SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Group C*-algebras of Heisenberg groups and C*-rigidity

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2022 - No M1

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Sven Raum

2022

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Mars 2022

Abstract

The theory of unitary representation of locally compact groups is closely related to the representation theory of the corresponding group C*-algebras. This thesis explores this relationship and the unitary representation theory of the Heisenberg groups will be worked out in detail. The representation theory of the Heisenberg groups will then be used to give a more concrete description of their corresponding group C*-algebras as C*-bundles. A driving motivation is C*-rigidity for connected, simply connected nilpotent Lie groups and the representation theory of the Heisenberg groups give concrete insight of what one might expect in the still unsolved 2-step nilpotent case.

Acknowledgements

I would like to thank my supervisor Sven Raum for all of his support, guidance, advice and patience that I received when working on the thesis. I would also want to thank him for taking me to the conference Non-commutativity in the North which both gave me valuable insights in the field of operator algebras and context and background for the thesis.

Contents

1 Introduction

Representation theory is the the study of how groups can act as linear transformations on a vector space and how these actions decompose. This more concrete description of a group allows algebraic problems to be reduced to problems of linear algebra which are often more tractable. On the other hand, representation theory also illuminates other results in mathematics. One example is Fourier series where Plancherel's theorem in the language of representation theory can be viewed as a type decomposition of the regular representation into irreducible subrepresentations. The theory of unitary representations started with G. Frobenius and I. Schur at the end of the 19th century with their work on how group representations of finite groups on complex vector spaces can be decomposed into irreducible representations. For non-finite topological groups one works with unitary representations which are strongly continuous group homomorphisms into the group of unitary operators on a Hilbert space. These conditions work as a substitute for the finiteness condition for finite groups. In the theory of unitary representations the main object of interest is that of the unitary dual of a group. The unitary dual is the set of all irreducible representations of the group up to equivalence. The results of finite groups could then be generalized for unitary representations on compact groups with the Peter-Weyl theorem in 1927 showing how the regular representation is decomposed into irreducible subrepresentations. For locally compact groups E. Wigner, motivated by quantum mechanics calculated the unitary dual of the Poincaré group in 1939. At the same time the theory of C^*-algebras started to emerge with (among other) the work of I. Gelfand and I. Segal and it was already suspected back then that theory of C^*-algebras were linked to the theory of unitary representations. The C^*-algebras are a particular class of Banach *-algebras such that $\|x^*x\| = \|x\|^2$ for any element in the algebra. This requirment and spectral radius formula for Banach algebras leads to the remarkable fact the analytic component of the C^*-algebra, its norm is completely determined by its algebraic structure. This property makes the C^*-algebras a particularly nice class to study and tools from both analysis and algebra can be used to study them. One exmaple of how well-behaved a C^* algebra is that any commutative C^*-algebra is isomorphic to space of continuous functions on its spectrum that vanish at infinity. This is not true in general for the Banach *-algebra $L^1(G)$ for some locally compact group G. In the 1960's the relationship between unitary representations of locally compact groups and C^*-algebras recieved more interest and it was shown that the representation theory of locally compact groups coincided with a type of C^*-algebras called the group C^*-algebras. This is the case since the unitary dual of a group and the dual object for the corresponding C^*-algebra can be equipped with natural topologies and the two spaces will be isomorphic to each other. More information on the early history of the subject can be found in J. Rosenberg [1]

In this thesis the correspondence between unitary representations of locally compact groups and the group C^*-algebras will be explored. In particular the unitary representation theory of the Heisenberg groups will be worked out in detail and a description for their group C^*-algebras will be given. A motivation for considering the Heisenberg group is to have concrete non-trivial examples in mind when working on rigidity questions. For discrete groups the representation theory can be translated in terms of the complex group algebra $\mathbb{C}[G]$ which is a finite analogue of the group C^*-algebra. A famous conjecture regarding group algebras is Kaplansky's conjecture which asserts that for any torsion-free group G the units of $\mathbb{C}[G]$ are of the form kf_g for some $k \in \mathbb{C}$ and $g \in G$. This conjecture implies the following rigidity result for torsion-free groups, if G and H are torsion-free groups such that $\mathbb{C}[G] \cong \mathbb{C}[H]$ then $G \cong H$ are isomorphic. For an arbitrary locally compact group G the same rigidity question can be asked and there are several types of group algebras to consider. Firstly there is the algebra $L^1(G)$ but there is also the maximal group C^*-algebra $C^*_{\max}(G)$ which was described above. This algebra is a completion of $L^1(G)$ with respect to special type of norm. There is also the reduced group C^*-algebra of G denoted by $C^*_{\min}(G)$ that is also a completion of $L^1(G)$ with the benefit that it is often more concrete to work with. The two group C^*-algebras are isomorphic when G is amenable and in that case one simply writes $C^*(G)$. The L^1-case was solved by Wendel, [15] in the 1950s when he showed that if G and H are locally compact groups and $L^1(G) \cong L^1(H)$ as Banach algebras then $G \cong H$. For the group C^*-algebras things does not work as nicely and a similar rigidity statement fails in the class of locally compact groups. One counter example are the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. For these groups the reduced and maximal group C^*-algebras are isomorphic and $C^*(\mathbb{Z}_4) \cong \mathbb{C}^4 \cong C^*(\mathbb{Z}_2 \times \mathbb{Z}_2)$ but clearly $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not ismorphic. Since the situation for the group C^*-algebras is more nuanced one has many types of rigidity statements to explore. A strong notion of rigidity is that of C^*-superrigidity of a group G. A locally compact group G is C^*-superrigid if $C_{\text{red}}(G) \cong C_{\text{red}}(H)$ for some locally compact group H implies that $G \cong H$. A classical result by Scheinberg, [14] shows that torsion-free abelian groups are C^*-superrigid. A more recent result is that of Eckhardt, C and Raum, S showing that every torsion-free, finitely generated 2-step nilpotent group is C^*-superrigid. It is conjectured that finitely generated nilpotent groups are C^*-superrigid. It is also not known if there exists a torsion-free group locally compact group that is not C^*-superrigid [17]. Guided by these results it is natural to examine which locally compact group G satisfies the following weaker rigidity result: If H is a connected, simply connected, nilpotent Lie group and $C_{\text{red}}(G) \cong C_{\text{red}}(H)$ then $G \cong H$. The collection of nilpotent connected, simply connected, nilpotent Lie groups is a good collection to consider rigidity results on. This is the case since if G is in this collection then $G/Z(G)$ and $Z(G)$ are also in this class and induction on the nilpotence class lets us use arguments that pass rigidity on G to rigidity on $G/Z(G)$ and $Z(G)$.

The overlying structure of the thesis is the following. In section 2 general results of locally compact groups are found and section 3 and 4 introduce the basic theory of unitary representations and C^*-algebras. In section 5 and 6 the relationship between unitary representations and representations of the group

C^*-algebra is given. In section 7 and 8 the nilpotent Lie groups are introduced and the representation theory of the Heisenberg group is worked out in detail as well as a description of its C^*-algebra. In the last section rigidity is discussed briefly building on the results for the heisenberg groups. A great reference for many of these things is the lecture notes in abstract harmonic analysis by S. Raum, [2].

2 Locally Compact Groups

We begin by introducing the topological groups and we will in particular be interested in the locally compact groups. The locally compact groups are nice to work with since they admit a measure, the Haar measure which allows one to do analysis on the group.

Definition 2.0.1 (Topological groups)**.** *A topological group G is a group G such that multiplication: $m : G \times G \to G$, $m(x, y) = xy$ and inversion $i : G \to G$, $i(x) = x^{-1}$ are continuous.*

Notation 2.0.2. *The unit element will be denoted with* 1*.*

Proposition 2.0.3. *The inversion map $i : G \to G$ and the translations maps $T_g : G \to G$, ${}_gT : G \to G$ defined for any fixed $g \in G$ by*

$$T_g(x) = xg$$
$${}_gT(x) = gx$$

are homeomorphisms.

Proof. For inversion: $i : G \to G$ note that i is continuous by definition of a topological group and is a bijection with continuous inverse since $i \circ i = \mathrm{id}_G$. The map ${}_gT$ is continuous since ${}_gT = m \circ \iota_{\{g\}\times G}$ where $\iota_{g\times G} : G \to \{g\} \times G$ is the injection $x \mapsto (g, x)$. The map ${}_gT$ is a bijection with continuous inverse since ${}_{g^{-1}}T \circ_g T = \mathrm{id}_G$. □

Notation 2.0.4. *Let $E \subseteq G$ and $g \in G$. The following notation will be used throughout the text,*

$$gE = \{gx \in G;\ x \in E\}$$
$$Eg = \{xg \in G;\ x \in E\}$$
$$E^{-1} = \{x^{-1};\ x \in E\}.$$

Since left translation, right translation and inversion are homeomorphisms it follows that gE, Eg, and E^{-1} are open, closed or compact whenever E is open, closed or compact respectively. A set $E \subseteq G$ is called symmetric if $E = E^{-1}$. If f is any function defined on G we define left- and right translates of f, $L_g f$ and $R_g f$ for any fixed $g \in G$ as the functions

$$(L_g f)(x) = f(g^{-1}x)$$
$$(R_g f)(x) = f(xg)$$

If f is a continuous function on G then $L_g f$ and $R_g f$ are continuous since they are a composition of f and either a left translation or a right translation, both being continuous. The reason for choosing g^{-1} instead of g in the definition of a left translate is in order to obtain a left action on function spaces.

Proposition 2.0.5. *For any open set V containing the identity 1 there exists an open set U containing 1 such that $U \subseteq V$ and $U^{-1} = U$.*

Proof. Assume that $V \subseteq G$ is open and contains 1. Define $U = V \cap V^{-1}$ then U is open since V is open and inversion is a homeomorphism. Since $1 \in V$ it follows that U contains 1 and $U = U^{-1}$. $\square$

Notation 2.0.6. *The Borel σ-algebra of a topological space, X will be denoted by $\mathcal{B}(X)$*

Proposition 2.0.7. *If $f : X \to Y$ is a homeomorphism then $f(\mathcal{B}(X)) = \mathcal{B}(Y)$*

Proof. Pick an open set $V \subseteq Y$ then $f^{-1}(V)$ is open and hence a Borel set in X. Since $V = f(f^{-1}(V)) \in f(\mathcal{B}(X))$ it follows that $f(\mathcal{B}(X))$ contains the open sets. Using the fact that f^{-1} is continuous it follows that $f(\mathcal{B}(X))$ is a σ-algebra and since it contains all the open sets it follows that $\mathcal{B}(X) \subseteq f(\mathcal{B}(X))$. Repeating the same arguments with the function f^{-1} it follows that $\mathcal{B}(X) \subseteq f^{-1}(\mathcal{B}(X))$ and therefore $f(\mathcal{B}(X)) \subseteq \mathcal{B}(X)$. $\square$

Definition 2.0.8. *An element x in a topological space has a compact neighbourhood if there exists an open set V containing x and a compact set K such that $V \subseteq K$.*

Definition 2.0.9 (Locally Compact Topological Space)**.** *A topological space X is locally compact if any $x \in X$ has a compact neighbourhood.*

Definition 2.0.10 (Locally Compact Group)**.** *A topological group that is also a locally compact Hausdorff space is called a locally compact group.*

Proposition 2.0.11 (Examples of Locally Compact Groups)**.** *Examples of locally compact groups are the following*

- *Any group G equipped with the trivial or discrete topology.*
- *$\mathbb{R}^n$ (under addition).*
- *The unit circle $\mathbb{T}$ under multiplication viewed as a subspace of $\mathbb{C}$.*
- *The general linear group $GL_n(\mathbb{R})$ consisting of real invertible $n \times n$-matrices viewed as a subspace of $\mathbb{R}^{n\times n}$.*

Proof. The first three examples are straightforward to show that they are locally compact Hausdorff groups and that group multiplication and inversion are continuous operations. For the general linear group note that we can identify the space of all real $n \times n$-matrices, $M(n \times n, \mathbb{R})$ with $\mathbb{R}^{n\times n}$ by stacking the columns of a given matrix on top of each other. After doing this identification we can give $\mathrm{GL}_n(\mathbb{R})$ the subspace topology induced by this identification. Since $\mathbb{R}^{n\times n}$ is locally compact and Hausdorff it follows that $GL_n(\mathbb{R})$ is locally compact and Hausdorff. Using the identification above a function mapping into $M(n \times n, \mathbb{R})$ is continuous if and only if each component is continuous and it follows that matrix multiplication $m : M(n\times n, \mathbb{R}) \times M(n\times n, \mathbb{R}) \to M(n\times n, \mathbb{R})$

is continuous since each entry in the matrix AB is a polynomial in the entries of A and B which is continuous. The restriction of m to $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$ is therefore continuous. By Cramer's rule the inverse of $A \in GL_n(\mathbb{R})$ is given by

$$A^{-1} = \frac{1}{\det A} C^\top.$$

Here $C^\top$ (the transpose of the cofactor matrix of A) is a polynomial in the entries of A hence continuous and $\det A$ is by the same reason continuous which shows that inversion is continuous. $\square$

2.1 Uniform Continuity

We will sometimes need a more restrictive version of uniform continuity for topological groups. The details are given below.

Definition 2.1.1. *Let G be a topological group. We say that a function $f : G \to \mathbb{C}$ is*

- *left uniformly continuous if for any $\epsilon > 0$ there exists an open set V containing the identity such that $|f(x) - f(y)| < \epsilon$ whenever $x \in Uy$*
- *right uniformly continuous if for any $\epsilon > 0$ there exists an open set V containing the identity such that $|f(x) - f(y)| < \epsilon$ whenever $y \in xU$*
- *uniformly continuous if it is left- and right uniformly continuous.*

Theorem 2.1.2. *If f is continuous compactly supported, complex valued function then f is uniformly continuous.*

Proof. See Lemma 1.3.7 in [3] $\square$

2.2 Nets

On well behaved topological spaces like metric spaces it is well known that a function $f : X \to Y$ is continuous if and only if for any sequence $(x_i)_i$ in X one has that if $x_i \to x$ then $f(x_i) \to f(x)$. The notion of nets for a topological space generalizes the notion of a sequence and enables one to check continuity of a function $f : X \to Y$ between arbitrary topological spaces by examining convergence of nets just like one does with sequences in the metric space setting.

Definition 2.2.1 (Directed Sets and Nets)**.** *A directed set I is a set I equipped with a reflexive and transitive relation $\leq$ such that for any $i, k \in I$ there exists $j \in I$ such that $i \leq j$ and $k \leq j$. Let X be a topological space and I a directed set. A net is a function $x_\bullet : I \to X$. One often writes x_i instead of $x_\bullet(i)$. A net $x_\bullet : I \to X$ converges to $x \in X$, written $x_\bullet \to x$ if for any open set V containing x there exists an $i_0 \in I$ such that $x_i \in V$ whenever $i_0 \leq i$.*

The following propeties of nets are useful and easy to verify

Proposition 2.2.2. *Let X and Y be topological spaces then the following holds*

- *A function $f : X \to Y$ is continuous if and only if $f(x_\bullet) \to f(x)$ whenever $x_\bullet \to x$ for any net in X.*
- *If X is Hausdorff then any convergent net in X has a unique limit.*
- *If $P = \prod_{j \in J} X_j$ has the product topology then the net $x_\bullet : I \to P$ converges to $x \in P$ if and only if $\pi_j(x_\bullet) \to \pi_j(x)$ for all $j \in J$.*

Definition 2.2.3 (The Topology of Pointwise Convergence). *Let X be a set, Y be a topological space and M be a collection of maps from X to Y. The topology of pointwise convergence on M is the topology of M inherited from the product space $\prod_X Y$.*

Remark 2.1. *Let M be a set of maps from a topological space X into another topological space. Using Proposition 2.2.2 it follows that a net $(f_\bullet)$ in M converges pointwise to $f \in M$ if and only if the net $\pi_x(f_\bullet) = f_\bullet(x)$ converges to $f(x)$ for each $x \in X$.*

Definition 2.2.4 (The General Linear Group). *If V is a topological vector space we can define $GL(V)$ as the set of all continuous vector space isomorphisms $L : V \to V$ with continuous inverse. The set $GL(V)$ is called the general linear group of V and as a topological space it is given the topology of pointwise convergence.*

Proposition 2.2.5. *The general linear group $GL(V)$ is a group in the ordinary algebraic sense.*

Proof. The proposition follows straight from the definition. □

2.3 The Haar Measure

In this section we will introduce and show existence of a measure called a Haar measure on any locally compact group that also interacts nicely with the group operation. A Haar measure is in particular a Radon measure which means that the measure interacts nicely with the topological properties of the group as well.

Definition 2.3.1 (Radon Measure). *Let X be a Hausdorff space. A measure $\mu : \mathcal{B}(X) \to [0, +\infty]$ is*

- *locally finite if any $x \in X$ has an open neighbourhood of finite measure*
- *outer regular on $E \subseteq G$ if $\mu(E) = \inf_{E \subseteq V} \mu(V)$*
- *inner regular on $E \subseteq G$ if $\mu(E) = \sup_{K \subseteq E} \mu(K)$*

where the infimum is taken over open supersets and the supremum is taken over compact subsets. A Radon measure $\mu : \mathcal{B}(X) \to [0, +\infty]$ is a locally finite measure that is outer regular on any Borel set and inner regular on any open set.

Remark 2.2. *Note that a locally finite measure is finite on any compact set. Conversely if X is locally compact and μ is a measure defined on $\mathcal{B}(X)$ that is finite on compact sets then, by definition of local compactness it is also locally finite.*

Remark 2.3. *Let X be a locally compact Hausdorff space and $C_c(X)$ be the space of complex valued, continuous, compactly supported functions on X. For any Radon measure μ on X the mapping $\Lambda_\mu : C_c(X) \to \mathbb{C}$*

$$\Lambda_\mu(f) = \int_X f d\mu \tag{1}$$

is a positive linear functional on $C_c(X)$, that is $\Lambda_\mu(f) \geq 0$ whenever $f \geq 0$. Conversely, the Riesz representation theorem for positive linear functionals, see Theorem 2.14 in [4] shows that any positive linear functional Λ on $C_c(X)$ is of the form (1) for some unique Radon measure. This correspondence gives us a bijection between the set of Radon measures on X and positive linear functionals on $C_c(X)$. This relationship will be used througout the text.

Definition 2.3.2 (Haar measure)**.** *Let G be a locally compact group. A non-zero Radon measure $\mu : \mathcal{B}(G) \to [0, +\infty]$ is*

- *a left Haar measure if $\mu(gE) = \mu(E)$ for any $g \in G$ and $E \in \mathcal{B}(G)$*
- *a right Haar measure if $\mu(Eg) = \mu(E)$ for any $g \in G$ and $E \in \mathcal{B}(G)$*

Any left Haar measure on G will be referred to as a Haar measure on G.

Remark 2.4. *Note that by Proposition 2.0.7 the sets gE and Eg are Borel sets whenever E is.*

Theorem 2.3.3 (Existence of Haar measures)**.** *For any locally compact group G there exists a left Haar measure and there exists a right Haar measure.*

Sketch of Proof. The idea behind the proof is to construct a positive linear function Λ on $C_c(G)$ such that

$$\Lambda(L_g f) = \Lambda(f)$$

for any $f \in C_c(G)$ and $g \in G$. By Riesz representation theorem there corresponds to this functional a Radon measure μ on G such that

$$\Lambda(f) = \int_G f d\mu$$

for any $f \in C_c(G)$. It remains to show (left) translation invariance of this measure, that is if $E \in \mathcal{B}(G)$ and $g \in G$ then

$$\mu(gE) = \mu(E). \tag{2}$$

Assume first that $E = K$ is a compact subset of G and pick $\epsilon > 0$. By outer regularity on K there exists an open set V such that $K \subseteq V$ and

$$\mu(K) \leq \mu(V) < \mu(K) + \epsilon.$$

By Urysohn's Lemma, see Theorem 2.12 in [4] there exists a compactly supported function $f \in C_c(G)$ such that

$$\chi_K \leq f \leq \chi_V.$$

It follows that

$$\begin{aligned}
\mu(gK) = \int_G \chi_{gK} d\mu = \int_G \chi_K(g^{-1}x) d\mu(x) \\
&\leq \int_G f(g^{-1}x) d\mu(x) = \Lambda(L_g f) \\
&= \Lambda(f) = \int_G f d\mu \\
&\leq \int_G \chi_V d\mu = \mu(V) \\
&< \mu(K) + \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we get

$$\mu(gK) \leq \mu(K).$$

A symmetrical argument shows that

$$\mu(K) \leq \mu(gK).$$

Hence (2) holds for any compact set. It then follows from inner regularity that (2) holds for any open set V. From this it follows from outer regularity and Proposition 2.0.7 that (1) holds for any Borel set. For the the proof of the right Haar measure one can now take a left Haar measure, μ and then consider $\mu'(E) = \mu(E^{-1})$ which is easily shown to be a right Haar measure. $\square$

Proposition 2.3.4. *If μ is a Haar measure on the locally compact group G then*

$$\int_G f(gx) d\mu(x) = \int_G f d\mu$$

for any $f \in L^1(\mu)$ and $g \in G$.

Proof. If E is any Borel set of G then for any fixed g one has

$$\begin{aligned}
\int_G \chi_E(gx) d\mu(x) &= \int_G \chi_{g^{-1}E}(x) d\mu(x) \\
&= \mu(g^{-1}E) = \mu(E) \\
&= \int_G \chi_E d\mu.
\end{aligned}$$

By linearity it follows that the proposition holds for any measurable simple function. Assume that $f \in L^1(\mu)$ is a positive function. There exists a sequence of simple measurable functions (s_n) pointwise monotonously increasing and converging everywhere to f by Theorem 1.17 in [4]. Then $s_n(gx)$ converges monotonously to $f(gx)$ for all $x \in G$ and by the monotone convergence theorem it follows that

$$\begin{aligned} \int_G f(gx)d\mu(x) &= \lim_{n\to\infty} \int_G s_n(gx)d\mu(x) \\ &= \lim_{n\to\infty} \int_G s_n d\mu = \int_G f d\mu. \end{aligned}$$

The proposition then follows after decomposing any element of $L^1(\mu)$ into its real positive, real negative, positive imaginary and negative imaginary parts. □

Proposition 2.3.5. *If μ is a Haar measure and V is a non-empty open subset of the locally compact group G then $\mu(V) > 0$.*

Proof. Assume that $\mu(V) = 0$ and pick any compact set K. Then K can be covered by sets of the form gV for $g \in V$. Since any set of the form gV is open the family $(gV)_{g\in G}$ is an open cover of and K. By compactness of K there exists $g_1, ..., g_n$ such that

$$\mu(K) \leq \mu(\bigcup_{i=1}^{n} g_i V) \leq \sum_{i=1}^{n} \mu(g_i V) = n\mu(V) = 0.$$

It follows that any compact set has measure 0 and by inner regularity it follows that any open set has measure 0. By outer regularity it follows that any Borel set has measure 0 which is a contradiction since $\mu = 0$ by definition is not a Haar measure. Hence $\mu(V) > 0$. □

Theorem 2.3.6 (Uniqueness of Haar measures)**.** *If μ and ν are two Haar measures on G then there exists a number $c \in \mathbb{R}_{>0}$ such that*

$$\mu(E) = c\nu(E)$$

for any $E \in \mathcal{B}(G)$.

