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Fundamental groups of schemes

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Abstract

The fundamental group of a topological space is the group of based homotopy classes at a point. The Zariski topology for schemes is not fine enough and lacks several desirable properties to construct such a group. Under suitable conditions the group of cover automorphisms of a universal covering is isomorphic to the topological fundamental group. Grothendieck introduced the étale topology and used finite étale covers to define an algebraic fundamental group of a scheme in [SGA71]. The goal of this thesis is to give an introduction to the étale fundamental group of schemes without going through Grothendieck's more general construction of Galois categories.

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Chapter 1

Introduction

In the early days of Galois theory one studied permutation groups of the roots to polynomials and their symmetries to gain information about their solubility. In modern language this corresponds to a permutation group action and through the study of Galois groups $Gal_k(L)$ of Galois extensions $k \subset L$. The group action only depends on the separable closure k_s of the base field k from which the polynomial has its coefficients and not the extension itself. After all, the set of morphisms $\mathrm{Hom}_k(L, k_s)$ where L is a finite extension of a field k and k_s the separable closure corresponds to the finite set of roots of a polynomial defined by L. If the group action is transitive on the finite set of roots then the extension must be a Galois extension.

Changing focus for a moment, a guiding principle in traditional algebraic topology is to understand topological spaces through the means of invariants[1] and combinatorics. One of the examples being the fundamental group $\pi_1(X, x)$ of a topological space X with a base point x. The group measures to what extent loops in X based at x can be continuously deformed. This fundamental group can be studied through topological covers $p : Y \to X$ of the space X and the group of automorphisms of such covers. We can, in fact, find a universal cover if the space is nice enough that has a similar role to the separable closure of a field in the Galois theory case.

Grothendieck's new theory in [SGA71] captured the essence of both examples by constructing what are called Galois categories, which unifies both while switching focus to the automorphism groups of a functor. The Galois theory example has a special fibre functor sending a separable extension L by

$$F_{k_s} : L \mapsto \mathrm{Hom}_k(L, k_s)$$

and in the algebraic topology the functor sends a cover to its fibre by

$$F_x : p \mapsto p^{-1}(x).$$

[1] Invariants are typically captured by functors because functors carry data between categories and their representations or simplifications.

He went further and through this formalism of Galois categories gave rise to the theory of algebraic fundamental group of schemes.

The natural thing to try to develop a theory of a fundamental group of schemes is to try to apply covers to a scheme over the Zariski topology. This approach fails, however, because there are not enough Zariski "locally trivial" covers, meaning that covers become simply copies of the object they cover after restricting to a sufficiently small set. To give an example, the complex manifold $\mathbb{C} \setminus \{0\}$ has fundamental group isomorphic to $\mathbb{Z}$ generated by the number of loops around the removed origin. The equivalent scheme $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ has no Zariski locally trivial cover that is connected. This is because the distinguished open sets in $\mathbb{C}^n$ are complements of hypersurfaces and are "too large".

The Zariski topology of schemes was also noted to not behave as expected in classical geometry in other ways. Some constructions of schemes over a base scheme $X \to S$ which normally gave vector bundles did not have the property of being locally trivial. Serre observed, see discussion in [DIE72], that in many cases an extension of the base $T \to S$, where T is an étale covering of S, was enough to make them so. This was one of the starting points for Grothendieck to replace the Zariski topology with the étale topology, which is not a topological space in the traditional sense but a Grothendieck topology. In the language of Grothendieck a Zariski covering, not to be confused by the topology notion of a covering which is different, of a scheme T is a covering of T with open immersions $\{T_i \to T\}$ that correspond to open subsets in a topology. The étale topology is defined by coverings with étale morphisms $\{E_i \to E\}$ which in a sense represent morphisms which are open and in a certain sense local isomorphisms. This results in a much finer Grothendieck topology which can "zoom in" even further than the Zariski topology.

We will see how these realizations help in constructing étale covers and allow us to study an algebraic version of the fundamental group, called the étale fundamental group. We will avoid the language of Galois categories by following [SZA09] to a large extent especially in the choice of topics. This thesis is expository in nature. We assume, nonetheless, familiarity with Galois theory, category theory and some exposure to algebraic geometry, although we will review the latter in the language of schemes. Knowledge of algebraic topology is helpful to draw parallels between the different fundamental group constructions. We also try to often show more details regarding the theory than would be otherwise prudent in a traditional text on the subject, and focus on examples that are directly applicable to the étale fundamental group.

Notation	Explanation	Reference
$\|S\|$	The underlying set of a categorical object	–
$\sharp\|S\|$	The cardinality of a finite set	–
$\bar{k}$	Algebraic closure of a field k	–
k_s	Separable closure of a field k	–
$D(f)$	Distinguished open set of a scheme at the element f.	2.2.4
$\mathscr{F}$	Presheaf or sheaf over a topological space X	2.3.1, 2.3.3
f_*	Pushforward of a morphism of sheaves	2.3.6
$\kappa(\mathfrak{p})$	Residue field of a scheme	2.3.13
A_f	Localization of a ring at set generated by powers of f	2.3.8
$\mathscr{F}_x$	Stalk of a presheaf at a point $x \in X$	2.3.10
$(X, \mathscr{O}_X)$	(locally) ringed space with the topology X and sheaf $\mathscr{O}_X$	2.3.13
$A_\mathfrak{p}$	Localization of a ring at the set evading the prime ideal $\mathfrak{p}$.	2.3.14
$X \times_S Y$	Fibre product of S-schemes X and Y	2.5.4
X_p	Scheme theoretic fibre of $f\colon X \to Y$ at the point $p \in Y$	2.5.10
$\bar{s}$	Geometric point of a scheme S.	2.5.14
$X_{\bar{s}}$	Geometric fibre at the geometric point $\bar{s}$.	2.5.14
$\Omega^1_{B\|A}$	Kähler differentials of the B-algebra A.	2.7.6
$\Omega^1_{X\|Y}$	Kähler differentials of the Y-scheme X.	2.7.14
$\mathscr{O}_X^G$	G-invariant sheaf of $\mathscr{O}_G$	2.8.6
L^G	The field of fixed elements of L under G	–
$Fib_{\bar{s}}$	Fibre functor over a geometric point $\bar{s}$	4.1.13

Category	Explanation
$\mathscr{R}ing$	Commutative and unital rings
$\mathscr{A}\mathscr{S}ch$	Affine schemes
$\mathscr{S}ch$, $\mathscr{S}ch/S$	Schemes and S-schemes, respectively
$\mathscr{T}op(X)^{\mathrm{op}}$	Opposite category of open sets on the topological space X with morphisms being the inclusions
$\mathscr{T}op\mathscr{G}rp$	Topological groups
$\mathscr{F}in\mathscr{E}t/S$	Finite étale morphisms over S
$\mathscr{S}et$, $\mathscr{F}in\mathscr{S}et$	Sets and finite sets, respectively
G-$\mathscr{S}et$	Sets with a group action of G

Chapter 2

Review of Algebraic Geometry

"The very idea of scheme is of infantile simplicity—so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really 'not serious'!"

— Alexander Grothendieck, *1985–1987 - [REC85]*

We begin by reviewing some aspects of algebraic geometry like the introduction of a scheme and other machinery that will be necessary to construct the étale fundamental group of a scheme. All rings will be assumed to be commutative and unital.

2.1 Limits and colimits

We quickly recall the definitions of limits and colimits based on cofiltered categories. The reader might use the familiar inverse and direct limits respectively from category theory if they choose and can skip this section. It will be needed for the last section when we study the étale fundamental group.

Definition 2.1.1. A cofiltered category is a category $\mathcal{J}$ for which the following hold:

(i) $\mathcal{C}$ has at least one object.

(ii) For every pair of objects $P_i, P_j \in \mathcal{J}$ there is an object $P_k \in \mathcal{J}$ such that the arrows $P_j \leftarrow P_k \rightarrow P_i$ exist.

(iii) For every pair of objects $P_i, P_j \in \mathcal{J}$ and every pair of arrows $f, g\colon P_i \rightarrow P_j$ there is an arrow $h\colon P \rightarrow P_i$ such that $f \circ h = f \circ h$.

A cofiltered diagram in a category $\mathcal{C}$ is a functor $P\colon \mathcal{J} \rightarrow \mathcal{C}$ where $\mathcal{J}$ is a cofiltered category.

Remark 2.1.2. We will sometimes denote the arrows $P_i \to P_j$ as $i \leq j$.

Definition 2.1.3. A limit of a cofiltered diagram $P\colon \mathcal{J} \to \mathcal{C}$ is denoted by $\varprojlim_{P_i \in P} P_i$ and is an object of $\mathcal{C}$ together with morphisms $p_j\colon \varprojlim_{P_i \in P} P_i \to P_j$ such that:

1. For $\phi\colon P_i \to P_j$ a morphism in $\mathcal{J}$ we have $p_j = P(\phi)p_i$,
2. For any object $W \in \mathcal{C}$ and a family of morphisms $q_i : W \to P_i$ such that for all morphisms $\phi\colon P_i \to P_j$ in $\mathcal{J}$ we have $q_j = P(\phi)q_i$, there exists a unique morphism $q\colon W \to \varprojlim_{P_i \in P} P_i$ such that $q_i = p_i q$ for all $i \in \mathcal{J}$.

Remark 2.1.4. With the cofiltered diagram having a target in $\mathcal{S}et$ the cofiltered limits are realized as the product of all P_i with "compatible" elements. That is

$$\varprojlim_{P_i \in \mathcal{J}} P_i = \{(x_i)_{P_i \in \mathcal{J}} \in \prod\nolimits_{P_i \in \mathcal{J}} P_i : \text{ for all } \phi : P_i \to P_j \text{ in } \mathcal{J},\ P(\phi)(x_i) = x_j\}.$$

The dual notion is that of colimits.

Definition 2.1.5. A colimit of a cofiltered diagram $P\colon \mathcal{J} \to \mathcal{C}$ is denoted by $\varinjlim_{P_i \in P} P_i$ and is an object of $\mathcal{C}$ together with morphisms $s_j\colon \varinjlim_{P_i \in P} P_i \to P_j$ such that:

1. For $\phi\colon P_i \to P_j$ a morphism in $\mathcal{J}$ we have $s_j = s_i P(\phi)$,
2. For any object $W \in \mathcal{C}$ and a family of morphisms $t_i : P_i \to W$ such that for all morphisms $\phi\colon P_i \to P_j$ in $\mathcal{J}$ we have $t_i = t_j P(\phi)$, there exists a unique morphism $t\colon \varinjlim_{P_i \in P} P_i \to W$ such that $t_i = ts_i$ for all $i \in \mathcal{J}$.

Remark 2.1.6. With the cofiltered diagram having a target in $\mathcal{S}et$ the cofiltered colimits are realized as the disjoint union subject to an equivalence relation. That is

$$\varinjlim_{P_i \in \mathcal{J}} P_i = \coprod_{P_i \in \mathcal{J}} P_i / \sim$$

where the equivalence relation is defined by two elements $x_i \in P_i$ and $x_j \in P_j$ being equivalent if there is a k such that $x_k \in P_k$ and there are morphisms $p_{ik} : P_i \to P_k$ and $p_{jk} : P_j \to P_k$ with $P(p_{ik})(x_i) = P(p_{jk})(x_j)$. So that two elements are equal if they are "eventually" equal down the colimit.

2.2 Spectrum of a ring

The classic point of view is to study the geometry over an algebraic closed field $k = \bar{k}$ and focus on a subset $M \subset k[x_1, \dots, x_n]$ of polynomials. One views irreducible varieties as the set of points on which all functions in M vanish, that is $V(M) := \{(x_1, \dots, x_n) \in k^n : f(x_1, \dots, x_n) = 0 \text{ for all } f \in M\}$.

A corollary of one of the main results in classic algebraic geometry is that if I is an ideal which makes $A = k[x_1, \dots, x_n]/I$ into a finitely generated k-algebra, then we can recover $V(I)$ from A by looking at its maximal ideals by the Nullstellensatz. This is because having a morphism $\bar{f}: A \longrightarrow k$ is the same as the one that factors through $f: k[x_1, \dots, x_n] \longrightarrow k$ with $I \subset \ker(f)$. In other words we have an equivalence between zero sets of polynomials over algebraically closed fields and finite type k-algebra morphisms.

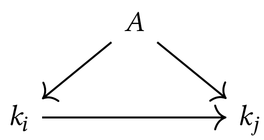
Proposition 2.2.1. *$V(I) = \mathrm{Hom}_k(A, k)$ where $A = k[x_1, \dots, x_n]/I$ is a finitely generated k-algebra.*

Curiosity can lead us to wonder what happens if we generalize this to ring homomorphisms between a commutative ring A and a field k, $\mathrm{Hom}_{\mathcal{R}ing}(A, k)$. For any prime ideal $\mathfrak{p} \subset A$ we have a natural map of rings

$$A \longrightarrow A/\mathfrak{p} \longrightarrow \mathrm{Frac}(A/\mathfrak{p}).$$

The inclusion of A into its field of fractions gives us a natural map. We denote the field of fractions by $\kappa(\mathfrak{p})$. Since there are awfully many morphisms in $\mathrm{Hom}_{\mathcal{R}ing}(A, k)$ we want to glue compatible ones by the following construction.

Definition 2.2.2. Define the spectrum $\operatorname{Spec} A$ of a ring A to be the collection of equivalence classes of ring morphisms $A \to k$ for k a field where two maps $A \to k_i$, $A \to k_j$ are identified if there exists a ring morphism $k_i \to k_j$ which makes the diagram commute.



We say, informally, that $\operatorname{Spec} A \coloneqq \varprojlim_{k_i} \mathrm{Hom}(A, k_i)$ even though this object might not exist as a set because of set theoretic size considerations. This construction doesn't mention prime ideals. But there is of course an identification between this and the classic definition of $\operatorname{Spec} A$ which is that of the set of prime ideals of A.

Proposition 2.2.3. *The set isomorphism $\operatorname{Spec} A \cong \{$prime ideals $\mathfrak{p} \subset A\}$ is given by the bijection of taking a map $(f: A \to k) \in \operatorname{Spec} A$ and sending it to its kernel $\ker(f) \subset A$.*

Proof. The map is well-defined by the commutativity of the diagram in the definition. If $f: A \to k$ is a map then $f(xy) = f(x)f(y) = 0$ are all elements of a field, which implies that either x or y are in $\ker(f)$ and so $\ker(f)$ is a prime ideal.

To construct the inverse we use our aforementioned inclusion into the field of fractions. That is with a prime ideal $\mathfrak{p}$ we can construct the natural inclusion

$$A \longrightarrow A/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p})$$

of A into its field of fractions, giving us a map we call $f_{\mathfrak{p}}$ with kernel $\mathfrak{p}$. If we have another map $f\colon A \to k$ with $\ker(f) = \ker(f_{\mathfrak{p}})$ we know that by the universal property of field of fractions that they must be in fact equal. □

Proposition 2.2.4. *The spectrum* Spec A *of* A *is a quasi-compact topological space, called the Zariski topology on* Spec A, *whose points are the prime ideals of* A *and a basis of open sets is given by the distinguished open sets* $D(f)$ *given by*

$$X \setminus V(f) = D(f) := \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal with } f \notin \mathfrak{p}\}$$

for all $f \in A$.

Proof. See [GW10, Proposition 2.5] □

These topologies Spec A will be the building blocks of schemes as we will see soon. First we must define what we mean by ”functions” on this topology. The key point is to give a notion of how we can restrict functions to subsets and glue them together on a topological space.

2.3 Ringed spaces

In a certain sense the minimal amount of structure necessary to speak of something that could have a resemblance of a ”geometrical space” is given by a sheaf $\mathcal{F}$ together with a topological space[1] X. These can be seen as functors from a category of open sets over a topological space[2] X to a category of interest $\mathcal{C}$. That is

$$\mathcal{F} : \mathcal{T}op(X)^{\mathrm{op}} \to \mathcal{C}.$$

The objects $\mathcal{F}(U)$, for an open set U, in $\mathcal{C}$ carry some resemblance as a collection of functions. Further we require to be able to ”glue” functions together, that is given an open covering $\{U_i \to U\}$ we need to be able to construct $\mathcal{F}(U)$ from the components $\mathcal{F}(U_i)$. This can be encoded in the limit of the following diagram (the limit is just the same as an equalizer diagram in this special case)

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

The first part says that if such a gluing exists then it is unique and the second shows that the glued object exists.

