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The Theorems of Engel, Lie, and Cartan in Lie Algebra

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Abstract

This paper aims to give an introduction to Lie algebras and extend some concepts from Group and Ring theory to Lie algebra without the need of studying toplogy beforehand. We will look at some fundamental theorems in Lie algebra, such as Lie's and Engel's theorems, and also prove Cartan's semisimplicity criterion. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ will be introduced early on and then continue to be used throughout the paper as a key example.

Contents

1 Basic definitions

To begin studying Lie algebras we first need to define an algebra over a field.

Definition 1.1 *Let K be a field. A **K-algebra** or an **algebra over K** is a ring A with identity s.t. $K \subset Z(A)$ and $1_K = 1_A$.*

Remark 1.2 *Multiplication in a ring A is a map $M : A \times A \Longrightarrow A$ usually shortened as ab instead of $a * b$.*

Remark 1.3 *The left- and right-distributivity over multiplication implies that m is bilinear over K.*

Definition 1.4 *A Lie algebra over K is a K-vector space $\mathfrak{g}$ paired with an operation called the **Lie bracket** $[,]$ satisfying the axioms for all $x, y, z \in \mathfrak{g}$*

a) Bilinearity,
$[ax + by, z] = a[x, z] + b[y, z]$,
$[z, ax + by] = a[z, x] + b[z, y]$

b) Anti-commutativity,
$[x, x] = 0$

c) The Jacobi identity,
$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Lemma 1.5 *Using bilinearity, we have that $[x, x] = 0$ (from Definition 1.4 b)) implies that $[x, y] = -[x, y]$ for all $x, y \in \mathfrak{g}$.*

Proof. $[x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] = 0$

Remark 1.6 *I define anti-commutativity as $[x, x] = 0$ because for rings with characteristic $\neq 2$ we have that $[x, y] = -[x, y]$ is the same as $[x, x] = xx - xx = 0$, so $2[x, x] = 0$. However for the special case of characteristic 2, $2[x, x] = 0$ is trivially true since $2 = 0$.*

Definition 1.7 *If R is an associative K-algebra then the Lie bracket is defined as the **commutator***

$$[x, y] = xy - yx.$$

Lemma 1.8 *$(R, [,])$ is a Lie algebra.*

The Lie bracket is also known as the commutator bracket because of this definition. From the commutator we can define an abelian Lie algebra.

Definition 1.9 *A Lie algebra* $\mathfrak{g}$ *is* ***abelian*** *if its bracket is identically 0:*

$$[x, y] = 0 \quad \forall x, y \in \mathfrak{g}.$$

Lemma 1.10 *In the setting of Definition 1.7 the Lie algebra* $(R, [,])$ *is abelian if and only if the associative algebra* R *is commutative.*

Proof. Using Definition 1.7, if $[x, y] = xy - yx$ and using that $[x, y] = 0$, then $xy - yx = 0 \Longleftrightarrow xy = yx$. □

The general linear group $GL_n(\mathbb{K})$ is the set of $n \times n$ invertible matrices paired with ordinary matrix multiplication well-known from linear algebra. The special linear group $SL_n(\mathbb{K})$ is the set of $n \times n$ matrices with entries from $\mathbb{K}$ having determinant 1. In other words, the group $SL_n(\mathbb{K})$ is the kernel of the determinant mapping

$$\det : GL_n(\mathbb{K}) \to \mathbb{K}^{\times}.$$

Both of these groups have Lie algebras defined over them.
The set of all $n \times n$ matrices with elements from $\mathbb{K}$ forms an associative ring commonly known as $M_n(\mathbb{K})$. Applying the construction of Definition 1.7 and using Lemma 1.8, the invertible matrices form a Lie algebra $(\mathfrak{gl}_n(\mathbb{K}), [,])$.
A subalgebra to this would be the Lie algebra we're interested in, namely $\mathfrak{sl}_2(\mathbb{K})$. We can define this not by the matrices with determinant 1 like in groups, but by trace equal to 0.

