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Some algebraic methods in knot theory

av

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Abstract

This thesis introduces some knot theory, focusing on the knot group and related ideas. Knots are defined as smooth embeddings of the circle in three-dimensional space, and the initial part of the thesis is dedicated to different notions of knot equivalence as well as certain geometric concepts. The knot group of a knot is defined as the fundamental group of the complement of the knot, and much of the thesis revolves around proving that knots are inequivalent by showing that they have non-isomorphic knot groups. A proof is given for a method of calculating a presentation of a knot group, the so-called 'Wirtinger presentation'. Next, an infinite class of knots called 'torus knots' is introduced, and their knot groups are analyzed and classified. The knot group fails to distinguish some knots that are nonetheless different, a problem adressed in the final part of the thesis through the so-called 'peripheral system', which augments the knot group with additional information about the knot, giving a more powerful invariant. Two applications of the peripheral system are presented: showing that no torus knot is equivalent to its mirror image, and proving two other specific knots inequivalent.

Sammanfattning

Denna uppsats introducerar delar av knutteori, med fokus på knutgrupper. Knutar definieras som släta (oändligt deriverbara) inbäddningar av en cirkel i tredimensionellt rum, och första delen av uppsatsen handlar om olika sorter av knutekvivalens och olika geometriska koncept. Knutgruppen av en knut definieras som fundamentalgruppen av knutens komplement, och en stor del av uppsatsen kretsar kring att bevisa att knutar är olika genom att bevisa att deras knutgrupper ej är isomorfa. Ett bevis presenteras för en metod för att beräkna en presentation av en knutgrupp, den så kallade 'Wirtingerpresentationen'. En oändlig klass av knutar, så kallade 'torusknutar', introduceras därnäst, och deras knutgrupper analyseras och klassificeras. Ett problem med knutgrupper är att de inte kan göra skillnad på alla knutar, vilket kan lösas genom att utöka knutgruppen med ytterligare information om knuten, i en mer kraftfull invariant som kallas 'periferalsystemet', som behandlas i sista delen av uppsatsen. Två tillämpningar av periferalsystemet presenteras: ett bevis att ingen torusknut är ekvivalent med sin spegelbild, och ett bevis att två andra specifika knutar ej är ekvivalenta.

1 Introduction and background

This thesis concerns knot theory. In this introductory section, some basic concepts are introduced. The latter sections focus on knot groups, which give a way of studying knots through group theory. Some familiarity with group theory (e.g. as presented in [DF91]) and topology (e.g. as presented in [Lee10]) is assumed throughout the thesis. Homology is sometimes referenced, though it is not required for understanding the thesis. Our basic definitions of knots rely on smooth (i.e. infinitely differentiable) functions and diffeomorphisms (i.e. smooth homeomorphisms whose inverses are also smooth). The theory of these functions, as well as some results we reference, can be found in [Hir76] and [Mil97], but as we do not explore this theory per se, the thesis is readable without that background.

1.1 Knots

The intuitive picture of a mathematical knot is that of a piece of string placed in three-dimensional space with both its ends tied together. In the literature on knot theory this notion has been made precise in many slightly different but practically equivalent ways. We will use the following definition.

Definition 1. A *knot* K is a smooth embedding of the circle S^1 in 3-dimensional Euclidean space: $K\colon S^1 \to \mathbb{R}^3$, or the image of such an embedding: $K \subset \mathbb{R}^3$.

The image of the embedding corresponds to the imagined loop of string. It will usually be irrelevant whether knots are considered as embeddings or simple subsets, but sometimes one is more convenient than the other. Another minor detail is the ambient space, which is sometimes taken as S^3 instead of $\mathbb{R}^3$, which is more convenient in some cases but has little overall effect on the theory. The role of smoothness in the definition is more important, however: it serves to avoid certain 'infinitely tangled' knots that could occur otherwise.[1]

One would like to think of two knots as identical if either one of them can be moved around freely without intersections, and in this way be transformed into the other. This is made precise in the following way.

Definition 2. Two knots $K, K'\colon S^1 \to \mathbb{R}^3$ are *smoothly isotopic* (s.i.) or simply *equivalent* if there exists a smooth function $f\colon S^1 \times [0,1] \to \mathbb{R}^3$ such that $f(x,0) = K(x)$ and $f(x,1) = K'(x)$ for all x, and $x \mapsto f(x,t)$ is an embedding for all t.

In definition 2 it is crucial that the isotopy as a whole is smooth. If arbitrary topological isotopies were allowed, even if all embeddings given by a fixed t were required to be smooth, *all* (smooth) knots would be isotopic: a knotted portion of a string could simply be 'pulled tight' to shrink into a point, which would be continuous but not differentiable in t.

Smooth isotopy will be proved to be an equivalence relation in section 1.3, allowing us to talk about equivalence classes of knots with respect to this relation. In that section we also introduce other notions of knot equivalence, which will let us better understand smooth isotopy. The questions investigated in this thesis all revolve around classifying knots up to this kind of equivalence.

We have yet to see any examples of knots, as we have not introduced our main tool for illustrating knots: knot diagrams.

[1] Such knots have been called *wild*, as opposed to the *tame* knots we study. For some examples of wild knots, see e.g. [BZ03, p. 3]. Instead of using smoothness, many authors instead require knots to be piecewise linear, which results in a similar theory.

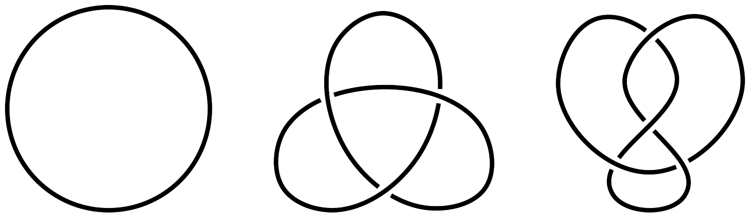

Figure 1: Diagrams of an unknot 0_1, a trefoil knot 3_1, and a figure-eight knot 4_1.

Definition 3. A diagram of a knot K is a projection of K onto a plane such that there are only finitely many points where the projection intersects itself, and each of these points results from exactly one strand of the knot crossing over another, where all crossings are marked to indicate which strand passes over the other.

This definition is somewhat informal, as the details will not be relevant to us. A more rigorous treatment can be found in [BZ03, Chapter 2]. Knot diagrams are initially useful in that they are easily drawn and unambiguously identify exactly one equivalence class (though each class is of course represented by infinitely many diagrams). However, some methods such as the Wirtinger presentation of a knot group (which we will later derive and use) also make use of the two-dimensional structure of knot diagrams to calculate certain properties of knots.

Example 1. Figure 1 shows three knot diagrams. To the left is shown the simplest possible knot diagram, without any crossings. This knot is called the unknot, and it is particular in many ways. The middle and right diagrams have three and four crossings respectively. We will later prove that all three of these knots are inequivalent.

Note that in any knot diagram with a nonzero number of crossings, each arc of the diagram has two endpoints at (not necessarily different) crossings, so the number of arcs equals the number of crossings. Diagrams with no crossings (representing the unknot) must be excluded as they nonetheless have one arc.

Proving that two knots are *equivalent* is usually done by simply manipulating a diagram in a series of comprehensible steps, going from one knot to the other, without giving any explicit formulas for isotopies.[2] Figure 2 gives an example of this, illustrating a figure-eight knot transforming into the same shape, but with all the orientation of all crossings reversed.

Any knot may be assigned a 'crossing number', namely the smallest number of crossings a diagram of that knot can have. The three knots just seen (including the mirror image of the trefoil, which is different as we will later show) are the only knots with crossing number ≤ 4, though we will not prove this. To find the equivalence classes of knots up to crossing number n, a simplistic method would be to draw every possible (essentially distinct) diagram with n crossings or fewer, exhibit equivalence proofs for the diagrams that are equivalent, and finally prove that the resulting classes of knots really are inequivalent. The number of distinct knots grows quickly, so this is only a feasible method for small n. Early tables of knots compiled in this way (though not always completely rigorously) have contributed a traditional notation system for knots: the knots with crossing number n are enumerated as $n_1, n_2, \ldots$, in an arbitrary order. This explains the notation in figure 1.

[2] There exists, in fact, a set of three operations on knot diagrams (so-called *Reidemeister moves*) such that not only do the operations produce equivalent diagrams, but *any* equivalent diagrams may be transformed into one another by a sequence of such operations. A proof of this can be found in [BZ03, Proposition 1.14].

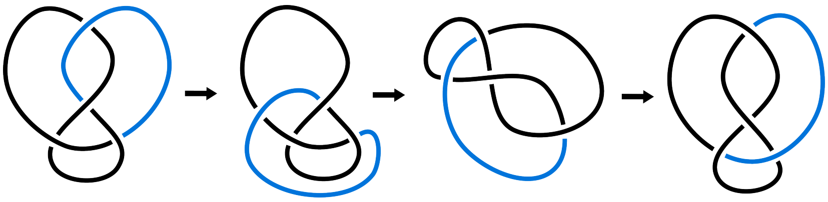

Figure 2: Illustration of a figure-eight knot turning into its mirror image. The first step moves only the blue strand, the second step only adjusts the shapes of the strands without affecting the crossings, and the last step is a rotation.

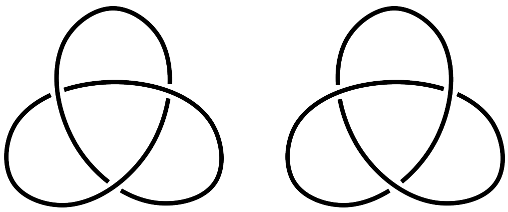

Figure 3: Diagrams of a trefoil knot 3_1 and its mirror image 3_1^*.

1.2 Geometric concepts

In this section we introduce some geometric operations and properties of knots.

Definition 4. Given a knot $K\colon S^1 \to \mathbb{R}^3$ we form its *mirror image* K^* as the composition of K with a reflection of $\mathbb{R}^3$ in some plane.

This definition is a well-defined operation on equivalence classes of knots, since all reflections may be smoothly transformed into each other. If the mirroring plane is taken to be the same as the one projected to when forming a knot diagram, it is clear that reversing the orientation of each crossing in a diagram yields a diagram of the mirror knot. Therefore figure 2 proves that the figure-eight knot is equivalent to its mirror image. We will call a knot *amphichiral* if it has this property, and *chiral* if does not.[3] We will later prove that the trefoil is chiral (as a special case of a more general fact: all torus knots are chiral), so the knots in figure 3 are inequivalent.

Definition 5. If an oprientation of the circle S^1 is chosen, a knot $K\colon S^1 \to \mathbb{R}^3$ can be said to be *oriented*. An *oriented diagram* of a knot marks the orientation of each strand (say, with arrows at the ends of strands).

