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Real numbers - Cantor's approach and infinite decimal expansions

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Michal Mnich

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Abstract

This paper presents two different constructions of the real numbers based on the existence of the field of rational numbers. The concepts of supremum, total orderings, fields and isomorphisms are introduced with basic set theory as a starting point. An intuitive approach to real numbers through infinite decimal expansions is presented and shown to result in a totally ordered field with the supremum property. A different approach by means of Cauchy sequences is also presented and shown to result in a totally ordered field with the supremum property. Finally, it is shown that any two totally ordered fields with the supremum property are isomorphic, and the real numbers are defined as any such field. It is argued that both presented constructions of the real numbers have their uses in different circumstances.

Contents

Introduction

Number systems have been used by humanity for thousands of years, first of them being of course the natural numbers. Counting required the possibility of adding numbers to obtain another number. However, this would not suffice. Sooner or later the negative integers had to be added to ensure that a natural number could be cancelled out. By such cancellation one would yield the mysterious number 0. To this day mathematicians have not come to a consensus whether 0 is actually a natural number. Either way, manipulations with the integers would allow to solve mathematical problems which could not be solved before, yet clearly still not all problems. Throughout history, the need to solve new mathematical problems has been a key driving force in the expansion of number systems. The simple equation $2x = 1$ cannot be solved by using only integers, clearly we need also the rational numbers $\frac{p}{q}$ to solve equations. Simple enough to define ratios of integers as numbers and set rules for multiplication and addition of ratios. This would allow to solve many more equations. But what about $x \cdot x = 2$? What is the diagonal in a square with a side of length 1? This is where expanding number systems suddenly became not so simple.

Mathematicians would try but for a long time not succeed to introduce the irrational numbers and give a precise definition of the real number system. Notably, attempts were made by known mathematicans as B. Bolzano (1781-1848), W. R. Hamilton (1805-1865) and K. Weierstrass (1815-1897), but the results were not entirely satisfactory, according to I. Weiss, [IW, p. 1]. This paper aims to show how the real numbers may be constructed and give a hint about why more abstract constructions of the reals might be preferred to more intuitive ones. In particular, this paper will make clear that the construction of real numbers requires several steps and cannot follow directly from rational numbers. The constructions are nontheless based on our knowledge about rational numbers.

Around the time of publication of first rigorous constructions of real numbers by R. Dedekind (1831-1916) and G. Cantor (1845-1918), efforts were also made to formalize natural numbers. Peano's axioms give a complete characterization of the natural numbers. Likewise, the Zermelo-Frankel set theory lays a solid foundation upon which the set of natural numbers can be strictly defined. These characterizations are worth noting, as any rigorous construction of real numbers cannot be fully satisfactory if it rests on the rational numbers, which are defined based on a less rigorous theory of natural numbers.

Preliminaries

Let us begin by laying a foundation upon which the real numbers may be constructed. This section aims to establish terminology and some basic concepts which will be used throughout this paper. Most of the section is based on Walter Rudin's *Principles of Mathematical analysis* [WR] and Hewitt and Stromberg's *Real and abstract analysis* [HS]. Many definitions and details that won't be used in this paper are however omitted for the sake of brevity. The unfamiliar reader is referred to those works for a more thorough introduction to the subject.

This section is mainly included to give the reader a more complete description of how the real numbers may be constructed. It should give the reader a fair understanding of how such a construction can be carried out relying only on basic set theory and the set $\mathbb{Q}$ of rational numbers, which are assumed to be known.

Notation. Sets will be denoted by capital letters, elements of sets by small letters. If an object x is an element of a set A, we will write $x \in A$, otherwise we write $x \notin A$. The set which contains no elements is denoted by $\varnothing$.

Definition 2.1. Let A and B be sets. If every $x \in A$ is also an element of B, then we say that A is a *subset* of B and write $A \subset B$. If both $A \subset B$ and $B \subset A$, we write $A = B$, else $A \neq B$.

2.1 Ordered sets

Definition 2.2. Let A and B be sets. The *Cartesian product* of A and B is the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. A *relation* is a subset of a Cartesian product of two sets.

Notation. If R is a relation and $(a, b) \in R$, we will often write aRb.

Definition 2.3. Let P be a set. A *total ordering* on P is a relation, denoted by $\leq$, which is a subset of $P \times P$ and satisfies

(i) $x \leq x$ [*reflexivity*];

(ii) $(x \leq y \text{ and } y \leq x) \implies x = y$ [*antisymmetry*];

(iii) $(x \leq y \text{ and } y \leq z) \implies x \leq z$ [*transitivity*];

(iv) $x, y \in P \implies (x \leq y$ or $y \leq x)$ [*trichotomy*].

If $\leq$ is a total ordering on P, then $(P, \leq)$ is called a *totally ordered set*. If x and y are elements of a totally ordered set such that $x \leq y$ and $x \neq y$, we sometimes write $x < y$. The expression $y \geq x$ means the same as $x \leq y$.

In the remainder of this paper, we will simply use the term *ordered set* to mean to totally ordered sets. Further, if $(P, \leq)$ is an ordered set and it is clear which total ordering $\leq$ is, we simply say that P is an ordered set. It may also be worth noting that every subset of an ordered set is also an ordered set.

Example 2.4. The reader is likely well acquainted with the symbol used for an ordering. In fact, the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ equipped with the usual *less or equal to* relation, denoted by $\leq$, is an ordered set.

Definition 2.5. Let X be an ordered set and E a subset of X. Suppose that there is an $\alpha \in X$ such that $e \leq \alpha$ for all $e \in E$. Then α is called an *upper bound* of E. Moreover, if α is such that

$$\beta < \alpha \implies \beta \text{ is not an upper bound of } E,$$

then α is called the *least upper bound*, or *supremum*, of E, and we write $\alpha = \sup E$. The terms *lower bound* and *greatest lower bound*, also called *infimum*, denoted $\inf E$, are defined analogously. If E has an upper/lower bound, then E is said to be *bounded from above/below*.

Definition 2.6. Suppose X is an ordered set such that every nonempty and bounded from above subset $E \subset X$ has a least upper bound that belongs to X. Then X is said to have the *least upper bound property*.

This will occasionally be called the *supremum property*. The definition of *greatest lower bound property* is analogous.

