SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Topological Fundamental Groups

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2022 - No K14

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

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2022

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Spring 2022

Abstract

This paper will examine a topological invariant known as the topological fundamental group. The first main theorem discussed gives conditions for exactly when the topological fundamental group is discrete, and the second one gives an example of a topological fundamental group which fails to be a topological group.

Acknowledgements

I would like to thank my supervisor Gregory Arone for suggesting the topic, for many interesting discussions, and for all of his support and guidance.

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1 Introduction

The fundamental group is a well-known topological invariant defined as the group of homotopy equivalence classes of loops in the space with path-class concatenation as its operation. This paper will focus on an even finer invariant than the usual fundamental group which is obtained by endowing the fundamental group with some extra structure, the structure of a topological space. This invariant is called the topological fundamental group. The topology on the topological fundamental group is defined as the quotient topology of the compact-open topology on the set of all loops in the space, which is a topology commonly used for function spaces.

The purpose of this paper is to define the topological fundamental group and investigate some of its properties. In particular we will discuss a result from [1] which gives necessary and sufficient conditions for the topological fundamental group of a space to be discrete, and we will also discuss a result from [2] which gives an example of a relatively nice space (a locally path connected and connected metric space) where the group multiplication of its topological fundamental group fails to be continuous, which means that there are cases where the topological fundamental group fails to be a topological group.

2 Preliminary Notions

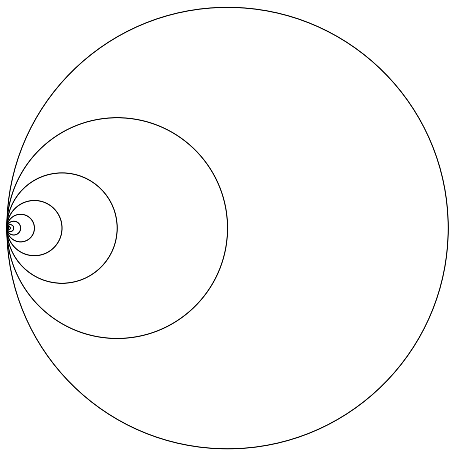
This section will review some important definitions, lemmas and notation that will be used throughout this paper.

Definition 2.1. Let I denote the unit interval $[0,1] \subset \mathbb{R}$ endowed with the subspace topology.

Definition 2.2. Consider the circles

$$\{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2\}$$

where $n \in \mathbb{Z}^+$. Define the **Hawaiian Earring**, H, as the union of all such circles endowed with the subspace topology from $\mathbb{R}^2$. Furthermore, let Y_n denote the union of the first n circles and let $p = (0,0)$. It will look something like this:



At first glance, one may think that the Hawaiian Earring looks just like a wedge sum of infinitely many circles, but in fact it is very different from it. The key difference is that the circles get smaller and smaller, which greatly alters the Hawaiian Earring's topological properties near the origin. For example we can have a loop in H that goes around infinitely many of the circles, since the fact that the circles get smaller and smaller will allow it to be continuous, in the wedge sum of circles a loop may only pass around finitely many of the circles. This makes the fundamental group of $\pi_1(H)$ drastically more complex than the fundamental group of the wedge sum of infinitely many circles.

Lemma 2.3. *There is a surjective homomorphism*

$$R : \pi_1(H,p) \to \prod_{n=1}^{\infty} \mathbb{Z}$$

and $\pi_1(H,p)$ *is uncountable.*

Proof. Let C_n be only the n:th circle in H, and let $r_n : H \to C_n$ be the retraction collapsing all circles in H except C_n to the origin. Each r_n induces a map $R_n : \pi_1(H,p) \to \pi_1(C_n,p) \cong \mathbb{Z}$ so the product of all such R_n will be a homomorphism $R : \pi_1(H,p) \to \prod_{n=1}^{\infty} \mathbb{Z}$.

To show R is surjective, let $x = (x_1, x_2, ...) \in \prod_{n=1}^{\infty}$. We will construct a loop $f : I \to H$ such that $R(f) = x$. To do this, let f wrap around C_n a total of x_n times in the interval $[1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$. It is clearly continuous on $[0, 1)$, and the fact that any neighborhood of the origin will contain all but finitely many of the circles ensures continuity at 1 as well. Since it wraps x_n times around C_n we will have $R_n(f) = x_n$, so $R(f) = x$ and R is surjective. It immediately follows that $\pi_1(H,p)$ is uncountable since the product of infinitely many copies of $\mathbb{Z}$ is. □

Lemma 2.4. *For each n we have that $\pi_1(Y_n,p) = F_n$ where F_n is the free group on n generators. Furthermore, there is an injection*

$$i_* : \pi_1(Y_n,p) \to \pi_1(H)$$

induced by the inclusion of Y_n into H.