Orienting a diagram divides its crossings into two essentially distinct types, depending on whether the undercrossing strand points to the left or to the right from the perspective of the overcrossing strand. Figure 4 shows an orientation of the 6_2 knot as well as the different kinds of crossings. Just as with mirror images, one might ask if the two orientations of a knot are the same. Knots whose two oriented versions can be transformed into each other by a smooth isotopy are called *invertible*. All knots studied in this thesis are invertible, but non-invertible knots do exist.

[3] In knot-theoretic literature, the word *amphichiral* is often encountered as *amphicheiral*. This is the form used by Peter Tait, who introduced the term in the 1870s [Tai77]. One might argue that the spelling with *-ei-* more closely matches the Ancient Greek χείρ (kheír) 'hand' from which the word derives, but all other English words derived from this root (e.g. *chirality*) are spelled with *-i-* in almost all sources, so we will use the spelling that is consistent with these other terms.

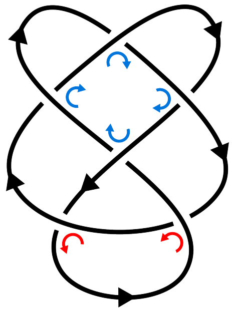

Figure 4: An oriented diagram of the 6_2 knot. The upper four crossings (blue) and the lower two (red) are different.

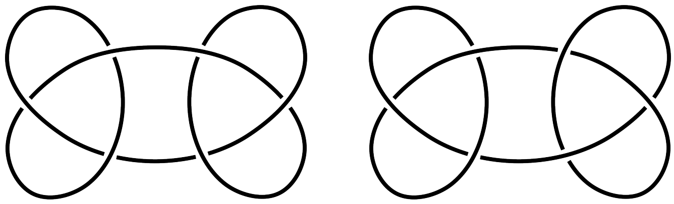

Figure 5: The square knot $3_1\#3_1^*$ and the granny knot $3_1\#3_1$.

Another operation commonly studied in knot theory is that of taking 'sums' of knots, which we define informally:

Definition 6. If K and K' are knots, their *sum* is denoted $K\#K'$, and is formed by removing a small arc of each of two knots, and connecting the two knots to each other with unknotted arcs between those endpoints.

Figure 5 shows the sum of a trefoil with its mirror image (a so-called 'square knot') as well as the sum of a trefoil with itself (a so-called 'granny knot'). It turns out that every knot has a unique decomposition into a sum of knots that themselves cannot be written as a knot sum (except that the sum may include the unknot, as taking a sum with the unknot never changes a knot). Knots which cannot be decomposed further are called 'prime' knots. All knots we will see in this thesis are prime, except for the square and granny knots (see e.g. [BZ03, Chapter 7]).

Finally, we introduce tubular neighborhoods of knots, which are necessary for many definitions and arguments throughout the thesis.

Definition 7. If $K \subset \mathbb{R}^3$ is a knot, a *tubular neighborhood* of K is the image of any embedding of a solid torus D in $\mathbb{R}^3$ such that the circle forming the core of D is embedded as K.

In effect, a tubular neighborhood of a knot looks like the knot (and deformation retracts onto it) but as a tube with nonzero width instead of just a curve. It should be noted that this is a special case of a more general definition of tubular neighborhood, which in [Hir76, p. 109] is shown to exist for all smooth submanifolds of $\mathbb{R}^3$. We can therefore take the existence of tubular neighborhoods of knots for granted. If knots were not required to be smooth, however, the existence of tubular neighborhoods would not be guaranteed.

1.3 Knot equivalences

We will now further study our main notion of knot equivalence, smooth isotopy, as well as other kinds of equivalence. We begin by proving that we are even justified in talking about equivalence classes under smooth isotopy.

Proposition 1. *Smooth isotopy is an equivalence relation.*

Proof. Let $K, K', K'' : S^1 \to \mathbb{R}^3$ be knots such that K is s.i. to K' through f, and K' is smoothly isotopic to K'' through f'.

Reflexivity and symmetry are simple: K is smoothly isotopic to itself through $(x,t) \mapsto K(x)$, and K' is smoothly isotopic to K through $(x,t) \mapsto f(x, 1-t)$.

Transitivity is more involved since we have to ensure that the resulting isotopy is smooth. We can choose (e.g. as the integral of a bump function) a smooth function $s : [0,1] \to [0,1]$ such that there is an $\varepsilon > 0$ such that $s(x) = 0$ if $0 \le x \le \varepsilon$ and $s(x) = 1$ if $1 - \varepsilon \le x \le 1$. Then K is isotopic to K'' through:

$$(x,t) \mapsto \begin{cases} f(x, s(2t)) & 0 \le x \le \frac{1}{2} \\ f'(x, s(1-2t)) & \frac{1}{2} \le x \le 1 \end{cases}$$

This isotopy is independent of t (agreeing with K') at an open interval around $t = \frac{1}{2}$, so it is smooth everywhere. □

We introduce two more equivalences, which we will use to better understand smooth isotopy:

Definition 8. Let $K, K' : S^1 \to \mathbb{R}^3$ be two knots.

- K and K' are *ambient diffeomorphic* if there exists a diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h \circ K = K'$.
- K and K' are *smoothly ambient isotopic* if there exists a smooth function $f : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that if $f_t : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by $f_t(x) = f(x,t)$, the following conditions hold: f_0 is the identity function, f_t is a diffeomorphism for all t, and $f_1 \circ K = K'$.

These terms and definitions are not common in knot-theoretic literature: usually, neither equivalence is required to be smooth. For isotopy of the knot itself, smoothness was important, but it turns out that requiring the isotopy to transform the ambient space solves the same problem, even without requiring smoothness. For this reason, authors not working with smooth knots usually take ambient isotopy as the basic notion of equivalence.

Proposition 2. *Ambient diffeomorphism and smooth ambient isotopy are both equivalence relations. Furthermore, smooth ambient isotopy is a refinement of ambient diffeomorphism.*

Proof. Let $K, K', K'' : S^1 \to \mathbb{R}^3$ be knots and suppose that K is ambient diffeomorphic to K' through h and that K' is ambient diffeomorphic to K'' through h'. Then K is ambient diffeomorphic to itself through the identity function, K' is ambient diffeomorphic to K since $h^{-1} \circ K' = h^{-1} \circ h \circ K = K$, and K is ambient diffeomorphic to K'' since $h' \circ h \circ K = h' \circ K' = K''$. Ambient diffeomorphism is therefore an equivalence relation. Smooth ambient isotopy may be proven to be an equivalence relation in the same way as in proposition 1. Finally, if K and K' are smoothly ambient isotopic through f, they are ambient diffeomorphic through f_1 by definition. □

It turns out that it is possible to describe with some precision just how much finer ambient isotopy is: each ambient diffeomorphism class consists of either one or two smooth ambient isotopy classes: the former case occurs for amphichiral knots, and in the latter case, the two isotopy classes are mirror images. The proof relies on the following fact about orientation-preserving maps:

Proposition 3. *If $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation-preserving diffeomorphism, then h is isotopic to the identity.*

A proof of this (in fact valid for all $\mathbb{R}^n$) can be found in [Mil97, p. 34]. The result about isotopies is a direct consequence:

Proposition 4. *If K and K' are ambient diffeomorphic, K is smoothly ambient isotopic either to K' or to the mirror image of K'.*

Proof. Let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism such that $h \circ K = K'$. If h is orientation-preserving, it is isotopic to the identity, which makes K smoothly ambient isotopic to K'. If h is orientation-reversing, choose a reflection $r\colon \mathbb{R}^3 \to \mathbb{R}^3$: then, $r \circ h$ is orientation-preserving and isotopic to the identity, and since $(r \circ h) \circ K = r \circ (h \circ K) = r \circ K'$, K is smoothly ambient isotopic to the mirror image of K'. $\square$

Finally we will see how this relates to smooth isotopy. We use the isotopy extension theorem, whose statement and proof can be found in [Hir76, p. 180].

Proposition 5. *Smooth ambient isotopy and smooth isotopy of knots are identical relations.*

Proof. If the knots K and K' are smoothly ambient isotopic through f, they are smoothly isotopic through $f \circ K$. The other direction requires the isotopy extension theorem, whose assumptions are satisfied since S^1 is compact and $\mathbb{R}^3$ lacks boundary. The theorem tells us that any smooth isotopy from K to K' can be extended to a smooth isotopy of $\mathbb{R}^3$. $\square$

2 Knot groups

Knot equivalence is a topological question, but considered as spaces, all knots are homeomorphic, by definition. Instead, the complement of a knot is a more useful object of study. In fact, it has been shown that two knots are equivalent (up to mirror images) if and only if their complements are homeomorphic. The forwards direction of this statement is an immediate consequence of ambient diffeomorphism (and is proved as part of proposition 6 below), but the fact that the complement determines the knot was a long-standing conjecture first proved in 1989 by Gordon and Luecke [GL89]. We will however not study the topology of knot complements directly, but initially limit our focus to their fundamental groups, which is a weaker invariant of knots, but it is more approachable. In section 4 we will see how the fundamental group can be augmented with some extra information to produce a stronger invariant.

2.1 Definition and calculation for the unknot

Definition 9. The *knot group* of a knot K refers to the fundamental group of its complement: $\pi_1(\mathbb{R}^3 - K, x)$, for some choice of $x \in \mathbb{R}^3 - K$.

We will usually omit the basepoint from the notation for the fundamental group, as the knot complement is path-connected.[4]

Proposition 6. *Equivalent knots have homeomorphic complements and isomorphic knot groups.*

Proof. Let K and K' be equivalent knots and let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism satisfying $h(K) = K'$ (which exists since smooth isotopy is stronger than ambient diffeomorphism). Since $h(x) \in K'$ if and only if $x \in K$ we may define a homeomorphism $g : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$ as a restriction of h. Since the fundamental group is a homeomorphism invariant (up to isomorphism), $\pi_1(\mathbb{R}^3 - K)$ and $\pi_1(\mathbb{R}^3 - K')$ are isomorphic. □

Our main tool for showing that two knots are inequivalent will be to show that they have non-isomorphic knot groups. The next sections are dedicated towards developing theory capable of giving presentations for knot groups as well as telling these presentations apart in certain cases. We should note that the fundamental group on its own cannot differentiate mirror images, as they have homeomorphic complements (but the methods of section 4 do not have this problem).

As an initial example we will directly compute the knot group of the unknot.

