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Discover arbitrage through the lens of linear programming

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Discover arbitrage through the lens of linear programming

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Abstract

Geometry being one of the first branches of mathematics it is early introduced in school to lay one of many foundations of modern mathematics. While linear programming trying to reach the best outcomes given constraints can be used separately and combined to discover economical models and assumptions, which this thesis aims to shed light on. This by presenting several mathematical theorems and discover its applications and similarities to economical models and lastly presenting a method for solving such problems.

Acknowledgements

I would like to thank my supervisor Yishao Zhou for her expertise, feedback and guidance during this project. But also for allowing and encouraging me to write about a subject I was truly interested in rather than a subject I beforehand has knowledge in. Working with Yishao has been a joy and this thesis would not have been possible without her help.

Contents

1 Introduction

'A finance professor and a normal person go on a walk and the normal person sees a €100 bill lying on the street. When the normal person wants to pick it up, the finance professor says: "Don't try to do that! It is absolutely impossible that there is a e 100 bill lying on the street. Indeed, if it were lying on the street, somebody else would already have picked it up before you".'

The classical joke originally mentioned in [7] but here quoted from [17] points on some aspects of arbitrage, mainly that it is more reasonable to assume that there don't exist any €100 bills lying around, however, this opportunity is what 'finance-people' calls arbitrage opportunity. The notion of arbitrage is essential in modern finance and mathematical finance, being the cornerstone for options pricing due to the work by F. Black, M. Scholes and R. Merton. The theory behind arbitrage is rather simple and basically consists of the idea that there is 'no free lunch' in financial markets and if there is they exist only for a brief time. This since if arbitrage opportunities were normal this would point to the fact that the market are not in any equilibrium. One example of arbitrage from financial markets could be if one could buy Apple-shares for $90 in USA and directly after sell them in London for 100$, obviously this is obscured.

Obviously arbitrage opportunities are seldom and today almost only something HFT (High Frequency Traders) can take advantage of by using large amount of data and place orders for just a few seconds before any one else can make the trade [11], however, this subject is out of the scope for this thesis. Instead this thesis aims to take a look at the theory of arbitrage in finance and it's connections to mathematics and linear programming, Farkas' Lemma, especially. In the end a discussion for it's practical uses will be presented.

Aim of the Study

This thesis aims to explore arbitrage in finance theory and explore it's connections to mathematics and linear programming in particular.

This thesis will only cover some parts of the more basic connections to, hopefully, shed light on the extensive use of mathematics in finance.

Structure of the Thesis

The thesis starts with mentioning some basic mathematical and economic concepts that lays the foundation for the rest of the thesis. However, this should not be seen as comprehensive but as a help. After the preliminaries, theorems get introduced and proved. They are later used in an economic setting and to prove the arbitrage theorem. Thenceforth linear programming is introduced and used to prove the fundamental theorem of asset pricing. Lastly, we go through a method for solving linear programming problems and discuss arbitrage in the real life.

2 Preliminaries

This section aims to highlight both important mathematical and economical/finance points that have to be grasped to understand the rest of the thesis. Especially the economical terms since the main group of readers have only a mathematical background. In the thesis to show transpose, $x^\top$ is used to not confuse with t meaning time for the economical interpretations.

2.1 Mathematical background

Here we remind the reader the concepts of distances and convex sets [4]. Recall that if a set $S \subseteq \mathbb{R}^n$ contains the limit point of each convergent sequence of points in S the set is closed. If we consider a nonempty and closed set $S \subseteq \mathbb{R}^n$ and a point $z \in \mathbb{R}^n$ we define the distance from z to S by z:
$d_s(z) = \inf\{\|s - z\| : s \in S\}$, where we use the euclidean norm. Further we say that $s_0 \in S$ is a nearest point of S to z if $d_s(z) = \|s_0 - z\|$.
Definition 1 (cone) A set $K \in \mathbb{R}^n$ is regarded as a cone, if
$x \in K \implies \alpha x \in K$ for any $\alpha \geq 0$ [3].

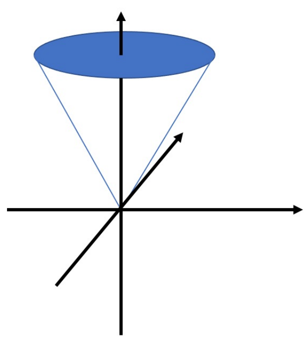

Figure 1: Cone

Definition 2 (conic hull) Given a set S, we can denote the conic hull of S as $cone(S)$ being the set of all non-negative combinations of the points in S, meaning [3]:

$$cone(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, x_i \in S \right\}.$$

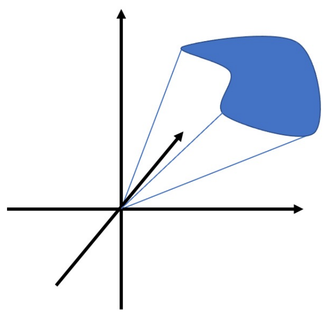

Figure 2: Conic hull

Definition 3 (affine combination) A affine combination is defined as [21]:

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

where we have $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Definition 4 (Convex set) If all line segments between any two points in the set C, the set C is convex, meaning: $\forall x, y \in C$, $\forall \theta \in [0, 1]$: $\theta x + (1 - \theta)y \in C$. We can generalize this definition to an arbitrary number of points where a convex combination of points $x_1, x_2, \ldots, x_n \in C$ is any point in the form $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$ and $\theta_i \geq 0$, $i = 1 \ldots n$ and $\sum_{i=1}^{n} \theta_i = 1$, then the set C is convex if any convex combination of points in C is in C [5].