We summarize this discussion.

Definition 2.3.1. A presheaf over a topological space X with values in a category $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{T}op(X)^{\mathrm{op}} \to \mathcal{C}$$

where $\mathcal{T}op(X)$ is the category of open sets of X with inclusion restrictions as morphisms. The image of an inclusion morphism is called a restriction morphism.

[1]In more generality over a general category with a Grothendieck topology, that is a site.

[2]The objects being open sets of a topological space X and the morphisms being inclusions.

Remark 2.3.2. For inclusions of open subsets $W \subset V \subset U$ of X this means that there are restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)$ and this morphism is the same as the one gotten from $W \subset U$ and $\mathcal{F}(U) \to \mathcal{F}(W)$.

Definition 2.3.3. A presheaf[3] $\mathcal{F}$ is a sheaf if for any open covering $U = \cup_i U_i$ of an open set of X the following diagram is commutative

$$0 \to \mathcal{F}(U) \xrightarrow{d_0} \prod_i \mathcal{F}(U_i) \xrightarrow{d_1} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

The morphisms are defined as $d_0 : s \mapsto (s_i)$ where s_i are the restriction maps coming from $U_i \to U$ and $d_1 : (s_i) \mapsto (s_i|_{U_i \cap U_j} - s_i|_{U_i \cap U_j})$.

Remark 2.3.4. Recall that a morphism of sheaves is a morphism of functors and thus respects the restriction maps.

Remark 2.3.5. The objects $\mathcal{F}(U)$ for a sheaf over X are called the sections of $\mathcal{F}$ over U. The elements of the objects are our "functions" on the sets U. For the case of $U = X$ we call them the global sections.

Definition 2.3.6. Let $f\colon X \to Y$ and let $\mathcal{F}, \mathcal{G}$ be sheaves on X and Y, respectively. The pushforward sheaf $f_*\mathcal{F}$ on Y is defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Definition 2.3.7. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of sheaf $\mathcal{O}_X$ of rings on the topological space X.

Morphisms $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces are pairs $(f, f^\#)$ where $f\colon X \to Y$ is a continuous morphism and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves.

Proposition 2.3.8. *There is a unique sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Spec} A$ satisfying*

$$\mathcal{O}_X(X) = A, \quad \mathcal{O}_X(D(f)) = A_f$$

for all $f \in A$ where A_f is the ring of fractions of A with denominators in the multiplicative set $\{1, f, f^2, ...\} \subset A$. We have, in particular, the global sections being equal to the ring $\mathcal{O}_X(X) = A$.

Proof. See [GW10, Theorem 2.33]. In particular the proof uses a ubiquitous "unity of partition" argument that is actually very deep and has roots in more modern treatments of "descent theory". □

Example 2.3.9. We will now show an example of how we can look at elements of a ring as functions that might be a bit confusing at first.

[3]With values in an abelian category so that we can simplify the equalizer/limit shown earlier to an exact sequence.

Let $X = \operatorname{Spec} A$. If f is a "global function" on X, that is an element of the ring $\mathcal{O}_X(X) = A$, and $x \in \operatorname{Spec} A$ is a topological point represented by the prime ideal $\mathfrak{p}_x$. Then the value $f(x)$ is the residue class of f given by the image of the canonical morphisms

$$f\colon\ A \longrightarrow A/\mathfrak{p}_x \longrightarrow \kappa(x).$$

We see that $f \in \mathfrak{p}_x$ is equivalent to $f(x) = 0$. Technically all the values $f(x)$ live inside different rings $\kappa(x)$ and hence we can identify $f \in A$ with the map

$$\operatorname{Spec} A \longrightarrow \coprod_{x \in \operatorname{Spec} A} \kappa(x).$$

If A is a k-algebra over an algebraically closed field we see that $\kappa(x) = k$ for all $x \in X$ and the function can be seen as a function in the regular sense $f\colon\ \operatorname{Spec} A \to k$.

For example if $A = \mathbb{R}[x]$ and $x = (x^2 + 1) \in \operatorname{Spec} A$ then the "function" $f = x^3 + x^2 + x + 1$ has values the values

$$f(x^2 + 1) = 0, f(x^2 - 1) = 2x + 2, f(x - 5) = 156,$$

which can be found by the euclidean algorithm.

Definition 2.3.10. The stalk of a presheaf $\mathcal{F}$ at a point $x \in X$ is defined as

$$\mathcal{F}_x \coloneqq \varinjlim_{x \in U} \mathcal{F}(U)$$

where the colimit is taken over all open sets $U \subset X$ that contain x with restriction morphisms as morphisms of the limit.

Remark 2.3.11. The stalk of a presheaf $\mathcal{F}_x$ therefore allows us to zoom in on a point x and consider all the sections that pass through x in an "infinitesimal" neighborhood.

Proposition 2.3.12. *A morphism of ringed spaces* $(f, f^\#)\colon\ (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *induces a morphism on the stalks*

$$f_x^\# :\ \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

Definition 2.3.13. A morphism of local rings $A \to B$ is called local if the image of the maximal ideal of A is a subset of the maximal ideal of B.

A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces such that the map on stalks $f_x^\# :\ \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring morphism, that is $f_X^\#(m_{f(x)}) \subset m_x$ where $m_{f(x)} \subset \mathcal{O}_{Y,f(x)}$ and $m_x \subset \mathcal{O}_{X,x}$ are the respective maximal ideals.

We call $\kappa(x) \coloneqq \mathcal{O}_{X,x}/m_x$ the residue field of X at x.

Proposition 2.3.14. *The ringed space* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a locally ringed space that has stalks satisfying* $\mathcal{O}_{\operatorname{Spec} A,x} = A_{\mathfrak{p}_x}$ *where* $\mathfrak{p}_x$ *is the prime ideal associated to* $x \in \operatorname{Spec} A$ *and the localization* $A_{\mathfrak{p}_x} \coloneqq S^{-1}A$ *with* $S = A \setminus \mathfrak{p}_x$.

Proof. See [GW10, Theorem 2.33]. □

2.4 Schemes

Definition 2.4.1. A locally ringed space $(X, \mathcal{O}_X)$ is called an affine scheme if there exists a ring such that $(X, \mathcal{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is called the structure sheaf.

Remark 2.4.2. From a map of rings $f\colon A \to B$ we have a morphism of affine schemes $\operatorname{Spec}(f)\colon \operatorname{Spec} B \to A$ by sending the prime ideal $\mathfrak{p} \in \operatorname{Spec} B$ to the prime ideal $f^{-1}(\mathfrak{p}) \in \operatorname{Spec} A$. This corresponds to the sheaf morphism above. We also have a map from affine schemes to rings given by $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$.

Proposition 2.4.3. *The functors* $\operatorname{Spec}(-)$ *and* $\mathcal{O}_{-}(-)$ *define an anti-equivalence between the category of rings and the category of affine schemes.*

$$\mathcal{SA}ff \underset{\mathcal{O}_X(X)}{\overset{\operatorname{Spec} R}{\rightleftarrows}} \mathcal{R}ing.$$

Proof. See [GW10, Theorem 2.35]. □

This is a hallmark result that shows one can translate problems in algebra to problems in geometry and vice-versa. It demonstrates how so many geometric problems can be translated to problems in commutative algebra.

Definition 2.4.4. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ having an open covering $X = \cup_i U_i$ such that all locally ringed spaces $(U_i, \mathcal{O}_{U_i})$ are affine schemes.

Proposition 2.4.5. *Let X, Y be schemes and $\{U_i\}_i$ an open affine covering of X. Then a family of morphisms $\{U_i \to Y\}_i$ glues to a morphism $X \to Y$ if and only if the morphisms coincide on the intersections $U_i \cap U_j$ and the resulting morphism $X \to Y$ is uniquely determined.*

Proof. See [GW10]. □

Proposition 2.4.6. *Let $(X, \mathcal{O}_X)$ be a scheme*[4] *and $Y = \operatorname{Spec} A$ an affine scheme. Then there is a natural bijection:*

$$\operatorname{Hom}(X, \operatorname{Spec} A) \to \operatorname{Hom}(A, \mathcal{O}_X(X)).$$

Proof. Let $X = \bigcup_i U_i$ be an affine open covering. We know from 2.4.3 that the natural map

$$\operatorname{Hom}(U_i, Y) \to \operatorname{Hom}(A, \mathcal{O}_X(X))$$

is a bijection. For a general affine $V \subset U_i \cap U_j$ the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Hom}(U_i, Y) & \longrightarrow & \operatorname{Hom}(A, \mathcal{O}_{U_i}(U_i)) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(V, Y) & \longrightarrow & \operatorname{Hom}(A, \mathcal{O}_V(V)). \end{array}$$

[4]This is proven in [EGA71, Proposition 1.6.3] for $(X, \mathcal{O}_X)$ a locally ringed space.

Because of 2.4.5 we can glue the morphisms like in the bottom row of the diagram into a global bijection because of the sheaf property. □

Example 2.4.7. Let $X_1, \dots, X_n$ be a finite number of affine schemes and $X_i = \operatorname{Spec} A_i$. Then the disjoint union $\coprod X_i$ is also an affine scheme which is isomorphic to $\operatorname{Spec}(\prod A_i)$ by 2.4.3.

Example 2.4.8. If k is a field then the underlying topological space $\operatorname{Spec} k$ consists of a single point, and the stalk of the structure sheaf at this point is k. While it's a boring object topologically we will see later that it is an incredibly rich object as a scheme through the étale fundamental group.

More generally if $A \cong \bigoplus L_i$ is a k-algebra for finite separable extensions $k \subset L_i$, then $\operatorname{Spec} A$ is the disjoint union of the one point schemes $\operatorname{Spec} L_i$.

Example 2.4.9. The affine scheme $\operatorname{Spec} \mathbb{Z}$ has closed points corresponding to the prime numbers and a non-closed point, the ideal (0) called the generic point. The generic point is dense in $\operatorname{Spec} \mathbb{Z}$. The structure sheaf over an open set U is the ring of rational numbers with denominator divisible by the primes not touching U.

Example 2.4.10. Let $A = B[x_1, \dots, x_n]$ be the polynomial ring of a ring B in n variables. The affine n-space over B is then $\mathbb{A}^n_B := \operatorname{Spec}(A)$.

Example 2.4.11. If k is a field and A is a finitely generated k-algebra then the closed points of $X = \operatorname{Spec} A$ constitute what is traditionally called an affine variety. See [HAR77, Proposition 2.6]. The difference between the variety and scheme is that X contains non-closed points corresponding to non-maximal prime ideals.

Example 2.4.12. Given a scheme X and an open subset $U \subset X$. Then the locally ringed space $(U, \mathcal{O}_{X|U})$ is a scheme, an open subscheme of X. Furthermore, let $j : U \subset X$ be the topological inclusion that also defines a morphism of schemes. Then the morphism of sheaves $\mathcal{O}_X \to j_* \mathcal{O}_U$ is called an open immersion.

Example 2.4.13. A morphism $Z \to X$ of affine schemes is a closed immersion if it corresponds to a quotient map $A \to A/I$ by 2.4.3 for some ideal I. A general morphism is a closed immersion if the induced topological map is a closed immersion (a homeomorphism onto a closed subset) and the map of sheaves $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Z$ is surjective.

2.5 Fibre product

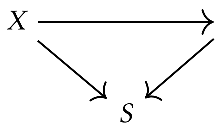
Let k be an algebraically closed field. Then the topological product of $\mathbb{A}^1_k \times \mathbb{A}^1_k$ is the product topology of two affine lines over k. As the open sets in a Zariski topology are complements of finite sets the product topology will be coarser than we would expect from $\mathbb{A}^2_k$. Indeed, the set $D(x - y)$ in is not open in the product topology. For this reason $\mathbb{A}^1_k \times \mathbb{A}^1_k$ is not isomorphic to $\mathbb{A}^2_k$ as schemes. For this reason we need another notion of products of schemes.

We will introduce a notion to be able to look at schemes relative to a base scheme.

Definition 2.5.1. Fix a scheme S. We define $\mathcal{S}ch/S$ to be the category of schemes over S. That is:

(i) Objects are morphisms $X \to S$ with the fixed target S.

(ii) A morphism from $X \to S$ to $Y \to S$ is a morphism $X \to Y$ that makes the diagram

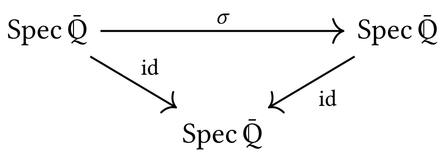


commute.

Remark 2.5.2. The schemes we've defined before are equivalent to $\mathcal{S}ch/\operatorname{Spec}\mathbb{Z}$. Generally we call a scheme in $\mathcal{S}ch/\operatorname{Spec}A$ an A-scheme. The advantage of working with schemes in the relative sense is that for an A-scheme the sections $\mathcal{O}_X(U)$ have a structure of an A-algebra.

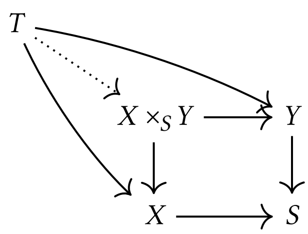
It is implicit that an S-scheme X comes with a structure morphism $X \to S$.

Remark 2.5.3. It is important to keep track of the base scheme. For example $\operatorname{Aut}_{\operatorname{Spec}(\bar{\mathbb{Q}})}(\bar{\mathbb{Q}})$ is trivial because there are no non-trivial σ such that



commutes. Meanwhile, $\operatorname{Aut}_{\operatorname{Spec}\mathbb{Q}}(\operatorname{Spec}\bar{\mathbb{Q}})$ is very rich.

Definition 2.5.4. Let X and Y be two S-schemes. Then the fibre product $X \times_S Y$ is the scheme defined by the universal property that given morphisms $Y \leftarrow T \to X$ that make the diagram commute:



there exists a unique morphism $T \to X \times_S Y$ that factors through the diagram.

Proposition 2.5.5. *The category of schemes has fibre products. In the affine case of* $X = \operatorname{Spec}A, Y = \operatorname{Spec}B, S = \operatorname{Spec}R$ *we have* $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$.

Proof. The idea of a proof is to show the proposition for affine schemes by traveling to the algebraic world through 2.4.3 and show the corresponding statement for the tensor product. The general case is done by using open covers $\{X_i\}, \{Y_i\}, \{S_i\}$ of X, Y, S and show that one can glue together the schemes $X_i \times_{S_i} Y_i$ into $X \times_S Y$ and have the diagram commute. See [HAR77] for the full proof. □

Remark 2.5.6. Note that the fibre product $X \times_S Y$ implicitly comes with morphisms $X \to S$ and $Y \to S$ that make the diagram above commute.

Remark 2.5.7. Recall that an S-scheme is a morphism of schemes $X \to S$. If we have a morphism $T \to S$ we would want to change the base of the S-scheme X to T. This is possible by pulling back $X \to S$ along $T \to S$ giving precisely $T \times_S X \to T$ which is often denoted as $X_T \to T$ and called the base change of X. This is one of the powerful features of schemes as opposed to classic varieties, where the fibre product does not exist.

To give examples of fibre product we will introduce the fibre of a scheme. But first we need some lemmas.

Lemma 2.5.8. *Let k be a field and X a scheme. Then giving a morphism* $\operatorname{Spec} k \to X$ *is equivalent to giving a point $x \in X$ and an extension $\kappa(x) \subset k$.*

Proof. Given $f\colon \operatorname{Spec} k \to X$ we obtain a point $x := f((0))$. The associated morphism $\mathcal{O}_{X,x}/m_x \to \mathcal{O}_{\operatorname{Spec} k,(0)}/m_{(0)}$ must be injective since $m_{(0)} \subset m_x$ by the local property of the morphism of locally ringed spaces. This gives injectivity of $\kappa(x) \to k$.