Example 2. Let $K = \{(x, y, 0) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be the unknot, and let

$$Z = \{(x, y, z) : x^2 + y^2 + z^2 = 4\} \cup \{(0, 0, z) : -2 < z < 2\}$$

be a sphere surrounding K, together with a diameter of the sphere along the z-axis, passing through the middle of K. Then by a construction presented in [Hat02] we may form a deformation retraction of $\mathbb{R}^3 - K$ onto Z by letting points outside of the sphere move directly onto it, and letting points inside the sphere move in a straight line away from the closest point of K onto either the sphere of Z or its diameter. This shows that $\mathbb{R}^3 - K$ is homotopy equivalent to a sphere with a diameter, which in turn is homotopy equivalent to the wedge sum $S^2 \vee S^1$ by moving one endpoint of the diameter onto the other. This final space has fundamental group $\pi_1(S^2 \vee S^1) = \pi_1(S^1) = \mathbb{Z}$, since $\pi_1(S^2) = 0$, proving that the knot group of the unknot K is infinite cyclic. □

[4] This is an intuitive fact but is not very easy to prove with the methods we have available, though it follows as a direct consequence of Alexander duality.

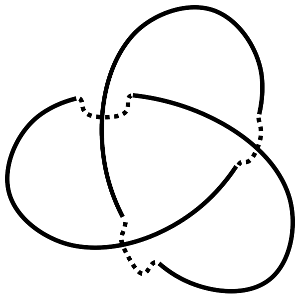

Figure 6: A trefoil knot in standard position.

We will not calculate other knot groups directly in this way, and in general, knot groups will not be as simple as $\mathbb{Z}$.[5] The most useful tool for computing knot groups is the Seifert–van Kampen theorem, specifically the version of the theorem which operates on group presentations (see e.g. [Lee10, Theorems 10.1 and 10.3]).

2.2 The Wirtinger presentation

There is a simple procedure to produce a presentation of a knot group from an oriented knot diagram, called the Wirtinger presentation. This procedure requires an orientation of the knot to be chosen: this has no effect on the group itself, but the relations are computed using an orientation which needs to be consistent. We will need knots to be positioned in a specific way:

Definition 10. A knot is in *standard position* if it lies wholly in the $z = 0$ plane, except for underpasses extending down to $z = -1$ at each crossing.

Any knot diagram may be turned into a knot in standard position and vice versa, with arcs in the diagram corresponding to sections in the $z = 0$ plane. Figure 6 shows a trefoil knot in standard position: the solid lines are at $z = 0$, and the dotted lines extend down to $z = -1$. Since the number of arcs (and the number of crossings) is finite for any knot in standard position, we can always choose an $\varepsilon > 0$ much smaller than the length of the shortest arc or the shortest distance between two arcs.

Given a knot in standard position we can now introduce the generators of the presentation and find their respective relations, and subsequently prove that those generators indeed generate the entire knot group.

Definition 11. Let K be an oriented knot in standard position with n arcs labeled $\alpha_1, \dots, \alpha_n$ and choose a point P in the $z = 0$ plane, $P \notin K$. Choose a small $\varepsilon > 0$ and encircle each α_i with a circle γ_i of radius ε. For each α_i form an element a_i of $\pi_1(\mathbb{R}^3 - K, P)$ as the following path:

1. Go from P to a point of γ_i, staying in the $z > 0$ half-space except at the start.
2. Go around γ_i once, clockwise from the perspective of the oriented arc α_i.
3. Go back to P by the same path.

The elements $\{a_1, \dots, a_n\} \subset \pi_1(\mathbb{R}^3 - K, P)$ are called *Wirtinger generators* of K.

[5] In fact, the unknot is the only knot whose group is $\mathbb{Z}$, but this result is relatively difficult; a proof can be found in [Rol03] where it is called the 'unknotting theorem'. This result (together with proposition 8 in this document) implies that all nontrivial knots have nonabelian groups.

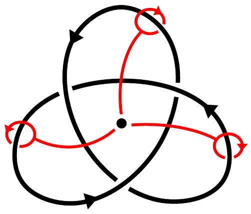

Figure 7: Wirtinger generators of the trefoil knot (in red).

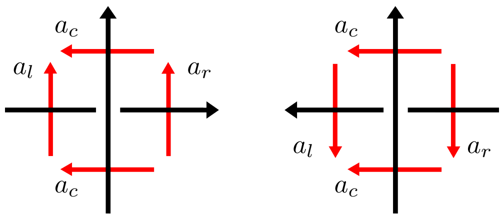

Figure 8: Wirtinger relations at two types of crossings.

Figure 7 shows Wirtinger generators of the trefoil knot. In all following diagrams, Wirtinger generators will be presented as arrows passing under their respective arcs, representing the oriented bottom half of γ_i, with the connections to and from a basepoint implicitly understood. It should be noted that this definition depends strongly on the projection chosen for the knot diagram. There are natural relations present between the Wirtinger generators at the crossings of the diagram:

Proposition 7. *The Wirtinger generators conform to the relation $a_c a_l = a_r a_c$ at each crossing, where a_c is the generator for the overcrossing arc, and a_r and a_l are the generators for the undercrossing arcs to the right and left of the directed overcrossing arc, respectively.*

Proof. There are two types of crossings we need to consider: those where the undercrossing strand comes from the left and those where it comes from the right. Both are shown in figure 8 with labels placed appropriately. By composing the arrows in the figure we see that the relation $a_c a_l = a_r a_c$ holds for both crossings. $\square$

It turns out that these generators and relations are sufficient to present the group, a result which has been credited to Wilhelm Wirtinger, though the original discovery was never published.

Theorem 1. *Given an oriented n-crossing diagram of a knot K, there is a group presentation*

$$\pi_1(\mathbb{R}^3 - K) = \langle\, a_1, \ldots, a_n \mid \text{crossing relations}\,\rangle,$$

where the generators a_i correspond to the n arcs in the diagram, and there are n crossing relations $a_c a_r = a_l a_c$ as in proposition 7.

Proof. The following argument is adapted from [Sti93, p. 146]. Place K in standard position according to the diagram, choosing a basepoint in the $z = 0$ plane, and let V be a tubular neighborhood of K consisting of the points that have distance at most ε to the knot, for some small $\varepsilon > 0$ and split the complement into two sections A and B:

$$A = \{(x, y, z) \in \mathbb{R}^3 - V : z > -1\}; \quad B = \{(x, y, z) \in \mathbb{R}^3 - V : z < -1 + \frac{\varepsilon}{2}\}$$

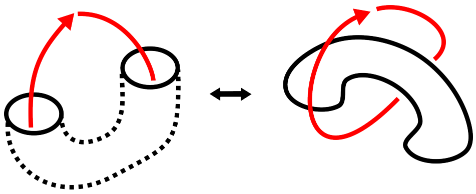

Figure 9: Homotopy between a half-space minus a handle and a half-space with a handle.

These are now open sets whose union is $\mathbb{R}^3 - T$. B is simply connected, but A has n tubular holes, one for each arc of the knot diagram. The fundamental group of A is free on n generators. To see this, push out the tubular holes according to figure 9 below to form a half-space with n handles, and contract the half-space to a ball or a point, forming a space clearly isotopic to a bouquet of n circles. Specifically, the generators of A can be chosen to coincide with the Wirtinger generators $a_1, \dots, a_n$.

The intersection $A \cap B$ is an infinite sheet of width $\frac{\varepsilon}{2}$ with a hole for each crossing in the diagram, so its fundamental group is also free on n generators. Generators for $A \cap B$ can be chosen to each go around one of the holes, making them equal to $a_c a_r a_c^{-1} a_l^{-1}$ in $\pi_1(A)$ at each crossing by proposition 7.

Applying the Seifert–van Kampen theorem to A and B then gives the desired presentation. □

The existence of the Wirtinger presentation gives some immediate insights into the possible structure of knot groups. The abelianization of a group is its quotient by its commutator subgroup, or equivalently by the relation generated by letting all elements of the group commute.

Proposition 8. *If G is a knot group, G is finitely presented, and its abelianization G^{Ab} is isomorphic to $\mathbb{Z}$.*

Proof. The Wirtinger presentation is always finite, proving the first assertion.

In G^{Ab}, all elements commute, so the Wirtinger relation $a_c a_r = a_l a_c$ implies $a_r = a_l$. Thus all generators are equal, and all relations are satisfied by default, leaving only a single generator and giving the presentation $G^{Ab} = \langle\, a \mid - \,\rangle = \mathbb{Z}$. □

The abelianization of the fundamental group is isomorphic to the first homology group of the same space, and it is not a coincidence why this is the same for all knots: it is known by Alexander duality that the homology of the complement of a knot does not depend on the embedding chosen for the knot, so the homology of knot complements is not a useful knot invariant.[6]

Example 3. By theorem 1, the figure-eight knot shown in figure 10 has the following presentation:

$$\langle\, a, b, c, d \mid ad = ba, bc = ab, cd = ac, dc = bd \,\rangle.$$

Using just the first three relations, we get that $dc = a^{-1}bac = a^{-1}bcd = bd$, so the fourth relation is redundant. This corresponds to deforming the loop represented by dc to the loop of bd by only passing under the crossings not represented by the fourth relation. In general, any one of the relations in the Wirtinger presentation may always be deduced from the others in this way, see e.g. [Rol03, p. 57].

[6] However, the homology of *covering spaces* of knot complements turns out to be useful; see e.g. [Rol03, Chapter 6].

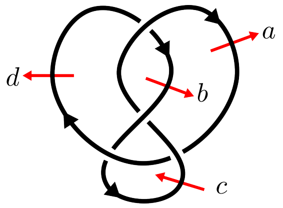

Figure 10: Oriented diagram of a figure-eight knot with named generators.

Simplifying further, we may make the substitutions $c = b^{-1}ab$ (relation 2) and $d = a^{-1}ba$ (relation 1), giving relation 3 as $b^{-1}aba^{-1}ba = ab^{-1}ab$, which after multiplying by b from the left gives the following presentation for the figure-eight knot group:

$$\langle\, a, b \mid aba^{-1}ba = bab^{-1}ab \,\rangle. \qquad \square$$

Example 4. The Wirtinger presentation of the trefoil knot is

$$\langle\, a, b, c \mid ab = ca, bc = ab, ca = bc \,\rangle.$$

Clearly, the third equation follows from the first two. From the second we substitute $c = b^{-1}ab$ in the first, giving the relation $ab = b^{-1}aba$. A presentation of the trefoil knot is therefore

$$G = \langle\, a, b \mid aba = bab \,\rangle.$$

In section 3.2, a different method of presenting the group of a trefoil knot is described (as a special case of a more general calculation). However, that method produces the presentation $H = \langle\, \alpha, \beta \mid \alpha^2 = \beta^3 \,\rangle$ (written here with different letters for clarity). These presentations must define isomorphic groups, and this can indeed be seen by letting $\alpha = aba$ and $\beta = ba$. This preserves the relation since $\alpha^2 = (bab)(aba) = \beta^3$, and α and β generate the whole group since $\alpha\beta^{-1} = aba(ba)^{-1} = a$ and $\beta^2\alpha^{-1} = baba(aba)^{-1} = b$.

