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The internal language of sheaves and applications to algebraic geometry

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Abstract

This thesis has two purposes. The first one is to introduce sheaves on spaces and present the Kripke-Joyal semantics for the internal language of sheaves. We will do this by first laying the ground work for the internal language, i.e. defining unary and binary operators on the subobject classifier corresponding to logical connectives and quantifiers. Then we define a forcing relation that translates internal properties to external properties and give a recursive way to unwind internal statements to external statements. In the second part we illustrate some translations of common properties and internal proofs of results from algebraic geometry presented in [Ble21], an example being proving that a sheaf of modules of finite type is internally a finitely generated module. The main portion of this part is dedicated to $\Box$-operators and proving that for a geometric formula the $\Box$'d version is equivalent to the $\Box$-translated version.

Contents

1 Introduction

In this thesis we will be introducing the internal language of the category of sheaves and the Kripke-Joyal semantics for formulae in the internal language, using this to work through some of the internal translations of notions in algebraic geometry from [Ble21]. We have also included a (very) short introduction to category theory, since much of the ground work for the internal language relies on category theory (we are after all working in the category of sheaves).

The idea behind the internal language is that the category of sheaves behaves almost like a universe of sets. It turns out that with the right definitions of the meet, join, implication and quantifiers as maps on the collection of subsheaves and truth values as subsheaves we get morphisms to the subobject classifier that act like the the connectives and quantifiers of first order logic. Consequently, with the help of the Kripke-Joyal semantics, we are able to translate proofs concerning sets into proofs concerning sheaves.

Why algebraic geometry? Algebraic geometry is hard, but since it is mainly about schemes and other locally ringed spaces we by translating the structure sheaf into the internal language obtain a ring and those are much simpler to work with. Generally the point of having an internal language is to reduce some complicated structures to simpler ones, this is quite useful since we can reduce some complicated proofs in algebraic geometry to ones in commutative algebra. This frankly sounds too good to be true, so of course there is a caveat, namely that the internal language is intutionistic. We can thus not make use of the classically valid statements $\varphi \vee \neg\varphi$ (law of excluded middle), $\varphi \Leftrightarrow \neg\neg\varphi$ and axiom of choice. But even if we can only work constructively internally there is till quite a lot we can do, we will for example give an internal proof of the statement "if $(X, \mathcal{O}_X)$ is locally ringed and $\mathcal{I} \subseteq \mathcal{O}_X$, then $(V(\mathcal{I}, \mathcal{O}_{V(\mathcal{I})})$ is a locally ringed space".

A large part of section 4 is dedicated to modal operators (actually Lawvere-Tierney topologies) which are maps that slightly weaken formulas and can be thought of as "property holds locally"-operators. These turn out to be helpful in understanding how properties that holds at a stalk or locally spread to the surrounding space, we will for example prove that for an O_X-module of finite type being the zero-module at the stalk at x is equivalent to it being the zero-module on some open neighbourhood of x.

2 Preliminaries

In this section we will list some definitions and results that are good to have in mind when reading this thesis.

Definition 2.1 (category)**.** A category $\mathbf{C}$ consists of

- a collection of objects, $\mathrm{ob}(\mathbf{C})$;
- for each $A, B \in \mathrm{ob}(\mathbf{C})$ a collection of morphisms from A to B, $\mathrm{Hom}(A, B)$;
- for each $A, B, C \in \mathrm{ob}(\mathbf{C})$ a function

$$\begin{array}{ccc} \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) & \to & \mathrm{Hom}(A, C) \\ (g, f) & \mapsto & g \circ f \end{array}$$

  satisfying associativity;
- for each object A in $\mathbf{C}$ there is a morphism id_A such that for each $f : A \to B$ we have that $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

Definition 2.2. A category is small if the collection of objects and every collection of morphisms are sets and locally small if each of its collections is a set.

Definition 2.3 (opposite category)**.** Given a category $\mathbf{C}$ the opposite category, $\mathbf{C}^{\mathrm{op}}$, is defined by reversing all morphisms (this is also a category).

Definition 2.4 (terminal object)**.** An object in a category is a terminal object if there from every other object is a unique morphism to the object.

Remark 2.5. The terminal object is often denoted by 1, we too shall follow this convention.

Definition 2.6 (functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories, a functor F between $\mathbf{C}$ and $\mathbf{D}$ consists of

- a function that associates each object $A \in \mathrm{ob}(\mathbf{C})$ to an object $F(A) \in \mathrm{ob}(\mathbf{D})$;
- for each pair of objects A, B in $\mathbf{C}$, a function which associates each morphism $f \in \mathrm{Hom}(A, B)$ to a morphism $F(f) \in \mathrm{Hom}(F(A), F(B))$ for each $A, B \in \mathrm{ob}(\mathbf{C})$, such that for $f : A \to B$ and $g : B \to C$ we have that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{ob}(\mathbf{C})$.

Remark 2.7. We will also call a contravariant functorn (a functor but with reversed composition) a functor.

Definition 2.8 (natural transformation)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories and F and G be functors from $\mathbf{C}$ to $\mathbf{D}$. A natural transformation α between F and G is a family of maps $(\alpha_A)_{A \in \mathrm{ob}(\mathbf{C})}$ such that for each $f : A \to B$ the following diagram commutes.

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
{\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_B} \\
G(A) & \xrightarrow[G(f)]{} & G(B)
\end{array}
$$

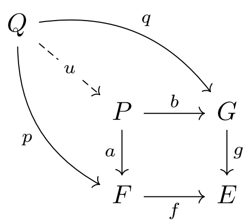
Definition 2.9 (pullback)**.** In a category $\mathbf{C}$ a pullback of two morphisms $f : F \to E$ and $g : G \to E$ is a pair of morphisms $a : P \to F$ and $b : P \to G$ such that the following diagram commutes

$$
\begin{array}{ccc}
P & \xrightarrow{b} & G \\
{\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
F & \xrightarrow[f]{} & E
\end{array}
$$

and for any other pair of morphisms p and q that makes this diagram

$$
\begin{array}{ccc}
Q & \xrightarrow{q} & G \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
F & \xrightarrow[f]{} & E
\end{array}
$$

commute there is a unique morphism $u : Q \to P$ which makes this diagram



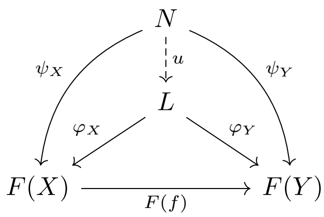
commute.

Remark 2.10. These are unique up to isomorphism by the Yoneda lemma 2.20.

Definition 2.11 (subobject)**.** A subobject of an object $F \in \mathrm{ob}(\mathbf{C})$ in a category $\mathbf{C}$ is an equivalence class of monics into F with equivalence of two such monomorphism $i : A \to F$ and $j : B \to F$ if there is isomorphism $h : A \to B$ such that $f = g \circ h$.

Definition 2.12 (Cone)**.** A cone on a functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$ is a pair (L, φ) such that $L \in \mathrm{ob}(\mathbf{C})$ and $\varphi = \{\varphi_X : L \to D(X)\}_{X \in \mathrm{ob}(\mathbf{J})}$.

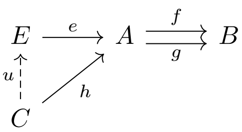
Definition 2.13 (limit)**.** Let $\mathbf{J}$ and $\mathbf{C}$ be categories and $D : \mathbf{J} \to \mathbf{C}$ be a functor, this is called a diagram of type $\mathbf{J}$ in $\mathbf{C}$. The limit of D is a cone (L, φ) such that there for each (N, ψ) is a unique $u : N \to L$ and



commutes for each $X, Y \in \mathrm{ob}(J)$ and $f \in \mathrm{Hom}(X, Y)$.

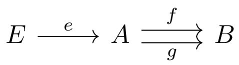
Definition 2.14 (finite limit)**.** Limit of a diagram of finite type.

Definition 2.15 (equalizer)**.** In any category **C**, given any two morphism $f, g : A \to B$ an equalizer of f and g (if it exists) is an object E and a morphism $e : E \to A$ such that $fe = ge$. Given any $h : C \to A$ with $fh = gh$, h factorizes uniquely through e, ie there is, for every h, a unique $u : C \to E$ such that



commutes.

Remark 2.16. For e, f and g as in the definition above the diagram

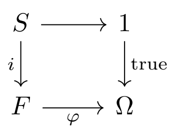


is called an equalizer diagram.

Lemma 2.17. *A category, C, that has all pullbacks and a terminal object has all finite limits.*

Proof. To prove this one first need to prove that having pullbacks and terminal object is equivalent to having finite products and equalizers, then that products and equalizers imply finite limits. For a proof we refer to Proposistion 5.23 [Awo10]. □

Definition 2.18 (subobject classifier)**.** A subobject classifier in a category **C** with finite limits is a monomorphism true : $1 \to \Omega$, such that for every monic $i : S \to F$ there is a unique morphism $\varphi : F \to \Omega$ making



is a pullback.

Remark 2.19. Since pullbacks are unique up to isomorphisms this implies that there is a bijection between subobjects of an object and morphisms φ as described in the definition.

Lemma 2.20 (Yoneda lemma)**.** *[Lei14, Thm. 4.2.1] In a locally small category* $\mathbf{C}$

$$Hom(Hom(-,F),X) \cong X(F)$$

naturally for $F \in ob(\mathbf{C})$ *and* $X : \mathbf{C} \to \textbf{\textit{Set}}$*.*

Remark 2.21. In particular, if α is a natural transformation from $\mathrm{Hom}(-,F)$ to X and $f : F \to G$ for some $G \in \mathrm{ob}(\mathbf{C})$, then $(X(f))(\alpha_F(\mathrm{id}_F) = \alpha_G(f)$ (cf [Lei14, p.97]), i.e. any natural transformation $\mathrm{Hom}(-,F)$ to X is determined by its value at id_F.

3 Sheaves and their internal language

3.1 Sheaves

Definition 3.1 (presheaf of sets)**.** For any category $\mathbf{C}$, a presheaf on $\mathbf{C}$ is a functor $\mathcal{F} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Remark 3.2. If $V \subseteq U \subseteq X$ are two open sets and $r \in \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(U,V)$, then for $x \in \mathcal{F}(U)$ we will denote $\mathcal{F}(r)(x)$ as $x|_V$.

Definition 3.3 (sheaf of sets)**.** Let, again, X be a topological space, a sheaf of sets on X is a presheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ such that for each open set $U \subseteq X$ and each open cover $\{U_i\}_{i \in I}$ of U

$$U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram, where for $t \in \mathcal{F}(U)$, $e(t) = \{t|_{U_i} |\ i \in I\}$ and for a family $t_i \in \mathcal{F}(U_i)$, $q(\{f_i\}) = \{f_i|_{(U_i \cap U_j)}\}$ and $p(\{f_i\}) = \{f_j|_{(U_i \cap U_j)}\}$.

Remark 3.4. The sheaf condition is equivalent to having the gluing property, i.e. if $U \subseteq X$ is open, $\{U_i\}_{i \in I}$ is an open cover of U and there is an $x_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$ then there is an $x \in \mathcal{F}(U)$ such that $x|_{U_i} = x_i$.

Example 3.5. We give some illustrative examples and non-examples of sheaves

(1) Let $\mathbb{R}$ with the usual topology be our space, then $F : \mathcal{O}(\mathbb{R})^{\mathrm{op}} \to \mathbf{Set}$ such that $F(U) = \mathbb{R}^U$ for any open $U \subseteq \mathbb{R}$ is a sheaf of sets. In fact F, with the usual addition and multiplication operators is a sheaf of rings (these we will talk more about later).

(2) Let R be a ring and $U \subseteq \mathrm{Spec}(R)$ be open, then $\mathcal{O}_{\mathrm{Spec}(R)}(U)$, the set of all regular functions on U, is a sheaf. Since these sets are rings $\mathcal{O}_{\mathrm{Spec}(R)}$ is also a sheaf of rings. We call this sheaf the structure sheaf of $\mathrm{Spec}(R)$.