Proof. Let A be a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and e^A its' exponential defined by $x \to e^x$, i.e.

$$e^A = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}.$$

We have that $\det e^A = e^{\lambda_1} * ... * e^{\lambda_n} = e^{\lambda_1 + ... + \lambda_n} = e^{trA}$, so if $trA = 0$ we get that $\det e^A = e^{trA} = e^0 = 1$. So for the case of defining the Lie algebras $\mathfrak{sl}_2(\mathbb{K})$, we require that the *trace* of the matrices to be equal to 0. □

Definition 1.11 *In the case of* $\mathbb{K} = \mathbb{C}$, *i.e.* $\mathfrak{sl}_2(\mathbb{C})$, *we use* h, e, f *as a standard basis*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Remark 1.12 *For this basis we have the following relations:*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

This basis along with the relations will be used extensively to prove various things regarding the properties of $\mathfrak{sl}_2(\mathbb{C})$.

2 Ideals and Homomorphisms

Definition 2.1 *A* ***subalgebra*** $\mathfrak{h}$ *of* $\mathfrak{g}$ *is a subspace which satisfies* $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. *A subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called an* ***ideal*** *if* $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Definition 2.2 *If a non-abelian Lie algebra has no non-trivial abelian ideals, we call it a* ***semi-simple*** *Lie algebra. Additionally we say a Lie algebra is* ***simple*** *if it has no non-trivial proper ideals.*

Based on these definitions, simplicity implies semisimplicity.

Lemma 2.3 *The sum of two ideals I and J is again an ideal.*

Proof. Since 0 is in both ideals, we have that $0 + 0 = 0 \in I + J$. Now for elements $x, y \in I + J$ we need to show that $x + y \in I + J$. We have $x = a + b$ for some $a \in I$ and $b \in J$, and similarly that $y = c + d$ for some $c \in I$ and $d \in J$. Then

$$x + y = (a + b) + (c + d) = (a + c) + (b + d).$$

The rearrangement makes it easy to see that the first term $a + c \in I$ and the second term $b + d \in J$. For any other integer n we have

$$nx = n(a + b) = na + nb.$$

Since na is a (scalar) multiple of a, we get that $na \in I$. Similarly we see that $nb \in J$. nx is then a sum of an element from I and an element from J. Then $nx \in I + J$. $\square$

Lemma 2.4 *The sum of k ideals, i.e. $I_1 + ... + I_k$, is also an ideal.*

Proof. I'm going to prove this using induction. For $n = 1$ the result is trivially true. We've seen the result holds for $n = 2$ above. Now suppose the result holds for $n = k$, and we'll need to prove it for $n = k + 1$.

$$I_1 + ... + I_{k+1} = (I_1 + ... + I_k) + I_{k+1}.$$

By our assumption the first term is an ideal and so is I_{k+1}, and by the previous lemma the sum of two ideals is again an ideal, and so if $J = (I_1 + ... + I_k)$ then $J + I_{k+1}$ is an ideal. $\square$

Lemma 2.5 *The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is simple.*

Proof. Suppose $\mathfrak{h} < \mathfrak{sl}_2(\mathbb{C})$ with $\mathfrak{h} \neq 0$ is an ideal. Let Z be an element in $\mathfrak{h}$, we can write this using our basis vectors as $Z = \alpha e + \beta f + \gamma h$ for some scalars α, β, γ. Then let $X = [h, [h, Z]] = [h, 2\alpha e - 2\beta f] = 4\alpha e + 4\beta f \in \mathfrak{h}$. Using that ideals are closed under addition(Lemma 5) and scalar multiplication, $Z - 1/4X = \gamma h \in \mathfrak{h}$.
Now, if $\gamma \neq 0$ then $h \in \mathfrak{h}$. Then also $[h, e] \implies e \in \mathfrak{h}$, and $[f, h] = 2f \implies f \in \mathfrak{h}$. We can thus see that $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$.
If instead $\gamma = 0$ then by the same reasoning as above, $[e, Z] = \beta h \in \mathfrak{h}$, and $[f, Z] = \alpha h \in \mathfrak{h}$. Then since $Z \neq 0$ we see that $h \in \mathfrak{h}$, and we can conclude that once again $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C})$. $\square$

A **Lie algebra homomorphism** is defined just like for groups, a linear mapping

$$\phi : \mathfrak{g} \to \mathfrak{g}^{'}, \quad \phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in \mathfrak{g}$.

