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Similarities and Hausdorff dimension in fractal geometry

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SIMILARITIES AND HAUSDORFF DIMENSION IN FRACTAL GEOMETRY

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Abstract. In this paper, we discuss key characteristics of fractals, we introduce a self-similar structure with the help of iterated function systems and Hausdorff dimension. We show that the attractor of an iterated function system is unique and then present the theory of Hausdorff measure, which provides a general notion of the size of a subset of $\mathbb{R}^n$. The main theorem provides a simple formula to compute the Hausdorff dimension of a self-similar set where the open set condition holds.

Contents

To see a world in a grain of sand,
And heaven in a wild flower.
Hold infinity in the palm of your hand,
And eternity in an hour.

William Blake

Date: January 19, 2022.

1. Introduction

Fractals are geometric figures where the smaller parts are in a way similar to the whole. Those similar patterns occur often in nature from the shape of immense spiral galaxies, to the pattern of a hurricane, to little eddies in a stream.

For a long time fractals were considered too complicated geometric shapes to mathematically analyze, since the classical mathematical tools available were not helpful with rough curves and surfaces. Until Mandelbrot finally put it in words in his book *The Fractal Geometry of Nature* in the 70's, see [Man82]. Mandelbrot was appreciated for taking the first broad attempt to investigate the ubiquitous notion of roughness. Ever since the science of fractals started to expand in mathematical foundations and applications in a variety of fields, like technology, medicine, finance, and computer science. Mandelbrot proposed over time multiple definitions of fractals, that later on turned out to be too restrictive, as they exclude interesting shapes that ought to be considered a fractal. At last he settled on to use a fractal without a pedantic definition [Edg07].

However, fractals have in common multiple properties, so when we refer to a set F as a fractal, we will typically have the following in mind.

- F has a fine structure, i.e. details on arbitrary small scales.
- F is too irregular to be described with the language of traditional mathematics. For example, the set F doesn't necessarily satisfy some simple geometric condition, nor it is the set of solutions for any simple equation.
- Often F has some form of self-similarity, could be determinant or statistical.
- Usually, the 'fractal dimension' of F is greater than its topological dimension.
- Although F has an intricate detailed structure, often F is defined in a simple way, perhaps by a recursive procedure.

In this work we use similarity mappings in form of contractions to define self similar sets and iterated function systems. Then we show the uniqueness property of the attractor for the iterated function system. We illustrate these concepts with the help of the middle fourth Cantor set example.

Of the variety of 'fractal dimension', we here define the Hausdorff measure and dimension of subsets of $\mathbb{R}^n$, to describe the roughness of the fractal. We visualize the non-integer dimension with the help of the von Koch curve and then calculate the dimension of this set.

We will see that it is often hard to estimate a lower and an upper bound for the dimension of a fractal set. The main theorem of this work solves this by providing a simple formula to calculate the dimension of a self-similar set for which the open set condition holds. To prove the main theorem we use all tools we introduced along the way. We take a closer look on the process of estimating the bounds of the dimension, through defining an outer measure on our fractal set.

The main resource in writing this bachelor thesis has been Falconer's *Fractal Geometry* [Fal04]. Thereby, the general approach and much content is based on this book. However, this thesis covers details concerning questions about measure theory that Falconer glosses over. A relevant source has been [Eij18].

The figures we use come from [Fal04] except for the middle fourth Cantor set, which is an edited version from the public domain.

2. Similarity mappings

When we see a fractal, the first thing we notice is how a certain geometrical shape is being repeated, in variety of sizes. In this thesis, we use contraction mappings from the set to its self, in order to describe the self similarity property of the fractal set. We start by introducing the following definitions and then we build on them. Throughout, let D be a closed subset of $\mathbb{R}^n$.

Definition 2.0.1. A mapping $S\colon D \to D$ is called a *contraction* on D if there is a number c with $0 < c < 1$ such that

$$|S(x) - S(y)| \leq c|x - y| \qquad \forall x, y \in D.$$

The scaling factor c is called the *ratio* of S. Clearly, contractions are continuous mappings.

Definition 2.0.2. If in a contraction S on D we have

$$|S(x) - S(y)| = c|x - y| \qquad \forall x, y \in D,$$

then the mapping is called a *similarity.*

A similarity S transform sets into geometrically similar sets, in the sense that the map preserves angels. So in the case of the triangle formed by the points $\{x, y, z\} \subseteq D$, with sides $x' = |x - y|, y' = |y - z|, z' = |z - x|$, we apply a similarity S to $\{x, y, z\}$ and obtain the points $\{S(x), S(y), S(z)\} \subseteq D$. The triangle formed by these points has sides of length cx', cy', cz'. The corresponding triangle is similar, which means the angles are the same.

Definition 2.0.3. The *diameter* of a set U is defined as

$$|U| = \sup\{|x - y|\colon x, y \in U\}.$$

We define $|\emptyset| = 0$.

Lemma 2.0.4. *If S is a similarity with ratio $c > 0$, and U is a non-empty subset of $\mathbb{R}^n$, then*

$$|S(U)| = c|U|.$$

Proof. By the definition of the diameter and the definition of similarity, it follows

$$\begin{aligned}
|S(U)| &= \sup\{|a - b|\colon a, b \in S(U)\} \\
&= \sup\{|S(x) - S(y)|\colon x, y \in U\} \\
&= \sup\{c|x - y|\colon x, y \in U\} \\
&= c \sup\{|x - y|\colon x, y \in U\} \\
&= c|U|.
\end{aligned}$$

□

3. Iterated Function Systems

In this section we look at finite collections of contractions, which are known as iterated function systems or IFS. Their fundamental property is that they determine a unique set called the attractor for the IFS. The attractor of an IFS is usually a fractal. We rely on attractors and their property in the proof of the main result.

Definition 3.0.1. A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$ with $m \geq 2$ is called *iterated function system.*

Definition 3.0.2. An *attractor* (or *invariant set*) for the *IFS* $\{S_1, \ldots, S_m\}$ is a non-empty compact subset F of D such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Definition 3.0.3. Let $\mathcal{S}$ denote the class of all non-empty compact subsets of $D \subseteq \mathbb{R}^n$, and let the IFS $\{S_1, \ldots, S_m\}$ be given. We define the corresponding transformation S on $\mathcal{S}$ by

$$S(E) = \bigcup_{i=1}^{m} S_i(E),$$

for any set E in $\mathcal{S}$.

Note that sets in $\mathcal{S}$ are transformed by S into other sets in $\mathcal{S}$, as shown in the proof of Theorem 3.3.1. By using a transformation we can express the attractor of an IFS as an element $F \in \mathcal{S}$ such that $F = S(F)$. We illustrate these new concepts with an example. We look at the middle fourth Cantor set which displays many fractal characteristics.

3.1. **Example: The middle forth Cantor set.**

FIGURE 1. The first seven iterations for the middle fourth Cantor set.

Formally as a self-similar fractal, the middle fourth cantor set can be defined via an iterated function system. We start first constructing the set F which represents the middle fourth Cantor set from a unit interval by a sequence of deletion operations, see Figure 1. Let E_0 be the interval $[0, 1]$ remove the middle fourth segment of this interval, i.e. $(\frac{3}{8}, \frac{5}{8})$, and let E_1 be the union of the intervals $[0, \frac{3}{8}]$ and $[\frac{5}{8}, 1]$. Remove the middle fourth segments of these two intervals and let E_2 be the union of the resulted four intervals as follows

$$E_2 = \left[0, \frac{9}{64}\right] \cup \left[\frac{15}{64}, \frac{24}{64}\right] \cup \left[\frac{40}{64}, \frac{49}{64}\right] \cup \left[\frac{55}{64}, 1\right].$$

Continuing in this way we obtain a sequence of compact sets E_n such that:

- $E_0 \supset E_1 \supset E_2 \cdots$
- E_n is the union of 2^n intervals.
- Since the middle intervals of length $(\frac{1}{4})^n$ are removed from $[0, 1]$ for each n^{th} iteration, the length of the set E_n is $(\frac{3}{4})^n$.

