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Generalisations of Interpretations of Fuss-Catalan numbers

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Generalisations of Interpretations of Fuss-Catalan numbers

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Abstract

The paper looks at the generalisation of the Catalan numbers $a_k(n)$ defined by $a_k(0) = 1$ and $a_k(n+1) = \sum_{j \geq 0, k|j} a_k(j)a_k(n-j)$ introduced by Per Alexandersson, Samuel Asefa Fufa, Frether Getachew, and Dun Qiu in 2022. We first look at combinatorial interpretations of the Fuss-Catalan numbers then we generalise these into interpretations of $a_k(n)$. Finally we give a combinatorial proof of the identity $a_k(n) = A_m(k+1, j+1)$ where $A_m(p, r)$ are the Raney-numbers.

Introduction

The Catalan numbers is the sequence of numbers $(C_n)_{n\geq 0}$ given by $C_0 = 1$ and $C_{n+1} = \sum_{k\geq 0} C_k C_{n-k}$. These numbers have many combinatorial interpretations (in fact at least 214 [6]) with applications within various fields of mathematics.

There are of course several generalisations of the Catalan numbers. An easy one is the Fuss-Catalan numbers which are given by the following:

$$C_0^k = 1, \qquad C_{n+1}^k = \sum_{\substack{j_1+\cdots+j_k=n \\ j_i\geq 0}} C_{j_1}^k \cdots C_{j_k}^k, \qquad \text{for } n \geq 0.$$

Here we see that when we let $k = 2$ we get the classical Catalan numbers. There are many interpretations and uses for the Fuss-Catalan numbers [2, 3, 4, 7, 5] they often lend themselves to generalizing interpretations of the Catalan numbers. For instance, one interpretation of the Catalan numbers is binary trees with n edges. The Fuss-Catalan numbers then counts the amount of k-ary trees with n edges.

A further generalisation are the Raney-numbers. They are usually introduced by their summation formula:

$$A_n(k,r) = \frac{r}{nk+r}\binom{nk+r}{n}.$$

These turn out to be the Fuss-Catalan numbers for $r = 1$. They also satisfy the relation that $A_n(k,k) = A_{n+1}(k,1)$.

Finally we have the sequences $(a_k(n))_{n\geq 0}$ which was recently introduced in [1], given by the following:

$$a_k(0) = 1 \qquad a_k(n+1) = \sum_{\substack{j\geq 0 \\ k|j}} a_k(j)a_k(n-j), \qquad \text{for } n \geq 0.$$

This sequence satisfies the identity

$$0 \leq j \leq k \qquad a_k(km+j) = A_m(k+1, j+1). \tag{1}$$

This sequence offers a new way to look at the Raney Numbers. The most interesting consequence of the identity is that $a_k(km) = A_m(k+1,1) = C_m^{k+1}$. This means that any combinatorial interpretation of this sequence also gives us a combinatorial interpretation of the Fuss-Catalan numbers.

For any interpretations of the Fuss-Catalan numbers we can attempt to consider it an interpretation of $a_k(n)$ for $n = km$ instead and see whether we can find interpretations for when $k \nmid n$.

In this paper we will see four interpretations of the Fuss-Catalan numbers. We will then extend these interpretations into interpretations of the sequence $a_k(n)$. Finally we'll use the interpretation of kx-avoiding paths to give a combinatorial proof of the identity (1), a previously open problem.

1 Combinatorial interpretations of the Fuss-Catalan numbers

These are generalisations of objects counted by the Catalan numbers. In Stanley's book [6] he has numbered the interpretations from 1 to 214. The index for each interpretation refers to the number of the interpretation of the classic Catalan numbers from the book.

To prove that objects are counted by a sequence defined by recursion, we can show that the objects themselves satisfy the same recurrence. This is the main method used in this thesis but otherwise one usually proves this either by finding a bijection to another interpretation or by showing that the objects are counted by the closed formula for the sequence.

1.1 k-ary trees

Defenition 1 *A k-ary tree is either empty or consists of a root vertex with k sub-trees, all of which are k-ary trees.*

Based on the recursion, k-ary trees with n vertices are in some sense the most natural interpretation of C_n^k. Each sub-tree is independent so the total number of ways for a given distribution of the nodes is a product of the possible ways to make k smaller trees. We sum is over the ways to distribute n vertices among the sub-trees with the additional final node being the root. When $k = 2$ these are binary trees[6, Item 4].