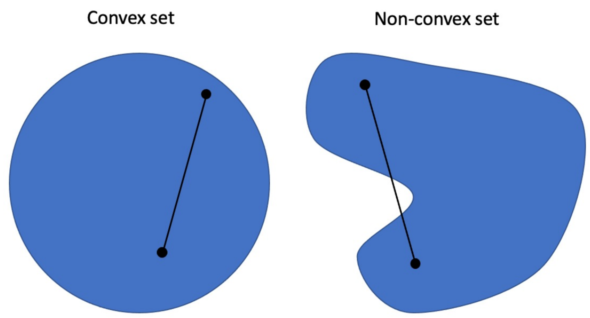

Figure 3: sets

Definition 5 (Supporting hyperplane) Consider a set S in Euclidian space $\mathbb{R}^n$, a supporting hyperplane is a hyperplane that has the two following properties [20]:

- The set is totally contained in one of the two closeed half-spaces limited by the hyperplane,
- The set has minimum one boundary-point on the hyperplane.

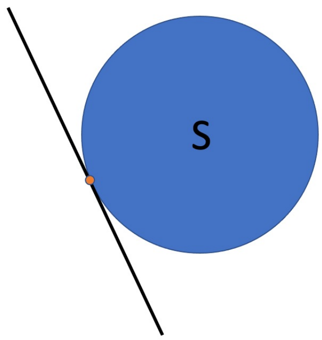

Figure 4: Supporting Hyperplane

Further, we can show that a nearest point always exist. Fix z and let ω be some point in C. Minimizing $\|z - x\|$ over all $x \in C$ is equivalent to minimizing the continuous function:

$$g(x) = \frac{1}{2}\|z - x\|^2$$

over the set of all $x \in C$ such that $\|z - x\| \leq \|x - \omega\|$, which is a bounded and closed set in $\mathbb{R}^n$ (thus compact). Then there is a minimizer. Accordingly, a closed non-empy set S assures that a nearest point exists. Note however that this may not be unique. This is true unless the set $C \subseteq \mathbb{R}^N$ is convex, that is, if: $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. This can be interpreted geometrically as whenever we choose two points in the set, all points on the line segment between these two points also lie in C' [20].Furthermore we can show that the nearest point for convex sets is unique. That is, if $C \subseteq \mathbb{R}^n$ is a closed and non-empty convex set. Then there is a unique nearest point c to z in C for every $z \in \mathbb{R}^n$. The proof goes as follows.

Proof. If we assume that c_0 and c_1 are nearest points to z in C and $d = d_C(z) = \|z - c_0\| = \|z - c_1\|$. This would led to both points being on the boundary of the closed ball B with its center in z and radius d where $B = \{y \in \mathbb{R}^n : \|y - z\| \leq d\}$. However, there is a contradiction here because the midpoint $c^* = \frac{1}{2}c_0 + \frac{1}{2}c_1$ lies in C since C is convex and c^* lies in the interior of B. Thus: $\|c^* - z\| < d$, hence a contradiction which proves that the nearest point must be unique.

2.2 Finance background

2.2.1 Payoff matrices

Consider an investor [14], having an array of investment opportunities including stocks, bonds, currency, options and so on, available. We can simply by assuming that there are n possible choices of investments available and the investor can at time t invest any amount of his finances in each of these possibilities. Consider time moving forward and at the end of a fixed time period every investment possibility will be worth some amount, meaning that changes in value of every investment can be represented as an n-dimensional vector. We can simplify further by assuming that there is a finite quantity of mutually exclusive scenarios that can occur and that each of these leads to a specific gain or loss for each investment. Because of these assumptions we can represent the returns, or changes in value, in an $m \times n$ payoff matrix where each row corresponds to each scenario and column to each investment. In the matrix, the $i,\ j$ entry $a_{i,j}$ gives us the return for investment i at the end of the time period for a unit invested under scenario j, see array 1.

$$\begin{array}{c|cccc} & I_1 & I_2 & \dots & I_n \\ \hline \text{Scenario 1} & a_{11} & a_{12} & \dots & a_{1n} \\ \text{Scenario 2} & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Scenario } m & a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \tag{1}$$

2.2.2 Arbitrage

There exists several definitions for arbitrage [9], being similar but different in some aspects (mainly if arbitrage should be guaranteed profit or at least 0 profit without risk), however, the implication is the same for all definitions: we can not make any profit if we do not take any risk. When there are mismatches in the markets we call them arbitrage opportunities.

As an example consider an asset, which costs 10 $ in Sweden but only 9 $ in Denmark, clearly we can make a profit by buying this asset in Denmark and sell it in Sweden. While we do this the price in Sweden will go down while it goes up in Denmark, but we will have made a nice profit without taking any risk. Clearly this shows that the market is not in equilibirum. Being able to do trades like this,i.e, to be guaranteed a profit without taking any risk is what economists calls an arbitrage trade. In theory this should not be possible since all assets should be equally priced in all markets, and if deviation occurs these will be quickly correlated and the markets is in equilibrium once again.

The assumption that arbitrage do not exist in markets is as fundamental as Newton's laws for physics and his third law [14]: for every action there is an equal and oppposite reaction bears semantic meaning as in finance; everything that can be bought can also be sold. This means that for every payoff column there exists an investment opportunity that has the exactly opposite effect, i.e, change the signs. This can seem counterintuintative but in finance there is something called short sell[1], the consequence is that a column in a payoff matrix can be replaced by any nonzero multiple of the column where the modified matrix represents the same investment opportunities.

Given a payoff matrix A including all possible investments and scenarios, we say that an arbitrage opportunity exists if there is a linear combination of columns of A with strictly positive entries. Hence, the arbitrage opportunity is a combination of buys and sells that for all scenarios yields a net gain.

Arbitrage opportunities is in theory possible everywhere, but to get a better understanding some examples are:

- Buying asset X in one market and selling in another market where the price is higher.
- Triangular arbitrage including three currencies where one currency is converted into another, which is then converted to a third currency and finally, back to the original currency.

This small list is in no way exhaustive but shows possible arbitrage opportunities.

2.2.3 The Arbitrage Theorem

The arbitrage theorem gives a convenient and interesting characterization of the no arbitrage condition [14]. If we have a payoff matrix A of the size $m \times n$ and an n-vector x, then result from investing x_i in investment i for $i = 1, 2, \ldots, n$ can be obtained by using the product Ax which is a payoff vector. Hence, all possible combinations of investments corresponds to the column space $col(A)$ for a payoff matrix A, representing the set of payoff vectors which construct the subspace of $\mathbb{R}^n$ of dimension at most n.