Proof. That $\pi_1(Y_n,p) = F_n$ follows from the Seifert-Van Kampen theorem. Since there is a retraction $r : H \to Y_n$ given by collapsing all the circles in H except for the first n ones to the origin, the inclusion $i : Y_n \to H$ will induce an injective homomorphism of fundamental groups, because since we have $r \circ i = \mathrm{id}_{Y_n}$ we get by functoriality that $r_* \circ i_* = \mathrm{id}_{\pi_1(Y_n,p)}$ so i_* has a left-inverse, which means that it is injective. □

We now define a couple of important topologies on function spaces.

Definition 2.5. Let X and Y be two topological spaces and $C(X,Y)$ the set of all continuous maps from X to Y. If K is a compact subset of X and U is an open subset of Y, we denote the set of all $f \in C(X,Y)$ such that $f(K) \subset U$ by $V(K,U)$. Then the collection of all $V(K,U)$ form a subbasis for the **compact-open topology** on $C(X,Y)$.

Definition 2.6. If Y is a metric space and X a compact space, define a metric on $C(X,Y)$ such that the distance between f and g is given by

$$\sup_{x \in X} d_Y(f(x), g(x)).$$

The topology on $C(X,Y)$ generated by this metric will be called the **topology of uniform convergence**.

The following lemma will be used in section 4.

Lemma 2.7. *If Y is a metric space and X is compact, the compact open topology on $C(X,Y)$ is equivalent to the topology of uniform convergence.*

Proof. We want to show that for every function $f \in C(X,Y)$ and for every open ball around f in the topology of uniform convergence there exists a basic set in the compact-open topology containing f which is contained in the ball, and that for every subbasic open neighborhood of f in the compact-open topology there exists an open ball around f in the topology of uniform convergence contained in the neighborhood.

To begin, let $B_\varepsilon(f)$ denote a ball around $f \in C(X,Y)$ of radius ε in the topology of uniform convergence. The balls $B_{\varepsilon/3}(f(x))$ for each $x \in X$ form an open cover of $f(X)$, since f is continuous $f(X)$ is compact, so $f(X)$ has a finite subcover of sets of the form $B_{\varepsilon/3}(f(x_n))$. Now let $K_n \subset X$ be the closure the preimage of $B_{\varepsilon/3}(f(x_n))$. It is then clear that each K_n is compact since they are closed subsets of a compact space, and they will also cover X. Furthermore, we have that $f(K_n) \subset B_{\varepsilon/2}(f(x_n)) = U_n$. So each subbasic open set of the form $V(K_n, U_n)$ will contain f, and thus so will their intersection which we will denote by A. Now suppose $g \in A$. Then for any $x \in K_n$ we have by the triangle inequality that

$$d_Y(f(x), g(x)) \leq d_Y(f(x), f(x_n)) + d_Y(f(x_n), g(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since $g(K_n) \subset U_n$ and $f(K_n) \subset U_n$ and U_n is a ball of radius $\frac{\varepsilon}{2}$. Since this holds for all x we have that $g \in B_\varepsilon(f)$, and therefore $A \subset B_\varepsilon(f)$.

Now let $V(K,U)$ be a subbasic open set in the compact-open topology. Let $f \in V(K,U)$. Since $f(K)$ is compact and contained in U, its minimum distance from $Y - U$ is $\varepsilon > 0$. Then if $d(f,g) < \frac{\varepsilon}{2}$ we have that $g(K) \subset U$ since $f(K) \subset U$, so $V(K,U)$ contains an open ball around f in the topology of uniform convergence. □

Now consider some topological space X, and let $C_x(X)$ denote the space of continuous loops with base point x endowed with the compact-open topology. Recall that the fundamental group $\pi_1(X,x)$ is defined as path homotopy classes of such loops with path class concatenation as its group operation. Therefore, we will make the following definition.