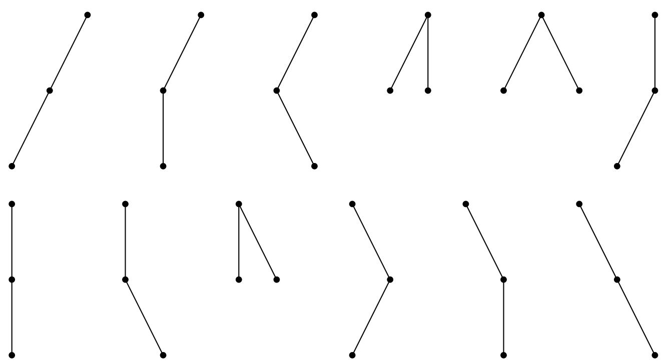

Figure 1: The 12 possible 3-ary trees with 3 vertices.

Proof. There is only one k-ary tree with 0 vertices, the empty tree. Let the children of the root be the sub trees $T_1, \ldots, T_k$. For a tree with $n+1$ vertices, let T_i have j_i vertices. Assuming that for all $n' \leq n$ the number of k-ary trees are counted by $C_{n'}^k$, the number of ways to create a tree with this exact distribution of vertices is $C_{j_1}^k \cdots C_{j_k}^k$. The total number of ways to create a k-ary tree with $n+1$ vertices is then given by

$$\sum_{j_1+\cdots+j_k=n} C_{j_1}^k \cdots C_{j_k}^k.$$

By induction, the amount of k-ary trees with n vertices is counted by C_n^k.

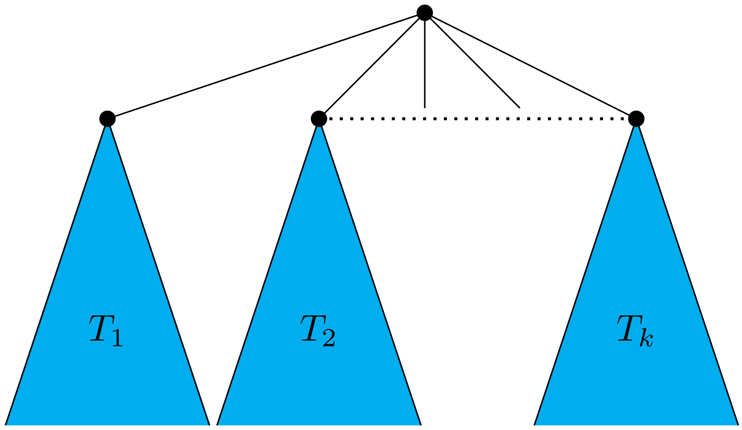

Figure 2: A k-ary divided into the sub-trees $T_1, \ldots, T_k$.

1.2 Divisions of convex polygons

Defenition 2 *Given a d-gon P, let a division D of P be a set of diagonals of P that do not cross within P. If every area within P in D borders exactly $k+1$ edges, D is said to divide P into $(k+1)$-gons.*

The Catalan numbers was first found as the number of triangulations of a $(n+2)$-gon [6, Item 1]. We wish to extend this into divisions into $(k+1)$-gons (since a triangle is a $(2+1)$-gon). For the divisions to be even we find that the polygon has to have $((k-1)n+2)$ vertices for some integer n. We now prove that the number of divisions of convex $((k-1)n+2)$-gons into $(k+1)$-gons is counted by C_n^k.

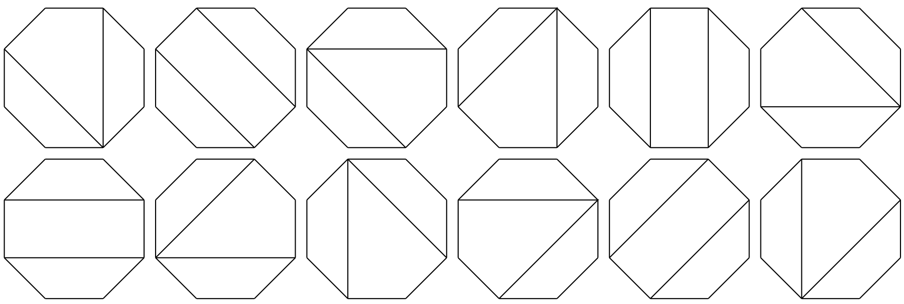

Figure 3: The 12 ways to divide a octagon into quadrilaterals.

Proof. For $n = 0$ there is one way to do this to a 2-gon (or line). Consider a $((n+1)(k-1)+2)$-gon and assume that the the divisions of polygons with fewer vertices follow the formula. Fix a edge in the polygon. Let the $(k+1)$-gon this edge neighbours be A. A then splits the polygon into k new polygons where the i:th one will have $j_i(k-1)+2$ edges (this since it is already divided into $(k+1)$-gons).

