin{cases} I = I_L + I_R \\ I = I_C + I_R \\ I_C = I_L \end{cases} \qquad \begin{cases} V = V_R \\ V = V_L + V_C. \end{cases}$$

These relations can be written in matrix form as

$$K\begin{pmatrix} V \\ I \end{pmatrix} = M\left(\frac{d}{dt}\right)\begin{pmatrix} V_R \\ V_L \\ V_C \\ I_R \\ I_L \\ I_C \end{pmatrix}$$

with

$$K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, M(\xi) = \begin{pmatrix} 1 & 0 & 0 & -R & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & L\xi \\ 0 & C\xi & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We would now like to eliminate the latent variables. For a small system of low order such as this it is perhaps easiest in practice to do this by substitutions. To start with, use $V = V_R$ and $I_C = I_L$ to eliminate V_R and I_C. The resulting system is then

$$\begin{cases} V = RI_R \\ L\frac{d}{dt}(I_L) = V_L \\ C\frac{d}{dt}(V_C) = I_L \\ I = I_L + I_R \\ V = V_L + V_C. \end{cases}$$

Next we use $V_C = V - V_L$ and $I_R = \frac{V}{R}$ to eliminate V_C and I_R giving us

$$\begin{cases} L\frac{d}{dt}(I_L) = V_L \\ C\frac{d}{dt}(V - V_L) = I_L \\ I = I_L + \frac{V}{R}. \end{cases}$$

Next we use $I_L = I - \frac{V}{R}$ to eliminate I_L, giving us

$$\begin{cases} L\frac{d}{dt}(I - \frac{V}{R}) = V_L \\ C\frac{d}{dt}(V - V_L) = I - \frac{V}{R}. \end{cases}$$

Finally, we use substitute the left hand side of the first equation of this system for V_L in the second to get the equation

$$C\frac{d}{dt}\left(V - L\frac{d}{dt}\left(I - \frac{V}{R}\right)\right) = I - \frac{V}{R}$$

which is equivalent to

$$CL\frac{d^2V}{dt^2} + CR\frac{dV}{dt} + V - CLR\frac{d^2I}{dt^2} - RI = 0. \tag{10}$$

We can of course also do this elimination by multiplication by a unimodular matrix. Let

$$U(\xi) = \begin{pmatrix} 1 + CL\xi^2 & R & -CR\xi & CLR\xi^2 & R & 0 & -1 - CL\xi^2 & -CR\xi \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{L\xi}{R} & 0 & 1 & -L\xi & 0 & 0 & \frac{L\xi}{R} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -C\xi \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -C\xi \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -C\xi \end{pmatrix},$$

which can easily be verified to be unimodular (it has determinant 1). This matrix is recovered by doing the elimination steps in matrix form. The matrix multiplication $U(\xi)\begin{pmatrix} K & -M(\xi) \end{pmatrix}$ yields

$$\begin{pmatrix} -1 - C\xi(R + L\xi) & R + CLR\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{L\xi}{R} & -L\xi & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -C\xi & 1 & 0 & -C\xi & 0 & 1 & 0 & 0 \\ -C\xi & 0 & 0 & -C\xi & 0 & 0 & -1 & 0 \\ -C\xi & 0 & 0 & -C\xi & 0 & 0 & 0 & -1 \end{pmatrix}$$

The first row gives us equation (10), and the second row is all zeros so it gives no information. The other equations give us

$$\tilde{K}\left(\frac{d}{dt}\right)\begin{pmatrix} V \\ I \end{pmatrix} = \tilde{M}\left(\frac{d}{dt}\right)\begin{pmatrix} V_R \\ V_L \\ V_C \\ I_R \\ I_L \\ I_C \end{pmatrix} \tag{11}$$

with

$$\tilde{K}(\xi) = \begin{pmatrix} 1 & 0 \\ \frac{L\xi}{R} & -L\xi \\ 1 & 0 \\ -C\xi & 1 \\ -C\xi & 0 \\ -C\xi & 0 \end{pmatrix}, \quad \tilde{M}(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -C\xi & 0 & 1 & 0 & 0 \\ 0 & -C\xi & 0 & 0 & -1 & 0 \\ 0 & -C\xi & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that $\tilde{M}(\xi)$ has determinant -1, and so is of full rank. Hence V and I can be chosen freely in equation (11), and so the behaviour of V and I are described totally by equation (10).

Remark 2.22. This does not work quite so smoothly in the L^1_{loc}-case. To apply Theorem 2.18 in this case we also need the columns of R_1 and $\tilde{M}$ to satisfy a properness condition, which we can not guarantee in the general case. Then (8) puts a smoothness condition on the possible solutions w, requiring that a certain number of (weak) derivatives of w belong to L^1_{loc} (a vacuous condition if $w \in \mathscr{C}^\infty$).

Let us demonstrate this with an example. Take the behaviour $\mathcal{B} \in \mathfrak{L}^2$ given by

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = l. \end{cases}$$

It is easily verified that

$$(w_1, w_2)(t) = \begin{cases} (t, 1), & t \geq 0 \\ (0, 0), & t < 0 \end{cases}$$

is an L^1_{loc} function that satisfies $\dot{w}_1 = w_2$. The weak derivative of w_2 is in this case the Dirac delta δ_0, not an L^1_{loc}-function, so this w_2 does not satisfy $\dot{w}_2 = l$. Therefore we cannot simply eliminate the latent variable when working with weak solutions in this case.

There are two specific types of latent variable representations that are of particular interest, image and state representations. Image representation is the case when R is just an identity matrix, so $\mathcal{B}$ is given by w such that

$$w = M\left(\frac{d}{dt}\right) l$$

for some $l \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^n)$. The name of course comes from viewing $M\left(\frac{d}{dt}\right)$ as a map from $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ to $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ so that $\mathcal{B} = \operatorname{im} M\left(\frac{d}{dt}\right)$. We will later show that not every behaviour in $\mathfrak{L}^d$ admits such a representation. We will see that it is precisely those behaviours that are controllable that admit such representations.

The other form of latent variable representation that is of particular interest is state space representation. The latent variable is in this case instead called the state, and is denoted x instead of l. For this definition, we first need to define the concatenation of functions $w_1, w_2 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$. The concatenation of w_1, w_2 at time t_0, denoted $w_1 \underset{t_0}{\wedge} w_2$ is given by

$$w_1 \underset{t_0}{\wedge} w_2(t) = \begin{cases} w_1(t), t < t_0 \\ w_2(t), t \geq t_0. \end{cases}$$

The following definition is very similar to Definition 2.1.

Definition 2.23. A dynamical system in state space form is a quadruple $\Sigma_s = (\mathcal{T}, \mathcal{W}, \mathcal{X}, \mathcal{B}_s)$ where $\mathcal{T} \subseteq \mathbb{R}$ is the time axis, $\mathcal{W}$ is a set called the signal space, $\mathcal{X}$ is a set called the state space and $\mathcal{B}_s \subseteq (\mathcal{W} \times \mathcal{X})^{\mathcal{T}}$ is the state behaviour, which satisfies the axiom of state:

$$\forall t_0 \in \mathcal{T}\big((w_1, x_1), (w_2, x_2) \in \mathcal{B}_s \wedge x_1(t_0) = x_2(t_0)\big) \implies (w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \overline{\mathcal{B}}_s.$$

We should note that if $\Sigma_s = (\mathcal{T}, \mathcal{W}, \mathcal{X}, \mathcal{B}_s)$ is a system in state space form, then $(\mathcal{T}, \mathcal{W} \times \mathcal{X}, \mathcal{B}_s)$ is a dynamical system in the sense of Definition 2.1. Also, if $P_{\mathcal{W}} : \mathcal{W} \times \mathcal{X} \to \mathcal{W}$ is the projection into $\mathcal{W}$, then $(\mathcal{T}, \mathcal{W}, P_{\mathcal{W}}\mathcal{B}_s)$ is also a dynamical system in the sense of Definition 2.1. This projection is in essence the elimination procedure for latent variables described above. In this case we say that Σ_s is a state space representation of the system $(\mathcal{T}, \mathcal{W}, P_{\mathcal{W}}\mathcal{B}_s)$, or for the behaviour $P_{\mathcal{W}}\mathcal{B}_s$.

The concatenation of two smooth functions will in general not be smooth, both the function values and derivatives may be different at the break point t_0. Hence we can not expect the concatenations of solutions to be strong solutions, but the axiom of state tells us that they are at least weak solutions. If we want to generalize state space representations to the L^1_{loc}-case, we need an additional condition in the antecedent, that x_1, x_2 are continuous at t_0. This is because L^1_{loc}-functions are defined up to almost everywhere equivalence, so the exact value at one specific point t_0 has no meaning.

The content of the axiom of state is that the state x at time t_0 determines which futures are possible for w. The state contains all information about the past of both x and w. In this sense the state acts as a kind of memory for the system. For $\mathfrak{L}^d$-behaviours the state space is chosen as $\mathcal{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$ (or sometimes $\mathbb{C}^n$). A state space representation is said to be minimal if n is minimal among all state space representations. State space representations of $\mathfrak{L}^d$-behaviours can also be characterized as those latent variable representations that can be given by a particularly simple form of equation.

Proposition 2.24. *Let $\mathcal{B} \in \mathfrak{L}^d$. Then $(\mathbb{R}, \mathbb{R}^d, \mathbb{R}^n, \mathcal{B}_s)$ is a state space representation of $\mathcal{B}$ only if there are matrices $E, F \in \mathbb{R}^{e \times n}, G \in \mathbb{R}^{e \times d}$ such that $\mathcal{B}_s$ is the solution set of*

$$E\dot{x} + Fx + Gw = 0.$$

The converse of this statement does also hold, however the proof is more involved. For a proof of the converse we instead refer to [4].

Proof. Take any $t_0 \in \mathbb{R}$ and any $(w_1, x_1), (w_2, x_2) \in \mathcal{B}$ such that $x_1(t_0) = x_2(t_0)$ and let

$$(w, x) := (w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2).$$

For any $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^e)$,

$$\begin{aligned}
\int_{\mathbb{R}} x^T E^T \dot{\varphi} dt &= \int_{-\infty}^{t_0} x_1^T E^T \dot{\varphi} dt + \int_{t_0}^{\infty} x_2^T E^T \dot{\varphi} dt = \\
&= \left[\varphi^T E x_1\right]_{-\infty}^{t_0} - \int_{-\infty}^{t_0} \varphi^T E \dot{x}_1 dt + \left[\varphi^T E x_2\right]_{t_0}^{\infty} - \int_{t_0}^{\infty} \varphi^T E \dot{x}_2 dt = \\
&= \varphi^T E(x_1(t_0) - x_2(t_0)) - \int_{-\infty}^{t_0} \varphi^T E \dot{x}_1 dt - \int_{t_0}^{\infty} \varphi^T E \dot{x}_2 dt = \\
&= -\int_{-\infty}^{t_0} \varphi^T E \dot{x}_1 dt - \int_{t_0}^{\infty} \varphi^T E \dot{x}_2 dt.
\end{aligned}$$

Hence for any $\varphi \in \mathscr{D}(\mathbb{R}, \mathbb{R}^e)$

$$\begin{aligned}
&\int_{\mathbb{R}} - x^T E^T \dot{\varphi} + (x^T F^T + w^T G^T)\varphi dt = \\
&= \int_{-\infty}^{t_0} \varphi^T E \dot{x}_1 dt + \int_{t_0}^{\infty} \varphi^T E \dot{x}_2 dt + \int_{\mathbb{R}} (x^T F^T + w^T G^T)\varphi dt = \\
&= \int_{-\infty}^{t_0} \varphi^T (E\dot{x}_1 + F x_1 + G w_1) dt + \int_{t_0}^{\infty} \varphi^T (E\dot{x}_2 + F x_2 + G w_2) dt = 0
\end{aligned}$$

so (w, x) solves the equation weakly, and so $\mathcal{B}_s$ is a state space representation. □

While we do not prove the converse, this proof can at least give us a hint of why state representations always can be given by such simple equations. In the first set of equalities we use that $\varphi^T E(x_1(t_0) - x_2(t_0)) = 0$. If we had higher order derivatives present and we tried this same approach of using integration by parts, we would get similar terms involving w_1, w_2 and derivatives of x_1, x_2 evaluated at t_0. We would still need these terms to vanish, and this in turn means that we will have some representation of the system where no higher order derivatives are present.

Combining state representation and input/output decomposition we arrive at an input/state/output (i/s/o) representation. Of particular interest are systems of the form

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n}, D \in \mathbb{R}^{q \times p}$. Note that this is of the form of Proposition 2.24 by setting

$$E = \begin{pmatrix} -I \\ 0 \end{pmatrix}, F = \begin{pmatrix} A \\ C \end{pmatrix}, G = \begin{pmatrix} B & 0 \\ D & -I \end{pmatrix}$$

so this really is a state representation. We can also write the system as

$$\begin{pmatrix} \xi I - A & 0 \\ -C & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} u.$$

Note that

$$\det \begin{pmatrix} \xi I - A & 0 \\ -C & I \end{pmatrix} = \det(\xi I - A)$$

which is not zero. Hence, by Theorem 2.18 u and y do indeed have the properties of input and output. The differential equation can be solved using the formula

$$x(t) = e^{At} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau.$$

The output y can then be written out explicitly as

$$y(t) = C e^{At} x(t_0) + \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

Here we can see the axiom of state in action. The possible values of $y(t)$ for $t > t_0$ are determined by just the value of $x(t_0)$ and $u(t)$ for $t > t_0$ (and the system paramters), and do not depend on the values of $u(t)$ or $y(t)$ for $t \leq t_0$. Since $u(t)$ is free in the first place, it is not limited by the state at t_0 anyway.

The state space framework, a more standard approach for dynamical systems, usually starts with this kind of i/s/o representation. In for example [8] a dynamical system (with outputs) is defined as a sextuple $(\mathcal{T}, \mathcal{U}, \mathcal{X}, \mathcal{Y}, \varphi, h)$, where $\mathcal{T}, \mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$ are the spaces of time, inputs, states and outputs respectively. The remaining two components, φ and h, are called the transition map and the readout map respectively, and they describe how the state and output change over time. They play essentially the same role as the behaviour $\mathcal{B}$ does in our behavioural definition. The properties of state, inputs, and outputs are then built into the definition of a dynamical system.

In some sense our definition is more general, as it has i/s/o systems as a special case, but the i/s/o approach can easily be adapted to handle for example i/o-behaviours. In some cases it may be useful to not have to assign roles to the variables from the start, this of course depends on what it is we are modelling. In practice the main difference between the different frameworks lie in perspective and some terminology.

We will now introduce two important concepts for dynamical systems, controllability and observability. The proofs of the following statements are from [6].

Definition 2.25. A time invariant system $(\mathcal{T}, \mathcal{W}, \mathcal{B})$ is called controllable if for any $w_1, w_2 \in \mathcal{B}$ there is $t_0 > 0$ and $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t), t < 0 \\ w_2(t - t_0), t \geq t_0. \end{cases}$$

Controllability means that we can transition between any two trajectories, though with a small delay. The delay t_0 does in fact not depend on w_1 or w_2. In general the delay must be positive, but it can be taken arbitrarily small. Two trivial examples of controllable behaviours are $\{0\}$ and $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$, the first since it only has one element and the second because smooth functions can always be interpolated in a smooth way.

Controllability can be thought of as a kind of opposite to autonomous systems. For an autonomous system the past completely determines the future, while controllability says that we can choose any future we want regardless of the past. Consequently an autonomous behaviour is not controllable, with the only exception being the trivial case $\mathcal{B} = \{0\}$.

Theorem 2.26. *The system $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$ is controllable if and only if the map $\lambda \mapsto \operatorname{rk}(R(\lambda))$ is constant for $\lambda \in \mathbb{C}$.*

Proof. We will first show the case when R is square and $\det R(\xi) \neq 0$. By Proposition 2.13 $\mathcal{B}$ is then autonomous and the solutions are given by the roots of $\det R(\xi)$. An autonomous behaviour is controllable if and only if it is trivial, which happens if and only if $\det R(\xi)$ has no roots, i.e. it is constant. This is equivalent to $\operatorname{rk}(R(\lambda))$ being the same for all $\lambda \in \mathbb{C}$.

Next we look at the general case for $R \in \mathbb{R}^{e \times d}[\xi]$. By Proposition 2.11 there are unimodular matrices U, V such that

$$URV(\xi) = \begin{pmatrix} D(\xi) & 0 \\ 0 & 0 \end{pmatrix}$$

where D is diagonal with $\det D(\xi) \neq 0$. We may have one, several or no zero rows under and to the right of $D(\xi)$ depending on the size and rank of R. Let $\tilde{\mathcal{B}} = V^{-1}\left(\frac{d}{dt}\right)\mathcal{B}$. Then

$$\tilde{\mathcal{B}} = \left\{ \tilde{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d) | D\left(\frac{d}{dt}\right) v_1 = 0 \right\}.$$

The second component v_2 can be chosen freely, so $\tilde{\mathcal{B}}$ is controllable if and only if $\ker D\left(\frac{d}{dt}\right)$ is. By the special case proven above it is controllable if and only if the rank of

$$\begin{pmatrix} D(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$$

is the same for all $\lambda \in \mathbb{C}$. Since U, V are unimodular this is equivalent to the map $\lambda \mapsto \operatorname{rk}(R(\lambda))$ being constant.

Next we show that $\tilde{\mathcal{B}}$ is controllable if and only if $\mathcal{B}$ is. Suppose $\tilde{\mathcal{B}}$ is controllable and that $w_1, w_2 \in \mathcal{B}$. Let $\tilde{w}_1 = V^{-1}\left(\frac{d}{dt}\right) w_1, \tilde{w}_2 = V^{-1}\left(\frac{d}{dt}\right) w_2$. Then since $\tilde{\mathcal{B}}$ is controllable there is $\tilde{w} \in \tilde{\mathcal{B}}$ such that

$$\tilde{w}(t) = \begin{cases} \tilde{w}_1(t), t < 0 \\ \tilde{w}_2(t - t_0), t \geq t_0. \end{cases}$$

for any given $t_0 > 0$. Then $w := V\left(\frac{d}{dt}\right)\tilde{w} \in \mathcal{B}$ and

$$w(t) = \begin{cases} w_1(t), t < 0 \\ w_2(t - t_0), t \geq t_0. \end{cases}$$

so $\mathcal{B}$ is controllable. The opposite direction is analogous, switching the roles of $V\left(\frac{d}{dt}\right)$ and $V^{-1}\left(\frac{d}{dt}\right)$. $\square$

This result extends to weak solutions as well. To see why we will look at the decomposition of a behaviour

$$\mathcal{B} = \mathcal{B}_{\text{aut}} + \mathcal{B}_{\text{cont}}$$

discussed earlier. We call $\mathcal{B}_{\text{cont}}$ the controllable part, and it can be shown that it is always controllable. Furthermore $\mathcal{B}$ is controllable if and only if $\mathcal{B}_{\text{aut}} = \{0\}$ (so that $\mathcal{B} = \mathcal{B}_{\text{cont}}$) and $\mathcal{B}$ is autonomous if and only if $\mathcal{B}_{\text{cont}} = \{0\}$ (so that $\mathcal{B} = \mathcal{B}_{\text{aut}}$). This decomposition is discussed in more detail in [6].