Proof. See Theorem 2.20 in [5] □

Notation 2.3.7. *Since any two Haar measures μ and ν are proportional to each other it follows that the two spaces $L^p(\mu)$ and $L^p(\nu)$, $p \in [1, +\infty]$ contain exactly the same elements. We will therefore for any Haar measure μ on G write $L^p(\mu)$ simply as $L^p(G)$. We will also denote integration with respect to a Haar measure μ by*

$$\int_G f d\mu = \int_G f dx$$

Example 2.3.8 (Examples of Haar measures)**.**

- *The counting measure is a Haar measure on any discrete group, in particular it is a Haar measure on* $\mathbb{Z}$*.*
- *The* n*-dimensional Lebesgue measure,* m_n *is a Haar measure on* $\mathbb{R}^n$*.*
- *The measure* $\mu(E) = m_1(\gamma^{-1}(E))$ *defines a Haar measure on the unit circle where* $\gamma : [0, 1) \to \mathbb{T}$ *given by* $\gamma(t) = e^{2\pi it}$*.*
- *When viewing* $GL_n(\mathbb{R})$ *as a subset of* $\mathbb{R}^{n\times n}$ *the set function*

$$\mu(E) = \int_E \frac{1}{|\det(X)|^n} dm_{n^2}(X)$$

defined on the Borel sets of $GL_n(\mathbb{R})$ *is a Haar measure.*

Proposition 2.3.9. *For any open set* $V \subseteq G$*, of finite Haar-measure there exists a sequence* (f_n) *of real valued compactly supported functions such that* $\lim_{n\to\infty} f_n = \chi_V$ *almost everywhere and* $f_n \leq f_{n+1} \leq \chi_V$*.*

Proof. Pick an open set V such that $\mu(V) < \infty$. By inner regularity of the Haar measure there exists for each integer $n > 0$ a compact set $K_n \subseteq V$ such that

$$\mu(V) - \frac{1}{n} < \mu(K_n) \leq \mu(V).$$

By Urysohn's lemma see Theorem 2.12 in [4] there exists for each $n > 0$ a compactly supported real valued function f_n such that

$$\chi_{K_n} \leq f_n \leq \chi_V.$$

After considering $g_n = \max_{1\leq i\leq n} f_i$ we can without loss of generality assume that $f_n \leq f_{n+1}$ for all $n > 0$. It is clear that $\lim_{n\to\infty} f(x) = \chi_V(x)$ except possibly on the set

$$E = V - \bigcup_n K_n$$

but

$$\mu(E) \leq \mu(V - K_n) = \mu(V) - \mu(K_n) < \frac{1}{n}$$

and it follows that $\mu(E) = 0$. □

2.4 The Modular Function

Let μ be a Haar measure on G, fix a $g \in G$ and consider the function defined on $\mathcal{B}(G)$ given by

$$\mu_g(E) = \mu(Eg^{-1}).$$

Since a Haar measure is assumed to be left invariant and not right invariant it is not always the case that $\mu_g = \mu$. However using left invariance of μ it is easy to show that μ_g is a (left) Haar measure. By the uniqueness Theorem 2.3.6 it follows that there exists a number $\Delta(g) > 0$ such that

$$\mu(Eg^{-1}) = \Delta(g)\mu(E).$$

From here we see that we get a function $\Delta : G \to \mathbb{R}_{>0}$ that in a sense measures to what extent our (left) Haar measure fails to be a right Haar measure. The reason why the inverse was choosen in $\mu_g(E) = \mu(Eg^{-1})$ was in order to make notation consistent with the theorem below.

Theorem 2.4.1 (The Modular Function). *For any locally compact group G there exists a continuous group homomorphism $\Delta : G \to \mathbb{R}_{>0}$ such that*

$$\int_G f(xg)d\mu(x) = \Delta(g) \int_G f d\mu$$

for any Haar measure μ and any $f \in L^1(G)$. Furthermore we have for any Borel set $E \subseteq G$ and any $g \in G$ the formula

$$\mu(Eg^{-1}) = \Delta(g)\mu(E).$$

Proof. Let μ be a Haar measure on G, select an element $g \in G$ and consider the functional defined on $C_c(G)$ by

$$\Lambda_g(f) = \int_G R_g f d\mu$$

It is easy to show that Λ_g is a positive linear functional. This functional also satisfies $\Lambda_g(L_h f) = \Lambda_g(f)$ for any $f \in C_c(G)$ and $h \in G$ since

$$\begin{aligned}
\Lambda_g(L_h f) &= \int_G R_g(L_h f) d\mu \\
&= \int_G f(h^{-1}xg) d\mu \\
&= \int_G L_h(R_g f) d\mu \\
&= \int_G R_g f d\mu \\
&= \Lambda_g(f)
\end{aligned}$$

Note that Proposition 2.3.4 has been used. It follows by the proof of the existence Theorem 2.3.3 that there exists a Haar measure μ_g on G such that

$$\Lambda_g(f) = \int_G f d\mu_g.$$

By the uniqueness Theorem 2.3.6 there exists a number $\Delta(g) > 0$ such that $\mu_g = \Delta(g)\mu$. Assume that ν is another Haar measure on G and the corresponding relation $\nu_g = \Delta'(g)\nu$ is given for some $\Delta'(g) > 0$. Pick a V such that $0 < \mu(V) < \infty$ then $\mu_g(V) = \Delta(g)\mu(V) \in (0, +\infty)$. Since there exists a $c > 0$ such that $\nu = c\mu$ it follows that $\nu(V) = c\mu(V)$ and $\nu_g(V) = c\mu_g(V)$ hence

$$\frac{\Delta(g)}{\Delta'(g)} = \frac{\mu_g(V)/\mu(V)}{\nu_g(V)/\nu(V)} = \frac{\mu_g(V)\nu(V)}{\nu_g(V)\mu(V)} = 1.$$

Thus $\Delta(g) = \Delta'(g)$ and it follows that the function $\Delta : G \to \mathbb{R}_{>0}$ is independent of the choice of Haar measure. We also have

$$\int_G f(xg)d\mu(x) = \Lambda_g(f) = \int_G f d\mu_g = \Delta(g)\int_G f d\mu.$$

for any $f \in C_c(G)$. Using this we see that Δ is a group homomorphism since for any $f \in C_c(G)$ and $g, h \in G$ we have

$$\begin{aligned}
\Delta(gh)\int_G f d\mu = \Lambda_{gh}(f) &= \int_G f(xgh)d\mu \\
&= \int_G (R_h f)(xg)d\mu(x) \\
&= \Delta(g)\int_G (R_h f)(x)d\mu(x) \\
&= \Delta(g)\int_G f(xh)d\mu(x) \\
&= \Delta(g)\Delta(h)\int_G f d\mu.
\end{aligned}$$

We now prove continuity of Δ. Since Δ is a group homomorphism it is enough to check continuity at the identity element $1 \in G$. Pick an $\epsilon > 0$ and choose an $f \in C_c(G)$ such that $\int_G f d\mu = 1$. Since G is locally compact there exists an open set U containing 1 such that $K = \overline{U}$ is compact. Let $\mathrm{supp}(f)$ be the (compact) support of f. The set $K\mathrm{supp}(f)$ is compact since it equals $m(K, \mathrm{supp}(f))$ where m is the multiplication function which is continuous. By uniform continuity there exists an open set $V \subseteq K$ containing 1 such that

$$|f(x) - f(y)| < \frac{\epsilon}{\mu(K\mathrm{supp}(f))}.$$

whenever $x^{-1}y \in V$. It follows that if g lies in the open set V^{-1} (which contains the identity) then

$$|f(xg) - f(x)| < \frac{\epsilon}{\mu(K\mathrm{supp}(f))}$$

for any $x \in G$. It follows that

$$\begin{aligned}
|\Delta(g) - 1| &\leq \int_G |f(xg) - f(x)| d\mu(x) \\
&= \int_{V\mathrm{supp}(f)} |f(xg) - f(x)| d\mu(x) \\
&< \mu(V\mathrm{supp}(f))) \frac{\epsilon}{\mu(K\mathrm{supp}(f))} \leq \epsilon
\end{aligned}$$

whenever $g \in V^{-1}$. Note that the support of the function $x \mapsto f(xg) - f(x)$ indeed is a subset of $V\mathrm{supp}(f)$ when $g \in V^{-1}$ since

$$\begin{aligned}
\mathrm{supp}(R_g f - f) &\subseteq \mathrm{supp}(R_g f) \cup \mathrm{supp}(f) \\
&\subseteq V\mathrm{supp}(f) \cup \mathrm{supp}(f) \\
&= V\mathrm{supp}(f)
\end{aligned}$$

where the last equality is true since $1 \in V$. We now prove the relation

$$\mu(Eg^{-1}) = \Delta(g)\mu(E)$$

Pick an open $V \subseteq G$ such that $\mu(V) < +\infty$. By Proposition 2.3.9 we can find a sequence of compactly supported function f_n such that $f_n \leq f_{n+1}$ for all n and $\lim_{n\to\infty} f_n = \chi_V$ almost everywhere with respect to μ. By uniqueness of left Haar measures we also have $\lim_{n\to+\infty} f_n = \chi_V$ almost everywhere with respect to μ_g. The monotone convergence theorem yields

$$\begin{aligned}
\mu_g(V) = \int_G \chi_V d\mu_g &= \lim_{n\to\infty} \int_G f_n d\mu_g \\
&= \lim_{n\to\infty} \int_G f_n(xg) d\mu(x) \\
&= \int_G \chi_V(xg) d\mu(x) \\
&= \int_G \chi_{Vg^{-1}} d\mu \\
&= \mu(Vg^{-1}).
\end{aligned}$$

Hence

$$\mu(Vg^{-1}) = \mu_g(V) = \Delta(g)\mu(V),$$

for any open set $V \subseteq G$ of finite measure and $g \in G$. From outer regularity it follows that the statement holds for any compact K hence for any open set by inner regularity and then for any Borel set by outer regularity. We next show that the formula

$$\int_G f(xg) d\mu(x) = \Delta(g) \int_G f d\mu$$

holds for any $f \in L^1(G)$. We pick an integrable function f and consider the function $(R_y f)(x) = f(xy)$ for a fixed $y \in G$. Since $(R_y f)$ is the composition of measurable functions it is measurable. If $f = g$ a.e. then

$$\begin{aligned}\mu(\{x \in G : \ R_y f(x) \neq R_y g(x)\}) &= \mu(\{x \in G : \ f(xy) \neq g(xy)\}) \\ &= \mu(\{x \in G : \ f(x) \neq g(x)\}y^{-1}) \\ &= \Delta(y)\mu(\{x \in G : \ f(x) \neq g(x)\}) = 0.\end{aligned}$$

It follows that $R_y f = R_y g$ almost everywhere for any $f, g \in L^1(G)$. It is clear that the formula holds for $f = \chi_E$ where E is a Borel set since $\mu(Eg^{-1}) = \Delta(g)\mu(E)$ and by approximation by simple functions it follows that the formula holds for any $f \in L^1(G)$. $\square$

Definition 2.4.2. *The function $\Delta : G \to \mathbb{R}_{>0}$ appearing in Theorem 2.4.1 is called the modular function of G and a locally compact group is called unimodular if $\Delta(g) = 1$ for all $g \in G$.*

Proposition 2.4.3. *Let G be a locally compact group then for any $f \in L^1(G)$ we have*

$$\int_G f(x)dx = \int_G \Delta(x) f(x^{-1})dx$$

and for any $f \in C_c(G)$ we have

$$\int_G f(x^{-1})dx = \int_G \Delta(x) f(x)dx.$$

Proof. Define $\Lambda : C_c(G) \to \mathbb{C}$ by

$$\Lambda(f) = \int_G \Delta(x) f(x^{-1})dx$$

Since $x \mapsto x^{-1}$ is continuous and f has compact support Λ is well defined. For any $g \in G$ and $f \in C_c(G)$ we have

$$\begin{aligned}\Lambda(L_g f) &= \int_G \Delta(x)(L_g f)(x^{-1})dx \\ &= \int_G \Delta(x) f(g^{-1}x^{-1})dx \\ &= \Delta(g^{-1}) \int_G \Delta(xg) f((xg)^{-1})dx \\ &= \Delta(g^{-1})\Delta(g) \int_G \Delta(x) f((x)^{-1})dx \\ &= \Lambda(f).\end{aligned}$$

It follows from the proof of construction of the Haar measure in Theorem 2.3.3 that Λ corresponds to a Haar measure and by the uniqueness theorem of Haar measures 2.3.6 it follows that there exists $c \in \mathbb{R}_{>0}$ such that $\Lambda(f) = cI(f)$ for all $f \in C_c(G)$ where $I(f) = \int_G f dx$. Pick an $\epsilon > 0$. By continuity of Δ there exists an open set V containing the identity such that $|\Delta(x) - 1| < \epsilon$ whenever $x \in V$. Let $U \subseteq V$ be a symmetric set containing the identity of G. Choose a function $f \in C_c(G)$ with support in U that is not zero on some open set. Define $g(x) = f(x)f(x^{-1})$. Then g is continuous with compact support contained in U and $g(x) = g(x^{-1})$ for all $x \in G$. By considering $g/I(|g|)$ we can assume that $I(|g|) = 1$ and it follows that

$$\begin{aligned}
|\Lambda(g) - I(g)| &\leq \int_G |\Delta(x)g(x^{-1}) - g(x)|dx \\
&= \int_U |\Delta(x) - 1| \; |g(x)|dx \\
&< \epsilon I(|g|) \\
&= \epsilon.
\end{aligned}$$

It follows that $\Lambda(g) = I(g)$ hence $c = 1$ and therefore

$$\int_G f(x)dx = \int_G \Delta(x)f(x^{-1})dx \tag{1}$$

for any $f \in C_c(G)$. If $V \subseteq G$ is open and of finite measure then by Proposition 2.3.9 there exists a sequence of monotonously increasing compactly supported functions (f_n) that converges to χ_V almost everywhere. By the monotone convergence theorem we then get

$$\begin{aligned}
\int_G \Delta(x)\chi_V(x^{-1})dx &= \lim_{n\to\infty} \int_G \Delta(x)f_n(x^{-1})dx \\
&= \lim_{n\to\infty} \int_G f_n(x)dx \\
&= \int_G \chi_V dx.
\end{aligned}$$

Hence (1) holds for $f = \chi_V$ where $\mu(V) < \infty$. Let E be a Borel set that is contained in some compact set K. By outer regularity for each $n > 0$ there exists an open set V containing E such that

$$\mu(E) \leq \mu(V) < \mu(E) + \frac{1}{n}$$

By considering $W_n = \bigcap_{i=1}^n V_i$ we can assume that $V_{n+1} \subseteq V_n$ and it follows that $\lim_{n\to+\infty} \chi_{V_n} = \chi_E$ almost everywhere and by the dominated convergence theorem we get

$$\begin{aligned}\int_G \chi_E dx &= \lim_{n\to\infty}\int_G \chi_{V_n} dx \\ &= \lim_{n\to\infty}\int_G \Delta(x)\chi_{V_n}(x^{-1})dx \\ &= \int_G \Delta(x)\chi_E(x^{-1})dx.\end{aligned}$$

The last equality holds because the continuous function Δ is bounded on K. By linearity (1) then holds for any linear combination of characteristic functions with compact supports. If f is a positive integrable function then it is an increasing limit of simple functions of compact support and by the monotone convergence theorem it follows that (1) holds for f. The general case clearly follows from the positive case. The statement

$$\int_G f(x^{-1})dx = \int_G \Delta(x)f(x)dx$$

for any $f \in L^1(G)$ follows from (1) by applying it to the integrable function $g(x) = f(x^{-1})$. $\square$

3 Group Representations

In this section group representations will be defined. We begin with a general definition for group representations on topological vector spaces and then restrict our attention to the special case of unitary representations. After this is done we introduce the so called left- and right regular representation of a locally compact group which appear in the contruction of the reduced group C^*-algebra.

3.1 Representations of Topological Vector Spaces

Definition 3.1.1 (Group Representation)**.** *Let G be a topological group and V a topological (Hausdorff) vector space. A representation of G on V is a (group) homomorphism $\pi : G \to GL(V)$ such that the map $(g, v) \mapsto \pi(g)v$ is continuous.*

Proposition 3.1.2. *If V is a Banach space then a group homomorphism $\pi : G \to GL(V)$ is a representation if and only if π is continuous and the map $g \mapsto \|\pi(g)\|$ defined on G is locally bounded.*

Proof. Let $\pi : G \to \mathrm{GL}(V)$ be a representation on the Banach space V. If $g_\bullet$ is a net in G that converges to $g \in G$ then by definition of the topology of pointwise convergence it follows that $\pi(g_\bullet)$ converges to $\pi(g)$ if and only if $\pi(g_\bullet)v$ converges to $\pi(g)v$ for all $v \in V$ which is true since π is a representation. It follows that π is continuous. Pick an element $g_0 \in G$. Since $(g, v) \mapsto \pi(g)v$ is continuous there exists an open sets $U \subseteq G$ and $W \subseteq V$ such that $g_0 \in U$, $0 \in W$ and

$$\|\pi(g)v\| \leq 1$$

whenever $g \in U$ and $v \in W$. Since there exists an $r > 0$ such that the open ball of radius r centered at 0 is contained in W. It follows that

$$\|\pi(g)\| = \sup_{\|v\| \leq 1} \frac{1}{r}\|\pi(g)rv\| \leq \frac{1}{r}$$

whenever $g \in U$ which means that π is locally bounded. Conversely assume that $\pi : G \to \mathrm{GL}(V)$ is a continuous homomorphism and is locally bounded. Pick elements $g_0 \in G$, $v_0 \in V$ and $\epsilon > 0$. Then by assumption g_0 has an open neighbourhood U such that $\|\pi(g)\| \leq 1$ for all $g \in U$ and

$$\|\pi(g) - \pi(g_0)\| < \frac{\epsilon}{2\|v_0\|}$$

whenever $g \in U$. Let W be the ball of radius $\frac{\epsilon}{2}$ centered at v_0. If $v \in W$ and $g \in U$ we have

$$\|\pi(g)v - \pi(g_0 v_0)\| \leq \|\pi(g)\|\|v - v_0\| + \|\pi(g) - \pi(g_0)\|\|v_0\| < \epsilon$$

which shows that the map $(g, v) \mapsto \pi(g)v$ is continuous. $\square$

3.2 Unitary Representations

Before defining the unitary representations we recall what Hilbert space adjoints and unitary operators are.

Proposition 3.2.1. *Let H_1, H_2 be two Hilbert spaces and $\mathscr{B}(H_1, H_2)$ be the set of all bounded linear operators from H_1, H_2. For any $A \in \mathscr{B}(H_1, H_2)$ there exists a unique element $A^* \in \mathscr{B}(H_2, H_1)$ such that*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for any $x \in H_1$ and $y \in H_2$. Furthermore we have $(A^*)^* = A$ and $\|A\| = \|A^*\|$.

Proof. Choose an $A \in \mathscr{B}(H_1, H_2)$. Fix a $y \in H_2$ and define $f_y : H_1 \to \mathbb{C}$ by

$$f_y(x) = \langle Ax, y\rangle.$$

It is clear that f_y is linear and since $|f_y(x)| \le \|A\|\|x\|\|y\|$ it follows that f_y is a bounded linear functional on H_1 of norm $\|f_y\| \le \|A\|\|y\|$. It follows that for each $y \in H_2$ there exists a unique element $A^*y \in H_1$ of norm $\|A^*y\| = \|f_y\|$ such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for all $x \in H_1$. For any $x \in H_1$, $y_1, y_2 \in H_2$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned}\langle x, A^*(\alpha y_1 + \beta y_2)\rangle &= \langle Ax, \alpha y_1 + \beta y_2\rangle \\ &= \overline{\alpha}\langle Ax, y_1\rangle + \overline{\beta}\langle Ax, y_2\rangle \\ &= \overline{\alpha}\langle x, A^*y_1\rangle + \overline{\beta}\langle x, A^*y_2\rangle \\ &= \langle x, \alpha A^*y_1 + \beta A^*y_2\rangle.\end{aligned}$$

This shows that A^* is linear. Since $\|A^*y\| \le \|A\|\|y\|$ it follows that $\|A^*\| \le \|A\|$ and we see that A^* is bounded. The relation $A^{**} = A$ follows from the calculation

$$\langle y, A^{**}x\rangle = \langle A^*y, x\rangle = \overline{\langle x, A^*y\rangle} = \overline{\langle Ax, y\rangle} = \langle y, Ax\rangle$$

Lastly to show that $\|A^*\| = \|A\|$ we recall that we already know that $\|A^*\| \le A$. Since $A^{**} = A$ we also have

$$\|A\| = \|A^{**}\| \le \|A^*\|,$$

which completes the proof □

Definition 3.2.2 (The Adjoint Operator)**.** *The operator $A^* \in \mathscr{B}(H_2, H_1)$ appearing in Theorem 3.2.1 for $A \in \mathscr{B}(H_1, H_2)$ is called the adjoint of A.*

Definition 3.2.3 (Unitary Operators and the Unitary Group)**.** *An operator $A \in \mathscr{B}(H_1, H_2)$ is called unitary if $A^*A = id_{H_1}$ and $AA^* = id_{H_2}$. The set of all unitary elements in $\mathscr{B}(H) = \mathscr{B}(H, H)$ will be denoted $\mathcal{U}(H)$ and is called the unitary group of H.*

Remark 3.1. *It is easy to show that $A \in \mathscr{B}(H_1, H_2)$ is unitary if and only if A is a surjective isometry.*

Proposition 3.2.4. *The unitary group $\mathcal{U}(H)$ with the topology of pointwise convergence is a topological group.*

Proof. To prove continuity of multiplication we need to show that if $F_\bullet : I \to \mathcal{U}(H) \times \mathcal{U}(H)$ is a net that converges to $(A, B) \in \mathcal{U}(H) \times \mathcal{U}(H)$ then $m(F_\bullet)$ converges to AB. The net $F_\bullet$ induces two nets $A_\bullet : I \to \mathcal{U}(H)$ and $B_\bullet : I \to \mathcal{U}(H)$ such that $A_\bullet$ converges to A and $B_\bullet$ converges to B. In order to avoid confusion of what $A_\bullet B_\bullet$ means we drop the dot notation in the next part which means that we need to show that $A_i B_i$ converges to AB. By definition of the topology of pointwise convergence $A_i B_i$ converges to AB if and only if $A_i B_i x$ converges to ABx for any $x \in H$. Pick an $x \in H$ then since $B_i \to B$ we have that $\|(B_i - B)x\| \to 0$ and since $A_i \to A$ we have that

$$\|(A_i - A)Bx\| \to 0.$$

Since unitary operators are isometric and hence have norm 1 it follows that

$$\begin{aligned}\|A_i B_i x - ABx\| &\leq \|A_i(B_i - B)x\| + \|(A_i - A)Bx\| \\ &\leq \|(B_i - B)x\| + \|(A_i - A)Bx\| \to 0.\end{aligned}$$

Which shows that multiplication is continuous. To show that inversion is continuous, assume that $A_\bullet \to A$ and pick $x \in H$. Set $y = A^{-1}x$ then $\|(A - A_\bullet)y\| \to 0$ and it follows that

$$\begin{aligned}\|A_\bullet^{-1}x - A^{-1}x\| &= \|(A_\bullet^{-1} - A^{-1})Ay\| \\ &= \|(A_\bullet^{-1}A - I)y\| \\ &= \|A_\bullet(A_\bullet^{-1}A - I)y\| \\ &= \|(A - A_\bullet)y\| \to 0.\end{aligned}$$

□

Definition 3.2.5. *Let G be a topological group and H be a Hilbert space. A unitary representation of G on H is a homomorphism of topological groups $\pi : G \to \mathcal{U}(H)$ and is denoted by (π, H).*

Remark 3.2. *By Propostion 3.1.2 a unitary representation is a representation in the sense of Definition 3.1.1.*

If $U : H_1 \to H_2$ is unitary then for any $x, y \in H_1$ we have

$$\langle Ax, Ay\rangle = \langle x, A^*Ay\rangle = \langle x, y\rangle$$

and it follows that U preserves the inner product of H_1. Since the inverse of a unitary map also is unitary it follows that the unitary maps can be seen as the Hilbert space isomorphisms. The notion of equivalence of unitary representations (π_1, H_1), (π_2, H_2) should therefore be a unitary map $U : H_1 \to H_2$ that interacts nicely with π_1 and π_2 as is formally defined now!

Definition 3.2.6 (Equivalence of Unitary Representations). *Let G be a topological group and (π_1, H_1) and (π_2, H_2) be two unitary representations of G. The two representations are unitarily equivalent if there exists a unitary map $U : H_1 \to H_2$ such that $U\pi_1(g) = \pi_2(g)U$ for any $g \in G$.*

If (π, H) is a unitary representation and K is a closed subspace H such that $\pi(g)K \subseteq K$ for all $g \in G$. Then $K \subseteq \pi(g^{-1})K$ for all $g \in G$ hence $\pi(g)K = K$ for all $g \in G$. The restrictions $\pi(g)|_K$ therefore map onto K and are isometries hence $\pi(g)|_K \in \mathcal{U}(K)$. It follows that we get a representation $g \mapsto \pi(g)|_K$ called a subrepresentation of (π, H).