Theorem 2.7. *A set X has the least upper bound property if and only if it has the greatest lower bound property.*

Proof. Let X be a set with the least upper bound property. Take any nonempty subset $E \subset X$ that is bounded from below. Let L be the set of all lower bounds of E. Then L is bounded from above, since we have

$$a \leq b \text{ for all } b \in E \text{ and for all } a \in L, \tag{1}$$

by the definition of L. Thus L has a least upper bound by the supremum property of X. Let $\alpha = \sup L$.

If α is not a lower bound for E, then there exists a $\beta \in E$ such that $\beta < \alpha$. But $\beta \in E$ is by (1) an upper bound for L, which contradicts that α is the *least* upper bound. Therefore α is a lower bound for E. Since $\alpha = \sup L \geq a$ for every $a \in L$, then α is the greatest lower bound for E, $\alpha = \inf E$. In particular, $\inf E$ exists in X.

We have shown that every set with the least upper bound property also has the greatest lower bound property. The proof of the converse is almost identical. □

Therefore no distinction must be made between the least upper bound and the greatest lower bound property of a set. We will henceforth for the sake of consistency choose to speak of the supremum property.

2.2 Groups and fields

In the following we will need to use *functions*. Hewitt and Stromberg give a complete description of how a funciton can be defined as a relation, see [HS, p. 9-10]. In this paper it will suffice to use a more informal definition, included mainly for the purpose of establishing notation.

Definition 2.8. A *function* f from a set X to a set Y is a rule that for every $x \in X$ assigns exactly one $y \in Y$, written as $f(x) = y$.

Sometimes we shall write $f \colon X \to Y$ and $x \mapsto f(x)$ to represent the function f.

Definition 2.9. Let A be a set. A *binary operation* $*$ on A is a function

$$* \colon A \times A \to A \colon (a, b) \mapsto a * b.$$

Onwards, the shorter notation ab will be used interchangeably with $a * b$ in contexts where it is clear which operation is being used. We continue with a definition of our first algebraic structure.

Definition 2.10. Let G be a set and $*$ a binary operation on G. If the operation satisfies

(i) $a(bc) = (ab)c \ \ \forall a, b, c \in G$ [*associativity*];

(ii) $\exists e \in G$ such that $ae = a, \forall a \in G$ [*existence of left identity*];

(iii) $\forall a \in G, \exists a^{-1} \in G$ such that $a^{-1}a = e$ for all e as in (ii) [*existence of left inverse*];

then $< G, * >$ is called a *group*. If moreover

(iv) $ab = ba \;\; \forall a, b \in G$ [*commutativity*],

then $< G, * >$ is said to be *Abelian*. In any group G, an element e as in (ii) is called a *left identity* in G and an element a^{-1} as in (iii) is called a *left inverse* of a.

Remark 2.11. It follows from the definition that a left identity e in a group is unique and must satisfy $ea = ae = a$ for all $a \in G$. Likewise, a left inverse a^{-1} is unique and must satisfy $a^{-1}a = aa^{-1} = e$. This is shown in [HS, p. 32-33]. Therefore such an $e \in G$ will be called *the* identity of G and each such a^{-1} will be called *the* inverse of a.

Definition 2.12. Suppose F is a set with two binary operations $+$ and $\cdot$, which we call *addition* and *multiplicaiton*, respectively. Then F is called a *field* provided that

(i) $< F, + >$ is an Abelian group, with the identity element 0;

(ii) $< F \setminus \{0\}, \cdot >$ is an Abelian group;

(iii) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in F$ [*distributivity*].

Notation. In order to avoid unnecessarily lenghty notation, we will from now on write ab instead of $a \cdot b$ and $ab + c$ instead of $(a \cdot b) + c$ whenever convenient. If a is an element in a field F then we write $-a$ and a^{-1} to denote the additive and multiplicative inverse of a, respectively. We will often write $a - b$ and $\frac{a}{b}$ instead of $a + (-b)$ and ab^{-1}.

The rational numbers $\mathbb{Q}$ form a field when combined with the conventional operations of addition and multiplication. For it is clear that 0 and 1 are identities, that every nonzero rational number has an additive and multiplicative inverse, and that the rules of associativity, commutativity and distributivity are precisely the rules we are used to in $\mathbb{Q}$.

Definition 2.13. Let F be a field and an ordered set. If for all $x, y, z \in F$

(i) $x < y \implies x + z < y + z$, and

(ii) $x > 0, y > 0 \implies xy > 0$,

then F is said to be an *ordered field.*

Some rules for working with inequalities in ordered fields can be derived directly from the above definition. We now state the results that will be relevant in the remainder of this text.

Proposition 2.14. *Let F be an ordered field. The following statements are true for every $x, y, z \in F$.*

(i) $x > 0 \iff -x < 0$.

(ii) If $y < z$ *then* $x > 0 \implies xy < xz$ *and* $x < 0 \implies xy > xz$.

(iii) $x \neq 0 \implies xx > 0$.

(iv) $0 < x < y \implies 0 < y^{-1} < x^{-1}$.

A proof is presented in [WR, p. 8]. We will not write it out, since it is rather technical and does not aid our discussion.

Theorem 2.15. *The field F is ordered if and only if there is a nonempty subset $P \subset F$ such that, with $N = \{-x : x \in P\} \subset F$,*

(i) $P \cup \{0\} \cup N = F$;

(ii) $P \cap N = \varnothing$;

(iii) if $x, y \in P$ *then* $xy \in P$ *and* $x + y \in P$.

Theorem 2.15 states an alternate definition of an ordered field. To show the equivalence, one can argue as follows.

Sketch of a proof. If F is an ordered field with a relation $<$, then let $P = \{x \in F : x > 0\}$. Then 2.13(i) and 2.13(ii) imply 2.15(iii) and the other points can be verified using that $x > 0 \iff -x < 0$ in an ordered field. Conversely, if F is a field with a subset P as in Theorem 2.15, then a total ordering on F can be defined by $x < y$ if $y - x \in P$. Then 2.13(ii) follows immediately from 2.15(iii) and 2.13(i) follows from the fact that $y + z - (x + z) = y - x$.

The elements $x \in P$, or equivalenlty $x > 0$, are called *positive* and the elements $y \in N$, or $y < 0$, are called *negative*. This definition is of course consistent with the customary meaning of the terms. As might have been expected, the field $\mathbb{Q}$ of rational numbers is ordered. Moreover, it is actually the smallest ordered field; see for instance [HS, p. 34-35]. Even though the integers $\mathbb{Z}$ do not form a field, we will continue to speak of positive and negative integers whenever it will be useful. Positive and negative integers satisfy all of the above rules which can applied to integers.