We have however already found that for an autonomous behaviour there are virtually no weak solutions that are not strong solutions. As such controllability is completely determined by the strong solutions, and hence the proposition above applies just as well if we consider weak solutions.

The most important use of controllability for us will be its relation to the existence of image representations. This will be particularly important in section 5 where we discuss dissipative systems.

Proposition 2.27. *A behaviour $\mathcal{B} \in \mathfrak{L}^d$ is controllable if and only if it has an image representation.*

Proof. Suppose first that $\mathcal{B}$ has an image representation $w = M\left(\frac{d}{dt}\right) l$, where $M \in \mathbb{R}^{d\times n}[\xi]$. We now carry out the latent variable elimination procedure described above. More precisely, there is by Proposition 2.10 a unimodular $U \in \mathbb{R}^{d\times d}[\xi]$ such that

$$U\left(\frac{d}{dt}\right) w = \begin{pmatrix} \tilde{M}\left(\frac{d}{dt}\right) \\ 0 \end{pmatrix} l$$

where $\tilde{M}$ has full row rank. If we decompose U as

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

so that U_1 has the same number of rows as $\tilde{M}$, then $\mathcal{B} = \ker U_2\left(\frac{d}{dt}\right)$. Since U is unimodular, the rows of $U(\lambda)$ are linearly independent for every $\lambda \in \mathbb{C}$ and so $\operatorname{rk}(U_2(\lambda))$ is the same for all $\lambda \in \mathbb{C}$. By Theorem 2.26 $\mathcal{B}$ is then controllable.

Next we suppose that $\mathcal{B}$ is controllable. There is then $R \in \mathbb{R}^{e\times d}[\xi]$ such that $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$, and by Proposition 2.10 we can assume R has full row rank. By Proposition 2.11 there are unimodular U, V such that $URV = \begin{pmatrix} D & 0 \end{pmatrix}$ where $D(\xi)$ is diagonal. Since $e = \operatorname{rk}(R(\lambda)) = \operatorname{rk}(D(\lambda))$ for every $\lambda \in \mathbb{C}$ the diagonal elements of D must be non-zero constants, and by suitable choices of U, V we can assume that D is an identity matrix. Hence we have $RV = \begin{pmatrix} U^{-1} & 0 \end{pmatrix}$. Now set

$$W(\xi) = V(\xi) \begin{pmatrix} U(\xi) & 0 \\ 0 & I \end{pmatrix}$$

where I is an identity matrix of size $d-e$ so that $W \in \mathbb{R}^{d \times d}[\xi]$. Note that since U and V are unimodular, so is W. Also,

$$RW = RV \begin{pmatrix} U(\xi) & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U^{-1} & 0 \end{pmatrix} \begin{pmatrix} U(\xi) & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix},$$

or equivalently $R = \begin{pmatrix} I & 0 \end{pmatrix} W^{-1}$. This means that R is the first e rows of the matrix W^{-1}. Let $\tilde{R}$ be the other $d-e$ rows of W and consider the behaviour given in latent variable form as

$$\begin{pmatrix} R \\ \tilde{R} \end{pmatrix} \left(\frac{d}{dt} \right) w = \begin{pmatrix} 0 \\ I \end{pmatrix} l$$

where the identity matrix in the right hand side is of size $d-e$. By the latent variable elimination procedure, this behaviour is $\ker R\left(\frac{d}{dt}\right) = \mathcal{B}$. Since W is unimodular, an equivalent representation of $\mathcal{B}$ is given by

$$w = W\left(\frac{d}{dt} \right) \begin{pmatrix} 0 \\ I \end{pmatrix} l,$$

giving us an image representation for $\mathcal{B}$. □

Next we define observability, which is used to describe if some sytem variables carry information about the other system variables.

Definition 2.28. Let $\mathcal{B} \in \mathfrak{L}^d$ and for $w \in \mathcal{B}$ consider a decomposition

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

We say that w_2 is observable from w_1 if

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ \tilde{w}_2 \end{pmatrix} \in \mathcal{B}$$

implies that $w_2 = \tilde{w}_2$.

If w_2 is observable from w_1 then w_2 does in some sense not add any more information about the system, w holds the same information about the system as w_1 does. Observability also means that there is an injection sending w_1 to w_2. The most important result is the following.

Theorem 2.29. *Let $R_1 \in \mathbb{R}^{e \times d_1}[\xi]$, $R_2 \in \mathbb{R}^{e \times d_2}[\xi]$ and consider the system $\mathcal{B}$ given by $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$. Then w_2 is observable from w_1 if and only if $\operatorname{rk}(R_2(\lambda)) = d_2$ for all $\lambda \in \mathbb{C}$.*

Proof. For any $(w_1, w_2), (w_1, \tilde{w}_2) \in \mathcal{B}$ we have by linearity that $(0, w_2 - \tilde{w}_2) \in \mathcal{B}$. Hence w_2 is observable from w_1 if and only if $R_2\left(\frac{d}{dt}\right) w_2 = 0$ implies $w_2 = 0$. Now let $\mathcal{B}_2 = \ker R_2\left(\frac{d}{dt}\right)$, so we have concluded that w_2 is observable from w_1 if and only if $\mathcal{B}_2 = \{0\}$.

Suppose w_2 is observable from w_1 so that $\mathcal{B}_2 = \{0\}$. By Proposition 2.10 there is unimodular U such that

$$UR_2 = \begin{pmatrix} \tilde{R}_2 \\ 0 \end{pmatrix}$$

where $\mathcal{B}_2 = \ker \tilde{R}_2$, $\mathrm{rk}(R_2) = \mathrm{rk}\left(\tilde{R}_2\right)$ and $\tilde{R}_2 \in \mathbb{R}^{c \times d_2}[\xi]$ has full row rank. We must then have $c \leq d_2$ and if $c < d_2$ then by Theorem 2.18 some components of w_2 could be chosen freely as inputs, contradicting $\mathcal{B}_2 = \{0\}$. Hence we must have $c = d_2$. Since $\mathcal{B}_2 = \{0\}$, $\tilde{R}_2$ must hence be unimodular, and so $\mathrm{rk}(R_2(\lambda)) = \mathrm{rk}\left(\tilde{R}_2(\lambda)\right) = d_2$ for all $\lambda \in \mathbb{C}$.

Next we suppose $\mathrm{rk}(R_2(\lambda)) = \mathrm{rk}\left(\tilde{R}_2(\lambda)\right) = d_2$ for all $\lambda \in \mathbb{C}$. Then we must have $c = d_2$ and $\det \tilde{R}_2(\xi)$ a non-zero constant. Hence $\tilde{R}_2$ is unimodular, so $\mathcal{B}_2 = \{0\}$ and w_2 is observable from w_1. $\square$

Using Theorem 2.29 it is easy to check that the image representation constructed in the proof of Proposition 2.27 is observable, giving us the following.

Corollary 2.30. *If $\mathcal{B} \in \mathfrak{L}^d$ is controllable, then it has an observable image representation.*

Let us look at controllability and observability of the i/s/o system given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

We will first look at controllability. To be clear we want to control both the external variables u, y as well as the state x. Note that the equations can be written in matrix form as

$$\begin{pmatrix} A - \frac{d}{dt}I & B & 0 \\ C & D & -I \end{pmatrix} \begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0.$$

By Theorem 2.26 this system is controllable if and only if

$$\mathrm{rk} \begin{pmatrix} A - \lambda I & B & 0 \\ C & D & -I \end{pmatrix}$$

is the same for all $\lambda \in \mathbb{C}$. If q is the size of y, then we have

$$\mathrm{rk} \begin{pmatrix} A - \lambda I & B & 0 \\ C & D & -I \end{pmatrix} = \mathrm{rk} \begin{pmatrix} A - \lambda I & B \end{pmatrix} + q$$

For any $\lambda \in \mathbb{C}$ that is not an eigenvalue of A, the matrix $A - \lambda I$ has rank n, where n is the size of x. Hence the system is controllable if and only if $\operatorname{rk}\begin{pmatrix} A - \lambda I & B \end{pmatrix} = n$ for every eigenvalue λ of A.

Next we look at observability. For i/s/o systems we are usually interested in whether the state x is observable from the external variables u and y. We can write the equations as

$$\begin{pmatrix} B & 0 \\ D & -I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} A - \frac{d}{dt} I \\ C \end{pmatrix} x.$$

By Theorem 2.29 x is observable from u and y if and only if

$$\operatorname{rk}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

is the same for all $\lambda \in \mathbb{C}$. Similar to controllability we have for λ that are not eigenvalues of A that $A - \lambda I$ is full rank (rank equal to n). Hence x is observable if and only if

$$\operatorname{rk}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n$$

for all eigenvalues λ of A.

It is thus quite easy to determine controllability and observability for this i/s/o system, provided that we can find the eigenvalues of A.

There is another system for which observability will be important to us, given in latent variable form as

$$\begin{aligned} 0 &= A\left(\frac{d}{dt}\right) l \\ w &= C\left(\frac{d}{dt}\right) l \end{aligned}$$

Here we are interested in if l is observable from w. From the theorem it is clear that this happens if and only if

$$\operatorname{rk}\begin{pmatrix} A(\lambda) \\ C(\lambda) \end{pmatrix} = n$$

for every $\lambda \in \mathbb{C}$ (here n is the size of l). If this holds we say that (A, C) is an observable pair.

3 Quadratic differential forms

When studying dynamical systems functionals of the variables are at times of interest. In many applications such functionals describe some form of energy, the total energy in the system, energy being delivered to the system, etc. We will look at functionals that are quadratic in the external signal w and its derivatives, so called quadratic differential forms, in the context of the linear systems discussed in the previous section. We will use them to study stability and dissipativeness of linear systems. The theory of this section is in large part based on [14].

Much like polynomial matrices in one variable are very useful to describe linear differential systems, we will see in this section that polynomial matrices in two variables are suitable to work with quadratic differential forms, and more generally bilinear differential forms. Before we get to those we study some useful properties of polynomial matrices in two variables.

We let $\mathbb{R}^{d_1 \times d_2}[\zeta, \eta]$ denote the set of $d_1 \times d_2$ matrices whose elements are polynomials in the two indeterminates ζ and η. Note that every $\Phi \in \mathbb{R}^{d_1 \times d_2}[\zeta, \eta]$ can be written

$$\Phi(\zeta, \eta) = \sum_{k,l} \Phi_{k,l} \zeta^k \eta^l$$

where the sum is over nonnegative k, l and finite, and $\Phi_{k,l} \in \mathbb{R}^{d_1 \times d_2}$ for every k, l. To $\Phi \in \mathbb{R}^{d_1 \times d_2}[\zeta, \eta]$ we then associate the infinite matrix

$$\widetilde{\Phi} = \begin{pmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that only finitely many elements of this matrix are non-zero. Some authors (e.g. [9]) take $\widetilde{\Phi}$ to be finite, its size then depends on the highest degrees of ζ and η that are present in Φ. Whether to treat $\widetilde{\Phi}$ as a finite or infinite matrix is only a matter of notation and taste since only the non-zero part of the infinite one matters. In this text we will stick to the infinite approach. Note that factorizations of finite matrices can essentially be applied to $\widetilde{\Phi}$ even though it is infinite, for example if $\widetilde{\Phi}$ is positive definite then we can write $\widetilde{\Phi} = D^T D$ for some infinite matrix D.

Let

$$E_d(\xi) = \begin{pmatrix} I_d \\ I_d \xi \\ I_d \xi^2 \\ \vdots \end{pmatrix}$$

where I_d is an identity matrix of size d. We can now write

$$\Phi(\zeta, \eta) = E_{d_1}^T(\zeta) \widetilde{\Phi} E_{d_2}(\eta).$$

Due to this relation we can use factorizations of $\widetilde{\Phi}$ to factorize $\Phi(\zeta, \eta)$. This will be particularly useful when we consider quadratic forms.

The polynomial matrix $\Phi \in \mathbb{R}^{d_1 \times d_2}[\zeta, \eta]$ induces a bilinear differential form (abbreviated as BLDF)

$$
\begin{aligned}
L_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{d_1}) \times \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{d_2}) &\to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}) \\
(v, w) &\mapsto \sum_{k,l} \left(\frac{d^k}{dt^k} v \right)^T \Phi_{k,l} \left(\frac{d^l}{dt^l} w \right)
\end{aligned}
$$

and if $d_1 = d_2 =: d$, a quadratic differential form (abbreviated as QDF)

$$
\begin{aligned}
Q_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d) &\to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}) \\
w &\mapsto L_\Phi(w, w).
\end{aligned}
$$

We will refer to the expression $L_\Phi(v, w)$ as L_Φ evaluated along v and w, and similarly $Q_\Phi(w)$ as Q_Φ evaluated along w. Both BLDFs and QDFs are defined here as operators on smooth functions. We will later study them in the setting where the functions v, w are not free, but rather belong to some behaviour $\mathcal{B} \in \mathfrak{L}^d$.

These definitions can of course easily be extended to functions v, w that are not completely smooth, but differentiable enough times that the expressions are well defined. We may also wish to extend this definition to L^1_{loc}-functions and interpreting the derivatives as weak derivatives. This is more complicated, since the derivative of an L^1_{loc}-function is not in general itself L^1_{loc}. Hence we either need some assumptions that we have enough nicely behaved derivatives or must involve distributions in the definition.

Just as linear systems can be induced by many different polynomial matrices, the same is the case for QDFs. Much like quadratic forms on $\mathbb{R}^d$, QDFs can always be induced by a matrix Φ that is symmetric in a certain sense. To see how, we first define the asterisk operator as

$$
\begin{aligned}
* : \mathbb{R}^{d_1 \times d_2}[\zeta, \eta] &\to \mathbb{R}^{d_2 \times d_1}[\zeta, \eta] \\
\Phi(\zeta, \eta) &\mapsto E^T_{d_2}(\zeta) \widetilde{\Phi}^T E_{d_1}(\eta).
\end{aligned}
$$

In words, the asterisk operator transposes the polynomial matrix, and then switches the indeterminates. In the case when $d_1 = d_2$ we say that $\Phi \in \mathbb{R}^{d \times d}[\zeta, \eta]$ is symmetric if $\Phi^* = \Phi$. We denote the subset of $\mathbb{R}^{d \times d}[\zeta, \eta]$ consisting of symmetric matrices by $\mathbb{R}^{d \times d}_{\text{s}}[\zeta, \eta]$. Note also that if $d_1 = d_2$, then $\Phi^* = \Phi$ if and only if $\tilde{\Phi}$ is symmetric. If $\Phi \in \mathbb{R}^{d \times d}_{\text{s}}[\zeta, \eta]$, then

$$
\begin{aligned}
Q_{\Phi^*}(w) &= \sum_{k,l} \left(\frac{d^k}{dt^k} w \right)^T \Phi^T_{k,l} \left(\frac{d^l}{dt^l} w \right) = \\
&= \left(\sum_{k,l} \left(\frac{d^l}{dt^l} w \right)^T \Phi_{k,l} \left(\frac{d^k}{dt^k} w \right) \right)^T = Q_\Phi(w).
\end{aligned}
$$

It follows that

$$
Q_{\Phi^*} = Q_\Phi = Q_{\frac{1}{2}(\Phi + \Phi^*)}
$$

and since $\widetilde{\Phi}^*$ is the transpose of $\widetilde{\Phi}$, if we let $\Psi = \Phi + \Phi^*$ then $\widetilde{\Psi}$ is symmetric. Hence every QDF can be induced by a symmetric matrix.

We can extend BLDFs to complex valued functions by

$$\begin{aligned} L_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{C}^{d_1}) \times \mathscr{C}^\infty(\mathbb{R}, \mathbb{C}^{d_2}) &\to \mathscr{C}^\infty(\mathbb{R}, \mathbb{C}) \\ (v, w) &\mapsto \sum_{k,l\geq 0} \left(\frac{d^k}{dt^k}\overline{v}\right)^T \Phi_{k,l} \left(\frac{d^l}{dt^l} w\right) \end{aligned}$$

where $\overline{v}$ denotes the complex conjugate of v. Just as before the definition of a QDF for complex w follows as $Q_\Phi(w) = L_\Phi(w, w)$. Suppose w splits into its real and imaginary parts as $w = w_r + iw_i$. Then we have

$$\begin{aligned} Q_\Phi(w) &= L_\Phi(w_r + iw_i, w_r + iw_i) \\ &= Q_\Phi(w_r) + Q_\Phi(w_i) + iL_\Phi(w_r, w_i) - iL_\Phi(w_i, w_r). \end{aligned}$$

Since we are interested in a QDF here we can assume that Φ is symmetric, and then $L_\Phi(w_r, w_i) = L_\Phi(w_i, w_r)$. It follows that

$$Q_\Phi(w) = Q_\Phi(w_r) + Q_\Phi(w_i).$$

This means that QDFs are real valued even for complex valued functions.

Since for any w, a BLDF L_Φ (or QDF) evaluated along v and w is itself a function of time it makes sense to differentiate it. The derivative is then itself a BLDF evaluated along v and w and the two BLDFs can be easily related using the dot operator defined as

$$\begin{aligned} \bullet : \mathbb{R}^{d_1\times d_2}[\zeta, \eta] &\to \mathbb{R}^{d_1\times d_2}[\zeta, \eta] \\ \overset{\bullet}{\Phi}(\zeta, \eta) &:= (\zeta + \eta)\Phi(\zeta, \eta). \end{aligned}$$

The induced BLDF of $\overset{\bullet}{\Phi}$ evaluated along some $v \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{d_1}), w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{d_2})$ is then

$$\begin{aligned} &L_{\overset{\bullet}{\Phi}}(v, w)(t) = \\ &= \sum_{k,l} \left(\frac{d^{k+1}}{dt^{k+1}} v(t)\right)^T \Phi_{k,l} \left(\frac{d^l}{dt^l} w(t)\right) + \left(\frac{d^k}{dt^k} v(t)\right)^T \Phi_{k,l} \left(\frac{d^{l+1}}{dt^{l+1}} w(t)\right) = \\ &= \frac{d}{dt}\left(\sum_{k,l} \left(\frac{d^k}{dt^k} v(t)\right)^T \Phi_{k,l} \left(\frac{d^l}{dt^l} w(t)\right)\right) = \frac{d}{dt}(L_\Phi(v, w))(t), \end{aligned}$$

so the dot operator encodes how differentiation affects the polynomial matrices. The case for QDFs is analogous.

Another operator on polynomial matrices that we will need is the delta operator, which relates polynomial matrices in two indeterminates to a matrix with only one indeterminate. It is defined as

$$\begin{aligned} \partial :&\mathbb{R}^{d_1\times d_2}[\zeta, \eta] \to \mathbb{R}^{d_1\times d_2}[\xi] \\ &\partial\Phi(\xi) := \Phi(-\xi, \xi). \end{aligned}$$

Just like we can differentiate BLDFs (or QDFs), we can also integrate them. We will assume that the external signals are compactly supported in this case to ensure that the integrals converge. Recall that we write $\mathscr{D}$ for smooth functions with compact support (see section 2.1). We define

$$\begin{aligned}\int L_\Phi :&\mathscr{D}(\mathbb{R},\mathbb{R}^{d_1}) \times \mathscr{D}(\mathbb{R},\mathbb{R}^{d_2}) \to \mathbb{R}\\ (v,w) \mapsto& \int_{\mathbb{R}} L_\Phi(v,w)dt.\end{aligned}$$

The integral of a QDF, denoted $\int Q_\Phi$ is defined analogously. For a finite interval $[t_0,t_1]$ we similarly define $\int_{t_0}^{t_1} L_\Phi(v,w)$. We say that $\int_{t_0}^{t_1} L_\Phi(v,w)$ is independent of path if it depends only on v,w and finitely many of their derivatives at times t_0 and t_1. These various operators on BLDFs are connected by the following result found in [14].