Conversely, we get the local morphisms $\mathcal{O}_{X,x} \to \kappa(x) \to k$ which induces a morphism of sheaves and hence our morphism $\operatorname{Spec} k \to X$ defined by inclusion. □

We get, in particular, the following result which allows us to speak of fibres of scheme morphisms since it is the natural scheme associated to a point of a scheme over a field.

Lemma 2.5.9. *Let X be a k-scheme. Then for any point $x \in X$ there is a canonical inclusion of schemes*

$$i\colon \operatorname{Spec} \kappa(x) \to X.$$

Definition 2.5.10. Let $f\colon X \to Y$ and a topological point $p \in Y$. The fibre of f at p is the scheme $X_p := X \times_Y \operatorname{Spec} \kappa(p)$.

Remark 2.5.11. The underlying topological space of the fibre of f at p is isomorphic to $f^{-1}(p)$ which also perhaps explains the name.

Example 2.5.12. Let k be a field of characteristic 0 and consider the morphism of schemes induced by $\phi_n\colon k[y] \to k[x]$ and $y \mapsto x^n$ for an integer $n > 1$. The corresponding map of schemes between $\phi_n^\#\colon X \to Y$ by letting $X = \operatorname{Spec} k[x], Y =$

Spec $k[y]$ gives raise to the fibres $X_p = X \times_Y \operatorname{Spec} \kappa(p)$. We have that at $k[x,y]/(x^n - y) \cong k[x]$ the fibre is given by

$$X_p = \operatorname{Spec} k[x,y]/(x^n - y) \times_{k[y]} \kappa(p)$$

for $p \in Y$.

If $p = (y)$ that is the prime ideal corresponding to $y = 0$ we get that $\kappa((y)) = k[y]/(y) = k$ and so $X_{(y)} = \operatorname{Spec} k[x]/(x^n)$ which is a non-reduced point of order n.

If we look at the situation over $k = \mathbb{Q}$ with a point $y \neq 0$ and $n = 2$ we get an interesting situation. The fibre of the point $y = 1$ is given by

$$\begin{aligned} X_{(y-1)} &= \operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - y) \times_{\mathbb{Q}[y]} \mathbb{Q}[y]/(y-1) \\ &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - y, y - 1) \\ &\cong \operatorname{Spec} \mathbb{Q}[x]/(x^2 - 1) \\ &\cong \operatorname{Spec} \mathbb{Q}[x]/(x-1) \coprod \operatorname{Spec} \mathbb{Q}[x]/(x+1) \end{aligned}$$

which is "unramified". However at $y = -1$ the fibre becomes

$$\begin{aligned} X_{(y+1)} &= \operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - y) \times_{\mathbb{Q}[y]} \mathbb{Q}[y]/(y+1) \\ &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - y, y + 1) \\ &\cong \operatorname{Spec} \mathbb{Q}[x]/(x^2 + 1) \\ &\cong \operatorname{Spec} \mathbb{Q}[i] \end{aligned}$$

which is a reduced point of which is a field extension of degree 2 over $\mathbb{Q}$ and thus "ramified".

This is an important example to keep in mind because it gives an example of an obstruction to the morphism ϕ_n being étale at the ramified point $y = 0$.

Example 2.5.13. The concept of a fibre allows us to see a morphism $f\colon X \to Y$ as a family of schemes X_p parametrized by Y or as a family of deformations X_p of some special fibre X_0.

A particularly enlightening case is when we have a family $X \longrightarrow \operatorname{Spec} \mathbb{Z}$. By taking the fibre over the generic point $(0) \in \operatorname{Spec} \mathbb{Z}$ we get a scheme $X_{(0)}$ defined over the residue field of (0), i.e. $\mathbb{Q}$. If we on the other hand take (p) for some prime number p we get a fibre $X_{(p)}$ defined over the finite field $\mathbb{Z}/p\mathbb{Z}$. This provides a glance into why schemes are important in number theory. The scheme X_p is called the reduction of X modulo p.

Definition 2.5.14. Let Ω be an algebraically closed field and X a scheme over S.

(i) (Geometric point) We call a morphism $\bar{s}\colon \operatorname{Spec} \Omega \to S$ a geometric point of S.

(ii) (Geometric point lying over) The unique image of $\bar{s}$ is a point $s \in S$ and $\bar{s}$ is the geometric point lying over s.

(iii) (Geometric fibre) The geometric fibre $X_{\bar{s}}$ is defined as the fibre product

$$X_{\bar{s}} := X \times_S \operatorname{Spec} \Omega$$

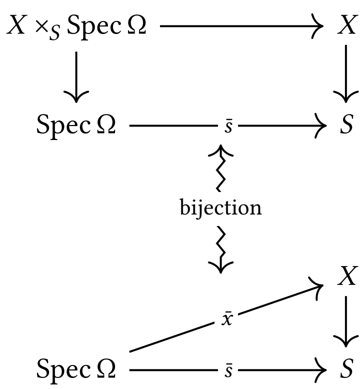
induced by $\bar{s}$ and the structure morphism of X as an S-scheme.

Remark 2.5.15. We will sometimes interchangeably view a geometric point $\bar{s}$ as a morphism and as the singleton set $\operatorname{Spec} \Omega$. Thus we will, by abuse of notation, denote a geometric morphism by $\bar{s} \to S$. Explicitly we have the identifications $k = \kappa(\bar{s})$ and $\bar{s} = \operatorname{Spec} \kappa(\bar{s})$ and $s = \operatorname{Im}(\bar{s})$.

Remark 2.5.16. We could define the geometric fibre as

$$\operatorname{Hom}_S(\operatorname{Spec} \Omega, X) = \{\bar{x} : \ \bar{x} \text{ is a lift of } \bar{s}\}$$

because of a natural bijection with the set of $X_{\bar{s}}$ using 2.5.12. The geometric fibre can be thus seen as the set of geometric points of X that are lifts of geometric points of S making the second diagram below commute:



Definition 2.5.17. The diagonal map coming from a morphism $X \to Y$ is defined to be $\Delta : X \to X \times_Y X$, defined by the identity morphisms $X \to X$. A morphism $X \to Y$ is separated if the diagonal map Δ is a closed immersion. The morphism $X \to Y$ is quasi separated if Δ is quasi compact.

2.6 Finiteness conditions

Finiteness conditions in algebraic geometry are frequently important as they are in commutative algebra. In a way they can be seen as abstractions of the notion of a variety in the classical sense being defined by a finite number of equations and variables.

We will review some of the notions that are useful in defining and understanding étale morphisms which will be our main point of study in later chapters.

Definition 2.6.1. We say that a scheme is *locally Noetherian* if it admits an affine open covering by the spectra of Noetherian rings.

We say that a scheme is *Noetherian* if it is locally Noetherian and quasi compact.

Definition 2.6.2. A ring homomorphism $A \to B$ is of *finite presentation* if B is isomorphic to $A[x_1, \dots, x_n]/I$ for some finitely generated ideal I. It is of *finite type* if there exists a surjection of A-algebras $A[x_1, \dots, x_n] \to B$.

A morphism of schemes $f\colon X \to Y$ is *locally of finite presentation* (respectively of *locally of finite type*) if for every affine open subscheme $V \subset Y$ and every affine open subscheme $U \subset f^{-1}(V)$ the induced morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation (respectively of finite type)

The morphism f is of finite type if in addition to being locally of finite type it is also quasi compact.

Example 2.6.3. The morphism $\coprod^{\infty} \mathbb{A}^1_k \to \mathbb{A}^1_k$ is of locally finite type but not finite type.

When can a morphism be defined by a finite number of polynomials? In the non-Noetherian situation it is often not enough to assume that morphisms are of finite type. Even if one can embed X as a closed subscheme of an affine space $\mathbb{A}^n_Y$ which is possible if $X \to Y$ is of finite type, it is possible that infinitely many equations are needed to define the subscheme X. But in the Noetherian situation they are thankfully equivalent.

Lemma 2.6.4. *For a morphism of schemes $f\colon X \to Y$ with Y locally Noetherian the following are equivalent:*

1. *f is locally of finite type*
2. *f is locally of finite presentation*

Proof. If f is locally of finite type then X is locally Noetherian, it follows that f is quasi-separated and hence of finite presentation. See [STA22, Lemma 01TX]. The other direction is clear from the definition. □

A notion that is slightly stronger than that of finite type is the following.

Definition 2.6.5. Let $f\colon X \to Y$ be a morphism of schemes. We call the morphism *affine* if Y has a covering by affine open subsets $\{V_i\}_i$ such that the open subschemes $f^{-1}(V_i)$ are affine schemes.

Definition 2.6.6. A ring morphism $A \to B$ is finite if B is finite as an A-module.

A morphism of schemes $f\colon X \to Y$ is finite if it is affine and the induced ring morphisms $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(f^{-1}(V_i))$ are finite for all i.

Example 2.6.7. For a field k the morphism $\operatorname{Spec}(k[x,y]/(x^n - y)) \to \operatorname{Spec} k[y]$ is a finite morphism since $k[x,y]/(x^n - y) \cong k[y] \oplus k[y][x] \oplus \dots \oplus k[y][x^{n-1}]$ as $k[y]$-modules.

Example 2.6.8. The structure morphisms $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^1_k \to \operatorname{Spec} k$ of finite type but not finite.

Example 2.6.9. Finite morphisms are affine by definition.

Remark 2.6.10. Intuitively finite type means that the fibres X_y of a morphism $f\colon X \to Y$ are finite dimensional and finite means they are of zero dimension. It is not true however that finite fibres implies the morphism being finite.

Proposition 2.6.11. *The notions of finite, (locally) finite type and locally finite presentation are all stable under composition and base change.*

Proof. See [GW10, Chapter 10] □

2.7 Kähler differentials

Our main goal of this section is to pave the way and define a form of calculus or differential structure on schemes based with the help of so called Kähler differentials. These will be a tool for the computation and determination of certain morphism of schemes being unramified which is at heart of this thesis. For that reason we need to make a slight detour and enlarge the category of locally ringed spaces in another direction.

2.7.1 Quasi-coherent Sheaves

Note that given a morphism of schemes $f\colon X \to Y$ the sheaf morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ makes $f_*\mathcal{O}_X(U)$ into a $\mathcal{O}_Y(U)$-module for each open subset $U \subset X$. For this and other reasons[5] we make the following definition.

Definition 2.7.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Then a sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{M}$ on X together with a morphism of sheaves $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ such that $\mathcal{M}(U)$ has a $\mathcal{O}_X(U)$-module structure for all open $U \subset X$.

In the case when $\mathcal{M}$ is an ideal in $\mathcal{O}_X(U)$ for all open $U \subset X$ we say it's a sheaf of ideals on X.

The following construction gives any affine scheme a module structure.

Definition 2.7.2. Let M be an A-module. Define the presheaf $\tilde{M}$ on $X = \operatorname{Spec} A$ on the basis of distinguished opens by

$$\tilde{M}(D(f)) := M \otimes_A A_f = M_f.$$

Proposition 2.7.3. *Let $\tilde{M}$ be as in the definition above. Then there is a unique $\mathcal{O}_X$-module $\tilde{M}$ satisfying*

$$\tilde{M}(D(f)) := M \otimes_A A_f = M_f.$$

Proof. One uses completely similar arguments to 2.3.8. □

[5]One of the strengths of the category of $\mathcal{O}_X$-modules is that it forms an abelian category just like the category of A-modules.

Definition 2.7.4. Let X be a scheme. A quasi-coherent sheaf $\mathscr{M}$ on X is a sheaf of $\mathscr{O}_X$-modules for which there is an open covering $\{U_i\}$ of X by open affines $U_i = \operatorname{Spec} A_i$ and A_i-modules M_i such that $\mathscr{M}_{|U_i} \cong \tilde{M}_i$.

If each M_i is finitely generated over A_i then $\mathscr{M}$ is a coherent sheaf. Moreover $\mathscr{M}$ is locally free if we can choose the datum of a quasi-coherent sheaf in such a way that M_i are free A_i-modules.

The class of affine morphisms is a sufficient restriction on the morphism for the pushforward $f_*\mathscr{O}_X$ to be a quasi-coherent sheaf as seen by this lemma for the case $\mathscr{F} = \mathscr{O}_X$.

Lemma 2.7.5. *Let $f\colon X \to Y$ be an affine morphism and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $f_*\mathscr{F}$ is a quasi-coherent sheaf on Y.*

Proof. This is true because the inverse image of a distinguished open set is a distinguished open. The one uses the affine condition to glue the sheaf together over general opens. □

2.7.2 Kähler Differentials

We are now ready to briefly introduce the algebraic analogue of differential forms found in differential geometry based on the exposition in [Qin02].

Definition 2.7.6. Let $A \to B$ be a morphism of rings. The B-module $\Omega^1_{B|A}$ of Kähler differentials is defined as the free B-module on the set of formal symbols $\{db\}_{b\in B}$ divided by the submodule generated by the following relations for $b_1, b_2 \in B$:

1. $d(b_1 + b_2) = d(b_1) + d(b_2)$.
2. $d(b_1 b_2) = b_1 d(b_2) + d(b_1) b_2$ which is the Leibniz rule.
3. $d(a) = 0$ for all $a \in A$.

Example 2.7.7. Let A be a ring and B be the polynomial ring $A[x_1, \dots, x_n]$. Then $\Omega^1_{B|A}$ is the free B-module generated by the symbols $\{dx_i\}$. To see that they are free we have a B-module morphism $f_i : \Omega^1_{B|A} \to B$ with $dx_j \mapsto \partial x_j/\partial x_i = \delta_{ij}$ for all i. Assume that $P = \sum P_i dx_i = 0$ then $f_i(P) = P_i = 0$ for all i.

For a polynomial we get $dP(x_1, \dots, x_n) = \sum \partial P/\partial x_i dx_i$ as expected.

Example 2.7.8. If B is a localization or a quotient of A then $\Omega^1_{B|A} = 0$. Indeed if $A \to B$ is surjective then $d(b) = ad(1) = 0$ for all $a \in A$ that is the inverse image of b. Suppose that $B = S^{-1}A$ is a localization of A then for any $b \in B$ there is a t such that $tb \in A$ and so $td(b) = d(tb) = 0$ and so $d(b) = 0$ because t is invertible in b.

Proposition 2.7.9. *We have the following properties for the module of Kähler differentials with $A \to B$ a ring morphism and A' an A-algebra.*

1. *(Base change)* $\Omega^1_{B\otimes_A A'|A'} \cong \Omega^1_{B|A} \otimes_A A'$

2. *(Localization) Let S be a multiplicative subset of B. Then* $\Omega^1_{B_S|A} \cong \Omega^1_{B|A} \otimes_B B_S$.

3. *Let $\psi\colon B \to C$ be a ring morphism, then the following sequence is exact*

$$\Omega^1_{B|A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C|A} \to \Omega^1_{C|B} \to 0$$

4. *Let $C = B/I$ then the following sequence is exact*

$$I/I^2 \to \Omega^1_{B|A} \otimes_B C \to \Omega^1_{C|A} \to 0$$

where the first morphism sends $b \mapsto db \otimes 1$.

Proof. The morphism α is defined as $d(b) \otimes c \mapsto cd(\psi(b))$ See [Qin02, Chapter 6. Proposition 1.8] for the full proof. □

We are interested in separable field extensions since they are the building blocks for étale algebras which in turn help us define étale morphisms. We will see that this generalizes when we define those.

Corollary 2.7.10. *Let $k \subset L$ be a finite extension with $L = k(\alpha) = k[x]/(f)$ for a minimal polynomial f of α. Then we have*

1. $\Omega^1_{L|k} \cong L$ *for L non-separable over k.*

2. $\Omega^1_{L|k} = 0$ *for L separable over k.*

Proof. By the last item in the proposition above we see that

$$(f)/(f)^2 \to Ldx \to \Omega^1_{L|k} \to 0.$$

The first morphism maps $f \mapsto \overline{f'(x)}dx = f'(\alpha)dx$ where $\overline{f'(x)}$ is the image of $f'(x)$ in L. If $k \subset L$ is separable then $f'(\alpha) \neq 0$ and thus $\Omega^1_{L|k} = 0$. If it is not separable then $f'(x) = 0$ and $\Omega^1_{L|k} = Ldx$. □

We can find a unique quasi-coherent sheaf $\tilde{\Omega}^1_{B|A}$ on $\operatorname{Spec} B$ that satisfies

$$\tilde{\Omega}^1_{B|A}(D(f)) \cong \Omega^1_{B_f|A}$$

for all distinguished open sets $D(f) \subset \operatorname{Spec} B$ from by the help of the localization part of 2.7.9. We will use this to travel from the local of affine schemes to the global of schemes as follows.