We have that

$$(j_1 + \cdots + j_k)(k-1) + 2k - k + 1 = (n+1)(k-1) + 2$$
$$j_1 + \cdots + j_k = n.$$

Fix A. The number of possible divisions of the i:th polygon is counted by $C^k_{j_i}$. Thus the number possible divisions containing A will be $C^k_{j_1} \cdots C^k_{j_k}$. All possible ways is then counted by

$$\sum_{j_1+\cdots+j_k=n} C^k_{j_1} \cdots C^k_{j_k}.$$

By induction the divisions of the polygons are counted by the sequence.

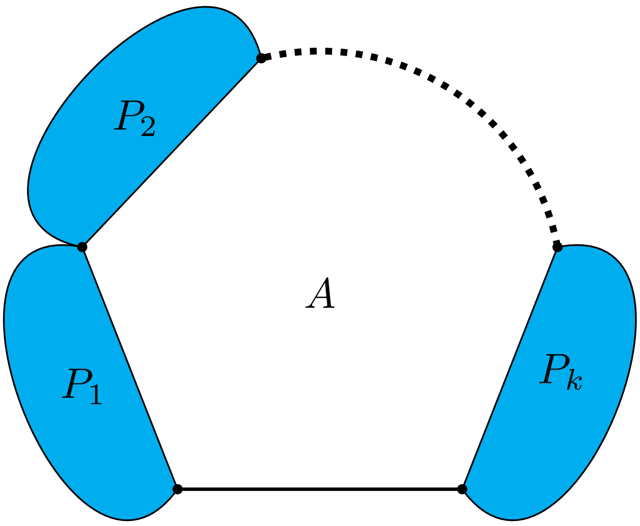

Figure 4: The polygon being being split by A into the k smaller polygons $P_1 \ldots P_k$.

1.3 Plane trees

Defenition 3 *A plane tree consists of a root vertex r and a sequence of sub-trees each of which is a plane tree. The root of each of the sub-trees are considered the children of r.*

The number of plane trees with n edges are counted by the classic Catalan numbers [6, Item 6]. What makes this generalisation to the Fuss-Catalan numbers interesting is that instead of extending or changing the objects we restrict them. We find that the number of plane trees with $(k-1)n$ edges where each vertex has a number of children divisible by $(k-1)$ are counted by the C_n^k. The interpretations that C_n^k counts are among the ones counted by $C_{(k-1)n}^2$.

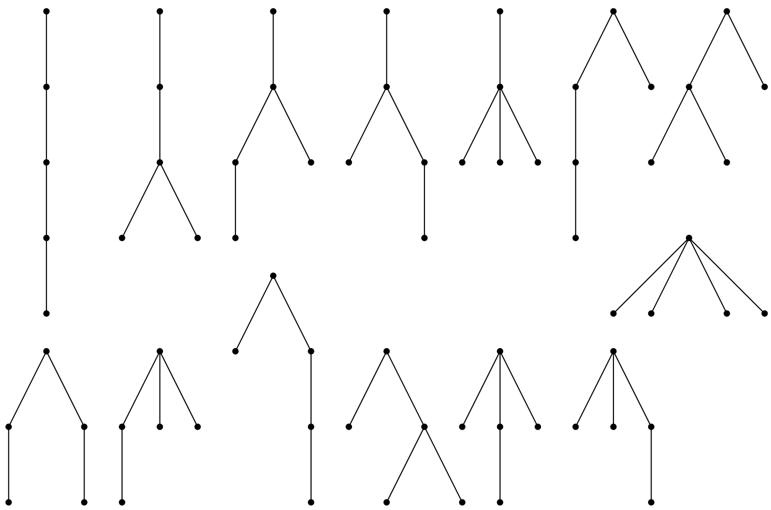

Figure 5: The 14 possible plane trees with four edges.
Note that there are exactly 3 for which all vertices have an even amount of children.

Proof. There is only one plane tree with 0 edges, which is the tree containing only the root. Otherwise the root has at least $(k-1)$ children. Consider the sub-trees of the first $(k-1)$ children of the root, let the i:th one have $(k-1)j_i$ edges. Then the edges in the remaining tree can be described with $(k-1)j_k$. We get that

$$(j_1 + \cdots + j_k)(k-1) = (n+1)(k-1) - (k-1)$$
$$j_1 + \cdots + j_k = n.$$

By inductive assumption each sub-tree can be drawn in $C_{j_i}^k$ ways. We then get that the total number of ways to draw the tree will be

$$\sum_{j_1+\cdots+j_k=n} C_{j_1}^k \cdots C_{j_k}^k.$$

By induction the trees are counted by the sequence.