Definition 2.8. If q denotes the map sending every element in $C_x(X)$ to its homotopy class in $\pi_1(X,x)$, then q is clearly surjective, so we can equip $\pi_1(X,x)$ with the quotient topology induced by q. We call this space the **topological fundamental group** of X with base point x, and we denote it by $\pi_1^{\text{top}}(X,x)$.

We will also discuss topological groups, which we define as follows.

Definition 2.9. Let G be a group endowed with a topology. If both the group operation which is a map from the product space $G \times G$ to G and the inversion map from G to itself are continuous, we say that G is a **topological group**. If we replace the condition that the group operation is continuous with a weaker condition, namely that we obtain a continuous map $G \to G$ by fixing either one of the arguments of the group operation and letting the other one vary, we get the definition of a **quasitopological group**.

Contrary to what one may believe from its name, the topological fundamental group is not always a topological group (although it is always a quasitopological group, as noted in [3]), we will see an example of a topological fundamental group that is not a topological group in section 4.

Example 2.10. The group $\mathbb{R}$ under usual addition endowed with the standard metric topology is a topological group, since both $x \mapsto -x$ and $(x, y) \mapsto x + y$ are continuous as functions from $\mathbb{R} \to \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, respectively.

Example 2.11. Any group G with the discrete topology is a topological group, since then both the operation and the inversion are trivially continuous.

Definition 2.12. Suppose G_k is a sequence of groups and that we have homomorphisms $f_k : G_{k+1} \to G_k$ for each $k \geq 0$, so we have a diagram

$$... \xrightarrow{f_3} G_3 \xrightarrow{f_2} G_2 \xrightarrow{f_1} G_1 \xrightarrow{f_0} G_0.$$

Then we define the **inverse limit** $\varprojlim G_k$ as the set of points $(x_0, x_1, ...) \in \prod_{k=0}^{\infty} G_k$ which satisfy $f_k(x_{k+1}) = x_k$ for all k. It is clear that $\varprojlim G_k$ is a group since G_k is a group and f_k is a homomorphism for all k.

Viewing the groups as discrete topological spaces, the inverse limit has a natural topology as a subspace of the infinite product $\prod_{k=0}^{\infty} G_k$. The topology on an infinite product of discrete spaces is not discrete, but it is Hausdorff, and a subspace of a Hausdorff space is always Hausdorff, therefore the inverse limit is a not necessarily discrete Hausdorff space.

If we consider the groups $\pi_1(Y_n, p) = F_n$ together with the maps induced by the retractions $Y_{n+1} \to Y_n$ which will map the generator corresponding to the innermost circle to the identity and fix the others, we can define $\lim_{\longleftarrow} F_n$. Furthermore, for each F_n we have a map $r_n : \pi_1(H, p) \to F_n$ induced by the retraction $H \to F_n$. Then the product of the maps r_n discussed above gives a continuous homomorphism $r : \pi_1^{\text{top}}(H, p) \to \varprojlim F_n$. The following result about this homomorphism is non-trivial, and a proof can be found in [4].

Theorem 2.13. *The homomorphism* $r : \pi_1^{top}(H, p) \to \varprojlim F_n$ *is injective.*

The following definition will be important in section 3.

Definition 2.14. Let X be a topological space and $x \in X$. A neighborhood of x is called **relatively inessential** if the homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is trivial. If every $x \in X$ has such a neigborhood we say that X is **semilocally simply connected**.

It is clear from the definition that every locally simply connected space is also semilocally simply connected, the converse is not true, however. We will illustrate the difference between the two with the following example which is briefly discussed in [5].

Example 2.15. Let H be the hawaiian earring as defined above and consider the cone $CH = (H \times I)/(H \times \{0\})$. Now we take $x = ((0,0),1)$. Then any small open neighborhood of x will contain non-trivial loops (since every such neighborhood will contain an infinite number of cylinders formed from the C_ns), so CH is not locally simply connected. However, since CH is a cone, which in particular means it is contractible, the inclusion of this loop into $\pi_1(CH, x) = 0$ will be trivial, so CH is in fact semilocally simply connected.

We also state the following standard result from [6] that will be used in section 3.