Definition 3.2.7 (Subrepresentations). *Let (π, H) be unitary representation of a topological group G. If there exists a closed subspace K of H such $\pi(g)K \subseteq K$ for all $g \in G$ then K is called π-stable or π-invariant. The correpsonding unitary representation denoted $(\pi|_K, K)$ of G defined by $\pi|_K(g) = \pi(g)|_K$ is called a subrepresentation of G.*

Definition 3.2.8 (Containment of Representations). *If (π, H_1) and (π_2, H_2) are unitary representations of G we say that π_1 is contained in π_2 if π_1 is unitarily equivalent to a subpresentation of π_2. We denote this relation by $\pi_1 \leq \pi_2$.*

Remark 3.3. *For any unitary representation (π, H) of G the two subrepresentations corresponding to the trivial subspaces $\{0\}$ and H of H are called the trivial subrepresentations of (π, H).*

After having defined the notion of a subrepresentation it is easy to define the notion of an irreducible representation

Definition 3.2.9. *A representation (π, H) of G is called irreducible if its only subrepresentations are trivial.*

Remark 3.4. *As one expects, unitary equivalence preserves irreducibility. This is the case since if (π_1, H_1) and (π_2, H_2) are unitarily equivalent by $U : H_1 \to H_2$ and $K_1 \subseteq H_1$ is a π_1-invariant subspace. Then UK_1 is a closed subspace of H_2 and is π_2-invariant since*

$$\pi_2(g)UK_1 = U\pi_1(g)U^{-1}UK_1 \subseteq UK_1.$$

This gives us a bijective correspondence between the π_1-invariant subspaces of H_1 and the π_2-invariant subspaces of H_2. This shows in particular that if π_1 is irreducible then π_2 is irreducible.

Just as one in linear algebra can take the direct sum of two vector spaces to get a new vector space one can in the theory of unitary representations get a new representation called the direct sum representation from two given representations. The direct sum representation acts on the direct sum of the underlying vector spaces and before we define it we give a general construction of the direct sum of Hilbert spaces.

Definition 3.2.10 (Direct Sum of Hilbert Spaces)**.** *Let $(H_i, \langle\cdot,\cdot\rangle_i)$ be a family of Hilbert spaces indexed by I. The Hilbert space direct sum $\bigoplus_{i\in I} H_i$ of the family (H_i) is the set of all $x \in \prod_{i\in I} H_i$ such that*

$$\sum_{i\in I} \langle x(i), x(i)\rangle_i < \infty$$

with componentwise addition and scalar multplication. The inner product of $\bigoplus_{i\in I} H_i$ is given by

$$\langle x, y\rangle = \sum_{i\in I} \langle x(i), y(i)\rangle_i$$

Remark 3.5. *The Cauchy-Schwartz inequality shows that the inner product is well defined and the rest of the defining properties of a Hilbert space are easily verified.*

Definition 3.2.11 (Direct Sum of Unitary Representations)**.** *If $(\pi_i, H_i)_i$ is a family of unitary representations of G indexed by I then the direct sum representation $\bigoplus_{i\in I} \pi_i$ is the unitary representation of G that acts on $\bigoplus_{i\in I} H_i$ by the relation*

$$(\pi(g)x)(i) = \pi_i(g)x(i)$$

for all $x \in \bigoplus_{i\in I} H_i$, $g \in G$ and $i \in I$.

Proposition 3.2.12. *The direct sum $\bigoplus_{i\in I} \pi_i$ of the unitary representations $(\pi_i, H)_{i\in I}$ of G is a unitary representation.*

Proof. For any $x \in \bigoplus_{i\in I} H_i$ and $g \in G$ we have

$$\begin{aligned}\langle(\oplus_{i\in I}\pi_i)(g)x, (\oplus_{i\in I}\pi_i)(g)x\rangle &= \sum_{i\in I} \langle \pi_i(g)x(i), \pi_i(g)x(i)\rangle \\ &= \sum_{i\in I} \langle x(i), x(i)\rangle = \langle x, x\rangle\end{aligned}$$

which shows that $(\oplus_{i\in I}\pi_i)(g)x$ is an element in $\bigoplus_{i\in I} H_i$ and that $(\oplus_{i\in I}\pi_i)(g)$ is an isometry. The operator $(\oplus_{i\in I}\pi_i)(g)$ is also surjective for any $g \in G$ since for any $y \in \bigoplus_{i\in I} H_i$ we can define the element $x \in \bigoplus_{i\in I} H_i$ by $x(i) = \pi_i(g)^* y(i)$ and then it follows that $(\oplus_{i\in I}\pi_i)(g)x = y$. This shows that $(\oplus_{i\in I}\pi_i)(g)$ is a unitary operator on $\bigoplus_{i\in I} H_i$. It remains to show that the map $g \mapsto (\oplus_{i\in I}\pi_i)(g) \in \mathcal{U}(\bigoplus_{i\in I} H_i)$ is a continuous group homomorphism. The homomorphism property is clear. For continuity of we need to show that $(\oplus_{i\in I}\pi_i)(g_\bullet)x \to (\oplus_{i\in I}\pi_i)(g)x$ for any $g \in G$, $x \in \oplus_{i\in I} H_i$ and any convergent net $g_\bullet \to g$. By definition of the inner product it follows that $(\oplus_{i\in I}\pi_i)(g_\bullet)x \to (\oplus_{i\in I}\pi_i)(g)x$ if and only if $((\oplus_{i\in I}\pi_i)(g_\bullet)x)(i) \to ((\oplus_{i\in I}\pi_i)(g)x)(i)$ for all $i \in I$. But

$$((\oplus_{i\in I}\pi_i)(g_\bullet)x)(i) = \pi_i(g_\bullet)x(i) \to \pi_i(g)x$$

since (π_i, H_i) is a unitary representation of G which proves continuity. $\square$

Example 3.2.13. *If (π, H) is a unitary representation of G that has a π-invariant subspace $K \subseteq H$ then it follows that the orthogonal complement of K, $K^{\perp}$ is also π-invariant since*

$$\langle x, \pi(g)y\rangle = \langle \pi(g)x, y\rangle = 0$$

for any $g \in G$, $x \in K$ and $y \in K^{\perp}$. It follows that π is equivalent to the direct sum representation $\pi_i|_K \oplus \pi_i|_{K^{\perp}}$. This shows that reducible unitary representations are decomposable.

Lastly we give the definition of a character in the context of abelian groups. The characters of an abelian group will be shown to have a close relationship to the irreducible representations of the group.

Definition 3.2.14 (Characters)**.** *If G is an abelian topological group we call any continuous group homomorphism $\chi : G \to \mathbb{S}^1$ a character of G.*

3.3 The Regular Representations

We will now define the left and right regular representation of a locally compact group G. Essentially the left representation of G acts on a given function space on G with left translation while the right regular representation acts by right translation. The most important case for us is the case when the space of functions is the Hilbert space $L^2(G)$ and in this case the left and right regular representation of G are unitarily equivalent.

Proposition 3.3.1. *Fix an element g in our locally compact group G and consider the maps L_g and R_g defined on $C_c(G)$ by $(L_g f)(x) = f(g^{-1}x)$ and $(R_g f)(x) = f(gx)$. The maps L_g and R_g extend uniquely to maps on $L^p(G)$, $p \in [1, +\infty)$ such that*

$$\|L_g f\|_p = \|f\|_p$$
$$\|R_g f\|_p = \Delta(g)^{1/p}\|f\|_p$$

for any $g \in G$, $f \in L^p(G)$ and $p \in [1, +\infty)$. For any $f \in L^p(G)$ the maps $g \mapsto L_g f$ and $g \mapsto R_g f$ are continuous. Furthermore the maps

$$g \mapsto L_g \in GL(L^p(G))$$
$$g \mapsto R_g \in GL(L^p(G))$$

are representations for any $p \in [1, +\infty)$.

Proof. For any $g \in G$, $f \in C_c(G)$ and $p \in [1, \infty)$ we have

$$\|L_g f\|_p^p = \int_G |f(g^{-1}x)|^p dx = \int_G |f(x)|^p dx = \|f\|^p$$
$$\|R_g f\|_p^p = \int_G |f(xg)|^p dx = \Delta(g) \int_G |f(x)|^p dx = \Delta(g)\|f\|^p$$

and since $C_c(G)$ is dense in $L^p(G)$ we can uniquely extend L_g and R_g to $L^p(G)$ with their norms preserved. We begin by showing that for fixed $f \in C_c(G)$ the map $g \mapsto L_g f$ is continuous at 1. Pick an open $V \subseteq G$ containing the identity such that $K = \overline{U}$ is compact. By uniform continuity of f we can find an open symmetric subset $U \subseteq K$ such that

$$|f(gx) - f(x)|^p < \frac{\epsilon}{\mu(K\text{supp}(f))}$$

whenever $g \in U$ and $x \in G$. Since the support of $x \mapsto f(gx) - f(x)$ is contained in $U\text{supp}(f)$ it follows that when $g \in U$ then

$$\|L_g f - f\|_p^p < \int_{U\text{supp}(f)} \frac{\epsilon}{\mu(K\text{supp}(f))} dx = \epsilon \frac{\mu(U\text{supp}(f))}{\mu(K\text{supp}(f))} \leq \epsilon.$$

It follows that $g \mapsto L_g f$ is continuous at 1. Assume now that $f \in L^p(G)$. Since $C_c(G)$ is dense in $L^p(G)$ we can find an $f_K \in C_c(G)$ such that $\|f - f_K\| < \frac{\epsilon}{3}$. By continuity of $g \mapsto L_g f_K$ at 1 there is an open set V containing 1 such that $\|L_g f_K - f_K\| < \frac{\epsilon}{3}$ whenever $g \in V$. If $g \in V$ then we have

$$\|L_g f - f\|_p \leq \|L_g f - L_g f_K\|_p + \|L_g f_K - f_K\|_p + \|f_K - f\|_p < \epsilon$$

Which shows that $g \mapsto L_g f$ is continuous at 1. Now pick an arbitrary point $y \in G$ and $f \in L^p(G)$. Since $g \mapsto L_g f$ is continuous at 1 there exists an open set V of 1 such that $\|L_g f - f\| < \epsilon$ whenever $g \in V$. If $y \in xV$ then

$$\|L_y f - L_x f\|_p = \|L_x(L_{x^{-1}y} f - f)\| = \|L_{x^{-1}y} f - f)\| < \epsilon$$

which shows that $g \mapsto L_g f$ is continuous. The proof that $g \mapsto R_g f$ is similar. It is trivial to show that $g \mapsto L_g$ and $g \mapsto R_g$ are group homomorphisms.

Note that the map $g \mapsto L_g$ is continuous if and only if for any net $x_\bullet$ converging to $x \in X$ we have $L_{x_\bullet} \to L_x$ but by properties of the topology of pointwise convergence this happens if and only if $L_{x_\bullet} f \to L_x f$ for any $f \in L^p G)$ which is true since we proved that the map $g \mapsto L_g f$ is continuous for any $f \in L^p(G)$. The same argument shows that $g \mapsto R_g$ is continuous. Since $\|L_g\| = 1$ and $\|R_g\| = \Delta(g)^{1/p}$ it is clear that the two functions $g \mapsto \|L_g\|$ and $g \mapsto \|R_g\|$ are locally bounded and by Proposition 3.1.2 it follows that $g \mapsto L_g$ and $g \mapsto R_g$ are representations. □

Definition 3.3.2. *Let G be a locally compact group. The unitary representations $\lambda : G \to \mathcal{U}(L^2(G))$ and $\rho : G \to \mathcal{U}(L^2(G))$ given by $\lambda(g) = L_g$ and $\rho(g) = \Delta(g)^{-1/2} R_g$ are called the left regular repsententation and right regular representation of G respectively.*

Remark 3.6. *Routine inner product calculations using Theorem 2.4.1 yield $\lambda(g)^* = \lambda(g)^{-1}$ and $\rho(g)^* = \rho(g)^{-1}$ which shows that λ and ρ indeed map into $\mathcal{U}(L^2(G))$ as claimed in the definition above.*

Proposition 3.3.3. *The left- and right regular representation of G are unitarily equivalent.*

Proof. Define $W : C_c(G) \to C_c(G)$ by $W(f)(x) = f(x^{-1})\Delta(x)^{1/2}$. For any $f \in C_c(G)$ we have that the function $g(x) = f(x^{-1})\Delta(x)^{1/2}$ satisfies $W(g) = f$ and it follows that W is surjective. We also have

$$\begin{aligned} \|Wf\|_2^2 &= \int_G |f(x^{-1})|^2 \Delta(x) dx \\ &= \int_G |f(x)|^2 dx \\ &= \|f\|_2^2 \end{aligned}$$

hence W is an isometry. The extension $W : L^2(G) \to L^2(G)$ of $W : C_c(G) \to C_c(G)$ is then also an isometry, where we slightly abuse notation. Since the image of W is dense in $L^2(G)$ and the image of W is complete and therefore closed it follows that $W : L^2(G) \to L^2(G)$ is surjective and therefore a unitary operator. If $f \in C_c(G)$ then

$$\begin{aligned} W(\lambda(g)f)(x) &= (\lambda(g)f)(x^{-1})\Delta(x)^{1/2} \\ &= f(g^{-1}x^{-1})\Delta(x)^{1/2} \\ &= f((xg)^{-1})\Delta(xg)^{1/2}\Delta(g)^{-1/2} \\ &= (Wf)(xg)\Delta(g)^{-1/2} \\ &= (\rho(g)(Wf))(x) \end{aligned}$$

hence $W\lambda(g)f = \rho(g)Wf$ for any $f \in C_c(G)$. If (f_n) is a sequence of compactly supported functions converging to f in $L^2(G)$ then $\|W\lambda(g)f - W\lambda(g)f_n\|_2 \to 0$ and $\|\rho(g)Wf - \rho(g)Wf_n\|_2 \to 0$ hence $\|W\lambda(g)f - \rho(g)Wf\|_2 = 0$ and the proposition follows. □

4 C*-Algebras

4.1 Basic Definitions and Properties

In this section we will give basic definitions of algebras and C^*-algebras in particular. A C^*-algebra A is a special type of a Banach algebra equipped with an involution $^* : A \to A$ such that the *-identity,

$$\|x^*x\| = \|x\|^2$$

holds for any $x \in A$. This rather innocent looking requirement has tremendous consequences for its underlying structure. The reason for this is that the above requirement implies by the spectral radius formula that $\|x\|^2 = \rho(x^*x)$ where $\rho(x)$ is the supremum of all $|\lambda|$ over the set of all complex λ such that $\lambda e - x$ is not invertible. Being invertible or not is an algebraic property and it follows that the analytical component of A, its norm is completely determined by its algebraic structure. This will, as we will see lead to many cases where algebraic properties of A will imply analytical consequences of A. Another example of how well-behaved a C^*-algebra is is that any commutative C^*-algebra is isomorphic to the the algebra of continuous functions on its structure space that vanish at infinity which will be discussed in section 6.

Definition 4.1.1 (Complex Algebra)**.** *A complex algebra A is a complex vector space equipped with a multiplication operation $A \times A \to A$ satisfying*

$$\begin{aligned} x(y+z) &= xy + xz \\ (x+y)z &= xz + yz \\ \alpha(xy) &= (\alpha x)y = x(\alpha y) \end{aligned}$$

for any $x, y, z \in A$ and $\alpha \in \mathbb{C}$ then A is a complex algebra. The algebra A is associative if

$$x(yz) = (xy)z$$

whenever x, y, z are elements of A. If there exists an element $e \in A$ such that

$$ex = xe = x$$

then A is said to be unital and e is called the identity element. If there for a given $x \in A$ exists an element x^{-1} such that $xx^{-1} = x^{-1}x = e$ then x is invertible and x^{-1} is an inverse of x. If for any $x, y \in A$ we have

$$xy = yx$$

Then A is commutative.

Definition 4.1.2 (Banach Algebra)**.** *If A is an associative algebra and a Banach space such that*

$$\|xy\| \leq \|x\|\|y\|$$

for any $x, y \in A$ then A is a Banach algebra. Furthermore, if A is unital it is also required that

$$\|e\| = 1$$

Definition 4.1.3 (Involution, *-Algebras and C^*-Algebras)**.** *Let A be a complex algebra. A mapping $A \to A$, $x \mapsto x^*$ is an involution if*

$$\begin{aligned}(x+y)^* &= x^* + y^* \\ (\alpha x)^* &= \overline{\alpha} x^* \\ (xy)^* &= y^* x^* \\ x^{**} &= x\end{aligned}$$

for any $x, y \in A$ and $\alpha \in \mathbb{C}$. A complex associative algebra equipped with an involution * is called a *-algebra. If A is a Banach algebra equipped with an involution such that

$$\|x^* x\| = \|x\|^2$$

then A is called a C^*-algebra.

Remark 4.1. *If A is a C^*-algebra then*

$$\|x\|^2 = \|x^* x\| \leq \|x^*\| \|x\|$$

and it follows that $\|x\| \leq \|x^*\|$. After replacing x with x^* in the previous argument the relation

$$\|x\| = \|x^*\| \tag{1}$$

for any $x \in A$ follows. This relation then implies

$$\|x x^*\| = \|x\| \|x^*\| \tag{2}$$

whenever $x \in A$. Conversely if A is a Banach algebra with an involution satisfying (1) *and* (2) *then A is a C^*-algebra since*

$$\|x x^*\| = \|x\| \|x^*\| = \|x\|^2.$$

The existence of an identity element e of a C^*-algebra A is not included in the definition. However the proposition below there always exists a unital C^*-algebra $\tilde{A}$ that contains A as a subalgebra and the norm on $\tilde{A}$ is an extension of the norm of A. This enables us in many cases to restrict our attention to unital subalgebras.

Proposition 4.1.4. *For any C^*-algebra A there exists a unital C^*-algebra $\tilde{A}$ such that A is a subalgebra of $\tilde{A}$ and $\|x\|_A = \|x\|_{\tilde{A}}$ for any $x \in A$.*

Proof. See Proposition 1.3.8 in [6] □

The prototypical examples of a C^*-algebra are given by following theorem.

Proposition 4.1.5. *The space $\mathscr{B}(H)$ of bounded linear operators on a hilbert space H is a C^*-algebra with involution given by the adjoint. Furthermore, any closed subalgebra of $\mathscr{B}(H)$ that is closed under taking adjoints is a C^*-algebra.*

Proof. It is well known that $\mathscr{B}(H)$ is a Banach algebra with multiplication given by composition. It is easy to show that the adjoint operator * on $\mathscr{B}(H)$ satisfies the properties of an involution. In order to prove that $\mathscr{B}(H)$ is a C^*-algebra it therefore remains to prove the equality

$$\|TT^*\| = \|T\|^2$$

for any $T \in \mathscr{B}(H)$. Since $\mathscr{B}(H)$ is a Banach algebra we have

$$\|TT^*\| \leq \|T\|\|T^*\| = \|T\|^2.$$

To prove the other direction Cauchy-Schwartz yields

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 = \sup_{\|x\|=1} \langle Tx, Tx\rangle \\ &= \sup_{\|x\|=1} \langle T^*Tx, x\rangle \leq \sup_{\|x\|=1} \|T^*T\|\|x\|^2 = \|T^*T\| \end{aligned}$$

which shows that $\mathscr{B}(H)$ is a C^*-algebra. Let A be a closed subalgebra of $\mathscr{B}(H)$ that is closed under taking adjoints. Since A is closed it it complete and therefore a Banach algebra since $\mathscr{B}(H)$ is a Banach Algebra. By assumption A is closed under taking adjoints and the adjoint therefore defines an involution on A and since the C^*-equality is satisfied in $\mathscr{B}(H)$ it is in particular also satisfied in A. □

Definition 4.1.6 (Algebra Homomorphisms and *-Homomorphisms)**.** *Let A and B be two complex algebras. A linear map $\phi : A \to B$ is an algebra homomorphism if*

$$\phi(xy) = \phi(x)\phi(y)$$

for any $x, y \in A$ then ϕ is called an algebra homomorphism. If A and B are *-algebras and $\phi : A \to B$ is an algebra homomorphism then ϕ is a *-homomorphism if

$$\phi(x^*) = \phi(x)^*$$

for any $x \in A$. If $\phi : A \to B$ is *-homomorphism with an inverse map $\psi : B \to A$ that is also a * -homomorphism then $\phi : A \to B$ is called a *-isomorphism.

For any algebra homomorphism $\phi : A \to B$ the kernel $\ker \phi$ is the set of all $x \in A$ such that $\phi(x) = 0$. The kernel is clearly a subspace of A and it also satisfies the absorption properties $yx \in \ker A$ and $yx \in \ker A$ for any $x \in I$ and $y \in A$. Subsets with these properties have been given the name two sided ideals.

Definition 4.1.7 (Ideals). *A subset I of a complex algebra A is called a left ideal if I is a subspace and if $yx \in I$ whenever $x \in I$ and $y \in A$. Similarly a subspace I is called a right ideal if $xy \in I$ whenever $x \in I$ and $y \in A$. If I is both a left ideal and a right ideal then I is called a two-sided ideal. A (left or right) ideal of a *-algebra A is called *-closed if $x^* \in I$ for any $x \in I$.*

Remark 4.2 (Quotient Algebras). *Let I be a closed two-sided ideal of a Banach algebra A and let A/I be the coset space. It is straightforward to show that the operations*

$$\begin{aligned} (x+I)+(y+I) &= x+y+I \\ \alpha(x+I) &= \alpha x + I \\ (x+I)(y+I) &= xy+I \end{aligned}$$

are well-defined and turns A/I into an associative algebra. Since A is a Banach-algebra we can also introduce a norm on A/I that turns it into a Banach-algebral, see section 11.4 in [7]. The canonical projection $\pi : A \to A/I$ will then be an algebra homomorphism such that $\|\pi(x)\| \leq \|x\|$ for all $x \in A$. If A is in particular a C^*-algebra and I is a closed two-sided ideal that is *-closed then we know that A/I is a Banach Algebra. It is also easy to show that the operation

$$(x+I)^* = x^* + I$$

is a well-defined involution on A/I and $\pi(x^*) = \pi(x)^*$. The norm identity for C^*-algebras does also hold on A/I, see Proposition 1.8.2 in [6] which means that A/I is a C^*-algebra. We summarize this below

Proposition 4.1.8. *If A is a C^*-algebra and I is closed, *-closed ideal of A then A/I is a C^*-algebra and $\pi : A \to A/I$ is a *-homomorphism.*

The C^*-algebras have a particularly nice structure to them. The following propositions are examples of this.

Proposition 4.1.9. *Let A be a Banach algebra equipped with an involution such that $\|x\| = \|x^*\|$ for all $x \in A$ and B be a C^*-algebra. If $\phi : A \to B$ is a *-homomorphism then A is contractive, that is $\|\phi\| \leq 1$.*

Proof. See 1.3.7 in [6] □

Proposition 4.1.10. *Let A be a C^*-algebra with norm $\|\cdot\|$. If $\|\cdot\|'$ is any other norm that makes A a C^*-algebra (with respect to the same algebraic operations) then $\|\cdot\| = \|\cdot\|'$*

Proof. We can without loss of generality assume that A has an identity e. Let $\|\cdot\|$ be a norm on A that makes it into a C^*-algebra. For any element $x \in A$ one has the relation $\|(x^*x)^2\| = \||(x^*x)\|^2$. This relation combined with induction then shows that for any positive integer n the relation $\|(x^*x)^{2^n}\|^{1/2^n} = \|x^*x\|$. The spectral radius formula see Theorem 10.13 in [7] then shows that

$$\|x\|^2 = \|x^*x\| = \lim_{n\to\infty} = \|(x^*x)^{2^n}\|^{1/2^n} = \sup\{\lambda \in \mathbb{C} \mid (\lambda e - x^*x)^{-1} \text{ exists}\}.$$

Define $\rho(x) = \sup\{\lambda \in \mathbb{C} \mid (\lambda e - x)^{-1} \text{ exists}\}$ then we have proved that $\|x\|^2 = \rho(x^*x)$ for any $x \in A$. But the number $\rho(x)$ is independent of the choice of norm on A and any other norm $\|\cdot\|^{'}$ that makes A a C^*-algebra with respect to the same operations must also satisfy $\|x\|'^2 = \rho(x^*x)$ for any $x \in A$ hence $\|\cdot\|' = \|\cdot\|$. $\square$

Proposition 4.1.11. *If $\phi : A \to B$ is an injective *-homomorphism between C^*-algebras then ϕ is an isometry*

Proof. Since *-homomorphisms are contractive we have that $\|\phi\| \leq 1$. On the other hand Proposition 1.8.3 in [6] shows that $\|\phi(x)\| \geq \|x\|$ for all $x \in A$ when ϕ is injective. It follows that $\|\phi(x)\| = \|x\|$ for all $x \in A$. $\square$

Proposition 4.1.12. *If $\phi : A \to B$ is a *-homomorphism then $\phi(A)$ is a closed C^*-algebra with respect to the norm and the operations of B.*

Proof. Assume first that the *-homomorphism $\phi : A \to B$ is also injective. Since ϕ is a *-homomorphism it is clear that $\phi(A)$ is closed under the operations of B. Since B is a C^*-algebra the C^*-identity also holds in $\phi(A)$. To show that $\phi(A)$ is complete we choose a Cauchy-sequence (y_n) in $\phi(B)$. There exists for each n an element $x_n \in A$ such that $\|\phi(x_n) = y_n\|$. The assumptions on ϕ implies that ϕ is an isometry from which we see that $(x_n)_n$ is a Cauchy sequence in A. By completeness of A there exists an element x such that $\lim_{n\to\infty} x_n = x$. Since y_n is a Cauchy sequence in B there exists a $y \in B$ such that $\lim_{n\to\infty} y_n = y$. It follows that $\phi(x) = y$ which shows that $\phi(A)$ is complete and hence a C^*-algebra. The C^*-algebra $\phi(A)$ is closed in B since $\phi : A \to B$ is an isometry.