The set $F = \bigcap_{i=1}^{\infty} E_n$ is the middle fourth Cantor set. It consists of the numbers that are in E_n for all n.

It can be shown that the middle fourth Cantor set has the following properties

- The set is a perfect set, i.e. it is closed and has no isolated points.
- The set is compact.
- The set is nowhere dense, i.e. the closure of the set does not contains any nonempty open intervals.

We now express this set in terms of similarity mappings. Let $S_1, S_2\colon [0,1] \to [0,1]$ be given by

$$S_1(x) = \frac{3}{8}x \tag{1}$$

$$S_2(x) = \frac{3}{8}x + \frac{5}{8}. \tag{2}$$

We see that $S_1(F)$ and $S_2(F)$ represent the left and right 'halves' of F. We have for all n

$$E_{n+1} = S_1(E_n) \cup S_2(E_n) = S(E_n)$$

and hence, by induction

$$F = \bigcap_{k=1}^{\infty} S^k(E_0).$$

We come back to this example multiple times, and add to it as we go.

3.2. **Hausdorff metric.** In this section we introduce the concept of δ-neighborhood and then define the Hausdorff metric. We will need these concepts to show the fundamental property of the IFS.

Definition 3.2.1. The δ-*neighborhood* of a set A is the set of points within distance δ of A, i.e. $A_\delta = \{x \in D : |x - a| \leq \delta \text{ for some } a \in A\}$.

Let $\mathcal{S}$ denote the class of all non-empty compact subsets of $D \subset \mathbb{R}^n$. We want to define a metric on $\mathcal{S}$, where the distance between two sets A and B in $\mathcal{S}$ is the least δ, such that the δ-neighborhood of A contains B, and the δ-neighborhood of B contains A. See Figure 2.

Definition 3.2.2. The *Hausdorff distance* on $\mathcal{S}$ is defined as

$$d(A,B) = \inf\{\delta \geq 0\colon A \subset B_\delta \text{ and } B \subset A_\delta\},$$

for $A, B \in \mathcal{S}$.

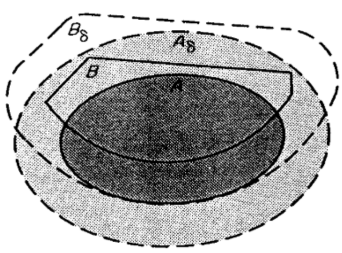

FIGURE 2. The Hausdorff distance between the sets A and B is the least $\delta \geq 0$ such that the δ-neighborhood A_δ of A contains B and vice versa.

Lemma 3.2.3. *The Hausdorff distance is a metric on $\mathcal{S}$.*

Proof. To show that d is a metric function, we need to show that it satisfies the three requirements of a metric.

(i) $d(A,B) > 0$ if $A \neq B$; $d(A,A) = 0$.
By definition it clearly holds $d(A,B) \geq 0$ since $d(A,B)$ is the infimum over non-negative numbers. Suppose $d(A,B) = 0$. Let $b \in B$ and note that for $n > 0$, we have $b \in B \subset A_{\frac{1}{n}}$ and so there exists $b_n \in A$, such that $|b - b_n| \leq \frac{1}{n}$, thus $b \in \overline{A}$. We know that A is compact, and hence closed we get $B \subseteq A$. Analogously, it can be shown $A \subseteq B$. We conclude $A = B$.

(ii) $d(A,B) = d(B,A)$. This is true by definition.

(iii) $d(A,B) \leq d(A,C) + d(C,B)$ where $A, B, C \in \mathcal{S}$.

Suppose $d(A,C) = \epsilon_1$, $d(C,B) = \epsilon_2$. Then for every $\delta_1 > \epsilon_1$ and for every $\delta_2 > \epsilon_2$ we have

$$A \subset C_{\delta_1}, C \subset A_{\delta_1}, C \subset B_{\delta_2}, B \subset C_{\delta_2}.$$

So it follows:

$$A \subset C_{\delta_1} \subseteq B_{\delta_1+\delta_2} \text{ and } B \subset C_{\delta_2} \subseteq A_{\delta_2+\delta_1},$$

where the second and last inclusions hold by the triangle inequality.
Thus $d(A,B) \leq \delta_1 + \delta_2$ for every δ_1, δ_2 with $\delta_1 > \epsilon_1$ and $\delta_2 > \epsilon_2$. So

$$d(A,B) \leq \epsilon_1 + \epsilon_2 = d(A,C) + d(C,B).$$

We conclude the Hausdorff distance is a metric on $\mathcal{S}$. $\square$

From hereon, we consider $\mathcal{S}$ as a metric space by means of the Hausdorff distance.

3.3. **Unique attractors.** We here show that iterated function systems define unique non-empty compact attractors, which are often fractals.

Theorem 3.3.1. *Let an iterated function system formed by contractions $\{S_1, \dots, S_m\}$ on $D \subseteq \mathbb{R}^n$ be given. Then this system has a unique attractor F.*
Moreover, if $E \in \mathcal{S}$ such that $S_i(E) \subset E$ for all $1 \leq i \leq m$, then the attractor can be expressed as

$$F = \bigcap_{k=0}^{\infty} S^k(E),$$

where S^k is the kth iterate of the transformation S corresponding to the given IFS.

Proof. By applying a translation if necessary, we can without loss of generality assume that $0 \in D$. The strategy will be as follows. First we show the existence of a set $E_0 \in \mathcal{S}$ such that $S_i(E_0) \subset E_0$ for all $1 \leq i \leq m$. Then we show that for any $E \in \mathcal{S}$ with $S_i(E) \subset E$ for all i, it holds that $\bigcap_{k=0}^{\infty} S^k(E)$ is an attractor. Finally, we show the uniqueness.

For the first part, we choose r such that it satisfies $\frac{|S_i(0)|}{1-c_i} \leq r$ for all i, where c_i denote the ratio of S_i. Then take $E_0 := D \cap B(0,r)$, where $B(0,r)$ is the closed ball of radius r around 0. Take $x \in E_0$, then by the triangle inequality we have $|S_i(x)| \leq |S_i(x) - S_i(0)| + |S_i(0)|$. Combining this with the definition of contraction gives $|S_i(x)| \leq c_i|x| + |S_i(0)|$. It follows $|S_i(x)| \leq c_i|x| + r(1 - c_i)$, and since $|x| \leq r$, we get $|S_i(x)| \leq c_i r + r - c_i r = r$. So we have found a set E_0 such that $S_i(E_0) \subset E_0$.

Since D is closed, subsets of D are closed in D if and only if they are closed in $\mathbb{R}^n$. Since we know that a subset of $\mathbb{R}^n$ is compact exactly when it is closed and bounded by [Rud64, Thm. 2.41], the same holds true for subsets of D. In particular, E_0 is compact, hence an element of $\mathcal{S}$.

Now let E be any element of $\mathcal{S}$, such that $S_i(E) \subset E$ for all $1 \leq i \leq m$. We want to show that if we iteratively apply S to the set E we will get the attractor of our IFS. As a consequence of the condition we have on E, we get $S^{k+1}(E) \subset S^k(E)$ for every $k \in \mathbb{N}$. We note that $S(E) = \bigcup_{i=1}^{m} S_i(E)$ is a compact set, since the continuous image of a compact set is compact, see [Rud64, Thm. 4.14], and a finite union of compact sets is compact. By induction, it follows that $S^k(E)$ is compact. We know that the intersection of a decreasing sequence of nonempty compact sets is nonempty, see [Rud64, Thm. 2.36], therefore $F := \bigcap_{k=0}^{\infty} S^k(E)$ is nonempty. Furthermore, as an intersection of closed and bounded sets, it is itself closed and bounded, and hence compact.