Definition 2.16. An ordered field F is said to have to *Archimedean property* if for all $x, y \in F$ with $x > 0$ there exists a positive integer n such that $nx > y$.

Laslty, we recall what is meant by the *absolute value* of an element in an ordered field and state a well-known result which will be useful for dealing with absolute values.

Definition 2.17. Let F be an ordered field and let $a \in F$. The *absolute value* of a, denoted by $|a|$, is defined as

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}.$$

Theorem 2.18 (Triangle inequality)**.** *If F is an ordered field and $x, y \in F$, then*

$$|a + b| \leq |a| + |b|.$$

2.3 Isomorphisms

It is often the case that we want to compare fields with eachother. In particular, sometimes we wish to determine whether two fields are "the same". Examinining whether they both contain exactly the same elements is often not practical and not particularily enlightening. For if an element in a field F is renamed to create a new field, then the new field no longer consists of the same elements as F. But clearly the elements in both fields behave identically under addition, multiplication and order, since definitions 2.12 and 2.13 are based only on relations between elements, not on how the elements are represented. That is, two ordered fields F_1 and F_2 cannot be distinguished as algebraic structures if the elements in F_1 are the elements in F_2 renamed. We will make use of functions to formalize the idea of renaming elements.

Definition 2.19. Let X and Y be sets and $f : X \to Y$ a function. Then f is said to be *injective* or *one-to-one* if

$$y_1 \neq y_2 \implies f(y_1) \neq f(y_2),$$

and f is said to be *surjective* or *onto* if

$$\forall y \in Y, \ \exists x \in X \text{ such that } y = f(x).$$

If f is both injective and surjective, then f is called *bijective*.

Definition 2.20. Let X and Y be ordered sets and $f\colon X \to Y$ a function. Then f is said to be *order-preserving* if for every $x \in X$

$$x > 0 \iff f(x) > 0.$$

Definition 2.21. Let F_1 and F_2 be fields. If $\varphi\colon F_1 \to F_2$ is a bijective mapping such that

$$\varphi(x + y) = \varphi(x) + \varphi(y),$$

$$\varphi(xy) = \varphi(x)\varphi(y),$$

then φ is called an *isomorphism*. If such an φ exists, we say that F_1 and F_2 are isomorphic and write $F_1 \cong F_2$.

If F_1 and F_2 are ordered fields and if there exists an order-preserving isomorphism φ between them, then by the definitions F_1 and F_2 have the same structure. In particular, φ sends the additive and multiplicative inverses of F_1 to the additive and multiplicative inverses of F_2, respectively.

Constructing the reals

Ever since the discovery of numbers which could not be represented as ratios mathematicians have tried to define the irrationals. The first complete constructions of the real numbers were published in the second half of the 19th century. Since then, many alternate constructions have been proposed. A summary of many of the known constructions has been provided in 2015 by Ittay Weiss, in his article *The real numbers - a survey of constructions*[]. In this paper we will only consider two different constructions, one intuitive construction by means of infinite decimal expansions, and another, perhaps less intuitive, but more elegent and far more common construction due to the German mathematician Georg Cantor (1845-1918).

3.1 Infinite decimal expansions

Encountering the real numbers for the first time, they are likely introduced as numbers whose decimal expansions need not be finite. This is a natural extension of the rational numbers and allows for using the same operations of addition and multiplication. Each infinite decimal expansion can be approximated arbitrarily well by a terminating decimal expansion, thus each sum or product of two real numbers can be approximated arbitrarily well by a sum or product of rational numbers. Depending on the desired accuracy of the result, the real number $\pi = 3.14159\ldots$ may be rounded to $\pi \approx 3$, $\pi \approx 3.14$ or even a number so close to π that a computer won't be able to tell them apart. While this is useful and sufficient for many applications, approximating real numbers is in essence reducing ourselves to rational numbers.

In the following we shall formalize the idea of infinite decimal expansions and define arithmetics that will be consistent with, yet not limited to, operations on rational numbers. The presented construction, along with most proofs, is largely inspired by [KK], wherein a similar practical construction is presented. We begin with a definition of the elements.

Definition 3.1. An *infinite decimal expansion* is a sequence $A = (a_n)_{n\geq 0}$ in $\mathbb{Z}$ such that $a_0 \in \mathbb{Z}$ and $0 \leq a_n \leq 9$ for all $n \geq 1$. If for some (a_n) there is an $N \in \mathbb{N}$ such that $N = 0$ or $a_N \neq 9$, and $a_n = 9$ for $n > N$, then we define

$$(a_0, a_1, \ldots, a_N, a_{N+1}, a_{N+2}, \ldots) = (a_0, a_1, \ldots, a_N + 1, 0, 0, \ldots). \tag{2}$$

Let R_d be the set of all infinite decimal expansions.

Notation.

(i) The reader may be more familiar with expressing the elements of R_d as $(a_0, a_1, a_2, \ldots) = a_0.a_1a_2\ldots$, we shall use this notation for non-negative a_0 whenever convenient. However, for $a_0 < 0$ we will think of the sequence $(a_0, a_1, a_2, \ldots)$ as the number $((a_0 + 1) + 0.a_1a_2\ldots)$. Thus $-0.333\ldots$ is represented by $(-1, 6, 6, 6, \ldots)$, $-3.141\ldots$ by $(-4, 8, 5, 8, \ldots)$ and so on.

(ii) Given an $A \in R_d$ where $a_m, \ldots, a_k$, $m \leq k$, is repeated for the rest of the sequence, we write

$$(a_0, a_1, ..., a_{m-1}, a_m, \ldots, a_k, a_m, \ldots) = (a_0, a_1, ..., a_{m-1}, \overline{a_m, \ldots, a_k}).$$

For instance, $0.171717\ldots = 0.\overline{17}$. In particular, (2) may now be written as

$$(a_0, a_1, \ldots, a_N, \bar{9}) = (a_0, a_1, \ldots, a_N + 1, \bar{0}).$$

Definition 3.2. Let $<$ be the relation on R_d defined by

$$A < B \iff a_i < b_i,$$

where i is the smallest integer such that $a_i \neq b_i$. If $a_i = b_i$ for all $i \in \mathbb{N}$, we write $A = B$. If either $A < B$ or $A = B$, we write $A \leq B$.

Note that by Definition 3.1, $0.\bar{9}$ and $1.\bar{0}$ represent the same element. To avoid any disambiguity, onwards we shall always choose to use the representation with repeating nines when dealing with such elements of R_d.