(3) Let X be a scheme, then the function mapping $U \subseteq X$ to the ring of regular functions on U is a sheaf of rings.

(4) Let again $\mathbb{R}$ be our space and $U \subseteq \mathbb{R}$ be open, then letting $\mathcal{B}(U)$ be the set of bounded functions from $\mathbb{R}$ to $\mathbb{R}$ gives us a presheaf that is not a sheaf.

All sheaves of sets over a fixed topological space X with natural transformations as morphisms form a category, which by convention is denoted $\mathrm{Sh}(X)$. As with functors in general we can for sheaves define subsheaves, which are subfunctors of sheaves which themselves are sheaves, we will use $\mathrm{Sub}(\mathcal{F})$ to denote the set of subsheaves of a sheaf $\mathcal{F}$. This may seem a bit misleading as $\mathrm{Sub}(\mathcal{F})$ is usually reserved for the subobjects of $\mathcal{F}$ but we can justify it with the following proposition.

Proposition 3.6. *[LM92, Prop. II.3.3] Each subobject of $\mathcal{F} \in Sh(X)$ is isomorphic to some subsheaf of $\mathcal{F}$.*

Proof. Let $m : \mathcal{A} \to \mathcal{F}$ be monic, then the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{A} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} & \xrightarrow[m]{} & \mathcal{F} \end{array}$$

must be a pullback (cf. [Lei14, Lemma 5.1.32]). By §II.2.2 in [LM92] it must also be a pullback in the presheaf category $\mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$, consequently it is a pullback pointwise and $\mathcal{A}(U)$ for each open $U \in X$ is isomorphic to some subset $\mathcal{S}(U)$ of $\mathcal{F}(U)$, therefore isomorphic to the subsheaf $\mathcal{S} \in \mathrm{Sub}(\mathcal{F})$. □

The set $\mathrm{Sub}(\mathcal{F})$ has some interesting properties.

Proposition 3.7. *[LM92, §III.8] For any sheaf $\mathcal{F}$ in the category of sheaves over a topological space X and $\mathcal{A}, \mathcal{B} \in Sub(\mathcal{F})$, $\{\mathcal{A}_i\} \subset Sub(\mathcal{F})$ and $U \in \mathcal{O}(X)$ the following are also elements in $Sub(\mathcal{F})$.*

- $(\mathcal{A} \wedge \mathcal{B})(U) := \mathcal{A}(U) \cap \mathcal{B}(U)$
- $(\mathcal{A} \vee \mathcal{B})(U) := \{x \in \mathcal{F}(U) \mid$ *for some open cover of* U, $\{U_i\}_{i \in I}$, $x|_{U_i} \in \mathcal{A}(U_i) \vee x|_{U_i} \in \mathcal{B}(U_i)\}$
- $(\mathcal{A} \Rightarrow \mathcal{B})(U) := \{x \in \mathcal{F}(U) \mid$ *for all open* $V \subseteq U$ $x|_V \in \mathcal{A}(V) \Rightarrow x|_V \in \mathcal{B}(V)\}$
- $(\bigwedge_i \mathcal{A}_i)(U) = \bigcap_i \mathcal{A}_i(U)$
- $(\bigvee_i \mathcal{A}_i)(U) = \{x \in \mathcal{F}(U) | x|_{U_j} \in \mathcal{A}_i(U_j)$ *for some* $i \in I$ *and some open cover of* $U, \{U_j\}_{j \in J}\}$
- $\top(U) = \mathcal{F}(U)$

- $\bot(U) = \varnothing$

Proof. These are all subfunctors because the restriction morphisms of subfunctors agree with that of the functor, i.e. if $V \subseteq U \subseteq X$ and $x \in \mathcal{A}(U)$ then $x|_V \in \mathcal{A}(V)$. We will prove that these indeed are sheaves, starting with conjunction.

Let $U \in X$ be open and $\{U_i\}_{i\in I}$ be an open cover of U. Assume that we for each $i \in I$ have an $x_i \in (\mathcal{A} \wedge \mathcal{B})(U_i)$ such that $x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j}$ for $i, j \in I$, then by definition $x_i \in \mathcal{A}(U_i)$ and $x_i \in \mathcal{A}(U_i)$ for all $i \in I$. Since both $\mathcal{A}$ and $\mathcal{B}$ are sheaves there must be some x that is in both $\mathcal{A}(U)$ and $\mathcal{B}(U)$ which satisfies $x|_{U_i} = x_i$.

For disjunction the premise is the same. By definition either x_i is in $\mathcal{A}(U_i)$ or in $\mathcal{B}(U_i)$, let $U_\mathcal{A}$ denote the union of all U_i's such that the corresponding x_i is in $\mathcal{A}(U_i)$ and $U_\mathcal{B}$ be the union of all U_i's such that x_i is in $\mathcal{B}(U_i)$. Since A is a sheaf there must be some $x_\mathcal{A} \in \mathcal{A}(U_\mathcal{A})$ such that $x_\mathcal{A}|_{U_i} = x_i$ for all $U_i \subseteq U_\mathcal{A}$, the analogous statment is true for $\mathcal{B}$. But then $x_\mathcal{A}|_{U_\mathcal{A}\cap U_\mathcal{B}} = x_\mathcal{B}|_{U_\mathcal{A}\cap U_\mathcal{B}}$ and by definition there must be some $x \in (\mathcal{A} \vee \mathcal{B})(U)$ such that $x|_{U_k} = x_k$ for $k = \mathcal{A}, \mathcal{B}$.

We will also use the same premise for implication, for $(\mathcal{A} \Rightarrow \mathcal{B})(U)$ to define a sheaf we need to prove that $x|_V \in \mathcal{A}(V) \Rightarrow x|_V \in \mathcal{B}(V)$ for the $x \in \mathcal{F}(U)$ that we get from the sheaf condition and the premise. By the premise we have that for any open $V \subseteq U$, $x_i|_{V\cap U_i} \in \mathcal{A}(V \cap U_i) \Rightarrow x_i|_{V\cap U_i} \in \mathcal{B}(V \cap U_i)$. Suppose $x_i|_{V\cap U_i} \in \mathcal{A}(V \cap U_i)$ for all $i \in I$, then because $x_i|_{U_i\cap U_j\cap V} = x_j|_{U_i\cap U_j\cap V}$ and because $\mathcal{A}$ is a sheaf there is some $x_V \in \mathcal{A}(V)$ such that $x_V|_{U_i\cap V} = x_i|_{U_i\cap V}$, this x_V must also be in $\mathcal{B}(V)$, since each $x_i|_{V\cap U_i}$ must also be in $\mathcal{B}(U_i \cap V)$, by the premise and the fact that $\mathcal{B}$ also is a sheaf. But $x_i|_{U_i\cap V} = x|_{U_i\cap V}$, thus $x_V = x|_V$ and $x|_V \in (\mathcal{A} \Rightarrow \mathcal{B})(V)$.

$\top$ is a sheaf by assumption and $\bot$ is a sheaf since the condition holds vacuously. We have omitted the proofs of arbitrary conjunctions and disjunction on purpose, they are quite similar to the binary versions. □

Now that we have all connectives of propositional logic, what about quantifiers? And indeed we can "use" quantifiers to define subsheaves, in fact we have the following proposition.

Proposition 3.8. *[LM92, §III.8] Any natural transformation $\varphi : \mathcal{E} \to \mathcal{F}$ between sheaves $\mathcal{E}, \mathcal{F} \in Sh(X)$ induces a functor $\varphi^{-1} : Sub(\mathcal{F}) \to Sub(\mathcal{E})$ and we have that for $\mathcal{A} \in Sub(\mathcal{E})$ both*

$$\exists_\varphi(\mathcal{A})(U) = \\ \{y \in \mathcal{F}(U) \mid \exists x \in \mathcal{A}(U_i)(\varphi_{U_i}(x) = y|_{U_i}) \text{ for some open cover } \{U_i\}_{i\in I} \text{ of } U\}$$

and

$$\forall_\varphi(\mathcal{A})(U) = \{y \in \mathcal{F}(U) \mid \forall V \subseteq U \ ((\varphi_V^{-1}(y|_v) \subseteq \mathcal{A}(V))\}$$

define subsheaves of $\mathcal{F}$.

Proof. Again these are all subfunctors because the restriction morphisms of subfunctors agree with that of the functor, so we only need to prove that they satisfy the sheaf condition.

Suppose $U \subseteq X$ is open, $\{U_i\}_{i \in I}$ is an open cover of U and that there is an $y_i \in \exists_\varphi(\mathcal{A})(U_i)$ for each $i \in I$ such that $y_i|_{U_i \cap U_J} = y_j|_{U_i \cap U_J}$. Then by definition there is an open cover of each $U_i = \cup_{n \in N_i} V_n$ such that there is an $x_n \in \mathcal{A}(V_n)$ and $\varphi_{V_n}(x_n) = y_i|_{V_n}$. But $\mathcal{F}$ is a sheaf, consequently there is some $y \in \mathcal{F}(U)$ such that $y|_{U_i} = y_i$ and thus $y_i|_{V_n} = y|_{V_n}$ and since $\{V_n\}_{n \in \cup_{i \in I} N_i}$ is an open cover of U it follows that $y \in \exists_\varphi(\mathcal{A})(U)$.

We have the same premise for the universal quantifier but with each $y_i \in \forall_\varphi(\mathcal{A})(U_i)$. Then by definition we have that for each open V, $\varphi^{-1}_{V \cap U_i}(y_i|_{V \cap U_i}) \subseteq \mathcal{A}(V \cap U_i)$. Since $\mathcal{F}$ is a sheaf there is a $y \in \mathcal{F}(U)$ such that $y|_{U_i} = y_i$. Suppose $\varphi^{-1}_V(y|_V) \neq \varnothing$, then for each $x \in \varphi^{-1}_V(y|_V)$, $x|_{V \cap U_i} \in \varphi^{-1}_{V \cap U_i}(y|_{V \cap U_i})$. But by assumption $\varphi^{-1}_{V \cap U_i}(y|_{V \cap U_i}) \subseteq \mathcal{A}(V \cap U_i)$, consequently $x \in \varphi^{-1}_V(y) \subseteq \mathcal{A}(V)$ since $\mathcal{A}$ is a sheaf. □

Proposition 3.9. *Sh(X) has all pullbacks and a terminal object.*

Proof. Since the terminal object also must be a terminal object pointwise (i.e. in **Set**) we can take the sheaf defined such that $1(U) = \{*\}$ for all open $U \subseteq X$.

For pullbacks, suppose we have $\mathcal{F}, \mathcal{G}, \mathcal{E} \in \mathrm{Sh}(X)$ and $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{E} \to \mathcal{G}$. For something to be a pullback in $\mathrm{Sh}(X)$, it has to be a pullback in the preasheaf category, and for this to be true it has to be a pullback pointwise. Thus we know that $(\mathcal{F} \times_\mathcal{G} \mathcal{E})(U) \cong \mathcal{F}(U) \times_{\mathcal{G}(U)} \mathcal{E}(U)$ for all open $U \subseteq X$, this is a sheaf. □

Proposition 3.10. *Sh(X) has all finite limits.*

Proof. A consequence of the previous proposition and 2.17. □

Proposition 3.11. *[LM92, §II.8] The sheaf $\Omega(U) = \{V \in \mathcal{O}(X) |\ V \subseteq U\}$ with* true $: 1 \to \Omega$ *defined by $1 \mapsto U$ is the subobject classifier in Sh(X).*

This proposition follows from the proof of the following proposition.