Theorem 3.1. *For any $\Phi \in \mathbb{R}^{d_1\times d_2}[\zeta,\eta]$, the following are equivalent:*

(i) $\int L_\Phi = 0$.

(ii) $\int_{t_0}^{t_1} L_\Phi$ *is independent of path.*

(iii) *There exists $\Psi \in \mathbb{R}^{d_1\times d_2}[\zeta,\eta]$ such that $\Phi = \overset{\bullet}{\Psi}$, in other words*

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta)}{\zeta+\eta}.$$

(iv) $\partial\Phi(\xi) = 0$.

If $d_1 = d_2$, then the statement also holds with L_Φ replaced by Q_Φ.

Proof. We will first show the implications (i) $\implies$ (iv) $\implies$ (iii) $\implies$ (i) and then (iii) $\implies$ (ii) $\implies$ (i).

(i) $\implies$ (iv): We first note that if $\int L_\Phi = 0$ then this in fact extends to complex valued functions. If v and w split into real and imaginary parts as $v = v_r + iv_i, w = w_r + iw_i$, then

$$L_\Phi(v,w) = L_\Phi(v_r,w_r) + L_\Phi(v_i,w_i) + iL_\Phi(v_r,w_i) - iL_\Phi(v_i,w_r).$$

Hence if $\int L_\Phi = 0$ for real v,w it also holds for complex valued functions. Now suppose for a contradiction that $\partial\Phi \neq 0$. Then there are $a \in \mathbb{C}^{d_1}, b \in \mathbb{C}^{d_2}, \omega \in \mathbb{R}$ such that $\overline{a}^T\Phi(-i\omega,i\omega)b \neq 0$. Let $v = ae^{i\omega t}$ and $w = be^{i\omega t}$. Since these functions do not have compact support, we will multiply them with a smooth transition function. There are many ways to construct smooth transition functions, the following is from [12]. Let

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0\\ 0, & t \leq 0,\end{cases}$$

which is easily verified to be a smooth function. Now let $g(t) = \frac{f(t)}{f(t)+f(1-t)}$, which is a smooth function that satisfies

$$g(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t \leq 0. \end{cases}$$

For $N = 1, 2, \ldots$ we define $h_N(t) = g(\frac{2\pi N}{\omega} + 1 + x)g(\frac{2\pi N}{\omega} + 1 - x)$. In the special case $\omega = 0$, we can replace $\frac{2\pi N}{\omega}$ with N and proceed the same way. The function h_N is then smooth and satisfies

$$h_N(t) = \begin{cases} 1, & |t| \leq \frac{2\pi N}{\omega} \\ 0, & |t| \geq \frac{2\pi N}{\omega} + 1. \end{cases}$$

Now we let $v_N(t) = v(t)h_N(t)$ and $w_N(t) = w(t)h_N(t)$. These functions are then smooth functions with compact support, v_N satisfies

$$v_N(t) = \begin{cases} ae^{i\omega t}, & |t| \leq \frac{2\pi N}{\omega} \\ 0, & |t| \geq \frac{2\pi N}{\omega} + 1. \end{cases}$$

Now consider the integral

$$\int_{\mathbb{R}} L_\Phi(v_N, w_N)dt = \int_{-\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}} L_\Phi(v_N, w_N)dt + \\ + \int_{-\infty}^{-\frac{2\pi N}{\omega}} L_\Phi(v_N, w_N)dt + \int_{\frac{2\pi N}{\omega}}^{\infty} L_\Phi(v_N, w_N)dt.$$

Since v and w have period $\frac{2\pi}{\omega}$, we have

$$v_N(t) = v(t)h_N(t) = v\left(t - \frac{2\pi(N-1)}{\omega}\right) h_1\left(t - \frac{2\pi(N-1)}{\omega}\right) = \\ = v_1\left(t - \frac{2\pi(N-1)}{\omega}\right)$$

and similar for w_N. By the variable change $s = t - \frac{2\pi(N-1)}{\omega}$ we have

$$\int_{\frac{2\pi N}{\omega}}^{\infty} L_\Phi(v_N, w_N)dt = \int_{\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}+1} L_\Phi(v_N, w_N)dt = \int_{\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}+1} L_\Phi(v_1, w_1)ds =: A.$$

The exact value of the integral A is not important, what matters is that it is independent of N. By symmetry we also have $\int_{-\infty}^{-\frac{2\pi N}{\omega}} L_\Phi(v_N, w_N)dt = A$.

Finally we have

$$
\begin{aligned}
&\int_{-\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}} L_\Phi(v_N, w_N)dt = \\
&= \int_{-\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}} \sum_{k,l} \left(\left(\frac{d}{dt}\right)^k \overline{a} e^{-i\omega t} \right)^T \Phi_{k,l} \left(\left(\frac{d}{dt}\right)^l b e^{i\omega t} \right) dt = \\
&= \int_{-\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}} \sum_{k,l} (-i\omega)^k e^{-i\omega t} \overline{a}^T \Phi_{k,l} b (i\omega)^l e^{i\omega t} dt = \\
&= \int_{-\frac{2\pi N}{\omega}}^{\frac{2\pi N}{\omega}} dt \sum_{k,l} \overline{a}^T (-i\omega)^k \Phi_{k,l} (i\omega)^l b = \\
&= \frac{4\pi N}{\omega} \overline{a}^T \Phi(-i\omega, i\omega) b.
\end{aligned}
$$

Hence we have

$$
\int_{\mathbb{R}} L_\Phi(v_N, w_N)dt = \frac{4\pi N}{\omega} \overline{a}^T \Phi(-i\omega, i\omega) b + 2A
$$

and since $\overline{a}^T \Phi(-i\omega, i\omega) b \neq 0$ and A is independent of N, there is some N such that $\int_{\mathbb{R}} L_\Phi(v_N, w_N)dt \neq 0$. Hence if (iv) is false, then so is (i). In other words (i) implies (iv).

(iv) $\implies$ (iii): Viewing $\Phi(\zeta, \eta)$ as polynomial in one variable ζ, we perform division by $\zeta + \eta$ with remainder (componentwise) and get

$$
\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta) + r(\zeta, \eta).
$$

for some $\Psi, r \in \mathbb{R}^{d_1 \times d_2}[\zeta, \eta]$. But then by (iv) we have

$$
0 = \partial\Phi(\xi) = r(-\xi, \xi)
$$

so all coefficients in r must be zero, and so $\Phi = \overset{\bullet}{\Psi}$.

(iii) $\implies$ (i): If $\Phi = \overset{\bullet}{\Psi}$, then

$$
\int_{\mathbb{R}} L_\Phi(v, w)dt = \int_{\mathbb{R}} \frac{d}{dt} L_\Psi(v, w)dt = [L_\Psi(v, w)]_{-\infty}^{\infty} = 0
$$

since v and w have compact support.

(iii) $\implies$ (ii): If $\Phi = \overset{\bullet}{\Psi}$, then

$$
\int_{t_1}^{t_2} L_\Phi(v, w)dt = \int_{t_1}^{t_2} \frac{d}{dt} L_\Psi(v, w)dt = L_\Psi(v, w)(t_2) - L_\Psi(v, w)(t_1)
$$

so the integral depends only on the values of v, w and their derivatives at the points t_1, t_2. Hence $\int_{t_1}^{t_2} L_\Phi$ is independent of path.

(ii) $\implies$ (i): For any $v \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{d_1}), w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{d_2})$, if we take t_1 small enough and t_2 big enough, t_1 and t_2 will lie outside the support of v and w. By independence of path $\int_{t_1}^{t_2} L_\Phi$ depends only on the function values and derivatives of v, w at t_1, t_2 which now all vanish. Hence

$$\int_{t_1}^{t_2} L_\Phi(v, w)dt = \int_{t_1}^{t_2} L_\Phi(0, 0)dt = 0.$$

Letting $t_1 \to -\infty, t_2 \to \infty$ the statement follows.

Proving the statements using Q_Φ when $d_1 = d_2$ is analogous. The most significant difference is in the proof that (i) $\implies$ (iv). The assumption that (iv) does not hold now gives that there is $a \in \mathbb{C}^d, \omega \in \mathbb{R}$ such that $\overline{a}\Phi(-i\omega, i\omega)a \neq 0$, and we only need to construct v_N, w_N is not needed. The proof then continues as above. $\square$

We are particularly interested in studying QDFs along behaviours in $\mathfrak{L}^d$, that is restricted to only $w \in \mathcal{B}$ for some $\mathcal{B} \in \mathfrak{L}^d$. To do so it is useful to define equivalence relations on polynomial matrices by the following. Let $\mathcal{B} \in \mathfrak{L}^d$, $D_1, D_2 \in \mathbb{R}^{e \times d}[\xi]$ and $\Phi_1, \Phi_2 \in \mathbb{R}_s^{d \times d}[\zeta, \eta]$. We write $D_1 \overset{\mathcal{B}}{=} D_2$ if $D_1\left(\frac{d}{dt}\right) w = D_2\left(\frac{d}{dt}\right) w$ for every $w \in \mathcal{B}$ and $\Phi_1 \overset{\mathcal{B}}{=} \Phi_2$ if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for every $w \in \mathcal{B}$. We will refer to these equivalences as $\mathcal{B}$-equivalence. They can be related to kernel representations of $\mathcal{B}$ by the following result from [14].

Proposition 3.2. *Let $\mathcal{B} \in \mathfrak{L}^d$. If $R \in \mathbb{R}^{e \times d}[\xi]$ gives a kernel representation of $\mathcal{B}$ and $D_1, D_2 \in \mathbb{R}^{e \times d}[\xi]$ and $\Phi_1, \Phi_2 \in \mathbb{R}_s^{d \times d}[\zeta, \eta]$, then $D_1 \overset{\mathcal{B}}{=} D_2$ if and only if there is $F \in \mathbb{R}^{e \times e}[\xi]$ such that*

$$D_2 = D_1 + FR$$

and $\Phi_1 \overset{\mathcal{B}}{=} \Phi_2$ if and only if there is $F \in \mathbb{R}^{e \times d}[\zeta, \eta]$ such that

$$\Phi_2(\zeta, \eta) = \Phi_1(\zeta, \eta) + R^T(\zeta)F(\zeta, \eta) + F^*(\zeta, \eta)R(\eta).$$

Proof. We consider D_1, D_2 first. If $D_2 = D_1 + FR$ then we have, for any $w \in \mathcal{B}$

$$D_2\left(\frac{d}{dt}\right) w = D_1\left(\frac{d}{dt}\right) w + F\left(\frac{d}{dt}\right) R\left(\frac{d}{dt}\right) w = D_1\left(\frac{d}{dt}\right) w$$

so $D_1 \overset{\mathcal{B}}{=} D_2$. Now let us assume that $D_1 \overset{\mathcal{B}}{=} D_2$. Choose unimodular U, V such that $URV =: \Delta$ is diagonal, and let $\tilde{\mathcal{B}} = V^{-1}\left(\frac{d}{dt}\right) \mathcal{B}$. Setting $D = D_2 - D_1$, we want to show that there is F such that $D = FR$, but by multiplication from the right with V this is equivalent to existence of $F' := FU^{-1}$ such that $D' = F'\Delta$, where $D' := DV$. By assumption we have for any $w \in \tilde{\mathcal{B}}$ that $D'\left(\frac{d}{dt}\right) w = 0$. Let $\delta_1, \delta_2, \ldots$ denote the diagonal elements of Δ. For any $w_1 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $\delta_1\left(\frac{d}{dt}\right) w_1 = 0$, we have

$$\begin{pmatrix} w_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \tilde{\mathcal{B}}$$

and consequently

$$D'\left(\frac{d}{dt}\right)\begin{pmatrix} w_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = d_1'\left(\frac{d}{dt}\right) w_1 = 0$$

where d_1' denotes the first column of D'. That $\delta_1\left(\frac{d}{dt}\right) w_1 = 0$ implies $d_1'\left(\frac{d}{dt}\right) w_1 = 0$ means that $\delta_1(\xi)$ must divide all elements of $d_1'(\xi)$, in other words there is a polynomial vector f_1 such that $d_1' = f_1\delta_1$. Carriying out the same argument for each diagonal element of Δ gives polynomial vectors $f_1, f_2, \ldots$.

Now we must consider two cases. If $e \leq d$, then Δ has e diagonal elements so we get e vectors $f_1, \ldots, f_e$. The last $d-e$ columns of Δ are zero,so the last $d-e$ components of $w \in \tilde{\mathcal{B}}$ are free. Hence the last $d-e$ columns of D' must also be zero. Setting $F' = \begin{pmatrix} f_1 & \cdots & f_e \end{pmatrix}$ we have $D' = F'\Delta$, so the statement is proven in this case.

If instead $e > d$, then we get d vectors $f_1, \ldots, f_d$. Now the last $e-d$ rows of Δ are zero, and so we can pick $f_{d+1}, \ldots, f_e$ arbitrarily, and setting $F' = \begin{pmatrix} f_1 & \cdots & f_e \end{pmatrix}$ we again have $D' = F'\Delta$, completing the proof for D_1, D_2.

Now consider Φ_1, Φ_2. If we have

$$\Phi_2(\zeta,\eta) = \Phi_1(\zeta,\eta) + R^T(\zeta)F(\zeta,\eta) + F^*(\zeta,\eta)R(\eta)$$

then for any $w \in \mathcal{B}$

$$\begin{aligned} Q_{\Phi_2}(w) &= Q_{\Phi_1}(w) + L_F\left(R\left(\frac{d}{dt}\right)w, w\right) + L_{F^*}\left(w, R\left(\frac{d}{dt}\right)w\right) = \\ &= Q_{\Phi_1}(w) + L_F(0,w) + L_{F^*}(w,0) = Q_{\Phi_1}(w) \end{aligned}$$

so $\Phi_1 \overset{\mathcal{B}}{=} \Phi_2$. Next we suppose $\Phi_1 \overset{\mathcal{B}}{=} \Phi_2$ and take U, V, Δ and $\tilde{\mathcal{B}}$ as above. Note that it is enough to show that there is F such that

$$\Phi'(\zeta,\eta) = \Delta^T(\zeta)F(\zeta,\eta) + F^*(\zeta,\eta)\Delta(\eta)$$

where $\Phi'(\zeta,\eta) = V(\zeta)\Phi_1(\zeta,\eta)V(\eta) - V(\zeta)\Phi_2(\zeta,\eta)V(\eta)$. If we let $\varphi_{k,l}'(\zeta,\eta)$ be the element of $\Phi'(\zeta,\eta)$ on the k:th row, l:th column and $f_{k,l}$ similar for F we can formulate this as the existence of F such that

$$\varphi_{k,l}'(\zeta,\eta) = \delta_k(\zeta)f_{k,l}(\zeta,\eta) + f_{l,k}(\eta,\zeta)\delta_l(\eta)$$

for every k, l.

To show this we will first show the BLDF version of the statement in the case when Φ' and Δ are polynomials. To be precise we will prove that if $\Phi' \in \mathbb{R}[\zeta,\eta]$ and $\delta_1, \delta_2 \in \mathbb{R}[\xi]$ such that $L_{\Phi'}(w_1, w_2) = 0$ whenever $\delta\left(\frac{d}{dt}\right) w_1 = \delta_2\left(\frac{d}{dt}\right) w_2 = 0$, then there are $f_1, f_2 \in \mathbb{R}[\zeta,\eta]$ such that

$$\Phi'(\zeta,\eta) = \delta_1(\zeta)f_1(\zeta,\eta) + f_2(\zeta,\eta)\delta_2(\eta).$$

To find such f_1, f_2 perform division with remainder on $\Phi'(\zeta,\eta)$, first dividing by $\delta_1(\zeta)$ and then $\delta_2(\eta)$. Hence we have

$$\Phi'(\zeta,\eta) = \delta_1(\zeta)f_1(\zeta,\eta) + f_2(\zeta,\eta)\delta_2(\eta) + \Psi(\eta,\eta)$$

where $\Psi(\zeta,\eta)$ has degree smaller than δ_1, δ_2 in ζ and η respectively. Take some $w_1 \in \ker \delta_1\left(\frac{d}{dt}\right), w_2 \in \ker \delta_2\left(\frac{d}{dt}\right)$ and note that

$$\begin{aligned} &L_\Psi(w_1,w_2) = \\ &= L_{\Phi'}(w_1,w_2) - L_{f_1}\left(\delta_1\left(\frac{d}{dt}\right)w_1, w_2\right) - L_{f_2}\left(w_1, \delta_2\left(\frac{d}{dt}\right)w_2\right) = 0. \end{aligned}$$

In particular $L_\Psi(w_1,w_2)(0) = 0$. If we let $n_1 = \deg \delta_1, n_2 = \deg \delta_2$ then $L_\Psi(w_1,w_2)(0)$ is a bilinear form in the the n_1 first derivatives of w_1, n_2 first of w_2 at $t = 0$. These can be chosen arbitrarily, so Ψ must be zero. This proves the 1D case for BLDFs. Setting $f(\zeta,\eta) = \frac{f_1(\zeta,\eta)+f_2(\eta,\zeta)}{2}$, we have

$$\Phi'(\zeta,\eta) = \delta_1(\zeta)f(\zeta,\eta) + f(\eta,\zeta)\delta_2(\eta).$$

which proves the 1D QDF case (that is, when $\delta_1 = \delta_2$).

For the general case, take w_k such that $\delta_k\left(\frac{d}{dt}\right)w_k = 0$ and let w be w_k on the k:th row and zero on the other rows. Then

$$Q_{\Phi'}(w) = Q_{\varphi'_{k,k}}(w_k) = 0.$$

By the 1D case there is hence f_{kk} such that

$$\varphi'_{k,k}(\zeta,\eta) = \delta_k(\zeta)f_{k,k}(\zeta,\eta) + f_{k,k}(\eta,\zeta)\delta_k(\eta).$$

Now take also w_l such that $\delta_l\left(\frac{d}{dt}\right)w_l = 0$ and let w be w_k on the k:th row, w_l on the l:th row and zero on the other rows. Then

$$\begin{aligned} 0 = Q_{\Phi'}(w) &= Q_{\varphi'_{k,k}}(w_k) + Q_{\varphi'_{l,l}}(w_l) + L_{\varphi'_{k,l}}(w_k,w_l) + L_{\varphi'_{l,k}}(w_l,w_k) = \\ &= 2L_{\varphi'_{k,l}}(w_k,w_l). \end{aligned}$$

By the 1D case there is then $f_{k,l}$ such that

$$\varphi'_{k,l}(\zeta,\eta) = \delta_k(\zeta)f_{k,l}(\zeta,\eta) + f_{l,k}(\eta,\zeta)\delta_l(\eta).$$

Since we can do this for every k and l, the proof is complete. $\square$

In practical terms, equivalence along a behaviour is more easily used by using the condition $R\left(\frac{d}{dt}\right)w = 0$ for substitutions. For example we can look at the system given by $w_1 + \dot{w}_2 = w_2$ and the QDF $w_1^2 - \dot{w}_2^2$. Clearly we then have

$$w_1^2 - \dot{w}_2^2 = (w_1 + \dot{w}_2)(w_1 - \dot{w}_2) \overset{\mathcal{B}}{=} w_2(w_1 - \dot{w}_2) = w_1 w_2 - w_2\dot{w}_2.$$

Working this way we do not really need to find the matrix F in the proposition.