Theorem 1 Given payoff matrix A, which is a an $m \times n$, exactly one of these two statements holds:

A1 Some payoff vector in $col(A)$ has all positive components, i.e. $Av > 0$ for some $v \in \mathbb{R}^n$.

A2 The probability vector $\pi = [\pi_1, \ldots, \pi_m]^\top$ exists that is orthogonal to every column of A, meaning that: $\pi^\top A = 0$, where $\pi^\top \geq 0$ and $\pi^\top = 1$, and **1** is

[1] Short sell is a strategy or investment that profit from the decline in a stock.[2]

an all one-vector.

This means that the absence of arbitrage and the existence of an assignment of scenarios with probabilities where every investment has en expected return of zero is equal.

To further clarify we take an example, here u stands for up meaning that the asset goes up in price while d means down and that the asset goes down in price, r is the risk-free asset which is the return we want to not lose the real value of money (this since we could invest in the risk-free asset we consider this the bottom line) consider a situation in which our payoff matrix contains a column of the form:

$$\begin{bmatrix} -1 + \frac{P_u}{(1+r)} \\ -1 + \frac{P_d}{(1+r)} \end{bmatrix}$$

Where P_u is the price if the asset goes up in value, P_d if it goes down and to calculate the change we divide with the price times 1 and the risk free rate, r during the period. This means that, according to the absence of arbitrage there is a probability vector:

$$\begin{bmatrix} \pi_u \\ \pi_d \end{bmatrix}$$

This vector is orthogonal to every column of the payoff matrix and it follows that: $\pi_u = \frac{1+r-d}{u-d}$ and $\pi_d = 1 - \pi_u = \frac{u-1-r}{u-d}$. This probability assignment is practical since under the assumption of no arbitrage all investments will have an expected payoff of zero. Hence, we can use these probabilities to establish the price of all securities where the payoff depends on the behavior of the stock. Further, we can suppose there exists an opportunity to invest in a security with the price P today and tomorrow P_u or P_d depending on it goes up or down, hence we can add a column to the pay off matrix:

$$\begin{bmatrix} -1 + \frac{P_u}{P(1+r)} \\ -1 + \frac{P_d}{P(1+r)} \end{bmatrix},$$

which describes the present value to a investment of one unit. With the absence of arbitrage we can now conclude:

$$\pi_u \left(-1 + \frac{P_u}{P(1+r)} \right) + \pi_d \left(-1 + \frac{P_d}{P(1+r)} \right).$$

If we now solve for P we obtain:

$$P = \frac{\pi_u P_u}{(1+r)} + \frac{\pi_d P_d}{(1+r)} = 0.$$

This price can be interpreted as the investments no-arbitrage price meaning that it is equal to its discounted payoff.

The Arbitrage Theorem is an example of a theorem of the alternative that appears in convex analysis in which one asserts the existence where a vector satisfy exactly one of two possible properties, one fundamental result of this type is Farkas' Lemma that we will prove later on.

3 The separating hyperplane theorem

A hyperplane is defined as a 'generalized plane' [4], more formally: a set $H \subseteq \mathbb{R}^n$ on the form $H = \{x \in \mathbb{R}^n : a^\top x = \alpha\}$ for some real number α and nonzero vector a. a is a normal vector of the hyperplane. In $\mathbb{R}^2$, the hyperplane is a line while in $\mathbb{R}^3$ it is a plane. Further, we denote $H_{a,\alpha} = H = \{x \in \mathbb{R}^n : a^\top x = \alpha\}$ while the halfspaces is defined as:

$$H^-_{a,\alpha} = \{x \in \mathbb{R}^n : a^\top x \leq \alpha\}$$

$$H^+_{a,\alpha} = \{x \in \mathbb{R}^n : a^\top x \geq \alpha\}$$

The halfspaces represents the two different sides of the hyperplane, as we easily can think of in $\mathbb{R}^2$ and $\mathbb{R}^3$, but for higher dimensions must trust our theory.

We say that the hyperplane $H_{a,\alpha}$ strongly separates the sets S and T if there exists an $\epsilon > 0$ such that $S \subseteq H^-_{a,\alpha-\epsilon}$ and $T \subseteq H^+_{a,\alpha+\epsilon}$ or the other way around. If we consider $\mathbb{R}^2$ this would mean that the sets S and T exists on different sides of the line $H_{a,\alpha}$ and that neither of the sets intersects the line.

3.1 Theorem

If we let $C \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$, where C is a nomnempty and closed convex set while $z \notin C$, then C and z can be strongly separated.

3.2 Proof

If we consider the sets C and z as above, further, let the unique nearest point to x in C be p, $x \in C$ and $0 < \lambda < 1$. Since C is convex, $(1-\lambda)p + \lambda(x-z) \in C$ and p is a nearest point, we subsequently have $\|(1-\lambda)p + \lambda(x-z)\| \geq \|p-z\|$, that is, $\|(p-z) + \lambda(x-p)\| \geq \|p-z\|$. If we square both sides and calculate their inner product we obtain: $\|p-z\|^2 + 2\lambda(p-z)^\top(x-p) + \lambda^2\|x-p\|^2 \geq \|p-z\|^2$. We now reduce by subtract $\|p-z\|^2$ on both sides, divide by λ, let $\lambda^+ \longrightarrow 0$ and lastly multiply by -1, which gives us the equality:

$$(z-p)^\top(x-p) \leq 0 \text{ for all } x \in C.$$

Now consider the hyperplane H which contains p and having $a := z - p$ as normal vector, that is, $H = \{x \in \mathbb{R}^n : a^\top x = \alpha\}$ where $\alpha = a^\top p$. Our inequality above shows that $C \subseteq H^-_{a,\alpha}$ and $z \notin H^-_{a,\alpha}$ as $z \neq p$ since $z \notin C$. If we now consider the paralell hyperplane H^* to H, having the same normal vector, which cointains the point $\frac{1}{2}(z + p)$, then we can see that H^* separates z and C as desired.

4 Farkas' Lemma

By making use of the separating hyperplane theorem (stated above) we can now prove Farkas' Lemma [4]. This lemma characterizes when systems of linear inequalities has a solution and is central in optimization.