We are now in a position to tackle the general case. Let $I = \ker\phi$ then I is a closed ,*-closed two-sided ideal of A. It follows that A/I is a C^*-algebra. Let $\pi : A \to A/I$ be the canonical projection map. Define for any $x + I \in A/I$ the map $\tilde{\phi}(x+I) = \phi(x)$. If $x+I = y+I$ then $\phi(x-y) = 0$ hence $\phi(x) = \phi(y)$ which shows that $\tilde{\phi}$ is well-defined. It is easy to show that $\tilde{\phi}$ is a *-homomorphism. If $\tilde{\phi}(x+I) = \tilde{\phi}(y+I)$ then $\tilde{\phi}(a-b+I) = 0$ hence $\phi(a-b) = 0$ which means that $a + I = b + I$. This shows that $\tilde{\phi}$ is injective. Since $\phi = \tilde{\phi}\circ\pi$ it follows that $\phi(A) = \tilde{\phi}(A/I)$. This shows that $\phi(A)$ is a C^*-algebra since $\tilde{\phi}$ is injective and A/I is a C^*-algebra. $\square$

We know that any closed, *-closed subalgebra of bounded operators on a Hilbert space is a C^*-algebra. One remarakable fact of C^*-algebras is that any C^*-algebra is isomorphorphic to a C^*-algebra of operators on a Hilbert space.

Theorem 4.1.13. *If A is a C^*-algebra then there exists isometric *-isomorphism of A onto a closed subset of $\mathscr{B}(H)$ for some Hilbert space H.*

Proof. See 12.41 in [7] □

One goal of the thesis is to define the maximal and reduced group C^*-algebras. These are defined as closures of $L^1(G)$ with respect to different C^*-norms. Up to this point only the Banach space structure of $L^1(G)$ has been used which means that we first need to define multiplication and a *-structure on $L^1(G)$. We also note that the same operations turns $C_c(G)$ into an associative *-algebra.

Proposition 4.1.14. *For $f, g \in C_c(G)$ the maps defined by $f^*(x) = \Delta(x)\overline{f(x^{-1})}$ and*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

turn $C_c(G)$ into an associative *-algebra. The operation $(f,g) \mapsto f * g$ is called convolution.

Proposition 4.1.15. *The involution map $^* : C_c(G) \to G$ and the convolution map $* : C_c(G) \times C_c(G) \to C_c(G)$ extend uniquely to $L^1(G)$ and make $L^1(G)$ a Banach *-algebra such that $\|f^*\|_1 = \|f\|_1$.*

Proof. Since $(x,y) \mapsto y^{-1}x$ and $(x,y) \mapsto y$ are continuous and products of measurable functions are measurable it follows that $(x,y) \mapsto f(y)g(y^{-1}x)$ is measurable when f and g are measurable. If f, g are integrable functions we also have

$$\begin{aligned}\int_G \int_G |f(y)g(y^{-1}x)|dxdy &= \int_G |f(y)| \int_G |g(y^{-1}x)|dxdy \\ &= \int_G |f(y)| \int_G |g(x)|dxdy = \|f\|_1\|g\|_1 < +\infty,\end{aligned}$$

and it follows that Fubini's theorem can be applied below. Note that σ-finiteness is not needed since the supports of f and g are σ-finite since they are integrable and our Haar measures can be replaced by σ-finite measures preserving the integrals. We therefore have for any integrable functions f, g

$$\int_G \int_G |f(y)g(y^{-1}x)|dydx = \int_G \int_G |f(y)g(y^{-1}x)|dxdy < +\infty$$

which shows that $(f * g)(x)$ is well defined for almost all x. Using our previous calculations it is also clear that $\|f * g\|_1 \le \|f\|_1\|g\|_1$. If $f_1 = f_2$ almost everywhere and $g_1 = g_2$ almost everywhere then

$$\begin{aligned}\|f_1 * g_1 - f_2 * g_2\|_1 &\le \|f_1 * (g_1 - g_2)\|_1 + \|(f_1 - f_2) * g_2\| \\ &\le \|f_1\|_1\|g_1 - g_2\| + \|f_1 - f_2\|_1\|g_2\| = 0\end{aligned}$$

and it follows that convolution is well defined as a function on $L^1(G)$ which clearly satisfies the properties of being a Banach-algebra multiplication in $L^1(G)$.

Is is easy to show that * is an involution on $C^c(G)$. For any $f \in C_c(G)$ we have by properties of the modular function

$$\|f^*\|_1 = \int_G \Delta(x)|f(x^{-1})|dx = \|f\|_1$$

and it follows that $^* : C_c(G) \to C_c(G)$ is an isometry on $C_c(G)$. By density of $C_c(G)$ in $L^1(G)$ we can extend this involution uniquely to an isometry $^* : L^1(G) \to L^1(G)$. Since * is an involution on the dense subspace $C_c(G)$ it follows that is an involution on $L^1(G)$. $\square$

5 Algebra Representations and the group C*-algebra

In this section we will develop the theory of representations of algebras and define the group C^*-algebra. This will give us a nice correspondence between unitary representations of G and a certain class of representations of the group algebra. What enables this correspondence is a Banach space integral called the Pettis integral which we first discuss.

5.1 The Pettis Integral

We would like to develop the notion of an integral

$$\int_X f d\mu$$

for a function $f : X \to V$ where V is a Banach space and $(X, \mathcal{M}, \mu)$ is a measure space. One way of defining $\int_X f d\mu \in V$ is to recall that the linear functionals on V separate points so we can define $\int_X f d\mu \in V$ in terms of how linear functionals act on it. Since integrals often are thought of as a kind of continuous sum a natural definition $\int_X f d\mu \in V$ would be that it is the unique element in V such that

$$\phi(\int_X f d\mu) = \int_X \phi \circ f d\mu$$

for all $\phi \in X^*$ given that such an element exists. Since the linear functionals separates points this element is unique if it exists. The theorem below provides a setting where this integral is well-defined. Details on the construction can be found in appendix B.6 in [3].

Theorem 5.1.1. *Let V be a Banach space, $(X, \mathcal{M}, \mu)$ a measure space and $f : X \to V$ be a measurable function. If the function $x \mapsto \|f(x)\|$ is in $L^1(\mu)$ then there exists a unique element denoted $\int_X f d\mu \in V$ such that*

$$\phi(\int_X f d\mu) = \int_X \phi \circ f d\mu$$

for any $\phi \in V^*$. The element $\int_X f d\mu$ also satisfies

$$\|\int_X f d\mu\| \leq \int_X \|f(x)\| d\mu(x)$$

Definition 5.1.2. *Let V be a Banach space, $(X, \mathcal{M}, \mu)$ a measure space and $f : X \to V$ a measurable function such that the function $x \mapsto \|f(x)\|$ is in $L^1(\mu)$. Then the unique element $\int_X f d\mu \in V$ as in Theorem 5.1.1 satisfying*

$$\phi(\int_X f d\mu) = \int_X \phi \circ f d\mu$$

for any $\phi \in V^*$ is called the Pettis integral of f with respect to μ.

Theorem 5.1.3. *Let (π, H) be a unitary representation and $f : G \to \mathbb{C}$ be an integrable function. Then the Pettis integral of $x \mapsto f(x)\pi(x) \in \mathscr{B}(H)$, $I_f = \int_G f(x)\pi(x)dx \in \mathscr{B}(H)$ is well defined and it defines a a map on $L^1(G) \to \mathscr{B}(H)$. For any $\xi \in H$ we have*

$$I_f\xi = \int_G f(x)\pi(x)\xi dx.$$

Where the right hand side is the Pettis integral of $x \mapsto f(x)\pi(x)\xi \in H$.

Proof. Let f be a complex-valued, integrable function defined on G and set $F(x) = f(x)\pi(x)$. For fixed $x \in G$ we have

$$\|F(x)\| = \sup_{\|\xi\|=1} \|F(x)\xi\| = \sup_{\|\xi\|=1} |f(x)|\|\pi(x)\xi\| = |f(x)|\|\pi(x)\| = |f(x)|$$

hence $F(x) \in \mathbb{B}(H)$. The function $F : G \to \mathscr{B}(H)$ is measurable since it equals the composition $x \mapsto (f(x), \pi(x)) \mapsto f(x)\pi(x)$ where the last function is scalar multiplication in $\mathscr{B}(H)$ which is continuous. We also have

$$\int_G \|F(x)\|dx = \int_G |f(x)|\|\pi(x)\|dx = \|f\|_1$$

and it follows that the Pettis integral of F is well defined. If f is an element of $L^1(G)$ then there exists an integrable function $g : G \to \mathbb{C}$ that represents f and we can define

$$I_f = \int_G g(x)\pi(x)dx.$$

This is well defined since if $h : G \to \mathbb{C}$ equals g almost everywhere and $\phi \in \mathscr{B}(H)^*$ then by definition of the Pettis integral we have

$$\begin{aligned}
\phi(\int_G g(x)\pi(x)dx) &= \int_G \phi(g(x)\pi(x))dx \\
&= \int_G \phi((g(x) - h(x) + h(x))\pi(x))dx \\
&= \int_G \phi(g(x) - h(x))\pi(x))dx + \int_G \phi(h(x))\pi(x)dx \\
&= 0 + \phi(\int_G h(x)\pi(x)dx).
\end{aligned}$$

By uniqueness of the Pettis integral it follows that

$$\int_G g(x)\pi(x)dx = \int_G h(x)\pi(x)dx$$

and therefore $f \mapsto I_f$ is well defined as a function on $L^1(G)$. Choose $\xi \in H$ and an integrable function $f : G \to \mathbb{C}$ and consider the map $A_f : G \to H$ given by $A(x) = f(x)\pi(x)\xi$ then A_f is measurable since it equals the composition $x \mapsto (f(x), \pi(x)\xi) \mapsto f(x)(\pi(x)\xi)$ where the last map is scalar multiplication in H. We also have

$$\int_G \|A_f(x)\|dx = \int_G |f(x)|\|\pi(x)\xi\|dx = \|\xi\|\|f\|_1$$

and it follows that the Pettis integral of A_f is well defined. A similar argument as before shows that the Pettis integral of A_f is well defined as a function on $L^1(G)$. Lastly, fix $\xi, \eta \in H$. Then since $A \mapsto \langle A\xi, \eta\rangle \in \mathbb{C}$ is a linear functional on $\mathscr{B}(H)$ and $\xi \mapsto \langle \xi, \eta\rangle$ is a linear functional on H we have by definition of the Pettis integral for any $f \in L^1(G)$ that

$$\langle I_f\xi, \eta\rangle = \int_G \langle f(x)\pi(x)\xi, \eta\rangle dx = \langle \int_G f(x)\pi\xi dx, \eta\rangle$$

Since η was arbitrary it follows that

$$I_f\xi = \int_G f(x)\pi(x)\xi dx.$$

□

Proposition 5.1.4. *For any f and g in $L^1(G)$ we have the following Pettis integral convolution formula*

$$f * g = \int_G f(x)L_x g dx.$$

Proof. Recall that the $L^\infty(G)$ can be isometrically embedded in the dual space of $L^1(G)$. This embedding is given by mapping $h \in L^\infty(G)$ to the functional $L_h : L^1(G) \to \mathbb{C}$ given by

$$L_h(f) = \int_G f(x)h(x)dx.$$

Using this identification it is easy to show that $L^\infty(G)$ separates points of $L^1(G)$. Define the pairing $(f, h) = L_h(f)$ for $f \in L^1(G)$ and $h \in L^\infty(G)$. Then by definition of the Pettis integral we have for any $f, g \in L^1(G)$ and $h \in L^\infty(G)$

$$\begin{aligned}
(\int_G f(x)L_x g dx, h) &= \int_G (f(x)L_x g, h)dx \\
&= \int_G \int_G f(x)(L_x g)(y)h(y)dydx \\
&= \int_G \int_G f(x)g(x^{-1}y)h(y)dydx \\
&= \int_G \int_G f(x)g(x^{-1}y)dxh(y)dy \\
&= \int_G (f * g)(y)h(y)dy \\
&= (f * g, h)
\end{aligned}$$

Since $L^\infty(G)$ separates points we have

$$\int_G f(x)L_x g dx = f * g.$$

□

5.2 Algebra Representations and Integrated Representations

Since the group structure of G is encoded both in the Haar measure of G and in the convolution formula for elements of $L^1(G)$ it is reasonable to expect that properties of G would carry over to properties of $L^1(G)$ and vice versa. Some examples of this that one can show are that G is discrete if and only if $L^1(G)$ is unital and that G is abelian if and only if $L^1(G)$ is abelian. This suggests that the the representation theory of G might be related to $L^1(G)$ in some way. Since unitary representations of a locally compact group are homomorphisms into $\mathcal{U}(H)$ for some Hilbert space H a natural candidate definition of a *-representation of a *-algebra would therefore be an algebra homomorphism into $\mathscr{B}(H)$. This is indeed a good definition since in this section we will show that there is correspondence between unitary representations of G and a well behaved collection of *-representations of $L^1(G)$.

Definition 5.2.1 (*-Representation). *A *-homomorphism $\pi : A \to \mathscr{B}(H)$ between a *-algebra A and the space of bounded linear operators on a Hilbert space H is called a *-representation of A. We will sometimes denote a *-representation by (π, H)*

Definition 5.2.2 (Non-Degenerate Representation). *If $\pi : A \to \mathscr{B}(H)$ is *-representation and the span of $\pi(A)H = \{\pi(a)\xi;\ a \in A,\ \xi \in H\}$ is dense in H then π is non-degenerate*

We now define some more important properties of *-representations.

Definition 5.2.3 (Equivalence of Representations). *We say that (π, H) and (π', H') are equivalent representations if there exists a unitary map $U : H \to H'$ such that $U\pi(x)U^{-1} = \pi(x)'$ for all $x \in A$.*

Definition 5.2.4 (Subrepresentations and Algebraic Subrepresentation). *A subrepresentation of a *-representation (π, H) of A is a closed subspace K of H such that $\pi(A)K \subseteq K$. An algebraic subrepresentation of π is a subspace K of H such that $\pi(A)K \subseteq K$.*

Definition 5.2.5 (Irreducibility and Algebraic Irreducibility). *A *-representation (π, H) of A is irreducible if its only subrepresentations are $K = H$ and $K = \{0\}$. The *-representation (π, H) of A is algebraically irreducible if its only algebraic subrepresentations are $K = H$ and $K = \{0\}$.*

Remark 5.1. *Algebraic irreducibility is far more restrictive than irreducibility. For C^*-algebras however they turn out to be the same.*

Proposition 5.2.6. *A *-representation of a C^* is irreducible if and only if it is algebraically irreducible.*

Proof. See 2.8.4 in [6]. $\square$

Proposition 5.2.7. *Let (π, H) be a unitary representation of a locally compact group G and define $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$ by*

$$\tilde{\pi}(f) = \int_G f(x)\pi(x)dx.$$

Then $\tilde{\pi}$ is non-degenerate contractive *-representation of $L^1(G)$.

Proof. Let (π, H) be unitary representation of our locally compact group G. It is easy to show that the Pettis integral is linear from which it follows that $\tilde{\pi}$ is linear. and $f \in L^1(G)$ we have

$$\|\tilde{\pi}f\| = \| \int_G f(x)\pi(x)dx\| \leq \int_G \|f(x)\pi(x)\|dx \leq \|f\|_1$$

which shows that $\tilde{\pi}$ is contractive. Recall that contractive means that $\|\tilde{\pi}\| \leq 1$. Choose $\xi, \eta \in H$ then since the map $A \mapsto \langle A\xi, \eta\rangle$ is a linear functional on $\mathscr{B}(H)$ we get for any $f, g \in L^1(G)$ the following (by the defining property of the Pettis integral)

$$\begin{aligned}
\langle \tilde{\pi}(f * g)\xi, \eta\rangle &= \langle \int_G (f * g)(x)\pi(x)dx\xi, \eta\rangle \\
&= \int_G \langle (f * g)(x)\pi(x)\xi, \eta\rangle dx \\
&= \int_G \int_G f(y)g(y^{-1}x)dy\langle \pi(x)\xi, \eta\rangle dx \\
&= \int_G \int_G f(y)g(y^{-1}x)\langle \pi(x)\xi, \eta\rangle dxdy \\
&= \int_G \int_G f(y)g(x)\langle \pi(y)\pi(x)\xi, \eta\rangle dxdy \\
&= \int_G f(y) \int_G \langle g(x)\pi(x)\xi, \pi(y)^*\eta\rangle dxdy \\
&= \int_G f(y)\langle \int_G g(x)\pi(x)dx\xi, \pi(y)^*\eta\rangle dy \\
&= \int_G f(y)\langle \tilde{\pi}(g)\xi, \pi(y)^*\eta\rangle dy \\
&= \int_G \langle f(y)\pi(y)\tilde{\pi}(g)\xi, \eta\rangle dy \\
&= \langle \tilde{\pi}(f)\tilde{\pi}(g)\xi, \eta\rangle.
\end{aligned}$$

It follows that $\tilde{\pi}(f * g) = \tilde{\pi}(f)\tilde{\pi}(g)$ hence $\tilde{\pi}$ is a homomorphism. To show that $\tilde{\pi}$ is a *-homomorphism we have

$$\begin{aligned}
\langle \tilde{\pi}(f^*)\xi, \eta\rangle &= \int_G \langle f^*(x)\pi(x)\xi, \eta\rangle dx \\
&= \int_G \Delta(x)\overline{f(x^{-1})}\langle \pi(x)\xi, \eta\rangle dx \\
&= \int_G \Delta(x)\overline{f(x^{-1})}\langle \xi, \pi(x)^*\eta\rangle dx \\
&= \int_G \overline{f(x)}\langle \xi, \pi(x^{-1})^*\eta\rangle dx \\
&= \int_G \overline{f(x)}\langle \xi, \pi(x)\eta\rangle dx \\
&= \overline{\int_G \langle f(x)\pi(x)\eta, \xi\rangle dx} \\
&= \overline{\langle \tilde{\pi}(f)\eta, \xi\rangle} \\
&= \langle \xi, \tilde{\pi}(f)\eta\rangle
\end{aligned}$$

which shows that $\tilde{\pi}(f^*)^* = \tilde{\pi}(f)$ hence $\tilde{\pi}(f^*) = \tilde{\pi}(f)^*$. It remains to show that $\tilde{\pi}$ is non-degenerate. Pick an element $\xi \in H$. Since (π, H) is a unitary representation we can find for any ϵ an open set $V \subseteq G$ of 1 such that $\|\pi(x)\xi - \xi\| < \epsilon$ for all $x \in V$. Define for any open set U of 1 of finite measure the function f_U on G by $f_U = \frac{1}{\mu(U)}\chi_U$ where μ is the Haar measure of G. For any $U \subseteq V$ and for any $\eta \in H$ we then have

$$\begin{aligned}
|\langle \tilde{\pi}(f_U)\xi - \xi, \eta\rangle| = |\int_G \langle f_U(x)\pi(x)\xi, \eta\rangle dx - \langle \xi, \eta\rangle| \quad &= |\frac{1}{\mu(U)}\int_U \langle \pi(x)\xi - \xi, \eta\rangle dx| \\
\leq \frac{1}{\mu(U)}\int_U |\langle \pi(x)\xi - \xi, \eta\rangle| dx. \\
\leq \epsilon\|\eta\|.
\end{aligned}$$

It follows that the net $\tilde{\pi}(f_\bullet)\xi$ converges weakly to ξ as the open sets gets smaller. To show that the net $\tilde{\pi}(f_\bullet)\xi$ converges in the norm of H to ξ we see by expanding the inner product that it is enough to show that $\langle \tilde{\pi}(f_\bullet)\xi, \tilde{\pi}(f_\bullet)\xi\rangle$ converges to $\|\xi\|^2$. Using the integral formula for the modular function and the definition of convolution show that

$$\langle \tilde{\pi}(f_U)\xi, \tilde{\pi}(f_U)\xi\rangle = \frac{1}{\mu(U)^2}\int_U\int_U \chi_U(x)\chi_{Uy^{-1}}(x)\langle \pi(y)\xi, \xi\rangle dy dx$$

There exists an open set V of 1 such that $\|\pi(y)\xi - \xi\| \leq \epsilon$ for all $y \in V$. Since the map $(x, y) \mapsto xy^{-1}$ is continuous there exists an open set U_0 of 1 such that $U_0U_0^{-1} \subseteq V$. For any open $U \subseteq U_0$ the support of the integrand above is then contained in V which shows

$$|\langle \tilde{\pi}(f_U)\xi, \tilde{\pi}(f_U)\xi\rangle - \langle \xi, \xi\rangle| \leq \epsilon\|\xi\|.$$

This means that $\overline{\pi}(f_U)\xi$ converges to ξ in the norm topology on H. It follows that ξ is in the closure of $\tilde{\pi}(L^1(G))H$ which shows that $\tilde{\pi}$ is non-degenerate. $\square$

Definition 5.2.8 (Integrated Representation)**.** *If (π, H) is a unitary representation of the locally compact group G then the non-degenerate, contractive *-representation, $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$ given by*

$$\tilde{\pi}(f) = \int_G f(x)\pi(x)dx$$

is called the integrated representation of π.