We also define positivity of QDFs. These definitions go very much hand in hand with definiteness of quadratic forms on $\mathbb{R}^d$. Let $\Phi \in \mathbb{R}^{d\times d}[\zeta,\eta]$. We say that Q_Φ is nonnegative, denoted $\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for every $w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^d)$. We say that Q_Φ is positive, denoted $\Phi > 0$, if $Q_\Phi \geq 0$ and $Q_\Phi(w) = 0$ only if $w = 0$. Nonpositive and negative QDFs are defined analogously. This naturally leads to a partial order of QDFs, we write $\Phi_1 \geq \Phi_2$ if $\Phi_1 - \Phi_2 \geq 0$. Positivity is also connected to factorization of QDFs much like factorizations of forms on $\mathbb{R}^d$. This is essentially done by factorization of $\widetilde{\Phi}$.

Proposition 3.3. *Let $\Phi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$. Then $\Phi \geq 0$ if and only if $\Phi(\zeta,\eta) = D^T(\zeta)D(\eta)$ for some polynomial matrix $D \in \mathbb{R}^{e\times d}[\xi]$. Also $\Phi > 0$ if and only if $\Phi(\zeta,\eta) = D^T(\zeta)D(\eta)$ where D additionally satisfies that $D(\lambda)$ has full rank for every $\lambda \in \mathbb{C}$.*

Proof. Recall the factorization $\Phi(\zeta,\eta) = E_d^T(\zeta)\widetilde{\Phi}E_d(\eta)$. If we let $W(t) = E_d\left(\frac{d}{dt}\right) w$, then we can write

$$Q_\Phi(w)(t) = W^T(t)\widetilde{\Phi}W(t).$$

Since $W(t)$ is an arbitrary (infinite) vector, $Q_\Phi(w)(t) \geq 0$ if and only if $\widetilde{\Phi}$ is positive semidefinite. That $\widetilde{\Phi}$ is positive semidefinite is equivalent to the existence of some matrix $\tilde{D}$ such that $\widetilde{\Phi} = \tilde{D}^T\tilde{D}$. This matrix $\tilde{D}$ can be taken with a finite number e rows and infinitely many columns, but with only a finite number of columns that are non-zero. Now take $D(\xi) = \tilde{D}E_d(\xi)$ which then satisfies $\Phi(\zeta,\eta) = D^T(\zeta)D(\eta)$

The requirement $\Phi > 0$ means that the D constructed above satisfies $D\left(\frac{d}{dt}\right) w \neq 0$ for $w \neq 0$. But this is equivalent to saying that the system

$$v = D\left(\frac{d}{dt}\right) w$$

is observable (w is observable from v) which, by Theorem 2.29 happens if and only if $D(\lambda)$ has full rank for every $\lambda \in \mathbb{C}$. □

Of particular interest to us is nonnegativity for w belonging to some behaviour $\mathcal{B}$. We say that Q_Φ is nonnegative along $\mathcal{B}$, denoted $\Phi \overset{\mathcal{B}}{\geq} 0$, if $\Phi(w) \geq 0$ for every $w \in \mathcal{B}$. We say that Q_Φ is positive along $\mathcal{B}$, denoted $\Phi \overset{\mathcal{B}}{>} 0$, if $\Phi \overset{\mathcal{B}}{\geq} 0$ and if $w \in \mathcal{B}$ satisfies $Q_\Phi(w) = 0$, then $w = 0$. Definitions for negative/nonpositive are analogous.

A QDF that is nonnegative along a behaviour is not necessarily nonnegative for all smooth functions. Take for example the QDF $Q_\Phi(w) = w_1^2 - w_2^2$ along the behaviour $\mathcal{B} \in \mathfrak{L}^2$ given by the equation $w_2 = 0$. Along $\mathcal{B}$ we have $Q_\Phi(w) = w_1^2 \geq 0$, but clearly Q_Φ is not nonnegative for arbitrary signals. Using $\mathcal{B}$-equivalence positivity along a behaviour can be brought back to the more general case, and so we can make use of factorizations in this case as well.

Proposition 3.4. *Let $\Phi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$, $R \in \mathbb{R}^{e\times d}[\xi]$ and $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$. Then*

(i) $\Phi \overset{\mathcal{B}}{\geq} 0$ *if and only if there is* $\Phi' \in \mathbb{R}_{\mathrm{s}}^{d\times d}[\zeta,\eta]$ *such that* $\Phi \overset{\mathcal{B}}{=} \Phi'$ *and* $\Phi' \geq 0$.

(ii) $\Phi \overset{\mathcal{B}}{>} 0$ *if and only if there is* $\Phi' \in \mathbb{R}_{\mathrm{s}}^{d\times d}[\zeta,\eta]$ *such that* $\Phi \overset{\mathcal{B}}{=} \Phi'$ *and* $\Phi' = D^T(\zeta)D(\eta)$ *for some* $D \in \mathbb{R}^{c\times d}[\xi]$ *such that* (R, D) *is an observable pair.*

The following proof builds on a proof in [14] where only the case when R is a polynomial is treated.

Proof. We show first the "if" part of both statements. If $\Phi \overset{\mathcal{B}}{=} \Phi'$ and $\Phi' \geq 0$ then for any $w \in \mathcal{B}$ we have

$$Q_\Phi(w) = Q_{\Phi'}(w) \geq 0$$

so $\Phi \overset{\mathcal{B}}{\geq} 0$. By Proposition 3.3 we have $\Phi'(\zeta,\eta) = D^T(\zeta)D(\eta)$. If we now further assume that (R, D) is an observable pair, then for any nonzero w such that $R\left(\frac{d}{dt}\right)w = 0$ (i.e. $w \in \mathcal{B}$) we must have $D\left(\frac{d}{dt}\right)w \neq 0$. Hence

$$Q_\Phi(w) = Q_{\Phi'}(w) = \left|D\left(\frac{d}{dt}\right)w\right|^2 \neq 0$$

for nonzero $w \in \mathcal{B}$, and the if part is proven.

Let us now look at the "only if" part, beginning with (i). Take unimodular matrices U, V such that $URV =: \Delta$ is diagonal and let $\Psi(\zeta,\eta) = V(\zeta)\Phi(\zeta,\eta)V(\eta)$ and $\tilde{\mathcal{B}} = V^{-1}\left(\frac{d}{dt}\right)\mathcal{B}$. It is then enough to show that there is some Ψ' such that $\Psi \overset{\tilde{\mathcal{B}}}{=} \Psi'$ and $\Psi' \overset{\tilde{\mathcal{B}}}{\geq} 0$. Denote the nonzero diagonal elements of Δ by $\delta_1, \ldots, \delta_m$ which we can assume are in the first m rows of Δ. Let us for the moment assume that $m < d$. Then the last $d - m$ components of $w \in \tilde{\mathcal{B}}$ are free. Decompose Ψ as

$$\Psi(\zeta,\eta) = \begin{pmatrix} \Psi_1(\zeta,\eta) & \Psi_2(\zeta,\eta) \\ \Psi_2^*(\zeta,\eta) & \Psi_3(\zeta,\eta) \end{pmatrix}$$

where Φ_1 is $m \times m$, Ψ_2 is $m \times (d-m)$ and Ψ_3 is $(d-m)\times(d-m)$. Now take

$$w = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{w} \end{pmatrix} \in \tilde{\mathcal{B}}$$

so that we have m zero rows and $\tilde{w}$ free. Then

$$0 \leq Q_\Phi(w) = Q_{\Phi_3}(\tilde{w})$$

and since $\tilde{w}$ is free this means $\Psi_3 \geq 0$. Next we will perform division with remainder on the first m rows and first m columns of Ψ. To be precise we divide the k:th row by $\delta_k(\zeta)$, for each $k = 1, \ldots, m$ and the l:th column by $\delta_l(\eta)$. This gives us $\Psi' \in \mathbb{R}^{d\times d}[\zeta, \eta]$ such that

$$\Psi(\zeta, \eta) = \Delta(\zeta)F_1(\zeta, \eta) + F_2(\zeta, \eta)\Delta(\eta) + \Psi'(\zeta, \eta).$$

The matrix Ψ' holds the remainder terms, so in the k:th row the degree of any element in ζ is less than the degree of δ_k and similar for the columns (in the first m rows and columns). The bottom right corner of Ψ' is unchanged, so equal to Ψ_3. Also, due to the symmetry of Ψ, division of the kth row by $\delta_k(\zeta)$ uses the same quotients and remainders as divsion of the k:th column by $\delta_k(\eta)$. It follows that $F_1 = F_2^*$ and so by Proposition 3.3 $\Psi \overset{\tilde{\mathcal{B}}}{=} \Psi'$.

Similar to before we decompose Ψ' as

$$\Psi' = \underbrace{\begin{pmatrix} \Psi_1' & \Psi_2' \\ (\Psi_2')^* & 0 \end{pmatrix}}_{\Psi_4} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \Psi_3 \end{pmatrix}}_{\Psi_5}.$$

Note that $\Psi_5 \geq 0$ because $\Psi_3 \geq 0$. Consider

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} \in \tilde{\mathcal{B}}.$$

Note that $Q_{\Psi_4}(w)(0)$ is a quadratic expression in $w_k(0), \frac{dw_k}{dt}(0), \ldots, \frac{d^{n_k-1}w}{dt^{n_k-1}}(0)$ where n_k is the degree of δ_k, for $k = 1, \ldots, d$. These initial conditions can be taken arbitrary for $w \in \tilde{\mathcal{B}}$, and hence the coefficient matrix $\widetilde{\Psi}_4$ must be positive definite. Hence $\Psi_4 \geq 0$, and so we can conclude that $\Psi' \geq 0$.

In the case $m = d$ we can simply forego the decomposition of Ψ and apply the division with remainder directly. The proof of part (i) is complete.

For the "only if" part of (ii), note that if $\Phi \overset{\mathcal{B}}{>} 0$, then by part (i) there is Φ' such that $\Phi'(\zeta, \eta) = D^T(\zeta)D(\eta)$ and $\Phi \overset{\mathcal{B}}{=} \Phi'$. If (R, D) is not an observable pair, then there is nonzero $w \in \mathcal{B}$ such that $D\left(\frac{d}{dt}\right) w = 0$. Then we have

$$Q_\Phi(w) = Q_{\Phi'}(w) = \left| D\left(\frac{d}{dt}\right) w \right|^2 = 0$$

which contradicts $\Phi \overset{\mathcal{B}}{>} 0$. Hence (R, D) must be an observable pair, proving part (ii). □

A key part of the proof above was to find a $\mathcal{B}$-equivalent representative of the QDF (Ψ' in this case) whose polynomial elements had degree lower than polynomials in the matrix giving the kernel representation. For autonomous behaviours this idea can be made more precise.

Consider an autonomous behaviour $\mathcal{B} \in \mathfrak{L}^d$ and $R \in \mathbb{R}^{d\times d}[\xi]$ such that $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$. A polynomial matrix $D \in \mathbb{R}^{e\times d}[\xi]$ is called R-canonical if $DR^{-1}(\xi)$

is a matrix of strictly proper rational functions (a rational function is strictly proper if the denominator has strictly higher degree than the numerator). Similarly a matrix $\Phi \in \mathbb{R}^{d\times d}[\zeta,\eta]$ is called R-canonical if $(R^T(\zeta))^{-1}\Phi(\zeta,\eta)(R(\eta))^{-1}$ is a matrix of strictly proper rational functions.

It can be proven (see [14]) that each equivalence class under $\mathcal{B}$-equivalence has a unique element that is R-canonical. This holds for polynomial matrices in both one or two indeterminates. This gives a quite natural and convenient representative for each equivalence class. Using the fact that the degrees of their elements are small, we can use arguments about for example arbitrariness of initial conditions, similar to above.

4 Stability

In stability analysis we are interested in how small perturbations of initial values can effect solutions to differential equations. Intuitively a system is stable if small changes in initial values only lead to small changes in the overall trajectory. For linear systems this is fairly simple, stability is essentially determined by how a solution $w(t)$ behaves as t goes to infinity. Since we are only interested in what happens for very large t, it suffices to look at t belonging to any interval that is not bounded above. By time invarience we may take the starting time as 0. Hence we will in this section focus on $w(t)$ for $t \in [0, \infty)$.

For linear behaviours we introduce three forms of stability.

Definition 4.1. Let $\mathcal{B} \in \mathfrak{L}^d$. Then $\mathcal{B}$ is (Lyapunov) stable if every $w \in \mathcal{B}$ is bounded on $[0, \infty)$, otherwise we say that $\mathcal{B}$ is unstable. $\mathcal{B}$ is semistable if for every $w \in \mathcal{B}$, $\lim_{t\to\infty} w(t)$ exists. Finally, $\mathcal{B}$ is asymptotically stable if for every $w \in \mathcal{B}$, $\lim_{t\to\infty} w(t) = 0$.

For nonlinear systems we usually speak of stability around equilibrium points, and the definition of Lyapunov stability is then more involved. For the linear systems we are studying the simpler definition above is in fact equivalent. Throughout this text we will refer to Lyapunov stability simply as stability.

Note that asymptotic stability implies semistability which in turn implies stability. Using Theorem 2.18, we can conclude that a necessary requirement for stability is that the behaviour is autonomous. If the system is not autonomous, then some component of the external signal could be chosen freely as input, and hence not bounded in general.

Consider an autonomous behaviour $\mathcal{B}$ with kernel representation given by some $R \in \mathbb{R}^{d\times d}[\xi]$, which hence must have full rank. We have seen in Proposition 2.13 that the solutions are specified by the roots of $\det R(\xi)$. Stability of $\mathcal{B}$ can also be determined from these roots. This is a classical result in stability theory.

Before stating the proposition, we make some definitions. For a square polynomial matrix R we define the roots of R as the roots of $\det R(\xi)$. If $\lambda \in \mathbb{C}$ is a root of R, then the algebraic multiplicity of λ is its multiplicity as a root of $\det R(\xi)$ and its geometric multiplicity is the dimension of $\ker R(\lambda)$. Note that the algebraic multiplicity of a root is always greater or equal than the geometric multiplicity. A root λ is said to be semisimple if the algebraic and geometric multiplicities are equal.

We can give an alternative characterization of what it means that a root λ of R is semisimple. It follows from Remark 2.16 that λ is semisimple if and only if the only terms involving λ in any solution to $R\left(\frac{d}{dt}\right) w = 0$ are of the form $ae^{\lambda t}$, that is, there are no terms of the form $at^j e^{\lambda t}$ with $j \geq 1$.

Proposition 4.2. *Let $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$ for some $R \in \mathbb{R}^{d\times d}[\xi]$. Then*

(i) *$\mathcal{B}$ is stable if and only if every root of R has nonpositive real part, and every root with real part zero is semisimple.*

(ii) *$\mathcal{B}$ is semistable if and only if $\mathcal{B}$ is stable and R has no non-zero purely imaginary root.*

(iii) *$\mathcal{B}$ is asymptotically stable if and only if every root of R has strictly negative real part.*

Proof. First we show that the criteria are necessary for each type of stability. Note that if λ is a root of R, then there is $a \in \mathbb{C}^d$ such that $R(\lambda)a = 0$. If $R(\xi) = \sum_{l=0}^{L} R_l \xi^l$ where $R_l \in \mathbb{R}^{d\times d}$, then we have

$$R\left(\frac{d}{dt}\right) ae^{\lambda t} = \sum_{l=0}^{L} R_l \frac{d^l}{dt^l} ae^{\lambda t} = \sum_{l=0}^{L} R_l \lambda^l ae^{\lambda t} = R(\lambda) ae^{\lambda t} = 0,$$

so $ae^{\lambda t} \in \mathcal{B}$.

(i) If λ is a root of R with positive real part, then $ae^{\lambda t}$ is not bounded so $\mathcal{B}$ cannot be stable in this case. Next suppose λ is a root of R with real part zero that is not semisimple. This means that there is some solution of the form $r(t)e^{\lambda t}$ where r is a non-constant polynomial. This solution is not bounded, hence $\mathcal{B}$ can not be stable in this case.

(ii) We have already remarked that a semistable behaviour is stable. If λ is a non-zero imaginary root, then there is $ae^{\lambda t} \in \mathcal{B}$ but this function has no limit as $t \to \infty$, and hence the system is not semistable.

(iii) We have already remarked that an asymptotically stable behaviour is semistable. Hence, by part (ii) we only need to consider the case of a root that is zero. If zero is a root then there is a constant solution $ae^{0\cdot t} = a \in \mathcal{B}$ that does not converge to zero, hence the behaviour is not asymptotically stable.

Next we show that the criteria are sufficient for each type of stability. By Proposition 2.13 all solutions are linear combinations of expressions of the form $t^j e^{\lambda t}$ where λ is a root.

(i) If λ is a root with negative real part, then $t^j e^{\lambda t}$ is bounded for any $j \geq 0$. If λ is a root of R with real part zero that is semisimple, then all terms involving that root are on the form $ae^{\lambda t}$ for some $a \in \mathbb{C}^d$ and hence bounded. Any $w \in \mathcal{B}$ is hence a linear combination of bounded functions, and hence itself bounded and so $\mathcal{B}$ is stable.

(ii) If λ is a root with negative real part, then $t^j e^{\lambda t}$ converges to zero for any j. If $\lambda = 0$ is a root, then since $\mathcal{B}$ is stable it is a semisimple root. All terms involving that root are thus constant solutions, which of course converge as $t \to \infty$. A linear combination of functions that converge also converges, so $\mathcal{B}$ is semisimple.

(iii) If λ is a root with negative real part, then $t^j e^{\lambda t}$ converges to zero for any $j \geq 0$. Since a general solution is a linear combination of such expressions, it too will converge to zero, so $\mathcal{B}$ is asymptotically zero. □

Example 4.3. In this example we will look at systems given by equations on the form

$$M\frac{d^2q}{dt^2} + D\frac{dq}{dt} + Kq = 0.$$

Such equations are often used to model viscoelastic mechanical systems. A commonly used constitutional model is a set of masses that are connected with springs and dampers. The viscous behaviour is modelled by the dampers and the elastic behaviour is modelled by the springs.

Here $q(t) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ describes the deviation in position of the masses relative to some state of equilibrium. The matrices $M, D, K \in \mathbb{R}^{d\times d}$ describe the masses, damping and stiffness of the springs respectively. From physical considerations we can usually assume that $M = M^T \geq 0, K = K^T \geq 0$ and $D + D^T \geq 0$.

We will now look at the case when q is scalar valued, we will return to the more general case later. We can write the equation above as $R\big(\frac{d}{dt}\big)\, q = 0$ where $R(\xi) = M\xi^2 + D\xi + K$. The roots of R are in this case

$$\frac{-D \pm \sqrt{D^2 - 4MK}}{2M}.$$

Here we will assume that $M, K > 0$ and $D \geq 0$. If $D^2 > 4MK$, then we get two distinct negative roots and by Proposition 4.2 the system is asymptotically stable. If $D^2 = 4MK$ then there is just one negative root (of multiplicity two) and again the system is asymptotically stable.