4.1 Theorem

In other words exactly one of these two statements holds: (i) There exists $x \geq 0$ so that $Ax = b$, (ii) There exists y so that $y^\top A \geq 0$ and $y^\top b < 0$.

4.2 Proof

If we denote the columns of the matrix A as $a^1, a^2, \ldots, a^n$, we can consider the set $C = \{\sum_{j=1}^{n} \lambda_j a^j : \lambda_j \geq 0 \text{ for } j = 1, \ldots, n\} \subseteq \mathbb{R}^m$ which is called the convex cone generated by $a^1, a^2, \ldots, a^n$, then C is closed. We also observe that only if $b \in C$ can we have a nonnegative solution x to $Ax = b$.

Now assume that $x \geq 0$ and that x satisfies $Ax = b$, if $y^\top A \geq 0$, $y^\top b = y^\top (Ax) = (y^\top A)x \geq 0$ follows as the inner product of two nonnegative vectors. In contrast, if we assume that there exists no nonnegative solutions to $Ax = b$, $b \notin C$ follows. But according to the separating hyperplane theorem, C and b can be strongly separated and, there is a nonzero vector $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ with $y^\top x \geq \alpha$ for every $x \in C$ and $y^\top b < \alpha$. As $o \in C$, we have $\alpha \leq 0$ and we can claim that for each $x \in C$ we have $y^\top x \geq 0$, this since, if $y^\top x < 0$ for some $x \in C$ it would exist a point $\lambda x \in C$ where $\lambda > 0$ such that $y^\top (\lambda x) < \alpha$ which is a contradiction. Subsequently as $a^j \in C$, $y^\top a^j \geq 0$ so $y^\top A \geq o$ and since $y^\top b < 0$ we have also proven the other direction of Farkas' Lemma. $\square$

4.3 Understanding Farkas' Lemma geometrically

Remember the defintions on a cone and conic hull from section 2.1 and the separating hyperplane theorem in section 2.1. Using these the geometric interpretation of Farkas Lemma is: Consider the matrix A and let $\tilde{a}_1, \ldots, \tilde{a}_n$ express the columns and let $\text{cone}(\tilde{a}_1, \ldots, \tilde{a}_n)$ be the cone of of all their possible nonnegative combinations. Then if we have $b \notin cone(\tilde{a}_1, \ldots, \tilde{a}_n)$ we can seperate it from the cone using a hyperplane.

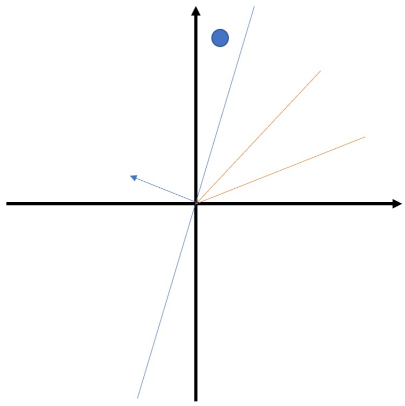

Figure 5: Farkas' Lemma Geometrically

4.4 Using Farkas' Lemma to prove the Arbitrage Theorem

Proof. If we assume that A1 and A2 from The Arbitrage Theorem section 2.2.3 are satisfied we have [14]:

$$\pi^\top Av = 0v = 0$$

But since $Av > 0$ we have $\pi^\top Av > 0$, this because the entries in π sum to one and are nonnegative, hence, it follows that A1 and A2 are mutually exclusive. If A2 does not hold we have no solution to

$$\begin{bmatrix} A^\top \\ 1 \end{bmatrix} \pi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with a nonnegative π since according to Farkas It follows that (i) fails for $\tilde{A} = \begin{bmatrix} A^\top \\ 1 \end{bmatrix}$ and $\tilde{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and since Farkas can guarantee that (ii) holds and hence there exists an $n+1$ vector which we can call y so that $y^\top \tilde{A} \geq 0$ and $y^\top \tilde{b} < 0$. If $v = [v_1, \ldots, v_n]^\top$, $s \in \mathbb{R}$ and we write $y = \begin{bmatrix} v \\ s \end{bmatrix}$ these conditions claim that $Av + s \geq 0$ and $s < 0$ and we can therefore conclude that $Av > 0$ leading to A1 being valid. □

5 One period economic model

5.1 Model setup

This one period model will allow us to state and prove two different fundamental theorems in mathematical finance without the need of technical detail, however, this theorem hold for more general models [18].

One period model means that we will only consider two points in time, called $t = 0$ and $t = 1$. The model lives on a finite space sample space $\Omega = \{\omega_1, ..., \omega_M\}$, we have a probability measure P defined on $\mathcal{F} = \mathcal{F}_1 = 2^\Omega$ and $P(\omega_i) = p_1 > 0$, $i = 1, ..., M$.

If we consider a market with $N + 1$ assets on the market and denote the price of asset i at time t by S_t^i, let:

$$S_t = \begin{pmatrix} S_t^0 \\ S_t^1 \\ \vdots \\ S_t^N \end{pmatrix}$$

Here we assume that $S_1 \in \mathcal{F}$ and that S_0 is deterministic. We will also assume that the asset number 0 is strictly positive: $S_0^0(\omega) > 0$ and $S_1^0(\omega) > 0$ for $\omega \in \Omega$, since it allows us to use it as a numeraire asset. A numeraire asset is acts as a benchmark when comparing the value of similar assets, allowing for comparisons between different assets, one such asset in real life is money [12].

5.2 Absence of arbitrage

Having asset 0 as the numerarie, allow us to compare money at different times, in this model, today where $t = 0$ and money in the future when $t = 1$.

Definition 6: We define the normalized price process as:

$$Z_t = \frac{S_t}{S_t^0} = \begin{pmatrix} \frac{S_t^0}{S_t^0} = 1 \\ \frac{S_t^1}{S_t^0} \\ \vdots \\ \frac{S_t^N}{S_t^0} \end{pmatrix}.$$

We normalize since it allows us to construct a Martingale process [19]. Note that if S^0 is a bank account we are only discounting everything to its present value, further, in the normalized economy we have $Z^0 \equiv 1$. This means that it corresponds to a bank with zero interest rate.