Proposition 3.12. *[LM92, Prop. I.3.1] There is a natural isomorphism*

$$\theta_\mathcal{F} : \mathit{Sub}(\mathcal{F}) \cong \mathit{Hom}(\mathcal{F}, \Omega).$$

Proof. To prove the bijection we define two functions and show that they are mutually inverse. Let $\theta_\mathcal{F} : \mathrm{Sub}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}, \Omega)$ send the equivalence class of a monic $i : \mathcal{A} \to \mathcal{F}$ to $\varphi : \mathcal{F} \to \Omega$ defined such that ψ_U sends $x \mapsto \bigcup\{U_i \subseteq U |\ x|_{U_i} \in \mathcal{A}(U_i)\}$, this ψ makes the following diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & 1 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \text{true}} \\ \mathcal{F} & \xrightarrow[\psi]{} & \Omega \end{array}$$

a pullback. Let $\eta_{\mathcal{F}} : \text{Hom}(\mathcal{F}, \Omega) \to \text{Sub}(\mathcal{F})$ send $\varphi \in \text{Hom}(\mathcal{F}, \Omega)$ to the collection of j which makes this

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & 1 \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \text{true}} \\ \mathcal{F} & \xrightarrow[\varphi]{} & \Omega \end{array}$$

a pullback (such a collection must exist by 3.9). That the function $\eta_{\mathcal{F}}$ indeed maps to $\text{Sub}(\mathcal{F})$ may need some justification, we begin by proving that any such j which makes the diagram a pullback is monic, this follows from the fact that it must pointwise be a pullback

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & 1 \\ \big\downarrow{\scriptstyle j_U} & & \big\downarrow{\scriptstyle \text{true}} \\ \mathcal{F}(U) & \xrightarrow[\varphi_U]{} & \Omega(U) \end{array}$$

and that $\mathcal{B}(U)$ must be isomorphic to the equalizer $\mathcal{F}(U) \times_{\Omega(U)} 1$ (since pullbacks are unique up to isomorphism) which in turn is isomorphic to some subset of $\mathcal{F}(U)$, thus j_U is injective, consequently j must be monic (cf. proof of 4.1). That the collection of such j's is a subobject follows from pullbacks being unique up to isomorphism, i.e. if j and j' both make the diagram a pullback there is some isomorphism of their domains h such that $j = j' \circ h$. Now suppose we have some subobject i of $\mathcal{F}$, we know that $\theta_{\mathcal{F}}(i) = \psi$ (as defined before), but since i is the pullback of ψ along true $\eta_{\mathcal{F}}$ must map ψ to i (by i we of course mean the equivalence class).

For the opposite composition let $\varphi \in \text{Hom}(\mathcal{F}, \Omega)$, then $\eta_{\mathcal{F}}$ maps φ to the subobject j of $\mathcal{F}$, with subsheaf $\mathcal{F}(U) \cong \mathcal{F}(U) \times_{\Omega(U)} 1$. $\theta_{\mathcal{F}}$ maps j to ψ as defined above, we need to show that $\psi = \phi$. Let $x \in \mathcal{F}(U)$ for some open $U \subseteq X$ and assume $\varphi_U(x) = V$, then by assumption $V \in U$ and by naturality of φ the following diagram must commute

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \Omega(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \Omega(V) \end{array}$$

this implies that $x|_V \in \mathcal{S}(V)$. To prove that V is the largest such subset consider any open subset V' of U such that $V \subseteq V'$ if $x|_{V'} \in S(V')$ then $\varphi_{V'}(x|_{V'}) = V'$ but this would contradict the naturality of φ. Because $\mathcal{S}$ is a sheaf, we can conclude that $\varphi_U = \psi_U$ for all open $U \subseteq X$, thus $\varphi = \psi$. Consequently both $\theta_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$ must be bijections.

Now to show that our θ is natural assume that we have some $f : \mathcal{E} \to \mathcal{F}$, then

$$\begin{array}{ccccc} \mathcal{B} & \longrightarrow & \mathcal{A} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} & \longrightarrow & \Omega \end{array}$$

is a pullback (rectangle) and since f induces a morphism $f^{-1} : \mathrm{Sub}(\mathcal{F}) \to \mathrm{Sub}(\mathcal{E})$ we have that $\theta_{\mathcal{E}} \circ f^{-1} = f^{-1} \circ \theta_{\mathcal{F}}$. □

Now that we have a decent understanding of $\mathrm{Sh}(X)$ with subobjects and all, here comes the punchline.

Proposition 3.13. *[LM92] We can define binary operators on Ω such that these correspond to the logical connectives on subsheaves.*

Before proving this we need the following lemma.

Lemma 3.14. *$\mathbf{Set}^{\mathbf{C}^{op}}$ is locally small if $\mathbf{C}$ is a small category.*

Proof. Let $\mathcal{F}, \mathcal{G} \in \mathrm{ob}(\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}})$. We know that each morphism $\alpha \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is just a family of morphisms $\{\alpha_C\}_{C \in \mathrm{ob}(\mathbf{C})}$. Since each α_C is a function from $\mathcal{F}(C)$ to $\mathcal{G}(C)$ there is an injection from $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ to the set $\prod_{C \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}} \mathcal{G}(C)^{\mathcal{F}(C)}$, consequently $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ must be a set too. □

By this lemma $\mathrm{Sh}(X)$ must be locally small, now for the proposition.

Proof of 3.13. We will prove this for disjunction, the proof for the others are similar. Define $\wedge_{\mathcal{F}}$ as the operation which makes the following diagram commute

$$\begin{array}{ccc} \mathrm{Sub}(\mathcal{F}) \times \mathrm{Sub}(\mathcal{F}) & \xrightarrow{\ \cap\ } & \mathrm{Sub}(\mathcal{F}) \\ \downarrow \wr & & \\ \mathrm{Hom}(\mathcal{F}, \Omega) \times \mathrm{Hom}(\mathcal{F}, \Omega) & & \downarrow \wr \\ \downarrow \wr & & \\ \mathrm{Hom}(\mathcal{F}, \Omega \times \Omega) & \xrightarrow[\wedge_{\mathcal{F}}]{} & \mathrm{Hom}(\mathcal{F}, \Omega) \end{array}$$

(the isomorphisms are by 3.12).

Since the following diagram commutes for any $f : \mathcal{E} \to \mathcal{F}$

$$\begin{array}{ccc} \mathrm{Sub}(\mathcal{F}) \times \mathrm{Sub}(\mathcal{F}) & \xrightarrow{\ \cap\ } & \mathrm{Sub}(\mathcal{F}) \\ {\scriptstyle f^{-1}} \downarrow & & \downarrow {\scriptstyle f^{-1}} \\ \mathrm{Sub}(\mathcal{E}) \times \mathrm{Sub}(\mathcal{E}) & \xrightarrow[\cap]{} & \mathrm{Sub}(\mathcal{E}) \end{array}$$

our $\wedge$ must be natural in $\mathcal{F}$. By the Yoneda lemma (which we are allowed to use because $\mathrm{Sh}(X)$ is locally small) any natural transformation from $\mathrm{Hom}(\mathcal{F}, \Omega\times\Omega)$ to $\mathrm{Hom}(\mathcal{F}, \Omega)$ is uniquely determined by its value at $\mathrm{id}_{\Omega\times\Omega}$ because for any $f : F \to \Omega \times \Omega$ we have that this diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}(\Omega\times\Omega, \Omega\times\Omega) & \xrightarrow{-\circ f} & \mathrm{Hom}(\mathcal{F}, \Omega\times\Omega) \\ {\scriptstyle \wedge_{\Omega\times\Omega}}\downarrow & & \downarrow{\scriptstyle \wedge_{\mathcal{F}}} \\ \mathrm{Hom}(\Omega\times\Omega, \Omega) & \xrightarrow[-\circ f]{} & \mathrm{Hom}(\mathcal{F}, \Omega) \end{array}$$

which implies that $\wedge_{\Omega\times\Omega}(\mathrm{id}_{\Omega\times\Omega})(f) = \wedge_{\mathcal{F}}(f)$ for any $\mathcal{F}$ and f. Consequently $\wedge_{\Omega\times\Omega}(\mathrm{id}_{\Omega\times\Omega}) : \Omega\times\Omega \to \Omega$ is the binary operation on Ω that corresponds to disjunction of subsheaves and we will denote it by $\wedge : \Omega \times \Omega \to \Omega$.

□

This is going to be useful later for the internal language

3.2 Internal language

We have already proved that it is in some sense possible to "apply" first order logic to $\mathrm{Sh}(X)$ and since we could almost say that a sheaf "believes" itself to be a set we should be able to view $\mathrm{Sh}(X)$ as a "universe of sets" equipped with some modified version of the language of sets. It turns out that this is possible and it is called internal language, in general an internal language consists of types, variables, terms and formulas. In the Mitchell-Bénabou language, which is one type of internal language of sheaves and the one that we are going to use, the types are the objects of $\mathrm{Sh}(X)$. We define variables and terms inductively

- Variables x of type $\mathcal{F}$ are interpreted as as the identity morphism $\mathrm{id} : \mathcal{F} \to \mathcal{F}$
- Terms σ of type $\mathcal{F}$ are interpreted as morphisms $\sigma = (\sigma_i)_{i\in I} : \mathcal{E}_i \to \mathcal{F}$ for some family $\{\mathcal{E}_i\}_{i\in I}$ of sheaves in $\mathrm{Sh}(X)$. (take look at this, maybe only binary prod?)

Formulas are terms of type Ω, we can apply the usual logical connectives because of 3.13, consequently from terms $\varphi : \mathcal{E} \to \Omega$ and $\psi : \mathcal{F} \to \Omega$ we can construct new terms

$$\varphi \wedge \psi : \mathcal{E} \times \mathcal{F} \xrightarrow{\langle\varphi,\psi\rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

$$\varphi \vee \psi : \mathcal{E} \times \mathcal{F} \xrightarrow{\langle\varphi,\psi\rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega$$

$$\varphi \Rightarrow \psi : \mathcal{E} \times \mathcal{F} \xrightarrow{\langle\varphi,\psi\rangle} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

We want to put special emphasis on the morphisms which correspond to true $\top$ and false $\bot$. Per 3.7 there is a subsheaf corresponding to $\top$ and one corresponding to $\bot$ for each sheaf $\mathcal{F}$ on X, now let instead $\top : \mathcal{F} \to \Omega$ and $\bot : \mathcal{F} \to \Omega$ denote the unique morphisms corresponding to the subsheaves. It may seem strange to have a different pair of $\top$ and $\bot$ for each sheaf, but this poses no issue for the semantics as we will see in the next section. We have also omitted $\neg$ on purpose since $\neg\varphi$ is defined as $\varphi \implies \bot$ and it is easier to apply the Kripke-Joyal sematics to this composition.

The usual description of the Mitchell-Bénabou language consists of a few other terms which are not relevant for our purpose, for a complete account cf. [LM92, §VI.5].

We promised that the language would behave like first order logic so there are of course also quantifiers, consider the formula $\varphi(x, \overline{y})$ with free variables x of type $\mathcal{X}$ and $\overline{y}$ of type $\mathcal{Y} = \mathcal{Y}_1 \times ...$ and denote by $\{(x, \overline{y})|\ \varphi(x, \overline{y})\}$ the subobject classified by the interpretation of $\varphi(x, \overline{y})$. Let $\pi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ be the projection of $\mathcal{X} \times \mathcal{Y}$ on $\mathcal{Y}$ then by 3.8 $\forall_\pi\{(x, \overline{y})|\varphi(x, \overline{y})\}$ and $\exists\pi\{(x, \overline{y})|\varphi(x, \overline{y})\}$ are subsheaves of Y and the interpretations of $\forall x\varphi(x, \overline{y})$ and $\exists x\varphi(x, \overline{y})$ must be the unique arrows from $\mathcal{Y}$ to Ω such that the following are pullbacks.

$$\begin{array}{ccccc}
\forall_\pi\{(x,\overline{y})|\varphi(x,\overline{y})\} & \longrightarrow & 1 & \longleftarrow & \exists_\pi\{(x,\overline{y})|\varphi(x,\overline{y})\} \\
\downarrow & & \downarrow{\scriptstyle \text{true}} & & \downarrow \\
\mathcal{Y} & \xrightarrow[\forall x\varphi(x,\overline{y})]{} & \Omega & \xleftarrow[\exists x\varphi(x,\overline{y})]{} & \mathcal{Y}
\end{array}$$

3.3 Kripke-Joyal Semantics

While the internal language provides us with a simple way to describe properties of some sheaf the Kripke-Joyal sematics provides a way to translate internal properties to the external language.