Proposition 2.7.11. *Let $f\colon X \to Y$ be a morphism of schemes. Then there exists a unique quasi-coherent sheaf $\Omega^1_{X|Y}$ on X such that for all affine open* $\operatorname{Spec} A \subset Y$ *and* $U \subset f^{-1}(\operatorname{Spec} A)$ *with $x \in U$ we have*

$$\Omega^1_{X|Y}|U \cong \tilde{\Omega}^1_{\mathcal{O}_X(U)/A} \quad \textit{and} \quad (\Omega^1_{X|Y})_x \cong (\tilde{\Omega}^1_{\mathcal{O}_X(U)/A})_x.$$

Proof. One first defines the stalks of the sheaf and then uses an étale space type argument by forcing the collection of stalks to be compatible and form a sheaf for arbitrary opens. See [Qin02, Chapter 6, Proposition 1.26] for the full proof. □

Example 2.7.12. If $X = \mathrm{Spec}(A[x_1, \ldots, x_n])$ over a ring A then $\Omega^1_{X|\,\mathrm{Spec}\,A} \cong \mathcal{O}_X^n$.

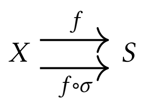
Remark 2.7.13. The diagonal morphism $\Delta : X \to X \times_Y X$ is locally a closed immersion and $\Omega(X)$ is closed in an open subset U of $X \times_Y X$. If $\mathcal{I} = \ker(\Delta^\#)$ is the sheaf of ideals defining the closed subset $\Delta(X)$ then one can show that $\Omega^1_{X|Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$ where Δ^* is the pullback of the $\mathcal{O}_{\Delta(X)}$-module $\mathcal{I}/\mathcal{I}^2$ along $X \xrightarrow{\sim} \Delta(X)$.

Definition 2.7.14. For $f\colon X \to Y$ the quasi-coherent sheaf $\Omega^1_{X|Y}$ defined in the proposition above is called the sheaf of Kähler differentials of X over Y.

2.8 Quotients of schemes

We need the theory of quotients of schemes to be able to speak of a Galois correspondence of Galois covers later on in the text.

Definition 2.8.1. Given a morphism of schemes $f\colon X \to S$ define $\mathrm{Aut}(X/S)$ be the group of automorphisms $\sigma\colon X \to X$ for which the diagram below commutes:



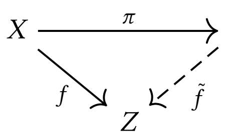
Remark 2.8.2. Let $\bar{s} : \mathrm{Spec}\,\Omega \to S$ be a geometric point, then there is a natural left action of $\mathrm{Aut}(X/S)$ on X by $\sigma \cdot x := \sigma(x)$. This extends to an action on any fibre $X_{\bar{s}} = X \times_S \mathrm{Spec}\,\Omega$ by acting on the first fibre product term.

Remark 2.8.3. There are many interesting actions on schemes. Let $k \subset \bar{k}$ be a Galois extension with $\bar{k}$ the algebraic or separable closure. For $\sigma \in Gal(k)$ we have a corresponding natural automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}\,\bar{k} \to \mathrm{Spec}\,\bar{k}$. Because $\mathrm{Spec}(\sigma_1) \circ \mathrm{Spec}(\sigma_2) = \mathrm{Spec}(\sigma_2 \circ \sigma_1)$ we have a left action of $Gal(k)^{\mathrm{op}}$ on $\mathrm{Spec}\,k$. This also extends to a $Gal(k)^{\mathrm{op}}$ action on the fibres $X_{\bar{s}}$.

Definition 2.8.4 (Categorical quotient)**.** Let G be a finite group acting on a scheme X. A quotient scheme X/G consists of a scheme Y and morphism $\pi\colon X \to Y$ satisfying the universal property:

(i) G-invariance. That is For every $\sigma \in G$ we have $\pi = \pi \circ \sigma$ as morphism of locally ringed spaces.

(ii) Any morphism $\psi\colon X \to Z$ satisfying (i) factors uniquely through π.



Proposition 2.8.5. *Let G be a finite group acting on a scheme X. The quotient scheme X/G exists if and only if every point $x \in X$ has an affine open neighborhood U that is stable under G. That is for all $\sigma \in G$ and $x \in X$ we have $\sigma \cdot x \in U$.*

Proof. [SGA71, Exposé V, Proposition 1.8]. □

The quotient scheme does not always need to exist in the category of schemes as is evident by the proposition above. This is in fact one of the reasons why algebraic spaces and later stacks were introduced. In the case of stacks one also keeps track of isomorphisms between objects in the category and not only the objects themselves. See [FAN] for an introduction to stacks.

A special case of the above theorem is when the group action on the scheme X is "nice enough" in a way that every point has a neighborhood where G won't act transitively across the orbits.

We won't need the full theorem above but are satisfied with the following weaker version for our purposes.

Remark 2.8.6. Let $f\colon X \to S$ be an affine and surjective morphism of schemes and $G \subset \mathrm{Aut}_S(X)$ a finite subgroup. We can define a ringed space $(X/G, \mathcal{O}_{X/G})$ by putting on the quotient topology on X by $\pi\colon X \to X/G$. G then acts on X. We also get a canonical group action of G on $\mathcal{O}_X$ by $\sigma \in G$ inducing a sheaf morphism $\mathcal{O}_X \to \sigma_* \mathcal{O}_X$. Define $\mathcal{O}_{X/G} := (\pi_* \mathcal{O}_X)^G$ where the right part are the G-invariant elements of $\pi_* \mathcal{O}_X$.

The property of G-invariance means explicitly that $\phi(x \cdot \sigma) = \phi(x)$ as topological spaces and $(\phi \cdot \sigma)^\# = (\phi_* \sigma^\#) \circ \phi^\#$ as maps of sheafs due to the slightly unexpected composition of the pushforward.

For a simple example $X = \operatorname{Spec} A$ an affine scheme of a ring A and $G \subset \mathrm{Aut}(A)$. The quotient scheme is simply $X/G = \operatorname{Spec}(A^G)$ where A^G are the elements of A that are invariant under the action of G.

Proposition 2.8.7. *Let $f\colon X \to S$ be an affine and surjective morphism and G a finite subgroup of the automorphism group $\mathrm{Aut}_S(X)$. Then the quotient scheme X/G exists with underlying set as the set of orbits of X under G and the structure sheaf the subsheaf of G-invariants of the pushforward $\mathcal{O}_{X/G} := (\pi_* \mathcal{O}_X)^G$ and $\pi\colon X \to X/G$ is the quotient topology morphism.*

Proof. [SZA09, Proposition 5.3.6] □

2.9 Representations of schemes

The famous Hilbert Nullstellensatz carries many forms and is central in geometry over algebraically closed fields. Let X is an affine variety over an algebraically closed field k. The functor[6] that sends X to its set of points over k, that is $X \mapsto \mathrm{Hom}(\operatorname{Spec} k, X)$, is faithfull due to the Nullstellensatz. Faithfullness allows us to

[6] Also called the functor of points.

talk about morphisms between affine varieties over k as functions between the sets of points of the varieties.

Definition 2.9.1. We call a category $\mathscr{C}$ with a faithful functor $F\colon \mathscr{C} \to \mathcal{S}et$ a concrete category.

The functor usually maps an object in $\mathscr{C}$ to the underlying set. Faithfulness means that morphisms in $\mathscr{C}$ can be identified with morphisms of sets. It is only satisfactory to call $\operatorname{Hom}(\operatorname{Spec} k, X)$ the set of points of X over k if the functor above is faithful. This is because in that case $\operatorname{Spec} k$ is a one point set and so $\operatorname{Hom}(\operatorname{Spec} k, X)$ can be identified with X by faithfulness. This fails badly as soon as we leave the space of algebraically closed fields. The affine scheme $X = \operatorname{Spec} \mathbb{R}[x]/(x^2+1)$ has no points over $\mathbb{R}$, this is equivalent to the functor of points not being able to distinguish into or out of X.

The functor $F\colon \mathscr{C} \to \mathcal{S}et$ can fail to be faithful if the morphisms in $\mathscr{C}$ have extra structure that is not visible at the level of the underlying sets, for example the category of CW-complexes or homotopy classes of morphisms which require a "higher level" of "bookkeeping".

Definition 2.9.2. Let $\mathscr{C}$ be a category. Fix an object $C \in \mathscr{C}$, a functor[7] $F\colon \mathscr{C}^{\mathrm{op}} \to \mathcal{S}et$ to the category of sets is representable if there is an object $C \in \mathscr{C}$ and a natural isomorphism of functors

$$\operatorname{Hom}(-, C) \xrightarrow{\sim} F.$$

We say that C represents F and that F is representable if it is represented by some C.

Example 2.9.3. The global sections functor $\mathcal{O}_{-}(-)$ is represented by $\mathbb{A}^1$. To see this let $X = \operatorname{Spec} \mathbb{Z}[x]$ then for every scheme T we have

$$\operatorname{Hom}(T, \mathbb{A}^1) \xrightarrow{\sim} \operatorname{Hom}(\mathbb{Z}[x], \mathcal{O}_T(T)) \cong \mathcal{O}_T(T)$$

by 2.4.6.

Example 2.9.4. More generally we see that if A is a ring and $f \in A[x]$ and let $X = \operatorname{Spec}(A[x]/(f))$ and T be a scheme over $\operatorname{Spec} A$, then we have

$$\operatorname{Hom}(T, X) \xrightarrow{\sim} \operatorname{Hom}(A[x]/(f), \mathcal{O}_T(T)) \cong \{t \in \mathcal{O}_T(T) : f(t) = 0\}$$

where the last isomorphism sends an A-algebra morphism ϕ to $\phi(x)$. The set $\operatorname{Hom}(T, X)$ is called the T-valued points of X and is denoted by $X(T)$. So the functor $X(-)$ that sends a scheme $T \mapsto X(T)$ is represented by $\operatorname{Spec}(A[x]/(f))$.

Lemma 2.9.5 (Yoneda's lemma)**.** *For a functor $F\colon \mathscr{C}^{op} \to \mathcal{S}et$ from a category $\mathscr{C}$ the map*

$$\operatorname{Hom}(\operatorname{Hom}_{\mathscr{C}}(-, X), F) \to F(X), \quad \phi \mapsto \phi(X)(id_X)$$

is bijective and functorial in X.

[7]Such a functor is by definition a presheaf

Proof. For $\phi \in F(X)$ define $\alpha_\phi(Y): Hom_{\mathscr{C}}(Y, X) \to F(Y)$ by $f \mapsto F(f)(\phi)$ for $f \in Hom_{\mathscr{C}}(Y, X)$. Then $\phi \mapsto \alpha_\phi$ is an inverse map. □

One of the hallmarks of category theory is the Yoneda lemma. It demonstrates why representability is so powerful because it shows that we can embedd the category of schemes into the category of such functors and thus allows us to consider schemes as functors instead. The schemes that lie in the essential image of the Yoneda embedding are the ones which are representable.

In fact one can show that the points of a scheme can be recovered from all k-valued points.

Proposition 2.9.6. *Let X be a scheme, then*

$$|X| = \coprod_{\text{Fields } k} \operatorname{Hom}(\operatorname{Spec} k, X)/\sim = \coprod_{\text{Fields } k} X(k)/\sim,$$

where two extensions $\operatorname{Hom}(\operatorname{Spec} k, X)$ *and* $\operatorname{Hom}(\operatorname{Spec} k', X)$ *are equivalent whenever there's a third field k'' such that* $\operatorname{Spec} k'' \cong \operatorname{Spec} k' \times \operatorname{Spec} k$.

Proof. See [STA22, Lemma 01J9]- □

It will be necessary that we allow for an enlargement for the category we work in, in a sense. We will see that we will want to allow for colimits of representable functors and arrive at the following definition due to Grothendieck.

Definition 2.9.7 (Pro-representable)**.** We say that the functor $F: \mathscr{C} \to \mathscr{S}et$ is pro-representable by $\tilde{P}$ if there is a cofiltered diagram $\tilde{P}: \mathscr{I} \to \mathscr{C}$ indexed by a directed set $\mathscr{I}$ and a natural isomorphism

$$\operatorname{Hom}(\tilde{P}, -) := \varinjlim_{i \in \mathscr{I}} \operatorname{Hom}(P_i, -) \xrightarrow{\sim} F.$$

Remark 2.9.8. Note that the colimit $\varprojlim_{i \in \mathscr{I}} P_i$ does not need to exist in the category $\mathscr{C}$ but the limit $\varinjlim_{i \in \mathscr{I}} \operatorname{Hom}(P_i, C)$ for $C \in \mathscr{C}$ does exist in $\mathscr{S}et$. If the colimit does exist and equals P then

$$\varinjlim_{i \in \mathscr{I}} \operatorname{Hom}(P_i, -) \cong \operatorname{Hom}(\varprojlim_{i \in \mathscr{I}} P_i, -) \cong \operatorname{Hom}(P, -) \xrightarrow{\sim} F$$

so that F is actually representable.

Chapter 3

Étale morphisms

3.1 Finite étale morphisms

In this section and going forward we require our schemes to be locally Noetherian to simplify some proofs and definitions, we will come back to this in a moment.

This section is based largely upon [SZA09, Chapter 5.2] but restructured and with additional material. Our aim is two-fold. On one hand to develop a corresponding notion to a local isomorphism in differential geometry and on the other to develop an algebraic analogue of a finite topological cover. It's good to keep those two goals in mind as we get to know our main protagonists, étale morphisms. We will see throughout the coming sections how these goals are gradually fulfilled. We will also demonstrate the connection to a more modern definition in terms of non-ramification and flatness.

Definition 3.1.1. Let $f\colon X \to S$ be a finite morphism of schemes. It is *finite locally free* if the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module.

Remark 3.1.2. This means that for open subsets $U \subset S$ we have $(f_*\mathcal{O}) \mid U \cong \mathcal{O}_U^n$ for some n depending on U.

Definition 3.1.3. Let A be an algebra that is finite dimensional over k. We say that A is étale over k if $A \cong \prod_i^n L_i$ where $k \subset L_i$ are separable extensions of k. When k_i are finite extensions we say that A is a finite étale algebra.

Remark 3.1.4. Étale algebras naturally occur during base change, specifically that of a geometric fibre.

Proposition 3.1.5. *Let A be a finite dimensional k-algebra. Then the following are equivalent:*

1. *A is étale over k.*
2. *$A \otimes_k \bar{k} \cong \bar{k}^n$.*
3. *$\Omega^1_{A|k} = 0$.*

Proof. Suppose that $A \cong \prod_i^n L_i$ for L_i finite and separable extensions of k. Further by the primitive element theorem we can find elements $\alpha_i \in L_i$ such that $L_i = k(\alpha_i)$ with p_i the minimal polynomial of α_i. For a fixed i we have

$$L_i \otimes_k \bar{k} \cong \bar{k}[x]/(p(x)) \cong \prod_i^n \bar{k}[x]/(x - a_i) \cong \prod_i^n \bar{k},\ a_i \in \bar{k}.$$

The second isomorphism is because of the "Chinese remainder theorem" and L_i being separable over k the minimal polynomial p_i has coprime factors. The other direction is found in [SZA09, Proposition 1.5.6]

For (1) ⇒ (3) we have $A \cong \prod_i^n L_i$ with L_i finite separable extensions of k. The maximal ideals m_i of A are the kernels of the projections onto each factor. Then $\Omega^1_{A|k} = 0$ if and only if $(\Omega^1_{A|k})_{m_i} = \Omega^1_{L_i|k} = 0$ for all m_i. This holds because the L_i are finite separable extensions, see 2.7.10.