We note

$$S(F) = S\big(\bigcap_{k=0}^{\infty} S^k(E)\big) \subseteq \bigcap_{k=0}^{\infty} S\big(S^k(E)\big),$$

which implies $S(F) \subseteq \bigcap_{k=0}^{\infty} S^{k+1}(E) \subseteq S(F)$, so they are equal. So

$$S(F) = \bigcap_{k=1}^{\infty} S^k(E) = F,$$

and we conclude that F is an attractor for the IFS.
To prove the uniqueness we assume that $H \neq F$ is another attractor. We know $S(H) = H$, and $S(F) = F$. We have

$$\begin{aligned} d(F,H) = d\big(S(F), S(H)\big) &= d\big(\bigcup_{i=1}^{m} S_i(F), \bigcup_{i=1}^{m} S_i(H)\big) \\ &\leq \max_{1 \leq i \leq m} d(S_i(F), S_i(H)). \end{aligned}$$

This is true since by the Hausdorff distance, if the δ-neighborhood $(S_i(F))_\delta$ contains $S_i(H)$ for every i, then $(\bigcup_{i=1}^{m} S_i(F))_\delta$ contains $\bigcup_{i=1}^{m} S_i(H)$ and vice versa. Using the contraction definition we obtain

$$d(F,H) = d(S(F), S(H)) \leq (\max_{1 \leq i \leq m} c_i) d(F,H).$$

By $0 < (\max_{1 \leq i \leq m} c_i) < 1$, the above implies that $d(F,H) = 0$. Since $\mathcal{S}$ is a metric space we conclude that $F = H$. □

Now we consider the middle fourth Cantor set example we looked at earlier in Subsection 3.1, and note that we can describe the set as the attractor of the similarities S_1, S_2 defined in Equations (1) and (2), so that

$$F = S_1(F) \cup S_2(F).$$

4. Outer Measure

When studying the mathematics of fractals, one will not get far before encountering measures in some form. For most fractal applications a few basic ideas of measure are needed. In our case we will only encounter the outer measure and the mass distribution on subsets of $\mathbb{R}^n$.

Definition 4.0.1 (Outer measure)**.** We call μ an *outer measure* on a set $\mathcal{X}$ if μ assigns a non-negative number, possibly ∞, to each subset of $\mathcal{X}$ such that:

(i) $\mu(\emptyset) = 0$
(ii) $\mu(A) \leq \mu(B)$ if $A \subset B$
(iii) If $A_1, A_2, \ldots$ is a countable sequence of sets then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

5. Hausdorff Measure

For a better understanding of the mathematics of fractals, familiarity with Hausdorff measure and dimension are essential. Of the variety of 'fractal dimension' in use, the definition of Hausdorff is one of the most important. It has the advantage of being defined on any set, and is mathematically convenient, as it is based on measures, which are relatively easy to manipulate.

The Hausdorff measure is a generalization for the traditional notion of length, area, and volume to non-integer dimensions. The idea is to cover a given set with small sets, for which we have an approximation of their size. Then we take these covering sets even smaller so the approximation becomes more accurate. The Hausdorff measure will roughly be the limit of those approximations. We first make precise what we mean by small covering sets.

Definition 5.0.1. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subseteq \bigcup_{i=1}^{\infty} U_i$ with $0 \leq |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F.

Definition 5.0.2 (Hausdorff measure). Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$. For every $\delta > 0$ we define

$$\mathcal{H}_{\delta}^{s}(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

Then the limit

$$\mathcal{H}^{s}(F) = \lim_{\delta \to 0} \mathcal{H}_{\delta}^{s}(F),$$

exists, and is called *the s-dimensional* ***Hausdorff measure of F***.

To unravel this definition, starting with $\mathcal{H}_{\delta}^{s}(F)$, we first look at all the possible covers of the set F with diameters at most δ and then we seek to minimize $\sum_{i=1}^{\infty} |U_i|^s$ by taking the infimum. Decreasing δ means permitting less covers, which results in an increase of $\mathcal{H}_{\delta}^{s}(F)$. Therefore the limit $\lim_{\delta \to 0} \mathcal{H}_{\delta}^{s}(F)$ always exists, although it may be infinite. Next we show that the Hausdorff measure indeed is an outer measure.

Theorem 5.0.3. *The function that sends $F \subset \mathbb{R}^n$ to the Hausdorff measure $\mathcal{H}^s(F)$ is an outer measure.*

Proof. We show that the Hausdorff measure satisfies the requirements of an outer measure.

(i). For every $\delta > 0$, we have $|\emptyset| = 0 < \delta$ and $\emptyset \subset \emptyset$, implying that $\{\emptyset\}$ is a δ-cover of $\emptyset$. Since we take the infima of all δ-covers we conclude $\mathcal{H}_{\delta}^{s}(\emptyset) = 0$ for any $\delta > 0$ and therefore, $\mathcal{H}^s(\emptyset) = 0$.

(ii). If $A \subset B$, then any cover of B is also a cover of A, implying that $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$.

(iii). We show that $\mathcal{H}_{\delta}^{s}(\cup_i A_i) \leq \sum_i \mathcal{H}_{\delta}^{s}(A_i)$. Let $\epsilon > 0$, write $\epsilon = \sum_i \epsilon_i$, where $\epsilon_i > 0$. Take a δ-cover $\{U_j^{(i)}\}$ of A_i such that

$$\sum_j |U_j^{(i)}|^s \leq \mathcal{H}_{\delta}^{s}(A_i) + \epsilon_i.$$

Then $\{U_j^{(i)}\}_{i,j}$ is a δ-cover of $U_i A_i$. So

$$\mathcal{H}_{\delta}^{s}(\cup_i A_i) \leq \sum_{i,j} |U_j^{(i)}|^s \leq \sum_i (\mathcal{H}_{\delta}^{s}(A_i) + \epsilon_i) = \sum_i \mathcal{H}_{\delta}^{s}(A_i) + \epsilon.$$

Letting $\epsilon \to 0$ gives $\mathcal{H}^s_\delta(U_i A_i) \le \sum_i H^s_\delta(A_i)$, and letting $\delta \to 0$ gives $\mathcal{H}^s(U_i A_i) \le \sum_i H^s(A_i)$.

Thus, the Hausdorff measure is indeed an outer measure. □

It can be shown that $\mathcal{H}^s(F)$ is a measure, which is a more specific notion than the outer measure, but we don't need this property of $\mathcal{H}^s(F)$ here.

5.1. **Some properties of the Hausdorff measure.** We note that the Hausdorff measure has multiple useful properties, we only need the scaling property and the fact that this measure function is non-increasing. We discuss these properties in the following paragraph.