Definition 3.15. Let $\mathrm{Sh}(X)$ be the category of sheaves of sets on X and $\varphi(x)$ be some formula $\varphi : \mathcal{F} \to \Omega$ in the internal language of $\mathrm{Sh}(X)$, then $U \vDash \varphi(\alpha)$ for $\alpha \in \mathcal{F}(U)$ iff $\alpha \in \{x|\varphi(x)\}(U)$.

This gives us the following theorem.

Theorem 3.16. *[LM92, p. VI.7.1] Let $Sh(X)$ be the category of sheaves of sets over the topological space X, $\mathcal{F}, \mathcal{E} \in Sh(X)$ and $\alpha \in \mathcal{F}(U)$, then*

(i) $U \vDash \top$ iff $U = U$ (i.e. always true)

(ii) $U \vDash \bot$ iff $U = \varnothing$

(iii) $U \vDash \varphi(\alpha) \wedge \psi(\alpha)$ iff $U \vDash \varphi(\alpha)$ and $U \vDash \psi(\alpha)$.

(iv) $U \vDash \varphi(\alpha) \vee \psi(\alpha)$ iff there is an open cover of U, $\{U_i\}_{i\in I}$, such that for all $i \in I$ either $U_i \vDash \varphi(\alpha|_{U_i})$ or $U_i \vDash \psi(\alpha|_{U_i})$.

(v) $U \vDash \bigwedge_{i \in I} \varphi_i(\alpha)$ *iff* $U \vDash \varphi_i(\alpha)$ *for every* $i \in I$.

(vi) $U \vDash \bigvee_{i \in I} \varphi_i(\alpha)$ *iff there is an open cover of* U, $\{U_j\}_{j \in J}$, *such that for each* $j \in J$ *there is an* i *such that* $U_j \vDash \varphi_i(\alpha|_{U_j})$.

(vii) $U \vDash \varphi(\alpha) \Rightarrow \psi(\alpha)$ *iff for all open* $V \subseteq U$ $V \vDash \varphi(\alpha|_V)$ *implies* $V \vDash \psi(\alpha|_V)$.

(viii) $U \vDash \neg\varphi(\alpha)$ *iff the only open* $V \subseteq U$ *such that* $V \vDash \varphi(x)$ *holds is* $V = \varnothing$.

(ix) $U \vDash \exists y : \mathcal{E}\ \chi(\alpha, y)$ *iff there is some open cover* $\{U_i\}_{i \in I}$ *such that there is some* $y_i \in \mathcal{E}(U_i)$ *such that* $U_i \vDash \chi(\alpha_{U_i}, y_i)$ *for all* $i \in I$.

(x) $U \vDash \forall y : \mathcal{E}\ \chi(\alpha, y)$ *iff for every open* $V \subseteq U$ *and for all* $\beta \in \mathcal{E}(V)$ $V \vDash \chi(\alpha|_V, \beta)$

Remark 3.17. $(viii)$ is just a special case of (vii).

Proof. Most of this proof relies on proposition 3.7 and 3.8 and each case is quite similar so we will only prove (i). Suppose $U \vDash \varphi(x) \wedge \psi(x)$, by definition this is equivalent to $\alpha \in \{x|\varphi(x) \wedge \psi(x)\}(U)$ by (prop 2.1) the meet is defined as $\{x|\varphi(x)\} \cap \{x|\psi(x)\}$ and thus $\alpha \in \{x|\varphi(x) \wedge \psi(x)\}(U)$ is equivalent to $\alpha \in \{x|\varphi(x)\}(U)$ and $\alpha \in \{x|\psi(x)\}(U)$. $\square$

To make things more readable we will instead of $\varphi(\alpha)$ write $\alpha \in \mathcal{G}$ if $\varphi : \mathcal{F} \to \Omega$ is the characteristic function of $\mathcal{G}$, thus

$$U \vDash \alpha \in \mathcal{G} \text{ iff } \alpha \in \mathcal{G}(U) \text{ for any section } \alpha \text{ of } \mathcal{F}$$

and if $\varphi(\alpha, \beta) = \delta_{\mathcal{F}}(\alpha, \beta)$ is the characteristic map for the diagonal of $\mathcal{F}$ then we will write $\alpha = \beta$ instead of $\delta_{\mathcal{F}}(\alpha, \beta)$, thus

$$U \vDash \alpha = \beta : \mathcal{F} \text{ iff } \alpha = \beta \in \mathcal{F}(U).$$

To illustrate how one would use 3.16 translate an internal statement to an external we give the following example:

Example 3.18. Let X be a topological space, $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$ and $f : \mathcal{F} \to \mathcal{G}$ and suppose $U \vDash \forall y : \mathcal{G}\ (\exists x : \mathcal{F}\ f(x) = y)$ for every open $U \subseteq X$. Then

$$\begin{aligned} &U \vDash \forall y : G\ (\exists x : F\ f(x) = y) \\ &\qquad \iff \quad \forall V \subseteq U\ \forall y \in \mathcal{G}(V)\ V \vDash \exists x : \mathcal{F}\ f(x) = y \\ &\qquad \iff \quad \forall V \subseteq U\ \forall y \in \mathcal{G}(V)\ \exists x_i \in \mathcal{F}(U_i)\ f_{U_i}(x_i) = y|_{U_i} \\ &\qquad\qquad\quad \text{for some open cover } \{U_i\}_{i \in I} \text{ of } V \end{aligned}$$

which we in section 4.1 will see is equivalent to f being an epimorphism.

3.4 Soundness

Before we ventures into the interesting world of internal proofs we need to make sure that the properties we would expect to hold indeed holds as to avoid any pitfalls on the way.

Theorem 3.19. *[Ble21, Prop. 2.5] The Kripke-Joyal semantics are sound with respect to intuitionistic logic, i.e. if φ implies ψ intuitionistically then $U \vDash \varphi$ implies $U \vDash \psi$.*

Proof. We can prove this by induction on the structure of intuisionistic proofs, these are all straight forward so we only give an example (complete list of rules can be found in the appendix). Let us prove this for the following rule:

$$\text{If } \varphi \text{ holds and } \varphi \wedge \psi \text{ imply } \chi \text{ then } \psi \Rightarrow \chi.$$

We have that

$$\begin{aligned} U \vDash \varphi \wedge \psi \Rightarrow \chi \quad &\iff \quad \forall V \subseteq U \ (V \vDash \varphi \wedge \psi \implies V \vDash \chi) \\ &\iff \quad \forall V \subseteq U \ ((V \vDash \varphi \text{ and } V \vDash \psi) \implies V \vDash \chi). \end{aligned}$$

But we assumed $U \vDash \varphi$ which implies that $\forall V \subseteq U \ (V \vDash \varphi)$ and thus $\forall V \subseteq U \ (V \vDash \psi \implies V \vDash \chi$ which is equivalent to $U \vDash \psi \Rightarrow \chi$. □

3.5 Sheaves of rings

For the purpose of this thesis we want be able to describe sheaves of rings, these are sheaves of sets with two binary operators $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, one unary $- : \mathcal{R} \to \mathcal{R}$ and two global sections 1 and 0 such that the usual ring axioms hold (i.e. instead of $\mathcal{R}(U)$ for some open $U \subseteq X$ being a set it is a ring). Now for our first look at an external property from the internal perspective:

Proposition 3.20. *[Ble21, Prop. 3.1] Let X be a topological space and $\mathcal{R}$ be a sheaf of sets over X with operators $+, \cdot, -$ and global sections 1 and 0 as described above, then $\mathcal{R}$ is a sheaf of rings if and only if $\mathcal{R}$ is a ring from the internal perspective.*

Proof. We will only prove that $\mathcal{R}$ is an abelian group under $+ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, the other conditions are proved in a similar way.

- Commutativity: $X \vDash \forall x, y : \mathcal{R} \ (x+y = y+x) \iff x+y = y+x \in \mathcal{R}(U)$ for any $U \subseteq X$ and any $x, y \in \mathcal{R}(U)$.
- Associativity: $X \vDash \forall x, y, z : \mathcal{R} \ (x+(y+z) = (x+y)+z) \iff x+(y+z) = (x+y)+z \in \mathcal{R}(U)$ for all open $U \subseteq X$ and any $x, y, z \in \mathcal{R}(U)$.
- Identity: $X \vDash \forall x : \mathcal{R} \ (0 + x = x) \iff$ there is a global section 0 such that $0|_U + x = x \in \mathcal{R}(U)$ for any $x \in \mathcal{R}(U)$ and all open $U \subseteq X$.
- Inverses: $X \vDash \forall x : \mathcal{R} \ (x + (-x) = 0) \iff$ for all open $U \subseteq X$ and any $x \in \mathcal{R}(U)$ we have that $x + (-x) = 0 \in \mathcal{R}(U)$.

□

Remark 3.21. An analogous statement holds for sheaves of modules.

We will leave the translations of some other properties that sheaves of rings can have for the section on translations of properties from algebraic geometry.

4 Some algebraic geometry

4.1 Some interesting properties we will use later

We saw in section 3.5 that a sheaf of rings is internally a ring, in this section we will translate some more useful external notions to internal ones.

Proposition 4.1. *[Ble21, Example 2.3] Let X be a topological space, $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X and $f : \mathcal{F} \to \mathcal{G}$ be a natural transformation, then f is an monomorphism (or epimorphism) if and only if f from the internal perspective is injective (or surjective).*

Proof. Suppose $X \vDash \forall x, y : \mathcal{F}\ f(x) = f(y) \Rightarrow x = y$, then

$$\begin{aligned} &X \vDash \forall x, y : F\ f(x) = f(y) \Rightarrow x = y \\ \iff &\text{ for all open } U \subseteq X \text{ and } \forall x, y \in \mathcal{F}(U)\ (U \vDash f(x) = f(y) : \mathcal{F} \Rightarrow U \vDash x = y : \mathcal{F}) \\ \iff &\text{ for all open } U \subseteq X \text{ and } \forall x, y \in \mathcal{F}(U)\ f_U(x) = f_U(y) \in \mathcal{F}(U) \Rightarrow x = y \in \mathcal{F}(U)) \end{aligned}$$

This means that f is injective pointwise, which we claim is equivalent to f being a monomorphism. To prove this claim suppose for the "only if"-direction that f is pointwise injective, then for any morphisms $g, h : \mathcal{G} \to \mathcal{F}$ such that $fg = fh$ and section $s \in \mathcal{G}(U)$ we have that $f_U g_U(s) = f_U h_U(s)$. But f is pointwise injective so $g_U = h_U$ for all $U \subseteq X$, thus $g = h$ and f is a monomorphism. For the "if"-direction suppose f is not injective, then for some $x \neq y\mathcal{F}(U)$ $f_U(x) = f_U(y)$. But then for some morphisms $g, h : \mathcal{G} \to \mathcal{F}$ such that g is equal to h everywhere except for some $s \in \mathcal{F}(U)$ where $g_U(s) = x$ and $h_U(s) = y$ we have $fg = fh$ but $g \neq h$. Thus f being a monomorphism implies that f is pointwise injective.

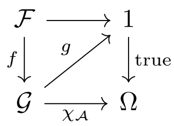
We determined that the internal surjective condition is equivalent to the following external condition in 3.18.

$$\forall V \subseteq U\ \forall y \in \mathcal{G}(V)\ \exists x_i \in \mathcal{F}(U_i)\ f_{U_i}(x_i) = y|_{U_i} \text{ for some open cover } \{U_i\}_{i \in I} \text{ of } V$$

We need to prove that this is equivalent to f being an epimorphism. For the "only if"-direction suppose f is right cancellable and define a subsheaf of $\mathcal{G}$ the following way

$$\mathcal{A}(U) = \{y \in \mathcal{F}(U) | \text{ for some open cover } \{U_i\} \text{ of } U\ \exists x_i \in \mathcal{F}(U_i)\ f(x_i) = y|_{U_i}\}\ (*)$$

Then the characteristic function $\chi_{\mathcal{A}}$ of $\mathcal{A}$, makes the following diagram commute



Since f is right cancellable we have that $\text{true} \circ g = \chi_{\mathcal{A}}$, which implies that $\mathcal{A} = \mathcal{G}$ and consequently f has property $(*)$.