Example 5. In section 1.2 we defined the square knot as $3_1\#3_1^*$ and the granny knot as $3_1\#3_1$. Knot groups do not distinguish mirror images from each other so we know that the trefoil itself has the same group as its mirror image, but even though (as we will see in section 4.3) the square and granny knots are not ambient isotopic or even mirror images, they still have the same group, which we now show.

We use a method for simplifying the calculation of the group of knot sums presented in [Rol03, p. 61]. Call one trefoil T_1 and the other T_2, and let $K = T_1\#T_2$ be their sum, regardless of orientation. Encase T_1 and T_2 in solid non-intersecting balls B_1 and B_2. Then,

$$\pi_1(\mathbb{R}^3 - K \cup B_1) = \pi_1(\mathbb{R}^3 - T_2) = \langle\, a_1, b_1 \mid a_1b_1a_1 = b_1a_1b_1 \,\rangle,$$
$$\pi_1(\mathbb{R}^3 - K \cup B_2) = \pi_1(\mathbb{R}^3 - T_1) = \langle\, a_2, b_2 \mid a_2b_2a_2 = b_2a_2b_2 \,\rangle,$$

as $K \cup B_1$ is just T_2 with a part of the knot 'thickened' where T_1 would be, and vice versa. We also have

$$\pi_1((\mathbb{R}^3 - K \cup B_1) \cap (\mathbb{R}^3 - K \cup B_2)) = \pi_1(\mathbb{R}^3 - K \cup B_1 \cup B_2) = \langle\, x \mid - \,\rangle.$$

We can choose the generators a_1, b_1, a_2, b_2 such that a_1 and b_2 correspond to the strand of the trefoil that is connected to the other in the knot sum, corresponding to

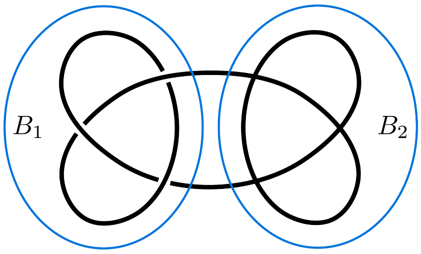

Figure 11: The construction for the group of the square and granny knots.

the generator x of the intersection. Finally we apply the Seifert–van Kampen theorem, which gives

$$\begin{aligned}\pi_1(\mathbb{R}^3 - K) &= \pi_1((\mathbb{R}^3 - K \cup B_1) \cup (\mathbb{R}^3 - K \cup B_2)) \\ &= \langle\, a_1, b_1, a_2, b_2 \mid a_1b_1a_1 = b_1a_1b_1, a_2b_2a_2 = b_2a_2b_2, b_1 = a_2 \,\rangle \\ &= \langle\, a, x, b \mid axa = xax, bxb = xbx \,\rangle\end{aligned}$$

As this method calculates the same knot group for K regardless of orientations, it is the group of both the square knot and the granny knot.

2.3 Homomorphisms to finite groups

Having a presentation of a knot group only helps us classify knots as different if we can actually prove that two presentations are not isomorphic. This is not an easy problem: the general problem of deciding whether two finite presentations define isomorphic groups is in fact not even decidable [Rab58].[7] There are, however, methods that are useful in many cases. One such method is to consider homomorphisms from a finitely presented group G into some finite group H. This is practical since such a homomorphism is always determined by the images of the generators of G. We will use this to prove that the three knots in figure 1 are all pairwise distinct.

Proposition 9. *The trefoil knot is not ambient isotopic to the unknot.*

Proof. We will consider a homomorphism into the finite nonabelian group

$$S_3 = \{e, (12), (13), (23), (123), (132)\}.$$

Let $G = \langle\, a, b \mid a^2 = b^3 \,\rangle$ be the trefoil group, and let $f\colon G \to S_3$ be generated by $f(a) = (12)$ and $f(b) = (123)$. Since $(12)^2 = (123)^3 = e$, f is a well-defined homomorphism, but (12) and (123) generate S_3, so $f(G) = S_3$. Since G has an epimorphism onto S_3 which is not cyclic, it cannot be $\mathbb{Z}$. The trefoil and the unknot therefore have non-isomorphic knot groups and cannot be ambient isotopic. □

Proposition 10. *The trefoil knot is not ambient isotopic to the figure-eight knot.*

Proof. We will prove that there is no epimorphism of the figure-eight knot group onto S_3, which together with the previous result shows that the trefoil and the figure-eight are distinct.[8] Let $H = \langle\, a, b \mid aba^{-1}ba = bab^{-1}ab \,\rangle$ be the figure-eight knot group and suppose that $f\colon H \to S_3$ is a homomorphism. Let $x = f(a)$ and $y = f(b)$.

[7] For the special case of knots and knot groups the problem is decidable but not by any practically useful method, according to [Sti93, p. 226], citing works by Friedhelm Waldhausen.

[8] This method is suggested without a computation in [CF77].

Then $xyx^{-1}yx = yxy^{-1}xy$. If x is an odd permutation and y even, or vice versa, one side of the equation will be odd and the other even, so x and y are both odd or both even. If they are both even, they are both part of the A_3 subgroup and cannot generate S_3. Finally, if x and y are both odd, they are both transpositions, and they are their own inverses. Any two transpotitions t_1 and t_2 in S_3 have the property that $t_1t_2t_1$ equals the third transposition that is neither t_1 nor t_2, which means that $xyx^{-1}yx = xyxyx = xzx = y$ for $z \neq x, y$, and likewise $yxy^{-1}xy = x$, so $x = y$ and the image of f is just the 2-element subgroup generated by a transposition. Therefore, f cannot be an epimorphism. □

Proposition 11. *The figure-eight knot is not ambient isotopic to the unknot.*

Proof. As the previous examples have shown, the figure-eight knot has the same homomorphisms into S_3 as the unknot, but we can make a similar argument to that of proposition 9 by instead considering the alternating group on 4 letters:

$$\begin{aligned} A_4 = \{e\} &\cup \{(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243)\} \\ &\cup \{(12)(34),\ (13)(24),\ (14)(23)\}. \end{aligned}$$

Let $H = \langle\, a, b \mid aba^{-1}ba = bab^{-1}ab \,\rangle$ be the figure-eight group, and let $f\colon H \to S_3$ be generated by $f(a) = (123)$ and $f(b) = (142)$. We get the following calculations:

$$(123)(142)(123)^{-1}(142)(123) = e = (142)(123)(142)^{-1}(123)(142)$$

$$(142)(123) = (143) \neq (234) = (123)(142).$$

This shows that f is a well-defined homomorphism and that $f(a)f(b) \neq f(b)(a)$. Therefore $ab \neq ba$, so $H \neq \mathbb{Z}$. □

It is not hard to program a computer to perform analyses such as the above for various given groups and knots (in the general case, by counting the total number of homomorphisms between the groups), and this method may be used to distinguish many different knots from each other.

3 Torus knots

In this section we investigate a certain class of knots in detail.

3.1 Definition and geometry

A torus knot is a knot lying on a torus surface embedded in $\mathbb{R}^3$, by which a 'standard' or 'unknotted' embedding of the torus is meant. An example of a torus knot is the trefoil knot, as shown in figure 12.

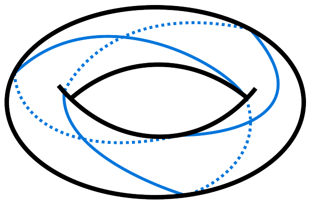

Figure 12: A trefoil knot on the surface of a torus.

We can view any torus knot as a map from the circle S^1 to the torus $S^1 \times S^1$, ignoring for the moment the question of precisely how the torus is embedded.

Proposition 12. *Taking S^1 to be represented by the complex unit circle and letting $p, q \in \mathbb{Z}$ be such that $\gcd(p,q) = 1$, the map $S^1 \to S^1 \times S^1$ given by $z \mapsto (z^p, z^q)$ is injective.*

Proof. Let $a, b \in S^1 \subset \mathbb{C}$ be such that $(a^p, a^q) = (b^p, b^q)$. This implies

$$p \arg a = p \arg b + 2\pi m; \quad q \arg a = q \arg b + 2\pi n$$

for $m, n \in \mathbb{Z}$. Letting $d = \arg a - \arg b$, we get $\frac{d}{2\pi} = \frac{m}{p} = \frac{n}{q} \in \mathbb{Q}$. If this rational number is written in reduced form as $\frac{r}{s}$, then $\frac{pr}{s} = m$ and $\frac{qr}{s} = n$ are integers, so $s|p$ and $s|q$, but $\gcd(p,q) = 1$, so $s = 1$. Thus $\arg a - \arg b = d = 2\pi r$ and since $r \in \mathbb{Z}$ and $|a| = |b| = 1$, this implies that $a = b$. $\square$

We now return to the question of how to embed the torus in $\mathbb{R}^3$. We choose an embedding such that $S^1 \times \{0\}$ maps to a *longitude*, i.e. a loop going around the 'outside' hole of the torus, and $\{0\} \times S^1$ maps to a *meridian*, i.e. a loop going around the 'inside' hole of the torus,[9] as in figure 13. This allows us to define a collection of knots:

Definition 12. A (p,q)-*torus knot* is given by the map $z \mapsto (z^p, z^q)$ for $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$, composed with the embedding of the torus described by figure 13.

We can view the longitude as a $(1,0)$-torus knot and the meridian as a $(0,1)$-torus knot (though in $\mathbb{R}^3$ they are ambient isotopic), and consequently a (p,q)-torus knot is one that wraps around the torus p times in the longitudinal direction and q times in the meridional direction. For example, the trefoil knot is a $(2,3)$-torus knot.

One thing might seem lacking from this definition: we have not specified a standard orientation of S^1 or the embedding, and we might as well have said that the trefoil

[9] This is the standard terminology, though it is perhaps confusing: on a globe, a meridian is a curve of constant longitude, while 'latitude' describes the perpendicular coordinate. At least one author, [Sti93], indeed talks about meridians and *latitudes* instead, but we will stick to the established convention.

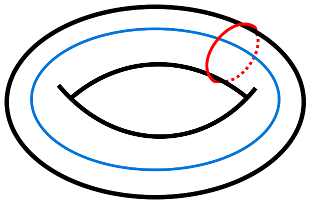

Figure 13: In blue: $S^1 \times \{0\}$, a longitude. In red: $\{0\} \times S^1$, a meridian.

knot in the picture is a $(-2, 3)$-torus knot for a different choice of orientation. For our purposes, this is not a problem, as long as we keep in mind that there is always an implicit choice of orientation.

So far we have only shown that (p, q)-torus knots actually exist: we have not shown that all knots on a torus surface are (p, q)-torus knots, and we have not shown which torus knots are isomorphic and which are not. We will focus on the latter question, but we first give an overview of the former. We may take the meridian μ and longitude λ as generators of $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$. Any closed curve on the torus is therefore homotopic to $\mu^a \lambda^b$ for integers a, b. The following facts about curves on tori form the core of the argument.