Lemma 5.1.1. *Let S be a similarity with ratio $c > 0$. If $F \subset \mathbb{R}^n$, then*

$$\mathcal{H}^s(S(F)) = c^s \mathcal{H}^s(F).$$

Proof. Let

$$A = \left\{ \sum_{i=1}^{m} c^s |U_i|^s \colon \{U_i\} \text{ is a } \delta\text{-cover of } F \right\},$$

$$B = \left\{ \sum_{i=1}^{m} |V_i|^s \colon \{V_i\} \text{ is a } c\delta\text{-cover of } S(F) \right\}.$$

Then we have $c^s \mathcal{H}^s_\delta(F) = \inf A$, and $\mathcal{H}^s_{c\delta}(S(F)) = \inf B$. We start by showing $c^s \mathcal{H}^s_\delta(F) \le \mathcal{H}^s_{c\delta} S(F)$. Let $\{U_i\}$ be a δ-cover of F. Then for $V_i := S(U_i)$ we have

$$\cup_i V_i = \cup_i S(U_i) = S(\cup_i U_i) \supseteq S(F),$$
$$\text{and } |V_i| = c|U_i| \le c\delta.$$

Furthermore, $\sum_{i=1}^{m} c^s |U_i|^s = \sum_{i=1}^{m} |S(U_i)|^s = \sum_{i=1}^{m} |V_i|^s$. It follows that $A \subseteq B$. Therefore, it holds

$$\mathcal{H}^s_{c\delta}(S(F)) = \inf B \le \inf A = c^s \mathcal{H}^s_\delta(F).$$

We let $\delta \to 0$ and obtain $\mathcal{H}^s(S(F)) \le c^s \mathcal{H}^s(F)$.
We replace S by S^{-1} and c by $\frac{1}{c}$, and F by $S(F)$ to get the other side of the inequality. □

Lemma 5.1.2. *For every $F \subset \mathbb{R}^n$ and $\delta < 1$ the function $\mathcal{H}^s_\delta(F)$ is non-increasing in $s > 0$.*

Proof. Observe for some $\alpha < 1$ the function $s \mapsto \alpha^s$ is a decreasing function of s. As long as $|U_i| < \delta < 1$ it follows that if $s_1 < s_2$ then

$$\sum_{i=1}^{\infty} |U_i|^{s_1} \ge \sum_{i=1}^{\infty} |U_i|^{s_2},$$

and hence,

$$\begin{aligned} \mathcal{H}^{s_1}_\delta(F) &= \inf \left\{ \sum_{i=1}^{\infty} |U_i|^{s_1} : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } |U_i| < \delta \right\} \\ &\ge \inf \left\{ \sum_{i=1}^{\infty} |U_i|^{s_2} : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } |U_i| < \delta \right\} \\ &= \mathcal{H}^{s_2}_\delta(U). \end{aligned}$$

□

Corollary 5.1.3. *The Hausdorff measure $\mathcal{H}^s(F)$ is a non-increasing function.*

6. Hausdorff dimension

The Hausdorff dimension is one key concept in studying fractals. It is a way to describe how rough (not smooth) the surface of the fractal is. Most interesting fractals have a non-integer dimension, we will use the Hausdorff dimension to make sense of the concept of fractional dimension. Before we introduce the definition, we first discover the behavior of the Hausdorff measure $\mathcal{H}^s(F)$ which is either zero or infinity for almost all values of s. In fact, only at one point the outer measure 'jumps' from one to the other. This unique value of s will then be the Hausdorff dimension of the fractal.

By taking the value of s too large the measure will always be zero, and by taking s too small the measure will tend to infinity, see Figure 3. The formal definition comes after the following lemma which supports our claim regarding this behavior.

Lemma 6.0.1. *Let $F \subset \mathbb{R}^n$ be given. If $\mathcal{H}^s(F) < \infty$, then for any $r > s$ we have $\mathcal{H}^r(F) = 0$. And if $\mathcal{H}^s(F) > 0$, then for any $r < s$ we have $\mathcal{H}^r(F) = \infty$.*

Proof. Assume $\mathcal{H}^s(F) < \infty$, take $r > s$, let $\delta > 0$, then for any δ-cover of F we have

$$\sum_{i=1}^{\infty}|U_i|^r = \sum_{i=1}^{\infty}|U_i|^{r-s}|U_i|^s \le \delta^{r-s}\sum_{i=1}^{\infty}|U_i|^s$$

and hence,

$$\begin{aligned}\mathcal{H}^r_\delta(F) &= \inf\left\{\sum_{i=1}^{\infty}|U_i|^r \colon \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}\\ &\le \inf\left\{\delta^{r-s}\sum_{i=1}^{\infty}|U_i|^s \colon \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}\\ &\le \delta^{r-s}\mathcal{H}^s_\delta(F).\end{aligned}$$

By letting $\delta \to 0$, we obtain

$$\mathcal{H}^r(F) = \lim_{\delta\to 0}\mathcal{H}^r_\delta(F) \le \lim_{\delta\to 0}\delta^{r-s}\mathcal{H}^s_\delta(F) = 0.$$

For the second claim we assume $\mathcal{H}^s(F) > 0$ and take $r < s$. Then we get

$$\sum_{i=1}^{\infty}|U_i|^r = \sum_{i=1}^{\infty}|U_i|^{r-s}|U_i|^s \ge \delta^{r-s}\sum_{i=1}^{\infty}|U_i|^s.$$

Analogously, letting $\delta \to 0$ gives

$$\mathcal{H}^r(F) = \lim_{\delta\to 0}\mathcal{H}^r_\delta(F) \ge \lim_{\delta\to 0}\delta^{r-s}\mathcal{H}^s(F) = \infty. \quad \square$$

This property of the Hausdorff measure justifies the following definition.

Definition 6.0.2. Hausdorff dimension For any $F \subset \mathbb{R}^n$, we define:

$$\dim_H(F) = \inf\{s \ge 0 \colon \mathcal{H}^s(F) = 0\} = \sup\{s \colon \mathcal{H}^s(F) = \infty\}.$$

We here define the supremum of the empty set to be 0.

By Lemma 6.0.1 we have

$$\mathcal{H}^s(F) = \begin{cases}\infty & \text{if } 0 \le s < \dim_H F\\ 0 & \text{if } s > \dim_H F.\end{cases}$$

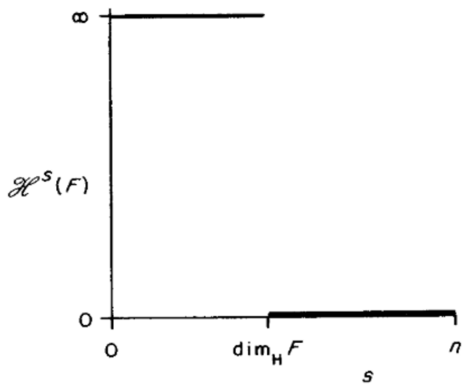

FIGURE 3. Graph of $\mathcal{H}^s(F)$ against s for a set F. The Hausdorff dimension is the value of s at which the jump from ∞ to 0 occurs.

Corollary 6.0.3. *It follows by Corollary 5.1.3 and Lemma 6.0.1 for any $F \subset \mathbb{R}^n$ that if*

$$0 < \mathcal{H}^s(F) < \infty,$$

then $s = \dim_H F$.

6.1. **Integer Hausdorff dimension.** We all have an intuitive notion of integer dimension for simple geometric shapes. For example, we know that a line should be one dimensional, a plane should be two dimensional, and so on. We here argue that the Hausdorff dimension agrees with this intuition.

One dimensional geometrical shapes: We here show that a line segment has Hausdorff dimension one. Let $L = [0, 2] \subset \mathbb{R}^1$. It suffices by Corollary 6.0.3 to show that $\mathcal{H}^1(L) = 2$, then $\dim_H(L) = 1$. First observe that, for any $\delta > 0$, and any δ-cover $\{U_i\}$ of L, it holds that $\sum_i |U_i|$ is the sum of the lengths of the intervals U_i. Now since $\{U_i\}$ covers L, it follows that $\sum_i |U_i| \geq 2$. Hence, $\mathcal{H}^1_\delta(L) \geq 2$ for all $\delta > 0$, so $\mathcal{H}^1(L) \geq 2$.

Now observe that $\mathcal{H}^s(L) = \lim_{n\to\infty} \mathcal{H}^s_{\frac{1}{n}}(L)$, where the limit is taken over $n \in \mathbb{N}$. Then take, for $n \in \mathbb{N}$, the sets $U_i := [\frac{i}{n}, \frac{i+1}{n}]$ with $0 \leq i < 2n - 1$, which covers L. It follows that $\mathcal{H}^1_{(\frac{1}{n})}(L) \leq \sum_i |U_i| = \frac{2n}{n} = 2$. This value represents the length of the interval L. By letting $n \to \infty$ we get $\mathcal{H}^1(L) = 2$ and hence, $\dim_H(L) = 1$. Remark that we chose to look at $L \subset \mathbb{R}^1$ so we can only cover L with intervals. In the case where $L \subset \mathbb{R}^n$, the process is quite similar, except we look at the intersection of the covering U_i with L. It follows directly that $|L \cap U_i| \leq |U_i|$, and $\sum_i |U_i| \geq \sum_i |L \cap U_i| \geq 2$.