Lemma 2.16 (The Lebesgue number lemma)**.** *If (X, d) is a compact metric space and we are given an open cover of X there exists a number $\lambda > 0$ (called a **Lebesgue number** of the cover) such that every subset of X with diameter less than λ is contained in some member of the cover.*

3 Discreteness

In this section we will explore that under some conditions on X, the space $\pi_1^{\text{top}}(X, x)$ is discrete (and as such is a topological group, even if it is not a very interesting example). We begin by stating the following theorem from [1].

Theorem 3.1. *Let X be a locally path connected topologcial space, then we have that $\pi_1^{top}(X, x)$ is discrete for every point $x \in X$ if and only if X is semilocally simply connected.*

Recall that X being semilocally simply connected means that every $x \in X$ has a relatively inessential neighborhood. In what follows, whenever f is a path $[f]$ will denote the set of all paths that are path-homotopic to f. The following two lemmas are from [1].

Lemma 3.2. *Let (X, x) be a pointed topological space. If the singleton $\{[c_x]\}$, where c_x denotes the constant map, is open in $\pi_1^{top}(X, x)$ then x has a relatively inessential neighborhood in X.*

Proof. The quotient map $q : C_x(X) \to \pi_1^{\text{top}}(X, x)$ is continuous and by assumption $\{[c_x]\} \subset \pi_1^{\text{top}}(X, x)$ is open, so $q^{-1}(\{[c_x]\}) = [c_x]$ is open in $C_x(X)$. Therefore c_x has a basic open neighborhood

$$c_x \in V = \bigcap_{n=1}^{N} V(K_n, U_n) \subset [c_x] \subset C_x(X)$$

where each $K_n \subset I$ is compact and each $U_n \subset X$ is open, so that $V(K_n, U_n)$ is a subbasic open set for the compact-open topology on $C_x(X)$ as defined in section 2. The goal is to show that

$$U = \bigcap_{n=1}^{N} U_n$$

is a relatively inessential neighborhood of x. Since U is the intersection of finitely many open sets it is clearly open in X, and by the definition of the compact-open topology each $f \in V(K_n, U_n)$ must satisfy $f(K_n) \subset U_n$, and since c_x is constantly x, we must have $x \in U_n$ for all n, so $x \in U$. Finally, let f be an arbitrary loop in U based at x. Then for each $1 \leq n \leq N$ we have

$$f(K_n) \subset U \subset U_n$$

which means $f \in V(K_n, U_n)$ and thus $f \in [c_x]$ so $[f] = [c_x]$ is trivial in $\pi_1(X, x)$, in other words the inclusion from $\pi_1(U, x)$ to $\pi_1(X, x)$ is trivial and therefore U is a relatively inessential neighborhood of x. □

Lemma 3.3. *Let (X, x) be a pointed topological space and let $f \in C_x(X)$. If X is locally path connected and semilocally simply connected then $\{[f]\}$ is open in $\pi_1^{top}(X, x)$.*

Proof. By the definition of the quotient topology, $\{[f]\}$ being open in $\pi_1^{\text{top}}(X, x)$ is equivalent to $q^{-1}(\{[f]\}) = [f]$ being open in $C_x(X)$. Let $g \in [f]$. We want to find a neighborhood of g contained in $[f]$. For each $t \in I$ let U_t denote a path connected relatively inessential neighborhood of $g(t)$ in X which exists by our assumptions. Then the collection of sets of the form $g^{-1}(U_t)$ for all $t \in I$ form an open cover of I. Let $\lambda > 0$ be a Lebesgue number for this cover. We choose an $N \in \mathbb{N}$ such that $1/N < \lambda$. For each $1 \leq n \leq N$ let

$$I_n = \left[\frac{n-1}{N}, \frac{n}{N}\right] \subset I$$

and then reindex the U_ts so that $g(I_n) \subset U_n$ for each $1 \leq n \leq N$. This is made possible by the Lebesgue number lemma. Now for each $1 \leq n \leq N$ let W_n denote the path component of $U_n \cap U_{n+1}$ containing $g(n/N)$ so we have

$$g\left(\frac{n}{N}\right) \in W_n \subset U_n \cap U_{n+1} \subset X.$$

No consider the basic open set

$$V = \left(\bigcap_{n=1}^{N} V(I_n, U_n)\right) \cap \left(\bigcap_{n=1}^{N-1} V\left(\left\{\frac{n}{N}\right\}, W_n\right)\right) \subset C_x(X).$$

Since $g(I_n) \subset U_n$ for all n and W_n is defined to contain $g(n/N)$, we have $g \in V$. We want to show that $V \subset [f]$. So let $h \in V$. As $[g] = [f]$, we just need to show that $[h] = [g]$ to conclude $h \in V$.