The **kernel** of a homomorphism ϕ, denoted by $ker(\phi)$, is the set of elements in the domain that map to 0. The first isomorphism theorem holds for Lie algebras as well. That is,

$$\mathfrak{g}/ker(\phi) \cong Im(\mathfrak{g}).$$

For ideals $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{g}$ s.t. $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$, the second isomorphism theorem for Lie algebras gives us that

$$\mathfrak{g}/\mathfrak{a} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong b/(\mathfrak{a} \cap \mathfrak{b}).$$

Definition 2.6 *Given subsets $A, B \subset \mathfrak{g}$, the commutator $[A, B]$ is the subspace*

$$[A, B] = \{span_k[a, b] | a, b \in \mathfrak{g}\}$$

Remark 2.7 *We need the span in this definition because if A, B are ideals, the set*

$$\{[a, b] | a, b \in \mathfrak{g}\}$$

wouldn't necessarily be closed under addition, nor an ideal. It would only be an ideal if we have

$$[a, b] + [c, d] = [e, f]$$

for some $c, d, e, f \in \mathfrak{g}$.

3 Central Series

Central series, upper and lower, can be used to define solvable and nilpotent Lie algebras.

Definition 3.1 *The **Upper Central Series** is defined recursively as*

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$$
$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq ...$$

Remark 3.2 *For $j \geq 1$, every $\mathfrak{g}^j$ is an ideal in $\mathfrak{g}^{j-1}$ called the **commutator ideal**.*

The subalgebra $\mathfrak{g}^1$ is called the **derived subalgebra** or **commutator subalgebra**, the set of all brackets $[a, b]$ with $a, b \in \mathfrak{g}$. This is analoguous to the commutator subgroup (or derived subgroup) in group theory; the smallest subgroup containing all commutators $aba^{-1}b^{-1} = [a, b]$.

Definition 3.3 *If $\mathfrak{g}^j = 0$ for some j then we call $\mathfrak{g}$ **solvable**.*

Solvable Lie algebras will be relevant later for Lie's theorem.

Lemma 3.4 *Any subalgebra or quotient algebra of a solvable Lie algebra is solvable.*

Proof. For $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, we have $\mathfrak{h}^k \subseteq \mathfrak{g}^k$. Then $\mathfrak{g}$ solvable implies $\mathfrak{h}$ solvable. Now if $\pi : \mathfrak{g} \to \mathfrak{h}$ is an onto homomorphism, i.e. an epimorphism, then $\pi(\mathfrak{g}^k) = \mathfrak{h}^k$. So again $\mathfrak{g}$ solvable implies $\mathfrak{h}$ solvable.

Lemma 3.5 *The sum $\mathfrak{a} + \mathfrak{b}$ of two solvable ideals $\mathfrak{a}, \mathfrak{b}$ is solvable.*

Proof. The second isomorphism theorem takes us there immediately; let $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$ and then we have

$$\mathfrak{h}/\mathfrak{a} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}).$$

The quotient $\mathfrak{h}/\mathfrak{a}$ is solvable by Lemma 3.4. Then also $\mathfrak{h}$ is solvable, and since $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$, we see that $\mathfrak{a} + \mathfrak{b}$ solvable. □

Lemma 3.6 *For a finite-dimensional Lie algebra $\mathfrak{g}$ there exists a unique solvable ideal rad $\mathfrak{g}$ of $\mathfrak{g}$ which contains all solvable ideals in $\mathfrak{g}$ and which is of maximal possible dimension.*

Proof. If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, then $\mathfrak{a}$+rad $\mathfrak{g}$ is again a solvable ideal by Lemma 8. Because rad $\mathfrak{g}$ is of maximal dimension; this sum is actually equal to rad $\mathfrak{g}$ and $\mathfrak{a} \subseteq$ rad $\mathfrak{g}$. If there exists two maximal solvable ideals rad$_1$ and rad$_2$ of $\mathfrak{g}$ then the sum rad$_1$+ rad$_2$ = rad$_1$ = rad$_2$ so they are the same ideal. Therefore there exists a unique maximal solvable ideal for a finite-dimensional Lie algebra $\mathfrak{g}$. □

The ideal rad $\mathfrak{g}$ is called the **radical** of $\mathfrak{g}$.
Using rad $\mathfrak{g}$ we can define semisimplicity in an alternative way; namely that a finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if rad $\mathfrak{g} = 0$. This is only one of the ways to show that a Lie algebra is semisimple, in chapter 6 we will look at two more ways to do this using the properties of ideals.

Definition 3.7 *The **Lower Central Series** is similarly defined recursively as*

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j]$$
$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$$

Definition 3.8 *We call $\mathfrak{g}$ **nilpotent** if $g_j = 0$ for some j.*

Nilpotency will play a central role in Engel's theorem.