Lemma 3.3. *The relation less or equal to on R_d, denoted $\leq$, is a total ordering.*

Proof. Reflexivity is clear, since $A = A$ for all $A \in R_d$. If $A \leq B$ and $B \leq A$ then $a_i = b_i$ for all $i \in \mathbb{N}$, i.e. $A = B$ and it follows that the relation is antisymmetric. If $A \leq B$ and $B \leq C$ and either $A = B$ or $B = C$, then obviously $A \leq C$. Suppose $A < B$ and $B < C$. Then $a_i < b_i$ for some i such that $a_k = b_k$ for all $k < i$, and if j is the smallest integer such that $b_j < c_j$, then $p = \min(i, j)$ is the smallest integer such that $a_p \neq c_p$, and we have $a_p < c_p$. Thus the relation is transitive. Lastly, every $A \in R_d$ is of the form (a_n), thus it can be compared with every other element of R_d, therefore trichotomy holds. □

Theorem 3.4. *Every nonempty subset of R_d which is bounded from above has a least upper bound in R_d.*

Proof. Let $X \subset R_d$ be nonempty and let $U = (u_n) \in R_d$ be an upper bound for E. If $U \in E$ then we are done. Suppose $U \notin E$. For every $A \in E$ we have $a_0 \leq u_0$. Note that $(\max_{A\in E} a_0, \bar{9}) \in R_d$ must also be an upper bound for E, since it's first element is greater than or equal to the first element of every $A \in E$. Define $s_0 = \max_{A\in E} a_0$. Next define s_n for $n \geq 1$ inductively as the greatest a_n in elements $A \in E$ of the form $(s_0, \ldots, s_{n-1}, a_n, \ldots)$. The sequence $S = (s_n) \in R_d$ defined in such a manner must be an upper bound of E, since $s_i \geq a_i$ for all $A \in E$ and $i \in \mathbb{N}$.
Now suppose that $U < S$, i.e. there is a $k \in \mathbb{N}$ such that $u_k < s_k$ and $u_i = s_i$ for all $i < k$. By definition of S, there is some element $A' = (s_0, \ldots, s_k, a_{k+1}, a_{k+2}, \ldots) \in E$. Then, clearly, $U < A'$, which contradicts that U is an upper bound for E. Thus S is the lowest upper bound, $S = \sup E$. □

Using the supremum property of R_d we can now proceed to define addition and multiplication on the set. However, we will first consider the rational number field $\mathbb{Q}$ and some of its connections to R_d. We will assume without proof that each $\frac{p}{q} \in \mathbb{Q}$ has a unique corresponding decimal representation that either terminates or after some point repeats digits.

Proposition 3.5. *There exists a one-to-one order-preserving mapping $\varphi\colon \mathbb{Q} \to R_d$. Moreover, $\varphi(\mathbb{Q})$ is dense in R_d.*

The mapping which we have in mind is that which sends every rational number to its decimal expansion. If the expansion is finite, simply add zeros to make it infinite. For negative rationals we need to make a minor correction due to the earlier remark that $-1.33\ldots$ should correspond to $(-2, 6, 6, \ldots)$ etc.. With such a choice of φ it becomes clear that the mapping is one-to-one and order-preserving, for the order on R_d is defined in the same manner that we would define an order on decimal expansions of rationals. Finally, it follows that $\varphi(\mathbb{Q})$ is dense in R_d, since two rationals can be chosen arbitrarily large or small and arbitrarily close to eachother. Therefore the same holds for $\varphi(\mathbb{Q})$, since φ is order-preserving. A formal proof would not provide much insight and will therefore not be presented.

This result allows us to regard $\mathbb{Q}$ as a subset of R_d. Therefore we may use our order on R_d to compare elements within and between the sets $\mathbb{Q}$ and R_d.

Lemma 3.6. *Let A be an element of R_d. Then there exist $A_n, A^n \in \mathbb{Q}$, for $n \geq 1$, such that* *(i)* $A_m \leq A_n \leq A \leq A^n \leq A^m$ *whenever* $m < n$*;*

(ii) $A^n - A_n \leq 10^{-n}$*;*

(iii) $\sup A_n = \inf A^n = A$.

Sketch of a proof. Each such A^n and A_n is a rational approximation of $A \in R_d$. One way to find a lower approximation A_n would be to cut off the decimal expansion of A after the nth decimal place. Then A^n can be obtained by letting $A^n = A_n + 10^{-n}$. With this choice of A_n and A^n, the result should be clear: 3.6(i) states that each A_n and A^n is in fact a lower, respectively upper approximation of A, and that cutting off the decimal expansion of A at place n cannot yield a worse approximation than cutting it off earlier, at place $m < n$. Next, (ii) states that A can be approximated arbitrarily well by A_n and A^n, if only n is chosen large enough. Finally, the lower/upper approximations of A are bounded from above/below by A, and so $\sup A_n$ and $\inf A^n$ exist in R_d. In fact they must be equal because of 3.6(ii), and $\sup A_n = \inf A^n = A$ because of (i).

Remark 3.7. It is to be understood that when performing arithmetics on the rational elements A_n and A^n, the operations used are addition and multiplication in $\mathbb{Q}$.

By Lemma 3.6(i) we have $A_n + B_n \leq A^n + B^n \leq A^1 + B^1$, so that $A_n + B_n$ is bounded from above for all $n \geq 1$. Since $A_n + B_n \in \mathbb{Q}$, the rational sum is an element in R_d and by the supremum property, $\sup(A_n + B_n)$ exists. A similar argument shows that $\inf(A^n + B^n)$ exists. In addition, we have $A_n + B_n \leq A^n + B^n$ by 3.6(i)] and $(A^n + B^n) - (A_n + B_n) \leq 2 \cdot 10^{-n}$ by 3.6(ii)], which implies $\sup(A_n + B_n) = \inf(A^n + B^n)$. Thus the following definition is legitimized.