Definition 7: The vector $h = (h^0, h^1, ..., h^N)^\top$ is a portfolio and the value process V^S corresponding to a portfolio h is defined as:

$$V^S(t) = \sum_{i=0}^{N} h^i S_t^i = h^\top S_t,$$

For a portfolio the normalized value process V^Z is defined as:

$$V^Z(t) = \sum_{i=o}^{N} h^i Z_t^i = h^\top Z_t = \frac{V^S(t)}{S_t^0}$$

Further, an portfolio that is an arbitrage portfolio such that:

$$V^S(0) = 0,$$

$$V^S(1) \geq 0 \text{ with probability } 1,$$
$$E^P[V^S(1)] > 0,$$

Where E^P stands for the expected i.e. the last row means that we expect the profit to be over zero. Given our assumption on S^0, we can also write this as:

$$V^Z(0) = 0,$$

$$V^Z(1) \geq 0 \text{ with probability } 1,$$
$$E^P[V^Z(1)] > 0.$$

Proposition 1.
We are now ready to state a fundamental theorem of mathematical finance [18]: that the market is free of arbitrage if and only if there exists a probability measure Q on Ω such that:

$$Q(\omega) > 0, \text{ for all } \omega \in \Omega, \tag{i}$$

$$\frac{S_0^i}{S_0^0} = E^Q \left[\frac{S_1^i}{S_1^0} \right] \quad i = 1, 2, ..., N. \tag{ii}$$

In other words, the condition on Q in (i) means that P and Q are equivalent, meaning:

$$P(A) = 0 \Longleftrightarrow Q(A) = 0,$$

which also could be written as:

$$P(A) = 1 \Longleftrightarrow Q(A) = 1.$$

So the measures agree on what happens with probability one and what happens when the probability is zero.

The second condition for (ii) in proposition 1 means that Z_t, the normalized price process, is martingale under Q. For formal definition of martingale and the martingale measure, which are beyond the scope of this text, we refer to [1].

Keep in mind that a different numeraire asset gives different martingale measures and that it is easier to find a risk-neutral measure Q when the sample space is larger, that is, M is larger. Now we prove the Proposition by construction of Q by the Farkas lemma.

As we know, only one of the following systems can be solved:

$$E1 : \begin{cases} A\lambda = g \\ \lambda \geq 0 \end{cases}, \quad E2 : \begin{cases} g^\top x < 0 \\ A^\top x \geq 0 \end{cases} \Longleftrightarrow E2 : \begin{cases} x^\top g < 0 \\ x^\top A \geq 0 \end{cases}$$

Proof. We are now going to prove proposition 1:

Let matrix D, a $(N + 1) \times M$ matrix be given by:

$$D = \begin{bmatrix} Z_1^0(\omega_1) & Z_1^0(\omega_2) & \dots & Z_1^0(\omega_M) \\ Z_1^1(\omega_1) & Z_1^1(\omega_2) & \dots & Z_1^1(\omega_M) \\ \vdots & \vdots & & \vdots \\ Z_1^N(\omega_1) & Z_1^N(\omega_2) & \dots & Z_1^N(\omega_M) \end{bmatrix}.$$

The first row consists of only ones since $Z_1^0 \equiv 1$, while the first column is the normalized price vector for the outcome ω_1, $Z_1(\omega_1)$ at time $t = 1$. So we can see what the other systems look like, we intend to write the arbitrage portfolio definition as either E1 or E2. To do this, we can start with rewrite $V^Z(0) = 0$ as:

$$V_0^Z \geq 0, \text{ and } V_0^Z \leq 0.$$

As earlier stated: $V_0^Z = h^\top Z_0$, we can use this to get:

$$h^\top Z_0 \geq 0, \text{ and } h^\top Z_0 \leq 0,$$

or, equivalently:

$$h^\top Z_0 \geq 0 \text{ and } -h^\top Z_0 \geq 0.$$

Further, we can rewrite the condition $V^Z(1) \geq 0$:

$$h^\top D \geq 0.$$

While the condition $E^P[V^Z(1)] > 0$ can be rewritten as:

$$-h^\top Dp < 0, \text{where } p = (P(\omega_1), \ldots, P(\omega_M))^\top .$$

If we let the portfolio h play the role of x there will not exist any arbitrage portfolios if we can't solve E2 with:

$$g = -Dp \text{ and } A = [Z_0 - Z_+ D].$$

Hence, A is an $(N+1) \times (M+2)$-matrix.

So if we can not solve the system E2 with the given matrices it means that there is no arbitrage since these statements are equivalent. Thus, we can conclude that no arbitrage is equivalent, according to Farkas' Lemma, to use the given matrices to solve the system E1, that is:

$$[Z_0 - Z_o D]\lambda = -Dp, \quad \lambda \geq 0 \text{ or } Z_0(\lambda_2 - \lambda_1) = D(p + \lambda^*).$$

Here $\lambda^* = (\lambda_3, \ldots, \lambda_{M+2})$, with $\lambda \geq 0$, the first equation is interpreted as:

$$\lambda_2 - \lambda_1 = \sum_{i=1}^{M}(p_i + \lambda_i^*) \Longrightarrow \frac{1}{\lambda_2 - \lambda_1}\sum_{i=1}^{M}(p_i + \lambda_i^*) = 1.$$

Now, if we let:

$$Q(\omega_i) = q_i = \frac{1}{\lambda_2 - \lambda_1} \sum_{i=1}^{M} (p_i + \lambda_i^*),$$

since then $q_i > 0$, $i = 1, \ldots, M$ and $Z_0 = Dq = E^Q[Z_1]$ where $q = (Q(\omega_1), \ldots, Q(\omega_M))^\top$ we are done. $\square$

6 Linear programming duality

Linear programming, shortened LP, or linear optimization, is the method to maximize a linear function in n variables that is subject to a finite number of linear constraints that are linear inequalitites and/or linear equations. A linear programming standard problem is often in the form [4]:

$$\sup\{c^\top x : Ax \leq b\}.$$

Further, the $m \times n$ matrix A, $c \in \mathbb{R}^n$, are given $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is a decision variable. This problem is called the primal problem, this is feasible if there exists an x satisfying $Ax \leq b$ and hence, such an x is called a feasible solution. If a solution x_0 exists and $c^\top x_0 = \sup\{c^\top x : Ax \leq b\}$ it is considered optimal, this also leads to supremum being attained and we can write max instead. In the problem above, if no solution exists we define supremum to be $-\infty$ and $+\infty$ if it is unbounded which means that there exists a sequence (x_k) of feasible solutions so that $c^\top x_k \to \infty$ as $k \to \infty$.