For the "if"-direction assume f has property $(*)$ and let $s \in \mathcal{G}(U)$, then for some open cover $\{U_i\}_{i \in I}$ of U such that there is some $x_i \in \mathcal{F}(U_i)$ and $f_{U_i}(x_i) = s_i$. Thus for any morphisms $g, h : \mathcal{G} \to \mathcal{H}$ such that $gf = hf$ we have that $g_{U_i}(s|_{U_i}) = h_{U_i}(s|_{U_i})$, but since this is a morphism of sheaves we know $g_{U_i}(s|_{U_i}) = g_U(s)|_{U_i}$, consequently $g_U(s)|_{U_i} = h_U(s)|_{U_i}$ for all $i \in I$ which implies that $g_U(s) = h_U(s)$.

□

Recall the definition of a sheaf of modules of finite type.

Definition 4.2. Let $(X, \mathcal{O}_X)$ and $\mathcal{F}$ be an $\mathcal{O}_X$-module, we say that $\mathcal{F}$ is of finite type if for each $x \in X$ there is an open neighbourhood U of x such that $F|_U$ is generated by finitely many sections.

$\mathcal{F}|_U$ being generated by finitely many sections means that for sections $x_1, ..., x_n \in \mathcal{F}(U)$ the map

$$\bigoplus_{1 \leq i \leq n} \mathcal{O}_X \to \mathcal{F}|_U \qquad (a_1, ..., a_n) \mapsto a_1 x_1 + ... + a_n x_n$$

is surjective.

Then, because of how surjective is defined for maps of sheaves, a sheaf of modules being of finite type is equivalent to it fullfilling the following condition:

"There is some open cover $\{U_i\}_{i \in I}$ of X such that for each $i \in I$ there is some $n \in \mathbb{N}$ and n sections of $\mathcal{F}(U_i)$, $(x_1, ..., x_n)$ such that for each $i \in I$ and each open $V \subseteq U_i$ and all sections $x \in \mathcal{F}(V)$, there is some open cover $\{V_j\}$ of V such that for each $x \in \mathcal{F}(V_i)$ there is $a_1, ..., a_n \in \mathcal{O}_X(V_i)$ such that $x_{V_i} = a_1 x_1|_{V_i} + ... a_n x_n|_{V_i}$".

Proposition 4.3. *[Ble21, Prop. 4.3] Let X be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite type if and only if internally $\mathcal{F}$ is finitely generated.*

Proof. Suppose $\mathcal{F}$ is finitely generated internally, i.e.

$$X \vDash \bigvee_{n \in \mathbb{N}} \exists x_1, ..., x_n : \mathcal{F} \forall x : \mathcal{F} \exists a_1, ..., a_n : \mathcal{O}_X (x = \sum_{i=0}^{n} a_i x_i))$$

Using the Kripke-Joyal semantics:

$$X \vDash \bigvee_{n\in\mathbb{N}} \exists x_1,...,x_n : \mathcal{F}\forall x : \mathcal{F}\exists a_1,...,a_n : \mathcal{O}_X(x = \textstyle\sum_{i=0}^n a_i x_i))$$

$\iff$ there is an open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$
$\exists n \in \mathbb{N}\exists(x_1,...,x_n) \in F(U_i)^n\ U_i \vDash \forall x : \mathcal{F}\exists a_1,...,a_n : \mathcal{O}_X(x = \sum_{i=0}^n a_i x_i))$

Unwinding the universal and existential quantifiers gives us

there is an open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$
$\exists n \in \mathbb{N}\ \exists(x_1,...,x_n) \in \mathcal{F}(U_i)^n$
$\forall V \subseteq U_i\ \forall x \in \mathcal{F}(V)$ there is an open cover $\{V_i\}_{i\in I}\exists(a_1,...,a_n) \in \mathcal{O}_X(V_i)$
$V_i \vDash x = \sum_{i=0}^n a_i x_i|_{V_i}$

which is equivalent to

there is an open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$
$\exists n \in \mathbb{N}\ \exists(x_1,...,x_n) \in \mathcal{F}(U_i)^n$
$\forall V \subseteq U_i\ \forall x \in \mathcal{F}(V)$ there is an open cover $\{V_i\}_{i\in I}\exists(a_1,...,a_n) \in \mathcal{O}_X(V_i)$
$x|_{V_i} = \sum_{i=0}^n a_i x_i|_{V_i}))$.

This is, as we concluded above, equivalent to $\mathcal{F}$ being of finite type. □

Remark 4.4. It may seem like an unnecessary detour to use the arbitrary disjunction, when we instead could have used $\exists n \in \mathbb{N}$ since these turn out to be equivalent. But there is a good reason behind this choice, the reason being that we want to avoid the issue of proving that there is a natural numbers object.

4.2 Modalities

We are not only interested in what property holds on an open set, but in what the properties tells us about the surrounding space and we thus introduce some thing called a modal operator, which is almost like a "weakening" of a formula.

4.2.1 Definitions and basic properties

Definition 4.5 (modal operator)**.** A modal operator is a map $\Box : \Omega \to \Omega$ such that

(1) $\varphi \implies \Box\varphi$

(2) $\Box\Box\varphi \implies \Box\varphi$

(3) $\Box(\varphi \wedge \psi) \iff \Box\varphi \wedge \Box\psi$

Remark 4.6. In particular, we have that $\Box(\varphi \Leftrightarrow \psi) \iff (\Box\varphi \Leftrightarrow \Box\psi)$, since $\varphi \Leftrightarrow \psi$ means $\varphi = \psi$, this is important to note for the proof of $\Box$ being monotonic.

But before presenting some interesting results about modal operator we believe it is helpful to give some examples of operators that are relevant to algebraic geometry.

(1) $\Box\varphi := (\alpha \Rightarrow \varphi)$

(2) $\Box\varphi := (\varphi \vee \alpha)$

(3) $\Box\varphi := \neg\neg\varphi$

(4) $\Box\varphi := ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$

In all of these α is a fixed proposition. Now for the promised lemma.

Lemma 4.7. *[Ble21, Lemma 6.3] $\Box$-operator is monotonic, i.e. if $\varphi \Rightarrow \psi$ then $\Box\varphi \Rightarrow \Box\psi$.*

Proof. We know that the assumption $\varphi \Rightarrow \psi$ is equivalent to the assumption $\varphi \wedge \psi \Leftrightarrow \varphi$, this implies $\Box(\varphi \wedge \psi \Leftrightarrow \varphi)$ by axiom (1), which in turn, by an earlier remark 4.6 about equivalence being the same as equals and axiom (3), is equivalent to $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$ which is equivalent $\Box\varphi \Rightarrow \Box\psi$. $\Box$

Corollary 4.8. *[Ble21, Lemma 6.3] If $\Box\varphi$ and $\varphi \Rightarrow \Box\psi$ then $\Box\psi$.*

Proof. By 4.7 we have that $\Box\varphi \Rightarrow \Box\Box\psi$ holds and by $\Box\varphi$ holding $\Box\Box\psi$ must hold too. Now by axiom (2) $\Box\psi$ must hold. $\Box$

We gave some examples of different types of modal operators before, now the following proposition gives them a geometric meaning for some choices of α. But let us first introduce some new notation.

By the fact that U is a global section of Ω we have that $V \vDash U \iff V \subseteq U$, we can thus make the following definition.

Definition 4.9. Let $!x$ denote $\text{int}(X \setminus \{x\})$ because $V \vDash \text{int}(X \setminus \{x\}) \iff x \notin V$.

Proposition 4.10. *[Ble21, Prop. 6.5] Let $U \subseteq X$ be open, $A \subseteq X$ be closed and $x \in X$, then for all open $V \subseteq U$*

(1) $V \vDash (U \Rightarrow \varphi) \iff V \cap U \vDash \varphi$

(2) $V \vDash (\varphi \vee A^c) \iff \exists W \subseteq V$ such that $A \cap V \subseteq W$ and $W \vDash \varphi$

(3) $V \vDash \neg\neg\varphi \iff \exists W \subseteq V$ which is open and dense such that $W \vDash \varphi$

(4) $V \vDash ((\varphi \Rightarrow !x) \Rightarrow !x) \iff x \notin V$ or $\exists W \subseteq V$ with $x \in W$ and $W \vDash \varphi$

Proof. (1)

$$\begin{aligned} V \vDash (U \Rightarrow \varphi) &\iff \forall W \subseteq V \; W \vDash U \Rightarrow W \vDash \varphi \\ &\iff \forall W \subseteq V \; (W \subseteq U) \Rightarrow W \vDash \varphi \\ &\iff V \cap U \vDash \varphi \end{aligned}$$

(2)

$$\begin{aligned}
V \vDash (\varphi \vee A^c) \quad &\iff \quad \text{there is some open cover of V} \\
&\quad\quad \{V_i\}_{i\in I} \text{ such that for each } i \in I \\
&\quad\quad \text{ either } V_i \vDash \varphi \text{ or } V_i \subseteq X \setminus A \\
&\iff \quad \bigcup\{V_i|\ V_i \vDash \varphi \text{ and } V_i \nsubseteq X \setminus A\} \vDash \varphi \\
&\iff \quad \exists W \subseteq V\ W \vDash \varphi \text{ and } V \cap A \subseteq W
\end{aligned}$$

(3) Let $W = \bigcup\{W' \subseteq V|\ W' \vDash \varphi\}$

$$\begin{aligned}
V \vDash \neg\neg\varphi \quad &\iff \quad V' \subseteq V\ (V' \vDash \neg\varphi \Rightarrow V' = \varnothing) \\
&\iff \quad V' \subseteq V\ ((V'' \subseteq V'\ (V'' \vDash \varphi) \Rightarrow (V'' = \varnothing)) \Rightarrow V' = \varnothing) \\
&\iff \quad V' \subseteq V\ (V'' \subseteq V'\ (V'' \subseteq W \Rightarrow V'' = \varnothing) \Rightarrow V' = \varnothing) \\
&\iff \quad V' \subseteq V\ (V \cap W = \varnothing) \Rightarrow V' = \varnothing \\
&\iff \quad W \text{ is dense in } V
\end{aligned}$$

(4)

$$\begin{aligned}
V \vDash ((\varphi \Rightarrow !x) \Rightarrow !x) \quad &\iff \quad \forall V' \subseteq V \text{ such that } V' \text{ is open} \\
&\quad\quad V' \vDash (\varphi \Rightarrow !x) \implies V' \vDash !x \\
&\iff \quad \forall V' \subseteq V \\
&\quad\quad (\forall V'' \subseteq V'\ (V'' \vDash \varphi \implies x \notin V'')) \\
&\quad\quad \implies x \notin V' \\
&\iff \quad x \notin V \text{ or } \exists W \subseteq V \text{ such that} \\
&\quad\quad x \in W \text{ and } W \vDash \varphi
\end{aligned}$$

□

A modal operator induces a map on the global sections of Ω (i.e. the open subsets of X), let us call this set $\mathcal{T}(X)$

$$\begin{aligned}
j : \mathcal{T}(X) \quad &\to \quad \mathcal{T}(X) \\
U \quad &\mapsto \quad \bigcup\{V \subseteq X|\ V \text{ open}, V \vDash \Box U\}
\end{aligned}$$

in other words there is a map j that maps U to the largest open subset of X on which $\Box U$ holds. In particular we have that for any open $U, V \subseteq X$

(1) $U \subseteq j(U)$

(2) $j(j(U)) \subseteq j(U)$

(3) $j(U \cap V) = j(U) \cap J(V)$

by the modal operator axioms. This in turn defines a subspace of X.