Definition 3.8. Let $A, B \in R_d$. Define addition on R_d by

$$A + B = \sup(A_n + B_n) = \inf(A^n + B^n).$$

Lemma 3.9. *Addition on* R_d *is commutative and satisfies, for* $A, B, C \in R_d$*,*

$$A < B \implies A + C \leq B + C. \tag{3}$$

Proof. Take $A, B \in R_d$. Commutativity follows immediately from the definition of addition on R_d and commutativity of rational numbers. That is, $A + B = B + A$. Suppose $A < B$. Then $\sup A_n < \sup B_n$, and the lower approximations must differ for sufficiently large n, hence there exists an $N \in \mathbb{N}$ such that $A_n < B_n$ for $n \geq N$. For any $C \in R_d$ we then have $A_n + C_n < B_n + C_n$ whenever $n \geq N$. Thus $A + C = \sup(A_n + C_n) \leq \sup(B_n + C_n) = B + C$ and we are done. □

The Lemma states a weaker version of that which is required for a field to be ordered, by Definition 2.13(i). We will later show strict inequaliy in (3).

Theorem 3.10. *The set R_d together with addition as defined above forms an Abelian group.*

Proof. Let $A, B, C \in R_d$. For any $n \geq 1$ we have

$$B^n + C^n \geq \inf(B^k + C^n) = B + C = \sup(B_k + C_k) \geq B_n + C_n.$$

Hence it follows, by (3), that $A + (B + C) \geq A_n + (B + C) \geq A_n + B_n + C_n$. Similiarily, $A + (B + C) \leq A^n + B^n + C^n$ for all n. By Lemma 3.6 (ii),

$$A^n + B^n + C^n - (A_n + B_n + C_n) \leq 3 \cdot 10^{-n},$$

thus $\sup(A_n + B_n + C_n) = \inf(A^n + B^n + C^n) = A + (B + C)$. The proof of $(A+B)+C = \sup(A_n+B_n+C_n)$ is identical. It follows that $(A+B)+C = A+(B+C)$.

We have already shown that R_d is Abelian [Lemma 3.9]. It is also clear that $0 \in R_d$ is the additive identity, since $A + 0 = 0 + A = \sup(0 + A_n) = \sup A_n = A$.

To show that $A \in R_d$ has an additive inverse, first we observe that since $A_n \leq A^m$ for all positive n and m, then $-A_n \geq -A^m$ for all positive n and m. This follows from Proposition 2.14 and the fact that $\mathbb{Q}$ is an ordered field. Now, if we let $A' = \sup(-A^k) = \inf(-A_k)$, then using (3) yields $A^n - A_m \geq A + A' \geq A_n - A^m$ for all positive n and m. In particular, for $n = m$ this means $10^{-n} \geq A + A' \geq -10^{-n}$, recalling that $A^n - A_n \leq 10^{-n}$. The equality holds for all positive n, hence it follows that $A + A' = \sup(-10^{-n}) = \inf(10^{-n}) = 0$ and it is clear that each $A \in R_d$ has an additive inverse $-A = A' = \sup(-A^n)$. □

Similiarily to addition in $\mathbb{Q}$, it is the case that multiplication of elements $A_n, B_n \in \mathbb{Q}$ yields a rational $A_n \cdot B_n \leq A^1 \cdot B^1$ for all n. Therefore $\sup(A_n \cdot B_n)$ exists in R_d, and may be used to define multiplication on R_d. As previously with addition, it can be shown that $\sup(A_n \cdot B_n) = \inf(A^n \cdot B^n)$. Also, note that by (3), $A < 0 \iff 0 < -A$, where strict inequality follows from $A = 0 \iff -A = 0$.

Definition 3.11. Let $f\colon R_d \times R_d \to R_d$ be given by $f(A,B) = \sup(A_n \cdot B_n) = \inf(A^n \cdot B^n)$. Define multiplication on R_d by

$$A \cdot B = \begin{cases} f(A,B) & A \geq 0, B \geq 0 \\ -f(-A,B) & A < 0, B \geq 0 \\ -f(A,-B) & A \geq 0, B < 0 \\ f(-A,-B) & A < 0, B < 0 \end{cases}.$$

Remark 3.12. Division into different cases is done to ensure that we only need to consider supremum for non-negative elements of R_d. Adding negative signs in front of the two middle cases is needed for Proposition 2.14(ii) to hold.

Theorem 3.13. *The set R_d with operations defined by 3.8 and 3.11 is a totally ordered field with the least upper bound property.*

Proof. We have already shown that R_d is an ordered set with the least upper bound property and that $< R_d, + >$ forms an Abelian group. It remains to verify that

(a) $< R_d, \cdot >$ is an Abelian group,

(b) multiplication is distributive over addition,

(c) $A < B \implies A + C < A + C$ for all $A, B, C \in R_d$,

(d) $A > 0, B > 0 \implies AB > 0$ for all $A, B \in R_d$.

(a). Commutativity of multiplication on R_d follows from commutativity of multiplication on $\mathbb{Q}$. Also, it is clear that $A \cdot 1 = 1 \cdot A = A$ for all $A \in R_d$. If either $A = 0, B = 0$ or $C = 0$, then $A(BC) = 0 = (AB)C$. Suppose $A > 0, B > 0$ and $C > 0$. Then $A < B \implies AC < BC$, proof is similar to the proof of the second part of Lemma 3.9. Now, since $BC = \sup(B_n C_n) > 0$, we obtain $A(BC) \geq A_n(BC) \geq A_n B_n C_n$ and $A(BC) \leq A^n B^n C^n$ for all positive n. The same steps for $(AB)C$ show that $A^n B^n C^n \geq (AB)C \geq A_n B_n C_n$ for all positive n. It follows that

$$A(BC) = \sup(A_n B_n C_n) = (AB)C.$$

The other cases are by definition reduced to the case $A \geq 0, B \geq 0, C \geq 0$. Multiplication on R_d is thus associative.

To find an inverse for $A > 0$, first note that $A_n \leq A^n \iff (A^n)^{-1} \leq (A_n)^{-1}$, which gives $(A^n)^{-1} \leq (A_1)^{-1}$. Hence $A' \coloneqq \sup((A^n)^{-1})$ exists and $(A^n)^{-1} \leq A' \leq (A_n)^{-1}$ for all n. This gives

$$A^n(A_n)^{-1} \geq A^nA' \geq AA' \geq A_nA' \geq A_n(A^n)^{-1}$$

for all n. But $A^n - A_n \leq 10^{-n}$ implies

$$(A_n)^{-1} \leq \frac{1}{A^n - 10^{-n}} \text{ and } (A^n)^{-1} \geq \frac{1}{A_n + 10^{-n}},$$

therefore we have

$$\begin{aligned} 1 + \frac{10^{-n}}{A_1 - 10^{-n}} \geq 1 + \frac{10^{-n}}{A^n - 10^{-n}} = \frac{A^n}{A^n - 10^{-n}} \geq A^n(A_n)^{-1} \geq AA' \\ \geq A_n(A^n)^{-1} \geq \frac{A_n}{A_n + 10^{-n}} = 1 - \frac{10^{-n}}{A_n + 10^{-n}} \geq 1 - \frac{10^{-n}}{A_1 + 10^{-n}} \end{aligned}$$

for all n. Since

$$\frac{10^{-n}}{A_1 \pm 10^{-n}} \longrightarrow 0, n \longrightarrow \infty,$$

we must have $AA' = 1$, so that $A^{-1} = A' = \sup((A^n)^{-1})$ is the inverse of A. If $A < 0$, similar calculations show that the inverse is given by $A^{-1} = -\sup(-A^n)^{-1}$.