If $D^2 < 4MK$ then we have two complex roots, both with real part $-\frac{D}{2M}$. If we assume $D > 0$, then we once again get asymptotic stability, but if $D = 0$ then we have two imaginary roots. Since they have multiplicity one they are semisimple roots, and so the system is stable, it is oscillating.

Intuitively this makes sense. So long as there is some nonzero damping, the system will tend to some equilibrium state no matter what initial state we put it in. Without any damping the spring system will simply oscillate indefinitely.

The result above characterizes these stability properties for linear systems using their roots. A common alternative approach to studying stability is through the use of Lyapunov functions. Consider a dynamical system $(\mathbb{R}, \mathbb{R}^d, \mathcal{B})$. A Lyapunov function is a differentiable function $V : \mathbb{R}^d \to \mathbb{R}$ such that $V(0) = 0, V(w_0) > 0$ for any nonzero $w_0 \in \mathbb{R}^d$ and along any $w \in \mathcal{B}$ we have

$$\frac{d}{dt}V(w(t)) < 0.$$

In applications the Lyapunov function typically describes some form of energy. The idea is that V evaluated along a trajectory $w \in \mathcal{B}$ is a strictly decreasing function that is always nonnegative. Since $V(w_0)$ is only zero if $w_0 = 0$, if we show that $V(w)$ converges to zero, then this implies that $w(t)$ must converge to zero as t goes to infinity. Hence we can use Lyapunov functions to show that a behaviour has certain stability properties. Intuitively, the system is losing energy, and so it tends to a state of minimal energy.

For linear systems it is suitable to take Lyapunov functions as quadratic functions. For a first order system we use a quadratic form in the external variable, for higher order systems we will find that QDFs fill this role nicely. We demonstrate the idea with a first order example.

Example 4.4. Let $\mathcal{B}$ be the solutions set of

$$\dot{w} = Aw$$

where $A \in \mathbb{R}^{d\times d}$. We will look for a Lyapunov function of the form $V(w) = w^T P w$ where $P = P^T \in \mathbb{R}^{d\times d}$ to study stability. We of course want $P > 0$ and it should also satisfy that

$$\frac{d}{dt}(w^T P w) = \dot{w}^T P w + w^T P \dot{w} = w^T (A^T P + PA) w < 0.$$

To find such P we study the matrix equation

$$A^T P + PA = Q, \tag{12}$$

which is called the Lyapunov equation. Here $Q = Q^T \in \mathbb{R}^{d\times d}$ is some given matrix, P is the sought unknown. The idea is that we choose some $Q < 0$ and try to find a symmetric $P > 0$ that satisfies the equation. If we can do so then $V(w) = w^T P w$ is a Lyapunov function which proves that $\mathcal{B}$ is asymptotically stable. On the other hand it can be shown that if $\mathcal{B}$ is asymptotically stable, then there is such a solution P for every $Q < 0$.

Depending on what type of stability we want to study it is possible to ease up the strict inequalities $P > 0$ and $Q < 0$ if we involve some observability conditions. We will not go into detail about first order systems. Instead we will now look at higher order systems, beginning with asymptotic stability.

Theorem 4.5. *Let $\mathcal{B} \in \mathfrak{L}^d$. Then $\mathcal{B}$ is asymptotically stable if and only if there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta, \eta]$ such that $\Psi \overset{\mathcal{B}}{\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$.*

For the only if part we will actually prove a slightly stronger statement, namely that if $\mathcal{B}$ is asymptotically stable, then there is for every $\Phi \in \mathbb{R}_s^{d\times d}[\zeta, \eta]$ such that $\Phi \overset{\mathcal{B}}{<} 0$, a $\Psi \in \mathbb{R}_s^{d\times d}[\zeta, \eta]$ such that $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{=} \Phi$ and $\Psi \overset{\mathcal{B}}{\geq} 0$. This Φ plays a similar role to Q in equation (12). Indeed, for the only if part we will use a polynomial Lyapunov equation to find Ψ.

Lemma 4.6. *Let $\mathcal{B} \in \mathfrak{L}^d$ be asymptotically stable, with kernel representation given by $R \in \mathbb{R}^{d\times d}[\xi]$. Then for every $\Phi \in \mathbb{R}_s^{d\times d}[\zeta, \eta]$ there is a solution $X \in \mathbb{R}^{d\times d}[\xi]$ to the equation*

$$X^T(-\xi)R(\xi) + R^T(-\xi)X(\xi) = \partial\Phi(\xi). \tag{13}$$

We refer to equation (13) as the polynomial Lyapunov equation. The proofs of Lemma 4.6 and Theorem 4.5 are adapted from [14], where similar but slightly stronger claims are proven.

Proof of Lemma 4.6. We will first prove the case when R is diagonal. Write the k:th diagonal element of $R(\xi)$ as $r_k(\xi)$, $\varphi_{k,l}(\zeta,\eta)$ for the component of Φ on the k:th row, l:th column and similarly $x_{k,l}(\xi)$ for $X(\xi)$. Then we want to show that we can find $x_{k,l}$ that solve

$$x_{l,k}(-\xi)r_k(\xi) + r_l(-\xi)x_{k,l}(\xi) = \varphi_{k,l}(-\xi,\xi) \tag{14}$$

for each $k = 1,\dots,d$ and $l = 1,\dots,d$. Note that due to the symmetry of Φ, it is enough to show that this is possible for $k \leq l$. The equations for $k > l$ are then automatically satisfied.

Consider first the case $k < l$. Since $\mathcal{B}$ is asymptotically stable, the roots of $r_k(\xi)$ and $r_l(\xi)$ are all in the left open half plane of $\mathbb{C}$. Hence $r_k(\xi)$ and $r_l(-\xi)$ have no common roots. By the Bezout identity there are $x_{k,l}, x_{l,k} \in \mathbb{R}[\xi]$ that satisfy equation (14).

Now we look at the case $k = l$. Then equation (14) reads

$$x_{k,k}(-\xi)r_k(\xi) + r_k(-\xi)x_{k,k}(\xi) = \varphi_{k,k}(-\xi,\xi).$$

It once again follows from the asymptotic stability of $\mathcal{B}$ that $r_k(\xi)$ and $r_k(-\xi)$ have no common roots. Hence there are by the Bezout identity $a, b \in \mathbb{R}[\xi]$ such that

$$a(\xi)r_k(\xi) + r_k(-\xi)b(\xi) = \varphi_{k,k}(-\xi,\xi).$$

Now set $x_{k,k}(\xi) = \frac{a(-\xi)+b(\xi)}{2}$. Then we have

$$\begin{aligned}
&x_{k,k}(-\xi)r_k(\xi) + r_k(-\xi)x_{k,k}(\xi) = \\
&= \frac{1}{2}\Big(a(\xi)r_k(\xi) + b(-\xi)r_k(\xi) + r_k(-\xi)a(-\xi) + r_k(-\xi)b(\xi)\Big) = \\
&= \frac{1}{2}\Big(\varphi_{k,k}(-\xi,\xi) + \varphi_{k,k}(\xi,-\xi)\Big) = \varphi_{k,k}(-\xi,\xi)
\end{aligned}$$

where the last step follows from the assumption that Φ is symmetric. This proves the statement when R is diagonal.

If R is not diagonal then the problem can be reduced to the diagonal case. To do so find unimodular U, V such that $URV =: \Delta$ is diagonal and let $\Phi'(\zeta,\eta) = V^T(\zeta)\Phi(\zeta,\eta)V(\eta)$. By the case proven above there is $Y \in \mathbb{R}^{d\times d}[\xi]$ such that

$$Y^T(-\xi)\Delta(\xi) + \Delta^T(-\xi)Y(\xi) = \partial\Phi'(\xi).$$

Now set $X(\xi) = U^T(-\xi)Y(\xi)V^{-1}(\xi)$. We then have

$$\begin{aligned}
&X^T(-\xi)R(\xi) + R^T(-\xi)X(\xi) = \\
&= V^{-T}(-\xi)Y^T(-\xi)U(\xi)R(\xi) + R^T(-\xi)U^T(-\xi)Y(\xi)V^{-1}(\xi) = \\
&= V^{-T}(-\xi)\left(Y^T(-\xi)\Delta(\xi) + \Delta^T(-\xi)Y(\xi)\right)V^{-1}(\xi) = \\
&= V^{-T}(-\xi)\Phi'(-\xi,\xi)V^{-1}(\xi) = \Phi(-\xi,\xi) = \partial\Phi(\xi)
\end{aligned}$$

which proves the general case. $\square$

Proof of Theorem 4.5. Suppose first that there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Psi \overset{\mathcal{B}}{\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$. Suppose also that $\mathcal{B} = \ker R(\frac{d}{dt})$, $R \in \mathbb{R}^{e\times d}[\xi]$. That $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$ means that there is some $D \in \mathbb{R}^{c\times d}[\xi]$ such that $\overset{\bullet}{\Psi}(\zeta,\eta) = D^T(\zeta)D(\eta)$ and (R, D) is an observable pair. We now have for any $w \in \mathcal{B}$ and $T \geq 0$

$$Q_\Psi(w)(T) - Q_\Psi(w)(0) = \int_0^T Q_{\overset{\bullet}{\Psi}}(w)(t)dt = -\int_0^T \left| D\left(\frac{d}{dt}\right) w(t)\right|^2 dt.$$

Since $\Psi \overset{\mathcal{B}}{\geq} 0$ we therefore have

$$\int_0^T \left| D\left(\frac{d}{dt}\right) w(t)\right|^2 dt = Q_\Psi(w)(0) - Q_\Psi(w)(T) \leq Q_\Psi(w)(0),$$

and so by letting $T \to \infty$ we have

$$\int_0^\infty \left| D\left(\frac{d}{dt}\right) w(t)\right|^2 dt < \infty$$

for every $w \in \mathcal{B}$. Now let λ be any root of R. Then there is some non-zero $a \in \mathbb{C}^d$ such that $ae^{\lambda t} \in \mathcal{B}$. We get in this case

$$\int_0^\infty \left| D\left(\frac{d}{dt}\right) ae^{\lambda t}\right|^2 dt = \int_0^\infty \left| D(\lambda)ae^{\lambda t}\right|^2 dt = |D(\lambda)a|^2 \int_0^\infty e^{2\Re(\lambda)t}dt. \quad (15)$$

Since $ae^{\lambda t} \in \mathcal{B}$ we have $R(\lambda)a = 0$ and so since (R, D) is an observable pair, $D(\lambda)a \neq 0$. For the integral in (15) to be finite we must have $\Re(\lambda) < 0$. Since this holds for every root of R, we have by Proposition 4.2 that $\mathcal{B}$ is asymptotically stable.

Next we assume that $\mathcal{B}$ is asymptotically stable. Take $\Phi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Phi \overset{\mathcal{B}}{<} 0$. By Lemma 4.6 there is $X \in \mathbb{R}^{d\times d}$ that solves the polynomial Lyapunov equation (13). Now consider the polynomial matrix

$$\Phi(\zeta,\eta) - X^T(\zeta)R(\eta) - R^T(\zeta)X(\eta).$$

By equation (13), applying the delta operator ∂ to this matrix we get the zero matrix, and hence by Proposition 3.1, there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that

$$\overset{\bullet}{\Psi}(\zeta,\eta) = \Phi(\zeta,\eta) - X^T(\zeta)R(\eta) - R^T(\zeta)X(\eta).$$

By Proposition 3.2 we then have $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{=} \Phi$, so $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$.

It remains to show that $\Psi \overset{\mathcal{B}}{\geq} 0$. For any $t_0, t_1 \in \mathbb{R}$ such that $t_0 < t_1$, we have since $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$, that

$$Q_\Psi(w)(t_1) - Q_\Psi(w)(t_0) = \int_{t_0}^{t_1} Q_{\overset{\bullet}{\Psi}}(w)(t)dt \leq 0$$

for any $w \in \mathcal{B}$. Hence $Q_\Psi(w)(t_1) \leq Q_\Psi(w)(t_0)$. Since $\mathcal{B}$ is asymptotically stable $w(t_1) \to 0$ as $t_1 \to \infty$, and so $Q_\Psi(w)(t_1) \to 0$ as well. Hence we have

$$Q_\Psi(w)(t_0) \geq 0$$

for any $t_0 \in \mathbb{R}$ and any $w \in \mathcal{B}$, so $\Psi \overset{\mathcal{B}}{\geq} 0$ and the proof is complete. □

Next we look at how QDFs can let us determine if a system is stable or unstable.

Theorem 4.7. *Let $\mathcal{B} \in \mathfrak{L}^d$.*

(i) *If there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Psi \overset{\mathcal{B}}{>} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{\leq} 0$, then $\mathcal{B}$ is stable.*

(ii) *If there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Psi \overset{\mathcal{B}}{\ngeq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$, then $\mathcal{B}$ is unstable.*

By $\Psi \overset{\mathcal{B}}{\ngeq} 0$ we mean that there is some $w \in \mathcal{B}$ and some $t \in \mathbb{R}$ such that $Q_\Psi(w)(t) < 0$.

Proof.

(i) Let $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$ for some $R \in \mathbb{R}^{e\times d}[\xi]$ and suppose there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Psi \overset{\mathcal{B}}{>} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{\leq} 0$. Let λ be a root of R. Then there is some nonzero $a \in \mathbb{C}^d$ such that $w = ae^{\lambda t} \in \mathcal{B}$. Since $\Psi \overset{\mathcal{B}}{>} 0$ there is E such that $\Psi(\zeta,\eta) \overset{\mathcal{B}}{=} E^T(\zeta)E(\eta)$ so that

$$Q_\Psi(w)(t) = \left| E\left(\frac{d}{dt}\right) ae^{\lambda t} \right|^2 = \left|E(\lambda)ae^{\lambda t}\right|^2 = e^{2\Re(\lambda)t} \left|E(\lambda)a\right|^2.$$

Since $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{\leq} 0$ we have for any $t \geq 0$

$$Q_\Psi(w)(t) - Q_\Psi(w)(0) = \int_0^t Q_{\overset{\bullet}{\Psi}}(w)(\tau)d\tau \leq 0$$

so $Q_\Psi(w)(t) \leq Q_\Psi(w)(0)$, meaning that $Q_\Psi(w)(t)$ is bounded as $t \to \infty$. We have

$$0 = R\left(\frac{d}{dt}\right) ae^{\lambda t} = R(\lambda)ae^{\lambda t},$$

so $R(\lambda)a = 0$. The assumption $\Psi \overset{\mathcal{B}}{>} 0$ implies that (R,E) is an observable pair, so we must then have $E(\lambda)a \neq 0$. For

$$Q_\Psi(w)(t) = e^{2\Re(\lambda)t} \left|E(\lambda)a\right|^2$$

to be bounded as $t \to \infty$, we must therefore have $\Re(\lambda) \leq 0$.

Now suppose $\lambda = i\omega$, $\omega \in \mathbb{R}$ and that

$$v = \sum_{l=0}^{L} a_l t^l e^{i\omega t} \in \mathcal{B}.$$

Then

$$E\left(\frac{d}{dt}\right) v = E(i\omega) a_L t^L e^{i\omega t} + p(t) e^{i\omega t}$$

where $p(t) \in \mathbb{C}^d[t]$ has polynomials of degree at most $L-1$. Then

$$\left| E\left(\frac{d}{dt}\right) v \right|^2 = t^{2L} |E(i\omega) a_L|^2 + |q(t)|^2$$

where $|q(t)|^2 \in \mathbb{R}[t]$ is of degree at most $2L-1$. For this to be bounded as $t \to \infty$ we must have $E(i\omega)a_L = 0$. But then $a_L e^{i\omega t} \in \mathcal{B}$ and

$$Q_\Psi(a_L e^{i\omega t}) = |E(i\omega) a_L|^2 = 0$$

which contradicts that $\Psi \overset{\mathcal{B}}{>} 0$. So if λ is a root on the imaginary axis, it must be semisimple. It then follows from Proposition 4.2 that $\mathcal{B}$ is stable.

(ii) If $\mathcal{B}$ is not autonomous, then $\mathcal{B}$ is unstable. For an autonomous $\mathcal{B}$ suppose we have $\Psi \in \mathbb{R}_s^{d\times d}[\zeta, \eta]$ such that $\Psi \overset{\mathcal{B}}{\ngeq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$. The assumption $\Psi \overset{\mathcal{B}}{\ngeq} 0$ means that there is some $w \in \mathcal{B}$ and $t_0 \in [0, \infty)$ such that $Q_\Psi(w)(t_0) < 0$. Suppose for a contradiction that $\mathcal{B}$ is stable. Then $Q_\Psi(w)(t_0) < 0$ is a sum of terms of the form

$$p_{k,l}(t) e^{(\lambda_k - \overline{\lambda}_l)t} \tag{16}$$

where $p_{k,l} \in \mathbb{C}^d[t]$ and λ_k, λ_l are roots of R. Furthermore λ_k, λ_l either have negative real part or have real part zero and is semisimple. If either $\Re(\lambda_k) < 0$ or $\Re(\lambda_l) < 0$ then the term (16) converges to zero as $t \to \infty$. If $\Re(\lambda_k) = \Re(\lambda_l) = 0$, then these are semisimpe roots, so $p_{k,l}$ is a constant vector. Hence the corresponding term (16) is periodic. We can therefore split $Q_\Psi(w)$ as

$$Q_\Psi(w)(t) = S_1(t) + S_2(t)$$

where $\lim_{t\to\infty} S_1(t) = 0$ and $S_2(t)$ is periodic, say with period T.

Since $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$ we have $Q_\Psi(w)(t) < Q_\Psi(w)(t_0)$ for all $t > t_0$. Note that if $S_1(t_0) \geq 0$, then we could pick a different $\tilde{w}$ with only the periodic terms so that

$$Q_\Psi(\tilde{w})(t_0) = S_2(t_0) < 0.$$

Then $Q_\Psi(\tilde{w})$ is periodic, contradicting that $Q_\Psi(\tilde{w})(t) < Q_\Psi(\tilde{w})(t_0)$ for $t > t_0$. Hence we must have $S_1(t_0) < 0$. However then there is $n \in \mathbb{N}$ such that

$$S_1(t_0 + Tn) > \frac{S_1(t_0)}{2}.$$

Then we have

$$
\begin{aligned}
&Q_\Psi(w)(t_0+Tn) - Q_\Psi(w)(t_0) = \\
&= S_1(t_0+Tn) + S_2(t_0+Tn) - S_1(t_0) - S_2(t_0) = \\
&= S_1(t_0+Tn) - S_1(t_0) > -\frac{S_1(t_0)}{2} > 0
\end{aligned}
$$

which is a contradiction. We can therefore conclude that $\mathcal{B}$ must be unstable. $\square$

In [14] it is claimed without proof that the converse of (i) also holds and if $\mathcal{B}$ is assumed to be autonomous, then the converse of (ii) also holds. It is indicated that one uses R-canonical representatives of the QDFs to do this. We can at least give an idea for how to prove a result similar to Lemma 4.6, though not quite as general. Consider the polynomial Lyapunov equation

$$X^T(-\xi)R(\xi) + R^T(-\xi)X(\xi) = \partial\Phi(\xi)$$

where we now assume that $\ker R\left(\frac{d}{dt}\right)$ is stable. The argument to diagonalize and reduce to the scalar case

$$x_{l,k}(-\xi)r_k(\xi) + r_l(-\xi)x_{k,l}(\xi) = \varphi_{k,l}(-\xi,\xi)$$

remains the same. The problem now is that since $\ker R\left(\frac{d}{dt}\right)$ is stable it is possible that $r_k(\xi)$ and $r_l(-\xi)$ have a common imaginary root. If this root is not present in the right hand side, then the equation can not be solved. The polynomial Lyapunov equation can therefore not be solved for arbitrary Φ, but to prove the converse of (i) in the theorem above we in fact only need this to be possible for some Φ. We can achieve this by replacing Φ with a matrix such that $\varphi_{k,l}(-\xi,\xi)$ has the desired roots. Let $\pm i\omega_1,\ldots,\pm i\omega_K$ be the imaginary roots of $\det R(\xi)$. If we replace $\Phi(\zeta,\eta)$ with

$$\left(\prod_{k=1}^{K}(\zeta^2+\omega_k^2)(\eta^2+\omega_k^2)\right)\Phi(\zeta,\eta)$$

in the Lyapunov equation, then stability is enough to conclude that a solution X exists.