Two dimensional geometrical shapes: We now want to show that the unit square $S \subset \mathbb{R}^2$ has Hausdorff dimension two. We define $S := [0, 1] \times [0, 1]$. We cover S by n^2 squares of sides $\frac{1}{n}$, so each small square has diameter $\frac{\sqrt{2}}{n}$. So for every $\delta \geq \frac{\sqrt{2}}{n}$ we have $\mathcal{H}^2_\delta(S) \leq \sum_{i=1}^{n^2} |\frac{\sqrt{2}}{n}|^2 = 2$. So $\mathcal{H}^2(S) \leq 2$. The process of finding a lower bound of $\mathcal{H}^2(S)$ is not easy (because it involves looking at $\sum_i |U_i|^2$ for all $\{U_i\}$ a δ-cover of S). However, it can be shown using [Fal86, Thm1.11], that for S measurable in $\mathbb{R}^2$, we have $\mathcal{H}^2(S) = c_n \mathrm{vol}^2(S)$, for c_n a constant depending on n. Using this results in $H^2(S) = 2$.

6.2. **The non-integer Hausdorff dimension of the von Koch curve.** We here study the von Koch curve to illustrate a non-integer Hausdorff dimension. In this paragraph we offer a closer look at fractional dimension and go through the process of estimating it to reveal the challenges it holds in practice.

We construct the von Koch curve V from a unit interval $E_0 = [0, 1]$. See Figure 4. The set E_1 is obtained by dividing E_0 into three equal segments and replacing the middle segment by the two sides of an equilateral triangle of the same side length as the segment being removed. Hence, the total length of E_1 is $\frac{4}{3}$. For E_2 we repeat, taking each of the four resulting segments, dividing each of them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle with sides of the same length as the length of the segment being removed. The total length of E_1 equals $(\frac{4}{3})^2$. Thus E_k comes from replacing the middle third of each straight line segment in E_{k-1} by the other two sides of the equilateral triangle. As k tends to infinity, it can be shown that E_k approaches a curve V, called the *von Koch curve*, in the sense that for all $\epsilon > 0$ there is a K such that for all $k > K$ the Hausdorff distance $d(E_k, V)$ is smaller than ϵ.

This curve V is self similar as it can be expressed as the union of four smaller copies of itself. Let $\alpha = \exp \frac{\pi i}{3}$ and let $S_1, S_2, S_3, S_4 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$S_1(x) = \frac{x}{3}; \qquad S_2(x) = \frac{\alpha x + 1}{3}; \qquad S_3(x) = \frac{\alpha^2 x + 2}{3}; \qquad S_4(x) = \frac{x + 2}{3}.$$

We observe that all these similarities scale the unit interval with a ratio $\frac{1}{3}$. S_2 and S_3 are similarities with a translation and a rotation. V is the attractor of these four similarities, so we can write $V = \cup_{i=1}^{4} S_i(V)$.

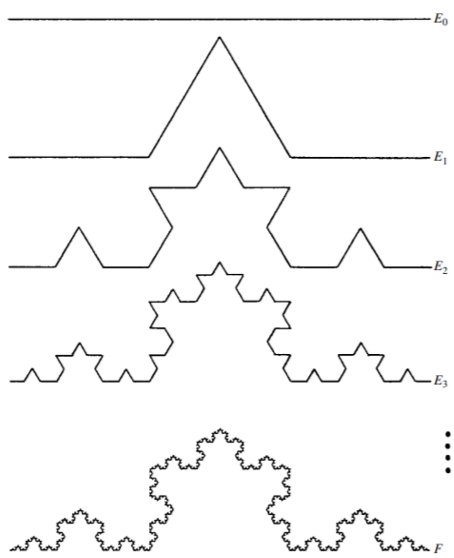

Figure 4. Construction of the von Koch curve

Calculating the dimension of this set with the tools we developed so far seems to be not possible. For example we define $V := \lim_{k\to\infty} E_k$, so in each finite step we can calculate $\mathcal{H}^s(E_k)$. However, we do not know whether $\mathcal{H}^s(\lim_{k\to\infty} E_k)$ is equal to $\lim_{k\to\infty} \mathcal{H}^s(E_k)$. For now we assume that for $s = \dim_H(V) > 0$, it holds that $0 < \mathcal{H}^s(V) < \infty$, and $s > 0$. (a big assumption, but we justify it in the last section of this paper). We want to write V as a disjoint union. To this end, we define V' as

$V \setminus \{(1,0)\}$, then we have that V' is a disjoint union of sets $S_i(V')$, and we have $\mathcal{H}^s(V) = \mathcal{H}^s(V') + \mathcal{H}^s(\{1,0\}) = \mathcal{H}^s(V')$. So we have

$$\begin{aligned} \mathcal{H}^s(V') &= \mathcal{H}^s(S_1(V') \cup S_2(V') \cup S_3(V') \cup S_4(V')) \\ &= \frac{1}{3^s}\mathcal{H}^s(V') + \frac{1}{3^s}\mathcal{H}^s(V') + \frac{1}{3^s}\mathcal{H}^s(V') + \frac{1}{3^s}\mathcal{H}^s(V') \\ &= \frac{4}{3^s}\mathcal{H}^s(V'). \end{aligned}$$

By assuming that the Hausdorff measure is finite and non-zero, we can obtain $1 = \frac{4}{3^s}$. This equation yields $s = \frac{\log 4}{\log 3} \approx 1.26$, which is the non-integer Hausdorff dimension of the von Koch curve.

6.3. **Mass distribution.** Roughly speaking, a mass distribution on a set is an outer measure that is finite and non-zero. We introduce this concept in order for us to estimate a concrete lower bound for the Hausdorff dimension of self-similar sets. We define a mass distribution on the set, and then use the mass distribution principle that we discuss in this paragraph.

Definition 6.3.1. An outer measure μ on a bounded subset F of $\mathbb{R}^n$ for which $0 < \mu(F) < \infty$ will be called a mass distribution on F.

Theorem 6.3.2 (The mass distribution principle)**.** *Let μ be a mass distribution on F, suppose that for some s there are numbers $c > 0$ and $\epsilon > 0$ such that*

$$\mu(U) \le c|U|^s$$

for all sets U with the property $|U| \le \epsilon$. Then $\mathcal{H}^s(F) \ge \frac{\mu(F)}{c}$ and

$$s \le \dim_H(F).$$

Proof. Let $\{U_i\}$ be any cover of F then

$$0 < \mu(F) \le \mu\left(\bigcup_i (U_i \cap F)\right) \le \sum_i \mu(U_i \cap F) \le c\sum_i |U_i|^s.$$

Recall that

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^m |U_i|^s \colon \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

and hence, $\mathcal{H}^s_\delta(F) \ge \frac{\mu(F)}{c}$. By letting $\delta \to 0$ we obtain $\mathcal{H}^s(F) \ge \frac{\mu(F)}{c}$. □