(b). The proof is much like the proof for associativity and leads to the conclusion

$$AB + AC = \sup(A_nB_n + A_nC_n) = A(B + C).$$

We will not write out the details.

(c). In any Abelian group we have $A + C = B + C \implies A = B$. The result now follows from (3).

(d). If $A > 0, B > 0$ then $A_n > 0, B_n > 0$ for all n, which means that $AB = \sup(A_nB_n) > 0$. $\square$

This is our final result about the properties of R_d. We took the effort to prove it with moderate rigor because it will be central to our conclusion in Section 3.3. But before that, let us for a moment set aside the infinite decimal expansions, and instead look into a more abstract extension of the rational numbers.

3.2 Cantor's construction

Cantor was one of the first to present a rigorous construction of the real numbers, in which he used *Cauchy sequences*. This section will give a description of Cantor's construction following [HS, p. 38-46]. Several proofs will be omitted in this paper, instead we suggest the interested reader to study the proofs presented in [HS]. The construction in essence fills up the holes in $\mathbb{Q}$, thus creating a new field which contains all the rationals, but also has a new, important property. This property will be explained and discussed in some depth after the construction has been presented.

Definition 3.14. Let F be an ordered field. A *Cauchy sequence* in F is a sequence (a_n) in F such that for every $\epsilon \in F, \epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that $i, j \geq N$ implies $|a_i - a_j| < \epsilon$.

Proposition 3.15. *Every Cauchy sequence (a_n) in an ordered field F is bounded, i.e. there exists an $M \in F$ such that $-M < a_n < M$ for all $n \in \mathbb{N}$.*

Proof. If (a_n) is Cauchy and we take $\epsilon = 1 > 0$, then there exits an $N(1) \in \mathbb{N}$ such that $|a_N - a_n| < 1$ for all $n \geq N$. Take $M = \max_{n \leq N}(|a_n|) + 1$. Then

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + \max_{n \leq N}(|a_n|) = M \text{ for all } n \geq N,$$

and clearly $|a_n| < M$ for all $n < N$. Hence $|a_n| < M$ for all $n \in \mathbb{N}$. □

Definition 3.16. Suppose (a_n) is a Cauchy sequence in an ordered field F. Define $[(a_n)]$ as the set of all Cauchy sequences (b_n) in F such that the sequence $(a_n - b_n)$ converges to 0. Let R_c be the set of all $[(a_n)]$ such that (a_n) is a Cauchy sequence in $\mathbb{Q}$.

Note that the representation for a given element in R_c is not unique. In fact, if $(b_n) \in [(a_n)]$, then $[(a_n)] = [(b_n)]$. This also implies that if $[(a_n)] \neq [(b_n)]$ then $[(a_n)] \cap [(b_n)] = \varnothing$.

Notation. For any $q \in \mathbb{Q}$, by $\bar{q}$ we mean the element $[(q, q, q, \ldots)]$ of R_c.

Definition 3.17. Let $[(a_n)], [(b_n)]$ be elements of R_c. Define *addition*,$+$ and *multiplication*,$\cdot$ in R_c by

$$[(a_n)] + [(b_n)] = [(a_n + b_n)]$$

$$[(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)].$$

Theorem 3.18. *Addition and multiplicaiton in R_c are binary operations. The set R_c together with addition and multiplicaiton is a field.*

We will accept this theorem without a proof. Most of the requirements for a field do not require lenghty proofs, as they follow from properties of rational numbers. As expected, the identities are $\bar{0} \in R_c$ and $\bar{1} \in R_c$. However, not all the field requirements are trivial. For a proof, see for instance [HS, p. 40f.].

Theorem 3.19. *The field R_c is totally ordered.*

The requirements for an ordered field can be verified by standard calculations if we define an order on R_c by setting $[(a_n)] > 0$ to mean that there is some $(b_n) \in [(a_n)]$ such that $b_n > 0$ for all n.

Lemma 3.20. *The mapping $\varphi\colon \mathbb{Q} \to R_c$ defined by $\varphi(q) = \bar{q}$ is one-to-one and order preserving.*

Proof. It is clear that φ is well defined, since $[(q, q, q, \ldots)]$ is well defined. Let $p, q \in \mathbb{Q}$ be such that $p \neq q$. Then $p - q = e > 0$ and $\bar{p} - \bar{q} = \bar{e}$. But $\bar{e}$ does not converge to 0, therefore $\varphi(p) = \bar{p} \neq \bar{q} = \varphi(q)$. Thus φ is one-to-one.

If $q \in \mathbb{Q}$ and $q > 0$, then $\varphi(q)$ contains the sequence $(q, q, q, \ldots)$ containing only positive elements of $\mathbb{Q}$. Hence $\varphi(q) > 0$. Similarily, if $q < 0$ then $\varphi(q) < 0$. It follows that φ is order preserving. $\square$

Theorem 3.21. *Every Cauchy sequence in R_c converges to some element $r \in R_c$.*

Proof. Let (x_n) be a Cauchy sequence in R_c. By 3.15, the sequence is bounded. Let $M \in R_c$ be such that $|x_n| < M - 1$ for all n, and take $y_1, z_1 \in \mathbb{Q}$ such that $-M \leq \bar{y}_1 \leq -M + 1$ and $M - 1 \leq \bar{z}_1 \leq M$. Hence

$$\bar{y}_1 \leq x_n \leq \bar{z}_1$$

for all n. For $i \geq 1$, let

$$\bar{\xi}_i = \frac{\bar{y}_i + \bar{z}_i}{2}.$$

If $\bar{y}_i \leq x_n \leq \bar{\xi}_i$ for infinitely many indices n, then let $\bar{y}_{i+1} = \bar{y}_i$ and $\bar{z}_{i+1} = \bar{\xi}_i$. Otherwise, let $\bar{y}_{i+1} = \bar{\xi}_i$ and $\bar{z}_{i+1} = \bar{z}_i$. By this construction, for every $i \geq 1$ we have that

$$0 \leq \bar{z}_i - \bar{y}_i \leq 2^{-i+1} M, \tag{4}$$

and

$$\bar{y}_i \leq x_n \leq \bar{z}_i$$

for infinitely many indices n. In particular, for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $\bar{y}_i \leq x_n \leq \bar{z}_i$ for all $i \geq 1$.