Remark 3.9 *Lemma 3.4 works precisely the same for nilpotent $\mathfrak{g}$ and $\mathfrak{h}$, i.e. that a subalgebra or quotient algebra of a nilpotent Lie algebra is nilpotent.*

Lemma 3.10 *Claim: $\mathfrak{g}_j \supset \mathfrak{g}^j$.*

Proof. I'm going to prove this by induction on j:
$j = 1$ is trivial, both series use the same definitions: $\mathfrak{g}_1 = \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$
$j + 1$: We have that $\mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j] = [\mathfrak{g}_j, \mathfrak{g}]$, $\mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$.
Hence by induction $\mathfrak{g}_j \supset \mathfrak{g}^j$. □

So using this Lemma we see that $\mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_{\mathrm{j}}] \supset [\mathfrak{g}, \mathfrak{g}^j] \supset [\mathfrak{g}^j, \mathfrak{g}^j] = g^{j+1}$ using the fact that $A \subset B \Longrightarrow [A, C] \subset [B, C]$ for all C.

Example 3.11 *Every abelian lie algebra $\mathfrak{a}$ is nilpotent since by Definition 1.9 the Lie bracket $\mathfrak{a}_1 = [\mathfrak{a}, \mathfrak{a}] = 0$.*

Corollary 3.12 *Nilpotent implies solvable.*

Proof. $\mathfrak{g}_j = 0$ and $\mathfrak{g}_j \supset \mathfrak{g}^j \Longrightarrow \mathfrak{g}^j = 0$.

Definition 3.13 *The **Heisenberg Lie algebra** is the Lie algebra of* 3×3 *upper triangular matrices with 0's on the diagonal.*

Lemma 3.14 *The Heisenberg Lie algebra is nilpotent, since*

$$\left[\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & af - cd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix will vanish when commuted with another Heisenberg matrix, hence $\mathfrak{g}_2 = 0$. $\square$

For any $n \geq 1$, the upper-triangular $n \times n$ matrices with 0's along the diagonal is a nilpotent Lie algebra. But the name "Heisenberg Lie algebra" is reserved specifically for the 3×3 case. Now let's prove that the Lie algebra is nilpotent for any n.

Proof. For some upper-triangular matrix with 0s on the diagonal A when we go from calculating A_i to A_{i+1} the zero-diagonal is expanding, one diagonal at a time, towards the top-right corner of the matrix. For a $n \times n$ matrix A there are $n - 1$ diagonals above the main diagonal. So by A_{n-1} the zero diagonal will have gone through the whole matrix and we have that $A_m = 0$ for all $m > n - 1$. $\square$

Example 3.15 *The Lie algebra of any matrices of the form*

$$\mathfrak{h} = \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

is an example of a solvable Lie algebra, which is not split-solvable.

Proof. We want to show that $\mathfrak{h}^j = 0$ for some j. We actually only need to calculate up to $\mathfrak{h}^2$ for this.

$$\mathfrak{h}^1 = (\mathfrak{h}, \mathfrak{h}) = \begin{pmatrix} 0 & \theta_1 & x_1 \\ -\theta_1 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \theta_2 & x_2 \\ -\theta_2 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \theta_2 & x_2 \\ -\theta_2 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \theta_1 & x_1 \\ -\theta_1 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & y_2\theta_1 - \theta_2 y_1 \\ 0 & 0 & -x_2\theta_1 + \theta_2 x_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{h}^2 = (\mathfrak{h}^1, \mathfrak{h}^1) = \mathfrak{h}^1\mathfrak{h}^1 - \mathfrak{h}^1\mathfrak{h}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \square$$

Definition 3.16 *For $\mathfrak{g}$ solvable, and a_i ideal in $\mathfrak{g}$ for all i, if there exists a sequence*

$$\mathfrak{g} = a_0 \supseteq a_1 \supseteq \dots \cdots \supseteq a_n = 0$$

then $\mathfrak{g}$ is called ***split-solvable****.*

Remark 3.17 *Split-solvable for Lie algebras is similar to supersolvable for groups. For supersolvable we require that the sequence of G_i only contains groups normal in G, whereas for Lie algebras we require that a_i is an ideal in $\mathfrak{g}$.*

Note how the requirements for these three properties (solvable, split-solvable, nilpotent) are similar but with slightly different requirements. We can describe the relation between these as follows

$$nilpotent \implies split - solvable \implies solvable.$$

So every nilpotent Lie algebra is also split-solvable, and every split-solvable Lie algebra is solvable.
Additionally, if we added abelian to this relation then by Example 3.11 $abelian \implies nilpotent$.