Modal operator	associated map	$X_\Box$
$\Box\varphi \equiv (U \Rightarrow \varphi)$	$j(V) = \mathrm{int}(U^c \cup V)$	U
$\Box\varphi \equiv (\varphi \vee A^c)$	$j(V) = V \cup A^c$	A
$\Box\varphi \equiv \neg\neg\varphi$	$j(V) = \mathrm{int}(\mathrm{cl}(V)$	smallest dense sublocale of X
$\Box\varphi \equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$j(V) = \begin{cases} X \text{ if } x \in V \\ X \setminus \mathrm{cl}(\{x\}) \text{ if } x \notin V \end{cases}$	$\{x\}$

Table 1: Subspaces and associated maps to some modal operators

Definition 4.11. Let j be a map on $\mathcal{T}(X)$ induced by some modal operator $\Box : \Omega \to \Omega$, then the subspace associated to $\Box$, $X_\Box$ (not to be confused with the game console), is given by the frame (partially ordered set with indexed union and finite intersections) $\mathcal{T}(X_\Box) := \{U \in \Omega(X) | j(U) = U\}$ (i.e. the set of open $U \subseteq X$ such that the largest subset on which $\Box U$ holds is U).

In general this is not a topological subspace but a sublocale since it does not know anything about the points in the underlying set, this is not a problem since sheaves only refer to open sets. The subspaces and associated maps of the four modal operators in 4.10 are listed in Table 1.

One thing one might notice is that it seems to be the case that $X_{\neg\neg}$ should coincide with $\{x\}$ if x is a generic point in X. Let us first recall the definition of a generic point.

Definition 4.12. Let X be a topological space and $x \in X$ be a point, we call x generic if $\mathrm{cl}(\{x\}) = X$.

Proposition 4.13 ([Ble21] Prop. 6.16)**.** *Let X be a scheme and ξ be a generic point of X then the modal operator $\Box := ((\Rightarrow !\xi) \Rightarrow !\xi)$ coincides with the double negation modality and $X_{\neg\neg} = \{\xi\}$.*

Proof. By definition $U \models !\xi \iff \xi \notin U$ but this cannot hold on any non-empty open U since ξ is dense in X and thus $U \models !\xi \iff U = \varnothing \iff U \models \bot$. Consequently

$$((\varphi \Rightarrow !\xi) \Rightarrow !\xi) \iff ((\varphi \Rightarrow \bot) \Rightarrow \bot) \iff \neg\neg\varphi.$$

The second claim can be verified by the Table 1. above. ∎

4.2.2 The $\Box$-translation

It is well known that the double negation translation (i.e. putting $\neg\neg$ before any subformula, denoted by $\varphi^{\neg\neg}$) has some interesting properties (such as φ is derivable classically iff $\varphi^{\neg\neg}$ is derivable intuitionistically), the following definition generalises the translation to any modal operator.

Definition 4.14 ($\Box$-translation)**.**

$$
\begin{array}{rcl}
(s = t)^{\Box} & := & \Box(s = t) \\
(s \in F)^{\Box} & := & \Box(s \in F) \\
\bot^{\Box} & := & \Box\bot \\
\top^{\Box} & := & \Box\top(\Leftrightarrow \top) \\
(\varphi \wedge \psi)^{\Box} & := & (\varphi^{\Box} \wedge \psi^{\Box}) \\
(\varphi \vee \psi)^{\Box} & := & \Box(\varphi^{\Box} \vee \psi^{\Box}) \\
(\varphi \Rightarrow \psi)^{\Box} & := & (\varphi^{\Box} \Rightarrow \psi^{\Box}) \\
(\forall x : F\varphi)^{\Box} & := & (\forall x : F\varphi^{\Box}) \\
(\exists x : F\varphi)^{\Box} & := & \Box(\exists x : F\varphi^{\Box})
\end{array}
\qquad
\begin{array}{rcl}
(\bigwedge_i \varphi_i)^{\Box} & := & \Box(\bigwedge_i \varphi_i^{\Box}) \\
(\bigvee_i \varphi_i)^{\Box} & := & \Box(\bigvee_i \varphi_i^{\Box})
\end{array}
$$

Definition 4.15. A formula φ is $\Box$-stable iff $\Box\varphi \Rightarrow \varphi$.

Lemma 4.16 ([Ble21] Lemma 6.22 (1))**.** *All $\Box$-translations of formulas are $\Box$-stable.*

Proof. The case for the formulas such that the translation puts a $\Box$ in front follows from 4.5 (2), for the other cases this needs to be proved by cases with induction on subformulas.

We present the case for implication

$$
\begin{array}{rcll}
\Box(\varphi \Rightarrow \psi)^{\Box} & \iff & \Box(\varphi^{\Box} \wedge \psi^{\Box} \Leftrightarrow \varphi^{\Box}) & \\
& \iff & (\Box\varphi^{\Box} \wedge \Box\psi^{\Box} \Leftrightarrow \Box\varphi^{\Box}) & \text{(by 4.6)} \\
& \iff & \Box\varphi^{\Box} \Rightarrow \Box\psi^{\Box} & \\
& \iff & \Box\varphi^{\Box} \Rightarrow \psi^{\Box} & \text{(by IH)} \\
& \iff & \varphi^{\Box} \Rightarrow \psi^{\Box} & \text{(by 4.8)} \\
& \iff & (\varphi \Rightarrow \psi)^{\Box} & \text{(by def.)}
\end{array}
$$

$\Box$

To again make sure that the properties we want to hold indeed holds we need to make sure that the $\Box$-translation is sound.

Proposition 4.17 ([Ble21] Lemma 6.23)**.** *Suppose there is some intuitionistic proof of $\varphi \Rightarrow \psi$ then there is an intutionistic proof of $\varphi^{\Box} \Rightarrow \psi^{\Box}$.*

Proof. We can prove this by induction on the structure of intuitionistic proofs, this is again very straight forward so we will again use the double rule for implication as an example.

Suppose we have $(\varphi \wedge \psi)^{\Box} \Rightarrow \chi^{\Box}$ and assume that $\varphi^{\Box}$ holds, we have that $(\varphi \wedge \psi)^{\Box}$ is equivalent to $\varphi^{\Box} \wedge \psi^{\Box}$ by definition, thus $\varphi^{\Box} \wedge \psi^{\Box} \Rightarrow \chi^{\Box}$. But $\varphi^{\Box}$ holds by assumption and consequently $\psi^{\Box} \Rightarrow \chi^{\Box}$ holds which is equivalent to $(\psi \Rightarrow \chi)^{\Box}$. $\Box$

4.2.3 $\Box$-sheaves and $\Box$-sheafification

Before we present one of the most important theorems of this section we need to take a closer look at $\mathrm{Sh}(X_\Box)$. We begin by defining the following map

$$i : \mathcal{T}(X) \to \mathcal{T}(X_\Box) \qquad U \mapsto j(U)$$

which induces a map

$$i^* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X) \qquad \mathcal{F} \mapsto (U \mapsto \mathcal{F}(j(U))).$$

It turns out that the image of i^* can be explicitly described, it consists of all $\Box$-sheafs (explained below), and thus to make things easier for us we identify $\mathrm{Sh}(X_\Box)$ with its image in $\mathrm{Sh}(X)$ (we can do this because of 4.21). The following is described from the internal perspective, thus a sheaf is a set.

Definition 4.18. A set $\mathcal{F}$ is $\Box$-separated if and only if

$$\forall s, t : \mathcal{F}(\Box(s = t) \implies s = t).$$

i.e $\Box$ does indeed see every element in $\mathcal{F}$ as different.

Definition 4.19. A set $\mathcal{F}$ is a $\Box$-sheaf if and only if it is $\Box$-separated and

$$\forall S \subseteq \mathcal{F}(\Box \ulcorner S \text{ is a singleton} \urcorner \implies \exists s : \mathcal{F}(\Box(s \in S))).$$

Remark 4.20. We can combine the two conditions into the following:

$$\forall S \subseteq \mathcal{F}(\Box \ulcorner S \text{ is a singleton} \urcorner \implies \exists! s : \mathcal{F}(\Box(s \in S))).$$

Proposition 4.21. *[Ble21, Prop. 6.13] Let X be a topological space and $\Box$ a modal operator in $Sh(X)$ and let i be defined as above then i^* induces an equivalence of categories $Sh(X_\Box) \simeq Sh_\Box(Sh(X))$, where $Sh_\Box(Sh(X))$ is the category of $\Box$-sheaves in $Sh(X)$.*

We won't prove this here cf. [Ble21] and [Joh02] for proof.

Definition 4.22. (Johnstone construction/plus construction [Vri87, Def. 2.3]) Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$ and $f : \mathcal{F} \to \mathcal{G}$, then

(1) $\mathcal{F}^+ = \{S \subseteq \mathcal{F} \ \Box(\ulcorner S \text{ is singleton} \urcorner)\}/\sim, \qquad \text{where } S \sim T :\Leftrightarrow \Box(S = T)$

(2) $f^+ : \mathcal{F}^+ \to \mathcal{G}^+, \ [S] \mapsto [\{f(x) | x \in S\}]$

(3) $\gamma : \mathcal{F} \to \mathcal{F}^+, \, x \mapsto [\{x\}]$

Proposition 4.23. *[Ble21, Prop. 6.15] The $\Box$-sheafification $\mathcal{F}^{++}$ of $\mathcal{F}$ is a $\Box$-sheaf.*

Proof. We divide this proof into two parts: first we prove that that $\mathcal{F}^+$ must be separated, then we prove that $\mathcal{F}^+$ is a $\Box$-sheaf if $\mathcal{F}$ is separated.
For the former, let $[S], [T] \in \mathcal{F}^+$, and suppose $\Box([S] = [T])$, but this is equivalent to $\Box(\Box(S = T))$. By definition of $\Box$ this implies $\Box(S = T)$ and thus $[S] = [T]$.

Now suppose $\mathcal{F}$ is separated, we want to prove that $\mathcal{F}^+$ is a $\Box$-sheaf. Let $S \subseteq \mathcal{F}^+$ such that $\Box\ulcorner S$ is singleton$\urcorner$ holds and take $t = \{x \in \mathcal{F} \mid \exists y \subseteq \mathcal{F}(\Box\ulcorner y \text{ is singleton}\urcorner \wedge x \in y \wedge [y] \in S\}$. It suffices to prove that $\Box([t] \in S)$ since we proved that $\mathcal{F}^+$ is separated. The statement is boxed so we can by 4.7 assume that S is an actual singleton i.e. $S = \{[u]\}$. What we want to prove then becomes $t = u$ and t is a singleton. By $\mathcal{F}$ being separated we have $t = \{x \in \mathcal{F} \mid \exists y \subseteq \mathcal{F}(\Box\ulcorner y \text{ is singleton}\urcorner \wedge x \in y \wedge [y] \in S\} = \{x \in \mathcal{F} | \ x \in u\}$, which tells us exactly that $t = u$ and since u is a singleton t must be too, thus $(t \in S)$. Consequently we have for an arbitrary $S \subseteq \mathcal{F}^+$ constructed some $t \in \mathcal{F}^+$ such that $\Box(t \in S)$ and $\forall S \subseteq \mathcal{F}(\Box\ulcorner S \text{ is a singleton}\urcorner \implies \exists t : \mathcal{F}^+(\Box(t \in S)))$ must hold. $\Box$

We might want a way to translate properties from $\mathrm{Sh}(X)$ into $\mathrm{Sh}(X_\Box)$ and we thus make the following definition.

Definition 4.24 (+-translation)**.**

- The context is changed from $\overline{x} \in \mathcal{F}$ to $\overline{x} \in \mathcal{F}^+$.
- Terms $x_1, ..., x_n$ and $f(\overline{y})$ are changed to $\gamma(x_1), ..., \gamma(x_n)$ and $f^+(\overline{y})$.
- φ^+ of a formula $\varphi : \mathcal{F} \to \Omega$ is attained by replacing all free variables with their γ-images and morphisms and domains of quantifications with their +-constructions, e.g. $(\forall x : \mathcal{F}\ f(x) = g(x))^+ := \forall x : \mathcal{F}^+\ f^+(x) = g^+(x)$.