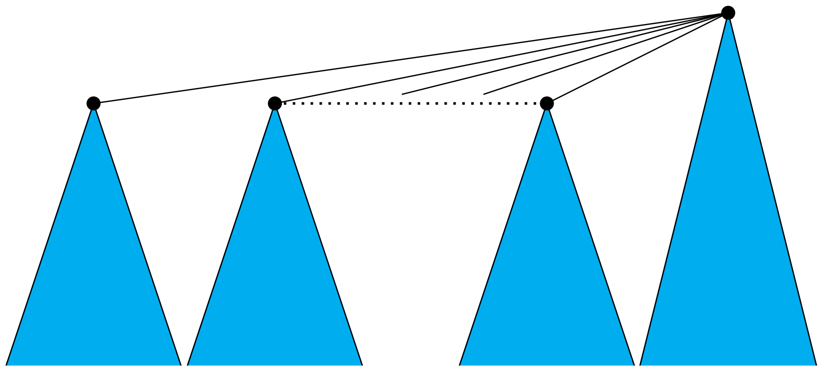

Figure 6: A plane tree with its $(k-1)$ first sub-trees separate from the rest of the tree

1.4 $(k-1)x$-avoiding paths

Defenition 4 *A kx-avoiding path from $(0,0)$ to (a,b) is a lattice path from $(0,0)$ to (a,b) consisting of the steps $(1,0)$ (east) and $(0,1)$ (north) with no step ending below the line $y = kx$*

The number of $1x$-avoiding pats from $(0,0)$ to (n,n) are counted by C_n[6, Item 24](Although item 24 stays below the line instead). To extend this we increase the slope of the line. We get that the number of $(k-1)x$-avoiding paths from $(0,0)$ to $(n, n(k-1))$ are counted by C_n^k.

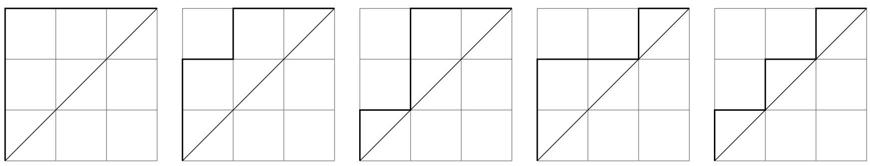

Figure 7: The 5 possible $1x$-avoiding paths from $(0,0)$ to $(3,3)$

Proof. When $n = 0$ there is only one path from $(0,0)$ to $(0,0)$.

Consider the path from $(0,0)$ to $(n+1, (k-1)(n+1))$. The final step will always be a east step since otherwise the previous step would have ended below the line $y = (k-1)x$.

Let P be a path from $(0,0)$ to $(n, (k-1)(n+1))$. The final point of P is at the line L_k, given by $y = (k-1)x + (k-1)$, meaning there is at least one north step leaving each of the lines L_i, given by $y = (k-1)x + (i-1)$ for $0 < i < k$. Let s_i be the last north step leaving L_i. For $1 < i < k$ let p_i be the subpath of P beginning at the end of s_{i-1} and ending at the beginning of s_i. Let p_1 be the subpath from $(0,0)$ to the beginning of s_1 and p_k the subpath from the end of s_{k-1} to $(n, (k-1)(n+1))$. Let j_i be the number of east steps in p_i. The path p_i

will both start and end at the line L_i and can not go below it since then either P is not $(k-1)x$-avoiding or there is a step leaving L_{i-1} in p_i, contradicting s_{i-1} being the final step to leave L_{i-1}.

Taking the steps of p_i starting at $(0,0)$ will thus give us a $(k-1)x$-avoiding path to $(j_i,(k-1)j_i)$. By the inductive assumption the number of ways to make such a path is $C^k_{j_i}$ and for a given placement of $s_1,\ldots,s_{k-1}$ we get that there are $C^k_{j_1}\cdots C^k_{j_k}$ paths with the placement. Since P contain n east steps we get that $j_1+\cdots+j_k=n$ so all possible paths are given by

$$\sum_{j_1+\cdots+j_k=n} C^k_{j_1}\cdots C^k_{j_k}.$$

By induction, the paths are counted by the sequence.