To each primal LP problem there is another associated LP problem with it, called its dual problem, this is for the primal problem above:

$$\inf\{b^\top y : A^\top y = c, y \geq O\}$$

For this problem we use the terms feasible solution, feasible problem, and optimal solution, just as for the primal problem. The infimum in the dual problem is defined as ∞ if the problem is not feasible and $-\infty$ if unbounded meaning there exists a sequence (y_k) of feasible solution so that $b^\top y_k \to \infty$ as $k \to \infty$. These two problems together make up one of the main theorems in optimization, the LP duality theorem.

6.1 Linear programming duality theorem

1. First assume that the primal problem has a finite solution that is optimal, this leads to the dual problem having a optimal solution as well [4] and

$$\max\{c^\top x : Ax \leq b\} = \min\{b^\top y : A^\top y = y, y \geq 0\}. \tag{1}$$

2. If either the primal or dual problem is unbounded it means that the other is not feasible. Thus we have
   $\max\{c^\top x : Ax \leq b\} = \min\{b^\top y : A^\top y = y, y \geq 0\}$ when at least one problem is feasible.

Proof

Let x and y be feasible in the primal and dual problem respectively, so $Ax \leq b$ and $A^\top y = c$ and $y \geq 0$, we have:

$$c^\top x = (A^\top y)^\top x = y^\top Ax \leq y^\top b = b^\top y.$$

The inequality is from $Ax \leq b$ as $y \geq 0$. If we now take the supremum for the feasible x and infiumum over feasible y in the inequality we get:

$$\sup\{c^\top x : Ax \leq b\} \leq \inf\{b^\top y : A^\top y = c, y \geq 0\} \tag{2}$$

If we let the optimal solution for the primal problem be x_0, $a_i^\top$ denote the i:th row in the matrix A, define $I = \{i \leq m : a_i^\top x_0 = b\}$ correspnding to the indices of inequalitites from $Ax \leq b$ that holds for $x = x_0$. We can now claim that for $z \in \mathbb{R}^n$ satisfying $a_i^\top z \leq 0$ for $i \in I$ leads to the inequality $c^\top z \leq 0$ holds. If this were not the case there would be a $z \in \mathbb{R}^n$ with $A_i^\top z \leq 0$ for all $i \in I$ and $c^\top z > 0$, leading to for a small $\epsilon > 0$, the point $x' = x_0 + \epsilon z$ satisfies $Ax' \leq b$ since: 2. we have for all $i \in I$; $a_i^\top x' = a_i^\top x_0 + \epsilon a_i^\top z = b_i + \epsilon a_i^\top \leq b_i$ as well as, 2. for $i \leq m$ with $i \notin I$ we have: $a_i^\top x' = a_i^\top x_0 + \epsilon a_i^\top z = b_i + \epsilon a_i^\top z \leq b$, and hence $a_i^\top x' \leq b_i$ for small ϵ . But since we have x_0 as the optimal solution we can not have: $c^\top x' = c^\top x_0 + \epsilon c^\top z > c^\top x_0$ and it is therefore a contradiction.

This claim makes it possible for us to apply Farkas' lemma (Section 4) to the matrix A where the vectors a_i for $i \in I$ is the column resulting in the fact that is must exist nonnegative numbers y_i for $i \in I$ so that $\sum_{i \in I} y_i a_i = c$. Hence, where $y \in \mathbb{R}^n$ $A^\top y = c$ is the vector consisting of y_i for $i \in I$ or $y_i = 0$. one feasible is y in the dual problem since $A^\top y = c$ and $y \geq 0$, further, if we use $y_i = 0$ for $i \notin I$ we obtain:

$$c^\top x_0 = y^\top Ax_0 = \sum_{i \in I} y_i(a_i^\top x_0) = \sum_{i \in I} y_i b_i = b^\top y.$$

Due to the inequality (2), this proves that y is for the dual problem an optimal solution and that the maximum in the first, primal, problem is equal to the minimum in the dual problem, leading to the equality (1) holds.

If we consider the second case in the theorem, that is when the primal problem is unbounded and the inequality (2), we can conclude that the dual problem is not feasible since this would mean that $b^\top y$ would be upper bound on $c^\top x$ because of the (2). This means that both sides would be ∞, the same is true if the dual problem is unbounded but the two sides of (2) would be $-\infty$. However, if the supremum is finite in (2) it is possible to prove that the supremum is attained and "sup=max=min=inf" due to the first part. □

To conclucde there exists three different possible solutions for an LP problem

1. Optimal finite solution exist for both problems and the optimal values are equal.
2. It is not feasible
3. The solution is unbounded

7 Proof of the fundamental theorem of asset pricing using LP

We will now apply LP to prove the fundamental theorem of asset pricing [4] (absence of arbitrage) and to find a dominant trading strategy. For this we use the following notations:

- K: number of scenarios, or states, n: number of assets
- $P = [p_{ij}]$: $K \times n$ payoff matrix, p_{ij} is payoff for asset k under state i
- $s \in \mathbb{R}^n$: a strategy where we buy s_j units of j which is an asset
- $x \in \mathbb{R}^K$: the payoff for a specific trading strategy for different states
- I: identity matrix with a suitable size, O: the zero vector and e: vector consisting of only ones
- Nul(A): nullspace and col(A): columnspace, both for matrix A

A vector y with positive components that has a sum that equals one so that the dot product of the vector and each column in P is zero means that it is a risk-neutral probability measure, this means that the expected payoff is zero for each asset.