By the construction of V, we have

$$h(I_n) \in U_n \quad \text{for each } 1 \leq n \leq N \text{ and}$$

$$h\left(\frac{n}{N}\right) \in W_n \quad \text{for each } 1 \leq n \leq N-1.$$

For each $1 \leq n \leq N-1$ we let $\gamma_n : I \to W_n$ be a continuous path starting at $h(n/N)$ and ending at $g(n/N)$. Such a path exists since W_n is path-connected. Let $\gamma_0 = \gamma N = c_x$. For each n let $s_n : I \to I_n$ be defined by

$$s(t) = \frac{1}{N}t + \frac{n-1}{N}$$

and let $g_n = g \circ s_n$ and $h_n = h \circ s_n$. So g_n and h_n will be the "parts" of g and h on I_n. Now for each n, let δ_n be a loop in U_n based at $g_n(0)$, defined by

$$\delta_n = g_n * \gamma_n^{-1} * h_n^{-1} * \gamma_{n-1}.$$

Since U_n is path-connected and relatively inessential, δ_n must be path homotopic to the constant path, so g_n is path homotopic to $\gamma_{n-1}^{-1} * h_n * \gamma_n$. This means that

$$\begin{aligned} [h] &= [h_1 * \ldots * h_N] \\ &= [\gamma_0^{-1} * h_1 * \gamma_1 * \gamma_1^{-1} * h_2 * \ldots * \gamma_{N-1}^{-1} * h_n * \gamma_N] \\ &= [g_1 * \ldots * g_N] \\ &= [g]. \end{aligned}$$

So $h \in V \subset [f]$, since for every $g \in [f]$ we have a neighborhood of g contained in $[f]$, we have that $[f]$ is open. □

Proof of theorem 3.1. With these two lemmas, we can finally prove theorem 3.1.

First we assume that $\pi_1^{\text{top}}(X, x)$ is discrete for all $x \in X$. Then clearly $\{[c_x]\}$ is open in $\pi_1^{\text{top}}(X, x)$. So by lemma 3.2 x has a relatively inessential neighborhood. Since this holds for all x, we have that X is semilocally simply connected.

For the other direction, assume X is semilocally simply connected. Singletons in $\pi_1^{\text{top}}(X, x)$ are open by lemma 3.3 which means $\pi_1^{\text{top}}(X, x)$ is discrete for all x. This concludes the proof.

□

4 The Hawaiian Earring

In this section we will discuss further the topological fundamental group of the Hawaiian Earring H as defined in 2.2. Throughout this section, p will denote the point $(0,0) \in H$. As before, $C_p(H)$ will denote the set of loops in H based at p endowed with the compact-open topology. Furthermore, Y_n will denote the union of the first n circles in H.

Note that H is not semilocally simply connected, so the main theorem of section 3 does not apply here. In fact, the topology on $\pi_1^{\text{top}}(H,p)$ doesn't at all behave nicely with the group operations, and it turns out that the group multiplication in $\pi_1^{\text{top}}(H,p)$ is discontinuous, so $\pi_1^{\text{top}}(H,p)$ is in fact not a topological group, which is the main theorem of this section.

As in [2], we will start by making the following definition

Definition 4.1. Let $q_n = (2/n, 0) \in C_n$. The **oscillation number** $O_n : C_p(H) \to \mathbb{N} \cup \{\infty\}$ will be defined as the maximum number m such that there exists a set $\{t_0, t_1, t_2, ..., t_{2m}\}$ such that $0 = t_0 < t_1 < t_2 < ... < t_{2m} = 1$ with $f(t_{2i}) = p$ and $f(t_{2i+1}) = q_n$ where $f \in C_p(X)$.

The oscillation number $O_n(f)$ thus represents the number of times that f loops around the n:th circle in H. The following is an important property of it.