7. Hausdorff dimension of self-similar sets

This section includes the main result of this work. We will show a simple formula for the dimension of self-similar sets. Throughout, we let F be the attractor of the IFS formed by the similarities $\{S_1, \dots, S_m\} \colon \mathbb{R}^n \to \mathbb{R}^n$ with

$$|S_i(x) - S_i(y)| = c_i|x - y| \qquad \forall x, y \in \mathbb{R}^n,$$

where c_i is the ratio of the similarity S_i satisfying $0 < c_i < 1$. We use the properties of the attractor F and the tools Hausdorff measure provides to show that, under certain conditions the dimension of the set F is the unique solution s for the equation

$$\sum_{i=1}^m c_i^s = 1. \tag{3}$$

In the proof of this result we estimate an upper and a lower bounds for the Hausdorff measure, so we can use that the attractor has a finite Hausdorff measure. This estimation requires a certain effort and uses all the tools we have introduced so far. We will need to define an outer measure on our set F so that the mass distribution principle conditions are satisfied. Furthermore, we ask of our set F to satisfy the open set condition (defined in Definition 7.1.6) which is a way of saying that we can cover the set without overlapping. We now introduce the following sets that we use from hereon, we let $\mathcal{I} = \{(i_1, i_2, \cdots) \colon 1 \leq i_j \leq m\}$ denote the set of infinite sequences with entries between 1 and m, and let $I_{i_1,\ldots,i_k} = \{(i_1, \ldots, i_k, q_{k+1,\ldots}) \colon 1 \leq q_j \leq m\}$ denote the cylinder set consisting of those sequences in $\mathcal{I}$ with initial terms $(i_1, \ldots, i_k)$. Let $\mathcal{J} = \{(i_1, \ldots, i_k) \colon 1 \leq i_j \leq m, k \in \mathbb{N}\}$ denote the set of finite sequences with entries between 1 and m. Lastly, let $\mathcal{J}_k = \{(i_1, \ldots, i_k) \colon 1 \leq i_j \leq m, k \in \mathbb{N}\}$ denote the set of finite sequences of length k with entries between 1 and m. The main proof requires a way to measure the attractor F, we do that by defining an outer measure on the set $\mathcal{I}$ and then extending it to the set F.

7.1. **Constructing the measure.** This subsection is dedicated to discussing details that Falconer glosses over in the main source of this thesis [Fal04]. We define an outer measure over our set, and then add more tools that we will later need in the main proof. We use the cylinder sets we introduced above to define a measure on $\mathcal{I}$. Let $\mu : \mathcal{P}(\mathcal{I}) \to \mathbb{R}$ be the function that assigns to a subset $I \subset \mathcal{I}$ the number

$$\mu(I) = \inf \left\{ \sum_{(i_1,\ldots,i_k)\in J} (c_{i_1} \cdots c_{i_k})^s \mid J \subseteq \mathcal{J}, I \subseteq \bigcup\nolimits_{(i_1,\ldots,i_k)\in J} I_{i_1,\ldots,i_k} \right\}. \tag{4}$$

Lemma 7.1.1. *The function μ is an outer measure on $\mathcal{I}$.*

Proof. We will check that μ satisfies the requirements of an outer measure.

(i). We evaluate μ at the empty set

$$\begin{aligned} \mu(\emptyset) &= \inf \Big\{ \sum_{(i_1,\ldots,i_k)\in J} (c_{i_1} \cdots c_{i_k})^s \mid J \subseteq \mathcal{J}, \emptyset \subseteq \bigcup\nolimits_{(i_1,\ldots,i_k)\in J} I_{i_1,\ldots,i_k} \Big\} \\ &= \inf \Big\{ \sum_{(i_1,\ldots,i_k)\in J} (c_{i_1} \cdots c_{i_k})^s \mid J \subseteq \mathcal{J} \Big\}. \end{aligned}$$

Let $\epsilon > 0$, claim $\mu(\emptyset) \leq \epsilon$. We know that for every c_i we have $c_i < 1$, in particular $c_1 < 1$. So for k big enough $(c_1^s)^k \leq \epsilon$. Take $J \coloneqq \{(1, 1, \ldots, 1)\} \subset \mathcal{J}$ where the length of this sequence is k. It follows

$$\sum_{(i_1,\ldots,i_k)\in J} (c_{i_1} \cdots c_{i_k})^s = (c_1 c_1 \cdots c_1)^s = (c_1^s)^k \leq \epsilon.$$

Since $\mu(\emptyset)$ equals the infima over all such expressions, it holds that $\mu(\emptyset) \leq \epsilon$. We let $\epsilon \to 0$, and obtain $\mu(\emptyset) = 0$.

(ii). We need that if $I \subset I'$, then $\mu(I) \leq \mu(I')$. This follows directly by the fact that any cover of I' is a cover of I.

(iii). Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of subsets of $\mathcal{I}$. We want to show $\mu(\cup_n A_n) \leq \sum_n \mu(A_n)$. We have by definition

$$\mu(A_n) = \inf \Big\{ \sum_{(i_1,\ldots,i_k)\in J_n} (c_{i_1} \cdots c_{i_k})^s \mid J_n \subseteq \mathcal{J}, A_n \subseteq \bigcup\nolimits_{(i_1,\ldots,i_k)\in J_n} I_{i_1,\ldots,i_k} \Big\}.$$

Let $\epsilon > 0$ be given, write $\epsilon = \sum_n \epsilon_n$ for certain $\epsilon_n > 0$. Choose one covering by choosing $J_n \in \mathcal{J}$ such that $\sum_{(i_1,\ldots,i_k)\in J_n} (c_{i_1} \cdots c_{i_k})^s \leq \mu(A_n) + \epsilon_n$. Then $\cup_n J_n$ give rise to a cover of $\cup_n A_n$, and hence, we have

$$\mu(\cup_n A_n) \leq \sum_n \sum_{(i_1,\ldots,i_k)\in J_n} (c_{i_1} \cdots c_{i_k})^s \leq \sum_n (\mu(A_n) + \epsilon_n) = \sum_n \mu(A_n) + \epsilon.$$

Since ϵ was arbitrary the inequality follows.

So we conclude that μ indeed is an outer measure on $\mathcal{I}$. $\square$

Lemma 7.1.2. *Let $I_{l_1,\ldots,l_n}$ be the cylinder set with initial terms $(l_1,\ldots,l_n)$, and let μ be the outer measure defined in Equation (4). Then we have*

$$\mu(I_{l_1,\ldots,l_n}) = (c_{l_1}\cdots c_{l_n})^s.$$

Proof. We have by definition of μ

$$\mu(I_{l_1,\ldots,l_n}) := \inf\left\{\sum_{(i_1,\ldots,i_k)\in J}(c_{i_1}\cdots c_{i_k})^s \mid J\subseteq\mathcal{J}, I_{l_1,\ldots,l_n}\subseteq\bigcup\nolimits_{(i_1,\ldots,i_k)\in J} I_{i_1,\ldots,i_k}\right\}$$

We start by showing $\mu(I_{l_1,\ldots,l_n}) \leq (c_{l_1}\cdots c_{l_n})^s$. Take $J := \{(l_1,\ldots,l_n)\}$, then we get $I_{l_1,\ldots,l_n}\subseteq\bigcup_{(i_1,\ldots,i_k)\in J} I_{i_1,\ldots,i_k} = I_{l_1,\ldots,l_n}$. It also follows $\sum_{(i_1,\ldots,i_k)\in J}(c_{i_1}\cdots c_{i_k})^s = (c_{l_1}\cdots c_{l_n})^s$. To show the opposite direction of the inequality we take $J\in\mathcal{J}$ where $I_{l_1,\ldots,l_n}\subseteq\bigcup_{(i_1,\ldots,i_k)\in J} I_{i_1,\ldots,i_k}$. Take $(j_1,\ldots,j_k)\in J$ such that $(l_1,\ldots,l_n,1,1,\ldots)\in I_{j_1,\ldots,j_k}$. Here we have two cases that we need to look at.
<u>*Case 1:*</u> If $k\geq n$, then we have $(l_1,\ldots,l_n,1,1,\ldots,1) = (j_1,\ldots,j_k)$ where the length of the first sequence is k. Then $(c_{l_1}\cdots c_{l_n})^s = (c_{l_1}\cdots c_{l_n}\cdot 1\cdots 1)^s = (c_{j_1}\cdots c_{j_k})^s \leq \sum_{(i_1,\cdots,i_k)\in J}(c_{i_1}\cdots c_{i_k})^s$.
<u>*Case 2:*</u> If $k<n$, then $(l_1\cdots l_k) = (j_1\cdots j_k)$. So we have $(c_{l_1}\cdots c_{l_n})^s \leq (c_{l_1}\cdots c_{l_k})^s = (c_{j_1}\cdots c_{j_k})^s \leq \sum_{(i_1,\ldots,i_n)\in J}(c_{i_1}\cdots c_{i_n})^s$, which proves the statement. $\square$

Next we transfer this outer measure μ to the set F.