Proposition 13. *Any simple closed curve on $S^1 \times S^1$ is null-homotopic or homotopic to $\mu^a \lambda^b$ for relatively prime integers a and b. Furthermore, any two homotopic simple closed curves on $S^1 \times S^1$ are ambient isotopic on $S^1 \times S^1$.*

These assertions are proven in the first chapter of [Rol03] by a series of geometric arguments which we will not repeat here. It follows that the isotopy classes of simple closed curves on $S^1 \times S^1$ are exactly the (p, q)-torus knots if a loop that is null-homotopic on S^1 is added as a '$(0, 0)$-torus knot' (our actual definition does not cover this case), but for knots in $\mathbb{R}^3$ this addition does not make a difference.

We now investigate which torus knots are equivalent as knots in $\mathbb{R}^3$:

Proposition 14. *The (p, q)-torus knot is ambient isotopic to torus knots of the types (q, p), $(-p, -q)$, and $(-q, -p)$, and the mirror image of the (p, q)-torus knot is ambient isotopic to the $(p, -q)$-torus knot.*

Proof. The easiest way to see that (p, q) is (q, p) is to work in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ instead of $\mathbb{R}^3$: in that case, the inside and the outside of the torus surface are both solid tori, which can be exchanged by an ambient isotopy.

Rotating the knot 180° around an axis passing through the torus twice turns the (p, q) type into $(-p, -q)$ (which implies that torus knots are invertible).

Mirroring the torus knot in the plane aligned with the core of the torus turns the meridian $(0, 1)$ into $(0, -1)$ but does not affect the longitude, so $(p, q)^* = (p, -q)$. $\square$

Proposition 15. *For all $n \in \mathbb{Z}$, the torus knots of types $(n, 1)$, $(n, -1)$, $(1, n)$, and $(-1, n)$ are all isotopic to the unknot.*

Proof. The $(1, n)$-torus knot, as seen from above, has no crossings, so it is an unknot. The others follow from proposition 14. $\square$

In our later investigation of torus knots, we will usually restrict ourselves to the case when $p, q \geq 2$.

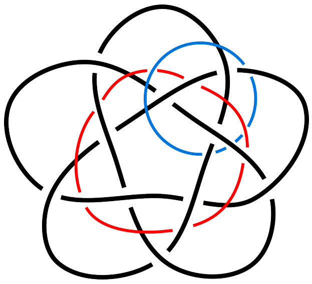

Figure 14: Generators for the group of the $(3, 5)$-torus knot.

3.2 Calculating a presentation

For torus knots, the Wirtinger method does not produce a very practical presentation, but applying the Seifert–van Kampen theorem directly turns out to be relatively simple, allowing us to prove the following presentation.

Theorem 2. *The knot group of a (p, q)-torus knot is presented by $\langle\, a, b \mid a^p = b^q \,\rangle$.*

Proof. Let T be an unknotted torus in $\mathbb{R}^3$ and let K be a (p, q)-torus knot on the surface of T. Let N_K be a closed tubular neighborhood of K, and let N_T be an open neighborhood of T thin enough that $N_T - N_K$ is homotopy equivalent to $T - K$, i.e. such that N_K extends outside N_T all along the knot on both sides of the torus surface. $\mathbb{R}^3 - T$ has two components: let A_0 be the bounded component inside T, and let B_0 be the unbounded component outside of T, and form

$$A = (A_0 \cup N_T) - N_K; \quad B = (B_0 \cup N_T) - N_K.$$

Now A and B are open sets in $\mathbb{R}^3$, $A \cup B = \mathbb{R}^3 - N_K$ and $A \cap B = N_T - N_K$. Since A is a solid torus, just with 'trenches' dug out along the surface where the knot is, it has $\pi_1(A) = \pi_1(S^1) = \mathbb{Z}$, and by the same argument as in example 2 we get $\pi_1(B) = \pi_1(S^2 \vee S^1) = \mathbb{Z}$. Let a and b be generators for $\pi_1(A)$ and $\pi_1(B)$ respectively. The intersection $A \cap B = N_T - N_K$ is homotopy equivalent to $T - K$, so it has fundamental group $\mathbb{Z}$ which we may give a generator c. The generator c wraps around the torus p times longitudinally, like K, so it equals a^p in $\pi_1(A)$. Similarly, it equals b^q in $\pi_1(B)$. By the Seifert–van Kampen theorem, we get a presentation for $\pi_1(\mathbb{R}^3 - K) = \pi_1(A \cup B)$ by joining the generators (a and b) and relations (none) of $\pi_1(A)$ and $\pi_1(B)$ together with the relation $a^p = b^q$, as desired. $\square$

3.3 Torus knots are different

The method of considering all homomorphisms into a suitably chosen finite group presented in section 2.3 is a powerful method of distinguishing any two given knots, but it is not easy to apply to the general case of all torus knots. In this section we will instead investigate the torus knot groups directly, and prove them non-isomorphic, establishing the existence of infinitely many inequivalent knots. This is the penultimate step in classifying all torus knots, leaving only the proof that torus knots are chiral, which is done in section 4.2.

Torus knot groups were first proven to all be inequivalent by Otto Schreier in [Sch24], whose analysis we will follow. We begin by proving a property of free products of groups which we will need.

Proposition 16. *If G and H are nontrivial groups, $G * H$ has trivial center.*

Proof. Let w be any non-identity element of $G * H$, as a reduced word, and assume that it begins with an element of G. It can be written either as $w = g_0 h_1 g_1 \dots h_n g_n$ or as $w = g_0 h_1 g_1 \dots h_n g_n h_{n+1}$ where all $g_i \in G - \{e\}$ and $h_i \in H - \{e\}$. Choose any non-identity element $h \in H$. The element hw is already in reduced form, and begins with h. The element wh may either be in reduced form, or be subject to a reduction of a product in H at the end. Either way, wh will begin with g_0, so $hw \neq wh$, and w cannot be central. Similarly, w cannot be central if it begins with an element of H, so $Z(G * H)$ is trivial. □

Theorem 2 gives a presentation of the group of a (p, q)-torus knot as $\langle\, a, b \mid a^p = b^q \,\rangle$, which we will denote as $G_{p,q}$ throughout this section and the next. We also denote the center of this group as $C_{p,q}$, the quotient by the center as $B_{p,q}$, and the abelianization of that quotient as $A_{p,q}$:

$$G_{p,q} = \langle\, a, b \mid a^p = b^q \,\rangle; \quad C_{p,q} = Z(G_{p,q}); \quad B_{p,q} = G_{p,q}/C_{p,q}; \quad A_{p,q} = (B_{p,q})^{Ab}$$

Our efforts will be focused on understanding these groups. We will start by using proposition 16 to calculate the center.

Proposition 17. *$C_{p,q}$ is the infinite cyclic subgroup of $G_{p,q}$ generated by a^p.*

Proof. Let $H \subset G_{p,q}$ be the cyclic subgroup generated by a^p, and let $h = a^{pn} = b^{qn}$ be any element of H. Then $ha = a^{pn+1} = ah$ and $hb = b^{qn+1} = bh$, so h commutes with the generators of $G_{p,q}$. Therefore, $H \subset C_{p,q}$, so H is a normal subgroup of $G_{p,q}$.

We can form the quotient $G_{p,q}/H = \langle\, [a], [b] \mid [a]^p, [b]^q \,\rangle$ which is equal to the free product $\langle\, [a] \mid [a]^p \,\rangle * \langle\, [b] \mid [b]^q \,\rangle$. Neither factor is trivial, so $G_{p,q}/H$ has only a trivial center per proposition 16. If x is a central element of $G_{p,q}$, $[x]$ must be central in $G_{p,q}/H$, so $C_{p,q} \subset H$, which implies that $C_{p,q}$ is exactly H. □

The result in proposition 17 is interesting in its own right, as it turns out that the center of all knot groups *not* belonging to torus knots is actually trivial, as proved by Burde and Zieschang in [BZ66].

We turn our attention to $B_{p,q}$, which can be represented as $\langle\, [a], [b] \mid [a]^p, [b]^q \,\rangle$ or equally as $\langle\, [a] \mid [a]^p \,\rangle * \langle\, [b] \mid [b]^q \,\rangle$, thanks to the calculation of $C_{p,q}$.

Proposition 18. *If $[x] \in B_{p,q}$ has finite order, $[x]$ is conjugate to a power of $[a]$ or to a power of $[b]$.*

Proof. Suppose $[g] \in B_{p,q} = \langle\, [a] \mid [a]^p \,\rangle * \langle\, [b] \mid [b]^q \,\rangle$ such that $[g]^r = 1$. This element has a normal form

$$[g] = [a]^{\alpha_1}[b]^{\beta_1} \dots [a]^{\alpha_m}[b]^{\beta_m}$$

where the only exponents allowed to be 0 are α_1 and β_m, for some integer m. Since the word $([a]^{\alpha_1}[b]^{\beta_1} \dots [a]^{\alpha_m}[b]^{\beta_m})^r$ cannot be reduced unless the normal form begins and ends with a power of the same element, either $\alpha_1 = 0$ or $\beta_m = 0$.

If $m = 1$, then $[g]$ must be a power of $[a]$ or $[b]$ (so it is of course also conjugate to one). Now assume the result for all elements of finite order whose normal form has fewer than m terms from either group. In that case we do two calculations depending on which of α_1 and β_m is zero:

$$\alpha_1 = 0 \implies [b]^{-\beta_1}[g][b]^{\beta_1} = [a]^{\alpha_2} \dots [b]^{\beta_m + \beta_1}$$
$$\beta_m = 0 \implies [a]^{\alpha_m}[g][a]^{-\alpha_m} = [a]^{\alpha_1 + \alpha_m} \dots [b]^{\beta_{m-1}}$$

In both cases, $[g]$ is conjugate to an element whose normal form has fewer than m terms from either group, and which has order r like $[g]$, making it conjugate to a power of $[a]$ or $[b]$ by assumption. The result follows from induction. □

We are now ready to prove the main theorem by finding an isomorphism invariant that distinguishes the torus knot groups.

Theorem 3 (Schreier)**.** *For the groups $G_{p,q}$, given that $0 < p < q$ and $\gcd(p, q) = 1$, the pair of integers (p, q) is an isomorphism invariant.*

Proof. This proof should be thought of as calculating p and q from the structure of $G_{p,q}$ using only information that is isomorphism-invariant. If an element of $B_{p,q}$ has finite order, the order divides p or q by proposition 18, so the largest finite order of an element in $B_{p,q}$ is q. Now consider the the abelianization $A_{p,q}$:

$$A_{p,q} = \langle\, [a], [b] \mid [a]^p = 1, [b]^q = 1, [a][b] = [b][a] \,\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_q.$$

Any isomorphic groups will have isomorphic quotients by their centers and isomorphic abelianizations, so q as well as $\mathbb{Z}_p \times \mathbb{Z}_q$ are invariantly derived from $G_{p,q}$, from which we can recover p as the order of $A_{p,q}$ (which is pq), divided by q. □

3.4 Group automorphisms

Schreier's analysis did not stop at proving the groups inequivalent: he also calculated all automorphisms of these groups. We will present Schreier's calculation of the automorphisms below, and use it to prove that torus knots are chiral in section 4.2. We start with finding the automorphisms of $B_{p,q}$.