For (3) ⇒ (1) see [SZA09, Proposition 5.1.31]. □

Definition 3.1.6. Let $f\colon X \to S$ be a finite locally free morphism of schemes. It is *finite étale* if each fibre X_s of f is the spectrum of a finite étale $\kappa(s)$-algebra.

A surjective finite étale morphism is called an étale cover.

Remark 3.1.7. In the literature one can frequently see finite étale morphisms and finite étale covers be used interchangeably. A cover in our terminology is strictly a surjective morphism.

The following lemma shows us that any finite étale morphism $X \to S$ is automatically an étale cover when S is connected and X is non-empty.

Lemma 3.1.8. *Let $f\colon X \to S$ be a finite and locally free morphism with S connected. Then $f(X)$ is both open and closed.*

Proof. Since f is finite $f(X)$ is closed. Furthermore f open because we can find a neighborhood N of $f(x)$ for $x \in X$ such that $(f_*\mathcal{O}_X)_s \cong \mathcal{O}_U^n \neq 0$ for all $s \in N$ and some $U \subset S$. If $s \notin f(X)$ then since the complement of $f(X)$ is open we can find a neighborhood $U \subset N$ such that $f^{-1}(U) = \emptyset$. Contradicting the non-zero of support assumption above. □

Corollary 3.1.9. *Let $f\colon X \to S$ be a finite étale morphism with S connected. Then f is an étale cover if and only if X is non-empty.*

Proof. Using above lemma this means that f is surjective if and only if X is non-empty because S is connected. □

Example 3.1.10. Let $K \subset L$ be a field extension. Then the corresponding morphism of affine schemes $\operatorname{Spec} L \to \operatorname{Spec} K$ is finite and locally free of degree $\dim_K(L)$. It is finite étale if and only if the field extension is finite and separable. This shows that finite étale morphisms generalizes the notion of finite and separable field extensions.

Example 3.1.11. Let k be a field of characteristic zero and $\phi_n : k[y] \to k[x]$ with $y \mapsto x^n$ for $n > 1$. Then the morphism of affine schemes is finite and locally free of degree n. But it's not étale because the fibre at the origin is not a finite étale $\kappa(0)$-algebra as we saw in 2.5.12.

Example 3.1.12. Let A be a ring and $B = A[x]/(f)$ and $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ for f a monic polynomial of degree d. B is patently a finitely generated A-module with the images of the monomials x^i so the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite and locally free. If we further assume that $(f, f') = A[x]$ then the morphism is finite étale. The fibre X_p over a prime $p \in S$ is by definition the spectrum of $B \otimes_A \kappa(p) \cong \kappa(p)[x]/(f)$. We only have, by assumption, simple roots and so by the Chinese Remainder Theorem it is a finite étale $\kappa(p)$-algebra.

Example 3.1.13. Let $x \mapsto x^2$ be a morphism. The associated morphism $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is an étale cover because we removed the ramified point at the origin. But it is not a covering in the Zariski topology.

3.2 Étale morphisms

We will now draw the connection to more modern definitions of finite étale morphisms that generalize to not necessarily finite étale morphisms easier which is used in the theory of étale cohomology. We do so to make it easier for the reader to compare the different notions presented here with other sources. Furthermore, we also explain the notion of ramification that we have only alluded to earlier.

3.2.1 Unramified morphisms

Recall that for a morphism of Riemann surfaces $f\colon X \to Y$ we have an associated map between the fields of meromorphic functions $\mathcal{M}(X) \to \mathcal{M}(Y)$. The fields contain stalks of holomorphic functions at x and $f(x)$. That is $\mathcal{O}_{X,x} \subset \mathcal{M}(X)$ and $\mathcal{O}_{Y,f(x)} \subset \mathcal{M}(Y)$, respectively. We get an induced morphism of stalks $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ with the maximal ideal of holomorphic functions vanishing at a point $m_{f(x)}$ mapping into the corresponding maximal ideal m_x. In fact considering the local structure of morphism between Riemann surfaces we have $m_{f(x)}\mathcal{O}_{X,x} = m_x^{e_x}$ where e_x is the ramification index of f at x which corresponds to the number of branches of f. The points for which $e_x > 1$ are called branching points. Topologically a branching point has no neighborhood where f is a local isomorphism. The function f is unramified at x if $e_x = 1$.

Another motivating example comes from the land of algebraic number theory. When $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$ are affine and the morphism comes from an integral extension of Dedekind domains $A \subset B$. It can then be shown that $f\colon X \to S$ is an étale cover, see [SZA09, Lemma 5.2.4]. The points of the fibre over $\mathfrak{p} \in S$ correspond to factors $\mathfrak{p}_i$ in the decomposition of the extension

$$\mathfrak{p}B = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

In particular $X_{\mathfrak{p}}$ is finite étale if and only if all the e_i are equal to 1. In algebraic number theory one says that the prime ideal $\mathfrak{p}$ is unramified in this case. We also see that localizing f at $\mathfrak{p}_i$ we get

$$\mathfrak{p}B_{\mathfrak{p}_i} = \mathfrak{p}_1 B_{\mathfrak{p}_1}^{e_1} \cdots \mathfrak{p}_r B_{\mathfrak{p}_r}^{e_r} = \mathfrak{p}_i B \mathfrak{p}_i{}^{e_i},$$

in other words f is unramified at $\mathfrak{p}_i$ if and only $e_i = 1$.

With these examples as motivation and adding a separability criterion we arrive at the following definition:

Definition 3.2.1. A morphism of locally of finite type $f\colon X \to S$ is unramified at $x \in X$ if for the induced map of local rings $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ we have:

1. The maximal ideal $m_{f(x)} \subset \mathcal{O}_{S,f(x)}$ generates the maximal ideal $m_x \subset \mathcal{O}_{X,x}$, that is $m_{f(x)}\mathcal{O}_{X,x} = m_x$.
2. The corresponding field extension $\kappa(x)/\kappa(f(x))$ is separable.

If f is unramified at all points in X we say that f is unramified.

Remark 3.2.2. In [EGA71] and several other sources the notion of an unramified morphism is required to be locally of finite presentation and not locally of finite type. The reason why we require our schemes to be locally Noetherian is so that we can forget about technicalities of some notions of unramified morphisms or étale morphisms only being true if they are assumed to be locally of finite type or locally of finite presentation[1]. Proposition 2.6.4 shows that under a locally Noetherian base we don't need to care about this distinction.

Proposition 3.2.3. *Let $f\colon X \to S$ be a finite and flat morphism. The following are equivalent:*

(i) The morphism f is étale.

(ii) The sheaf of relative differentials vanishes, i.e. $\Omega^1_{X/S} = 0$.

(iii) The diagonal morphism $\Delta\colon X \to X \times_S X$ coming from f is an isomorphism of X onto an open and closed subscheme of $X \times_S X$.

Proof. We outline the proof for (i) ⇒ (ii), the rest of the directions are found in [SZA09, Proposition 5.2.7]. It suffices to show that $(\Omega^1_{X|S})_x = 0$ for all $x \in X$. Since stalks are local we may assume that $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$ and the sheaf morphism makes $\Omega^1_{X|S}$ into a finitely generated B-module. Now we have reduced the problem to that of $M := \Omega^1_{B_q|A_p}$ being zero for every $q \in \operatorname{Spec} B$. The ring B_q is local with the maximal ideal qB_q and by Nakayama's lemma

$$M \otimes_{B_q} \kappa(q) = M/qM = 0$$

[1]For example the description of unramified morphism in terms of fibres or in terms of vanishing of Kähler differentials require locally of finite type

if and only if $M = 0$. So we have reduced it to prove $\Omega^1_{B_q|A_p} \otimes_{B_q} \kappa(q) = 0$. But because the morphism f is ramified we must have $B_q \otimes_{A_p} \kappa(p) = \kappa(q)$. This leads to the chain

$$\Omega^1_{B_q|A_p} \otimes_{B_q} \kappa(q) = \Omega^1_{B_q|A_p} \otimes_{B_q} (B_q \otimes_{A_p} \kappa(p)) = \Omega^1_{B_q \otimes_{A_p} \kappa(p)|\kappa(p)} = \Omega^1_{\kappa(q)|\kappa(p)} = 0.$$

The last equality is due to 2.7.10 and the second is due to 2.7.9. □

The third condition gives us a practical criterion to check for a map being finite étale.

Example 3.2.4. Let $f\colon \operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ be the natural map. Then since $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$ we have that by the short exact sequence in 2.7.9

$$\Omega^1_{\mathbb{Z}[i]/\mathbb{Z}} \cong \mathbb{Z}[i]dx/\mathbb{Z}[i]df \cong \mathbb{Z}[i]/(2\mathbb{Z}[i])$$

because $df = d(x^2+1) = 2dx$ and by the exact sequence in 2.7.9.

3.2.2 Flat morphisms

One doesn't get very far in algebraic geometry before hearing the notion of "flatness". It's often what one has in mind when one looks for a "nice" morphism of schemes. The definition is somewhat algebraic and non-geometric. But miracles happen when you observe it in the geometric world, after all for a morphism $X \to Y$ it ensures that the fibres X_y are "deformed" continuously as $y \in Y$ varies.

Definition 3.2.5. An A-module B is flat if the functor $- \otimes_A B$ is exact. A ring morphism $A \to B$ if B is flat as an A-module.

A morphism of schemes $f\colon X \to Y$ is flat at $x \in X$ if the corresponding map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say that f is flat if it is flat at all points $x \in X$.

Example 3.2.6. Flatness show up naturally in the nature, like:

1. Most things over a field are flat, like field extensions seen as vector spaces.
2. Free and thus also projective modules are flat.
3. The localization of a ring $S^{-1}R$ has a natural map from R which makes it a flat R-module and thus gives rise to a flat morphism of schemes.

Definition 3.2.7. We call a finite morphism of schemes $f\colon X \to Y$ étale if it is flat and unramified.

Remark 3.2.8. We need Y to be locally Noetherian for this notion of étale and finite together be equivalent to *finite étale* above in 3.1.6. But as long as we assume Y is so, we can move interchangeably between the definitions, as the following proposition shows.

Proposition 3.2.9. *A morphism $f\colon X \to Y$ where Y is locally Noetherian is finite étale (as in 3.1.6) if and only if it is finite and étale (as in 3.2.7).*

Proof. See [LEN, Proposition 6.9]. □

Schemes over a field k are automatically flat. This allows for an easier description of étale morphisms for schemes over k. Compare it with the definition of an étale algebra over k in 3.1.3.

Proposition 3.2.10. *Let k be a field and X a scheme over k. Then the following are equivalent:*

(i) $X \to \operatorname{Spec} k$ *is étale.*

(ii) $X \to \operatorname{Spec} k$ *is unramified.*

(iii) $X \cong \coprod^n \operatorname{Spec} L_i$*, for $k \subset L_i$ finite and separable field extensions.*

Proof. Over a field flatness is automatic, so we have equality of the first two points. It is clear that the last implies unramified.

So we need to show that being étale implies the last. X is of locally of finite type. Since the property is local on X we may reduce it to $X = \operatorname{Spec} A$ for a finitely generated k-algebra A. By the definition of unramified we have that the local rings $\mathcal{O}_{X,x}$ are finite separable extensions of k and so $\dim(A) = 0$. For that reason A is Artinian and a finite product of local Artinian rings each of which is a finite separable extension of k. See [AM69, Theorems 8.5 and 8.7]. □

3.2.3 Comparison to topological covers

For the notion of étale cover to be an algebraic generalization of its topological cousin we need to check some geometric properties.

Proposition 3.2.11. *Let $f\colon X \to S$ be a finite étale morphism of schemes and $\bar{s}$ a geometric point of S. The cardinality of the geometric fibre $X_{\bar{s}}$ is equal to the rank of the stalk $(f_*\mathcal{O}_X)_{\bar{s}} \cong \mathcal{O}_{S,s}^{n_s}$. The rank is a locally constant function on S, it is in particular constant if S is connected.*

Proof. By the definition of a finite locally free morphism we see that there exists a neighborhood for every point in S, that is $s \in U \subset S$, such that $(f_*\mathcal{O}_X)$ restricted to U is free of rank n_s. The stalk of every point in U will be free of rank n_s over $\mathcal{O}_S$ and thus locally constant. It is constant if S is connected by the definition of a connected topological set.

By 3.1.5 we see that the cardinality of $X_{\bar{s}}$ is equal to the rank of the stalk, n_s, at any geometric point of S. □

Definition 3.2.12. Let $f\colon X \to S$ be an étale cover we call it a trivial cover if:

1. X is isomorphic to a finite disjoint union of copies of S as an S-scheme.

2. f restricts to the identity map on each component.

If a scheme S has no non-trivial étale covers we say it is simply connected.

Proposition 3.2.13. *Let S be a connected scheme and $\phi\colon X \to S$ an affine surjective morphism. Then ϕ is a finite étale cover if and only if there is a finite locally free and surjective morphism $\psi\colon Y \to S$ such that $X \times_S Y$ is a trivial cover of Y.*

Proof. See [SZA09, Proposition 5.2.9]. □

We have thus shown that étale covers $f\colon X \to S$ over S connected behave similarly to finite topological covers and as we would expect geometrically. Étale morphisms are after all a good approximation of what a "local isomorphism" should mean in terms of a flat and unramified morphism. Also that the geometric fibre over each geometric point has a constant finite number of points in 3.2.11. And finally that étale covers over a connected set are locally trivial 3.2.13, in other words there's a morphism $Y \to S$ so that $X \times_S Y$ becomes a trivial cover of Y for a sufficiently "small" morphism $Y \to S$.

3.2.4 Category of finite étale morphisms

The prior section showed us that the notion of an étale morphism is interesting in its own right, and we will now present some technical lemmas so that collection of finite étale morphisms constitutes a category as a step closer to the étale fundamental group.

Lemma 3.2.14. *Let $f\colon X \to S$ and $g\colon Y \to X$ be morphisms of schemes.*

1. *If $g \circ f$ is finite and f is separated, then g is finite.*
2. *If, additionally, $g \circ f$ and f are finite étale, then so is g.*

Remark 3.2.15. The lemma above shows that in any commutative triangle of morphism of schemes if any two of the morphisms are finite étale then so is the third.

Proof. [SZA09, Lemma 5.3.2] □

Lemma 3.2.16. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be finite étale morphisms and X an S-scheme. Then*

(i) (Closed under composition) $g \circ f$ is finite étale

(ii) (Stable under base change) Let $T \to S$ be a morphism, then the induced morphism

$$X \times_S T \to Y$$

is also finite étale.

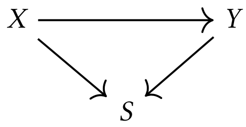
Proof. This is somewhat straightforward from the definition in terms of finite locally free morphisms. ☐

Using these two propositions and lemmas we can see that we can form a category of finite étale morphisms over a scheme S.

Definition 3.2.17. Fix a scheme S. We define $\mathcal{F}in\mathcal{E}t/S$ to be the category of finite étale morphisms over S. That is:

(i) Objects are finite étale morphisms $X \to S$ with the fixed target S.

(ii) A morphism from $X \to S$ to $Y \to S$ is a morphism $X \to Y$ that makes the diagram



commute.

Note that this is also the full subcategory of $\mathcal{S}ch/S$ where the objects are restricted to be finite étale. $\mathcal{S}ch/S$ is also known as the category of relative schemes over S or the slice category of $\mathcal{S}ch$ over S.

Example 3.2.18. In the most atomic example let k be an algebraically closed field and $S = \operatorname{Spec} k$. Then $\mathcal{F}in\mathcal{E}t/S$ is equivalent to the category of finite sets $\mathcal{F}in\mathcal{S}et$. This is because a finite étale scheme over k is the disjoint union of spectra of finite separable fields extensions of k. See Proposition 3.2.10.

Proposition 3.2.19. *Let $f\colon X \to S$ be a étale cover and $s\colon S \to X$ a section of f. Then s is an isomorphism of S onto a closed and open subscheme of X.*

Proof. By Lemma 3.2.14 we see that s is finite étale. A section must be injective because $f \circ s = \mathrm{id}_S$ and so an isomorphism onto its image. By 3.1.8 the image of s is open and closed. ☐

Remark 3.2.20. Note that in particular if S is connected then the section maps S isomorphically onto a whole connected component of X.