By making some minor adjustments to the proofs of Theorems 4.5 and 4.7 we can get a similar condition for semistability.

Corollary 4.8. *Let $\mathcal{B} \in \mathfrak{L}^d$ be autonomous and suppose there is $\Psi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\Psi \overset{\mathcal{B}}{=} E^T(\zeta)E(\eta)$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{=} -D^T(\zeta)D(\eta)$ for some $E \in \mathbb{R}^{e_1\times d}[\xi], D \in \mathbb{R}^{e_2\times d}[\xi]$. If further $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$, where $R \in \mathbb{R}^{d\times d}[\xi]$,*

$$\operatorname{rk}\begin{pmatrix} R(\lambda) \\ D(\lambda) \end{pmatrix} = d \tag{17}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ *and*

$$\operatorname{rk}\begin{pmatrix} R(0) \\ E(0) \end{pmatrix} = d, \tag{18}$$

then $\mathcal{B}$ *is semistable.*

Proof. The proof is similar to those of Theorems 4.5 and 4.7 so we will only briefly cover the differences. In the proof of Theorem 4.5, we used $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{<} 0$ to show that if λ is a root of R, then $\Re(\lambda) < 0$. Since equation (17) need not hold for $\lambda = 0$, we instead conclude that if λ is a root of R then either $\Re(\lambda) < 0$ or $\lambda = 0$.

We can then show that if $\lambda = 0$ is a root of R then it is semisimple the same way we showed imaginary roots are semisimple in the proof of Theorem 4.7, using equation (18) instead of the assumption $\Psi \overset{\mathcal{B}}{>} 0$. It then follows from Proposition 4.2 that $\mathcal{B}$ is semisimple. $\square$

Example 4.9. We will now return to the spring-damper system we looked at in Example 4.3, given by the equation $R\left(\frac{d}{dt}\right) q = 0$ where $R(\xi) = M\xi^2 + D\xi + K$. We will use QDFs to study the case when q is vector valued. Recall that from the outset we had the assumption $M = M^T \geq 0, K = K^T \geq 0$ and $D + D^T \geq 0$. We will use a QDF induced by

$$\Psi(\zeta, \eta) = M\zeta\eta + K.$$

This QDF describes the total energy in the system, the first term gives the kinetic energy due to movement and the second gives the potential energy stored in the springs. Note that

$$\begin{aligned}\overset{\bullet}{\Psi}(\zeta, \eta) = M(\zeta^2\eta + \zeta\eta^2) + K(\zeta + \eta) = R^T(\zeta)\eta I + \zeta I R(\eta) - (D + D^T)\zeta\eta \overset{\mathcal{B}}{=} \\ \overset{\mathcal{B}}{=} -(D + D^T)\zeta\eta.\end{aligned}$$

Clearly we have $\Psi \geq 0$ and $\overset{\bullet}{\Psi} \overset{\mathcal{B}}{\leq} 0$, to conclude something about stability we need to sharpen at least one of these inequalities.

Note that if $\ker K$ is nontrivial, then there is a constant solution q_0 to the equation $R\left(\frac{d}{dt}\right) q = 0$. This means that there is more than one (and by scaling the vector, in fact infinitely many) equilibrium states. If we on the other hand assume that there is only one equilibrium point, the origin, then $K > 0$. This means that $\Psi > 0$ and so by Theorem 4.7 the system is stable.

If we want asymptotic stability, then we need to make sure that $(R(\xi), \sqrt{D + D^T}\xi)$ is an observable pair, equivalently that

$$\operatorname{rk}\begin{pmatrix} M\lambda^2 + D\lambda + K \\ \sqrt{D + D^T}\lambda \end{pmatrix} = d$$

for all $\lambda \in \mathbb{C}$. A condition for this to hold found in [14] is that $K > 0$ and $\ker(D + D^T) \subseteq \ker M$. To see this suppose we have $x \in \mathbb{C}^d$ such that

$$\begin{pmatrix} M\lambda^2 + D\lambda + K \\ \sqrt{D + D^T}\lambda \end{pmatrix} x = 0. \tag{19}$$

If $\lambda = 0$ then we have $Kx = 0$ and since $K > 0$ this implies that $x = 0$. For nonzero λ we must have $(D + D^T)x = 0$ and consequently $Mx = 0$. It follows that we must also have $D\lambda x + Kx = 0$. It follows that

$$0 = \overline{x}^T Dx + \frac{1}{\lambda}\overline{x}^T Kx = \frac{1}{2}\overline{x}^T(D + D^T)x + \frac{1}{\lambda}\overline{x}^T Kx = \frac{1}{\lambda}\overline{x}^T Kx$$

and so we must have $x = 0$. It follows that $(R(\xi), \sqrt{D + D^T}\xi)$ is an observable pair, and by Theorem 4.5 the system is asymptotically stable.

This condition can be slightly adjusted to get the semistable case instead. We once again assume $\ker(D + D^T) \subseteq \ker M$, but instead of assuming $K > 0$ we will use the weaker condition $\ker K \cap \ker(D + D^T) = \emptyset$. For nonzero λ, equation (19) once again leads us to $(D + D^T)x = Kx = 0$, which under these new assumptions again gives us $x = 0$. If $\ker K$ is nontrivial, then as noted above there is some nonzero constant solution to $R\left(\frac{d}{dt}\right) q = 0$. We still need to check that that there are no solutions of the form $q(t) = \alpha + \beta t$ with $\beta \neq 0$ (in other words, that 0 is a semisimple root of R). We must then have

$$R\left(\frac{d}{dt}\right)(\alpha + \beta t) = K\alpha + K\beta t + D\beta = 0.$$

It follows that $\beta \in \ker K$, giving us that $K\alpha + D\beta = 0$. Multiplication by β^T gives

$$0 = \beta^T K\alpha + \beta^T D\beta = \frac{1}{2}\beta^T(D + D^T)\beta.$$

Now since $\ker K \cap \ker(D + D^T) = \emptyset$ this implies $\beta = 0$. Hence the system is semistable in this case.

As noted earlier a system that is not autonomous cannot be stable because some components could be chosen as inputs, which we can choose freely and hence is not bounded in general. But this raises the question, what if we choose the input to be bounded, will the output be bounded? This idea leads to another form of stability.

Definition 4.10. Let $\mathcal{B} \in \mathfrak{L}^d$ be a behaviour in i/o-form. We say that $\mathcal{B}$ is bounded input-bounded output (BIBO) stable if $(u, y) \in \mathcal{B}$ and u bounded implies that y is bounded.

Sometimes this form of stability is called L^∞ i/o-stability, as the input and output have finite L^∞ norm in this case. We can similarly define L^p i/o-stability by using L^p norms instead.

The main result regarding BIBO stability is the following, which is from [6].

Proposition 4.11. *Let $\mathcal{B}$ be the i/o-behaviour given by $P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u$ with $P^{-1}Q(\xi)$ a matrix of proper rational functions. Then $\mathcal{B}$ is BIBO stable if and only if each root of P either has negative real part or its real part is zero, it is semisimple and not a pole of $P^{-1}Q(\xi)$.*

Proof. We begin by showing that each of the conditions on the roots of P are necessary. Note first that if any root has positive real part or has real part zero but is not semisimple, then by Proposition 4.2 the system $\ker P\big(\frac{d}{dt}\big)$ is unstable. Then there is an unbounded solution $y \in \ker P\big(\frac{d}{dt}\big)$, and if we combine it with the (bounded) input $u(t) = 0$, we get an i/o pair where the input is bounded but the output is not, so the system is not BIBO stable.

Next we look at the case when a root $\lambda = i\omega$ of P is a pole of $P^{-1}Q(\xi)$. Consider first the case when both u and y are scalar, so that P and Q are polynomials. Then we can find a bounded input with unbounded output as follows. Let us consider i/o pairs where the input is on the form $u = ae^{i\omega t}$ for some $a \in \mathbb{C} \setminus \{0\}$. We can write this in matrix form as

$$R\left(\frac{d}{dt}\right)\begin{pmatrix} u \\ y \end{pmatrix} = 0, \quad R(\xi) := \begin{pmatrix} -Q(\xi) & P(\xi) \\ \xi - i\omega & 0 \end{pmatrix},$$

the second row of R ensuring the input is on the desired form. Note that $\ker R\big(\frac{d}{dt}\big)$ is now an autonomous system since $\det R(\xi) = (\xi - i\omega)\det P(\xi) \neq 0$, so we can use Proposition 4.2 to show that it is unstable. Suppose the algebraic multiplicity of $i\omega$ in $\det P(\xi)$ is n. Then the algebraic multiplicity of $i\omega$ in $\det R(\xi)$ is $n+1$.

If $n = 1$, then since $i\omega$ is a pole of $P^{-1}Q(\xi)$, $i\omega$ can not be a root of $Q(\xi)$. Then the kernel of

$$R(i\omega) = \begin{pmatrix} -Q(i\omega) & 0 \\ 0 & 0 \end{pmatrix}$$

has dimension 1, and so $i\omega$ is not a semisimple root of R. If $n > 1$, then the algebraic multiplicity of $i\omega$ in $\det R(\xi)$ is $n+1 > 2$, and since the kernel of $R(i\omega)$ can have dimension at most 2, $i\omega$ is not a semisimple root of R in this case either. It then follows from Proposition 4.2, that $\ker R(\xi)$ is unstable. Hence there is an unbounded solution pair u, y and we know $u = ae^{i\omega t}$ for some $a \in \mathbb{C}$, which is bounded. Hence y must be unbounded, and so $\mathcal{B}$ is not BIBO stable in this case.

Now let us return to the general case of input u of size p and output y of size q. By Proposition 2.11 there are unimodular U, V such that UPV is diagonal. Let $\tilde{P} = UPV$, $\tilde{Q} = UQ$ and consider

$$\tilde{P}^{-1}\tilde{Q}(\xi) = V^{-1}P^{-1}Q(\xi).$$

We now claim that if $i\omega$ is a pole of $P^{-1}Q(\xi)$, then it is also a pole of $\tilde{P}^{-1}\tilde{Q}(\xi)$. To see why, suppose $i\omega$ is not a pole of $\tilde{P}^{-1}\tilde{Q}(\xi)$, so that $\tilde{P}^{-1}\tilde{Q}(i\omega)$ is well defined. Then $V(i\omega)\tilde{P}^{-1}\tilde{Q}(i\omega)$ is also well defined, but this is equal to $P^{-1}Q(i\omega)$ which is not well defined. Hence $i\omega$ must be a pole of $\tilde{P}^{-1}\tilde{Q}(\xi)$.

Now suppose $i\omega$ is a pole in the k:th row, l:th column of $\tilde{P}^{-1}\tilde{Q}(\xi)$ and consider the system given by

$$\tilde{P}\left(\frac{d}{dt}\right)y = \tilde{Q}\left(\frac{d}{dt}\right)u.$$

Take an input u that is of the form $ae^{i\omega t}$ on the l:th row, and zero on the other rows. There is then a corresponding output y, with its k:th component given by the equation

$$\tilde{p}_k\left(\frac{d}{dt}\right) y_k = \tilde{q}_{k,l}\left(\frac{d}{dt}\right) u_l.$$

This is the 1D case handled earlier, so there is an unbounded solution for y_k on the form $y_k = (b+ct)e^{i\omega t}$. Hence we have a bounded input, unbounded output pair u, y for the equation $\tilde{P}\big(\frac{d}{dt}\big)\, y = \tilde{Q}\big(\frac{d}{dt}\big)\, u$. The pair $u, V\big(\frac{d}{dt}\big)\, y$ is then a bounded input, unbounded output pair for $P\big(\frac{d}{dt}\big)\, y = Q\big(\frac{d}{dt}\big)\, u$, so $\mathcal{B}$ is not BIBO stable.

To show that these requirements on the roots are sufficient for BIBO stability we use Lemma 2.17, giving us that given any input u, any output y solving the equation can be written as a sum $y = y_{\mathrm{h}} + y_{\mathrm{p}}$ where $y_{\mathrm{h}} \in \ker P\big(\frac{d}{dt}\big)$ and y_{p} is given by the formula

$$y_{\mathrm{p}}(t) = A_0 u(t) + \sum_{k=1}^{N}\sum_{l=1}^{n_k} A_{k,l}\int_0^t \frac{(t-\tau)^{l-1}}{(l-1)!} e^{\lambda_k(t-\tau)} u(\tau)d\tau$$

where

$$P^{-1}(\xi)Q(\xi) = A_0 + \sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{A_{k,l}}{(\xi-\lambda_k)^l},$$

is the partial fraction decomposition. The conditions on the roots guarantee that $\ker P\big(\frac{d}{dt}\big)$ is stable (via Proposition 4.2), and consequently y_{h} is bounded.

That the imaginary roots of P are not poles of $P^{-1}Q$ means that $A_{k,l} = 0$ for k such that $\Re(\lambda_k) = 0$. If $\lambda_1, \dots, \lambda_{N^-}$ are the roots with $\Re(\lambda_k) < 0$, then we can write y_{p} as

$$y_{\mathrm{p}}(t) = A_0 u(t) + \sum_{k=1}^{N^-}\sum_{l=1}^{n_k} A_{k,l}\int_0^t \frac{(t-\tau)^{l-1}}{(l-1)!} e^{\lambda_k(t-\tau)} u(\tau)d\tau.$$

Now suppose u is a bounded input, say $\sup_{t\geq 0}|u(t)| =: M \in \mathbb{R}$. Then

$$\begin{aligned} |y_{\mathrm{p}}| &\leq \|A_0\|\, M + \sum_{k=1}^{N^-}\sum_{l=1}^{n_k} \|A_{k,l}\| \int_0^t \frac{(t-\tau)^{l-1}}{(l-1)!} |e^{\lambda_k(t-\tau)}| M d\tau = \\ &= M\left(\|A_0\| + \sum_{k=1}^{N^-}\sum_{l=1}^{n_k} \frac{\|A_{k,l}\|}{(l-1)!}\int_0^t (t-\tau)^{l-1} e^{\Re(\lambda_k)(t-\tau)} d\tau\right) \end{aligned}$$

where $\|A\| = \sup\{|Ax| : x \in \mathbb{R}^p, |x| = 1\}$ is a matrix operator norm. We note that for any $\mu < 0$ we have

$$\int_0^t (t-\tau)^{l-1} e^{\mu(t-\tau)} d\tau = \int_0^t s^{l-1} e^{\mu s} ds.$$

Iterated integration by parts shows that this is equal to a polynomial in t times $e^{\mu t}$. This last expression is bounded as $t \to \infty$, and it follows that $|y_{\mathrm{p}}|$ in the formula above is also bounded, and hence $\mathcal{B}$ is BIBO stable. $\square$

The assumption that $P^{-1}Q(\xi)$ is a matrix of proper rational functions is needed to use Lemma 2.17. While this condition is not necessary to get an i/o-behaviour, the process we used to recover an output for a given input u when proving Theorem 2.18 involved differentiating a solution similar to y_{p} above. This in turn means differentiating the input u. This means that without the properness condition on $P^{-1}Q(\xi)$, even if the input is bounded we may not get a bounded output.

We illustrate with a simple example, if $P = 1$ and $Q(\xi) = \xi$, i.e. the equation

$$y = \frac{d}{dt}u.$$

A solution to this is $u(t) = \sin(t^2), y(t) = 2t\cos(t^2)$ which has bounded input, but the output is not bounded. Hence this system is not BIBO stable.

Example 4.12. We will once again look at the spring-damper system discussed in Examples 4.3 and 4.9, but now with an external input. The system will now be given by the equation $R\left(\frac{d}{dt}\right) q = F$ where $R(\xi) = M\xi^2 + D\xi + K$ and F is a (time dependant) input. In the asymptotically stable case discussed above all roots of R have strictly negative real part, and so by Proposition 4.11 the system is also BIBO stable in this case. If R has roots with real part zero, then one would need more details on what the matrices M, D and K look like if one wants to see which may be poles of the transfer function.

5 Dissipative systems and LQ-control

In this section we will look at another application of QDFs in the area of dissipative systems and how this relates to LQ-control problems. When looking at issues of stability we used positivity and negativity of certain QDFs. For dissipative systems we will instead use a form of average positivity.

To get an intuitive idea of what dissipative systems are it is useful to put them in contrast with closed systems. A closed system usually refers to a system that has no interaction with the outside, for example no energy enters or leaves the system. In other words the total energy in the system always remains the same. A dissipative system is in contrast one where (on average) for example energy, is being supplied to the system. We could of course also have systems where the energy is on average leaving the system, but mathematically this makes little difference since adding a minus sign to the quantity of study would give us a quantity that is on average increasing.

The basis of a dissipative system is a supply rate, describing how the supply enters the system. In this text we will look at supply rates that are given by QDFs, which is often suitable when working with linear systems. A more general approach to supply rates can be found in [2].

Definition 5.1. Let $\mathcal{B} \in \mathfrak{L}^d$ and let $\Phi, \Psi, \Delta \in \mathbb{R}^{d \times d}[\zeta, \eta]$. We say that $\mathcal{B}$ is dissipative with respect to the supply rate Q_Φ, if

$$\int_{\mathbb{R}} Q_\Phi(w)dt \geq 0 \tag{20}$$

for all $w \in \mathcal{B} \cap \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. The QDF Q_Ψ is called a storage function for Q_Φ if

$$Q_{\overset{\bullet}{\Psi}}(w) \leq Q_\Phi(w) \tag{21}$$

for all $w \in \mathcal{B}$. The QDF Q_Δ is called a dissipation rate for Q_Φ if $\Delta \overset{\mathcal{B}}{\geq} 0$ and

$$\int Q_\Phi(w)dt = \int Q_\Delta(w)dt$$

for all $w \in \mathcal{B} \cap \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$.

The supply rate Q_Φ describes the rate at which supply enters the system for a given external signal. In equation (20) we consider only signals with compact support. We can motivate this in two ways. First, this ensures that the integral is finite. Second, equation (20) can then be interpreted as saying that for any external signal that begins with the system at rest and eventually takes the system back to being at rest, the net flow of supply is into the system.

At any time t, the supply that has entered the system must either still be stored in the system or it has dissipated. A storage function Q_Ψ describes how much supply is stored in the system at any given time. The inequality (21) is called the dissipation inequality and it tells us that the change in storage is always limited by the supply rate. We do not in general have equality because some of the supply that enters the system does not remain in storage, but instead dissipates.

The dissipation rate of course describes how fast the dissipation is happening. The dissipation rate is nonnegative since the dissipation only works one way, dissipation only causes supply to leave the system. That the integrals of Q_Φ and Q_Δ are equal represent that for a system that is initially and finally at rest, the net supply change must be zero. Hence all supply that enters must leave either through the supply rate or through dissipation.