Theorem 5.2.9. *If G is a locally compact group then there exists a bijective correspondence between unitary representations (π, H) of G and non-degenerate, contractive *-representations $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$ of $L^1(G)$. The bijection is given by associating any unitary representation with its integrated representation.*

Proof. In proposition 5.2.7 it was shown that the integrated representation $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$ of a unitary representation (π, H) is a non-degenerate, contractive *-representation. To show the converse pick a non-degenerate contractive *-representation $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$ of $L^1(G)$. We begin by noting that the definition of convolution shows that

$$(g^* * L_x f)(y) = ((L_{x^{-1}} g)^* * f)(y)$$

for all f and g in $C_c(G)$ and $x, y \in G$. By density of $C_c(G)$ it follows that $g^* * L_x f = (L_{x^{-1}} g)^* * f$ holds for all $f, g \in L^1(G)$ and any $x \in G$. Pick an element $v = \sum_{i=1}^n \alpha_i \tilde{\pi}(f_i)\xi_i$ in the span of $\tilde{\pi}(L^1(G))H$ and define

$$\pi(x)v = \sum_{i=1}^n \alpha_i \tilde{\pi}(L_x f_i)\xi_i$$

for any $x \in G$. The calculation

$$\begin{aligned}
\|\langle \pi(x)v \rangle\|^2 &= \sum_{i,j=1}^n \langle \alpha_i \tilde{\pi}(L_x f_i)\xi_i, \alpha_j \tilde{\pi}(L_x f_j)\xi_j \rangle \\
&= \sum_{i,j=1}^n \langle \alpha_i \tilde{\pi}((L_x f_j)^* * L_x f_i)\xi_i, \alpha_j \xi_j \rangle \\
&= \sum_{i,j=1}^n \langle \alpha_i \tilde{\pi}((L_{x^{-1}} L_x f_j)^* * f_i)\xi_i, \alpha_j \xi_j \rangle \\
&= \sum_{i,j=1}^n \langle \alpha_i \tilde{\pi}(f_i)\xi_i, \alpha_j \tilde{\pi}(f_j)\xi_j \rangle \\
&= \|v\|^2
\end{aligned}$$

This relation shows that if also $v = \sum_{i=1}^{n} \beta_i \tilde{\pi}(g_i)\xi_i$ then $\pi(x)$ evaluted on the difference of these 2 sums is zero. It follows that $\pi(x)$ is well-defined on the span of $\tilde{\pi}(L^1(G))H$. The relation above also shows that $\pi(x)$ is an isometry on this space. Since $\pi(x)$ is clearly surjective it follows that $\pi(x)$ is unitary. Since the span of $\tilde{\pi}(L^1(G))H$ is dense in H it follows that we can extend $\tilde{\pi}$ to a unitary map on H. It is clear that $\pi(xy)v = \pi(x)\pi(y)v$ for any v on the span $\tilde{\pi}(L^1(G))H$ and by density it follows that $\pi(xy) = \pi(x)\pi(y)$. Choose an $\epsilon > 0$ and pick for $1 \leq i \leq n$ elements α_i,ξ_i of H and f_i of $L^1(G)$. Since the map $x \mapsto L_x f_i$ is continuous for any i we can find an open set V of 1 such that $\|L_x f_i - f_i\| \leq \frac{\epsilon}{C}$ for any $x \in V$ where $C = \sum_i |\alpha_i| \|xi_i\|$. Since $\tilde{\pi}$ is contractive it follows that

$$\begin{aligned}
\|\pi(x) \sum_i \alpha_i \tilde{\pi}(f_i)\xi_i - \sum_i \alpha_i \tilde{\pi}(f_i)\xi_i\| &= \| \sum_i \alpha_i (\tilde{\pi}(L_x f_i) - \tilde{\pi}(f_i))\xi_i\| \\
&= \| \sum_i \alpha_i \tilde{\pi}(L_x f_i - f_i)\xi_i\| \\
&\leq \sum_i |\alpha_i| \frac{\epsilon}{C} \|\xi_i\| \\
&\leq \epsilon.
\end{aligned}$$

This shows that $\pi(x)$ is a continous map on the span $\tilde{\pi}(L^1(G))H$. By density it is continuous on H. We have thus showed that (π, H) is a unitary representation. To finish the bijection it remains to show that the integrated representation of π, now denoted by $\overline{\pi}$ is equal to $\tilde{\pi}$ and if two integrated representations are equal then the corresponding unitary representations are equal. By definition of π we have for any $f, g\ inL^1(G)$ and $\xi \in H$

$$\begin{aligned}
\overline{\pi}(f)\tilde{\pi}(g)\xi &= \int_G f(x)\pi(x) dx \tilde{\pi}(g)\xi \\
&= \int_G f(x)\pi(x)\tilde{\pi}(g)\xi dx \\
&= \int_G f(x)\tilde{\pi}(L_x g)\xi dx \\
&= \int_G \tilde{\pi}(f(x)L_x g) dx \xi.
\end{aligned}$$

It is shown in Appendix B.6 in [3] that the Pettis integral commutes with bounded linear maps from which it follows using our convolution formula that

$$\begin{aligned}
\overline{\pi}(f)\tilde{\pi}(g)\xi &= \tilde{\pi}(\int_G f(x) L_x g dx)\xi \\
&= \tilde{\pi}(f * g)\xi = \tilde{\pi}(f)\tilde{\pi}(g)\xi.
\end{aligned}$$

This shows that $\overline{\pi} = \tilde{\pi}$ on the span of $\tilde{\pi}(L^1(G))H$. By density they agree for all $\xi \in H$. If π_1 and π_2 are two unitary representations on H of G such that

$\tilde{\pi_1} = \tilde{\pi_2}$ then for any $f \in L^1(G)$ and $\xi, \eta \in H$ it follows that

$$\int_G f(x)\langle(\pi_1(x) - \pi_2(x))\xi, \eta\rangle dx = 0$$

Since this is true for any $f \in L^1(G)$ it follows that $\langle(\pi_1(x) - \pi_2(x))\xi, \eta\rangle = 0$ for any $x \in G$ and $\xi, \eta \in H$. This shows that $\pi_1(x) = \pi_2(x)$ for all $x \in G$. □

Proposition 5.2.10. *Let (π, H) be a unitary representation of G. Then (π, H) is irreducible if and only if the integrated representation $\tilde{\pi}$ is irreducible. If (π', H) is another unitary representation of G then (π, H) and (π', H) are equivalent if and only if their integrated representations are equivalent.*

Proof. Assume the (π, H) is an irreducible unitary representation of G and that K is a closed subspace of H such that $\tilde{\pi}(f)K \subseteq K$ for all $f \in L^1(G)$. Take an element η from the orthogonal complement $K^\top$. For any $\xi \in K$ and $f \in L^1(G)$ we then have

$$0 = \langle\tilde{\pi}(f)\xi, \eta\rangle = \int_G f(x)\langle\pi(x)\xi, \eta\rangle dx.$$

Since above holds for any $f \in L^1(G)$ it follows that $\langle\pi(x)\xi, \eta\rangle = 0$ for any $\xi \in K$ and $x \in G$. This implies that $\pi(x)\xi$ is in $(K^\top)^\top$ for any $x \in G$ and $\xi \in K$. But $(K^\top)^\top = K$ and it follows that K is a π-invariant subspace. This shows that $K = H$ or $K = \{0\}$ showing that $\tilde{\pi}$ is irreducible. A similar but easier argument shows that $\tilde{\pi}$ is irreducible when $\tilde{\pi}$ is irreducible. Let (π', H) is another unitary representation of G. If π and π' are equivalent then there exists a unitary map such that $U\pi(x)U^* = \pi'(x)$ for any $x \in G$. For any $\xi, \eta \in H$ and $f \in L^1(G)$ we then have

$$\begin{aligned}\langle U\tilde{\pi}(f)U^*\xi, \eta\rangle &= \int_G f(x)\langle\pi(x)U^*\xi, U^*\eta\rangle dx \\ &= \langle\tilde{\pi}(f)U^*, U^*\eta\rangle = \langle\tilde{\pi}'(f), \eta\rangle\end{aligned}$$

which shows that $\tilde{\pi}$ and $\tilde{\pi}'$ are equivalent. Conversely if $\tilde{\pi}$ and $\tilde{\pi}'$ are equivalent then there exists a unitary map such that $U\tilde{\pi}(f)U^* = \tilde{\pi}'(f)$ for all $f \in L^1(G)$. The formula for integrated representations then shows that

$$\int_G f(x)\langle\pi'(x)\xi, \eta\rangle = \int_G f(x)\langle U\pi(x)U^*\xi, \eta\rangle$$

for any $f \in L^1(G)$ and $\xi, \eta \in H$. This implies that $U\pi(x)U^* = \pi'(x)$ for all $x \in G$ showing that π and π' are equivalent. □

5.3 Maximal and Reduced Group C^*-Algebras of a Locally Compact Group

For a locally compact group G we have obtained a bijective correspondence between unitary representations of G and non-degenerate, contractive *-representations of $L^1(G)$. However $L^1(G)$ is in general not a C^*-algebra and the C^*-algebras have a particularly nice structure to them. It is therefore natural to try to find a way to transfer this result to the C^*-setting. One way of doing this is to define the maximal group C^*-algebra of G. Another C^*-algebra which is also useful in this setting is the reduced group C^* of G.

Definition 5.3.1 (The Reduced Group C^*-Algebra)**.** *If G is a locally compact group and $\lambda : L^1(G) \to \mathscr{B}(L^2(G))$ is the integrated representation of the left regular representation of G then $C^*_{red}(G) = \overline{\lambda(G)}^{\|\cdot\|}$ is called the the reduced group C^*-algebra of G.*

Proposition 5.3.2. *The reduced group C^*-algebra of a locally compact group is a C^*-algebra.*

Proof. By Proposition 4.1.5 we need to show that $C^*_{\mathrm{red}}(G) = \overline{\lambda(G)}^{\|\cdot\|}$ is a closed subalgebra that is closed under taking adjoints. By definition $C^*_{\mathrm{red}}(G)$ is closed in $\mathscr{B}(H)$. If $a, b \in C^*_{\mathrm{red}}(G)$ then there exists sequences $(f_n), (g_n)$ such that $\|a - \lambda(f_n)\| \to 0$ and $\|b - \lambda(g_n)\| \to 0$. Since λ is a homomorphism it follows that $\lambda(f_n)\lambda(g_n) = \lambda(f_n * g_n) \in C^*_{\mathrm{red}}(G)$ for all n. We also have

$$\|ab - \lambda(f_n)\lambda(g_n)\| \leq \|a - \lambda(f_n)\|\|b\| + \|\lambda(f_n)\|\|b - \lambda(g_n)\| \to 0$$

hence $ab \in C^*_{\mathrm{red}}(G)$. An easier proof shows that $C^*_{\mathrm{red}}(G)$ is closed under taking adjoints and it follows that $C^*_{\mathrm{red}}(G)$ is a C^*-algebra. ☐

Proposition 5.3.3. *The integrated representation $\lambda : L^1(G) \to C^*_{red}(G)$ of the left regular representation is injective.*

Proof. Define for any open set U of finite Haar measure measure that contains 1 the function $f_U = \frac{1}{\mu(U)}\chi_U$ where μ is the Haar measure on G. Then it can be shown that for any $f \in L^1(G)$ the net $f * f_\bullet$ converges to f in the L^1-norm as U gets small. Assume that $\lambda(f) = 0$ for some $f \in L^1(G)$. By the proposition we have proved for convolution and the Pettis integral it follows that for any open U of finite measure containing 1

$$0 = \lambda(f)f_U = \int_G f(x)L_x dx f_U = \int_G f(x)L_x f_U dx = f * f_U$$

but $f * f_U$ tends to f in $L^1(G)$ as U gets small. It follows that $f = 0$ which shows that λ is injective. ☐

Proposition 5.3.4. *Let G be a locally compact group. Then the map $\|\cdot\|_{max}$ defined by*

$$\|f\|_{max} = \sup_{(\pi,G)} \|\tilde{\pi}(f)\|$$

(where (π, H) is any unitary representation of G) is a norm on $L^1(G)$. Furthermore this norm satisfies the identity

$$\|f^* f\|_{max} = \|f\|^2_{max}$$

Proof. The supremum is over the set of all $\|\pi(f)\|$ where (π, G) is a unitary representation. This set is a non-empty set of real numbers and it follows that the supremum is well defined. Since the integrated representation of a unitary representation is contractive it follows that $\|f\|_{\max} \leq \|f\|_1$ for any $f \in L^1(G)$. For any f and g in $L^1(G)$ it is also clear that $\|cf\|_{max} = |c|\|f\|_{max}$ for $c \in \mathbb{C}$ and that $\|f+g\|_{max} \leq \|f\|_{max} + \|g\|_{max}$. If $\|f\|_{max} = 0$ then $\lambda(f) = 0$ and since the integrated representation of the left-regular representation is injective it follows that $f = 0$. Lastly, since $\mathscr{B}(H)$ is a C^*-algebra for any Hilbert space H we have

$$\begin{aligned}\|ff^*\|_{\max} &= \sup_{(\pi,H)} \|\tilde{\pi}(ff^*)\| = \sup_{(\pi,H)} \|\tilde{\pi}(f)\tilde{\pi}(f)^*\| \\ &= \sup_{(\pi,H)} \|\tilde{\pi}(f)\tilde{\pi}(f)^*\| = \sup_{(\pi,H)} \|\tilde{\pi}(f)\|^2 \\ &= \|f\|^2_{\max}\end{aligned}$$

□

Definition 5.3.5 (The Maximal Group C^*-algebra)**.** *For a locally compact group G we let $\|\cdot\|_{max}$ be the norm in Proposition 5.3.4 of $L^1(G)$. The maximal C^*-algebra of G, $C_{max}(G)$ is then the completion of $L^1(G)$ with respect to the norm $\|\cdot\|_{max}$.*

We will now show that the correspondence betweeen unitary representations of G and non-degenerate contractive *-representations of $L^1(G)$ extends to the maximal C^*-algebra.

Theorem 5.3.6. *There is a bijective correspondence between unitary representations of a locally compact group G and non-degenerate *-representations of $C^*_{max}(G)$ on Hilbert spaces. The bijection sends a unitary representation (π, H) of G to the unique extension of the integrated representation $\tilde{\pi}$ to $C^*_{max}(G)$.*

Proof. By Theorem 5.2.9 there is a bijective correpondence between unitary representations of G and non-degenerate, contractive representations of $L^1(G)$ which sends a unitary representation (π, H) to its integrated representation $\tilde{\pi} : L^1(G) \to \mathscr{B}(H)$. By definition of the norm $\|\cdot\|_{\max}$ it follows that $\|\tilde{\pi}(f)\| \leq \|f\|_{\max}$ for any $f \in L^1(G)$. This inequality implies that we can extend $\tilde{\pi}$ uniquely to a *-representation of $C^*_{\max}(G)$. Since $\tilde{\pi}$ is non-degenerate on $L^1(G)$ it follows that its extension is also non-degenerate. Conversely let $\pi : C^*_{\max}(G) \to \mathscr{B}(H)$ be a non-degenerate *-representation on a Hilbert space H and consider the restriction $\pi|_{L^1(G)} : (L^1(G), \|\cdot\|_1) \to \mathscr{B}(H)$. The restriction is clearly a *-homomorphism. Pick an $\epsilon > 0$. Since $\pi : C^*(G) \to \mathscr{B}(H)$ is non-degenerate there exists for each $h_0 \in H$ an integer N, elements $(h_n)_{1 \leq n \leq N}$ of H and elements $(a_n)_{1 \leq n \leq N}$ of $C^*(G)$ such that

$$\|\sum_{n=1}^{N} \pi(a_n)h_n - h_0\| \leq \frac{\epsilon}{2}.$$

Since $L^1(G)$ is dense in $C^*(G)$ (under $\|\cdot\|_{\max}$) there exists for each n, $1 \leq n \leq N$ an element $f_n \in L^1(G)$ such that

$$\|f_n - a_n\|_{\max} < \frac{\epsilon}{2N\|h_n\|}.$$

It follows that

$$\begin{aligned}\|\sum_{n=1}^{N} \pi(f_n)h_n - h_0\| &\leq \|\sum_{n=1}^{N} \pi(f_n - a_n)h_n\| + \|\sum_{n=1}^{N} \pi(a_n)h_n - h_0\| \\ &\leq \sum_{n=1}^{N} \|f_n - a_n\|_{\max}\|h_n\| + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

which shows that $\pi|_{L^1(G)}$ is non-degenerate. Since $\pi|_{L^1(G)}$ is a *-homomorphism from a Banach algebra (with $\|f^*\|_1 = \|f\|_1$) into a C^*-algebra it follows that $\pi|_{L^1(G)}$ is contractive. Using the bijective correspondence between non-degenerate, contractive *-homomorphisms of $L^1(G)$ and unitary representations of G the theorem then follows. □

Remark 5.2 (Amenability)**.** *We have defined two C^*-algebras of a locally compact group G, namely the reduced C^*-algebra $C^*_{red}(G)$ of G and the maximal C^*-algebra $C^*_{max}(G)$ of G. One might ask if these two algebras are isomorphic to each other. The integrated representation $\tilde{\lambda} : L^1(G) \to C^*_{red}(G) \subseteq \mathscr{B}(L^2(G))$ of the left-regular representation of G extends to *-homomorphism*

$$\tilde{\lambda} : C^*_{max}(G) \to C^*_{red}(G).$$

Since the image $\tilde{\lambda}(C^*_{max}(G))$ is a C^*-algebra it is complete and therefore closed in $C_{red}(G)$. It also contains the dense subset $\tilde{\lambda}(L^1(G))$ and it follows that $\tilde{\lambda} : C^*_{max}(G) \to C^*_{red}(G)$ is surjective. However this map is not in general injective. One can show that when G is an amenable group then $\tilde{\lambda} : C^*_{max}(G) \to C^*_{red}(G)$ is injective and hence an isometric *-isomorphism. This is what we present next.

Definition 5.3.7 (Amenability)**.** *If G is a locally compact group then a linear functional m on $L^\infty(G)$ is called a mean if $f(x) \geq 0$ for almost all x implies that $m(f) \geq 0$. The mean m is said to be left-invariant if $m(L_x f) = m(f)$ for all $x \in G$ and right-invariant $m(R_x f) = m(f)$ for all $x \in G$. The group G is said to be amenable if it has a left or right-invariant mean.*

Theorem 5.3.8. *If G is amenable then $C^*_{max}(G)$ is isometrically *-isomorphic to $C^*_{red}(G)$.*

Proof. See above discussion and Corollary G.39 in [8]. □

Notation 5.3.9. *When we are in a situation where $C^*_{max}(G)$ is isometrically *-isomorphic to $C^*_{red}(G)$ we denote either of these objects with $C^*(G)$.*

Theorem 5.3.10. *Abelian groups are amenable.*

Proof. See Theorem G.2.1 in [8] □

Remark 5.3. *We will later show that nilpotent groups are amenable.*

5.4 The Group C^*-Algebra of a Locally Compact Abelian Group

In this section we will calculate the group C^*-algebra of a locally compact abelian group G. Since G is abelian we know that $C^*_{\text{max}}(G)$ is isometrically *-isomorphic to $C^*_{\text{red}}(G)$ so we can restrict our attention to $C^*_{\text{red}}(G)$. Before we begin we need to recall some results from Fourier analysis which can all be found in the first chapter of [9]. For any locally compact abelian group we can define its dual group $\widehat{G}$ as the set of continuous homomorphisms $G \to \mathbb{T}$ (the unit circle) with multiplication given by pointwise multiplication of functions. The Fourier transform of $f \in L^1(G)$ is a function $\hat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)dx.$$

Since G is unimodular any left Haar measure is also a right Haar measure and vice versa. If we equip $\widehat{G}$ with the weak topology induced by the set of Fourier transforms, $\{\hat{f}, f \in L^1(G)\}$ then one can show that $\widehat{G}$ is also a locally compact abelian group and $\hat{f} \in C_0(\widehat{G})$ whenever $f \in L^1(G)$. Furthermore if the Fourier transform is viewed as a map $\widehat{\ }: L^1(G) \to C_0(\widehat{G})$ then $\widehat{\ }$ is a bounded *-homomorphism and the image of $\widehat{\ }$ is dense in $C_0(\widehat{G})$. Lastly Plancherel's theorem tells us that one can extend $\widehat{\ }$ viewed as a map on $L^1(G) \cap L^2(G)$ to a unitary map $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ called the Fourier-Plancherel transform. Note that since $\hat{G}$ is also a locally compact group it also has a Haar measure from which it is clear what $L^2(\widehat{G})$ means. The following proposition will be used when we calculate the reduced group C^*-algebra of a locally compact abelian group.

Proposition 5.4.1. *Let G be a locally compact group, $f \in C_0(G)$ and define the multiplication operator $M_f : L^2(G) \to L^2(G)$ by $(M_f g)(x) = f(x)g(x)$. Then M_f is well defined bounded linear operator on $L^2(G)$ and the map $f \mapsto M_f$ is a *-homomorphism and an isometry of $C_0(G)$ into $\mathscr{B}(L^2(G))$.*

Proof. Choose an $f \in C_0(G)$. It is easy to show that $M_f g$ is well defined as an element of $L^2(G)$ and it is clearly linear. We also have

$$\|M_f g\|_2^2 \leq \int_G |f(x)g(x)|^2 dx \leq \|f\|_\infty^2 \int_G |g(x)|^2 dx = \|f\|_\infty^2 \|g\|_2^2$$

From which is clear that $M_f \in \mathscr{B}(L^2(G))$ and $\|M_f\| \leq \|f\|_\infty$. On the other hand we can choose $\epsilon > 0$ and since f vanishes at infinity it has a global maximum $\|f\|_\infty$ attained at a point $x_0 \in G$. Pick an open neighbourhood V of x_0 with compact closure. Let $W = |f|^{-1}((\|f\|_\infty - \epsilon, +\infty))$ which is open and set $U = W \cap V$ then $0 < m(U) < +\infty$ where m is the Haar measure and

$$\|M_f \frac{\chi_U}{\sqrt{m(U)}}\|_2^2 = \frac{1}{m(U)} \int_U |f(x)|^2 dx \geq (\|f\|_\infty - \epsilon)^2.$$

Hence $\|M_f\| = \|f\|_\infty$. Since $M_{\alpha f+\beta g} = \alpha M_f + \beta M_g$, $M_{fg} = M_f M_g$ and $M_f^* = M_{f^*}$ where $f^*(x) = \overline{f(x)}$ for any $\alpha, \beta \in \mathbb{C}$ and $g, f \in C_0(G)$ the proposition follows. $\square$

Remark 5.4. *If we replace G and a Haar-measure with a locally compact Hausdorff space and a Radon measure we see that the only place where the proof breaks is that the Radon measure of an open set can be* 0*. A Radon measure is said to be fully supported if the Radon measure is non-zero on any open set. It follows that the proposition can be generalized to any locally compact Hausdorff space with a fully supported Radon measure.*

Remark 5.5. *Since $\widehat{G}$ is also a locally compact group the above proposition is also true for $C_0(\widehat{G})$.*

We will now prove that $C^*_{\text{red}}(G)$ is isomorphic to $C_0(\widehat{G})$ when G is a locally compact abelian group. Since we know that abelian groups are amenable the same result holds for $C^*_{\max}(G)$.