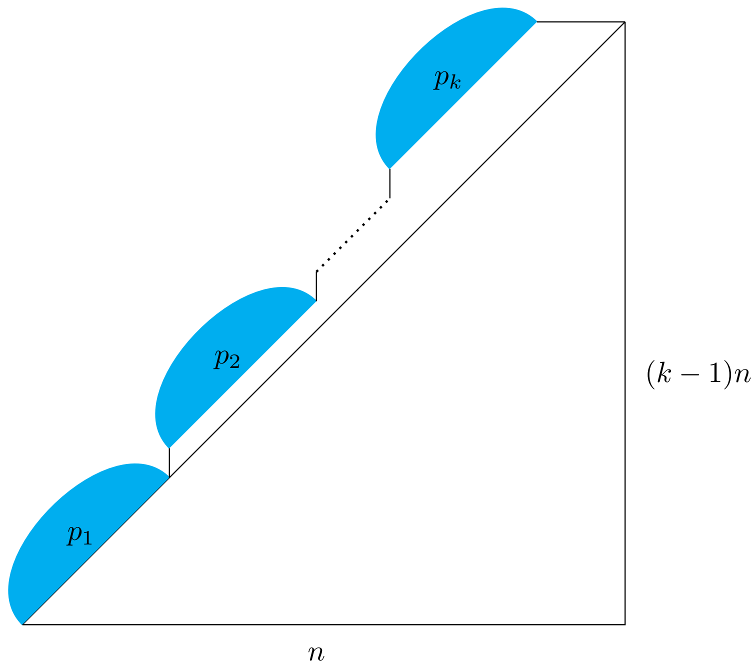

Figure 8: The path being split up into the sub-paths $p_1,\ldots,p_k$ by $s_1,\ldots,s_{k-1}$.

2 Combinatorial interpretations of $a_k(n)$

We will now look at interpretations of the sequence $a_k(n)$ such that the interpretation specializes to our previous interpretations of the Fuss-Catalan numbers.

Let $n = km + j$ where $0 \leq j \leq k$. We note here that when $k|n$ we might have two interpretations, either $n = km + k$ or $n = k(m+1) + 0$. In these cases we'll show that either the objects are identical or that they are equinumerous.

2.1 $(k+1)$-ary trees

To turn these into an interpretation of $a_k(n)$ we in some sense add the sub-trees one at a time. We let the trees have $m+1$ edges and only allow nodes in the first $j+1$ sub-trees.

Proof. Given $j = k$ we get a $(k+1)$-ary tree with $m+1$ vertices. For $j = 0$ we instead get a root with one child which is a $(k+1)$-ary tree containing $m+1$ vertices, the bijection between the two is obvious.

For $n = 0$ there is the empty tree and also the tree where the first child is empty. We wish to show that the amount of trees for $n+1 = km + j + 1$ is counted by $a_k(n+1)$. Consider the $(j+2)$th sub-tree, let it have l nodes. The later sub-trees are empty and since this will be a proper $(k+1)$-ary tree it will be counted by $a_k(lk)$. The remaining tree can be constructed from $n - l$ nodes and a root which only has $j+1$ children, thus it is counted by $a_k(n - lk)$. We then get that the total number of ways to create the tree is

$$\sum_l a_k(kl)a_k(n-kl) = \sum_{k|j} a_k(j)a_k(n-j).$$

By induction the trees are counted by $a_k(n)$.

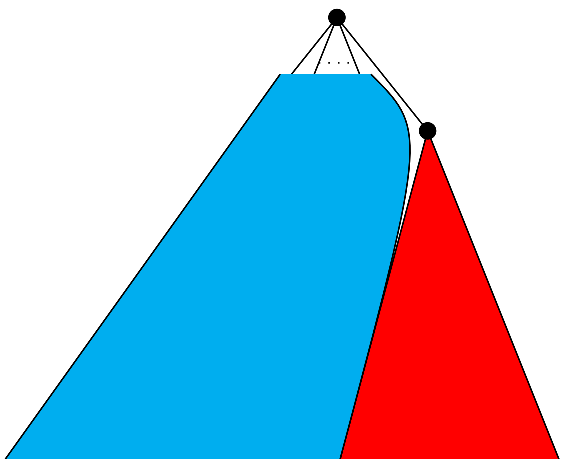

Figure 9: The tree being split into the full $(k+1)$-ary in red and the remaining tree in blue.

2.2 Divisions of convex polygons

For these to specialize into the Fuss-Catalan interpretation we wish for them to be $(n+2)$-gons divisible into $(k+2)$-gons. However, if $k \not| n$, the $(n+2)$-gon will not be divisible into $(k+2)$-gons. If we let our division contain one $(j+2)$-gon this is solved. Restricting the $(j+2)$-gon to neighbour the a specific edge of the polygon we get that they are counted by $a_k(n)$. We now state it properly.