4 Killing form

Definition 4.1 *For a finite-dimensional Lie Algebra* $\mathfrak{g}$*, the* ***Killing form***

$$B : g \otimes g \mapsto k$$

named after Wilhelm Killing is a symmetric bilinear form on $\mathfrak{g}$ *with* $X, Y \in \mathfrak{g}$ *given by the formula*

$$B(x, y) = Tr(adXadY).$$

An example using the basis of $\mathfrak{sl}_2\mathbb{C}$ should clarify what this means. We first need to define what the adjoint representation is.

Definition 4.2 *The* ***Adjoint representation*** *of a Lie algebra is the mapping* $ad : \mathfrak{g} \to End(\mathfrak{g})$ *defined by* $ad_x(y) = [x, y]$*.*

Example 4.3 *The adjoint representation of* $\mathfrak{sl}_2(\mathbb{C})$ *has dimension 3, using the basis from Definition 9. The calculation of these is relatively simple: for* ad_e *we compute the commutators* $[e, e], [e, f], [e, h]$ *and use these as column vectors in a* 3×3 *matrix. The same process is used for* ad_f *and* ad_h*. Note that the basis-relations previously calculated can help us here.*
Firstly we know that any commutator with itself is zero by definition, since $[e, e] = ee - ee = 0$*. Onto the second one,* $[e, f] = h$*. And for the third, we can use anti-commutativity of the Lie bracket to see that* $[e, h] = -[h, e] = -2e$*.*
So the resulting matrix

$$ad_e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We can follow the same procedure for computing ad_f *and* ad_h*, and we get*

$$ad_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad ad_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now let's apply the Killing form to these, by checking the trace of all vectors with themselves, and then all of the combinations. We should have $3^2 = 9$

calculations here.
$B(h,h) = tr(ad_h ad_h) = 8$
$B(e,f) = B(f,e) = tr(ad_e ad_f) = 4.$

So the resulting matrix we get is

$$B = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Lemma 4.4 *The Killing form is* ***invariant*** *under automorphisms of* $\mathfrak{g}$*, i.e.*

$$B([X,Y],Z) = B(X,[Y,Z]).$$

Proof. I will start by expanding the left side from the equation above. We get

$$B([X,Y],Z) = B(XY - YX, Z) = tr((XY - YX)Z) = tr(XYZ - YXZ). (1)$$

We can now use the cyclic invariance of the trace mapping, i.e. that

$$tr(AB) = tr(BA)$$

or for more variables,

$$tr(ABCD) = tr(DABC) = tr(CDAB) = tr(BCDA).$$

We can then rewrite (1) as

$$tr(XYZ - YXZ) = tr(XYZ - XZY) = tr(X(YZ - ZY))$$
$$= tr(X[Y,Z]) = B(X,[Y,Z]). \quad \square$$

Definition 4.5 *For Lie algebras* $\mathfrak{a}, \mathfrak{b}$*, let* $\mathfrak{g}$ *be the external direct sum of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *as vector spaces. That is, sets of ordered pairs with coordinate-wise addition and scalar multiplication. We can then define a bracket operation for* $\mathfrak{g}$ *so that* $\mathfrak{a}$ *brackets with* $\mathfrak{a}$ *as before, and* $\mathfrak{b}$ *brackets with* $\mathfrak{b}$ *as before, and* $[\mathfrak{a}, \mathfrak{b}] = 0$*. We say that* $\mathfrak{g}$ *is the Lie algebra* ***direct sum*** *of* $\mathfrak{a}$ *and* $\mathfrak{b}$*, and we write this as*

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}.$$

In fact, $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals in* $\mathfrak{g}$*.*

5 The theorems of Engel and Lie

Now onto some of the most important theorems in the study of Lie algebras regarding solvability and nilpotency, from Sophus Lie and Friedrich Engel respectively.

Theorem 5.1 (Lie's Theorem) *For $\mathfrak{g}$ solvable, there exists a basis for a finite-dimensional vector space over a field K s.t. all linear transformations*

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

are represented by upper triangular matrices.