Lemma 4.2. *Let $f \in C_p(H)$, then $O_n(f)$ is finite.*

Proof. Let $U = H - \{q_n\}$ and $V = H - \{p\}$. The sets $f^{-1}(U)$ and $f^{-1}(V)$ form an open cover of I. By the Lebesgue number lemma, we can subdivide I into finitely many subintervals with $0 = x_0 < x_1 < ...x_k = 1$ such that for each i the interval $[x_{i-1}, x_i]$ is contained in either $f^{-1}(U)$ or $f^{-1}(V)$, then if $0 = t_0 < t_1 < t_2 < ... < t_{2m} = 1$ such that $f(t_{2i}) = p$ and $f(t_{2i+1}) = q_n$, no two t_j, t_{j+1} can belong to the same interval, so $2m \leq k+1$, which proves the claim. □

This means that a loop in $C_p(H)$ can't go around any given circle infinitely many times (although it may still go around infinitely many *different* circles).

The following standard lemma will be used in the proof of lemma 4.4.

Lemma 4.3. *If $f_k \to f$ uniformly and $a_k \to a$ then $f_k(a_k) \to f(a)$.*

Proof. We know by the assumption of uniform convergence that for each $\varepsilon > 0$ there exists some natural number N such that $k > N$ implies $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ for all x, and there exists some natural number M and $\delta > 0$ such that $k > M$ implies $|a_k - a| < \delta$ which in turn implies $|f(a_k) - f(a)| < \frac{\varepsilon}{2}$, since $a_k \to a$ and f is continuous. But then if $k > \max(N, M)$ we have by the triangle inequality that

$$|f_k(a_k) - f(a)| \leq |f_k(a_k) - f(a_k)| + |f(a_k) - f(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since f is continuous, therefore $f_k(a_k) \to f(a)$. □

The next three lemmas, based on remark 1, 2 and 3 of [2], will be very important in the proof of the main theorem of this section.

Lemma 4.4. *If $f_k \to f$ uniformly in $C_p(H)$ and $O_n(f_k) \geq m$, then $O_n(f) \geq m$.*

Proof. That $O_n(f_k) \geq m$ means that for each k we have set of the form $\{t_0^k, t_1^k, t_2^k, ..., t_{2m}^k\}$ where $t_0^k = 0$, $t_{2m}^k = 1$, $f_k(t_{2i}^k) = p$ and $f_k(t_{2i+1}^k) = q_n$. Also, we have $t_0^k < t_1^k < ... < t_{2m}^k$. We construct the set $\{0, t_1, t_2, ..., t_{2m-1}, 1\}$ as follows. Take t_1 to be the limit of a convergent subsequence $t_1^{m_k}$ which exists by compactness, and taking t_2 to be the limit of a convergent subsequence $t_2^{m_{k_l}}$, and so on. Since $f_k \to f$ uniformly and by the previous lemma we will have that $f_k(t_1^{m_k}) \to f(t_1)$ and so on for each t_i, so $f(t_{2i}) = p$ and $f(t_{2i+1}) = q_n$. Note as well that $t_i < t_{i+1}$ for each i, we can't have $t_i = t_{i+1}$ since $f(t_i) \neq f(t_{i+1})$ and f is continuous. This shows that $O_n(f) \geq m$. □

That g corresponds to a maximally reduced finite word in the free group on n generators means that if $x_1, x_2, ..., x_n$ are the generators of F_n (which is the fundamental group of Y_n), and we have a finite word in F_n with k letters, we subdivide $[0, 1]$ into k equal parts, one for each letter x_i or x_i^{-1}, and for each part we let g travel once around the i:th circle clockwise if that element is an x_i, and counterclockwise if it is an x_i^{-1}. So if the word is $x_1x_3x_1^{-1}$, for instance, g will go around the first circle clockwise on $[0, \frac{1}{3}]$, go around the third circle once clockwise on $[\frac{1}{3}, \frac{2}{3}]$ and go around the first circle once counterclockwise on $[\frac{2}{3}, 1]$.

Lemma 4.5. *Let f and g be in the same path component of $C_p(H)$ and let the image of g be contained in some Y_n. Then if g corresponds to a maximally reduced finite word w in the free group on n generators, we have $O_n(f) \geq O_n(g)$.*

Proof. Let $R_m : H \to Y_m$ denote the retraction collapsing all the circles not in Y_m to p and let $f_1 = R_m(f)$. Then $O_n(f_1) = 0$ if $n > m$ and $O_n(f_1) = O_n(f)$ if $n \leq m$, so we always have $O_n(f) \geq O_n(f_1)$.