LP model to find arbitrage

Consider the problem:

$$\max \sum_{i=1}^{K} x_i \tag{3}$$

$$\text{subject to: } x = Ph, \; x \geq 0$$

In this problem $x = Ph$ relates the trading strategy s and payoff x where the linear equation tells us that x is a linear combination of the columns in P. The nonnegative x is desirable since this means that we will not lose money under any state, the objective is to maximize the sum of the payoffs when we sum over all states, or in other words, look for positive payoffs for at least one scenario, that is, an arbitrage possibility. An arbitrage exists if and only if the

LP problem has a optimal value that is positive.

Proof for the arbitrage theorem using LP

We can prove this theorem by the duality theory [4] to the problem (3), we start by writing it in the form of a primal problem by using $Ph = x$ is equivalent to $Ph - x \leq O$, $-Ph + x \leq O$, this results in:

$$\max\left\{\begin{bmatrix} O \\ e \end{bmatrix}^\top \begin{bmatrix} h \\ x \end{bmatrix} : \begin{bmatrix} P & -I \\ -P & I \\ O & -I \end{bmatrix} \begin{bmatrix} h \\ x \end{bmatrix} \leq \begin{bmatrix} O \\ O \\ O \end{bmatrix}\right\}$$

The dual problem is:

$$\min\left\{\begin{bmatrix} O \\ O \\ O \end{bmatrix}^\top \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} : \begin{bmatrix} P^\top & -P^\top & O \\ -I & I & -I \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} O \\ e \end{bmatrix}, y^1, y^2, y^3 \geq O\right\}$$

Observe that the objective function is 0 and by substituting $y = y^2 - y^1$, $z = y^3$ the dual problem becomes:

$$\min\{0 : P^\top y = O,\ y = z + e,\ z \geq O\}.$$

If and only if there exist a vector $y \in \mathrm{Nul}(P^\top) = \mathrm{Col}(P^\perp)$ such as $y \geq e$ the problem above has a feasible solution. This is equivalent to the fact that there exists a $y \in \mathrm{Nul}(P^\top)$ with $y_i > 0$ $(i \leq K)$ and $\sum_i y_i = 1$ which follows by y being scaled. We have now proved that the following points are equivalent:

- The LP problem has a optimal value of zero
- there exists a vector $y \in \mathrm{Nul}(P^\top)$ with $\sum_i y_i = 1$ which is strictly positive, further this is a risk-neutral probability measure
- there exists no arbitrage.

The proof is therefore complete.□

LP to find a dominant trading strategy [2] This problem contains the variable for the former problem and $\epsilon \in \mathbb{R}$, we can write it as:

$$\max \epsilon \tag{4}$$

[2]Dominant trading strategy is a portfolio that has the same cost as another but is guaranteed to out-perform it [15]

$$\text{subject to } x = Ph,\ x \geq \epsilon e$$

The second constraint implies that for each $j \leq n$ we have $x_j \geq \epsilon$, also note that objective is to find the optimal ϵ, i.e, a trading strategy that maximizes the minimum outcome, also notice that the problem has a feasible solution in the zero vector. This means that there exists a dominant trading strategy if and only if the optimal value of (4) is positive.

Theorem. There exists no dominant trading strategy if and only if it exist a linear pricing measure.

Proof. We can prove this by a similar method as above, we start to rewrite the LP problem (4) as a primal problem:

$$\max \left\{ \begin{bmatrix} O \\ O \\ 1 \end{bmatrix}^\top \begin{bmatrix} h \\ x \\ \epsilon \end{bmatrix} : \begin{bmatrix} P & -I & O \\ -P & I & O \\ O & -I & e \end{bmatrix} \begin{bmatrix} h \\ x \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} O \\ O \\ O \end{bmatrix} \right\}$$

Hence, the dual problem is:

$$\min\{0 : P^\top(y^1 - y^2) = 0,\ -y^1 + y^2 - y^3 = 0,\ e^\top y^3 = 1,\ y^1, y^2, y^3 \geq O\}$$

By substituting $y = y^2 - y^1$ and $\pi = y^3$ we can eliminate y and simplify to:

$$\min\{0 : P^\top \pi = 0,\ \sum_j \pi_j = 1,\ \pi \geq O\}$$

Since a feasible solution is a linear pricing measure we have proved that the following statements are equivalent:

- there exist no dominant trading strategy
- there is a pricing measure that is linear
- the optimal value to (4) is zero

So the proof is complete. □

8 Methods for solving linear programming problems

There exist several methods for solving linear programming problems divided into two different classes, simplex or interior. Here we will focus on one method that is simple to understand but for problems with a large number of unknown variables and/or constraints are not always practical or possible in real world time to use since the complexity is very high compared to other methods like the simplex algorithm. This method, called Fourier-Motzkin eliminiation [10], is however simple to understand and for small problems an alternative.

The idea behind the method is very simple, we have a n-variable problem to an equivalent $(n-1)$-problem, here equivalent means that the first system of inequalities has a feasible solution if and only if the second one does. This procedure is then iterated to eliminate one variable at a time until we only have one left which is easy to solve since we only need to check if there exists any number between the greatest lower bound and least upper bound. We will then be able to trace back our steps and by using a solution (if it exists) to the 1-variable problem to find solutions to the 2-variable problem and so on to the n-variable problem. The proof for the method is omitted in this text but if of interest please see *"Fourier-Motzkin Elimination and Its Dual*"* by George B. Dantzig and B. Curtis Eaves [6].