Theorem 5.4.2. *Let G be a locally compact abelian group. Use proposition 5.4.1 to view $C_0(\widehat{G})$ as a closed, *-closed subalgebra of $\mathscr{B}(L^2(\widehat{G}))$. Define the map ad $\mathcal{F}$ on $C^*_{red}(G)$ by ad $\mathcal{F}(a) = \mathcal{F}a\mathcal{F}^*$ then ad $\mathcal{F}$ maps into $C_0(\widehat{G})$ and ad $\mathcal{F} : C^*_{red} \to C_0(\widehat{G})$ is a surjective isometric *-homomorphism.*

Proof. Pick $f \in L^1(G)$ and consider the map $\mathcal{F}\lambda(f) : L^2(G) \to L^2(\widehat{G})$. Since $\|\lambda(f)\| \leq \|f\|_1$ we have

$$\|\mathcal{F}\lambda(f)g\|_2 \leq \|f\|_1\|g\|_2$$

hence $\|\mathcal{F}\lambda(f)\| \leq \|f\|_1$ and $\mathcal{F}\lambda(f) : L^2(G) \to L^2(\widehat{G})$ is a bounded map. If $g \in C_c(G)$ then $\lambda(f)g \in L^2(G)$. Since $g \in L^1(G) \cap L^2(G)$ we also have by Proposition 5.1.3 and 5.1.4

$$\lambda(f)g = (\int_G f(x)L_x dx)g = \int_G f(x)L_x g dx = f * g$$

and it follows that $\lambda(f)g \in L^1(G)$. Since $\mathcal{F}$ coincides with the Fourier transform, $\widehat{\ }$ on $L^1(G) \cap L^2(G)$ and $\widehat{\ }$: $L^1(G) \to C_0(\widehat{G})$ is a homomorphism it follows that

$$\mathcal{F}\lambda(f)g = \mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

and $\mathcal{F}(f) \in C_0(\widehat{G})$. If $g \in L^2(G)$ and (g_n) is a sequence of compactly supported functions converging to g in $L^2(G)$ then $\mathcal{F}\lambda(f)g_n \to \mathcal{F}\lambda(f)g$ in $L^2(\widehat{G})$ since $\mathcal{F}\lambda(f)$ is bounded. We also have

$$\begin{aligned}\|\mathcal{F}(f)\mathcal{F}(g_n - g)\|_2 &\leq \|M_{\mathcal{F}(f)}\|\|\mathcal{F}(g_n - g)\|_2 \\ &= \|\mathcal{F}(f)\|_\infty\|g_n - g\|_2 \to 0\end{aligned}$$

hence $\mathcal{F}(f)\mathcal{F}(g_n) \to \mathcal{F}(f)\mathcal{F}(g)$ in $L^2(\widehat{G})$. It follows that

$$\mathcal{F}\lambda(f)g = \lim_{n\to+\infty} \mathcal{F}\lambda(f)g_n = \lim_{n\to+\infty} \mathcal{F}(f)\mathcal{F}(g_n) = \mathcal{F}(f)\mathcal{F}(g).$$

If $g \in L^2(\widehat{G})$ and $f \in L^1(G)$ then since $\mathcal{F}$ is unitary we have

$$\mathcal{F}\lambda(f)\mathcal{F}^*g = \mathcal{F}(f)\mathcal{F}(\mathcal{F}^*(g)) = \mathcal{F}(f)g = M_{\mathcal{F}(f)}g.$$

In other words, if $f \in L^1(G)$ then ad $\mathcal{F}(\lambda(f)) \in C_0(\widehat{G})$. Pick any $a \in C^*_{\text{red}}(G)$ then there exists a sequence of $(f_n) \in L^1(G)$ such that

$$\lim_{n\to+\infty} \|a - \lambda(f_n)\| \to 0$$

hence

$$\lim_{n\to+\infty} \|\mathcal{F}a\mathcal{F}^* - \mathcal{F}\lambda(f_n)\mathcal{F}^*\| \to 0$$

since $\mathcal{F}$ is an isometry. From this we see

$$\mathcal{F}a\mathcal{F}^* = \lim_{n\to+\infty} \mathcal{F}\lambda(f_n)\mathcal{F}^* = \lim_{n\to+\infty} M_{\mathcal{F}(f_n)}.$$

But the set of multiplication operators is a closed subset of $\mathscr{B}(L^2(\widehat{G}))$ and therefore we must have that $\mathcal{F}a\mathcal{F}^* \in C_0(\widehat{G})$. Using the fact that $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is a surjective isometry we get

$$\sup_{\|g\|_2=1} \|\mathcal{F}a\mathcal{F}^*g\|_2 = \sup_{\|g\|_2=1} \|a\mathcal{F}^*g\|_2 = \sup_{\|f\|_2=1} \|af\|_2 = ||a||$$

hence ad $\mathcal{F}$ is an isometry. It is clear that ad $\mathcal{F}$ is a linear map. Since $C^*_{\text{red}}(G)$ is complete and ad $\mathcal{F}$ is an isometry it follows that ad $\mathcal{F}(C^*_{\text{red}}(G))$ is complete and hence closed in $C_0(\widehat{G})$. Since ad $\mathcal{F}(C^*_{\text{red}}(G))$ contains the dense subspace $\{\hat{f}, f \in L^1(G)\}$ it follows that ad $\mathcal{F}(C^*_{\text{red}}(G)) = C_0(\widehat{G})$ hence ad $\mathcal{F}$ is surjective. It remains to show that ad $\mathcal{F}$ is a *-homomorphism which follows from calculations

$$\text{ad } \mathcal{F}(ab) = \mathcal{F}ab\mathcal{F}^* = \mathcal{F}a\mathcal{F}^*\mathcal{F}b\mathcal{F}^* = \text{ad } \mathcal{F}(a)\text{ad } \mathcal{F}(b)$$

and

$$\text{ad } \mathcal{F}(a^*) = \mathcal{F}a^*\mathcal{F}^* = \mathcal{F}^{**}a^*\mathcal{F}^* = (\mathcal{F}a\mathcal{F}^*)^* = \text{ad } \mathcal{F}(a)^*$$

□

6 Duals and Spectral Theory

Given a unitary representation of a group G a natural question to ask is whether it is possible to decompose this representations into a direct sum of irreducible representations. For compact groups the Peter-Weyl theorem (Theorem A.5.2 [8]) shows that any unitary representation can be decomposed as a direct sum of its irreducible subrepresentations. For general locally compact groups this might not be possible since being reducible only implies that it has a subrepresentation and we don't know if it is irreducible or not. One example of this is the regular representation for any locally compact, but not compact group. In this case the regular representation has no irreducible subrepresentations (see Corollary C.47 in [8]). However it is possible to decompose it with a direct integral. This explains the interest in irreducible representations. Since equivalent representations are the same from a representation theoretical perspective it follows that one would like to study the class of all equivalence classes of irreducible representations, $\widehat{G}$ of a given group G. It is not clear that $\widehat{G}$, called the unitary dual of G is well-defined as a set but this will be proven in section 6.3 with the GNS-construction. For a *-algebra A one would similarly want to study its set of equivalence classes of irreducible representations $\widehat{A}$ called the dual or spectrum of A. These duals, $\widehat{G}$ and $\widehat{A}$ will not only be considered as sets but they will also be equipped with natural topologies. For a locally compact G we know already that we have a correspondence between $\widehat{G}$ and $\widehat{C^*_{\max}(G)}$ through the integrated representation and after equipping the topologies on these two spaces this map will be a homeomorphism. This shows that one can equally well study the representation theory of G through its C^*-algebra. The term spectrum is an overloaded word in analysis and there are also notion of the gelfand spectrum (which we choose to call the structure space for this reason) and the spectrum of a point. These spectrums are also important and in the first section we give an overview of them.

6.1 The Gelfand Transform and Spectral Theory

In this section some basic facts of the Gelfand transform and the spectrum of an element will be given. These facts will be needed to prove Schur's Lemma which we will prove in the next section. The facts mentioned here are well known and can be found in many books such as in [3] and in [7].

Definition 6.1.1 (The Gelfand Transform and the Structure Space)**.** *Let A be a commutative Banach algebra and define Δ to be the set of all non-trivial complex homomorphisms $h : A \to \mathbb{C}$. For any $x \in A$ the map $\hat{x} : \Delta \to \mathbb{C}$ given by $\hat{x}(h) = h(x)$ is called called the Gelfand transform of x. Denote by $\hat{A}$ the set of all Gelfand transforms and give Δ the weak topology induced by $\hat{A}$. The topological space Δ is called the structure space of A.*

Remark 6.1. *The trivial homomorphism is of course the homomorphism $\phi(x) = 0$ for all $x \in A$. If the commutative Banach algebra A has an identity element e then a complex homomorphism $\phi : A \to \mathbb{C}$ is non-trivial if and only if $\phi(e) = 1$.*

Proposition 6.1.2. *Let A be a a commutative Banach algebra. Then the following hold*

- *Δ is a locally compact Hausdorff space.*
- *If A is unital then Δ is compact.*
- *The Gelfand transform $\hat{x}$ is in $C_0(\Delta)$ for any $x \in A$.*
- *The map $\widehat{\ }: A \to C_0(\Delta)$ is a continuous algebra homomorphism.*

Proof. See Theorem 2.4.5 in [3] □

If A is a commutative C^*-algebra one might not expect that the Gelfand transform will respect the *-structure on A since the elements of Δ were not required to be *-homomorphisms. However the remarkable theorem of Gelfand-Naimark shows not only that the Gelfand transform is a *-homomorphism but also that it is an isomorphism.

Theorem 6.1.3 (Gelfand-Naimark)**.** *If A is a commutative C^*-algebra then the Gelfand transform $\widehat{\ }: A \to C_0(\Delta)$ is an isometric *-isomorphism. Furthermore Δ is compact if and only if A is unital.*

Proof. See Theorem 2.6.7 in [3]. □

For a unital Banach algebra A we can also define the notion of the spectrum $\sigma(x)$ of an element $x \in A$.

Definition 6.1.4. *Let A be a unital Banach algebra with identity element e. The spectrum $\sigma(x)$ of an element x in A is defined as the set of all $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is not invertible.*

Proposition 6.1.5. *Let A be a unital Banach algebra and x be any element of A. Then the spectrum $\sigma(x)$ of x is compact and non-empty. If A is commutative then the range of $\hat{x}$ is the spectrum $\sigma(x)$.*

Proof. See Theorem 10.13 and 11.5 in [7]. □

Restricting our attention to commutative and unital C^*-algebras we get a particularly nice result that enables us to do continuous functional calculus.

Theorem 6.1.6. *Let A be a commutative unital C^*-algebra which contains an element x such that the set of polynomials in the variables x and x^* are dense in A. Then $C(\sigma(x))$ is isometrically *-isomorphic to $C(\Delta)$ where the isomorphism is given by sending $f \in C(\sigma(x))$ to $f \circ \hat{x} \in C(\Delta)$.*

Proof. See Theorem 11.19 in [7]. □

Remark 6.2 (Continuous Functional Calculus)**.** *Let A be a commutative unital C^*-algebra and $x \in A$ be an element that satisfies the assumptions of Theorem 6.1.6. We can then combine the Gelfand-Naimark Theorem and Theorem 6.1.6 to obtain that for any $f \in C(\sigma(x))$ there exists an element $y \in A$ such that $\hat{y} = f \circ \hat{x}$. If we denote this element y by $f(x)$ it follows that $\|f(x)\| = \|f\|_\infty$. This identification between continuous functions on $\sigma(x)$ and elements of A is called continuous functional calculus and it also satisfies the following properties,*

$$
\begin{aligned}
(\alpha f + \beta g)(x) &= \alpha f(x) + \beta g(x) \\
(fg)(x) &= f(x)g(x) \\
(\overline{f})(x) &= f(x)^*
\end{aligned}
$$

for any $f, g \in C(\sigma(x))$ and $\alpha, \beta \in \mathbb{C}$.

6.2 Schur's Lemma

In this section Schur's Lemma will be proved which gives a characterization for irreducible unitary representations. To prove it we introduce the following notions. A set of operators $\mathcal{S} \subseteq \mathscr{B}(H)$ in the Hilbert space H is called self-adjoint if $S \in \mathcal{S}$ implies that $S^* \in \mathcal{S}$. The commutant of $\mathcal{S}$ is

$$\mathcal{S}' = \{T \in \mathscr{B}(H) |\ TS = ST\}.$$

Finally $\mathcal{S}$ is said to act topologically irreducible if the only closed $\mathcal{S}$-invariant subspaces of H are $\{0\}$ and H. Schur's lemma in the context of operators is then the following.

Theorem 6.2.1 (Schur's Lemma for Operators)**.** *Let $\mathcal{S} \subseteq \mathscr{B}(H)$ be a self-adjoint set of operators on the Hilbert space H. Then $\mathcal{S}$ acts topologically irreducible if and only if $\mathcal{S}' = \mathbb{C}id_H$.*

Proof. Assume first that $\mathcal{S}' = \mathbb{C}\mathrm{id}_H$ and let $K \subseteq H$ be an $\mathcal{S}$-invariant closed subspace of H. Since K is closed we can decompose H as $H = K \oplus K^\perp$ where $K^\perp$ is the orthogonal complement of K. Let $P : H \to K$ be the natural projection onto K, then any $x \in H$ can be written on the form $x = Px + Qx$ where $Qx \in K^\perp$. If x and y are elements of H we have

$$
\begin{aligned}
\langle Px, y\rangle &= \langle Px, Py + Qy\rangle = \langle Px, Py\rangle \\
&= \langle Px + Qx, Py\rangle = \langle x, Py\rangle
\end{aligned}
$$

and it follows that P is self-adjoint. By definition of P is also clear that $PSP = SP$ for any $S \in \mathcal{S}$ since K is $\mathcal{S}$-stable. Since $\mathcal{S}$ is self-adjoint we get for any $S \in \mathcal{S}$

$$SP = PSP = (PS^*P)^* = (S^*P)^* = PS$$

and it follows that $P \in \mathcal{S}' = \mathbb{C}\mathrm{id}_H$. Since $P^2 = P$ it is clear that either $P = \mathrm{id}_H$ or $P = 0$ hence we either have $K = H$ or $K = \{0\}$ and it follows that $\mathcal{S}$ is topologically irreducible.

For the converse assume that $\mathcal{S}$ is topologically irreducible and pick an element $T \in \mathcal{S}'$. Since $\mathcal{S}$ is self-adjoint it follows that $\mathcal{S}'$ is closed under taking adjoints. Let $\Re T = \frac{1}{2}(T+T^*)$ and $\Im T = \frac{1}{2i}(T-T^*)$ then since $\mathcal{S}'$ is closed under addition and adjoints it follows that $\Im T$ and $\Re T$ are elements of $\mathcal{S}'$. It is also straightforward to check that $\Re T$ and $\Im T$ are self-adjoint and $T = \Re T + i\Im T$. Hence, if we can show that $A \in \mathcal{S}'$ and $A^* = A$ implies that $A \in \mathbb{C}\mathrm{id}_H$ then it follows that $T \in \mathbb{C}\mathrm{id}_H$ and we are done. Assume that A is self-adjoint element of $\mathcal{S}'$. Let $M = \mathbb{C}[A] \subseteq \mathscr{B}(H)$ be the set of all complex polynomials in A where we let $A^0 = \mathrm{id}_H$. Then M is a subalgebra of $\mathscr{B}(H)$ that is closed under taking adjoints and it follows that the closure $\mathscr{A} = \overline{M}$ is a C^*-algebra by Proposition 4.1.5. Since M is commutative it follows that $\mathscr{A}$ is commutative. By construction the polynomials in A are dense in $\mathscr{A}$ and the Theorem of Gelfand-Naimark combined with Theorem 6.1.6 shows that $\mathscr{A}$ is isometrically *-isomorphic to $C(\sigma(A))$. If the spectrum $\sigma(A)$ consists of only one point then in particular $C(\sigma(A))$ is a one-dimensional vector space and it follows that $\mathscr{A}$ is a one dimensional vector space hence there exists $\alpha \in \mathbb{C}$ such that $A = \alpha\mathrm{id}_H \in \mathbb{C}\mathrm{id}_H$. We are therefore done if we show that the spectrum $\sigma(A)$ only consists of one point. If $\sigma(A)$ does not consist of one point then since it is non-empty it follows that there exists two distinct points x, y in $\sigma(A)$. Since $\sigma(A)$ is Hausdorff there exists open and disjoint sets V and W of $\sigma(A)$ containing x and y respectively. By Urysohn's Lemma, 2.12 in [4] there exist continuous functions $f, g \in C(\sigma(A))$ such that $f(x) \neq 0$, $g(y) \neq 0$ and f is supported in V and g is supported in W hence $fg = 0$. Functional calculus then gives us non-zero elements $f(A)$ and $g(A)$ of $\mathscr{A}$ such that $f(A)g(A) = 0$. Since A is an element of $\mathcal{S}'$ and $\mathcal{S}'$ is closed it follows that $\mathscr{A} \subseteq \mathcal{S}'$. The subspace $\overline{f(A)H}$ is then $\mathcal{S}$-stable since

$$S\overline{f(A)H} \subseteq \overline{Sf(A)H} = \overline{f(A)SH} \subseteq \overline{f(A)H}$$

for any $S \in \mathcal{S}$. By assummtion $\mathcal{S}$ is topologically irreducible hence $\overline{f(A)H} = H$ since $f(A) \neq 0$. But this implies the contradiction $g(A) = 0$ since

$$g(A)H = g(A)\overline{f(A)H} \subseteq \overline{g(A)f(A)H} = \{0\}$$

and it follows that $\sigma(x)$ consists of exactly one point. □

Schur's Lemma in the context of representation theory is then an easy corollary

Theorem 6.2.2 (Schur's Lemma). *Let (π, H) be a unitary representation of G and set $\pi(G) = \{\pi(g) |\ g \in G\}$. Then (π, H) is irreducible if and only if the commutant satisifies $\pi(G)' = \mathbb{C}id_H$.*

Proof. Let (π, H) be a unitary representation of G. Since $\pi(g)^* = \pi(g^{-1})$ it follows that $\pi(G)$ is a self-adjoint set of operators. It is also clear that (π, H) is irreducible if and only if $\pi(G)$ is topologically irreducible which by Schur's lemma for operators happens if and only if $\pi(G)' = \mathbb{C}\mathrm{id}_H$. □

Schur's lemma yields the following important description of irreducible representations of abelian groups.

Proposition 6.2.3. *If G is an abelian topological group then any irreducible unitary representation (π, H) of G is 1-dimensional. Thus there exists a character $\chi : G \to \mathbb{S}^1$ such that $\pi(g) = \chi(g)id_H$ for any $g \in G$.*

Proof. Let (π, H) be an irreducible unitary representation of our abelian group G. Fix any $h \in G$ then for any $g \in G$ we have

$$\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$$

hence $\pi(G) \subseteq \pi(G)'$. Schur's lemma then shows us that $\pi(G) \subseteq \mathbb{C}\mathrm{id}(H)$. Pick a non-zero x in H and consider the closed subspace $X = \mathrm{span}\{x\}$ of H. Since $\pi(G) \subseteq \mathbb{C}\mathrm{id}_H$ is irreducible it follows that $H = X$, that is (π, H) is one-dimensional. Since $\pi(g) \in \mathbb{C}\mathrm{id}(H)$ and is unitary there exists for each $g \in G$ a number $\chi(g) \in \mathbb{S}^1$ such that $\pi(g) = \chi(g)\mathrm{id}_H$. The map $\chi : G \to \mathbb{S}^1$ is clearly a homomorphism of groups since π is. Since H is one-dimensional it follows that $\mathcal{U}(H)$ is homeomorphic to $\mathbb{S}^1$ and this shows that $\chi : G \to \mathbb{S}^1$ is continuous. □

6.3 The Unitary Dual

In representation theory the primary object of study for a given topological group is its irreducible representations. In this section we will, as said in the introduction of chapter 6 show that the set $\widehat{G}$ of all equivalence classes of irreducible representations G actually is a set which we call the unitary dual of G. For abelian groups Proposition 6.2.3 already shows that this is the case, since it shows that any equivalence class of irreducible representations can be identified with exactly one character and any character is identified in this way. We summarize the definition below

Definition 6.3.1 (The Unitary Dual)**.** *For any topological group G the set of equivalence classes of irreducible representations of G is called the unitary dual of G which is denoted by $\widehat{G}$*

Remark 6.3. *The notion of equivalence discussed above is of course the equivalence relation given by unitary equivalence. Two unitary representations of a group G are equivalent to each other if they are unitarily equivalent.*

The fact that $\widehat{G}$ is a set was explained above in the abelian case. For the general case we first need to introduce functions of positive type and the GNS construction (Gelfand-Naimark-Segal). The idea of the proof that $\widehat{G}$ is a set is to show that it can be identified with a subset of functions of positive type on G. There is a related notion of kernels of positive types for a topological space X which we define first. The GNS construction is also well explained in Appendix C in [8].

Definition 6.3.2 (Kernels of Positive Type). *A kernel of positive type on a topological space X is a continuous function $\Phi : X \times X \to \mathbb{C}$ such that*

$$\sum_{n=1}^{N} \sum_{m=1}^{N} c_n \overline{c_m} \Phi(x_n, x_m) \geq 0,$$

for any $N \in \mathbb{N}$, any $c_1, ..., c_N$ in $\mathbb{C}$ and any $x_1, ..., x_N$ in G.

A reason for considering these functions is that for any Hilbert space H and continuous function $f : X \to H$ the continuous function $\Phi(x, y) = \langle f(x), f(y) \rangle$ is of positive type. This is the case since

$$\begin{aligned}\sum_{n=1}^{N} \sum_{m=1}^{N} c_n \overline{c_m} \langle f(x_n)\xi, f(x_m)\xi \rangle &= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle c_n f(x_n)\xi, c_m f(x_m)\xi \rangle \\ &= \langle \sum_{n=1}^{N} c_n f(x_n)\xi, \sum_{m=1}^{N} c_m f(x_m)\xi \rangle \geq 0\end{aligned}$$

The GNS construction shows that the converse is true.

Theorem 6.3.3 (GNS Construction). *Let Φ be a kernel of positive type on the topological space X. Then there exists a Hilbert space H and a continuous function $f : X \to H$ such that the following properties hold,*

- *$\Phi(x, y) = \langle f(x), f(y) \rangle$ for all $x, y \in X$.*
- *The span of the image $f(X)$ is dense in H.*

Furthermore, if the pair (H', f') H' where H' is Hilbert space and $f' : X \to H'$ a continuous function satisfies the same properties as (H, f) then there exists a unique unitary map $U : H \to H'$ such that $f' = U \circ f$.

Proof. Let Φ be a kernel of positive type and define for any $x \in X$ the function $\Phi_x \in C(X)$ by $\Phi_x(y) = \Phi(x, y)$. Let V be the span of $\{\Phi_x | x \in X\} \subseteq C(X)$. Set $\phi = \sum_{i=1}^{n} a_i \Phi_{x_i}$ and $\psi = \sum_{j=1}^{m} b_j \Phi_{x_j}$ and define

$$\langle \phi, \psi \rangle_V = \sum_{i}^{n} \sum_{j}^{m} a_i \bar{b}_j \Phi(x_i, x_j).$$

The equalities

$$\langle \phi, \psi \rangle_V = \sum_{i}^{n} \sum_{j}^{m} a_i \bar{b}_j \Phi(x_i, x_j) = \sum_{j=1}^{m} \bar{b}_j \phi(x_j) = \sum_{i=1}^{n} a_i \psi(x_i)$$

shows that the value of $\langle \phi, \psi \rangle_V$ does not depend on the choice of representation of ϕ or ψ as a sum. By definition of $\langle \cdot, \cdot \rangle_V$ it is clear that $\langle \cdot, \psi \rangle_V$ is additive for any fixed $\psi \in V$. Since Φ is a kernel of positive type it is easy to show that $\Phi(x, y) = \overline{\Phi(y, x)}$ for all $x, y \in X$ which implies that $\langle \phi, \psi \rangle_V = \overline{\langle \psi, \phi \rangle}_V$. If $\phi = \sum_{i=1}^n a_i \Phi_{x_i}$ then

$$\langle \phi, \phi \rangle_V = \sum_{i=1}^n \sum_{j=1}^m a_i \overline{a_j} \Phi(x_i, x_j) \geq 0.$$

These properties are the ones needed for showing the Cauchy-Schwartz inequality $|\langle \phi, \psi \rangle_V|^2 \leq \langle \phi, \phi \rangle_V \langle \psi, \psi \rangle_V$. Since $\langle \phi, \Phi_x \rangle_V = \phi(x)$ for any $\phi \in V$ and $x \in X$ the Cauchy-Schwartz inequality shows that

$$|\phi(x)|^2 \leq \Phi(x, x) \langle \phi, \phi \rangle_V$$

for any $x \in X$ hence if $\langle \phi, \phi \rangle_V = 0$ then $\phi = 0$. These facts show that $\langle \cdot, \cdot \rangle$ is an inner product on V. Let $(H, \langle \cdot, \cdot \rangle)$ be the Hilbert space completion of $(V, \langle \cdot, \cdot \rangle_V)$. The bound

$$|\phi(x)|^2 \leq \Phi(x, x) \|\phi, \phi\|_V$$

shows that for any Cauchy sequence (ϕ_n) in V the limit $\lim_{n \to \infty} \phi_n(x)$ exists for any $x \in X$. This means that H can be realized as the space of complex functions on X that are pointwise limits of functions of V. If we define $f : X \to H$ by $f(x) = \Phi_x$ then $\langle f(x), f(y) \rangle = \Phi(x, y)$ for any $x, y \in X$. Since $\|\psi\|^2 = \langle \psi, \psi \rangle$ for any $\psi \in H$ it follows that

$$\|f(x) - f(y)\|^2 = \|f(x)\|^2 - 2\Re(\langle f(x), f(y) \rangle) + \|f(y)\|^2.$$

The right side of the equality above are continuous functions which means that $f : X \to H$ is continuous. This establishes the existence of the pair (H, f) that we were looking for. If (H', f') is another pair, $f' : X \to H'$ that satisfies the same properties as (H, f) then it follows that

$$\begin{aligned}
\|\sum_{i=1}^n a_i \Phi_{x_i}\|^2 &= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a}_j \Phi(x_i, x_j) \\
&= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a}_j \langle f'(x_i), f'(x_j) \rangle \\
&= \|\sum_{i=1}^n a_i f'(x_i)\|^2.
\end{aligned}$$

This shows that the map $\tilde{U} : V \to H'$ given by $\sum_{i=1}^n a_i \Phi_{x_i} \mapsto \sum_{i=1}^n a_i f'(x_i)$ is well defined and is an isometry. The extension of $\tilde{U}$ to $U : H \to H'$ is then also an isometry. Since the image $U(V)$ is the span of $\{g(x) | x \in X\}$ which is dense in H' by assumption it follows that U is surjective and hence an isometry. It is clear that $U(f(x)) = g(x)$ for all $x \in X$ and this relation determines U uniquely. □

We now turn the attention to positive functions on a topological group G.

Definition 6.3.4 (Functions of Positive Type)**.** *A function of positve type on a topological group G is a continuous function $\phi : G \to \mathbb{C}$ such that*

$$\sum_{n=1}^{N}\sum_{m=1}^{N} c_n \overline{c_m} \phi(g_m^{-1} g_n) \geq 0,$$

for any $N \in \mathbb{N}$, any $c_1, ..., c_N$ in $\mathbb{C}$ and any $g_1, ..., g_N$ in G.

Remark 6.4. *If $\phi : G \to \mathbb{C}$ is a function of positive type then $\Phi(x, y) = \phi(y^{-1}x)$ is a kernel of positive type on G.*

Another construction that is also called the GNS construction will show that to any function of positive type on G we can associate a unitary cyclic representation.