Notation 7.1.3. For any arbitrary set $E\subseteq\mathbb{R}^n$ and $(i_1,\ldots,i_k)\in\mathcal{J}_k$, we denote $E_{i_1,\ldots,i_k} = S_{i_1}\circ\cdots\circ S_{i_k}(E)$.

Remark that by [Fri82, Thm. 3.4.1], the intersection of $F_{i_1,\ldots,i_k}$ for a given sequence consists of one point that we denote as $x_{i_1,i_2,\ldots}$, so that

$$\{x_{i_1,i_2,\ldots}\} = \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}.$$

Definition 7.1.4. Let $A\subseteq F$, Let μ be defined as in Equation (4). We define

$$\tilde{\mu}(A) = \mu\big(\big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq A\big\}\big).$$

Lemma 7.1.5. *The function $\tilde{\mu}$ is an outer measure on F.*

Proof. We check the three requirements of an outer measure to verify this claim.

(i). We evaluate $\tilde{\mu}$ at the empty set $\tilde{\mu}(\emptyset) = \mu\big(\big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq\emptyset\big\}\big)$. For any $(i_1,i_2,\cdots)$ we have $\bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}$ is nonempty, meaning that no such sequences exist. We conclude $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$.

(ii). Let $A\subset B$, we want to show $\tilde{\mu}(A)\leq\tilde{\mu}(B)$. We have by assumption

$$\big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq A\big\} \subset \big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq B\big\}.$$

This implies

$$\mu\big(\big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq A\big\}\big) \leq \mu\big(\big\{(i_1,i_2,\cdots)\colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}\subseteq B\big\}\big),$$

which results in $\tilde{\mu}(A)\leq\tilde{\mu}(B)$.

(iii). To show $\tilde{\mu}(\cup_j A_j) \leq \sum_j \tilde{\mu}(A_j)$, we start by observing

$$\Big\{(i_1, i_2, \cdots) \colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k} \subseteq \cup_j A_j\Big\} = \bigcup_j \Big\{(i_1, i_2, \cdots) \colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k} \subseteq A_j\Big\},$$

and then evaluate $\tilde{\mu}$ at this set

$$\begin{aligned}\tilde{\mu}(\cup_j A_j) &= \mu\bigg(\bigcup_j \Big\{(i_1, i_2, \cdots) \colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k} \subseteq A_j\Big\}\bigg) \\ &\leq \sum_j \mu\Big(\Big\{(i_1, i_2, \cdots) \colon \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k} \subseteq A_j\Big\}\Big) \\ &= \sum_j \tilde{\mu}(A_j).\end{aligned}$$

We conclude that $\tilde{\mu}$ is an outer measure on F. □

Definition 7.1.6. Let $S_1, \ldots, S_m$ be an IFS. This system satisfies the open set condition if there is a non-empty, bounded and open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V$$

with this union being disjoint.

This condition assures that the components $S_i(F)$ of the iterated function system do not overlap 'too much'. In the middle fourth Cantor set example, the open set condition holds for S_1 and S_2 with V as the open interval $(0, 1)$.

We want to show that, provided that S_i satisfies the open set condition, the Hausdorff dimension of the attractor is given by Equation (3). For that we require the following geometrical lemma.

Lemma 7.1.7. *Let $\{V_i\}$ be a collection of open disjoint subsets of $\mathbb{R}^n$, let $a_1, a_2, r \in \mathbb{R}$ with $r > 0$ and $0 < a_1 < a_2$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball with radius $a_2 r$. Then any ball B of radius r intersects at most $(1 + 2a_2)^n a_1^{-n}$ points of the closures $\overline{V}_i$.*

Proof. By assumption, we know $|V_i| \leq 2a_2 r$ for all V_i. So if some $\overline{V}_i$ intersects the ball with radius r, then $\overline{V}_i$ must be contained in the ball with radius $(2a_2 + 1)r$ concentric with the ball with the radius r. Note that $(a_1 r)^n$ is less or equal to the volume of V_i for all V_i. Let q denote the number of closures $\overline{V}_i$ intersecting with the ball with radius r. Then there exists q disjoint balls with radius $a_1 r$ that are contained in the ball with radius $(2a_2 + 1)r$. It follows that $q(a_1 r)^n \leq (2a_2 + 1)^n r^n$, giving us directly the stated bound for q. □

7.2. **Main theorem on the Hausdorff dimension of self-similar sets.** Now we have introduced and collected all we need in order to show the main theorem of this work.

Theorem 7.2.1. *Let $\{S_1, \ldots, S_m\}$ with ratios $0 < c_i < 1$ for $1 < i < m$ be an IFS with attractor F. Suppose the open set condition holds for this system, then the Hausdorff dimension of F is equal to s, where s is the unique solution for*

$$\sum_{i=1}^{m} c_i^s = 1.$$

Proof. We denote by $s \geq 0$ the unique solution for $\sum_{i=1}^{m} c_i^s = 1$. Observe that such an s exists uniquely since the function $s \mapsto \sum_i c_i^s$ is strictly decreasing and has value $m \geq 1$ at $s = 0$, and has limit 0. We show that $\dim_H(F) = s$ by showing that the Hausdorff measure satisfies $0 < \mathcal{H}^s(F) < \infty$. For the latter, we need a suitable upper and a lower estimate of the Hausdorff measure.

We first show that $\mathcal{H}^s(F) < \infty$. Recall, for any arbitrary set $A \subseteq \mathbb{R}^n$ and $(i_1, \ldots, i_k) \in \mathcal{J}_k$ we denote $A_{i_1,\ldots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(A)$. We recall the attractor is the unique set that satisfies $F = \bigcup_{i=1}^{m} S_i(F)$, we can iterate the attractor as follows

$$
\begin{aligned}
F = \bigcup_{i=1}^{m} S_i(F) &= \bigcup_{i=1}^{m} S_i \left(\bigcup_{j=1}^{m} S_j(F) \right) = \bigcup_{i,j=1}^{m} S_i \circ S_j(F) \\
&= \cdots = \bigcup_{i_1,\ldots,i_k \in \mathcal{J}_k} S_{i_1,\ldots,i_k}(F) = \bigcup_{\mathcal{J}_k} F_{i_1,\ldots,i_k}.
\end{aligned}
$$

Hence the unions $\bigcup_{\mathcal{J}_k} F_{i_1,\ldots,i_k}$ form a cover of F. These covers of F will provide a suitable upper estimate for the Hausdorff measure. We use Lemma 2.0.4 to obtain $|A_j| = |S_j(A)| = c_j|A|$, and hence

$$
|F_{i_1,\cdots,i_k}| = (c_{i_1} \cdots c_{i_k})|F| \leq (\max_{1 \leq i \leq k} c_i)^k |F|.
$$