Proposition 19. *All automorphisms f of $B_{p,q} = \langle\, [a], [b] \mid [a]^p = [b]^q = [1] \,\rangle$ are given by $f([a]) = [t][a]^r[t]^{-1}$ and $f(b) = [t][b]^s[t]^{-1}$, for integers r, s such that $\gcd(p, r) = \gcd(q, s) = 1$, and an element $[t] \in B_{p,q}$.*

Proof. Since $f([a])^p = 1$, the order of $f([a])$ must divide p (and is finite). By proposition 18, $f([a])$ must then be conjugate to a power of $[a]$ or $[b]$. Since (conjugates of) powers of $[b]$ have have orders dividing q and $\gcd(p, q) = 1$, and $f([a])$ cannot have order 1 as $[a] \neq 1$, we must have $f([a]) = \rho[a]^r\rho^{-1}$ for some integer r and some $\rho \in B_{p,q}$. Similarly, we must have $f([b]) = \sigma[b]^s\sigma^{-1}$ for some integer s and some $\sigma \in B_{p,q}$. It remains to prove that r and s are relatively prime to p and q, and that there is a single element $[t] \in B_{p,q}$ that can replace both conjugating factors.

Consider now the abelianization $A_{p,q} \cong \mathbb{Z}_p \times \mathbb{Z}_q$ with the abelianizing homomorphism $g\colon B_{p,q} \to A_{p,q}$ having $g([a]) = (1, 0)$ and $g([b]) = (0, 1)$. Since f and g are surjective, $g \circ f$ must also be surjective. Under g, conjugating factors cancel, giving $g \circ f([a]) = g([a]^r) = (r, 0)$ and $g \circ f([b]) = f([b]^s) = (0, s)$, assuming that $0 \le r < p$ and $0 \le s < q$ are chosen. The elements $(r, 0)$ and $(0, s)$ must generate the whole group $\mathbb{Z}_p \times \mathbb{Z}_q$, so we must have $\gcd(p, r) = \gcd(q, s) = 1$.

Consider now conjugation by ρ^{-1}, which is an (inner) automorphism of $B_{p,q}$. Letting $\tau = \rho^{-1}\sigma$, the elements $\rho^{-1}f([a])\rho = [a]^r$ and $\rho^{-1}f([b])\rho = \tau[b]^s\tau^{-1}$ are generators of $B_{p,q}$. If a reduced word of τ were to contain the sequence $[b]^\beta[a]^\alpha$ for nonzero α, β, then no word on $[a]^r$ and $\tau[b]^s\tau^{-1}$ including $[b]$ could be cancelled to have length 1, so $[b]$ cannot be generated. This implies that $\tau = [a]^x[b]^y$ (allowing one or both of x and y to be 0), so $\sigma = \rho\tau = \rho[a]^x[b]^y$. Letting $[t] = \rho[a]^x$ we get:

$$\begin{aligned} f([a]) &= \rho[a]^r\rho^{-1} = \rho[a]^x[a]^r[a]^{-x}\rho^{-1} = [t][a]^r[t]^{-1} \\ f([b]) &= \sigma[b]^s\sigma^{-1} = \rho[a]^x[b]^y[b]^s[b]^{-y}[a]^{-x}\rho^{-1} = [t][b]^s[t]^{-1} \end{aligned}$$

This has the required form. □

Finally we prove Schreier's characterization of the automorphisms of the full group.

Proposition 20. *All automorphisms f of $G_{p,q} = \langle\, a, b \mid a^p = b^q \,\rangle$ are given by $f(a) = ta^{\varepsilon}t^{-1}$ and $f(b) = tb^{\varepsilon}t^{-1}$, where $\varepsilon \in \{-1, 1\}$ and $t \in G_{p,q}$.*

Proof. An automorphism of $G_{p,q}$ induces automorphisms of the center $C_{p,q}$ and of $G_{p,q}/C_{p,q} = B_{p,q}$. Since the center is infinite cyclic, generated by a^p, we must have

$$f(a)^p = f(a^p) = a^{\varepsilon p} = b^{\varepsilon q} = f(b^q) = f(b)^q$$

for some choice of $\varepsilon \in \{-1, 1\}$. We know that in $B_{p,q}$ we must have $[f(a)] = [t][a]^r[t]^{-1}$ and $[f(b)] = [t][b]^s[t]^{-1}$, corresponding to:

$$\begin{aligned} f(a) \in [ta^r t^{-1}] &\implies f(a) = ta^{r+np}t^{-1} \\ f(b) \in [tb^s t^{-1}] &\implies f(b) = tb^{s+mq}t^{-1} \end{aligned}$$

for some integers m, n. Combining these results, $a^{\varepsilon p} = f(a)^p = ta^{(r+np)p}t^{-1}$, but $a^{(r+np)p}$ is central, so $a^{\varepsilon p} = a^{(r+np)p}$, implying $\varepsilon = r + np$ as a^p has infinite order. Similarly, $\varepsilon = s + mq$. $\square$

4 The peripheral system

The knot group has proven useful for distinguishing multiple different knots, but it has two apparent flaws: it cannot distinguish mirror images from each other, and it cannot distinguish certain knot sums like the square and granny knots from each other. Both of these problems are solved by augmenting the knot group with information about certain curves on the boundary of a tubular neighborhood of a knot. Variations on these methods go back to [Deh14] in which they are used to prove that the trefoil knot is chiral, and were also used (in the form of 'peripheral subgroups') in [Fox52] to distinguish the square and granny knots. We will use the specific notion of a *peripheral system* (presented in [BZ03, Section 3C]) to give a generalization of Dehn's result to all torus knots (following [Sch24], [Sti93] and [BZ03]), as well as to present Fox's result in terms of the peripheral system.

4.1 Meridian and longitude

We introduced meridian and longitude curves on a standard torus in section 3.1. Curves on the tubular neighborhood of a knot can be identified by the same principle, though our definition no longer relies on any particular embedding of the torus.

Definition 13. Let K be an oriented knot with tubular neighborhood V.

- A *meridian* of K is a simple closed curve μ on δV that is null-homotopic in V but not in δV.
- A *longitude* of K is a simple closed curve λ on δV that is homotopic to K in V.

The meridian and longitude identified in section 3.1 as generators of $\pi_1(S^1 \times S^1)$ satisfy definition 13 (after embedding the torus $S^1 \times S^1$ as V), so there exists at least one meridian and at least one longitude, which we may call μ_0 and λ_0, respectively.

These are not uniquely determined: the orientation of the meridian is ill-defined, and while the longitude always has its orientation aligned with the knot, it is nonetheless more ambiguous than the meridian. Any number of 'twists' may be added to a longitude, as in the bottom right of figure 15. This twisting is equivalent to repeated composing with a meridian (after an isotopy). There is a natural way to choose an unambiguous meridian-longitude pair, however:

Definition 14. A *preferred meridian-longitude pair* of an oriented knot K is a meridian μ and a longitude λ such that $\mu \cap \lambda$ is a single point, μ goes clockwise around the knot, and λ, when viewed as an element of $G = \pi_1(\mathbb{R}^3 - K, \mu \cap \lambda)$, represents 0 in the abelianization of G.

The requirement that λ abelianizes to 0 has many equivalent statements, some of them perhaps more intuitive, using e.g. linking numbers or homology, but we have not developed the theory needed for those; details may be found in [Rol03, p. 132], for example.

Proposition 21. *Any two preferred meridians or longitudes are related by an ambient isotopy of δV.*

Proof. Taking μ_0 and λ_0 to be fixed (not necessarily preferred) meridian and longitude generators of $\pi_1(\delta V)$, we know that since μ_0 is null-homotopic in V, a curve homotopic to $\mu_0^a \lambda_0^b$ on δV is homotopic to λ_0^b in V.

This implies that all meridians are (nonzero) powers of μ_0^a, but only μ_0 and μ_0^{-1} are represented by simple curves by proposition 13, so the choice of orientation is enough to disambiguate the meridian.

Since every longitude must be homotopic to K in V, all longitudes are homotopic to $\mu_0^a\lambda_0$. Since μ_0 represents a Wirtinger generator, it abelianizes to 1 or -1, which means that there is exactly one a such that $\mu_0^a\lambda_0$ abelianizes to 0. Hence, all preferred longitudes are homotopic on δV, and thus ambient isotopic on δV by proposition 13.

We have not proven that the same isotopy may be chosen to map both curves onto their counterparts simultaneously, but we will not need that, either. $\square$

Definition 15. Let K be an oriented knot with knot group $G = \pi_1(\mathbb{R}^3 - K, x)$ for any choice of basepoint $x \in \mathbb{R}^3 - K$, and a preferred meridian-longitude pair (μ, λ), and let γ be any path from x to $\mu \cap \lambda$. Then, the triple $(G, [\gamma\mu\gamma^{-1}], [\gamma\lambda\gamma^{-1}])$ is called a *peripheral system* of K.

We will think of the peripheral system as the group G together with two distinguished elements, corresponding to a meridian and a longitude. Later in the section we will have no need to consider paths explicitly, and we will then simply write peripheral systems as (G, μ, λ), using μ and λ to denote elements of G. For the next proposition, showing that the peripheral system is unique up to conjugation by a common element, we stick to the bracket notation:

Proposition 22. *If K is a knot with knot group $G = \pi_1(\mathbb{R}^3 - K, x)$ and two peripheral systems:*

$$(G, [\gamma\mu\gamma^{-1}], [\gamma\lambda\gamma^{-1}]) \quad \textit{and} \quad (G, [\gamma'\mu'\gamma'^{-1}], [\gamma'\lambda'\gamma'^{-1}]),$$

there is an inner automorphism of G (i.e. conjugation by a fixed element of G) mapping one peripheral system onto the other.

Proof. By proposition 21, there is an isotopy of δV taking λ to λ'. Let δ be the path taken by $\mu \cap \lambda$ on δV in this isotopy, ending at $\mu' \cap \lambda'$. We then know that $\delta^{-1}\mu\delta$ and $\delta^{-1}\lambda\delta$ are homotopic in δV to the images under the isotopy of μ and λ, respectively. For the longitude, this image is exactly λ'. For the meridian, the image of μ is a meridian with the same orientation, which means that it is homotopic to μ'.