Now let U be a contractible open subspace of Y_m containing p. And let $J_1, J_2, ...$ be the open interval components of $f_1^{-1}(Y_m - \{p\})$. Note that $J_i \subset (0, 1)$ since $f_1(0) = f_1(1) = p$. Then $f_1^{-1}(U), J_1, J_2, ...$ is an open cover of $[0, 1]$ which is compact, and therefore the open cover has a finite subcover so all but finitely many of the J_is are contained in $f_1^{-1}(U)$. And since the value of f_1 will be p at the endpoints of $\overline{J_i}$, f_1 must be nullhomotopic on all of the $\overline{J_i}$s that are contained in $f_1^{-1}(U)$.

So f_1 is path-homotopic to f_2 which we define by replacing f_1 with the constant function on each J_i that is contained in $f_1^{-1}(U)$. The gluing lemma for locally finite closed covers ensures that f_2 is continuous, since $\overline{f_1^{-1}(U)}, \overline{J_1}, \overline{J_2}, ...$ is a locally finite closed cover of $[0, 1]$ and since the value of f_1 will be p at the endpoints of $\overline{J_i}$.

Furthermore, $O_n(f_1) \geq O_n(f_2)$. Then $[f_2]$ can be interpreted as a finite word v in $\pi_1(Y_m, p)$. And as noted before $[f_1] = [f_2]$. Note that for a path corresponding to a word, the oscillation number will just be the number of occurrences of the generator corresponding to that particular circle. We now know that $[g]$ and $[f_2]$ represents the same element in the free group F_m and that $[g]$ represents a reduced word, so $[g]$ is obtained from $[f_2]$ by eliminating redundancies. This can only reduce the oscillation number. Therefore we have found that $O_n(f) \geq O_n(f_1) \geq O_n(f_2) \geq O_n(g)$. □

Lemma 4.6. *The path components of $C_p(H)$ are closed subspaces of $C_p(H)$.*

Proof. By Theorem 2.13 there is a continuous injection from $\pi_1^{\text{top}}(H,p)$ to $\lim_{\leftarrow} \pi_1(Y_n, p)$, and as noted in [3], $\lim_{\leftarrow} \pi_1(Y_n, p)$ is Hausdorff It follows that $\pi_1^{\text{top}}(H,p)$ is Hausdorff and in particular that singletons in $\pi_1^{\text{top}}(H,p)$ are closed. Then by the definition of the quotient topology we have that the path components of $C_p(H)$, which are the preimages of singletons in $\pi_1^{\text{top}}(H,p)$, are closed. □

The following theorem and its proof is based on a part of theorem 1 in [2].

Theorem 4.7. *The standard multiplication $M : \pi_1^{top}(H,p) \times \pi_1^{top}(H,p) \to \pi_1^{top}(H,p)$ defined by path class concatenation is discontinuous and therefore $\pi_1^{top}(H,p)$ fails to be a topological group.*

Proof. Let $x_n \in C_p(H)$ orbit C_n once counterclockwise. Let $c_p \in C_p(H)$ denote the constant loop. For integers $n \geq 2$ and $k \geq 2$ let $a(n,k) \in C_p(H)$ be a loop corresponding to the finite word $(x_n x_k x_n^{-1} x_k^{-1})^{k+n}$ and let $w(n,k) \in C_p(H)$ be a loop corresponding to the finite word $(x_1 x_k x_1^{-1} x_k^{-1})^n$. Furthermore, let $F \subset \pi_1^{\text{top}}(H,p) \times \pi_1^{\text{top}}(H,p)$ be the set of all ordered pairs $([a(n,k)], [w(n,k)])$. To show that M is discontinuous, we will exhibit a closed set $A \in \pi_1^{\text{top}}(H,p)$ such that $M^{-1}(A)$ is not closed in $\pi_1^{\text{top}}(H,p) \times \pi_1^{\text{top}}(H,p)$.

Consider the set $A \subset \pi_1^{\text{top}}(H,p)$ such that each element of A is of the form $[a(n,k)] * [w(n,k)]$.

Firstly, notice that an element in A will be of the form $[(x_n x_k x_n^{-1} x_k^{-1})^{k+n}] * [(x_1 x_k x_1^{-1} x_k^{-1})^n]$. None of these are trivial (since by lemma 2.4 a non-zero element in the fundamental group of a wedge sum of circles corresponds to a non-zero element in the fundamental group of H), and when we take their product nothing will cancel out since the first one ends with x_k and the second one starts with x_1. Therefore the product is not trivial, so $[c_p] \notin A$.