Let $e > 0$ be in $\mathbb{Q}$. It follows from (4) that $|\bar{y}_i - \bar{y}_j| < 2^{-N+1}M < \bar{e}$ if $N > \frac{2M}{\bar{e}}$ and $i, j \geq N$. This implies $|y_i - y_j| < e$ for $i, j \geq N$, since there is an order preserving one-to-one mapping from $\mathbb{Q}$ to R_c. Therefore (y_n) is a Cauchy sequence and we may define $r = [(y_n)] \in R_c$. We now wish to show that (x_n) converges to r.

Let $\epsilon > 0$ be in R_c. Then there is an $N(\frac{\epsilon}{2}) \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\epsilon}{2}$ for all $n, m \geq N$. Moreover, $\bar{y}_i \leq r \leq \bar{z}_i$ and there exists an $n \geq N$ such that $\bar{y}_i \leq x_n \leq \bar{z}_i$ for all $i \geq 1$. Thus $|x_n - r| < 2^{-i+1}M < \frac{\epsilon}{2}$ if we choose $i > \frac{4M}{\epsilon}$. Therefore, for all $m \geq N(\frac{\epsilon}{2})$ we have

$$|x_m - r| \leq |x_m - x_n| + |x_n - r| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Theorem 3.22. *The field R_c has the least upper bound property.*

Proof. Let E be a nonempty subset of R_c that is bounded from above. Take an element $x_1 \in E$ and an upper bound $u_1 \in R_c$ of E. For $i \geq 1$, define

$$\xi_i = \frac{x_i + u_i}{2}.$$

If $\xi_i \in E$, then let $x_{i+1} = \xi_i$ and $u_{i+1} = u_i$. Otherwise, let $x_{i+1} = x_i$ and $u_{i+1} = \xi_i$.

Note that $x_i \leq u_i$ and $u_i - x_i = 2^{-i+1}(u_1 - x_1)$ for all $i \geq 1$. Therefore $|x_i - x_j| \leq 2^{-N+1}(u_1 - x_1)$ for all $i, j \geq N$. For every $\epsilon > 0$ in R_c we then have $|x_i - x_j| < \epsilon$ if $i, j \geq N$, provided that $N > \frac{2(u_1 - x_1)}{\epsilon}$. The same holds for (u_n). Thus the sequences (u_n) and (x_n) are Cauchy, and by Theorem 3.21, the sequences converge to some u and x in R_c, respectively. Further, it follows from the construction of the sequences that $x_i \leq x$ and $u \leq u_i$ for every $i \geq 1$. Therefore $|x - u_i| \leq |x_i - u_i| = 2^{-i+1}(u_1 - x_1)$. By the same argument as previously, it follows that (u_n) converges to x, so that $u = x$.

Since every u_i is an upper bound, u is also an upper bound. For if u is not an upper bound, then there exists some $s \in E$ such that $u < s \leq u_i$ for all $i \geq 1$, which contradicts that (u_n) converges to u. Similarly, there cannot exist a lower upper bound than $u = x$ since every x_i is in E, so the existence of a lower upper bound than x would contradict the fact that (x_n) converges to x. It follows that $x = u = \sup E$. □

Proofs of Theorems 3.21 and 3.22 are mostly inspired by [CP, p. 18-19], along with certain ideas from [HS, p.41-43]. We will conclude this section by pointing out that the described construction is but a special case of the more general construction presented by Hewitt and Stromberg. Following their steps, we have shown that R_c contains $\mathbb{Q}$, is ordered and has the least upper bound property. However, the same steps may be applied to *any* ordered field F, creating a completion $\bar{F}$ which can be shown to be Archimedean ordered provided that F is Archimedean ordered. In such $\bar{F}$, every Cauchy sequence will converge. Furthermore, it can be shown that any two completions $\bar{F}_1, \bar{F}_2$ of Archimedean ordered fields have the same structure, i.e. they are indistinguishable as algebraic structures. A version of this result will be formulated and exploited in the following section.

3.3 Unification of definitions

To show that R_d and R_c represent the same field one needs to show that there exists an order-preserving isomorphism between the two. An isomorphism can be constructed explicitly by sending each $(a_n) \in R_d$ to $[(a_0, a_0.a_1, a_0.a_1a_2, \ldots)] \in R_c$, and then verifying the function's properties. However, this section will present a different approach. The existence of such an isomorphism between R_d and R_c will be asserted by a more general result about all ordered fields with the least upper bound property.

Definition 3.23. Let F be an ordered field with multiplicative identity 1. If n is an integer then the element $n \in F$ is defined as the sum of n multiplicative identities and $-n$ is defined as the sum of n additive inverses of 1. If an element $r \in F$ is of the form nm^{-1}, where $n, m \in \mathbb{Z}$ and $m \neq 0$, then r is called a *fraction*.

Lemma 3.24. *Suppose that F is an ordered field with the least upper bound property. Then F has the Archimedean property and for all $x, y \in F$ such that $x < y$ there exists a fraction $\frac{n}{m} \in F$ such that*

$$x < \frac{n}{m} < y.$$

Proof. (Based on [WR, p. 9]). Let $x, y \in F$ and $x > 0$. Suppose $y \geq nx$ for all positive integers n. Then the set $S = \{xn \colon n \in \mathbb{Z}\}$ is bounded from above and

$s := \sup S$ exists in F. Further, $x > 0$ implies $-x < 0$ so that $s - x < s - 0 = s$. Therefore $s - x$ is not an upper bound of S, and there exists some positive integer n such that $s - x < nx$. But $s - x < nx$ implies $s < nx + x = (n+1)x$. Clearly, $(n+1)$ is a poitive integer, which contradicts that $s = \sup S$. Hence our assumption was wrong, and F is Archimedean.