When working with dissipative systems it is preferable to work with controllable systems. In fact what dissipativeness should mean for non controllable behaviour has been the subject of some discussion. In for example [14] it is argued that the existence of a storage function is a more suitable definition for dissipativeness in a more general context, while [9] uses the inequality (20) as we do. The reason controllable systems are easier to handle is because we can make use of image representations. Due to Corollary 2.30 we can assume without loss of generality that we have an observable image representation. We can then make use of the following fact.

Lemma 5.2. *If $w = M\left(\frac{d}{dt}\right) l$, where $M \in \mathbb{R}^{d\times n}[\xi]$, is an observable image representation, then $M(\xi)$ has a polynomial left inverse*

$$M^\dagger(\xi) := (M^T M)^{-1} M^T(\xi).$$

Proof. Since $w = M\left(\frac{d}{dt}\right) l$ is observable we have $\mathrm{rk}(M(\lambda)) = n$ for all $\lambda \in \mathbb{C}$ and so $\mathrm{rk}\big(M^T(\lambda)M(\lambda)\big) = n$ for all $\lambda \in \mathbb{C}$. Hence $\det M^T M(\xi)$ has no roots, so it must be a non-zero constant. This means that $M^T M(\xi)$ is a unimodular matrix and hence $(M^T M)^{-1}(\xi)$ is a polynomial matrix. It follows that $M^\dagger(\xi)$ is also a polynomial matrix and

$$M^\dagger M(\xi) = (M^T M)^{-1} M^T M(\xi) = I$$

so $M^\dagger$ is a left inverse for M. □

Let us suppose $\mathcal{B} \in \mathfrak{L}^d$ is controllable and has an observable image representation given by $w = M\left(\frac{d}{dt}\right) l$. For $\Phi \in \mathbb{R}^{d\times d}[\zeta,\eta]$, let

$$\Phi'(\zeta,\eta) = M^T(\zeta)\Phi(\zeta,\eta)M(\eta).$$

Since the image representation is observable we have a one-to-one correspondence between $w \in \mathcal{B}$ and $l \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^n)$ such that $w = M\left(\frac{d}{dt}\right) l$, and we have

$$Q_\Phi(w) = Q_{\Phi'}(l).$$

Hence looking at Q_Φ on $w \in \mathcal{B}$ can be reduced to looking at $Q_{\Phi'}$ on $l \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^n)$. Consequently dissipativeness of $\mathcal{B}$ with respect to Q_Φ is equivalent to dissipativeness of $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^n)$ with respect to Φ'. Also, if $Q_{\Psi'}, Q_{\Delta'}$ are a storage function and a dissipation rate for $Q_{\Phi'}$ respectively, then

$$\Psi(\zeta,\eta) = (M^\dagger(\zeta))^T \Psi'(\zeta,\eta) M^\dagger(\eta)$$

$$\Delta(\zeta,\eta) = (M^\dagger(\zeta))^T \Delta'(\zeta,\eta) M^\dagger(\eta)$$

are a storage function and a dissipation rate for Q_Φ. It is therefore enough for controllable behaviours to check the free case when considering dissipativness and the existence of storage function or dissipation rate. For controllable systems these notions are in fact equivalent. For this reason the question of what dissipativeness should be defined as is not so important in the controllable case.

Theorem 5.3. *Let $\mathcal{B} \in \mathfrak{L}^d$ be controllable and $\Phi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$. Then the following are equivalent:*

(i) *$\mathcal{B}$ is dissipative with respect to Q_Φ.*

(ii) *There is a storage function Q_Ψ for Q_Φ.*

(iii) *There is a dissipation rate Q_Δ for Q_Φ.*

If the above conditions hold then there is a one to one pairing of storage functions and dissipation rates given by the equation

$$\overset{\bullet}{\Psi} = \Phi - \Delta.$$

In the proof of this theorem, in particular the existence of a dissipation rate, we will need to factor the matrix $\partial\Phi$ in a particular way. The details are given by the following lemma which we will prove first.

Lemma 5.4. *Let $\Phi \in \mathbb{R}_s^{d\times d}[\zeta,\eta]$ such that $\int_\mathbb{R} Q_\Phi(w)dt \geq 0$ for all $w \in \mathscr{D}(\mathbb{R},\mathbb{R}^d)$. Then there is a matrix $F \in \mathbb{R}^{d\times d}[\xi]$ such that $\partial\Phi(\xi) = F^T(-\xi)F(\xi)$.*

The proof of Lemma 5.4 is based on [3] where the case when Φ is a polynomial is proven. The proof of Theorem 5.3 is from [14].

Proof of Lemma 5.4. The first step of the proof is to show that $\int_\mathbb{R} Q_\Phi(w)dt \geq 0$ for all $w \in \mathscr{D}(\mathbb{R},\mathbb{R}^d)$ implies that $\overline{a}\Phi(-i\omega,i\omega)a \geq 0$ for all $a \in \mathbb{C}^d, \omega \in \mathbb{R}$. The proof is very similar to the proof of Theorem 3.1 so we will be brief. Assume for a contradiction that $\overline{a}^T\Phi(-i\omega,i\omega)a < 0$ for some $a \in \mathbb{C}^d, \omega \in \mathbb{R}$. For $N = 1, 2, \ldots$ we construct functions v_N just as in the proof of Theorem 3.1 where we furthermore found that

$$\int_\mathbb{R} Q_\Phi(v_N)dt = \frac{4\pi N}{\omega}\overline{a}^T\Phi(-i\omega,i\omega)a + 2A$$

where A is a constant that does not depend on N. It follows that $\int_\mathbb{R} Q_\Phi(v_N)dt < 0$ for large enough N. Therefore $\int_\mathbb{R} Q_\Phi(v_N)dt \geq 0$ for all $w \in \mathscr{D}(\mathbb{R},\mathbb{R}^d)$ implies that $\overline{a}^T\Phi(-i\omega,i\omega)a \geq 0$ for all $a \in \mathbb{C}^d, \omega \in \mathbb{R}$.

By Proposition 2.11 there are unimodular matrices U, V such that

$$U(\xi)\partial\Phi(\xi)V(\xi) =: R(\xi) \tag{22}$$

is diagonal. Transposing equation (22) and switching ξ for $-\xi$ we have

$$V^T(-\xi)(\partial\Phi)^T(-\xi)U^T(-\xi) = R(-\xi). \tag{23}$$

Since Φ is symmetric we have $\partial\Phi(\xi) = \partial(\Phi^*)(\xi) = (\partial\Phi)^T(-\xi)$, and so equations (22) and (23) describe the same diagonalization. Hence we have $U(\xi) = V^T(-\xi)$ and $R(\xi) = R(-\xi)$.

Note that we now have for any $a \in \mathbb{C}^d, \omega \in \mathbb{R}$ that

$$\overline{a}^T R(i\omega)a = \overline{(V(i\omega)a)}^T \Phi(-i\omega, i\omega)(V(i\omega)a) \geq 0.$$

If we let $R(\xi) = \operatorname{diag}(r_1(\xi), \dots, r_d(\xi))$ and

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix},$$

then we have

$$\overline{a}_k^T r_k(i\omega)a_k \geq 0$$

for each $k = 1, \dots, d$. This implies $r_k(i\omega) \geq 0$ for all $k = 1, \dots, d$. If $i\omega_0$ is an imaginary root of r_k, then it must have even multiplicity. To see this note that if $i\omega_0$ is a root with odd multiplicity, then $r_k(i\omega)$ switches sign as ω varies from values smaller than ω_0 to larger, which would contradict that $r_k(i\omega) \geq 0$.

We also note that since $R(\xi) = R(-\xi)$ we also have that r_k is even. Hence for any root λ of r_k that is not on the imaginary axis, $-\lambda$ is another root. Since r_k is a real polynomial it follows that we also have $\pm\overline{\lambda}$ as roots. All roots, counted with multiplicity, can hence be put in groups of four of the form $\pm\lambda_l, \pm\overline{\lambda}_l$. If we now let

$$\gamma_k(\xi) = \prod_l (\xi - \lambda_l)(\xi - \overline{\lambda}_l)$$

then $r_k(\xi) = \gamma_k(-\xi)\gamma_k(\xi)$. Setting $\Gamma(\xi) = \operatorname{diag}(\gamma_1(\xi), \dots, \gamma_d(\xi))$, we have $R(\xi) = \Gamma(-\xi)\Gamma(\xi)$. Finally, setting $F(\xi) = \Gamma(\xi)V^{-1}(\xi)$ we have

$$F^T(-\xi)F(\xi) = V^{-T}(-\xi)\Gamma(-\xi)\Gamma(\xi)V^{-1}(\xi) = V^{-T}(-\xi)R(\xi)V^{-1}(\xi) = \partial\Phi(\xi)$$

and the proof is finished. □

Proof of Theorem 5.3. We will show the equivalences by proving the chain (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). Since $\mathcal{B}$ is controllable it suffices to show the statement in the case $\mathcal{B} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$.

(i) $\Longrightarrow$ (iii): We use Lemma 5.4, giving us F such that $\partial\Phi(\xi) = F^T(-\xi)F(\xi)$. Letting $\Delta(\zeta, \eta) = F^T(\zeta)F(\eta)$, we will show that this is a dissipation rate. We have, for any $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$

$$Q_\Delta(w) = \left| F\left(\frac{d}{dt}\right) w \right|^2 \geq 0$$

so $\Delta \geq 0$. We also have $\partial(\Phi - \Delta) = 0$, and so by Theorem 3.1, $\int Q_{\Phi-\Delta}(w) = 0$ for all $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. Hence $\int_{\mathbb{R}} Q_\Phi(w)dt = \int_{\mathbb{R}} Q_\Delta(w)dt$, and so Q_Δ is a dissipation rate for Q_Φ.

(iii) $\Longrightarrow$ (ii): If $\int Q_\Phi = \int Q_\Delta$, then $\int_{\mathbb{R}} Q_{\Phi-\Delta}(w)dt = 0$ for all $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$. By Theorem 3.1 there is then Ψ such that $\overset{\bullet}{\Psi} = \Phi - \Delta$. Since $\Delta \geq 0$ it follows that $\overset{\bullet}{\Psi} \leq \Phi$.

(ii) $\Longrightarrow$ (i): For any $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$ we have

$$\int_{\mathbb{R}} Q_\Phi(w)dt \geq \int_{\mathbb{R}} Q_{\overset{\bullet}{\Psi}}(w)dt = [Q_\Psi(w)]_{-\infty}^{\infty} = 0,$$

since w has compact support and hence $\lim_{t\to\pm\infty} Q_\Psi(w) = 0$.

If (iii) holds, then it is clear from the proof that there is a storage function Ψ that satisfies $\overset{\bullet}{\Psi} = \Phi - \Delta$, moreover this Ψ is unique. □

Storage functions for a certain supply rate are in general not unique. If $\int Q_\Phi = 0$, then there is a unique storage function given by $\overset{\bullet}{\Psi} = \Phi$, but otherwise we can in general find several.

The storage function describes how much has been stored in the system up to the time t, depending on the external signal. As such we may guess that it has some connection to the state, which is a kind of memory and should hold all information about the past of the external signal. Indeed, by the following proposition, if our behaviour is in state space form then any storage function can be expressed as a function of the state. Additionally this expression can be taken as a quadratic form, no derivatives of the state involved.

Proposition 5.5. *Suppose $\mathcal{B} \in \mathfrak{L}^d$ is controllable, has a state space representation $\mathcal{B}_s$ and that $\mathcal{B}$ is dissipative with respect to Q_Φ. If Q_Ψ is a storage function for Q_Φ, then there is a matrix $K = K^T \in \mathbb{R}^{n\times n}$ such that for any $(w, x) \in \mathcal{B}_s$*

$$Q_\Psi(w)(t) = x^T(t)Kx(t).$$

A proof of this statement can be found in [14]. The set of storage functions has some further nice properties. It is both convex and has a maximal and minimal element.

Proposition 5.6. *Let $\mathcal{B} \in \mathfrak{L}^d$ be controllable and dissipative with respect to Q_Φ. Then the set of storage functions for Q_Φ is convex.*

Proof. Let us suppose for simplicity that $\mathcal{B} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^d)$. If Q_{Ψ_1}, Q_{Ψ_2} induce two storage functions for Q_Φ and $s \in (0, 1)$, then

$$\begin{aligned}\frac{d}{dt}\left(sQ_{\Psi_1}(w) + (1-s)Q_{\Psi_2}(w)\right) &= sQ_{\overset{\bullet}{\Psi}_1}(w) + (1-s)Q_{\overset{\bullet}{\Psi}_2}(w) \leq \\ &\leq sQ_\Phi(w) + (1-s)Q_\Phi(w) = Q_\Phi(w)\end{aligned}$$

so $sQ_{\Psi_1} + (1-s)Q_{\Psi_2}$ is a storage function for Q_Φ. Consequently the set of storage functions is convex. □

Theorem 5.7. *Let $\mathcal{B} \in \mathfrak{L}^d$ be controllable and dissipative with respect to Q_Φ. Then there are storage functions Ψ^-, Ψ^+ such that for any storage function Ψ we have*

$$\Psi^- \leq \Psi \leq \Psi^+.$$

For a proof of this statement we refer to [14], where it is also shown how to find these Ψ^-, Ψ^+. We will not go into detail here but only give the basic idea, which starts with factoring $\partial\Phi(\xi)$. We use one factorization $\partial\Phi(\xi) = A^T(-\xi)A(\xi)$, where the roots of A are in the closed right half plane, and $\partial\Phi(\xi) = H^T(-\xi)H(\xi)$, where the roots of A are in the closed left half plane. These different factorizations are given by choosing the γ_k:s in different ways in the proof of Lemma 5.4.

The maximal and minimal storage functions are then induced by

$$\Psi^+ = \frac{\Phi(\zeta,\eta) - A^T(\zeta)A(\eta)}{\zeta+\eta}$$
$$\Psi^- = \frac{\Phi(\zeta,\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta}.$$

By Proposition 5.5, if we have a state space representation of $\mathcal{B}$, then Ψ_-, Ψ_+ can be expressed as

$$Q_{\Psi^+}(w) = x^T K^+ x$$
$$Q_{\Psi^-}(w) = x^T K^- x.$$

for some matrices $K^+, K^- \in \mathbb{R}^{n\times n}$.

There is another interpretation of these two storage functions that perhaps give a more intuitive idea of what they are. Again we consider a behaviour with state behaviour $\mathcal{B}_s$ and $(w,x) \in \mathcal{B}_s$. The storage functions are given as

$$Q_{\Psi^+}(w)(t) = \inf\left\{\int_{-\infty}^{t} Q_\Phi(\tilde{w})d\tau | (\tilde{w},\tilde{x}) \in \mathcal{B}_s, x(t) = \tilde{x}(t), \tilde{w} \in \mathscr{D}(\mathbb{R},\mathbb{R}^d)\right\}$$
$$Q_{\Psi^-}(w)(t) = \sup\left\{-\int_{t}^{\infty} Q_\Phi(\tilde{w})d\tau | (\tilde{w},\tilde{x}) \in \mathcal{B}_s, x(t) = \tilde{x}(t), \tilde{w} \in \mathscr{D}(\mathbb{R},\mathbb{R}^d)\right\}.$$

At first glance it may not be entirely clear that these expressions are QDFs, or that they can be expressed as quadratic functions of the state. In fact the finiteness of this infimum and supremum is a direct consequence of the dissipativeness of $\mathcal{B}$ (see [2]). If we accept that they are storage functions, showing that they are the maximal and minimal storage functions is not so difficult.

Indeed, for any storage function $Q_\Psi, w \in \mathcal{B} \cap \mathscr{D}(\mathbb{R},\mathbb{R}^d)$ and $t_0 < t_1$ we have by the dissipation inequality that

$$\int_{t_0}^{t_1} Q_\Phi(w)d\tau \geq Q_\Psi(w)(t_1) - Q_\Psi(w)(t_0).$$

If we first let $t_1 \to \infty$ and $t_0 = t$ we have

$$Q_\Psi(w)(t) \geq -\int_t^\infty Q_\Phi(w)d\tau.$$

For any $(\tilde{w}, \tilde{x}) \in \mathcal{B}_s$ such that $x(t) = \tilde{x}(t)$ we have $Q_\Psi(w)(t) = Q_\Psi(\tilde{w})(t)$, since Q_Ψ depends only on the state at t. Taking supremum over all such $\tilde{w}$ we have that $\Psi \geq \Psi^-$.

Similarly, if we let $t_1 = t$ and $t_0 \to -\infty$, then we have

$$Q_\Psi(w)(t) \leq \int_{-\infty}^{t} Q_\Phi(w) d\tau$$

and taking infimum over all $\tilde{w}$ such that $x(t) = \tilde{x}(t)$ gives $\Psi \leq \Psi^+$.

Due to these last formulas for Q_{Ψ^+} is called the required supply and Q_{Ψ^-} is called available storage. For Q_{Ψ^+} the idea is that it describes the smallest amount of supply that needs to be stored to take the system from rest to the state $x(t)$. On the other hand Q_{Ψ^-} describes the largest amount of supply that can be extracted from the system in state $x(t)$ when bringing it to rest.

Since storage functions can be represented as quadratic forms of the state, dissipativeness is equivalent to the existence of quadratic forms with certain properties. This allows us to link dissipativeness with certain matrix inequalities, which will be very important when we look at LQ-control problems. The following result is called the Kalman-Yakubovich-Popov Lemma (often abbreviated as KYP Lemma), here formulated for generalized first order systems. The proof is from [9], where a more general version for higher order system is also proven. For the higher order version the idea is to introduce some latent variables to give a first order representation of the behaviour, and then apply the first order version.

Before we state the Proposition we need to introduce the notion of a trim behaviour. For a dynamical system $(\mathcal{T}, \mathcal{W}, \mathcal{B})$ the set of consistent points is

$$W_0 = \{w_0 \in \mathcal{W} | \exists w \in \mathcal{B}, w(0) = w_0\}.$$

We say that $\mathcal{B}$ is trim if $W_0 = \mathcal{W}$, meaning that at time $t = 0$ we can have any signal in the signal space. In some sense this means that the signal space is as small as possible. The reason we need this is that we want be able to conclude from a statement like $w^T K w \geq 0$ for all $w \in \mathcal{B}$, that the matrix K is positive semidefinite. If the behaviour is not trim, then we can not make this conclusion.

Proposition 5.8 (Kalman-Yakubovich-Popov Lemma).
Let $\mathcal{B}$ be the solution set of

$$G\dot{w} = Fw$$

where $F, G \in \mathbb{R}^{e \times d}$. Assume that $\mathcal{B}$ is controllable and trim and let $M = M^T \in \mathbb{R}^d$. Then the following are equivalent

(i) $\int_{\mathbb{R}} w^T M w dt \geq 0$ *for all* $w \in \mathcal{B} \cap \mathscr{D}(\mathbb{R}, \mathbb{R}^d)$.

(ii) *There is* $P = P^T \in \mathbb{R}^{e \times e}$ *such that* $M + F^T P G + G^T P F \geq 0$.