The following corollary will be one of the central pieces of building an algebraic fundamental group.

Corollary 3.2.21. *If $Z \to S$ is a connected S-scheme and $f_1, f_2\colon Z \to X$ are two S-morphism to a finite étale S-scheme X with $f_1(\bar{z}) = f_2(\bar{z})$ for some geometric point $\bar{z} \to Z$ then $f_1 = f_2$.*

Proof. We may assume that $S = Z$ by passing to the fibre product $Z \times_S X \to Z$ and use that étale morphisms are stable under base change. Then we prove that if two sections of an étale cover $X \to S$ of a connected scheme coincide at a geometric point, then they are equal. But this follows from 3.2.19 as each section is an isomorphism of S onto a connected component of X and thus determined by the image of a geometric point. ☐

Remark 3.2.22. The corollary generalizes an algebraic topology fact that says if two "path lifts" of a point are equal then the paths f_1, f_2 must be equal.

Remark 3.2.23. It also generalizes the following fact from Galois theory that, for $k \subset L$ a finite and separable extension, the morphism $\mathrm{Aut}_k(L) \to \mathrm{Hom}_K(L, K_s)$ is injective. The reader might remember that $k \subset L$ is Galois whenever the morphism is bijective.

In the world of geometry the proposition above shows that for $f\colon X \to S$ a connected étale cover the "evaluation morphism" $\mathrm{Aut}_S(X) \to X_{\bar{s}}$ which sends $\sigma \mapsto \sigma(\bar{x})$ where $\bar{x} \in X_{\bar{s}}$ is injective. We will see when this is bijective when we touch upon the Galois theory of étale covers.

Chapter 4

Fundamental groups of schemes

> *"A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet it finally surrounds the resistant substance."*
>
> — Alexander Grothendieck, *1985–1987 - [REC85]*

4.1 Galois theory of étale covers

We will now define a generalization of Galois extensions to that of étale covers. The topologically inclined reader will note that this also corresponds to regular covers.

Definition 4.1.1. Let X be an S-scheme with structure morphism $\phi\colon X \to S$. Define $\mathrm{Aut}_S(X)$ to be the group of scheme automorphisms of X preserving ϕ. That is $\sigma \in \mathrm{Aut}_S(X)$ if $\sigma \circ \phi = \sigma$.

Remark 4.1.2. Given a finite étale morphism $f\colon X \to S$ and a geometric point $\bar{s} \in S$ there is a natural left action on $X_{\bar{s}}$ by $\mathrm{Aut}_S(X)$ by base change of the action on X.

Remark 4.1.3. Note that we can also define an action on $X_{\bar{s}}$ by the action on geometric points by the alternative definition of the geometric fibre as in 2.5.16 as follows. Given $\sigma \in \mathrm{Aut}_S(X)$ and $\bar{x}$ define $\sigma \cdot \bar{x} := \sigma \circ \bar{x}$.

Proposition 4.1.4. *Let $f\colon X \to S$ be a connected étale cover. Then* $\mathrm{Aut}_S(X)$ *is finite and acts freely on $X_{\bar{s}}$.*

Proof. Let σ be an element of $\mathrm{Aut}_S(X)$ not the identity. Suppose that σ fixes a geometric point $\bar{x}$. Then Corollary 3.2.21 applied to σ and id_X gives that $\sigma = \mathrm{id}_X$ which is a contradiction. Let $f_1 = f$ and $f_2 = f \circ \sigma$, then by the above we must find some geometric point of X that maps to different points. So the action on any geometric fibre is free, or the permutation representation of $\mathrm{Aut}(X/S)$ is faithful. Since the morphism f is finite étale each geometric fibre is finite as a set, implying that $\mathrm{Aut}(X/S)$ is itself finite. □

Remark 4.1.5. For a connected $X \in \mathcal{F}in\mathcal{E}t/S$ and $\bar{s}$ a geometric point we have that the morphism

$$\mathrm{Aut}_S(X) \to X_{\bar{s}}$$

is injective as touched upon in 3.2.23. Now let $S = \mathrm{Spec}\, k$ for an algebraically closed field k. Then $X \cong \mathrm{Spec}\, L$ where L is a finite separable extension of k by 3.2.10. In the algebraic world we get a map

$$\mathrm{Aut}_k(L) \to L \otimes_k \bar{k}$$

with the choice of a geometric point $\bar{s} \in S$ corresponding to a choice of separable closure. If $k \subset L$ is a Galois extension then we have a bijection

$$\mathrm{Aut}_k(L) \to L \otimes_k \bar{k} \cong \prod_{i=1}^{|\mathrm{Aut}_k(L)|} \bar{k}.$$

What it actually tells us is that $\mathrm{Aut}_S(X)$ acts transitively on $X_{\bar{s}}$. For this reason we have the following definition of a Galois cover.

Definition 4.1.6. A connected étale cover $f\colon X \to S$ is defined to be Galois cover, or regular, if $\mathrm{Aut}_S(X)$ acts transitively on geometric fibres $X_{\bar{s}}$ for a geometric point $\bar{s}\colon \mathrm{Spec}\,\Omega \to S$.

Lemma 4.1.7. *Let $\phi\colon X \to S$ be a finite étale morphism and $G \subset \mathrm{Aut}_S(X)$ a subgroup. Let $Y \to S$ be a flat morphism of schemes. Then there is a canonical isomorphism $(X \times_S Y)/G \cong (X/G) \times_S Y$.*

Proof. The natural map $X \times_S Y \to (X/G) \times_S Y$ is constant on G-orbits. It therefore induces a map $(X \times_S Y)/G \to (X/G) \times_S Y$.

To see that this is an isomorphism we can argue that over an affine neighborhood $\mathrm{Spec}\, A$ of each point of S with preimages $\mathrm{Spec}\, B$ in X and $\mathrm{Spec}\, C$ in Y. Note that C is flat A-algebra by assumption. The isomorphism then reduces to $B^G \otimes_A C \cong (B \otimes_A C)^G$ as we pass to the algebraic world by 2.4.3.

The sequence of B-modules

$$0 \to B^G \to B \to \bigoplus_{g \in G} B$$

with rightmost morphism defined by mapping $b \mapsto (g(b)-b)_{g\in G}$ is exact. Tensoring by C gives us the exact sequence

$$0 \to B^G \otimes_A C \to B \otimes_A C \to \bigoplus_{g \in G} B \otimes_A C.$$

Because $(B \otimes_A C)^G$ is the kernel of the right most map in the exact sequence we are done. □

Lemma 4.1.8. *Let $f\colon X \to S$ be a connected finite étale cover and $G \subset \operatorname{Aut}(X/S)$ a subgroup. Then the quotient scheme X/G exists and all morphisms in the diagram below are étale covers.*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/G \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & S & \end{array}$$

Proof. [SZA09, Proposition 5.3.7] □

Lemma 4.1.9. *Let $f\colon X \to S$ be a connected étale cover and $\bar{s}$ a geometric point in S. It is Galois if and only if one of the following conditions are satisfied:*

(i) The evaluation morphism $\operatorname{Aut}_S(X) \to X_{\bar{s}}$ is bijective.

(ii) $|\operatorname{Aut}_S(X)| = |X_{\bar{s}}|$

(iii) The canonical morphism $X/\operatorname{Aut}_S(X) \to S$ is an isomorphism.

(iv) $\operatorname{Aut}_S(X)$ acts transitively on one geometric fibre $X_{\bar{s}}$.

In particular, the notion of a Galois cover is, a posteriori, independent of the fibre functor.

Proof. The first condition is clear from our discussions in the previous remarks, the second follows from the first. For the third note that the final objects in $\mathcal{F}in\mathcal{E}t/S$ are morphisms $S \to S$ and in $\mathcal{F}in\mathcal{S}et$ the single element sets. By 4.1.7 we have

$$S_{\bar{s}} \cong (X/\operatorname{Aut}_S(X))_{\bar{s}} \cong X_{\bar{s}}/\operatorname{Aut}_S(X)$$

with the left hand being a final object in $\mathcal{F}in\mathcal{S}et$ by the isomorphism assumption. It follows that the right-hand object is a final object in $\mathcal{F}in\mathcal{S}et$ so that $\operatorname{Aut}_S(X)$ acts transitively on $X_{\bar{s}}$. For the last condition we assume that $\operatorname{Aut}_S(X)$ acts transitively on $X_{\bar{s}}$ for a specific point geometric $\bar{s}$. Choose an element in $\bar{x} \in X_{\bar{s}}$ and define a map $\operatorname{Aut}_S(X) \to X_{\bar{s}}$ by evaluation of $\sigma \mapsto \sigma(\bar{x})$. Since the action is free and transitive we have a bijection. □

Proposition 4.1.10 (Galois correspondence of étale covers)**.** *Let $f\colon X \to S$ be a Galois cover and suppose $g\colon Z \to S$ is an intermediate connected étale cover and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

commutes. Then the following holds

(i) $\pi\colon X \to Z$ is a Galois cover and in fact $Z \cong X/H$ for some subgroup $H \subset \operatorname{Aut}_S(X)$.

(ii) There is a bijection between subgroups of $\mathrm{Aut}_S(X)$ *and intermediate covers*

$$\{H \subset \mathrm{Aut}_S(X)\} \longleftrightarrow \{X \to Z \to X\}$$

(iii) The correspondence is bijection on normal subgroups of $\mathrm{Aut}_S(X)$ *and Galois covers* $Z \to S$*, in this case* $\mathrm{Aut}_S(Z) \cong \mathrm{Aut}_S(X)/H$.

Proof. Use 3.2.21, 3.2.19, 4.1.8 and 4.1.9. □

Remark 4.1.11. Note that for a Galois cover $f\colon \operatorname{Spec} L \to \operatorname{Spec} k$ we have by earlier discussions that $k \subset L$ is a Galois extension. Then the proposition above is exactly the classical Galois correspondence.

Proposition 4.1.12 (Galois closure)**.** *Let* $f\colon X \to S$ *be a connected étale cover. Then there exists a morphism* $\pi\colon P \to X$ *such that* $f \circ \pi\colon P \to S$ *is a Galois cover.*

Proof. This proof is due to Serre represented from [Méz00]. Let $\bar{s}$ be a geometric point in S and let $F = \{\bar{x}_1, \dots, \bar{x}_n\}$ be the points of $X_{\bar{s}}$. Any ordering of the points $\bar{x}_i$ induces a geometric point $\bar{x}$ in $X^n := X \times_S \cdots \times_S X$ with $\bar{x}_i$ in the i-th component. This is because of the natural bijection

$$X^n \times_S \operatorname{Spec}\Omega \to (X \times_S \operatorname{Spec}\Omega)^n.$$

Let $\pi\colon P \to X^n \to X$ be the composition of the embedding of the connected component P of X^n containing $\bar{x}$ and the projection onto the first coordinate. Both π and $f \circ pi$ are finite étale by 3.2.16 and P is an étale cover of S.

Let $\pi_{ij}\colon X^n \to X \times_S X$ be the projection onto the product of factors i and j. Let

$$\Delta := \bigcup \pi_{ij}^{-1}(\Delta(X))$$

where $\Delta(X)$ is the diagonal of image $\pi_{ij}(X)$ in $X \times_S X$. By 3.2.3 the diagonal of X is open and closed and therefore so is its inverse image. If $P \cap \Delta$ is not empty then $P \subset \Delta$ as P is connected. This can not occur since $\bar{x} \notin \Delta$. Whence $P \cap \Delta$ is empty and all elements of $P_{\bar{s}}$ is represented by an n-tuple with distinct elements.

To see that P is Galois we argue as follows. Every permutation σ of the $\bar{x}_i$ induces an S-automorphism f_σ of X^n permuting the components. Hence, $\mathrm{Aut}_S(P)$ can be identified with a subgroup G of the symmetric group on n elements. If $f_\sigma(\bar{x}) \in P_{\bar{s}}$ then since $\sigma(P) \cap P$ is not empty and P is connected then $\sigma(P) = P$ and $\sigma \in \mathrm{Aut}_S(P)$. Hence, the action of $\mathrm{Aut}_S(P)$ is transitive on one geometric fibre and by 4.1.9 P is Galois. □

Definition 4.1.13. Let S be a scheme and fix a geometric point $\bar{s}\colon \operatorname{Spec}\Omega \to S$. Define the fibre functor $Fib_{\bar{s}}$ as the composition of the geometric fibre functor $X \mapsto X \times_S \operatorname{Spec}\Omega$ and the forgetful functor $\mathcal{S}ch/S \to \mathcal{S}et$ which sends a scheme to its underlying set. That is:

$$Fib_{\bar{s}}\colon \mathcal{F}in\mathcal{E}t/S \to \mathcal{F}in\mathcal{S}et$$

$$X \mapsto |X_{\bar{s}}| = |X \times_S \operatorname{Spec}\Omega|.$$

Remark 4.1.14. Note how we need to fix a geometric point $\bar{s}$ in the definition of the fibre functor similarly to how one specifies a base point in the topological fundamental group.

Remark 4.1.15. Note that for $X \in \mathcal{F}in\mathcal{E}t/S$ we have defined an action of $\mathrm{Aut}_S(X)$ on $X_{\bar{s}}$ before. We can define an action of $\mathrm{Aut}_S(X)$ on $Fib_{\bar{s}}(X)$ in the same manner by $\sigma \cdot \bar{x} \mapsto Fib_{\bar{s}}(\sigma)(\bar{x})$. Although we could denote that element $\sigma(\bar{x})$ we choose the former to be more explicit that it's the functor *Fib* and not merely the object in the target we're interested in.

To define an algebraic monodromy action we need to introduce the concept of an automorphism of the fibre functor.

Definition 4.1.16. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between two categories. Let $\mathrm{Aut}(F)$ be the automorphism group of F which consists of invertible natural transformations under composition.

In effect this means that an automorphism $\sigma \in \mathrm{Aut}(F)$ is a collection of bijections $\sigma_X\colon F(X) \to F(X)$ for each object $X \in \mathcal{C}$. Given morphisms $f\colon X \to Y$ these bijections satisfy the commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\sigma_X}\downarrow & & \downarrow{\scriptstyle\sigma_Y} \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

4.2 Topological groups

We need to make a quick detour to the world of topological groups before we take our penultimate step in the journey.

Definition 4.2.1. A topological group $G \in \mathcal{T}op\mathcal{G}rp$ is a group where the underlying set has a topology and group structure that is continuous with respect to this topology.

Example 4.2.2. A finite group with the discrete topology is a topological group because the multiplication and inverse operations are automatically continuous.

Example 4.2.3. Let $G = \mathbb{C}$ with the standard topology. Then the additive group structure is continuous and $\mathbb{C}$ is a topological group. Similarly, $G = \mathbb{C}^\times$ the complex numbers with the origin removed is a topological group with the multiplicative group structure.

Remark 4.2.4. If N is a neighborhood of an element g in a topological group G then $g^{-1}N$ is a neighborhood of the identity element. N is open if and only if $g^{-1}N$ is open. This is a common way of translating things to neighborhoods of the identity elements when considering topological groups.

Definition 4.2.5. An action of a group G on a topological space X is continuous if the associated maps are continuous.

Definition 4.2.6 (Pro-finite group)**.** Let $G\colon \mathcal{J} \to \mathcal{TopGrp}$ be a cofiltered diagram where $\mathcal{J}$ is the cofiltered subcategory of finite discrete topological groups. Then a topological group is called pro-finite if it is a limit of a cofiltered diagram G.

Example 4.2.7. Take $\mathcal{J}$ to be the category of all finite subgroups of the group of integers $G = \mathbb{Z}$ with morphisms between $n\mathbb{Z} \to m\mathbb{Z}$ if and only if $n\mathbb{Z} \subset m\mathbb{Z}$. Then the pro-finite group is denoted $\mathbb{Z}^\wedge$.