The number of divisions of a convex $(n+2)$-gon with a bottom edge into m $(k+2)$-gons and one $(j+2)$-gon that neighbours the bottom edge is counted by $a_k(n)$.

Proof. For $j = k$ the shape consists of m $(k+2)$-gons and one $(k+2)$-gon that neighbours the bottom edge. This is the same as $m+1$ $(k+2)$-gons and a 2-gon which neighbours (and is) the bottom edge.

A line (or 2-gon) can be divided like this in exactly one way. Assume that $a_k(n')$ counts the number of divisions for for all $n' \leq n$.

Given a convex $(n+3)$-gon, index the vertices clockwise around the polygon such that the bottom edge joins vertices 0 and $n+2$. Consider the vertices with indexes on the form $kl+1$ such that $kl+1 < n+2$. The triangle given by the bottom edge and one of these vertices split the polygon into two new convex polygons, A with $kl+2$ edges and B with $k(m-l)+j+1$ edges. Let the bottom edge of each be the one neighbouring the triangle. The amount of ways to divide these are $a_k(kl)$ and $a_k(n-kl)$ for A and B respectively.

A, when divided, will only consist l $(k+2)$-gons while B will have a $(j+2)$-gon neighbouring the triangle and $(m-l)$ $(k+2)$-gons. The $(j+2)$-gon together with the triangle creates a $(j+3)$-gon.

By induction the divisions are counted by $a_k(n)$.

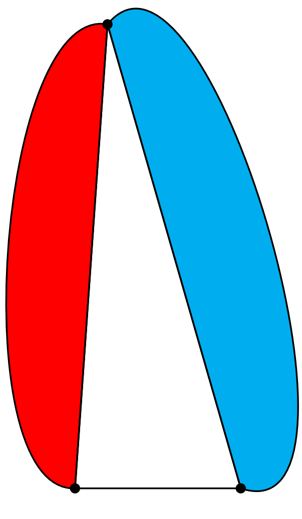

Figure 10: The polygon being split by the triangle into A in red and B in blue.

2.3 Plane trees

For the specialisation we want the trees to have n edges. Similar to the polygons we find that if every vertex has a number of children divisible by k, the number of edges will be a multiple of k. To fix this, we add an exception for the root which should have a number of children congruent to j mod k. Properly stated:

The number of plane trees with n edges where each node has a number of children divisible by k except for the root which has a number of children congruent to n mod k is given by $a_k(n)$.

Proof. Since $0 \equiv k \mod k$, both $j = 0$ and $j = k$ result in the same objects.

There is one plane tree with 0 edges, the tree containing only the root.

Consider the rightmost child of the root in a tree with $n + 1$ edges. The sub-tree under it will contain an amount of edges divisible by k which we can represent as lk. The ways to make this sub-tree will be $a_k(lk)$. Consider the remaining part of this tree, it will contain $n + 1 - lk - 1 = n - lk$ edges and the root will have an amount of children congruent to n mod k. The ways to make this tree is thus given by $a_k(lk)a_k(n - lk)$ summing over all the possible values of l we get the amount of ways to make one of these trees with $n + 1$ vertices. By induction the trees are counted by $a_k(n)$.

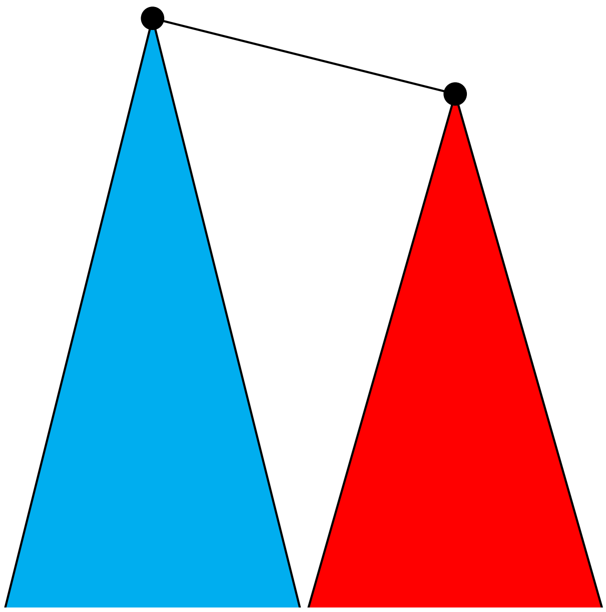

Figure 11: The plane tree split into the last child in read and the remaining tree in blue.

2.4 kx-avoiding paths

When $j = 0$ we want the interpretation to be the number of kx-avoiding paths from $(0, 0)$ to (m, n) for the specialisation. It turns out that this definition is enough for when $j > 0$ as well.