8.1 Procedure

If we suppose we have variables $x_1,\ x_1,\ \ldots,\ x_n$ and we want to eliminate x_n we start with solving for x_n, for each inequality as:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

there is two possible inequalities we can get:

$$x_n \leq \frac{b - aa_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1}}{a_n}$$

or

$$x_n \geq \frac{b - aa_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1}}{a_n}$$

this depending on whether a_n is greater than or smaller than 0, if $a_n = 0$ we don't include the inequality since it doesn't include x_n.
After this we will have a collection of lower and upper bounds:

$$x_n \geq L_1,\ x_n \geq L_2,\ \ldots,\ x_n \geq L_k$$

$$x_n \leq U_1, /x_n/leU_2,\ \ldots,\ x_n \leq U_m.$$

Here L_i and U_i is expressions for the $n-1$ variables $x_1,\ x_2, \ldots,\ n_{n-1}$. Now if it is possible to pick a value for x_n that satisfies all these bounds all inequalitites hold. This is the case if and only if:

$$\max\{L_1,\ L_2,\ \ldots,\ L_k\} \leq \min\{U_1,\ U_2, \ldots,\ U_m\}$$

since then there is something in between all the upper bounds and all the lower bounds. However, we can not write this in a single equation since the lower and upper bounds exists of unknown values so we can not say which ones are larger or smaller. Instead we need to compromise by writing down $k \times m$ inequalities:

$$\begin{array}{cccc} L_1 \leq U_1 & L_1 \leq U_2 & \cdots & L_1 \leq U_m, \\ L_2 \leq U_1 & L_2 \leq U_2 & \cdots & L_1 \leq U_m, \\ \vdots & \vdots & \ddots & \vdots, \\ L_k \leq U_1 & L_k \leq U_1 & \cdots & L_k \leq U_m. \end{array}$$

Now if an x_n exists that satisfies all of these constraints, we have for every i and j, $L_1 \leq x_m \leq U_j$, meaning that all of these inequalities hold. Further, if all holds it means that the max-min inequality holds as well and we can pick x_n between $\max\{L_1, \ldots, L_k\}$ and $\min\{U_1, \ldots, U_m\}$. This results in a new system that consists of $n-1$ variables and consist of the $k \times m$ inequalities above, plus any of the original inequalities which didn't involve x_n in them to begin with.

As one can understand this method can lead to the number of inequalities growing very quickly. If we start with 8 inequalities in n variables, it is possible that we get 2^{2^k+2} inequalities in $n-k$ variables until we get $2^{2^{n-1}+2}$ inequalities in 1 variable on the last step. This implies that it is worse than exponential and hence for larger systems a different method is advised.

8.2 Example

To get a better understanding of the method we next go through an example of system of inequalities that includes two variables.

$$\begin{cases} x - y \geq 1 \\ -x + 2y \geq 1 \\ 3x - 5y \geq 1 \\ x, y \geq 0 \end{cases}$$

We start with eliminating y, this by solving for y:

$$\begin{cases} x - y \geq 1 \Longrightarrow y \leq x - 1 \\ -x + 2y \geq 1 \Longrightarrow y \geq \frac{x+1}{2} \\ 3x - 5y \geq 1 \Longrightarrow y \leq \frac{3x-1}{5} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Now we pair the lower bounds with the higher bounds which gives us five inequalitites on x:

$$\begin{cases} \frac{x+1}{2} \leq x - 1 \\ 0 \leq x - 1 \\ \frac{x+1}{2} \leq \frac{3x-1}{5} \\ 0 \leq \frac{3x-1}{5} \\ x \geq 0 \end{cases}$$

These can be simplified into lower and upper bounds on x:

$$\begin{cases} x \geq 3 \\ x \geq 1 \\ x \geq 7 \\ x \geq \frac{1}{3} \\ x \geq 0 \end{cases}$$

In this example, it turned out that they all were lower bounds which means that we know that they can be satisfied, the only constraint is that x needs to be at least 7. Now, depending on which value we put in as x we get different possible values on y. For instance, if we take $x = 7$ we get: $y \leq 6$, $y \geq 4$, $y \leq 4$, and $y \geq 0$ meaning that $y = 4$ is a feasible solution.

9 Arbitrage in real life

So to shortly recap, arbitrage is a term in finance that describes the possibility to make a profit without taking the risk of losing money. As we have seen, this idea can be shown and proved by the use of mathematics, but does arbitrage exist in real life?

According to economic theory, no, according to the markets it is more difficult to asses since there exist different definitions on arbitrage and market participants are not willing to show their models. But with the technical development of faster computers small inequalities in price are smoothed out (often by high frequency traders) faster than ever, resulting in markets to some extent moving closer to the economic theory. However, if we consider the high frequency traders from the outside we still can not say anything regarding arbitrage since we do not have a complete view of how they operate. But for the ordinary person who buys and sells on the markets, they should not see arbitrage as anything they can accomplish but as a economic concept to help build models to gain an understanding of how economy works in theory.

To circle back to the aim of the thesis, exploring the connection between arbitrage and mathematics, it is interesting to see how the economic theory can be build up by using mathematics, mainly linear programming. Even though its practical use should be seen as insignificant it can spark an interest and idea of how to develop the models to improve on this point. This especially since many economists tend to think using economic reasoning rather than mathematical that has the possibility to deepen the knowledge further. However, the use of mathematics in finance is increasing and today there exist several funds consisting of only mathematicians trying to beat the market with widely different result, for example Renessaince capital [22] who can be considered one of if not the best hedge fund ever and Long-term capital management [13] resulting in catastrophe.

For tips on further reading on the subject, I would recommend: Arbitrage and Geometry (2017 Daniel Q. Naiman and Edward R. Schneinerman [14] who discuss the geometric meaning and the notion of arbitrage including a collection of investments and payoff matrices that describes, the return for an investor under different scenarios. In the end they also ask themselves the question "given a random payoff-matrix, what is the probability of an arbitrage opportunity?". As the article do not include any analysis on a real data-set there is still blanks to fill. But the discussion and intuition are nonetheless interesting from a theoretical standpoint.

I would also recommend: Arbitrage opportunities on derivatives: A linear programming approach by S. Herzel [8] who proposes a test on how to check a given market for arbitrage opportunities using the LP approach. He also test the SP500 index traded at CBOE for both call and put options [3] with maturity [4] in August 1999. He finds that with this data he was not able to find an arbitrage opportunity. For a further explanation of his method we refer to his article where he also explains why this works and gives a better mathematical background.

[3] A option is a financial instrument that gives the holder the right to buy (put) or sell (call) a underlying instrument connected to the option [16]

[4] Maturity date is the last day to exercise the option

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