Since $[c_p] \notin A$ we have that $([c_p], [c_p]) \notin M^{-1}(A)$. Note that $F \subset M^{-1}(A)$. We will first show that $M^{-1}(A)$ is not closed.

Suppose $[c_p] \in U$ and U is open in $\pi_1^{\text{top}}(H,p)$. Then $V = q^{-1}(U)$ is open in $C_p(H)$ where q denotes the standard quotient map $C_p(H) \to \pi_1^{\text{top}}(H,p)$. Since $c_p \in V$, there exists N and K such that $n \geq N$ and $k \geq K$ implies $a(n,k) \in V$, since by lemma 2.7, V must contain a basic open ball with the uniform metric around c_p, if we choose n and k large enough, $a(n,k)$ will be contained in such a ball.

Note that $(x_1x_1^{-1})^N$ is path homotopic to c_p and hence contained in V. Also, $w(N,k) \to (x_1x_1^{-1})^N$ uniformly as $k \to \infty$. Thus there exists $K' > K$ such that if $k \geq K'$ then $w(N,k) \in V$ Therefore $([a(N,K')], [w(N,K')]) \in U \times U$.

We have shown that every open neighborhood of $([c_p],[c_p])$ contains a point in F, therefore $([c_p],[c_p])$ is a limit point of F. This means in particular that $([c_p],[c_p])$ is also a limit point of $M^{-1}(A)$ that is not contained in $M^{-1}(A)$, so $M^{-1}(A)$ is not closed in $\pi_1^{\text{top}}(H,p) \times \pi_1^{\text{top}}(H,p)$.

It remains to prove that A is closed in $\pi_1^{\text{top}}(H,p)$. This is equivalent to $q^{-1}(A)$ being closed in in $C_p(H)$. Suppose $f_m \to f \in C_p(H)$ uniformly and $f_m \in q^{-1}(A)$.

That f_m is in $q^{-1}(A)$ means that each f_m is homotopic to some $a(n,k)$ concatenated with some $w(n,k)$. So we can find sequences $\{n_m\}$ and $\{k_n\}$ such that $f_m \in [a(n_m,k_m)] * [w(n_m,k_m)]$.

Assume for contradiction that n_m is unbounded. Then for each integer K, there is a subsequence $\{n_{m_i}\}$ such that $n_{m_i} \geq K$ for each i. Also, since $f_m \to f$, the subsequence f_{m_i} also converges to f. Then we get from lemma 4.4 that $O_1(f) \geq K$ since $O_1(f_m) \geq O_1(w(n_m,k_m)) \geq n_m$ by lemma 4.5. Since this is true for any K we get that $O_1(f)$ is unbounded which contradicts lemma 4.2. So $\{n_m\}$ must be bounded.

Next, assume that $\{n_m\}$ is bounded but $\{k_m\}$ is not. Then by essentially the same argument (just looking at $a(n_m,k_m)$ instead of $w(n_m,k_m)$) we find that there is some integer N such that $O_N(f) > M$ for any integer K, which again contradicts lemma 4.2. So $\{k_m\}$ must be bounded.

Since both $\{n_m\}$ and $\{k_m\}$ are bounded, and since $q^{-1}(A)$ consists of a union of path components in $C_p(H)$, it follows from the pigeon hole principle that some subsequence of $\{f_m\}$ is contained in a single path component of $C_p(H)$. Then by lemma 4.6 we get that f is also in this path component, and in particular $f \in q^{-1}(A)$. Since $q^{-1}(A)$ contains all its limit points, it is closed, and by the definition of the quotient topology this implies that A is closed.

Since we have found a closed set whose preimage is not closed, the multiplication in $\pi_1^{\text{top}}(H,p)$ is discontinuous which means that it is not a topological group. $\square$

In a very similar fashion as the above proof, one can also show that the product of two quotient maps, in particular the map $q \times q : C_p(H) \times C_p(H) \to \pi_1^{\text{top}}(H_p) \times \pi_1^{\text{top}}(H,p)$, fails to be a quotient map. For more details on this, see the full proof of theorem 1 in [2].

5 References

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