To make the translation from $\mathrm{Sh}(X)$ into $\mathrm{Sh}(X_\Box)$ we would need to apply the +-translation twice.

Theorem 4.25 ([Ble21] Thm. 6.31)**.** *Let X be a topological space and $\Box$ be a modal operator on $Sh(X)$ then for any formula φ*

$$Sh(X) \vDash \varphi^\Box \iff Sh(X_\Box) \vDash \varphi$$

where all parameters on the right side are pulled back to $X_\Box$ along $X_\Box \hookrightarrow X$.

Remark 4.26. We have written $\mathrm{Sh}(X)$ instead of X and $\mathrm{Sh}(X_\Box)$ instead of just $X_\Box$, to mark that we work in different contexts.

Before proving this we prove the following lemma.

Lemma 4.27 ([Ble21], Lemma 6.38)**.** *Let $\Box$ be a modal operator, φ be some formula, Then $\varphi \Leftrightarrow ((\varphi^\Box)^+)^+$ intuitionistically.*

Proof. For any formula φ, let $\varphi^{\boxplus}$ denote the formula we get after applying $\Box$-translation and then substituting all domains of quantification with their plus construction. For each set F there is a canonical map from $\mathcal{F}$ to $\mathcal{F}^+$, $x \mapsto [\{x\}]$, thus it suffices to show that $\varphi^{\Box}(x_1, ..., x_n)$ is equivalent to $\varphi^{\boxplus}([\{x_1\}, ..., [\{x_n\}])$. We prove this by induction on formula structure (predictable). The cases for any formula without any bounded quantifiers or $=$ and $\in$ are uninteresting, since nothing is changed.
We begin with "$\in$", suppose $\varphi = x \in \mathcal{G}$, then $\varphi^{\Box} = \Box(x \in \mathcal{G})$ and $\varphi^{\boxplus} = \Box([\{x\} \in \mathcal{G}^+)$. By 4.7 it suffices to show that $x \in \mathcal{G} \Leftrightarrow [\{x\}] \in \mathcal{G}^+$, this holds by the definition of the plus construction 4.22. The case for "$=$" is similar (cf. [Ble21]).

Now for bounded quantifiers, suppose $\varphi = \forall x : \mathcal{F}\varphi(x)$, then $\varphi^{\Box} = \forall x : \mathcal{F}\varphi^{\Box}(x)$ and $\varphi^{\boxplus} = \forall x' : \mathcal{F}^+\varphi^{\boxplus}(x')$. For the "if"-direction suppose $\forall x : \mathcal{F}\varphi^{\boxplus}(x)$ holds, this implies that $\varphi^{\boxplus}(x)$ holds for any $x' : \mathcal{F}^+$ such that $x' = [\{x\}]$ for some $x : \mathcal{F}$, but by assumption $\varphi^{\boxplus}([\{x\}]) \Leftrightarrow \varphi^{\Box}(x)$ thus $\forall x' : \mathcal{F}^+\varphi^{\boxplus}(x') \Rightarrow \forall x : \mathcal{F}\varphi^{\Box}(x)$. For the "only if"-direction suppose $\forall x : \mathcal{F}\varphi^{\Box}(x)$ and let $y \in \mathcal{F}^+$, then there is some $z \subseteq \mathcal{F}$ such that $\Box\ulcorner z$ is singleton$\urcorner$ and $y = [z]$. Since we want to prove a boxed statement we can assume that z is a singleton, and thus $z = \{x'\}$ for some $x' \in \mathcal{F}$, but then by assumption $\varphi^{\Box}(x')$ holds and by hypothesis $\varphi^{\boxplus}([\{x'\}])$ must also hold so for every z that is a singleton $\varphi^{\boxplus}([z])$ holds and thus $\forall y : \mathcal{F}^+\Box\varphi^{\boxplus}(y)$ which is equivalent to $\forall y : \mathcal{F}^+\varphi^{\boxplus}(y)$. The proof for existential quantifier is similar (cf. [Ble21]). $\square$

Proof of 4.25. We prove this by induction on formula structure (for a nicer proof cf. [Ble21]), all cases are similar similar we present the one for implication as an example.

$$\begin{aligned} U \vDash (\chi \Rightarrow \psi)^{\Box} &\iff \forall V \subseteq U\ (V \vDash \chi^{\Box} \implies V \vDash \psi^{\Box}) && \text{(by def.)} \\ &\iff \forall V \subseteq U\ (j(V) \vDash \chi \implies j(V) \vDash \psi) && \text{(by IH)} \\ &\iff j(U) \vDash (\chi \Rightarrow \psi) && \text{(by def.)} \end{aligned}$$

$\square$

Now by applying this to the modal operators in 4.10 we get the following corollary.

Corollary 4.28. *[Ble21, Cor. 6.32] Let X be a topological space, then*

(1) Let $U \subseteq X$ be open and $\Box\varphi :\equiv (U \Rightarrow \varphi)$, then

$$X \vDash \varphi^{\Box} \iff U \vDash \varphi$$

(2) Let $A \subset X$ be closed and $\Box\varphi :\equiv (\varphi \vee A^c)$, then

$$X \vDash \varphi^{\Box} \iff A \vDash \varphi$$

(3) Let $\Box\varphi :\equiv \neg\neg\varphi$, then

$$X \vDash \varphi^{\Box} \iff X_{\neg\neg} \vDash \varphi$$

(4) Let $x \in X$ and $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$, then

$$X \vDash \varphi^{\Box} \iff \varphi \text{ holds at } x$$

Remark 4.29. by φ holding at a point x we mean that φ holds if we substitute all parameters with their stalks at x (as in thm. 4.25).

4.2.4 $\Box$ vs. $\Box$-translation

It is well known that for a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ properties like $s_x = 0 \in \mathcal{F}_x$ are equivalent to $s = 0 \in \mathcal{F}(U)$ for some neighbourhood $U \subseteq X$ around x, in the internal language translates to

$$\begin{aligned} X \vDash \Box(s = 0 : \mathcal{F}) &\iff \exists V \subseteq X (U \vDash s = 0 : F) \\ &\iff (s = 0 : \mathcal{F}) \text{ holds at } x \\ &\iff X \vDash (s = 0 : \mathcal{F})^{\Box} \end{aligned}$$

with $\Box\varphi := ((\varphi \Rightarrow !x) \Rightarrow !x)$. It is quite interesting that for this particular choice of formula $\Box(s = 0 : \mathcal{F}) \iff (s = 0 : \mathcal{F})^{\Box}$, this poses questions about what conditions we need impose on φ to make $\Box\varphi \iff \varphi^{\Box}$ hold. Since any modal operator is a pullback along some geometric morphism this reduces to a well known property in categorical logic, namely that the meaning of any geometric formula is preserved under pullback along any geometric morphism (this is quite a handwavy explanation, but it serves the purpose of justifying how we know that the condition we are seeking indeed is geometricity, cf. [Ble21] and [nLa] for a more thorough treatment of the subject). This seems a bit circular since a geometric formula is defined as a formula whose meaning is preserved under pullbacks along geometric morphisms, but since we will not discuss geometric morphisms we believe the choice of using the proposition about the structure of geometric formulas as a definition is justified.

Definition 4.30. A formula φ is geometric if and only if it consists of only

$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists.$$

Definition 4.31. A geometric implication is a formula of the form

$$\forall \cdots \forall (...) \implies (...)$$

where the subformulas (...) are geometric.

Lemma 4.32 ([Ble21] Lemma 6.25)**.** *Let φ be a formula such that for any subformula ψ that is an antecedent of an implication it holds that $\psi^{\Box} \Rightarrow \Box\psi$, then $\Box\varphi \Rightarrow \varphi^{\Box}$.*

Proof. We prove this by induction on formula structure, all cases except for implication are direct.

We know that $(\psi^{\Box} \Rightarrow \chi^{\Box})$ is $\Box$-stable so by 4.8 we can assume $(\psi \Rightarrow \chi)$ instead, but then by assumption and induction hypothesis

$$\psi^{\Box} \implies \Box\psi \implies \Box\chi \implies \chi^{\Box}.$$

$\Box$

An important consequence of this lemma is that $\Box\varphi \Rightarrow \varphi^\Box$ if φ is geometric, we will later see that this also holds if φ is a geometric implication.

Lemma 4.33. *[Ble21, Lemma 6.26] Let φ be geometric, then $\varphi^\Box \Rightarrow \Box\varphi$.*

Proof. This proof is also by induction on formula structure, again the cases are similar and we thus only give an example.

$$\begin{array}{rcll} (\psi \wedge \chi)^\Box & \iff & (\psi^\Box \wedge \chi^\Box) & \text{(by def.)} \\ & \implies & (\Box\psi \wedge \Box\chi) & \text{(by IH)} \\ & \iff & \Box(\psi \wedge \chi) & \text{(by 4.5)} \end{array}$$

□

Remark 4.34. From this we can indeed deduce that $\Box\varphi \Leftrightarrow \varphi^\Box$ if φ is geometric and also that the right implication also holds if φ is a geometric implication.

Proposition 4.35. *[Ble21, Lemma 6.28] Let φ, φ' and ψ be formulas and assume:*

(1) The formula φ' is geometric.

(2) φ and φ' are equivalent given that ψ holds.

(3) Both $\Box\psi$ and $\psi^\Box$ hold.

Then $\Box\varphi \Leftrightarrow \varphi^\Box$.

Proof. Assume $\varphi^\Box$, then $(\varphi \wedge \psi)^\Box$ (which is equivalent to $\varphi^\Box \wedge \psi^\Box$) holds. Since the $\Box$-translation is sound with respect to intuitionistic logic by 4.17 $(\varphi')^\Box$ must hold. We know $\Box\varphi'$ must hold by 4.33 and thus by monotonicity (4.7) $\Box\varphi$ holds. □

Proposition 4.36. *[Ble21, Cor. 6.34] Let φ be geometric, then φ holds at some open neighbourhood around x if and only if φ holds at x.*

Proof. This is a consequence of 4.32, 4.33, 4.28 and 4.10. □

A well known fact in algbraic geometry is that a morphism of sheaves on a space is injective (or surjective) if and only if the induced map on stalks is injective (or surjective) for every stalk, note that injectivety (and surjectivity) are geometric implications, it turns out that this can be generalised to all geometric implications.

Proposition 4.37 ([Ble21] cor. 2.11)**.** *A geometric implication holds on X is and only if it holds on all points of x.*

Proof. The "only if" direction follows from remark 4.34.

For the "if" direction we will for notational simplicity assume that the geometric implication is of the form $\forall s : F\varphi(s) \Rightarrow \psi(s)$ (like Blechschmidt did). Now assume that $\forall s : F\varphi(s) \Rightarrow \psi(s)$ holds for every $x \in X$. Let $U \subseteq X$ be open and $s \in F(U)$ be a section such that $\varphi(s)$ holds on U, then by 4.36 it suffices to show that $\psi(s)$ holds at every point of U. But by 4.36 $\varphi(s)$ holds at every point in U, consequently $\psi(s)$ must hold at every point in U by assumption. □

Using this proposition we get results like:

Proposition 4.38. *[Ble21, Prop. 3.3] A scheme X is reduced if and only if $\mathcal{O}_X$ is reduced from the internal perspective.*

Proof. Reducedness is a geometric implication

$$\forall s : \mathcal{O}_X \left(\bigvee_{n \in \mathbb{N}} s^n = 0 \right) \Rightarrow (s = 0)$$

and the proposition follows directly by 4.37. □

Proposition 4.39. *A locally ringed space is internally a local ring.*

Proof. We will make use of the fact that the definition of a local ring is classically equivalent to the ring having the properties: (1) $1 \neq 0$ and (2) $\ulcorner x + y$ is invertible$\urcorner \implies \ulcorner x$ is invertible$\urcorner \vee \ulcorner y$ is invertible$\urcorner$. These are both geometric implications and the proposition follows from 4.37. □

4.2.5 Some internal proofs

It is easy to lose sight of the purpose when there is quite a lot of technical ground work so let us briefly remind ourselves that we promised the reader that this was useful for algebraic geometry. Now let us present some internal proofs of some results from algebraic geometry.