Example 4.2.8. Take $\mathcal{J}$ to be the category of all finite subgroups, with cardinality p^k for some k, of the group of integers $G = \mathbb{Z}$ with morphisms between $p^k\mathbb{Z} \to p^l\mathbb{Z}$ if and only if $k \leq l$. Then the pro-finite group is $\mathbb{Z}_p$, the additive group of p-adic integers.

Proposition 4.2.9. *Let $k \subset L$ a Galois extension of fields and let $\mathcal{J}$ be the cofiltered category of finite Galois subextensions $k \subset L_i \subset L$. Then we have the isomorphism*

$$Gal_k(L) \cong \varprojlim_{k \subset L_i \subset L} Gal_k(L_i).$$

Proof. Let $\phi\colon\ Gal_k(L) \to \prod Gal_k(L_i)$ be a group homomorphism that sends a k-automorphism σ to its restriction. Because all subextensions are Galois we see that $\sigma(L_i) \subset L_i$ for all i. The morphism ϕ must be injective because if it's not then there's an element α that's not fixed by σ and for which the restriction of σ to $k(\alpha) \subset L_k$ is nontrivial, for some k. To show that the image of ϕ is the limit we can argue as follows. Take an element (σ_{L_i}) of $\varprojlim Gal_k(L_i)$ and define a k-automorphism σ of L by $\sigma(\alpha) := \sigma_{L_k}(\alpha)$ where L_k is the extension which contains $k(\alpha)$. This is well-defined because $\mathcal{J}$ constitutes a cofiltered diagram. □

Remark 4.2.10. If G is a pro-finite group then we can endow each G_i with the discrete topology, their product with the product topology and $G \subset \prod G_i$ with the subspace topology. This is the weakest topology for which the projection maps $G \to G_i$ are continuous. In fact the following is true

Lemma 4.2.11. *Let $G\colon\ \mathcal{J} \to \mathcal{TopGrp}$ be a cofiltered diagram with every G_i equipped with the discrete topology. The limit $\varprojlim_{G_i \in \mathcal{J}} G_i$ is a closed topological subgroup of $\prod G_i$.*

Proof. See [SZA09, Lemma 1.3.8]. □

Proposition 4.2.12. *A profinite group is compact and totally disconnected. The open subgroups correspond precisely to the closed subgroups of finite index.*

Proof. See [SZA09, Corrollary 1.3.9]. □

Definition 4.2.13 (Pro-finite completion)**.** Let G be a topological group and let $\mathcal{I}$ be the category with objects the open normal subgroups G_i of finite index in G. Define the morphism $G_i \to G_j$ exist if and only if $G_i \subset G_j$. This constitutes a cofiltered category and the functor $G\colon \mathcal{I} \to \mathcal{T}op\mathcal{G}rp$ defined by sending $G_i \mapsto G/G_i$ is a cofiltered diagram. The pro-finite completion of G is defined as the limit

$$G^\wedge := \varprojlim_{G_i \in \mathcal{I}} G/G_i.$$

Definition 4.2.14. Let G be a group. A $G-$set, S, is a set S along with a group action of G on S. A G-equivariant map is a set map between $G-$sets $f\colon S \to S'$ for which $g \cdot f(s) = f(g \cdot s)$ for all $g \in G$ and $s \in S$

The category of G-sets is denoted by $G - \mathcal{S}et$. The full subcategory of finite G-sets is denoted by $G - \mathcal{F}in\mathcal{S}et$.

Lemma 4.2.15. *Let X be a topological space with the discrete topology and G a group that acts continuously on X. Then the stabilizer around a point x defined by $G_x := \{g \in G\colon\ gx = x,\ \text{for all}\ x \in X\}$ is an open subgroup of G.*

Proof. The stabilizer is the inverse image of x under the composition of the inclusion and multiplication map and thus open. □

Lemma 4.2.16 (Orbit-stabilizer theorem)**.** *Let $S \in G$-$\mathcal{S}et$. Then the orbit $Gx := \{gx\colon\ g \in G\}$ of $x \in G$ is isomorphic to G/G_x in the category of G-$\mathcal{S}et$.*

Proof. The G-set morphism $\phi\colon\ G/G_x \to Gx$ defined by $gG_x \to gx$ for $g \in G$ is well-defined and an isomorphism. □

Definition 4.2.17. The action of G on S is transitive if S isomorphic to a G-set of the form G/H.

Remark 4.2.18. The category G-$\mathcal{S}et$ has all limits and colimits like $\mathcal{S}et$. The coproduct is the disjoint union. Therefore, every G-set is the disjoint union of orbits. The morphisms in G-$\mathcal{S}et$ are then by the orbit-stabilizer lemma 4.2.16 determined by morphisms between orbits.

Lemma 4.2.19. *Let G be a pro-finite group with the natural projections $\pi_k\colon\ G \to G_k$. Then the set $\{\ker(\pi_k)\}_{k\in\mathcal{K}}$ forms a basis of open neighborhoods of the identity element in G.*

Proof. The projections are continuous and the image of each non-identity element $g \in G$ must have a non-trivial image in some π_k by the definition of a limit. □

4.3 Étale fundamental group and main theorem

As mentioned before the overarching goal is to be able to describe étale covers which we will be able to in a way through the étale fundamental group. In the case of a topological fundamental group the universal cover makes the fibre functor representable. The algebraic fibre functor defined earlier is not representable. The situation is salvageable by expanding the category and allowing for so called pro-objects which makes it pro-representable. There are similarities to pro-finite groups, and we shall see that the somewhat abstract definition below will be describable by a more tractable and familiar object as we put both a pro-finite and topological structure on it.

Definition 4.3.1. The étale fundamental group, or algebraic fundamental group, of a connected scheme S at the geometric point $\bar{s}$ is defined to be the automorphism group of the fibre functor over $\bar{s}$, namely:

$$\pi_1^{\acute{e}t}(S, \bar{s}) := \mathrm{Aut}(Fib_{\bar{s}})$$

The choice of a base point $\bar{s}$ is important and corresponds in the classical case to choosing a separable closure of the base field. By 3.2.21 we know that it is sufficient to know the behaviour of the étale covers and their geometric fibres over a single point.

Remark 4.3.2. For a functor F the group of automorphisms $\mathrm{Aut}(F)$ acts naturally on the objects $F(c)$ for $c \in C$ by $\sigma \cdot c := \sigma_C(F(c))$ for $\sigma \in \mathrm{Aut}(F)$ and where $\sigma_C : F(C) \to F(C)$ is the induced component of σ.

In the case of $Fib_{\bar{s}} : \mathcal{F}in\mathcal{E}t/S \to \mathcal{S}et$ we have a natural action of $\pi_1^{\acute{e}t}(S, \bar{s}) = \mathrm{Aut}(Fib_{\bar{s}})$ on $Fib_{\bar{s}}(X)$ for every finite étale scheme X over S. Therefore $Fib_{\bar{s}}$ can be enriched as a functor to $\pi_1^{\acute{e}t}$-$\mathcal{F}in\mathcal{S}et$. It maps Galois covers $X \in \mathcal{F}in\mathcal{E}t/S$ to $\pi_1^{\acute{e}t}(S, \bar{s})$-sets of the form $G = \pi_1^{\acute{e}t}(S, \bar{s})/H$ for H a normal open subgroup. This is because the action of $\pi_1^{\acute{e}t}(S, \bar{s})$ is transitive on $Fib_{\bar{s}}(X)$.

In order to understand the étale fundamental group of a connected scheme S at a point $\bar{s}$ we need to understand the automorphisms of the fibre functor. A first step is to find a representation of the fibre functor.

Definition 4.3.3. Let S be a connected scheme and $\bar{s}$ a geometric point. Define the cofiltered category $\mathcal{J}$ and the functor $\tilde{P} : \mathcal{J} \to \mathcal{F}in\mathcal{E}t/S$ by

(i) The objects $(P_i, \bar{p}_i) \in \mathcal{J}$ where the first factor comes from a subcategory of $\mathcal{F}in\mathcal{E}t/S$ with the objects P_i connected S and the second from a set of distinguished geometric points $\bar{p}_i \in F_{\bar{s}}(P_i)$ for every P_i.

(ii) The morphisms between $\phi_{ij} : (P_i, \bar{p}_i) \to (P_j, \bar{p}_j)$ satisfying $F_{\bar{s}}(\phi_{ij})(\bar{p}_j) = \bar{p}_i$ are denoted by the partial order $i \leq j$, if such morphisms exist.

Remark 4.3.4. We will denote the objects of $\mathcal{J}$ by P_i instead of $(P_i, \bar{p}_i)$ when possible without causing confusion, but we also keep in mind that they are ”pointed”.

Lemma 4.3.5. *Let S be a connected scheme and $\bar{s}$ a geometric point. Further, let $f : X \to S$ be an étale cover with a fixed geometric point $\bar{x} \in Fib_{\bar{s}}(X)$. Then there exists a $P_k \in \mathcal{J}$ that is a Galois cover over S with a morphism $\pi : P_k \to X$ such that $\bar{x}$ is the pullback of the distinguished point $\bar{p}_k \in P_k$ by π. In other words we have $Fib_{\bar{s}}(\pi)(\bar{p}_k) = \bar{x}$.*

Proof. Let Z be the connected component $f\colon Z \to X$ that lies over $\bar{x}$. Proposition 4.1.12 gives us a Galois closure, that is a $P_k \in \mathcal{I}$ and $\pi\colon P_k \to Z$ that factors through f by $f \circ \pi\colon P_k \to X$. As π is a Galois cover it is connected and hence surjective, this means we can find a $\bar{p} \in P_k$ such that $Fib_{\bar{s}}(f \circ \pi)(\bar{p}) = \bar{x}$. As $\mathrm{Aut}(P_k)$ is transitive we can find a $\sigma \in \mathrm{Aut}(P_k)$ such that $\sigma(\bar{p}) = \bar{p}_k$, this is in fact unique by 3.2.21. We thus have $Fib_{\bar{s}}(f \circ \pi \circ \sigma)(\bar{p}_k) = \bar{x}$ as wanted. □

Lemma 4.3.6. *The functor $\tilde{P}$ defined in 4.3.3 constitutes a cofiltered diagram for a set of distinguished geometric points $\bar{p}_i \in F_{\bar{s}}(P_i)$ for all $P_i \in \mathcal{I}$. For each $X \in \mathcal{F}in\mathcal{E}t/S$ we have a colimit*

$$\mathrm{Hom}(\tilde{P}, X) := \varinjlim_{P_i \in \mathcal{I}} \mathrm{Hom}(P_i, X).$$

Proof. It is clear that the set $\mathcal{I}$ is partially ordered. We show it is also directed.

If $P_i, P_j \in \mathcal{I}$ then $f\colon P_i \times_S P_j \to S$ is finite étale. We can then apply 4.3.5 to f and get a $P_k \in \mathcal{I}$ an S-morphism $\pi\colon P_k \to P_i \times_S P_j$ that satisfies $Fib_{\bar{s}}(\bar{p}_k) = (\bar{p}_i, \bar{p}_j)$. Compose π with the projections of the fibre product onto respective factor to get that $i \leq k$ and $j \leq k$. □

Lemma 4.3.7. *Let $\mathcal{K}$ be the subcategory of $\mathcal{I}$ restricting to the objects P_i that are Galois covers. $\mathcal{K}$ is then a cofinal set which that there is a natural equivalence*

$$\varinjlim_{P_i \in \mathcal{I}} \mathrm{Hom}(P_i, X) \cong \varinjlim_{P_k \in \mathcal{K}} \mathrm{Hom}(P_k, X).$$

The transition maps of the rightmost colimit $\phi_{ij}\colon P_j \to P_i$ are unique if they exist.

Proof. Let $P_i \in \mathcal{I}$. Then by 4.3.5 we can find a $P_k \in \mathcal{K}$ such that $P_i \leq P_k$. Uniqueness of the transition morphism comes from 3.2.21. □

Without loss of generality we will therefore assume henceforth that the cofiltered category in the functor $\tilde{P}$ runs over Galois covers instead of simply connected covers.

Proposition 4.3.8. *Let S be a connected scheme and $\bar{s}$ a geometric point. The functor $Fib_{\bar{s}}$ is pro-representable by the co-filtered diagram $\tilde{P}\colon \mathcal{K} \to \mathcal{F}in\mathcal{E}t/S$ defined in 4.3.3. That is we have the natural isomorphism of functors for every $X \in \mathcal{F}in\mathcal{E}t/S$.*

$$\mathrm{Hom}(\tilde{P}, X) := \varinjlim_{P_k \in \mathcal{K}} \mathrm{Hom}(P_k, X) \xrightarrow{\sim} Fib_{\bar{s}}(X).$$

Proof. For each $P_k \in \mathcal{K}$ and $X \in \mathcal{F}in\mathcal{E}t/S$ we have a map of sets defined by

$$\mathrm{Hom}(P_k, X) \to Fib_{\bar{s}}(X), \quad \phi \mapsto Fib_{\bar{s}}(\phi)(\bar{p}_k).$$

We claim that this map is injective. Assume $f, g\colon P_k \to X$ be two Galois covers such that $Fib_{\bar{s}}(f)(\bar{p}_k) = Fib_{\bar{s}}(g)(\bar{p}_k)$ then by 4.3.5 we see that $f = g$ as required.

This map is "functorial" in the sense that it respects the diagram $\bar{P}$ by sending a morphism $\theta_{kl}\colon P_l \to P_k$ to $\mathrm{Hom}(P_l, X) \to \mathrm{Hom}(P_k, X)$ by composition. By the universal property of the colimit $\mathrm{Hom}(\tilde{P}, X)$ we have a unique morphism Φ of functors defined on the objects by

$$\Phi_X\colon \ \mathrm{Hom}(\tilde{P}, X) \to Fib_{\bar{s}}(X), \quad [\phi] \mapsto Fib_{\bar{s}}(\phi)(\bar{p}_k),$$

where the representative of $[\phi]$ is $\phi\colon P_k \to X$. The map Φ is injective because the maps Φ_X are.

For surjectivity we note that given $\bar{x} \in Fib_{\bar{s}}(X)$ for an étale cover X we can find a $P_k \in \mathcal{K}$ and an S morphism $\pi\colon P_k \to X$ such that $Fib_{\bar{s}}(\pi)(\bar{p}_k) = \bar{x}$. This implies surjectivity of Φ.

Further if $f\colon X \to Y$ is a morphism in $\mathcal{F}in\mathcal{E}t/S$ then the induced maps defined by $G_k(f)\colon \ \mathrm{Hom}(P_k, X) \to \mathrm{Hom}(P_k, Y)$ for all $P_k \in \mathcal{K}$ gives a map $G(f)$ between the colimits and the diagram below commutes

$$\begin{array}{ccc}
\mathrm{Hom}(\tilde{P}, X) & \xrightarrow{\Phi_X} & Fib_{\bar{s}}(X) \\
{\scriptstyle G(f)}\big\downarrow & & \big\downarrow{\scriptstyle Fib_{\bar{s}}(f)} \\
\mathrm{Hom}(\tilde{P}, Y) & \xrightarrow[\Phi_Y]{} & Fib_{\bar{s}}(Y)
\end{array}$$

As Φ_X are maps of sets and bijection is equivalent to isomorphism in the category of sets. Therefore the functors $\mathrm{Hom}(\tilde{P}, -)$ and $Fib_{\bar{s}}(-)$ are naturally equivalent and $Fib_{\bar{s}}$ is pro-representable. □

Theorem 4.3.9 (Grothendieck)**.** *The étale fundamental group of a connected scheme S with a geometric point $\bar{s}$ is a profinite group with the isomorphism of groups*

$$\pi_1^{ét}(S, \bar{s}) \cong \varprojlim_{P_k \in \mathcal{K}} Gal_S(P_k).$$

where $Gal_S(P_k) := \mathrm{Aut}_S(P_k)^{op}$. The transition maps ψ_{ij} are, in particular, surjective.