Proof. The difference between the cases $j = 0$ and $j = k$ is that when $j = 0$ we go one extra step to the right. However, this extra step to the right has to be the last since otherwise the path would go below the line $y = kx$. Thus each path in the $j = 0$ case is a path in the $j = k$ case with an extra step to the right at the end.

Assume that $a_k(n')$ counts the objects for $n' \leq n$. For a path from $(0, 0)$ to $(m, km + j + 1)$ that stays above $y = kx$, consider the last step that the path leaves the line $y = kx$. The step begins at (l, kl) so the path before it will be counted by $a_k(kl)$ the remaining path will go from $(l, kl + 1)$ to $(m, km + j + 1)$ which will stay above the line $y = kx + 1$. This is the same as a path from $(0, 0)$ to $(m - l, n - kl)$ staying above $y = kx$ which is counted by $a_k(n - kl)$. All the ways to create the path is then counted by the sum of $a_k(kl)a_k(n - kl)$ for the potential values of l from $1, \ldots, m$. Thus by induction the number of paths are counted by $a_k(n)$.

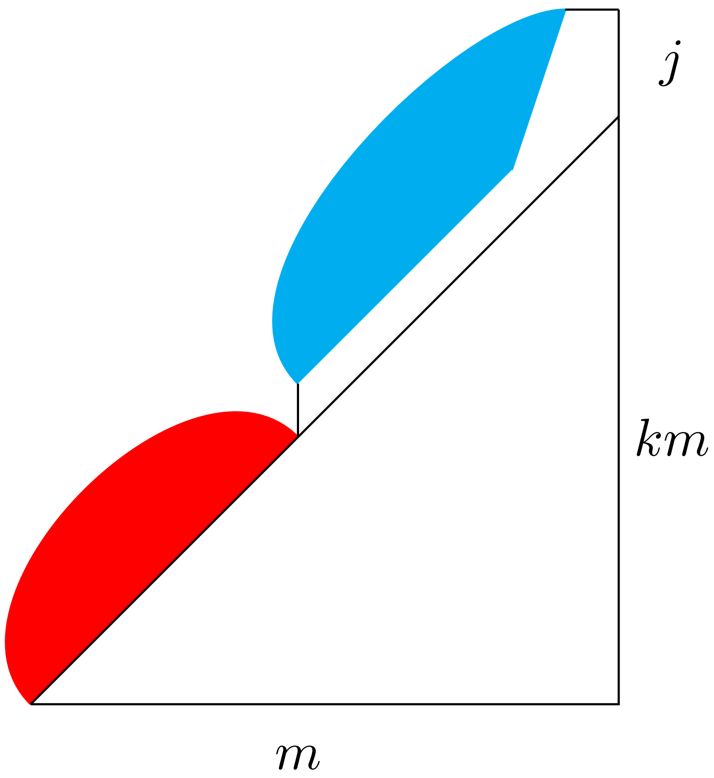

Figure 12: The path split up by the last step leaving $y = kx$.

3 Combinatorial proof of the closed formula for $a_k(n)$

We now give a combinatorial proof of the closed formula for the sequence $a_k(n)$. This proof is based on the one for the classic Catalan numbers in [6]. The original proof uses ballot strings (strings of $+1$s and -1s such that all partial sums are non-negative) these are analogous to $1x$-avoiding paths so we'll use our interpretations of kx-avoiding paths for the proof. Firstly we need to define rotating a path and strict $kx-$avoiding paths.

Defenition 5 *Given a lattice path P of length n with steps s_i, $0 \leq i < n$. Let a rotation P' for some integer $0 \leq k < n$ be a path such that the i:th step is s_j where $j \equiv i + k \mod n$.*

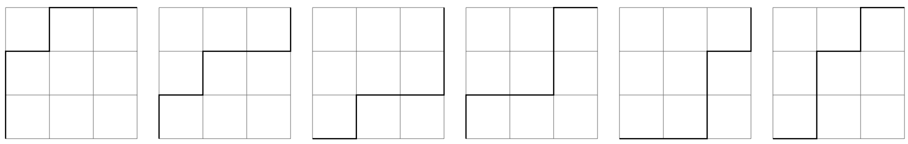

Figure 13: The 6 rotations of the path north-north-east-north-east-east

Defenition 6 *Let a strict kx-avoiding path be a kx-avoiding path such that the path never touches the line $y = kx$ except at the point $(0, 0)$.*

Next, the proof finds the proportion of strict kx-avoiding from $(0, 0)$ to $(m, n + 1)$. It finds this by considering the equivalence classes along rotation. This is where the proof becomes more complicated. In the original case there where exactly one rotation in each class which resulted in a strict path. Additionally all the equivalence classes where the same size. In our cases we may have both equivalence classes of different sizes and several paths within them being strict, complicating the proof somewhat.