Theorem 5.2 (Engel's Theorem) *A finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent iff for each $X \in \mathfrak{g}$,*

$$ad(X) : \mathfrak{g} \to \mathfrak{g}$$

is a nilpotent endomorphism on $\mathfrak{g}$, i.e. $ad(X)^k = 0$ for some k.

To later be able to show the irreducibility of a representation, we should first define what a representation actually is.

Definition 5.3 *For a vector space V over a field K, a **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\pi : \mathfrak{g} \to End_K V$$

where $End_k V$ are the endomorphisms, i.e. mappings from V to itself. This means that π should satisfy

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

for any $X, Y \in \mathfrak{g}$.

Remark 5.4 *If π is a representation of a solvable Lie algebra $\mathfrak{g}$, $\pi(\mathfrak{g})$ is solvable.*

6 Cartan's semisimplicity and Representations

Before we get into Cartan's theorem, we should start by defining what it menans for a form to be non-degenerate.

Definition 6.1 *Let K be a field, V be a K-vector space, and f a bilinear form*

$$f : V \times V \to K.$$

We say f is non-degenerate if $f(x, y) = 0$ for all y implies that $x = 0$.

Example 6.2 *Over the same vector space V as in Definition 6.1, with a basis e_i of V, if we have*

$$f : V \times V \to K$$

then that's defined by $A_{ij} = f(e_i, e_j)$. The bilinear form f is non-degenerate if and only if its matrix A is invertible.

Remark 6.3 *The matrix characterization of non-degeneracy is also independent of the choice of basis because changing basis by an invertible matrix B changes the matrix of f from A to B^TAB. Note that A is invertible if and only if B^TAB is, since*

$$det(B^T)det(A)det(B) = det(B)det(A)det(B) = det(B)^2det(A)$$

and since B invertible; $det(B) \neq 0$ so $det(B)^2detA = 0$ if and only if $det(A) = 0$.

Let ϕ be a representation of a Lie algebra $\mathfrak{g}$. An **invariant subspace** is a vector subspace U s.t. $\phi(X)U \subseteq U$ for all $X \in \mathfrak{g}$. A representation is called **irreducible** if there are exactly two such invariant subspaces; namely 0 and $\mathfrak{g}$ itself.

Theorem 6.4 (Cartan's Criterion for Semisimplicity) *The Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate.*

Corollary 6.5 *When the Killing form of a Lie algebra $\mathfrak{g}$ is nondegenerate, $\mathfrak{g}$ has no non-trivial abelian ideals.*

Proof. As a contradiction, assume $\mathfrak{a}$ is an abelian non-trivial ideal to $\mathfrak{g}$. If $A \in \mathfrak{a}$, then we have

$$(ad(A)ad(X))^2 = 0$$

for any $X \in \mathfrak{g}$. Hence

$$B(A, X) = Tr(ad(A)ad(X)) = 0$$

for all $X \in \mathfrak{g}$ since B is non-degenerate; so A has to be trivial. $\square$

We therefore have a more general result from which Cartan's semisimplicity can be derived:

Lemma 6.6 *If the Lie algebra $\mathfrak{g}$ does not contain any non-trivial abelian ideal, and if there exists a symmetric invariant non-degenerate bilinear form $\phi(X, Y)$ on $\mathfrak{g} \times \mathfrak{g}$, then $\mathfrak{g}$ is a direct sum of simple nonabelian subalgebras.*

Proof. Let $\mathfrak{m}$ be a minimal non-zero ideal in $\mathfrak{g}$. Then $[\mathfrak{m}, \mathfrak{m}]$ is also an ideal in $\mathfrak{g}$ contained in $\mathfrak{m}$. Since $\mathfrak{m}$ is minimal, $[\mathfrak{m}, \mathfrak{m}]$ is either all of $\mathfrak{m}$ or (0). However, it can't be the trivial ideal since then $\mathfrak{m}$ would be abelian, so $[\mathfrak{m}, \mathfrak{m}]$=$\mathfrak{m}$.
Now let $\mathfrak{m}^{'}$ be the subspace of $\mathfrak{g}$ orthogonal for ϕ to $\mathfrak{m}$. Since ϕ is $\mathfrak{g}$-invariant, $\mathfrak{m}^{'}$ is an ideal in $\mathfrak{g}$. If we have elements $X \in \mathfrak{m}$, $Y \in \mathfrak{m}^{'}$, $Z \in \mathfrak{g}$ we get