Now, if $x < y$ then $y - x > 0$. By the Archimedean property, there exists a positive integer m such that

$$m(y - x) > 1 \tag{5}$$

and a positive integer k such that $k > mx$ and $-k < mx$. The set of all such k is bounded from below by 0, hence it has a least element, call it k'. If $k' - 1 \leq nx$, let $n = k'$. Otherwise, let $n = -k'$. Then we have $n - 1 \leq nx < n$, and together with (5) gives

$$mx < n \leq mx + 1 < my.$$

But m is positive, so this gives

$$x < \frac{n}{m} < y.$$

□

Theorem 3.25. *Suppose that F_1 and F_2 are ordered fields with the least upper bound property. Then there exists an order-preserving isomorphism $\varphi\colon F_1 \to F_2$.*

Proof. Denote the additive and multiplicative identities in F_i by 0_i and 1_i, for $i \in \{1, 2\}$. Let $\varphi(0_1) = 0_2$ and $\varphi(1_1) = 1_2$. By n_1 and n_2 we mean the sum of n multiplicative identities in F_1 and F_2, respectively. For fractions $\frac{n_1}{m_1} \in F_1$, define

$$\varphi(\frac{n_1}{m_1}) = \frac{n_2}{m_2}.$$

For all other $x \in F_1$, define $\varphi(x) = \sup\{\varphi(\frac{n_1}{m_1})\colon \frac{n_1}{m_1} < x\}$. Then φ is defined for every element of F_1 and the definition is unambiguous because an $r \in F_1$ is either a fraction or it is not. We will from now on drop the subscripts when it is clear which field's elements we mean.

If $r = 0$ then $\varphi(r) = 0$. If $r = \frac{n}{m} > 0$ then either both n and m are positive, or they are both negative, by Proposition 2.14. Then both $\varphi(n) = n$ and $\varphi(m) = m$ are positive or both negative. Thus $\frac{n_2}{m_2} > 0$, so φ sends positive fractions to positive fractions. Similarily, φ send negative fractions to negative fractions, i.e. φ preserves order on fractions. By this fact, we may write

$$\varphi(x) = \sup\{\varphi(\frac{n_1}{m_1})\colon \frac{n_1}{m_1} \leq x\}$$

for all $x \in F_1$. Now, if x is any positive element in F_1, then there exists some fraction $r \in F_1$ such that $0 < r < x$, therefore $0 < r < \varphi(x)$. The same argument holds for negative $x \in F_1$. It follows that φ preserves order on all elements of F_1.

Let $x, y \in F_1$ such that $x \neq y$. Suppose without loss of generality that $x < y$. Then there exists a fraction r such that $x < r < y$. Hence $\varphi(x) < \varphi(r) \leq \varphi(y)$ and it is clear that $\varphi(x) \neq \varphi(y)$. Injectivity follows.

Next, suppose χ is in F_2. Then $\chi = \sup\{\frac{n}{m}\colon \frac{n}{m} \leq \chi\}$. But the set of fractions $\frac{n_2}{m_2}$ in F_2 which are less than χ is bounded, therefore the set of fractions $E = \{\frac{n_1}{m_1}\colon \varphi(\frac{n_1}{m_1}) \leq \chi\}$ is bounded. If we let $x = \sup E$, then $\varphi(x) = \chi$. Surjectivity follows.

If $x, y \in F_1$ then

$$\varphi(x+y) = \sup\{\varphi(\frac{n_1}{m_1})\colon \frac{n_1}{m_1} \leq x+y\} = \sup\{\varphi(\frac{n_1}{m_1})\colon \frac{n_1}{m_1} \leq x\}+\sup\{\varphi(\frac{n_1}{m_1})\colon \frac{n_1}{m_1} \leq y\}$$

$$= \varphi(x) + \varphi(y);$$

and similarly $\varphi(xy) = \varphi(x)\varphi(y)$. □

Now the following definition is legitimized.

Definition 3.26. The *real number field*, denoted $\mathbb{R}$, is any ordered field with the least upper bound property.

Theorem 3.27. *The field of real numbers* $\mathbb{R}$ *is algebraically and order isomorphic with* R_d *and* R_c*, respectively.*

Proof. The result follows immediately from Theroems 3.13, 3.22 and 3.25. □

Corollary 3.28. *There is an order-preserving isomorphism between* R_d *and* R_c*.*

Thus it is clear that the elements of R_d are the same as the elements of R_c, merely renamed. We may therefore view them as the same field, $R_d \cong R_c \cong \mathbb{R}$.

Conclusions

Much is to be said about the impact of real numbers on mathematics. The concepts of continuous functions, derivatives and integrals in $\mathbb{R}$ all rely on the least upper bound property, which differentiates $\mathbb{R}$ from $\mathbb{Q}$.

The real numbers are also used in dealing with complex numbers, in fact it can be shown that $\mathbb{C}$ is isomorphic to $\mathbb{R} \times \mathbb{R}$, or simply defined as $\mathbb{R} \times \mathbb{R}$ with adequate operations, see for instance [BMPS, p. 2] or [WR, p. 12]. By the Fundamental Theorem of Algebra, every polynomial of degree n in $\mathbb{C}$ has n complex roots, see [BMPS, p. 41]. The complex field is then a final number system which no longer needs to be expanded in search for roots for polynomials. Because of the isomorphism between $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$, many of the properties of $\mathbb{R}$ also hold in the complex number field.

Viewing real numbers as infinite decimal expansions might not have many practical uses, but it is nonetheless worth noting that the decimal expansions give a perfectly valid description of $\mathbb{R}$. Because of this, infinite decimal expansions need not be abandonded and forgotten when we speak of real numbers more generally and rigorously. Thus the decimal expansions may still serve as a tool for creating understanding whenever they are likely to give more insight than abstract representations.

The second construction which we considered has another strenght: it is more general. Not only can one complete the field $\mathbb{Q}$ of rational numbers to obtain $\mathbb{R}$, but the same can be done for any ordered field, as is shown in [HS]. Furthermore, a completion by means of Cauchy sequences actually does not require that we begin with a field. It is enough to have a set M with a notion of distance between its elements, i.e. a *metric space*; see for instance [HN, p. 2-8] on metric spaces. Then, instead of using the absolute value, which is defined in fields, we may use the distance function to define Cauchy sequences in M. Similar steps to those presented in Section 3.2 then lead to the creation of a complete metric space $\overline{M}$, an outline of such a completion is described in [BN]. Complete metric spaces have applications in computer security and engineering, among others [NO].

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