Proof. We begin by showing that (i) implies (ii). The statement (i) means that $\mathcal{B}$ is dissipative with respect to $w^T M w$ and so by Theorem 5.3 there is some storage function for this supply rate. Note that Gw has the properties of a state variable, and so by Proposition 5.5 this storage function can be expressed as $w^T G^T K G w$ for some $K \in \mathbb{R}^{d \times d}$. It satisfies the dissipation inequality

$$\frac{d}{dt}(w^T G^T K G w) \leq w^T M w \iff$$
$$w^T(F^T K G + G^T K F)w \leq w^T M w.$$

Setting $P = -K$ we have

$$w^T(M + F^T P G + G^T P F)w \geq 0,$$

and since $\mathcal{B}$ is trim this implies (ii).

To show (ii) implies (i) we simply need to show that $-w^T G^T P G w$ is a storage function for $w^T M w$. Indeed, it satisfies the dissipation inequality

$$\frac{d}{dt}(-w^T G^T P G w) = -w^T F^T P G w - w^T G^T P F w \leq w^T M w$$

so it is a storage function. By Theorem 5.3 $\mathcal{B}$ is dissipative with respect to $w^T M w$, i.e. (i) holds. □

5.1 LQ-control problems

Dissipative systems and QDFs have a close connection to LQ-control problems, a type of optimization problems. LQ-control problems can be formulated in a variety of ways, here we give one of the simpler formulations.

In this section we will look at an i/s/o system on the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= x \end{aligned} \tag{24}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}$. Of course we do not need both the variables x and y, so we will just write x and use equation (24). Note that since x is now also the output, we will write the external signal as

$$w = \begin{pmatrix} u \\ x \end{pmatrix}.$$

For the remainder of this section $\mathcal{B}$ will refer to the solution set of (24). We will also assume that this state space representation is minimal.

Here "LQ" is short for linear-quadratic, the linear part refers to a linear system, in our case given by $\mathcal{B}$. The quadratic part refers to a quadratic cost functional (sometimes called performance index), in our case given by

$$\mathcal{J}(u, x_0) = \int_0^\infty x^T Q x + u^T R u dt$$

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{p \times p}$. Here $x_0 \in \mathbb{R}^n$ is an initial value for x. The LQ-control problem is to find a control u that minimizes $\mathcal{J}$ under the conditions that (u, x) satisfy equation (24) and $x(0) = x_0$.

The term LQ-control problem refers more generally to problems where we want to minimize or maximize some quadratic cost expression subject to some linear equations. The linear equation could for example be higher order, the usual approach is then to introduce new variables to reduce to a first order problem. We could also have a different cost expression, for example including a mixed term $u^T Sx$. These cases can essentially be reduced to the one we will look at through a change of variables. Hence if we can solve the problem presented above, these slightly altered problems have solutions that can be attained through a change of variables.

The problem we have formulated is the infinite horizon case, referring to the fact that we integrate to infinite time horizon in $\mathcal{J}$. Finite horizon versions of the problem can also be studied, see for example [8]. Curiously the finite horizon LQ-problem is actually more complicated than the infinite horizon problem.

We note that we have written the cost functional $\mathcal{J}$ as a function of u and x_0 but not of x even though x is included in the expression defining $\mathcal{J}$. The motivation is that given an initial condition x_0 and having chosen a control u, the initial value problem

$$\dot{x} = Ax + Bu, x(0) = x_0$$

has a unique solution if u is chosen sufficiently smooth. As a consequence $\mathcal{J}$ is completely determined once we have chosen x_0 and u. The smoothness condition on u turns out to not be a problem, as we will find that under suitable assumptions there is always an optimal solution that is smooth.

Next we need to consider a few questions regarding well-posedness of the problem. For the problem to be well-posed we of course need the integral $\mathcal{J}$ to converge for some input u, so in particular we need $x^T Qx \to 0$ as $t \to \infty$. If $\mathcal{B}$ is not controllable then there may be initial values x_0 for which $x^T Qx$ does not converge to zero regardless of input. Hence we will assume going forward that $\mathcal{B}$ is controllable.

To give us a clue to what the optimal cost might be, we will use the theory of dissipative systems. We take the supply rate as $Q_\Phi(w) = x^T Qx + u^T Ru$, the integrand in the cost function $\mathcal{J}$. This integrand is usually called the instantaneous cost. Since we have assumed $Q \geq 0, R > 0$ we have that $Q_\Phi \geq 0$ so $\mathcal{B}$ is dissipative with respect to this supply rate. We can now make use of storage functions, in particular the available storage which we found to be equal to

$$\begin{aligned} Q_{\Psi^-}(w)(0) &= \sup \left\{ -\int_0^\infty Q_\Phi(\tilde{w}) d\tau | (\tilde{w}, \tilde{x}) \in \mathcal{B}_s, x(0) = \tilde{x}(0), \tilde{w} \in \mathscr{D}(\mathbb{R}, \mathbb{R}^d) \right\} \\ &= -\inf \left\{ \mathcal{J}(\tilde{u}, x_0) | (\tilde{u}, \tilde{x}) \in \mathcal{B}, \tilde{x}(0) = x_0, (\tilde{u}, \tilde{x}) \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{p+n}) \right\}. \end{aligned}$$

We see that the available storage is essentially minus the sought minimum of $\mathcal{J}$.

There are some details that need clearing up though. First of all the available storage is given here by an infimum while we seek a minimum, so it is a priori not clear if this minimum is ever attained. Second, the infimum is taken over external signals with compact support. It turns out that the optimal control will be a smooth function, but it will not have compact support.

We will find that the restriction of compact support can not simply be removed. Compact support implies that the state always goes to zero as $t \to \infty$, so called zero endpoint solutions. It turns out that if the state does not converge to zero, the free endpoint case, then the situation is more complicated.

The key to solving this problem is choosing u as a linear state feedback, that is we choose u as a linear function of x. For this kind of approach to be feasible the state at time t of course needs to be known to us at time t if we are to feed it back into the system. In our case the state is also the output so it is justified to feed it back to itself. We will hence look for a suitable control of the form $u = Fx$ where $F \in \mathbb{R}^{p\times n}$.

It turns out that a suitable feedback is given by setting $u = -R^{-1}B^TPx$ where $P \in \mathbb{R}^{n\times n}$ is a symmetric solution to the equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0. \tag{ARE}$$

This matrix equation is called the algebraic Riccati equation (abbreviated as ARE). We will use the theory of dissipative systems to give a motivation for where this equation comes from. The first step is to use the KYP Lemma. To do so note that we can write the equations for our system as

$$\begin{pmatrix} 0 & I \end{pmatrix} \dot{w} = \begin{pmatrix} B & A \end{pmatrix} w$$

and that since our state space representation is minimal, $\mathcal{B}$ is also trim. By the KYP Lemma there is hence a matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{pmatrix} R & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} B^T \\ A^T \end{pmatrix} P \begin{pmatrix} 0 & I \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} P \begin{pmatrix} B & A \end{pmatrix} \geq 0$$

which we can rewrite as

$$\begin{pmatrix} R & B^TP \\ PB & A^TP + PA + Q \end{pmatrix} \geq 0. \tag{25}$$

This matrix can be factored as

$$\begin{pmatrix} R & B^TP \\ PB & A^TP + PA + Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ PBR^{-1} & I \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & \mathscr{R}(P) \end{pmatrix} \begin{pmatrix} I & R^{-1}B^TP \\ 0 & I \end{pmatrix}$$

where $\mathscr{R}(P) = A^TP + PA + Q - PBR^{-1}B^TP$. Since $R > 0$ the inequality (25) is equivalent to $\mathscr{R}(P) \geq 0$. Note that $\mathscr{R}(P) = 0$ is the ARE, so clearly any solution to the ARE also solves the inequality $\mathscr{R}(P) \geq 0$. Hence if P is a solution to the ARE, then $-x^TPx$ is a storage function. It can be shown that the inequality $\mathscr{R}(P) \geq 0$ has a solution only if $\mathscr{R}(P) = 0$ has a solution. In our case we know $\mathscr{R}(P) \geq 0$ has some solution since we know there exists at least one storage function. In particular it can be shown that the maximal and minimal storage functions given by Ψ^+ and Ψ^- do correspond to solutions of the ARE.

It is worth noting that not all storage functions correspond to solutions to the ARE. We found that the set of storage functions is convex, but the set of solutions to the ARE is not convex in general. In fact the solution set of the ARE is isomorphic to an algebraic variety, often just a finite set. A proof of this as well as several other topological and geometric considerations about the solutions to the ARE can be found in [7].

Now let P be a solution to the ARE. Differentiating the form $x^T P x$ we get

$$\begin{aligned}
\frac{d}{dt}\left(x^T P x\right) &= (\dot{x})^T P x + x^T P \dot{x} = (Ax + Bu)^T P x + x^T P (Ax + Bu) = \\
&= x^T (A^T P + PA) x + u^T B^T P x + x^T P B u = \\
&= x^T (P B R^{-1} B^T P - Q) x + u^T B^T P x + x^T P B u = \\
&= (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) - x^T Q x - u^T R u.
\end{aligned}$$

From here we can actually see that $(u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x)$ is the dissipation rate that corresponds to the storage function given by $-x^T P x$. Integrating the first and last expressions in t from 0 to ∞ gives

$$\begin{aligned}
&\lim_{t\to\infty}\left(x^T P x\right) - x_0^T P x_0 = \\
&= \int_0^\infty (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) dt - \int_0^\infty x^T Q x + u^T R u dt
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathcal{J}(u, x_0) = x_0^T P x_0 &- \lim_{t\to\infty}\left(x^T P x\right) + \\
&+ \int_0^\infty (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) dt.
\end{aligned}$$

Clearly if we had $\lim_{t\to\infty}\left(x^T P x\right) = 0$, then since $R > 0$ we would get the minimal cost if we set $u = -R^{-1} B^T P x$. The minimal cost itself would then be $x_0^T P x_0$. To ensure that we have $\lim_{t\to\infty}\left(x^T P x\right) = 0$ we will need an observability condition, that $(\xi I - A, Q)$ is an observable pair.

Proposition 5.9. *If $(\xi I - A, Q)$ is an observable pair and u is a control such that $\mathcal{J}(u, x_0)$ is finite, then the resulting state trajectory converges to zero as $t \to \infty$.*

Proof. If $\mathcal{J}(u, x_0)$ is finite, then we must have $u \to 0$ and $Qx \to 0$ as $t \to \infty$. Since $u \to 0$ we also have

$$\dot{x} - Ax = Bu \to 0.$$

We can therefore decompose x as $x(t) = x_1(t) + x_2(t)$ where $\dot{x}_1 = Ax_1$ and $x_2 \to 0$. We also have

$$Qx_1 = Qx - Qx_2 \to 0.$$

Since $\dot{x}_1 = Ax_1$, we can write

$$x_1(t) = \sum_{k=1}^{N} \Re(e^{\lambda_k t}) p_k(t) b_k$$

where $\lambda_1, \dots, \lambda_N$ are the unique eigenvalues of A, $p_k(t) \in \mathbb{R}[t]$ and $b_k \in \mathbb{R}^n$. Because $(\xi I - A, Q)$ is an observable pair $Qb_k \neq 0$ whenever $b_k \neq 0$. Hence

$$Qx_1(t) = \sum_{k=1}^{N} \Re(e^{\lambda_k t}) p_k(t) Q b_k$$

can only converge to zero if $\Re(\lambda_k) < 0$ for all k. This in turn means that $\lim_{t\to\infty} x_1(t) = 0$, proving the statement. □

Recall that the available storage can be written $x^T K^- x$, and this is the minimal storage function. Setting $P^+ = -K^-$, we thus get a maximal solution of the ARE. Using the control $u^* = -R^{-1}B^T P^+ x$, the final cost is

$$\mathcal{J}(u^*, x_0) = x_0^T P^+ x_0 - \lim_{t\to\infty} \left(x^T P^+ x\right) \leq x_0^T P^+ x_0,$$

which is clearly finite. The following result is therefore immediate.

Corollary 5.10. *If $(\xi I - A, Q)$ is an observable pair and we set the control to $u^* = -R^{-1}B^T P^+ x$, then the resulting state trajectory converges to zero as $t \to \infty$.*

Note that this result could be stated as, if $(\xi I - A, Q)$ is an observable pair, then the behaviour $\tilde{\mathcal{B}}$ given by $\dot{x} = (A - BR^{-1}B^T P^+)x$ is asymptotically stable. We can actually get this result in another way, using a Lyapunov function and Theorem 4.5. The Lyapunov function in this case is $x^T P^+ x$. Indeed, this function is nonnegative and its derivative is

$$\frac{d}{dt}\left(x^T P^+ x\right) = -x^T Q x - (u^*)^T R u^* = -x^T Q x - x^T P^+ B R^{-1} B^T P^+ x.$$

Clearly $\frac{d}{dt}\left(x^T P^+ x\right)$ is nonpositive. Furthermore if $x \in \tilde{\mathcal{B}}$ is not identically zero but is such that $-x^T Q x - x^T P^+ B R^{-1} B^T P^+ x$ is identically zero, then in particular we have $Qx(t) = 0$ and $R^{-1}B^T P^+ x = 0$ and so $u^* = 0$. Hence such x must satisfy the equation $\dot{x} = Ax$. However this contradicts that $(\xi I - A, Q)$ is an observable pair.

Example 5.11. We will now look at an example of how Corollary 5.10 can fail if $(\xi I - A, Q)$ is not an observable pair. In this example we take $n = p = 2$, $A = 0$, $B = R = I$ and

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case the ARE reads as $Q - P^2 = 0$ which has only two solutions,

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly the maximal solution is

$$P^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the feedback $u^* = -P^+x$, the state should satisfy $\dot{x} = -P^+x$. With the initial condition $x(0) = x_0$ we have

$$x(t) = e^{-P^+t}x_0 = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} x_0$$

and

$$u^*(t) = -P^+x(t) = \begin{pmatrix} -e^{-t} & 0 \\ 0 & 0 \end{pmatrix} x_0.$$

We can now see that for initial conditions on the form

$$x_0 = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

both the control and state trajectory will be non-zero but approach zero as $t \to \infty$, and the total cost will be a^2. On the other hand for initial conditions on the form

$$x_0 = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

the control will be zero, the state constant equal to x_0 and the total cost will be zero. In this case this is the optimal control, even if the resulting state behaviour is not asymptotically stable.

With the observability condition we can now prove that u^* really gives the optimal control.

Proposition 5.12. *If $(\xi I - A, Q)$ is an observable pair, then the control $u^* = -R^{-1}B^TP^+x$ is the unique optimal control.*

Proof. Let $u = v + u^*$ be a control such that $\mathcal{J}(u, x_0)$ is finite. By Proposition 5.9 the resulting state trajectory from using the control u converges to zero as $t \to \infty$. We then have

$$\begin{aligned} \mathcal{J}(u, x_0) = x_0^TP^+x_0 - \lim_{t\to\infty} \left(x^TP^+x\right) + \\ + \int_0^\infty (v + u^* + R^{-1}B^TP^+x)^T R(v + u^* + R^{-1}B^TP^+x)dt = \\ = x_0^TP^+x_0 + \int_0^\infty v^TRvdt \end{aligned}$$

Since $R > 0$, this has a unique minimum when $v = 0$, in other words when the control is equal to u^*. □

Under the assumption that $(\xi I - A, Q)$ is an observable pair it also holds that P^+ is positive definite. To see this note that if $x_0 \in \ker P^+$, then the optimal cost is given by

$$\mathcal{J}(0, x_0) = x_0^T P^+ x_0 = 0.$$

The resulting state trajectory would have to satisfy $Qx = 0$ and $\dot{x} = Ax$, and by the observability assumption this implies that $x_0 = 0$.

Example 5.13. Let us now look at an example of why the observability condition is necessary in Proposition 5.12.

In this example we take $n = p = 2$,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$B = R = I$ and

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case the ARE has solutions

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, P_4 = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$P_{\gamma^+} = \begin{pmatrix} \sqrt{1-\gamma^2} & \gamma \\ \gamma & -1-\sqrt{1-\gamma^2} \end{pmatrix}, P_{\gamma^-} = \begin{pmatrix} -\sqrt{1-\gamma^2} & \gamma \\ \gamma & -1+\sqrt{1-\gamma^2} \end{pmatrix}$$

where $\gamma \in [-1, 1]$. The maximal solution is in this case P_3, to see this note that only P_1 and P_3 have no negative eigenvalues and clearly P_3 is maximal of the two. However the optimal control is not $u = -P_3 x$ in this case, but instead $u = -P_1 x$ is the optimal control.

If we use the control $u = -P_3 x$, then the resulting state trajectory is given by

$$x(t) = e^{(A-P_3)t} x_0 = e^{-t} x_0$$

and the cost

$$\mathcal{J}(-P_3 x, x_0) = x_0^T P_3 x_0 - \lim_{t \to \infty} (x^T P_3 x) = x_0^T P_3 x_0.$$

The control $u = -P_1 x$ gives the state trajectory

$$x(t) = e^{(A-P_1)t} x_0 = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} x_0$$

and the cost

$$\mathcal{J}(-P_1 x, x_0) = x_0^T P_1 x_0 - \lim_{t \to \infty} (x^T P_1 x).$$

If we write the initial condition as

$$x_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then

$$\lim_{t\to\infty}(x^T P_1 x) = \lim_{t\to\infty} e^{-2t} x_1^2 = 0$$

and hence

$$\mathcal{J}(-P_1 x, x_0) = x_0^T P_1 x_0.$$

However, if $x_2 \neq 0$, then $x_0^T P_1 x_0 < x_0^T P_3 x_0$ so using P_1 for the control gives a smaller total cost, even if P_3 is the maximal solution to the ARE.

We can also note that in this case when using the optimal control, the resulting state trajectory is unbounded as $t \to \infty$ whenever $x_2 \neq 0$. It is the fact that $\lim_{t\to\infty}(x^T P_1 x) = 0$ despite x not converging to zero that allows us to use a smaller solution of the ARE.

The above example shows that without the observability assumption $u^* = -R^{-1}B^T P^+ x$ is not necessarily the optimal control. The key thing to note is that with the optimal control in the example, the resulting state behaviour is unstable. If we restricted ourselves to only those controls for which the state goes to zero, the zero endpoint case, then we would in fact find that it is the control given by the maximal solution P_3 that gives the optimal cost.

The free end point case is in fact always solved by a solution to the ARE, but not necessarily the maximal solution. This is discussed in for example [13]. It can be shown that the free end point case is solved using the smallest positive semidefinite solution of the ARE (see [2]). What we have seen above is that if $(A - \xi I, Q)$ is an observable pair, then the maximal solution P^+ is in fact the only positive semidefinite solution and it gives the optimal control in both the free end point and zero end point case (the two cases coincide).

It can also be shown that P^+ is the only solution to the ARE for which all eigenvalues of $A - BR^{-1}B^T P$ lie in the closed left half plane of $\mathbb{C}$ (see [2]). For any other solution to the ARE at least one eigenvalue of $A - BR^{-1}B^T P$ has positive real part, which means that the resulting state behaviour is necessarily unstable. In the example above we indeed saw that the solution given by P_1 is unstable.

The ARE itself can also be used to draw conclusions about stability due to its similarity with the Lyapunov equation. If P is a solution to the ARE and we set $A_P = A - BR^{-1}B^T P$, then the ARE implies that

$$A_P^T P + P A_P = -Q - PBR^{-1}B^T P.$$

This is a Lyapunov equation for the system $\dot{x} = A_P x$, so it can be used to discuss the stability of this system.

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