$$\phi(X, [Z, Y]) = \phi([X, Z], Y) = 0$$

since $[X, Z] \in \mathfrak{m}$ and $[m, m^{'}] = 0$ because of the orthogonality between them. The intersection $\mathfrak{m} \cap \mathfrak{m}^{'}$ can either be all of $\mathfrak{m}$ or (0). Since $\mathfrak{m}$ is minimal, $\mathfrak{m} \cap \mathfrak{m}^{'} \neq \mathfrak{m}$ since then $\mathfrak{m} \subset \mathfrak{m}^{'}$ and then we would get that

$$\phi(X, Y) = 0$$

for any $X, Y \in \mathfrak{m}$. However if $A \in \mathfrak{m}$, and for $B_i, C_i \in \mathfrak{m}$, we can write

$$A = \sum_i [B_i, C_i].$$

Then for any $X \in \mathfrak{g}$ we should have

$$\phi(A, X) = \sum_i \phi([B_i, C_i], X) = \sum_i \phi(B_i, [C_i, X]) = 0$$

because $[C_i, X] \in \mathfrak{m}$, but this contradicts ϕ being non-degenerate. Therefore we must have $\mathfrak{m} \cap \mathfrak{m}^{'} = 0$.
Now since ϕ is non-degenerate, $\mathfrak{g}$ is the direct sum of two ideals $\mathfrak{m}$ and $\mathfrak{m}^{'}$. However the restriction to $\mathfrak{m}^{'} \times \mathfrak{m}^{'}$ for the form ϕ is a symmetric invariant nondegenerate bilinear form and $\mathfrak{m}^{'}$ can't contain any nontrivial abelian ideal since such an ideal would also be an ideal in $\mathfrak{g}$. $\square$

Theorem 6.7 *If we have a complex representation ϕ of $\mathfrak{sl}_2(\mathbb{C})$ on a finite-dimensional complex vector space V, then V is* ***completely reducible*** *in the sense that there exists invariant subspaces $U_1, ..., U_r$ of V where*

$$V = U_1 \oplus ... \oplus U_r$$

and each representation to U_i is irreducible.

Corollary 6.8 *Let ϕ be a representation of $\mathfrak{sl}_2(\mathbb{C})$ on a complex vector space V, and suppose that each vector $v \in V$ lies in a finite-dimensional invariant subspace. Then V is the direct sum of finite-dimensional invariant subspaces on which $\mathfrak{sl}_2(\mathbb{C})$ acts irreducibly.*

Proof. Using Theorem 6.7, each member of V lies in a finite direct sum of irreducible invariant subspaces. So $V = \sum_{s \in S} U_s$, for S some index set and each U_s an irreducible invariant subspace.
For a subset R to S, call it independent if the sum $\sum_{r \in R} U_r$ is direct. This condition implies that for every finite subset $r_1, ..., r_n$ of R and every set of elements $u_i \in U$, that for the equation

$$u_1 + ... + u_n = 0$$

each $u_i = 0$. It follows from this that the union of any increasing chain of independent subsets of S is itself independent.
For the index set S, by Zorn's lemma, there is a maximal independent subset T of S. By definition the sum $V_0 = \sum_{t \in T} U_t$ is direct. Now if we can show that $V_0 = V$ then the proof is finished; it has then been shown that V is the direct sum of finite-dimensional invariant subspaces.
Now for each $s \in S$, we should have $U_s \subseteq V_0$.
For $s \in T$ this is obvious.
For $s \notin T$, the maximality of T implies that $T \cup s$ is not independent. So the sum $U_s + V_0$ is not a direct sum, and we should have $U_s \cap V_0 \neq 0$. However, this intersection is an invariant subspace of U_s. Since each U_s is irreducible and the intersection is not equal to 0, the intersection must be just U_s.
It then follows that $U_s \subseteq V_0$, which is what we wanted to show. $\square$

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References

[1] Knapp, A., 1996. *Lie Groups Beyond an Introduction.* Boston, Mass.: Birkhäuser.
[2] Dummit, D. and Foote, R., 2004. *Abstract algebra.* Danvers, Mass.: John Wiley & Sons.
[3] Artin, M., 2001. *Algebra.* Boston, Mass.: Prentice Hall.
[4] Dieudonné, J., 1953. *On semi-simple Lie algebras.* Ann Arbor, Michigan: American Mathematical Society.