Now consider $\gamma'\delta^{-1}\gamma^{-1}$. This is a loop based at x, so it generates an element of G. We conjugate by this element:

$$[\gamma'\delta^{-1}\gamma^{-1}][\gamma\mu\gamma^{-1}][\gamma'\delta^{-1}\gamma^{-1}]^{-1} = [\gamma'\delta^{-1}\mu\delta\gamma'^{-1}] = [\gamma'\mu'\gamma'^{-1}]$$
$$[\gamma'\delta^{-1}\gamma^{-1}][\gamma\lambda\gamma^{-1}][\gamma'\delta^{-1}\gamma^{-1}]^{-1} = [\gamma'\delta^{-1}\lambda\delta\gamma'^{-1}] = [\gamma'\lambda'\gamma'^{-1}]$$

This maps one peripheral system onto the other. $\square$

The following proposition gives a way of calculating a peripheral system of a knot, given a diagram.

Proposition 23. *Let $a_1, \dots, a_n$ be Wirtinger generators for an oriented knot K with a given diagram. The generator a_1 may be taken as a meridian element of $\pi_1(\mathbb{R}^3 - K)$. An expression for a corresponding longitude element is constructed as follows: travel along the knot starting at the arc corresponding to a_1. At each undercrossing, letting a_c be the generator for the overcrossing strand, write down a_c if the overcrossing strand points to the right, and let a_c^{-1} if it points to the left. After completing a circuit around the knot, write down the power of a_1 that makes the exponents add to 0.*

Proof. Any Wirtinger generator fulfills the requirements for a meridian element by definition. The choice of a_1 as a meridian determines where the longitude must begin and end. The procedure described in the proposition amounts to composing the arrows representing Wirtinger generators, with the directions fixed so that they follow the knot, so some longitude must be produced in this way. The final step ensures that the longitude abelianizes to 0. $\square$

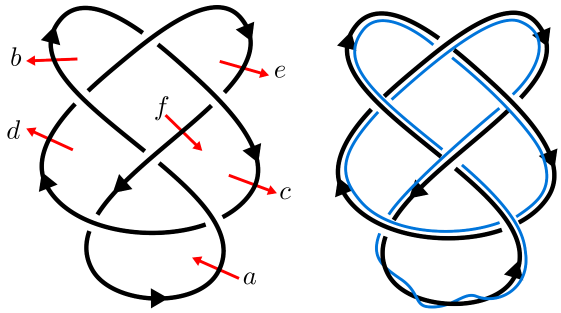

Figure 15: The 6_2 knot with Wirtinger generators and with a longitude.

Example 6. To illustrate the process, figure 15 shows the so-called 6_2 knot and red Wirtinger generators $(a, \ldots, f)$ to the left. We start at the arc of the generator a and move in the direction of b. At the first crossing, f has the opposite orientation to the one needed to follow the knot, so we write down f^{-1}, and so on. We initially get $f^{-1}e^{-1}ab^{-1}c^{-1}d$ in this way, to which we add a^2 to get an element corresponding to the preferred longitude $f^{-1}e^{-1}ab^{-1}c^{-1}da^2$ shown to the right in figure 15.

We can now investigate how the peripheral system may be useful when we have more than one knot.

Proposition 24. *If f is an ambient isotopy between the knots K and K', a preferred meridian-longitude pair (μ, λ) of K gets mapped by f onto a preferred meridian-longitude pair (μ', λ') of K'.*

Proof. We must check that the isotopy preserves the properties in definitions 13 and 14. The intersection of the curves being a single point is preserved by any bijective function. The isotopy induces isomorphisms of all relevant fundamental groups, so the homotopic properties are also preserved. The ambient isotopy is orientation-preserving, so μ' is a preferred meridian. Finally, the image under the abelianization map is preserved under an isomorphism, so λ' is a preferred longitude. □

Proposition 25. *If K and K' are equivalent knots with peripheral systems (G, μ, λ) and (G', μ', λ'), then there is an isomorphism $f \colon G \to G'$ such that $f(\mu) = \mu'$ and $f(\lambda) = \lambda'$.*

Proof. The equivalence can be taken as an ambient isotopy, which maps (G, μ, λ) onto *some* peripheral system of K', and since all peripheral systems of the same knot are related by conjugation, the isomorphism induced by the ambient isotopy can be composed with conjugation to form f. □

Proposition 26. *Let the knot K have peripheral system (G, μ, λ). If K is amphichiral, there exists an automorphism $f \colon G \to G$ such that $f(\mu) = \mu^{-1}$ and $f(\lambda) = \lambda$.*

Proof. A knot and its mirror image have isotopic groups, as mirroring in a plane is a homomorphism. If the intersection between the meridian and the longitude is taken as the basepoint, the mirroring reverses the orientation of a meridian with respect to the knot, but maintains the longitude (figure 16). □

Equally, invertible knots necessarily have an automorphism reversing both μ and λ, though we will not use this.

A fact of theoretical interest is that the implication of proposition 25 has a converse (see e.g. [BZ03] on the peripheral system), so the peripheral system is a complete invariant of knots, but the proof of this is far beyond our scope.

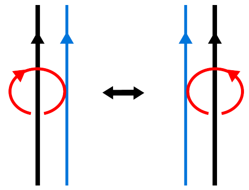

Figure 16: Effect of mirroring on a meridian and a longitude.

4.2 Torus knots are chiral

Max Dehn was the first to prove that the trefoil knot is chiral, through showing that an ambient isotopy between the trefoil and its mirror image would lead to an impossible automorphism of its knot group. The group-theoretic part of that work was simplified and generalized by Schreier, whose results we have presented in section 3.4.

We will apply the peripheral system to torus knots in general. To do this, we need to express meridian and longitude curves in terms of the generators of the knot group. We begin by proving a geometric fact about curves on tori, along the lines of proposition 12.

Proposition 27. *If p, q, r, s are integers such that $ps - qr \in \{-1, 1\}$, the closed curves $S^1 \to S^1 \times S^1$ defined by $z \mapsto (z^p, z^q)$ and $z \mapsto (z^r, z^s)$ intersect in only one point.*

Proof. Let $a, b \in S^1 \subset \mathbb{C}$ be such that $(a^p, a^q) = (b^r, b^s)$, i.e.

$$p \arg a = r \arg b + 2\pi m; \quad q \arg a = s \arg b + 2\pi n$$

for $m, n \in \mathbb{Z}$. Assuming that $\arg a \neq 0$, we get:

$$\begin{aligned}(p \arg a)(s \arg b + 2\pi n) &= (q \arg a)(r \arg b + 2\pi m)\\ ps \arg b + 2\pi pn &= qr \arg b + 2\pi qm\\ (ps - qr) \arg b &= 2\pi(qm - pn)\\ \arg b &= \pm 2\pi(qm - pn)\end{aligned}$$

Therefore at least one of $\arg a$ and $\arg b$ is an integer multiple of 2π, giving either $a = 1$ or $b = 1$. If $a = 1$ we have $(b^r, b^s) = (1, 1)$ and if $b = 1$ we have $(a^r, a^s) = (1, 1)$ but since $ps - qr = 1$ implies both $\gcd(p, q) = 1$ and $\gcd(r, s) = 1$, proposition 12 gives $a = b = 1$ in both these cases. □

We may now use these curves to calculate a peripheral system of torus knots, following an argument presented in [BZ03] for knots on any handlebodies, which this is a special case of.

Proposition 28. *If the torus knot group $G_{p,q}$ is presented as $\langle\, a, b \mid a^p = b^q \,\rangle$ in the usual way, and r, s are integers such that $ps - qr = 1$, then $\mu = a^{-r} b^s$ and $\lambda = a^p \mu^{-pq}$ are a meridian and a longitude of the knot.*

Proof. We take the knot to be the curve $\kappa = z \mapsto (z^p, z^q)$, and overlay the curve $\chi = z \mapsto (z^r, z^s)$, as shown in figure 17. Choose a meridian μ intersecting χ twice and split it into two parts: v inside the torus (dashed in the figure) and v' outside the torus (solid in the figure) such that $\mu = vv'$. For all relevant spaces we take the start of v as the basepoint. Denote the section of χ outside the tubular neighborhood as χ'. We

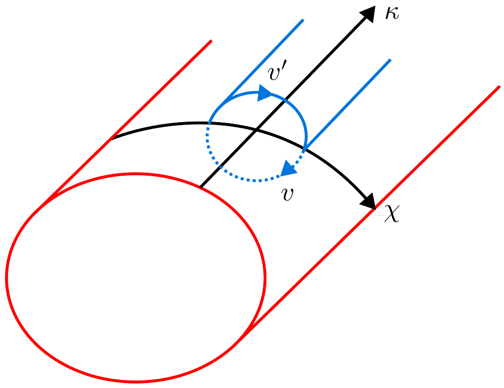

Figure 17: A torus cross-section (red) with a torus knot κ (black) and a once-intersecting curve χ (black), as well as a tubular neighborhood of κ (blue).

then have $[\chi' v^{-1}] = a^r$ in π_1(inside of torus) and $[\chi' v'] = b^s$ in π_1(outside of torus), so v is homotopic to $a^{-r}\chi'$ and v' is homotopic to $\chi'^{-1}b^s$. Hence,

$$\mu = vv' = a^{-r}\chi'\chi'^{-1}b^s = a^{-r}b^s.$$

The fact that κ and χ only intersect once is crucial, as otherwise the latter could not extend into both the inside and the outside of the torus in this way.

Having calculated an expression for the meridian, we know that all longitudes are of the form $a^p\mu^n$. We may take an abelianization (to $\mathbb{Z}$) of $G_{p,q}$ as $a \mapsto q$ and $b \mapsto p$, in which case setting $n = -pq$ gives the abelianization of $a^p\mu^n$ as

$$pq + (-qr + ps)(-pq) = pq - pq = 0. \qquad \square$$

We have previously calculated all automorphisms, in section 3.4, and now we only need to show that none of them can satisfy the relations required by the peripheral system.