Let $\delta > 0$, to find a suitable δ-cover, take k_δ big enough such that $(\max_{1 \leq i \leq k_\delta} c_i)^{k_\delta}|F| < \delta$. Then $\{F_{i_1,\ldots,i_{k_\delta}} \mid (i_1, \ldots, i_{k_\delta}) \in \mathcal{J}_{k_\delta}\}$ forms a δ-cover of F. Therefore,

$$
\mathcal{H}^s_\delta(F) \leq \sum_{\mathcal{J}_{k_\delta}} |F_{i_1,\ldots,i_{k_\delta}}|^s,
$$

and hence

$$
\begin{aligned}
\mathcal{H}^s(F) &\leq \lim_{\delta \to 0} \sum_{\mathcal{J}_{k_\delta}} |F_{i_1,\ldots,i_{k_\delta}}|^s \\
&= \lim_{\delta \to 0} \sum_{\mathcal{J}_{k_\delta}} (c_{i_1} \cdots c_{i_{k_\delta}})^s |F|^s \\
&= \lim_{\delta \to 0} \left(\sum_{i_1=1}^{m} c_{i_1}^s \right) \cdots \left(\sum_{i_{k_\delta}=1}^{m} c_{i_{k_\delta}}^s \right) |F|^s \\
&= |F|^s < \infty.
\end{aligned}
$$

Now we aim towards showing $\mathcal{H}^s(F) > 0$. We use the outer measure $\tilde{\mu}$ we defined in Definition 7.1.4 to estimate a suitable lower bound for the Hausdorff measure over F. It can be shown by Lemma 7.1.2 that $\tilde{\mu}(F) = 1$, and so we obtain that $\tilde{\mu}(F)$ is a mass distribution, see Definition 6.3.1.

We now want show that $\tilde{\mu}$ satisfies the mass distribution principal in Theorem 6.3.2. By the open set condition 7.1.6, there exists a nonempty bounded open set such that it satisfies $\bigcup_i S_i(V) \subset V$, with $S_i(V)$ are disjoint. By continuity of similarities $\overline{V} \supset S(\overline{V}) = \bigcup_{i=1}^{m} S_i(\overline{V})$. By Theorem 3.3.1, it follows that the decreasing sequence of iterations $S^k(\overline{V})$ converges to F. In particular we have $\overline{V} \supset F$ and $\overline{V}_{i_1,\ldots,i_k} \supset F_{i_1,\ldots,i_k}$ for each finite sequence $(i_1, \ldots, i_k) \in \mathcal{J}$.

We fix $0 < r < 1$ a radius of a ball B_r. We want to estimate $\tilde{\mu}(B_r)$ by considering the sets $V_{i_1,\ldots,i_k}$ with diameters comparable with that of B_r and with closures intersecting $F \cap B_r$. Let $(i_1, i_2, \ldots)$ be an infinite sequences in $\mathcal{I}$. The function $\mathbb{N} \to \mathbb{R}\colon k \mapsto c_{i_1} \cdots c_{i_k}$ has limit 0, so there exists the smallest k', depending on our choice of sequence, such that

$$
c_{i_1} \cdots c_{i_{k'}} < r. \tag{5}
$$

Now for such a k', it holds that $c_{i_1} \cdots c_{i_{k'-1}} \geq r$, this follows by minimality of k'. Therefore, $c_{i_{k'}} r \leq c_{i_1} \cdots c_{i_{k'}}$, and so

$$(\min_{1 \leq i \leq m} c_i).r \leq c_{i_{k'}} r \leq c_{i_1} \cdots c_{i_{k'}}. \tag{6}$$

We denote by Q the set of all such sequences $(i_1, \dots, i_{k'})$ where k' satisfies the condition above. Then for every infinite sequence in $\mathcal{I}$ there is exactly one value of k such that $(i_1, \dots, i_k) \in Q$. Recall that $V_1, \dots, V_m$ are disjoint, implying that $V_{i_1,\dots,i_k,1}, V_{i_1,\dots,i_k,2}, \dots, V_{i_1,\dots,i_k,m}$ are disjoint as well. Since $\overline{V}$ is compact we can use Theorem 3.3.1 to express the attractor F as $F = \bigcap_{k=0}^{\infty} S^k(\overline{V})$.

Take $x \in F$, then there exists $(i_1, \dots, i_k, \cdots)$ such that $x \in \bigcap_{e=1}^{\infty} F_{i_1,\dots,i_e} \subseteq F_{i_1,\dots,i_k}$ with $(i_1, \dots, i_k) \in Q$. So $F \subseteq \bigcup F_{(i)} \subseteq \bigcup \overline{V}_{(i)}$ with $(i) \in Q$. Now we choose $0 > a_1 > a_2$ such that $B_{a_1} \subseteq \overline{V} \subseteq B_{a_2}$. This is possible since both balls are open. We have

$$B_{a_1 r(\min_i c_i)} \subseteq B_{a_1 c_{i_1} \cdots c_{i_k}} \subseteq \overline{V}_{i_1,\dots,i_k} \subseteq B_{a_2 c_{i_1} \cdots c_{i_k}} \subseteq B_{a_2 r},$$

where the first inclusion follows by Equation (6), and the last inclusion follows by Equation (5). We define Q_1 to be the set of all sequences $(i_1, \dots, i_k) \in Q$ such that the given ball B_r intersects $\overline{V}_{i_1,\dots,i_k}$. We now use Lemma 7.1.7 to obtain $|Q_1| \leq \frac{(1+2a_2)^n}{(a_2 \min c_i)^n} =: q > 0$. So we have

$$\begin{aligned}
\tilde{\mu}(B_r) &= \tilde{\mu}(F \cap B_r), \text{ since the mass is distributed over } F \\
&\leq \tilde{\mu}\left((\cup_{(i) \in Q} \overline{V}_{(i)}) \cap B_r\right), \text{ since it covers } F \\
&\leq \tilde{\mu}\left(\cup_{(i) \in Q_1} \overline{V}_{(i)}\right) \\
&\leq \sum_{(i) \in Q_1} (c_{i_1}, \dots, c_{i_k})^s \\
&\leq \sum_{Q_1} r^s \\
&\leq q.r^s.
\end{aligned}$$

Lastly we take a set $U \subset F$, we have $U \subset B_{|U|}$. This implies $\tilde{\mu}(U) \leq \tilde{\mu}(B_{|U|}) \leq q.|U|^s$, as we estimated. By the mass distribution principal Theorem 6.3.2 it follows

$$\mathcal{H}^s(F) \geq \frac{\tilde{\mu}(F)}{q} > 0.$$

We now have $0 < \mathcal{H}^s(F) < \infty$, so by Corollary 6.0.3 we conclude $\dim_H(F) = s$. $\square$

7.3. **Examples of determining the Hausdorff dimension.** We have seen that it is not easy to determine the Hausdorff dimension of a fractal by estimating upper and lower bounds for the measure on the set. Theorem 7.2.1 offers us a straight forward formula to compute the dimension of the set. In fact, it reduces the problem to a simple calculation. We know from earlier *the middle fourth Cantor set* is the attractor F of the similarities $S_1, S_2 \colon [0,1] \to [0,1]$ with $S_1 \colon x \mapsto \frac{3}{8}x$ and $S_2 \colon x \mapsto \frac{3}{8}x + \frac{5}{8}$. The open set condition holds by considering the open interval $(0,1)$. We have $\sum_{i=1}^{2} c_i^s = \frac{8}{3}^s + \frac{8}{3}^s = 2(\frac{8}{3})^s = 1$. This equation yields $\dim_H(F) = \frac{\log 2}{\log \frac{8}{3}}$.

In the *von Koch curve* V we have seen that the attractor is the union of four similarities each of ratio $\frac{1}{3}$. We now calculate it's dimension using Theorem 7.2.1. We consider the open equilateral triangle in $\mathbb{R}^2$, with vertices $(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})$ so the open set condition is satisfied. Therefore, a straight forward calculation gives us $\dim_H(V) = \frac{\log 4}{\log 3}$.

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