Definition 6.3.5 (Cyclic representations)**.** *A unitary representation (π, H) on a topological group G is called cyclic if there exist a $\xi \in H$ such that the span of $\pi(G)\xi$ is dense in H. The element ξ is called a cyclic vector.*

Remark 6.5. *For any irreducible representation (π, H) of G any element $\xi \in H$ must be a cyclic vector since otherwise the closure of the span of $\pi(G)\xi$ would be a π-invariant subspace of H.*

Remark 6.6. *If (π, H) and (π', H') are unitarily equivalent representations given by $U : H \to H'$ and (π, H) is cyclic with cyclic vector ξ then (π', H') is cyclic with cyclic vector $\xi' = U\xi$.*

Theorem 6.3.6 (GNS Construction)**.** *For any function ϕ of positive type on G there exists a cyclic unitary representation (π, H) of G with cyclic vector ξ such that*

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle$$

for any $x \in G$. Furthermore if (π', H') is another cyclic unitary representation of G with cyclic vector G such that $\phi(x) = \langle \pi'(x)\xi', \xi' \rangle$ for all $x \in G$ then (π, H) and (π', H') are equivalent with unitary map $U : H \to H'$ such that $U\xi = \xi'$.

Proof. Let $\phi : G \to \mathbb{C}$ be a function of positive type on G and consider the kernel of positive type $\Phi(x, y) = \phi(y^{-1}x)$ on G. By the other GNS-construction there exists a hilbert space H and a continuous map $f : X \to H$ range such that

$$\langle f(x), f(y) \rangle = \Phi(x, y) = \phi(y^{-1}x)$$

for all $x, y \in G$. Furthermore, the span of the range of f is dense in H. Consider for any fixed $g \in G$ the function $f_g(x) = f(gx)$. It satisfies

$$\langle f(gx), f(gy) \rangle = \Phi(gx, gy) = \phi(y^{-1}x).$$

By uniqueness of the GNS construction there exists a unitary map $\pi(g) : H \to H$ such that $f_g(x) = \pi(g)f(x)$ for any $x \in G$. It follows that

$$\pi(gh)f(x) = f(ghx) = \pi(g)f(hx) = \pi(g)\pi(h)f(x),$$

for any $x \in G$. Since the span of the range of f is dense in H it follows that $\pi(gh) = \pi(g)\pi(h)$ for any $g, h \in G$. The relation $\pi(g)f(x) = f(gx)$ combined with continuity of f and density shows that the map $g \mapsto \pi(g)\eta$ is continuous for any $\eta \in H$. This shows that (π, H) is a unitary representation of G. Set $\xi = f(1)$. Then $\pi(G)\xi = f(G)$ and it follows that (π, H) is a cyclic representation. We also have

$$\langle \pi(x)\xi, \xi\rangle = \langle \pi(x)f(1), f(1)\rangle = \langle f(x), f(1)\rangle = \Phi(x, 1) = \phi(x).$$

If (π', H') is another cyclic unitary representation with cyclic element ξ that also satisfies

$$\phi(x) = \langle \pi'(x)\xi', \xi'\rangle$$

for any $x \in G$ then

$$\langle \pi'(x)\xi', \pi'(y)\xi'\rangle = \langle \pi'(y^{-1}x)\xi', \xi'\rangle = \phi(y^{-1}x) = \Phi(x, y)$$

for any x, y. By uniqueness of the GNS consctruction there exists a unitary map $U : H \to H'$ such that $\pi'(x)\xi' = Uf(x) = U\pi(x)\xi$ for any $x \in G$. Setting $x = 1$ yields $U\xi = \xi'$. It then follows that

$$U\pi(x)U^{-1}\xi' = U\pi(x)\xi = \pi'(x)\xi$$

for any $x \in G$. It follows that $U\pi(x)U^{-1} = \pi'(x)$ on $\pi'(G)\xi'$ since

$$\begin{aligned}
U\pi(x)U^{-1}\pi'(y)\xi' &= U\pi(x)U^{-1}\pi'(y)UU^{-1}\xi' \\
&= U\pi(x)\pi(y)U^{-1}\xi' = U\pi(xy)U^{-1}\xi' \\
&= \pi'(x)\pi'(y)\xi'.
\end{aligned}$$

This implies that $U\pi(x)U^{-1} = \pi'(x)$ for all $x \in G$ on the span of $\pi'(G)\xi'$ hence on H' since ξ' is a cyclic vector. $\square$

Remark 6.7. *By the GNS-construction there corresponds to any function of positive type $\phi : G \to \mathbb{C}$ a cyclic representation (π_ϕ, H_ϕ) with cyclic vector ξ_ϕ. If π is a cyclic representation of G with cyclic vector ξ. Set $\phi(x) = \langle \pi(x)\xi, \xi\rangle$ then it follows by the GNS-construction that π is unitarily equivalent by (U) to π_ϕ and $U\xi = \xi_\phi$. This means that the equivalence classes of cyclic representations can be parametrised as a set by If (π_ϕ, H_ϕ) which shows that the equivalence classes of cyclic representations is a set. Since any irreducible representation is cyclic it follows that $\widehat{G}$ is a set.*

Corollary 6.3.6.1. *The collection of equivalence classes of unitary cyclic representations of a topological group G is a set. It follows that $\widehat{G}$ is a set.*

6.4 Topology on the Unitary Dual

For any irreducible unitary representation (π, H) its diagonal matrix coefficients are given by

$$\phi_\xi(x) = \langle \pi(g)\xi, \xi \rangle$$

for $x \in G$. These are the functions of positive type associated to π and any equivalent representation will have the same functions of positive type associated to them. A way of defining a topology on $\widehat{G}$ could therefore be to let a point $\pi \in \widehat{G}$ be close to π' if functions of positive type associated to π' can be approximated by functions of positive type associated to π in some sense. This is what the Fell topology does.

Definition 6.4.1. *For any equivalence class $[\pi]$ in $\widehat{G}$ the functions of positive type associated to $[\pi]$ are the functions of the form $\phi_\xi(x) = \langle \pi(x)\xi, \xi \rangle$ where ξ is any vector in the Hilbert space associated to H.*

Remark 6.8. *The functions of positive type associated to $[\pi] \in \widehat{G}$ do not depend on the choice of representative (π, H). Due to this we will often abuse notation and simply write π for the equivalence class $[\pi]$.*

Definition 6.4.2 (The Fell Topology on $\widehat{G}$)**.** *For any $\pi \in \widehat{G}$ let $\phi_1, ..., \phi_n$ be some collection of functions of positive type associated to π, let K be a compact subset of G and let $\epsilon > 0$. Define $W(\pi, \phi_1, ..., \phi_n, K, \epsilon)$ to be the set of all $\pi' \in \widehat{G}$ such that there exists for any ϕ_j a function of positive type ψ_j associated to π' such that*

$$|\phi_j(x) - \psi_j(x)| < \epsilon$$

for all $x \in K$. The set of all $W(\pi, \phi_1, ..., \phi_n, Q, \epsilon)$ is a basis of a topology on $\widehat{G}$ which is called the Fell topology.

Proposition 6.4.3. *The set of all $W(\pi, \phi_1, ..., \phi_n, K, \epsilon)$ is a basis of a topology on $\widehat{G}$.*

Proof. It is clear that the union of all such sets cover $\widehat{G}$ since π is in $W(\pi, \cdot, ..., \cdot, \cdot, \cdot)$ for any $\pi \in \widehat{G}$. Pick two such sets $W_1 = W(\pi, \phi_1, ..., \phi_n, K, \epsilon)$ and $W_2 = W(\pi', \psi_1, ..., \psi_n, Q, \delta)$ with ρ lying in their intersection. Then there are functions $(f_i)_{1 \leq i \leq n}$ $(g_i)_{1 \leq i \leq m}$ of positive type associated to ρ such that

$$a_i = \max_{x \in K} |\phi_i(x) - f_i(x)| < \epsilon$$
$$b_i = \max_{x \in Q} |\psi_i(x) - g_i(x)| < \delta$$

Let $c = \min(\epsilon - a_i)$ and $d = \min(\delta)$ and set $C = \min_{c,d}$ and define

$$W = W(\rho, (f_i)_{1 \leq i \leq n}, (g_i)_{1 \leq i \leq m}, K \cup Q, C).$$

Pick an element σ in W and consider the function ϕ_i. There exists a function h_i of positive type associated to σ such that

$$|f_i(x) - h_i(x)| < C$$

for all $x \in K \cup Q$. It follows that

$$\begin{aligned}|\phi_i(x) - h_i(x)| &\leq |\phi_i(x) - f_i(x)| + |f_i(x) - h_i(x)| \\ &< a_i + \min(\epsilon - a_i) \leq \epsilon\end{aligned}$$

for any $x \in K$. The same argument can be used for any other function $(\phi_i)_i$ or $(\psi_i)_i$ which means that $W \subseteq W_1 \cap W_2$. This shows that elements on the form $W(\pi, \phi_1, ..., \phi_n, K, \epsilon)$ defines a basis for $\widehat{G}$. □

Example 6.4.4 (Locally Compact Abelian Groups)**.** *Let G be a locally compact abelian group. Then any irreducible representation (π, H) is of the form $\chi(x) id_H$ where $\chi : G \to \mathbb{T}$ is a character. It follows that any function of positive type is of the form*

$$\langle \pi(x)\xi, \xi \rangle = \langle \chi(x)\xi, \xi \rangle = \chi(x)\|\xi\|^2$$

for some $\xi \in H$. It follows that a subbase for the topology on $\widehat{G}$ when viewed as the set of characters is generated by sets of the form

$$V_{K,U} = \{\chi \in \widehat{G} | \chi(K) \subseteq U\}$$

for some open set $U \in \mathbb{T}$ and a compact set $K \in G$. This topology is called the compact open topology on $\widehat{G}$ and is shown in [9] to be equivalent to the weak topology induced by the set of Fourier transforms which is the topology that we previously gave it.

6.5 The Dual of a C^*-Algebra

Just as with unitary representations we can consider irreducible representations for a *-algebra A and by analogy we would like to have a dual object $\widehat{A}$ of equivalence classes of irreducible representations of A as well. For the maximal group C^*-algebra of a locally compact group G the integrated representation gives a bijection between $\widehat{G}$ and $\widehat{C^*_{\max}(G)}$ which shows that $\widehat{C^*_{\max}(G)}$ is a set. The dual (or spectrum as it is also called) of a general *-algebra is also well-defined as a set. The reason for this is that there is correspondence (similar to our GNS-constructions) between equivalence classes of *-representations of A and a subset of functions of the form $x \mapsto \langle \pi(x)\xi, \xi \rangle$ where (π, H) is a *-representation and $\xi \in H$, see Proposition 2.4.1 in [6]. This correspondence has a uniqueness property for cyclic representations just as in the GNS-construction. This uniqueness property can be used show that these collections of equivalence classes are indeed sets. Functions of the $x \mapsto \langle \pi(x)\xi, \xi \rangle$ form are called positive forms on A. There is also another notion of a dual of A called the primitive ideal space of A which is the set of kernels of irreducible representations of A. The primitive ideal space, $\mathrm{Prim}(A)$ is sometimes more managable than $\widehat{A}$ but in many cases they are in bijection.

Definition 6.5.1 (The Dual of a *-Algebra)**.** *For any involutive algebra A we let $\widehat{A}$ denote the set of all equivalence classes of irreducible *-representations of A.*

Definition 6.5.2 (The Primitive Ideal Space)**.** *Let A be a *-algebra. We denote by $Prim(A)$ the set of all subsets in I of A such that $I = \ker\pi$ for some irreducible *-representation π of A.*

We will begin with putting a topological structure on the primitive ideal space.

Definition 6.5.3 (Hull-Kernel Closure and Jacobson topology)**.** *Let A be a *-algebra. For any subset $S \subseteq Prim(A)$ we define*

$$I(S) = \bigcap_{\pi \in S} \ker \pi$$

and let $\overline{S}$ be the set of all elements π of $Prim(A)$ such that $I(S) \subseteq \ker\pi$. The operation $S \mapsto \overline{S}$ is called hull-kernel closure. The hull-kernel closure satisfies the Kuratowski's closure axioms and therefore defines a topology on Prim called the Jacobson topology.

Remark 6.9. *Since equivalent irreducible representations have the same kernel it follows that we get a canonical surjective map $k : \widehat{A} \to Prim(A)$. This map will be used to define a topology on $\widehat{A}$.*

Definition 6.5.4. *We define a topology on $\widehat{A}$ by declaring the open sets to be the ones on the form $V = k^{-}(W)$ for some open W in $Prim(A)$.*

Remark 6.10. *For a locally compact group G we know that there is a bijection between $\widehat{G}$ and $\widehat{C^*}(G)$ given by the taking the integrated representation. This bijection is also natural candidate to use in order to define a topology on $\widehat{C^*}(G)$. However the content of Proposition 18.1.5 in [6] is that these two topologies will coincide.*

The following is a nice class of C^*-algebras which is also important when considering the canonical map $k : \widehat{A} \to \mathrm{Prim}(A)$ as we soon shall see.

Definition 6.5.5 (Liminal C^*-Algebras)**.** *A C^*-algebra, A is said to be liminal if for any irreducible *-representation $\pi : A \to \mathscr{B}(H)$ the evaluation $\pi(x)$ is a compact operator for any $x \in A$.*

For us it will be of interest that the group C^*-algebras of connected nilpotent Lie groups are liminal. Note that the reduced and maximal group C^*-algebra coincides since nilpotent groups are amenable. This fact will be proved in the next section.

Proposition 6.5.6. *The group algebra $C^*(G)$ is liminal if G is a connected nilpotent Lie group.*

Proof. See *13.11.12 in [6]. $\square$

Proposition 6.5.7. *If $C^*(G)$ is liminal for some locally compact group G then the canonical map $k : \widehat{A} \to Prim(A)$ is a homeomorphism.*

Proof. This is proposition 2.1 in [18] $\square$

Proposition 6.5.8 (Central Characters)**.** *The map $res : \widehat{G} \to \widehat{Z(G)}$ given by $\pi \mapsto \pi|_{Z(G)}$ is a quotient map.*

Proof. Let (π, H) be an irreducible unitary representation on G. By Schur's lemma we have $\pi(Z(G)) \subseteq \pi(G)' \subseteq \mathbb{C}\mathrm{id}_H$ which shows that $\mathrm{res}(\pi)$ is irreducible. The fact $\mathrm{res} : \widehat{G} \to \widehat{Z(G)}$ is continuous is clear from the definition of the Fell topology. One can induce representations from subgroups, see appendix E in [8] $\square$

The following fact is now straightforward to prove.

Proposition 6.5.9. *Any isometric *-isomorphism $\Phi : C^*(G) \to C^*(H)$ for some connected nilpotent Lie groups G and H induces a homeomorphism of $\widehat{C^*(H)} \to \widehat{C^*(G)}$*

6.6 Projection Valued Measures and Unitary Representations

In this section we will define projection valued measures and apply them in the context of unitary representations. All the details on projection valued measures can be found in chapter 12 in [7].

Definition 6.6.1. *Projection Valued Measures Let $(X, \mathcal{M})$ be a measureable space and H a Hilbert space. A projection valued measure E is a map $E : \mathfrak{M} \to \mathscr{B}(H)$ such that for any elements ω and ω' in $\mathfrak{M}$*

- *$E(\varnothing) = 0$ and $E(X) = id_H$.*
- *$E(\omega)$ is a self-adjoint projection.*
- *$E(\omega \cap \omega') = E(\omega)E(\omega')$.*
- *$E(\omega \cup \omega') = E(\omega) + E(\omega')$ whenever $\omega \cap \omega' = \varnothing$.*
- *For any x and y in H the function $E_{x,y} : \mathfrak{M} \to \mathbb{C}$ defined by*

$$E_{x,y}(\omega) = \langle E(\omega)x, y\rangle$$

is a complex measure on X.

When X is a locally compact Hausdorff space $\mathfrak{M}$ is the σ-algebra of Borel sets then it is also customary to add the requirement that $E_{x,y}$ should be a regular measure for any $x, y \in H$.

6.6.2. *The spaces $B(X, \mathfrak{M})$ and $L^\infty(E)$*
Let $(X, \mathfrak{M})$ be a measurable space, $E : \mathfrak{M} \to \mathscr{B}(H)$ a projection valued measure and $f : X \to \mathbb{C}$ a $\mathfrak{M}$-measurable function. There exists a countable collection of open balls $\{B_i\}$ that is a basis for the topology of $\mathbb{C}$. Let V be the union of all the B_i such that $E(f^{-1}(B_i)) = 0$. Then one can show that $E(f^{-1}(V)) = 0$ and V is the largest open set with this property. The essential range of f is the complement of V and f is said to be essentially bounded if the essential range is a bounded set. If f is essentially bounded then its essential range is a compact set and we can therefore define $\|f\|_\infty$ to be the maximum of f over its essential range.

Define $B(X, \mathfrak{M})$ to be the C^*-algebra of all $\mathfrak{M}$-measurable bounded functions $f : X \to \mathbb{C}$ normed by the supremum norm. The set

$$N = \{f \in B(X, \mathfrak{M}) | \; ||f||_\infty = 0\}$$

is a closed, *-closed ideal of $B(X, \mathfrak{M})$ and it follows that $B(X, \mathfrak{M})/N$ is a C^*-algebra which we denote by $L^\infty(E)$.

Theorem 6.6.3. *For any projection valued measure $E : \mathfrak{M} \to \mathscr{B}(H)$ on some measurable space $(X, \mathfrak{M})$ and some Hilbert space H there exists an isometric $*$-isomorphism Φ of $L^\infty(E)$ onto a commutative sub-C^*-algebra A of $\mathscr{B}(H)$. The element $\Phi(f)$, $f \in L^\infty(E)$ is the unique element of $\mathscr{B}(H)$ satisfying*

$$\langle \Phi(f)x, y \rangle = \int_X f dE_{x,y}$$

for all $x, y \in H$.

Proof. See Theorem 12.21 in [7] □

Definition 6.6.4. *Integrals of Projection Valued Measures*
With the theorem above in mind it is natural to introduce the notation

$$\Phi(f) = \int_X f dE$$

for $f \in L^\infty(E)$ which then means that the integral $\int_X f dE$ is the unique element in $\mathscr{B}(H)$ such that

$$\langle (\int_X f dE)x, y \rangle = \int_X f dE_{x,y}$$

holds for all $x, y \in H$.

Theorem 6.6.5. *If G is a locally compact abelian group then there is a bijection between unitary representations of G and projection valued measures defined on the Borel sets of the dual group $\widehat{G}$. If (π, H) is a unitary representation of G then there exists a projection valued measure $E : \mathcal{B}(\widehat{G}) \to \mathscr{B}(H)$ such that*

$$\pi(x) = \int_{\widehat{G}} \gamma(x) dE(\gamma)$$
$$\tilde{\pi}(f) = \int_{\widehat{G}} \overline{\hat{f}(\gamma)} dE(\gamma)$$

For all $x \in G$ and $f \in L^1(G)$ where $\tilde{\pi}$ is the integrated representation of π. Conversely, if $E : \mathcal{B}(\widehat{G}) \to \mathscr{B}(H)$ is a projection valued measure on the Borel sets of the dual group then

$$\pi(x) = \int_{\widehat{G}} \gamma(x) dE(\gamma)$$

defines a unitary representation of G.

Sketch. The proof is quite long but the construction of a projection valued measure from a unitary representation of a locally compact abelian group ties in nicely with the rest of the chapter and therefore I include that part. The rest of the proof can be found in Theorem D.31 in [8]. Assume that (π, H) is a unitary representation of G. Fix any ξ in H and define $f : G \to \mathbb{C}$ by

$$f(x) = \langle \pi(x)\xi, \xi \rangle.$$

Then f is a function of positive type. By Bochner's Theorem, see Theorem 1.4.3 in [9] there exists a regular finite positive measure μ defined on the Borel sets of $\widehat{G}$ such that

$$f(x) = \int_{\widehat{G}} \gamma(x) d\mu(\gamma)$$

for all $x \in G$. Now define for any ξ and η in H the function $\phi_{\xi,\eta} : G \to \mathbb{C}$ by

$$\phi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle$$

Then the polarization identity

$$\begin{aligned} \langle \pi(x)\xi, \eta \rangle &= \langle \pi(x)(\xi + \eta), \xi + \eta \rangle - \langle \pi(x)(\xi - \eta), \xi - \eta \rangle \\ &\quad + i\langle \pi(x)(\xi + i\eta), \xi + i\eta \rangle - i\langle \pi(x)(\xi - i\eta), \xi - i\eta \rangle \end{aligned}$$

shows that $\phi_{\xi,\eta}(x)$ is a linear combination of functions of positive type and Bochner's Theorem then implies that there exists a complex regular measure $\mu_{\xi,\eta}$ on the Borel sets of $\widehat{G}$ such that

$$\langle \pi(x)\xi, \eta \rangle = \int_{\widehat{G}} \gamma(x) d\mu_{\xi,\eta}(\gamma) \tag{1}$$

for all $x \in G$. Since the integral representation in Bochners Theorem is unique and each of the four measures corresponding to each term in the polarization identity is exactly one of the following types: real-positive, real-negative, imaginary-positive and imaginary-negative it follows that the measure in (1) is unique. The calculation

$$\int_{\widehat{G}} \gamma(x) d\mu_{\alpha\xi+\beta\nu,\eta} = \langle \pi(x)(\alpha\xi + \beta\nu), \eta \rangle$$
$$= \alpha\langle \pi(x)\xi, \eta \rangle + \beta\langle \pi(x)(\nu), \eta \rangle = \int_{\widehat{G}} \gamma(x) d(\alpha\mu_{\xi,\eta} + \beta\mu_{\nu,\eta})(\gamma)$$

combined with uniqueness shows that the function $(\xi, \eta) \mapsto \mu_{\xi,\eta}(B)$ is linear in the first argument for any fixed Borel set $B \subseteq \widehat{G}$. A similar calculation on the second argument shows that $(\xi, \eta) \mapsto \mu_{\xi,\eta}(B)$ is a sesquilinear form for any fixed Borel set B. Since

$$|\mu_{\xi,\eta}(\widehat{G})| = |\langle \xi, \eta \rangle| \leq \|\xi\|\|\eta\|$$

it follows that $\|\mu_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$ and therefore $|\mu_{\xi,\eta}(B)| \leq \|\xi\|\|\eta\|$ which shows that $(\xi, \eta) \mapsto \mu_{\xi,\eta}(B)$ is bounded for any Borel set B. By Theorem 12.8 in [7] there exists for each Borel set $B \subseteq \widehat{G}$ a unique operator $E(B) \in \mathscr{B}(H)$ such that

$$(E(B)\xi, \eta) = \mu_{\xi,\eta}(B)$$

for all ξ and η in H. The mapping $E : \mathcal{B}(\widehat{G}) \to \mathscr{B}(H)$ can then be verified to be a projection valued measure. By (1) we also know that

$$\pi(x) = \int_{\widehat{G}} \gamma(x) dE(\gamma)$$

If m is the Haar measure on $\widehat{G}$ then the integrated representation is given by

$$\begin{aligned}\langle \tilde{\pi}(f)\xi, \eta \rangle &= \int_G f(x) \langle \pi(x)\xi, \eta \rangle dm(x) \\ &= \int_G f(x) \int_{\widehat{G}} \gamma(x) d\mu_{\xi,\eta}(\gamma) dm(x) \\ &= \int_{\widehat{G}} \overline{\hat{f}(\gamma)} d\mu_{\xi,\eta}(\gamma).\end{aligned}$$

This means by definition that

$$\tilde{\pi}(f) = \int_{\widehat{G}} \overline{\hat{f}(\gamma)} dE(\gamma)$$

□

Theorem 6.6.6. *If $(X, \mathfrak{M})$ is a measurable space then there is a bijective correspondence between continuous *-representations of $B(X, \mathfrak{M}) \to \mathscr{B}(H)$ and projection valued measures $E : \mathfrak{M} \to B(H)$.*

Proof. This is corollary 1.55 [5]. □

7 Nilpotent Groups and Matrix Lie Groups

We have proved that the reduced group C^*-algebra of a locally compact abelian group G is isomorphic to the space $C_0(\widehat{G})$ of continuous functions vanishing at infinity on the dual group $\widehat{G}$. On the algebraic side, the class of nilpotent groups is a generalization of the abelian groups and is natural class of groups to consider next. On the topological side the Lie groups have a nice theory associated them and some examples of matrix Lie groups will be given below.

7.1 Matrix Lie Groups

A Lie group G is a group and a smooth manifold such that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth maps. A subclass of the Lie groups are the matrix Lie groups, which essentially are the Lie groups that can be realized as groups of matrices. Being matrix groups, the matrix Lie groups are much more concrete to work with and can therefore be studied without going to deep into the realm of differential geometry.