Theorem 4. *All (p, q)-torus knots for $p, q \geq 2$ are chiral.*

Proof. Assume that such a knot is amphichiral. Then there exists an automorphism f of the group $G_{p,q} = \langle\, a, b \mid a^p = b^q \,\rangle$ such that

$$f(a^{-r}b^s) = b^{-s}a^r; \quad f(a^p(a^{-r}b^s)^{-pq}) = a^p(a^{-r}b^s)^{-pq}$$

for integers r, s such that $ps - qr = 1$. We know that f, like any automorphism of $G_{p,q}$, must have $f(a) = ta^{\varepsilon}t^{-1}$ and $f(b) = tb^{\varepsilon}t^{-1}$ for some $\varepsilon \in \{-1, 1\}$ and $t \in G_{p,q}$. Putting this into the equation for the meridian we get

$$b^{-s}a^r = f(a^{-r}b^s) = f(a)^{-r}f(b)^s = ta^{-\varepsilon r}b^{\varepsilon s}t^{-1}$$

The first expression abelianizes to $-ps+qr = -1$, and the last expression abelianizes to $-\varepsilon qr + \varepsilon ps = \varepsilon$, so $\varepsilon = -1$. Using $f(a^p) = a^{-p}$ and $f(a^{-r}b^s) = (a^{-r}b^s)^{-1}$, we get

$$a^p(a^{-r}b^s)^{-pq} = f(a^p(a^{-r}b^s)^{-pq}) = f(a)^p f(a^{-r}b^s)^{-pq} = a^{-p}(a^{-r}b^s)^{pq}.$$

This implies $a^{2p} = (a^{-r}b^s)^{2pq}$, but this is impossible: the left-hand side is in the center, but the right-hand side is not (it does not commute with a, for example). $\square$

A consequence of this is that any amphichiral knot cannot be a torus knot. In particular, we have proved that the figure-eight knot is not a torus knot.

4.3 The square and granny knots

We showed that the square and granny knots have isomorphic groups in example 5. In this section we will prove that they are nonetheless not ambient isotopic. The argument roughly follows [Fox52].

The main idea is to show that the peripheral systems for the two knots are inequivalent by considering homomorphisms into the symmetric group S_5. As a preliminary step, we consider the trefoil on its own.

Proposition 29. *Let $G = \langle\, a, b \mid aba = bab \,\rangle$ be the group of the trefoil knot. If $f\colon G \to S_5$ is a homomorphism such that $f(a) = (12345)$, then either $f(b) = (12345)$ or $f(b) = (12345)^k(13254)(12345)^{-k}$ for some integer k.*

Proof. For convenience let $\alpha = f(a) = (12345)$ and $\beta = f(b)$, so $\alpha\beta\alpha = \beta\alpha\beta$. We know that β is also a 5-cycle since α and β are conjugate:

$$(ab)a(ab)^{-1} = bab(ab)^{-1} = b.$$

If $\beta = \alpha^k$ for some k, then $\alpha\beta\alpha = \beta\alpha\beta$ implies $\alpha^{k+2} = \alpha^{2k+1}$, so $k = 5n + 1$ for some integer n, giving the only solution $\alpha = \beta = (12345)$.

If β is not a power of α, there are 20 other 5-cycles it may map to. However, conjugation of β by a power of α respects the group relation as shown just below, so we only need to check one representative from each group of cycles related by such a conjugation.

$$\begin{aligned} \alpha\beta\alpha = \beta\alpha\beta &\iff \alpha^n\alpha\beta\alpha\alpha^{-n} = \alpha^n\beta\alpha\beta\alpha^{-n} \\ &\iff \alpha(\alpha^n\beta\alpha^{-n})\alpha = (\alpha^n\beta\alpha^{-n})\alpha(\alpha^n\beta\alpha^{-n}). \end{aligned}$$

Conjugation by α^k is easy to calculate, as it simply adds k (cyclically) to each letter. The table below shows the resulting groups of 5-cycles.

β	(12354)	(12453)	(12543)	(13254)
$\alpha\beta\alpha^{-1}$	(23415)	(23514)	(23154)	(24315)
$\alpha^2\beta\alpha^{-2}$	(34521)	(34125)	(34215)	(35421)
$\alpha^3\beta\alpha^{-3}$	(45132)	(45231)	(45231)	(41532)
$\alpha^4\beta\alpha^{-4}$	(51243)	(51342)	(51432)	(52143)

We check one element of each group to see if it can be chosen as β:

$$\begin{aligned} \beta = (12354) &\implies \beta\alpha\beta = (15432) \neq (14532) = \alpha\beta\alpha \\ \beta = (12453) &\implies \beta\alpha\beta = (243) \neq (153) = \alpha\beta\alpha \\ \beta = (12543) &\implies \beta\alpha\beta = (354) \neq (345) = \alpha\beta\alpha \\ \beta = (13254) &\implies \beta\alpha\beta = (23)(45) = \alpha\beta\alpha \end{aligned}$$

From this we get the other class of possible homomorphisms. □

We now prove by example that the square knot has a certain property (which we later show that the granny knot does not).

Proposition 30. *Let (G, μ, λ) be a peripheral system for the square knot. Then there is a homomorphism $f\colon G \to S_5$ such that the image $f(G)$ is nonabelian, μ is mapped to a 5-cycle, and λ is mapped to the identity.*

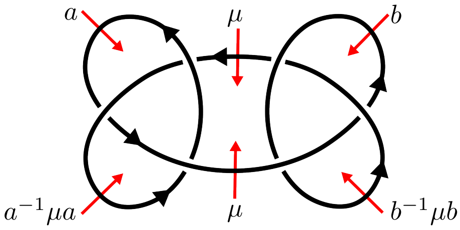

Figure 18: Generators for the square knot.

Proof. Let $G = \langle a, \mu, b \mid a\mu a = \mu a\mu, b\mu b = \mu b\mu \rangle$ as in figure 18, where we take the Wirtinger generator μ as the meridian element of the peripheral system. There are two generators labeled μ in the picture, as they represent the same homotopy class in the fundamental group. The expression $a^{-1}\mu a$ follows from the Wirtinger relation at the crossing at beginning of its arc, and $b^{-1}\mu b$ follows from the relation at the crossing at the end of its arc. By proposition 23, the longitude corresponding to μ is

$$\lambda = a\mu(a^{-1}\mu a)(b^{-1}\mu b)^{-1}\mu^{-1}b^{-1}$$

Consider the homomorphism generated by $f(\mu) = (12345)$ and $f(a) = f(b) = (13254)$, which is well-defined by the same calculation as in the previous proof. We need to check that it satisfies the necessary properties. The image is nonabelian:

$$f(a\mu) = (13254)(12345) = (153)$$
$$f(\mu a) = (12345)(13254) = (142)$$

The meridian μ is mapped onto a 5-cycle by definition, and we calculate the longitude using $f(a) = f(b)$:

$$f(\lambda) = f(a\mu(a^{-1}\mu a)(b^{-1}\mu b)^{-1}\mu^{-1}b^{-1}) = f(a\mu a^{-1}\mu a)f(b\mu b^{-1}\mu b)^{-1} = e.$$

We have shown that f has all the required properties. □

We finally prove that the granny knot does not have the property just confirmed for the square knot, completing the proof that they are inequivalent.

Proposition 31. *Let (G, μ, λ) be a peripheral system of the granny knot. Then there is no homomorphism $f\colon G \to S_5$ such that the image $f(G)$ is nonabelian, μ is mapped to a 5-cycle, and λ is mapped to the identity.*

Proof. To begin, it should be noted that if such a homomorphism did exist, composing it (from the right) with an inner automorphism of G would not change the relevant properties, so a proof for a specific choice of peripheral system is equally valid for all peripheral systems of the same knot. Composition (from the left) with an inner automorphism of S_5 also preserves the relevant properties, so we only need to consider classes of homomorphisms up to inner automorphism.

Let now $G = \langle a, \mu, b \mid a\mu a = \mu a\mu, b\mu b = \mu b\mu \rangle$ as in figure 19. This is similar, but not identical to the situation for the square knot. We again choose the Wirtinger generator μ as the meridian element of the peripheral system, and we get the corresponding longitude:

$$\lambda = a\mu(a^{-1}\mu a)b\mu(b^{-1}\mu b)\mu^{-6}.$$

We will find all homomorphisms mapping μ to a 5-cycle, and show that none of them satisfy the other two properties. Since we only need to consider mappings up to

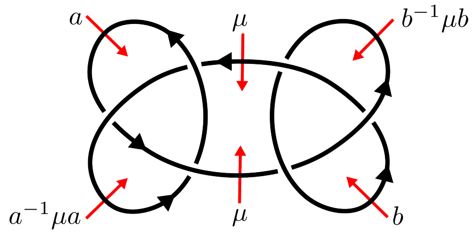

Figure 19: Generators for the granny knot.

inner automorphism of S_5, so we may choose that $f(\mu) = (12345)$, remembering that this still allows us to freely conjugate by any power of (12345).

The subgroup $A = \langle a, \mu \mid a\mu a = \mu a\mu \rangle$ of G is isomorphic to the trefoil group, and so is $B = \langle \mu, b \mid b\mu b = \mu b\mu \rangle$. For each subgroup separately, there are two distinct options for how it might be mapped into S_5, as shown in proposition 29: given that μ maps to (12345), the other generator either also maps to (12345) or to (13254) conjugated by a power of (12345). Any combination of choices for the two subgroups gives a well-defined homomorphism. We therefore have four cases to consider. For the sake of readability we let $\tau = (12345)$:

1. The first case is when $f(a) = f(b) = \tau$. In this case, all the generators of G map to the same 5-cycle, so $f(G)$ is abelian.

2. Second, we consider $f(a) = \tau$ and $f(b) = \tau^n(13254)\tau^{-n}$, for some integer n. We can choose an inner automorphism so that $n = 0$. The longitude must then map to the following:

$$f(\lambda) = \tau^3(13254)\tau(13254)^{-1}\tau(13254)\tau^{-6} = (15432)$$

3. Third, the opposite scenario: $f(a) = \tau^m(13254)\tau^{-m}$ and $f(b) = \tau$, for some integer m. We can choose an inner automorphism so that $m = 0$. The longitude must then map to the following:

$$f(\lambda) = (13254)\tau(13254)^{-1}\tau(13254)\tau^{-3} = (15432)$$

4. Fourth: $f(a) = \tau^m(13254)\tau^{-m}$ and $f(b) = \tau^n(13254)\tau^{-n}$, for integers m and n. We can choose an inner automorphism so that $m = 0$, but n is arbitrary. We perform an intermediate calculation:

$$\begin{aligned} f(b\mu b^{-1}\mu b) &= \tau^n(13254)\tau^{-n}\tau(\tau^n(13254)\tau^{-n})^{-1}\tau\tau^n(13254)\tau^{-n} \\ &= \tau^n(13254)\tau(13254)^{-1}\tau(13254)\tau^{-n} \\ &= \tau^n(13524)\tau^{-n} \end{aligned}$$

We now get the meridian as the following:

$$f(\lambda) = (13524)\tau^n(13524)\tau^{-n-6}$$

There are 5 values of n that can produce distinct products, and for all values the product is (14253).

The first case is rejected because $f(G)$ is abelian, and the others because the longitude λ is not mapped to the identity element, concluding the proof. □

The above propositions demonstrate that the square and granny knots are not ambient isotopic. It should be noted that since the square knot is amphichiral (and it is not just ambient isotopic to its mirror image, but identical to it, if the reflection is taken to map one trefoil onto the other) this also shows that the square and granny knots are not each other's mirror images, which in turn shows that the knot group on its own fails to distinguish more ambient isotopy classes than just mirror images.

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