Proposition 4.40 ([Ble21] lemma 6.40)**.** *Let X be a scheme (or a ringed space) and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

(1) Let $x \in X$, then $\mathcal{F}_x$ is the zero module iff $\mathcal{F}$ is zero on some open neighbourhood of x.

(2) Let $A \subseteq X$ be closed, then the restriction $\mathcal{F}|_A$ is zero if and only iff it is zero on some open subset of X containing A.

Proof. We know that $\mathcal{F}$ being of finite type is equivalent to F being finitely generated from the internal perspective, by 4.3, let $(m_1, ..., m_n)$ be the generators. Consequently the non geometric formula $\forall x : \mathcal{F}(x = 0)$ is equivalent to the geometric $\bigwedge_{i=1}^{n}(m_i = 0)$ internally, thus by 4.35 for any modal operator $\Box$ we have that $(\forall x : \mathcal{F}(x = 0))^{\Box} \Rightarrow \Box(\forall x : \mathcal{F}(x = 0))$ and by 4.32 $\Box(\forall x : \mathcal{F}(x = 0)) \Rightarrow (\forall x : \mathcal{F}(x = 0))^{\Box}$. Using $\Box\varphi := ((\varphi \Rightarrow !x) \Rightarrow !x)$, the established equivalence and the equivalences in 4.10 and 4.32 proves (1) and using $\Box\varphi := (\varphi \vee A^c)$ and the same equivalences proves (2). □

Proposition 4.41 ([Ble21] lemma 6.42)**.** *Let X be a scheme (or ringed space), $x \in X$, $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type and $n \in \mathbb{N}$, then the following are equivalent:*

(1) There is a generating family for $\mathcal{F}_x$ consisting of n elements.

(2) There is an open neighbourhood U of x such that

$$U \models \ulcorner \text{there is a collection of } n \text{ elements such that these generate } \mathcal{F}\urcorner$$

Proof. Let $\Box\varphi \equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$, we need to show that

$$X \models \Box(\exists x_1, ..., x_n : \mathcal{F}\ \forall x : \mathcal{F}\ \exists a_1, ..., a_n : \mathcal{O}_X(x = \sum_{i=1}^{n} a_i x_i))$$

holds if and only if

$$X \models (\exists x_1, ..., x_n : \mathcal{F}\ \forall x : \mathcal{F}\ \exists a_1, ..., a_n : \mathcal{O}_X(x = \sum_{i=1}^{n} a_i x_i))^{\Box}$$

holds. The "only if"-direction follows from 4.33. For the "if"-direction we need to show that $(\exists x_1, ..., x_n : \mathcal{F}\forall x : \mathcal{F}\exists a_1, ..., a_n : \mathcal{O}_X(x = \sum_{i=1}^{n} a_i x_i))$ is equivalent to some geometric formula under the assumption that $\mathcal{F}$ is of finite type. This reduces the problem to one about "linear transformations", we know that for some $m \in \mathbb{N}$ it holds internally that $\ulcorner y_1, ..., y_m$ generates $\mathcal{F}\urcorner$ then $(\exists x_1, ..., x_n : \mathcal{F}\forall x : \mathcal{F}\exists a_1, ..., a_n : \mathcal{O}_X(x = \sum_{i=1}^{n} a_i x_i)$ implies that there for these $x_1, .., x_n$ exists an $A \in \mathcal{O}_X^{m\times n}$ such that $\overline{y} = A\overline{x}$, the reverse implication also holds under the assumption. More formally, given that $\ulcorner y_1, ..., y_m$ generates $\mathcal{F}\urcorner$ holds

$$(\exists x_1, ..., x_n : \mathcal{F}\ \forall x : \mathcal{F}\ \exists a_1, ..., a_n : \mathcal{O}_X(x = \textstyle\sum_{i=1}^{n} a_i x_i)$$

$$\iff \exists x_1, ..., x_n : \mathcal{F}\ \exists A : \mathcal{O}_X^{m\times n} \ulcorner \overline{y} = A\overline{x}\urcorner.$$

Since the right hand side is geometric we by 4.35 get the desired result. □

Observe that we did not use the finite type assumption for the "only if"-direction and we consequently get the following corollary.

Corollary 4.42. *The right implication in 4.41 still holds if we remove the assumption of F being of finite type.*

Proposition 4.43. *[Ble21, Prop. 10.4] Let X be a locally ringed space and $\mathcal{I} \subseteq \mathcal{O}_X$ a sheaf of ideals, then the subspace associated to $\Box\varphi := (\varphi \vee (1 \in \mathcal{I}))$ is $V(\mathcal{I}) = \{x \in X | \mathcal{I}_x \neq (1) \subseteq \mathcal{O}_{X,x}\}$.*

Proof. By the fact that $(1 \in \mathcal{I}) \Leftrightarrow D(\mathcal{I})$ internally we have that for all open $U \subseteq X$

$$U \models (1 \in \mathcal{I}) \iff U \subseteq D(\mathcal{I})$$

And thus according to the Table 1 we have that the subspace is $D(\mathcal{I})^c = V(\mathcal{I})$. □

Proposition 4.44. *[Ble21, Prop. 10.5] Let X be locally ringed, $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals, then $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ is also locally ringed.*

Proof. By 4.39 we need to show

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a local ring} \urcorner$$

which is equivalent to

$$\mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X/\mathcal{I} \text{ is a local ring} \urcorner^{\Box}$$

with $\Box\varphi := (\varphi \vee (1 \in \mathcal{I}))$ by 4.25. Consequently we only need to give an intuitionistic proof of

$$\forall x, y : \mathcal{O}_X/\mathcal{I}\ (\ulcorner x + y \text{ is invertible} \urcorner \Rightarrow \Box(\ulcorner x \text{ is inv.} \urcorner \vee \ulcorner y \text{ is inv.} \urcorner))$$

by 4.8. We know that $x = r + \mathcal{I}$ and $y = s + \mathcal{I}$ for some $r, s \in \mathcal{O}_X$, for $x + y \equiv 1 \pmod{\mathcal{I}}$ there must be some $t \in \mathcal{O}_X$ and some $i \in \mathcal{I}$ such that $tr + ts + i = 1 \in \mathcal{O}_X$, but by assumption $\mathcal{O}_X$ is local and thus either tr, ts or i is invertible in $\mathcal{O}_X$. If tr is invertible then x is invertible in $\mathcal{O}_X/\mathcal{I}$ and same goes for ts and y, thus $\ulcorner x \text{ is inv.} \urcorner \vee \ulcorner y \text{ is inv.} \urcorner \vee (1 \in \mathcal{I})$ holds, i.e. $\Box(\ulcorner x \text{ is inv.} \urcorner \vee \ulcorner y \text{ is inv.} \urcorner)$ holds. □

Proposition 4.45. *[Ble21, Prop. 10.6] Let X be a locally ringed space and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals, then $V(\mathcal{I})$ is reduced (as a ringed space) if and only if $\mathcal{I}$ is radical internally.*

Proof. We know from 4.38 that $V(\mathcal{I})$ being reduced is equivalent to $O_{V(\mathcal{I})}$ being a reduced ring internally, i.e.

$$\mathrm{Sh}(V(\mathcal{I})) \models \forall s : \mathcal{O}_{V(\mathcal{I})} \left(\bigvee_{n \in \mathbb{N}} s^n = 0 \right) \Rightarrow (s = 0)$$

Which is equivalent to

$$\mathrm{Sh}(V(\mathcal{I})) \models \bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_{V(\mathcal{I})}(s^n = 0) \Rightarrow (s = 0)$$

By previous results and some commutative algebra the following series of equivalences holds

$$\begin{aligned} & \mathrm{Sh}(V(\mathcal{I})) \models \textstyle\bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_{V(\mathcal{I})}(s^n = 0) \Rightarrow (s = 0) \\ \iff\quad & \mathrm{Sh}(X) \models (\textstyle\bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_X/\mathcal{I}\ (s^n = 0) \Rightarrow (s = 0))^{\Box} \\ \iff\quad & \mathrm{Sh}(X) \models \textstyle\bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_X/\mathcal{I}\ (s^n = 0) \Rightarrow \Box(s = 0) \\ \iff\quad & \mathrm{Sh}(X) \models \textstyle\bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_X\ (s^n \in \mathcal{I}) \Rightarrow \Box(s \in \mathcal{I}) \\ \iff\quad & \mathrm{Sh}(X) \models \textstyle\bigwedge_{n \in \mathbb{N}} \forall s : \mathcal{O}_X\ (s^n \in \mathcal{I}) \Rightarrow (s \in \mathcal{I}). \end{aligned}$$

□

5 Conclusion

We want to conclude by revisiting the ideas and purpose mentioned in the introduction. The goal of this thesis was to present the internal language of the category of sheaves and the Kripke-Joyal semantics for formulas in the internal language. We began by defining $\top$, $\bot$, meets, joins, implication and quantifiers as nullary, binary or unary operators on subobjects of some sheaf, then letting the corresponding morphism from Ω to Ω act on the characteristic functions of the subobjects gives us formulas in the internal language. Next we needed a way to translate the internal formulas into the external language, this was achieved by making the following definition: for any formula $\varphi(x)$ in the internal language of $\mathrm{Sh}(X)$ and open set $U \subseteq X$

$$U \vDash \varphi(\alpha) \iff \alpha \in \{x|\ \varphi(x)\}(U).$$

By combining this definition and the definition of the morphisms corresponding to the logical connectives we get a method to recursively translate internal formulas to external positions.

In the second part of the thesis we started with translating some internal properties to external ones and then examined some types of "modal operators" (actually a Lawvere-Tierney topologies) in $\mathrm{Sh}(X)$, placing our focus on when $\Box$'d statements are equivalent to their $\Box$-translated versions. The conclusion being that equivalence holds when the statement is a geometric formula, this gives us insight as to why common facts such as "A morphism of locally ringed spaces is an isomorphism if and only if it is an isomorphism at each stalk" or why "an O_X-module of finite type is zero at some open neighborhood at a point if and only if it's stalk at that point is zero".

Appendix

Structural rules

$$\frac{}{\varphi \vdash \varphi} \qquad \frac{\psi \vdash \psi}{\varphi[\overline{s}/\overline{x}] \vdash \psi[\overline{s}/\overline{x}]} \qquad \frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}$$

Rules for nullary and binary conjunction

$$\frac{}{\varphi \vdash \top} \qquad \frac{}{\varphi \wedge \psi \vdash \varphi} \qquad \frac{}{\varphi \wedge \psi \vdash \psi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

Rules for nullary and binary disjunction

$$\frac{}{\bot \vdash \varphi} \qquad \frac{}{\varphi \vdash \psi \vee \varphi} \qquad \frac{}{\psi \vdash \psi \vee \psi} \qquad \frac{\varphi \vdash \chi \quad \varphi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

Rules for arbitrary set-indexed conjunction and disjunction

$$\frac{}{\bigwedge_{i \in I} \varphi_i \vdash \varphi_j \text{ for all } j \in J} \qquad \frac{\varphi \vdash \psi_j \text{ for all } j \in I}{\varphi \vdash \bigwedge_{i \in I} \psi_i}$$

$$\frac{}{\varphi_j \vdash \bigvee_{i \in I} \varphi_i \text{ for all } j \in J} \qquad \frac{\varphi_j \vdash \psi \text{ for all } j \in I}{\bigvee_{i \in I} \varphi_i \vdash \psi}$$

Double rule for implication

$$\frac{\varphi \wedge \psi \vdash \chi}{\varphi \vdash \psi \Rightarrow \chi}$$

Double rules for bounded quantifiers

$$\frac{\varphi \vdash \psi}{\exists y : Y \ \varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi \vdash \forall y : Y \psi}$$

Rules for equality

$$\frac{}{\top \vdash x = x} \qquad \frac{}{(\overline{x} = \overline{y} \wedge \varphi \vdash \varphi[\overline{y}/\overline{x}]}$$

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