Lemma 7 *For any given path P from $(0, 0)$ to $(a, ak + b)$, where a, b positive integers, there are exactly b rotations which are strict kx-avoiding paths.*

Proof. Let s_i be the i:th step in P. Let c be the smallest integer such that the path intersects $kx + c$ at some point. Let s_j be the last step that leaves the line. Let P' be P rotated j steps. Then P' is a strict kx-avoiding path since

- All steps from s_j and onwards in P will not hit the line that s_j leaves.
- All steps before s_j that used to hit $kx + c$ will now instead hit $kx + (b - c)$. Since $(b - c)$ is positive, this will be larger than kx and since these are the lowest points, all points in this part stays strictly above kx as well.

For P' consider the steps that are the last to leave the lines $kx + 1, \ldots, kx + b - 1$. Rotating to begin with these steps will by the same argumentation as before create a strict kx-avoiding path since $(b - c)$ is positive for $1 \leq c \leq b - 1$. Rotating to begin with any other step (that isn't the starting step) will either give us that:

- If the step is leaving one of the lines, it will hit the line again before the end of P', making the rotation non-strict.
- If the step is not leaving one of the lines, it must be leaving $kx + b$ or higher, but $kx + b$ is hit at the last step, making the rotation non-strict.

Thus for a strict kx-avoiding path from $(0, 0)$ to $(ak + b)$ there are b steps to rotate to (including step 0) such that the result is a strict kx-avoiding path. Since any path has a rotation that is a strict kx-avoiding path, the lemma holds for all paths from $(0, 0)$ to $(a, ak + b)$. $\square$

Lemma 8 *Given all unique paths from* $(0, 0)$ *to* $(a, ak + b)$ *the proportion of strict* kx*-avoiding paths is*

$$\frac{b}{a(k+1)+b}.$$

Proof. Two paths are considered *equivalent* if you can rotate one to create the other. For a given equivalence class, suppose there are $a(k+1)+b$ elements. Then, by lemma 7 we have that there are exactly b of them which are strict kx-avoiding paths.

We note that for $a = 1$ or $b = 1$ this is always the case since then

$$\gcd(a, ka + b) = \gcd(a, b) = 1.$$

Assume that the lemma holds for all equivalence classes of paths from $(0, 0)$ to (a', b') where $a' < a$ and $b' < b$.

Suppose the equivalence class has fewer than $a(k+1)+b$ elements. Then the path in this equivalence class must be repeating, since the same pattern occurs when beginning at different steps. If the path repeats m times then $m|a$ and $m|(ak + b)$ thus, we'll also have that $m|b$. Let $a = a'm$ and $b = b'm$. Since the first $a'(k + 1) + b'$ steps repeat m times, checking if these steps stays strictly above kx is equivalent to checking if the entire path stays strictly above kx. By our inductive assumption, we have that the proportion of strict kx-avoiding paths in the equivalence class is

$$\frac{b'}{a'(k+1)+b'} = \frac{b'm}{(a'(k+1)+b)m} = \frac{b}{a(k+1)+b}.$$

Since every equivalence class has the same proportion of strict paths, the lemma holds. $\square$

We are now ready to state the main result of this thesis.

Theorem 9 *A combinatorial interpretation of the Raney numbers* $A_m(p, r)$ *is given by all* kx*-avoiding paths from* $(0, 0)$ *to* $(m, (p-1)m + r - 1)$.

Proof. Consider all strict kx-avoiding paths from $(0, 0)$ to $(m, (p-1)m+r)$. All these paths must begin with a step to the north and removing this step, these paths are exactly the kx-avoiding paths from $(0, 0)$ to $(m, (p-1)m+r-1)$. By lemma 8 we have that the proportion of strict kx-avoiding paths is $r/(mp+r)$, thus we get the number of kx-avoiding paths as

$$\frac{r}{mp+r}\binom{mp+r}{m} = A_m(p, r).$$

□

In particular we get the following corollary, which now gives a combinatorial proof of the identity in [1]. This identity was proven therein using Lagrange inversion and generating functions, it was an open problem to provide a combinatorial proof of the following.

Corollary 10 *For* $0 \leq j \leq k$

$$a_k(km + j) = A_m(k + 1, j + 1).$$

Proof. Since they both share the interpretation of the number of kx-avoiding paths from $(0, 0)$ to $(m, km + j)$, the equality must hold. □

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