Proof. Recall that $\mathcal{K}$ is the restriction of the cofiltered category defined in 4.3.3 to Galois covers. The associated diagram has unique morphisms $\phi_{ij}\colon (P_i, \bar{p}_i) \to (P_j, \bar{p}_j)$ that satisfy $Fib_{\bar{s}}(\phi_{ij})(\bar{p}_j) = \bar{p}_i$. We have bijections $\lambda_k\colon \ \mathrm{Aut}_S(P_k) \to Fib_{\bar{s}}(P_k)$ defined by $\sigma \mapsto Fib_{\bar{s}}(\sigma)(\bar{p}_k)$. For any $j \leq i$ we define $\psi_{ij}\colon \ \mathrm{Aut}_S(P_j) \to \mathrm{Aut}_S(P_i)$ as the composition $\mathrm{Aut}_S(P_j) \to \mathrm{Aut}_S(P_i)$ in the following diagram

$$\begin{array}{ccc}
\mathrm{Aut}_S(P_j) & \xrightarrow[\lambda_j]{} & Fib_{\bar{s}}(P_j) \\
\vdots & & \big\downarrow{\scriptstyle Fib_{\bar{s}}(\psi_{ij})} \\
\mathrm{Aut}_S(P_i) & \xleftarrow{\lambda_i^{-1}} & Fib_{\bar{s}}(P_i)
\end{array}$$

For $\sigma \in \mathrm{Aut}_S(P_j)$ the element $\psi_{ij}(\sigma)$ is determined automorphism of P_i that satisfies the commutativity of the diagram

$$\begin{array}{ccc} P_j & \xrightarrow{\phi_{ij}} & P_i \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \psi_{ij}(\sigma)} \\ P_j & \xrightarrow{\phi_{ij}} & P_i \end{array}$$

It is unique due to the transitivity of the action on fibres, that is

$$Fib_{\bar{s}}(\psi_{ij}(\sigma))(\bar{p}_j) = Fib_{\bar{s}}(\phi_{ij} \circ \sigma)(\bar{p}_i).$$

The map transition maps ψ_{ij} are clearly group homomorphisms. Furthermore they are clearly surjective since by the transitivity proved by the above diagram each $\psi_{ij} \circ \phi_{ij}$ is of the form $\phi_{ij} \circ \sigma$. We therefore have a limit $\varprojlim_{P_k \in \mathcal{K}} Gal_S(P_k)$. The contravariant Yoneda embedding $P_k \mapsto \mathrm{Hom}(P_k, -)$ gives a morphism

$$\varprojlim_{P_k \in \mathcal{K}} Gal_S(P_k) \to \mathrm{Aut}(\mathrm{Hom}(\tilde{P}, -))$$

that is seen to be a bijective group morphism by the uniqueness of the transition maps for both (co)limits. □

Remark 4.3.10. This theorem shows that for a geometric point $\bar{x} \in Fib_{\bar{s}}(X)$ that is dominated by a Galois cover $P_k \in \mathcal{K}$ then the action of $\pi_1^{ét}(S, \bar{s})$ on $\bar{x}$ is factored through and action from $Gal_S(P_k)$. This means that the action is continuous and the above isomorphism is in fact an isomorphism of topological groups.

Theorem 4.3.11. *(Grothendieck) Let S be a connected scheme and $\bar{s}$ a geometric point. The functor $Fib_{\bar{s}}$ induces an equivalence of $\mathcal{F}in\mathcal{E}t/S$ and the category of continuous and finite $\pi_1^{ét}(S, \bar{s})$-sets. Furthermore we have*

1. *Connected covers correspond to sets with transitive $\pi_1^{ét}(S, \bar{s})$-action and*
2. *Galois covers correspond to finite quotients $\pi_1^{ét}(S, \bar{s})/H$ with H a normal subgroup.*

Proof. (Step 1. Fully faithfull.) Let $f, g\colon X \to Y \in \mathcal{F}in\mathcal{E}t/S$ be two étale covers such that $Fib_{\bar{s}}(f) = Fib_{\bar{s}}(g)$. Pick out a connected component through $i\colon Z \to X$. Then obviously $Fib_{\bar{s}}(f \circ i) = Fib_{\bar{s}}(g \circ i)$ and by 3.2.21 we have $f \circ i = g \circ i$. Doing this for all connected components we come to the conclusion that $f = g$. This shows the functor is faithful.

The "full" part of the proof is based on an adaptation of [LEN, Section 3.18]. Let X, Y be étale covers over S and $F\colon Fib_{\bar{s}}(X) \to Fib_{\bar{s}}(Y)$ a morphism. We need to construct a morphism $f\colon X \to Y$ which gets sent to F by the fibre functor to show it's full. We start by showing that we can reduce the problem to X, Y

both being connected. Let $X = \sqcup^s X_i$ be a decomposition of X into connected components. Then $\mathrm{Hom}(X, Y) \cong \sqcup^s \mathrm{Hom}(X_i, Y)$ and similarly for $Fib_{\bar{s}}$ since it preserves coproducts. In similar fashion we can assume that Y is connected.

We know that $\pi_1^{ét}(S, \bar{s})$ acts transitively on both $Fib_{\bar{s}}(X)$ and $Fib_{\bar{s}}(Y)$. Therefore the map F will be determined by the image of $\bar{x} \in X$, denote it by $\bar{y} := F(\bar{x})$. Let $\pi\colon P_k \to X \times_S Y$ be a Galois cover that maps $Fib_{\bar{s}}(\pi)(\bar{p_k}) = (\bar{x}, \bar{y})$, one exists by 4.3.5. Let π_X, π_Y be the respective projections onto each factor then $Fib_{\bar{s}}(\pi_X)(\bar{p_k}) = \bar{x}$ and $Fib_{\bar{s}}(\pi_Y)(\bar{p_k}) = \bar{y}$.

Note that X is a quotient $P_k / \mathrm{Aut}_X(P_k)$ and an intermediate cover of $P_k \to Y$. We therefore have a unique morphism $f\colon X \to Y$ such that $\pi_X \circ f = \pi_Y$.

$$\begin{array}{ccc} P_k & \overset{\sigma}{\dashrightarrow} & P_k \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Now choose an element $\bar{q_k} \in Fib_{\bar{s}}(P_k)$ with $Fib_{\bar{s}}(\pi_Y)(\bar{q_k}) = Fib_{\bar{s}}(f \circ \pi_X)(\bar{p_k})$ and a σ that maps $Fib_{\bar{s}}(\sigma)(\bar{p_k}) = \bar{q_k}$. It is then clear by use of 3.2.21 that the commutative diagram above commutes and $Fib_{\bar{s}}(f)(\bar{x}) = \bar{y}$ and so $Fib_{\bar{s}}(f) = F$.

(Step 2. Essentially surjective.) This part of the proof is based on an adaptation of [SZA09] and [STA22, 0BN4] Let E be a finite continuous $\pi_1^{ét}$-set. We can decompose E into a finite number of orbits and with each orbit being a transitive $\pi_1^{ét}$-set. This is because the functor $Fib_{\bar{s}}$ preserves finite coproducts because the fibre product of schemes does. So without loss of generality we may assume that E is a transitive $\pi_1^{ét}$-set. Let $G := \pi_1^{ét}(S, \bar{s})$ then the stabilizer $H := G_x$ of a point $x \in E$ is open by 4.2.15 and $E \cong G/H$ as $\pi_1^{ét}$-sets. We need to find an étale cover X such that $Fib_{\bar{s}}(X) \cong E$.

Looking at the natural projections $\pi_k\colon \pi_1^{ét}(S, \bar{s}) \to Gal_S(P_k)$ from the pro-finite group structure we see that the finite subset of $\{\ker \pi_k\}_{k \in \mathcal{K}}$ forms a neighborhood of 1 in $\pi_1^{ét}(S, \bar{s})$ by 4.2.19. Then H contains some $U = \ker \pi_k$ for some k. Since P_k is Galois we have that $Fib_{\bar{s}}(P_k) \cong G/U$ as $\pi_1^{ét}$-sets for some open $U \subset G$. Again because P_k is Galois $Gal_S(P_k)$ acts transitively on $Fib_{\bar{s}}(P_k)$ which implies that U is a normal subgroup and $U \subset H$. Full faithfullness gives us the following

$$\mathrm{Aut}_S(P_k) \cong \mathrm{Aut}_{\pi_1}(Fib_{\bar{s}}(P_k)) \cong \mathrm{Aut}_{\pi_1}(G/U) \cong G/U$$

and further $H/U \subset G/U$. Finally we define the étale cover $X := P_k/(H/U)$ then

$$Fib_{\bar{s}}(X) = Fib_{\bar{s}}(P_k/(H/U)) \cong Fib_{\bar{s}}(P_k)/(H/U) \cong (G/U)/(H/U) \cong G/H$$

by 4.1.7 and we are done. □

We will now very briefly touch upon how the étale fundamental group is a common generalization of the Galois theory and the topological fundamental group.

4.3.1 Galois theory

We recall that an extension $k \subset L$ is Galois if $L^{\mathrm{Aut}_k(L)} = k$, that is $\mathrm{Aut}_k(L)$ fixes k nothing else. Recall the classic theorem of Galois theory.

Theorem 4.3.12. *(Galois theory for finite extensions) Let $k \subset L$ be a finite Galois extension with Galois group G. The maps that send subfields M to $H := \mathrm{Aut}_M(L)$ and subgroups H to fixed fields $M := L^H$ is an inclusion reversing bijection.*

Let k be a field and A a finite étale k-algebra. For a fixed separable closure k_s of k there is a $Gal(k) := Gal_k(k_s)$ action on k_s that translates to an action on $\mathrm{Hom}_k(A, k_s)$. Grothendieck's version that includes infinite Galois extensions is the following.

Theorem 4.3.13. *The contravariant functor $F := A \mapsto \mathrm{Hom}_k(L, k_s)$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finite sets with continuous and transitive left-$Gal(k)$-action. Galois extensions give rise to finite sets isomorphic to some finite quotient of $Gal(k)$.*

Lemma 4.3.14. *The separable closure k_s of a field k is the colimit over all of its finite Galois subextensions L. We have the isomorphisms*

$$Gal(k) \cong \varprojlim_L \mathrm{Hom}_k(L, k_s) \cong \varprojlim_L Gal_k(L)$$

which means that the absolute Galois group is profinite.

Proof. The first part is just a reformulation of the definition of separable closure, that is $k_s \cong \varinjlim_L L$. Further the endomorphisms of a separable k-extension is an automorphism so that

$$Gal(k) = \mathrm{Hom}_k(k_s, k_s) = \mathrm{Hom}_k(\varinjlim_L L, k_s) \cong \varprojlim_L \mathrm{Hom}_k(L, k_s).$$

We also have that the image of such an extension $L \to k_s$ must necessarily be L itself. On the other hand any morphism $L \to L$ that fixes k can be extended to $L \to k_s$ by the properties of separable closure. Therefore we can identify $\mathrm{Hom}_k(L, k_s)$ with $\mathrm{Hom}_k(L, L) = Gal_k(L)$ and we get the desired result. □

Now to the proof of 4.3.13.

Proof. (Theorem 4.3.13) Let $S = \mathrm{Spec}\, k$ where k is a field. Then the automorphism group of $Fib_{\bar{s}}$ is the absolute Galois group of k. That is

$$\mathrm{Aut}(Fib_{\bar{s}}) \cong Gal_k(k_s).$$

After all, an étale cover X of $S = \mathrm{Spec}\, k$ is $\mathrm{Spec}\, A$ for A a finite étale k-algebra like we've seen several times before. The functor $Fib_{\bar{s}}$ maps $X \mapsto \mathrm{Spec}(A \otimes_k \Omega)$. Since we may restrict ourselves to connected covers which correspond to finite separable extensions $k \subset L$ with $[L : k] = \#|\mathrm{Hom}_k(L, \bar{k})| = n$, all distinct morphisms. But since $\mathrm{Spec}(A \otimes_k \Omega) \cong \coprod^n \Omega$ the spectrum has precisely n points so we have a bijection. The image of those morphisms lie in the separable closure and thus $Fib_{\bar{s}}(X) \cong \mathrm{Hom}_k(L, k_s)$. As a mere curiosity we also see that the fibre functor consists of the roots of the minimal polynomial defining L and that the automorphism group of the fibre functor acts on these roots. By the above theorems we see that $\pi_1^{ét}(S, \bar{s}) \cong Gal(k)$. □

4.3.2 Topological coverings

We start by giving a quick introduction to topological coverings and the topological fundamental group. Let X be a connected, locally connected and locally simply connected topological space.

Definition 4.3.15. A cover of X is a topological space Y with a continuous morphism $p\colon Y \to X$ such that each point of X has a neighborhood V for which the fibre $p^{-1}(V)$ decomposes as a disjoint union of subsets $U_i \subset Y$ and such that $p(U_i) \cong V$. A cover with finite fibres is called a finite cover.

Remark 4.3.16. Note that the fibres are discrete sets and that given $x \in X$ the fundamental group $\pi_1(X, x)$ acts on the fibre $p^{-1}(x)$ by the so called *monodromy action* by lifting a loop representative $[\gamma]\colon [0,1] \to X$ from $\pi_1(X, x)$ to a path $[\bar{\gamma}]\colon [0,1] \to Y$ of Y by starting at $y := p^{-1}(x)$.

Definition 4.3.17. A covering $p\colon Y \to X$ with a transitive group action of $\mathrm{Aut}_X(Y)$ is a Galois covering.

Remark 4.3.18. Note that this is in similarity to a Galois extension in how a Galois extension has a transitive action by $Gal(k)$ and to that of an Galois cover of schemes.

Theorem 4.3.19. *The fibre functor*

$$Fib_x\colon (p\colon Y \to X) \mapsto p^{-1}(x)$$

equipped with the monodromy action gives an equivalence of the category of covers of X with the category of sets with left $\pi_1(X, x)$ action. Galois covers correspond to coset spaces of normal subgroups of $\pi_1(X, x)$.

Proof. See [SZA09, Theorem 2.3.4]. □

Remark 4.3.20. The fibre functor is in fact representable by a universal cover $\pi\colon \tilde{X}_x \to X$ so that $Fib_x \cong \mathrm{Hom}(\tilde{X}_x, -)$. The existence of the universal cover $\tilde{X}_x$ corresponds in analogy to the existence of the separable closure of a base field. The choice of a point x corresponds to the choice of a separable closure, and finally the fundamental group is analogous to the absolute Galois group $Gal(k)$.

Because of the representability one can see that

$$\mathrm{Aut}(\tilde{X}_x) \cong \mathrm{Aut}(Fib_x) \cong \pi_1(X, x).$$

Corollary 4.3.21. *The functor Fib_x induces an equivalence of the category of finite covers of X with the category of finite continuous sets with left $\pi_1(X,x)^\wedge$ action. $\pi_1(X,x)^\wedge$ denotes the profinite completion of $\pi_1(X,x)$.*

Proof. See [SZA09, Corollary 2.3.9] □

For X a scheme of finite type over $\mathbb{C}$ one can define an analytic space X^{an} associated with X by embedding the $\mathbb{C}$-points of X into $\mathbb{C}^m$. This is a topological space. See [HAR77, Appendix B].

A morphism of $\mathbb{C}$-schemes of finite type $\phi\colon Y \to X$ induces a morphism of analytic spaces $\phi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$. In fact it can be shown that if ϕ is a finite étale morphism then ϕ^{an} is a local isomorphism.

The étale fundamental group is, while powerful, also notoriously difficult to compute in practice. Grothendieck proved the following theorem that in some cases allows one to compute the étale fundamental group through the pro-finite completion of the topological fundamental group.

Theorem 4.3.22. *Let X be a connected scheme of finite type over $\mathbb{C}$ and $\bar{x}$: $\mathrm{Spec}\,\mathbb{C} \to X$ a geometric point. The functor $(Y \to X) \mapsto (Y^{an} \to X^{an})$ induces an equivalence of the category of finite étale covers of X with that of finite topological covers of X^{an}.*

This functor induces an isomorphism

$$\pi_1(X^{an}, \bar{x})^\wedge \cong \pi_1^{\acute{e}t}(X, \bar{x}).$$

The left-hand side is the pro-finite completion of the topological fundamental group of X with the base point $\bar{x}$.

Proof. The theorem relies on deep theorem of algebraisation of finite topological covers. See [SGA71, Exposé XII, Corollary 5.2]. □

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