Definition 7.1.1 (Matrix Lie groups)**.** *Any closed subgroup H of the general linear group $GL_n(\mathbb{R})$ is a matrix Lie group.*

Remark 7.1. *Recall that the topology of $GL_n(\mathbb{R})$ is given by the subspace topology induced by $M(n \times n, \mathbb{R})$ which is identified with $\mathbb{R}^{n\times n}$. Since the determinant function is continuous on $M(n \times n, \mathbb{R})$ and $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ it follows that $GL_n(\mathbb{R})$ is an open (smooth) submanifold of $M(n \times n, \mathbb{R})$. Now the same argument as the proof that the general linear group is a topological group, proposition 2.0.11 with the word continuous replaced with the word smooth shows that inversion and multiplication in $GL_n(\mathbb{R})$ are smooth operations which shows that $GL_n(\mathbb{R})$ is a Lie group.*

It is not easy to show that a matrix Lie group in fact is a Lie group since it is not clear apriori that a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ is a smooth manifold and a proof that this is indeed the case uses the theory of Lie Algebras which we will not discuss here.

Proposition 7.1.2. *Any matrix Lie group is a Lie group*

Proof. See Corollary 3.45 in [10]. □

Proposition 7.1.3 (Examples of matrix Lie groups)**.** *The following groups are matrix Lie groups*

- *The general linear group $GL_n(\mathbb{R})$.*
- *The special linear group $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ of matrices with determinant* 1*.*
- *The orthogonal group $O(n) \subseteq GL_n(\mathbb{R})$ of matrices such that $A^\top = A^{-1}$*
- *The special orthogonal group $SO(n) = O(n) \cap SL_n(\mathbb{R})$*

Proof. The group $\mathrm{GL}_n(\mathbb{R})$ is trivially a matrix Lie group. The set $\mathrm{SL}_n(\mathbb{R})$ is a group since the determinant is multiplicative and it a closed subset of $\mathrm{GL}_n(\mathbb{R})$ since it is the inverse image $\det^{-1}(\{1\}) = \mathrm{SL}_n(\mathbb{R})$ of the continuous determinant function. Since $(AB)^\top = B^\top A^\top$ it follows that $O(n)$ is a group. Define the map $\iota : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ by $\iota(A) = (A, A^\top)$ and consider the composition map $\Phi = m \circ \iota$ where m is multiplication. If $\Phi(A) = I_n$ then using the determinant it follows that A is invertible and $A^\top = A^{-1}$. Since Φ is continuous and $\Phi^{-1}(\{I_n\}) = O(n)$ it follows that O_n is closed. Being the intersection of two closed groups it is clear that $SO(n)$ is also a closed group. $\square$

The unitriangular matrix group and the Heisenberg group are two other examples of matrix Lie groups and will be defined below.

Definition 7.1.4 (The Unitriangular Matrix group)**.** *Define the set $\mathbb{U}_n(\mathbb{R})$ as the set of all real $n \times n$-matrices of the form*

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Here any $*$ (to be thought of as different entities) is a placeholder for a real number. The set $\mathbb{U}_n(\mathbb{R})$ is called the n-th (real) unitriangular group.

Proposition 7.1.5. *$\mathbb{U}_n(\mathbb{R})$ is a matrix Lie group.*

Proof. The proof that $\mathbb{U}_n(\mathbb{R})$ is a group proceeds by induction over n. The set $\mathbb{U}_1(\mathbb{R}) = \{I_1\}$ is clearly a group. Assume that $\mathbb{U}_{n-1}(\mathbb{R})$ is a group and pick $A_n, B_n \in \mathbb{U}_n(\mathbb{R})$. Then A_n can be written as the block matrix

$$A_n = \begin{pmatrix} A_{n-1} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

where $A_{n-1} \in \mathbb{U}_{n-1}(\mathbb{R})$ and $\mathbf{a}$ is real column matrix of length $n-1$ and $\mathbf{0}$ is a row matrix of length $n-1$. Doing the similar decomposition for any $B_n \in \mathbb{U}_n$ we see

$$\begin{pmatrix} A_{n-1} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B_{n-1} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} A_{n-1}B_{n-1} & A_{n-1}\mathbf{b} + \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}.$$

Hence $A_nB_n \in \mathbb{U}_n(\mathbb{R})$. It is straightforward to check that the inverse of A_n is given by

$$A_n^{-1} = \begin{pmatrix} A_{n-1}^{-1} & -A_{n-1}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathbb{U}_n(\mathbb{R}).$$

Which shows that $\mathbb{U}_n(\mathbb{R})$ is a group. To show that $\mathbb{U}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is closed note that any sequence $(x_n)_n$ in $\mathbb{R}^m$ converges to $x \in \mathbb{R}^m$ if and only if each of its component sequences converge to the corresponding component of x. Since the topology of $GL_n(\mathbb{R})$ is given by viewing it as a subspace of $\mathbb{R}^{n\times n}$ it follows that if $(A_n)_n$ is a sequence in $\mathbb{U}_n(\mathbb{R})$ that converges to $A \in \mathrm{GL}_n(\mathbb{R})$ then any component a of A correspondong to an element below the main diagonal is given by $a = \lim_{n\to\infty} 0 = 0$. Similarly any component a of A corresponding to a component on the main diagonal is given by $a = \lim_{n\to\infty} 1 = 1$ which shows that $A \in \mathbb{U}_n(\mathbb{R})$. $\square$

Definition 7.1.6 (The Heisenberg Group)**.** *Let n be a non-negative integer and let H_{2n+1} be the set of all matrices of the form*

$$\begin{pmatrix} 1 & \boldsymbol{a} & c \\ \boldsymbol{0} & I_n & \boldsymbol{b} \\ 0 & \boldsymbol{0} & 1 \end{pmatrix}$$

where $\boldsymbol{a} \in M(1 \times n, \mathbb{R})$, $\boldsymbol{b} \in M(n \times 1, \mathbb{R})$ and $c \in \mathbb{R}$ and the boldface zeros are suitably chosen zero-matrices. The set $H_{2n+1}(\mathbb{R})$ is called the Heisenberg group with $2n+1$ degrees of freedom.

Proposition 7.1.7. *The Heisenberg group is a matrix Lie group.*

Proof. Clearly $H_{2n+1}(\mathbb{R})$ contains the identity. A quick calculation shows that

$$\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{d} & f \\ \mathbf{0} & I_n & \mathbf{e} \\ 0 & \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{a}+\mathbf{d} & c+f+\mathbf{ae} \\ \mathbf{0} & I_n & \mathbf{b}+\mathbf{e} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \tag{1}$$

and it follows that $H_{2n+1}(\mathbb{R})$ is closed under multiplication. From formula (1) it is also easily seen that

$$\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\mathbf{a} & \mathbf{ab}-c \\ \mathbf{0} & I_n & -\mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \in H_{2n+1}(\mathbb{R})$$

which shows that $H_{2n+1}(\mathbb{R})$ is a group. A similar argument as the one showing that $\mathbb{U}_n(\mathbb{R})$ is closed shows that $H_{2n+1}(\mathbb{R})$ is closed in $GL_n(\mathbb{R})$ $\square$

7.2 Nilpotent Groups

The upper central series of a group G is a sequence of normal subgroups Z_i that in a sense measures failure of centrality in G. This is the case since $Z_1 = Z(G)$, the center of G and if $x \in Z_{i+1}$ then $x^{-1}y^{-1}xy \in Z_i$ for all $y \in G$. A central series is similar to the upper central series but with flexibility on how the sequence of subgroups is chosen. A nilpotent group is a group where the upper central series or equivalently some central series eventually equals G. Before the definitions are given it is useful to recall the lattice theorem for groups which states that if $N \leq G$ is a normal subgroup of G then there is an inclusion preserving bijection between subgroups H such that $N \leq H \leq G$ and subgroups

$Q \leq G/N$. Furthermore if $N \leq H \leq G$ then H is normal in G if and only if the corresponding group is normal in G/N. The bijection is given by $H \mapsto \pi(H)$ where $\pi : G \to G/N$ is the natural projection.

Definition 7.2.1 (Cental Series)**.** *Let G be a group. A sequence of subgroups*

$$1 = G_0 \leq G_1 \leq ... \leq G$$

where each G_i is normal in G and $G_{i+1}/G_i \leq Z(G/G_i)$ is called a central series for G.

Definition 7.2.2 (Upper Central Series and Nilpotence)**.** *Let G be a group and set $Z_0 = 1 \leq G$. Define recursively Z_{i+1} to be the normal group in G such that $Z_{i+1}/Z_i = Z(G/Z_i)$. This is well-defined since Z_0 is normal in G and if Z_i is normal in G then Z_{i+1} is normal in G since it corresponds to the normal subgroup $Z(G/G_i)$ in G/G_i. The series*

$$1 = Z_0 \leq Z_1 \leq ... \leq G$$

is called the upper central series of G. If there exists an integer n such that

$$1 = Z_0 \leq Z_1 \leq ... \leq Z_n = G,$$

then G is called nilpotent and the smallest such n is called the nilpotency class of G.

Remark 7.2. *It is easily seen that $Z_1 = Z(G)$ which shows that abelian groups are nilpotent with nilpotency class not greater than 1. Conversely any group of nilotency class not greater than 1 is abelian. The trivial group is the only group of nilotency class of 0.*

Proposition 7.2.3. *A group G is nilpotent if and only if there exists a central series that terminates at G, that is if there exists a central series (G_i) for G such that*

$$1 = G_0 \leq G_1 \leq ... \leq G_n = G. \tag{1}$$

for some integer n. Moreover for any central series (G_i) of G we have

$$G_i \leq Z_i \tag{2}$$

for all i. Hence if G is nilpotent then the upper central series Z_i is the central series of G that terminates at G in the least number of steps. It follows that the nilpotency class of G equals the smallest possible n in (1) over all central series for G.

Proof. It is clear that the upper central series for G is a central series for G hence if G is nilpotent then this central series for G terminates. If (2) is true then it follows that the upper central series for G terminates if some central series terminates. It therefore remains to prove (2) which we will do by induction. Let

(G_i) be a central series of G. The base case is clear since $G_0 = 1 = Z_0$. Assume that $G_i \leq Z_i$ and define $\phi : G/G_i \to G/Z_i$ by

$$\phi(xG_i) = xZ_i$$

for $x \in G$. If $xG_i = yG_i$ for $x, y \in G$ then $y^{-1}x \in G_i \subseteq Z_i$ hence $xZ_i = yZ_i$ and it follows that ϕ is well defined. It is clear that $\phi(H/G_i) = H/Z_i$ for any $G_i \leq H \leq G$ and we therefore get

$$\begin{aligned} G_{i+1}/Z_i = \phi(G_{i+1}/G_i) &\leq \phi(Z(G/G_i)) \\ &\leq Z(\phi(G/G_i)) = Z(G/Z_i) = Z_{i+1}/Z_i \end{aligned}$$

hence $G_{i+1} \leq Z_{i+1}$ □

Proposition 7.2.4. *Subgroups of nilpotent groups are nilpotent*

Proof. Let G be a group and H be a subgroup of G. If G is nilpotent then we have a series

$$1 \leq G_1 \leq ... \leq G_n = G$$

where each G_i is normal in G and $G_{i+1}/G_i \leq Z(G/G_i)$. Define $H_i = G_i \cap H$. It is clear that H_i is normal in H since G_i is normal in G. The relation $H_{i+1}/H_i \leq Z(H/H_i)$ is equivalent to showing that

$$x^{-1}h_{i+1}^{-1}xh_{i+1} \in H_i$$

for all $x \in H$ and $h_{i+1} \in H_{i+1}$. Fix an i and choose $x \in H$ and $h_{i+1} \in H_{i+1}$. It is clear that $x^{-1}h_{i+1}^{-1}xh_{i+1} \in H$ since H_{i+1} is a subgroup of H. Since (G_i) is a central series and $h_{i+1} \in G_{i+1}$ we also have that

$$x^{-1}h_{i+1}^{-1}xh_{i+1} \in G_i.$$

By definition of H_i it follows that $x^{-1}h_{i+1}^{-1}xh_{i+1} \in H_i$. □

Proposition 7.2.5. *Nilpotent locally compact groups are amenable.*

Proof. A group is nilpotent if and only if it has a finite nilpotency class. A group with nilpotency class 1 or 0 is amenable since it is abelian. We will use Proposition G.2.2 in [8] that states if G is topological group with a closed normal subgroup N such that G/N and N are amenable then G is amenable. Pick a positive integer n and assume that any Nilpotent topological group with nilpotency class strictly less than n is amenable. Let G be a topological group with nilpotency class $n > 1$ and consider its center $Z(G)$. Since $Z(G)$ is abelian it is nilpotent. Fix a $g \in G$ and consider the map $f_g(x) = gxg^{-1}x^{-1}$. This map is clearly continuous and it follows that $Z(G)$ is closed since

$$Z(G) = \bigcap_{g \in G} f_g^{-1}(\{1\})$$

Consider now the quotient $G/Z(G)$. Since G is nilpotent with nilpotency class $n > 1$ it is clear that $Z(G)$ is not the trivial group. We can therefore quotient each group in the upper central series of G with the center and obtain a central series of $G/Z(G)$ of length $n-1$. It follows that the nilpotency class of $G/Z(G)$ is at most $n-1$. Since the canonical projection $G \to G/Z(G)$ is surjective, continuous and open it follows that $G/Z(G)$ is locally compact. The hypothesis therefore implies that $G/Z(G)$ is amenable. This implies that G is amenable. $\square$

Proposition 7.2.6.

- *The permutation group S_n is nilpotent if and only if $n \leq 2$.*
- *The group $GL_n(\mathbb{R})$ is nilpotent if and only if $n = 1$.*

Proof. It is clear that S_1 and S_2 are abelian and hence nilpotent. Assume that $\sigma \in Z(S_3)$ is non trivial. Then it contains a cycle. If the cycle is of the form $\sigma = (1\ 2\ 3)$ then we can let $\tau = (1\ 2)$ and we see that $\sigma\tau = (1\ 3)$ but $\tau\sigma = (2\ 3)$ and it follows that σ is not in the center in that case. Otherwise σ is of the form $\sigma = (a\ b)$ where a, b are distinct element of $\{1, 2, 3\}$ and if c is the third element we have $(b\ c)(a\ b) = (a\ c\ b)$ but $(a\ b)(b\ c) = (a\ b\ c)$. It follows that $Z(S_3) = 1$ is trivial. The group Z_2 in the upper central series of S_3 therefore satisfies $Z_2/1 = Z(S_3/1) \cong 1$ hence $Z_2 = Z_1 = 1$ and by induction it follows that the upper central series does not terminate at S_3 which shows that S_3 is not nilpotent. Since S_3 is a subgroup of S_n whenever $n \geq 3$ it follows that S_n is not nilpotent when $n \geq 3$.

For the general linear group it is clear that the group $GL_1(\mathbb{R}) \cong \mathbb{R}^\times$ is nilpotent. Asssume that $n \geq 2$ and define the map $\Phi : GL_2(\mathbb{R}) \to GL_n(\mathbb{R})$ by

$$\Phi(A) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{pmatrix}$$

where $\mathbf{0}$ denotes suitable zero-matrices and I_n is the $n \times n$-identity matrix. Since $\det(\Phi(A)) = \det(A) \neq 0$ it follows that Φ is well defined. It is clear that Φ is injective and it is a homomorphism since

$$\Phi(A)\Phi(B) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{pmatrix} \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{pmatrix} = \begin{pmatrix} AB & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{pmatrix} = \Phi(AB).$$

It follows that $GL_n(\mathbb{R})$ where $n \geq 2$ is not nilpotent if $GL_2(\mathbb{R})$ is not nilpotent. We therefore only need to prove that $GL_2(\mathbb{R})$ is not nilpotent which we will do by showing that the subgroup $GL_2^+(\mathbb{R})$, consisting of invertible matrices with positive determinant is not nilpotent. Choose a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $Z(GL_2^+(\mathbb{R}))$. Then A needs to commute with the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in GL_2^+(\mathbb{R})$$

from which it follows that $a = d$ and $b = -c$. The matrix A also needs to commute with the invertible matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2^+(\mathbb{R})$$

from which it follows that $b = 0$ hence

$$Z(GL_2^+(\mathbb{R})) = \{cI;\ c > 0\}.$$

Define the map $\phi : GL_2^+(\mathbb{R}) \to SL_2(\mathbb{R})$ given by

$$\phi(A) = \frac{1}{\sqrt{\det(A)}}A.$$

It is clear that ϕ is well-defined since $\det(A) > 0$ whenever $A \in GL_2^+(\mathbb{R})$ and

$$\det(\phi(A)) = \det(\frac{1}{\sqrt{\det(A)}}A) = (\frac{1}{\sqrt{\det(A)}})^2\det(A) = 1.$$

It is easy to check that ϕ is a surjective homomorphism. The kernel of ϕ is $\{cI,\ c > 0\} = Z(GL_2^+(\mathbb{R}))$ and it follows that

$$Z(GL_2^+(\mathbb{R})/Z(GL_2^+(\mathbb{R}))) \cong Z(SL_2(\mathbb{R}))$$

Since the commutativity calculations above hold in $SL_2(\mathbb{R})$ it follows that $Z(SL_2(\mathbb{R})) = 1$ and we have that Z_2 must satisfy

$$Z_2/Z(GL_2^+(\mathbb{R}))) = Z(GL_2^+(\mathbb{R})/Z(GL_2^+(\mathbb{R}))) \cong 1.$$

Thus Z_2 corresponds to the trivial subgroup of $GL_2^+(\mathbb{R})/Z(GL_2^+(\mathbb{R}))$ and it follows that $Z_2 = Z(GL_2^+(\mathbb{R})) = Z_1$ which shows that $GL_2(\mathbb{R})$ is not nilpotent. □

Proposition 7.2.7. $\mathbb{U}_n(\mathbb{R})$ *is nilpotent with nilpotency class* $n - 1$.

Proof. Fix an integer $n > 1$ and define U_n^m to be $\{I_n\}$ if $m \leq 0$ and if $0 < m < n$ to be set of all elements of the form

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\
0 & 1 & 0 & \cdots & 0 & 0 & * & * & \cdots & * \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}.$$

Where the number of non-zero elements of the first row, not counting the first 1 is m. Furthermore define $U_0^m = \{I_1\}$ for all m. It is clear that any U_n^m is a subgroup of $\mathbb{U}_n(\mathbb{R})$. We will show by induction on n that the upper central series of $\mathbb{U}_n(\mathbb{R})$ is given by

$$1 = U_n^0 \leq U_n^1 \leq ... \leq U_n^{n-1} = \mathbb{U}_n(\mathbb{R})$$

from which it follows that $\mathbb{U}_n(\mathbb{R})$ is nilpotent of nilpotency class $n-1$. The base case is clear since $\mathbb{U}_1(\mathbb{R}) = \{I_1\} = U_0^0$. Pick an integer $n > 0$ and assume that the proposition holds for this $n-1$. We then want to show that

$$U_n^{m+1}/U_n^m = Z(\mathbb{U}_n(\mathbb{R})/U_n^m).$$

where $0 \leq m \leq n-2$. This is equivalent to showing that $A_n \in U_n^{m+1}$ if and only if

$$A_n^{-1} B_n^{-1} A_n B_n \in U_n^m \tag{1}$$

for all $B \in \mathbb{U}_n(\mathbb{R})$ and $0 \leq m \leq n-2$. Using the formulas for multiplication and inversion in $\mathbb{U}_n(\mathbb{R})$ we have

$$\begin{aligned}
&A_n^{-1} B_n^{-1} A_n B_n \\
&= \begin{pmatrix} A_{n-1} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}^{-1} \begin{pmatrix} B_{n-1} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_{n-1} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B_{n-1} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} A_{n-1}^{-1} & -A_{n-1}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B_{n-1}^{-1} & -B_{n-1}^{-1}\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} A_{n-1}B_{n-1} & A_{n-1}\mathbf{b} + \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} A_{n-1}^{-1}B_{n-1}^{-1} & -A_{n-1}^{-1}B_{n-1}^{-1}\mathbf{b} - A_{n-1}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} A_{n-1}B_{n-1} & A_{n-1}\mathbf{b} + \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} A_{n-1}^{-1}B_{n-1}^{-1}A_{n-1}B_{n-1} & A_{n-1}^{-1}B_{n-1}^{-1}(A_{n-1}\mathbf{b} + \mathbf{a}) - A_{n-1}^{-1}B_{n-1}^{-1}\mathbf{b} - A_{n-1}^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}.
\end{aligned}$$

It follows by the induction hypothesis that (1) holds for any $B_n \in \mathbb{U}_n(\mathbb{R})$ if and only if $A_{n-1} \in U_{n-1}^{m-1}$ and any element after the m:th-row of the column matrix

$$A_{n-1}^{-1}B_{n-1}^{-1}(A_{n-1}\mathbf{b} + \mathbf{a}) - A_{n-1}^{-1}B_{n-1}^{-1}\mathbf{b} - A_{n-1}^{-1}\mathbf{a} \tag{2}$$

is zero for any $B_{n-1} \in \mathbb{U}_{n-1}(\mathbb{R})$ and any column vector $\mathbf{b}$ of length $n-1$. Note that (2) equals

$$A_{n-1}^{-1}B_{n-1}^{-1}(A_{n-1} - I_{n-1})\mathbf{b} + A_{n-1}^{-1}(B_{n-1}^{-1} - I_{n-1})\mathbf{a} \tag{3}$$

and we analyse each term in (3) separately. Since $A_{n-1} \in U_{n-1}^{m-1}$ it follows that any row after the m:th row in $A_{n-1} - I_{n-1}$ is zero . Consider the m':th row, $m' > m$ of $A_{n-1}^{-1}B_{n-1}^{-1}$, since $A_{n-1}^{-1}B_{n-1}^{-1}$ is unitriangular any element of row m' on a column before m' is zero and it follows that the m':th row of $A_{n-1}^{-1}B_{n-1}^{-1}(A_{n-1} - I_{n-1})$ is zero hence the m':th element of

$$A_{n-1}^{-1}B_{n-1}^{-1}(A_{n-1} - I_{n-1})\mathbf{b}$$

is zero for any B_{n-1} and $\mathbf{b}$. If $\mathbf{a}$ is any column vector of length $n-1$ that is 0 after row m then it is clear that $A_n \in U_n^m$. Conversely if $B_{n-1} \in \mathbb{U}_{n-1}(\mathbb{R})$ is the matrix whose inverse are all 0 except on the main diagonal and any element of the last column after row m equals 1 then the elements after row m of

$$A_{n-1}^{-1}(B_{n-1}^{-1} - I_{n-1})\mathbf{a}$$

equals the corresponding elements of $\mathbf{a}$ which shows that

$$U_n^{m+1}/U_n^m = Z(\mathbb{U}_n(\mathbb{R})/U_n^m)$$

$\square$

Proposition 7.2.8. *The Heisenberg group $H_{2n+1}(\mathbb{R})$ is nilpotent with nilpotency class* 2.

Proof. Define

$$A = \begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & \mathbf{d} & f \\ \mathbf{0} & I_n & \mathbf{e} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

then the multiplication formula

$$\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{d} & f \\ \mathbf{0} & I_n & \mathbf{e} \\ 0 & \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{a}+\mathbf{d} & c+f+\mathbf{ae} \\ \mathbf{0} & I_n & \mathbf{b}+\mathbf{e} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \tag{1}$$

shows that $AB = BA$ if and only if $\mathbf{ae} = \mathbf{db}$. It is then routine to show that $Z(H_{2n+1}(\mathbb{R}))$ consists of all elements of the form

$$\begin{pmatrix} 1 & \mathbf{0} & c \\ \mathbf{0} & I_n & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

where $c \in \mathbb{R}$. Define the map $\phi : H_{2n+1} \to \mathbb{R}^{2n}$ by

$$\phi(\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & I_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}) = (\mathbf{a}, \mathbf{b}^\top) \in \mathbb{R}^{2n}$$

From the multiplicationn formula (1) it is clear that ϕ is a homomorphism and it is clearly surjective. From our calulation of the center of H_{2n+1} we see that the kernel of ϕ coincides with $Z(H_{2n+1}(\mathbb{R}))$ and we therefore have

$$H_{2n+1}(\mathbb{R})/Z(H_{2n+1}(\mathbb{R})) \cong \mathbb{R}^{2n}.$$

It follows that Z_2 satisfies

$$Z_2/Z(H_{2n+1}(\mathbb{R})) = Z(H_{2n+1}(\mathbb{R})/Z(H_{2n+1}(\mathbb{R}))) = H_{2n+1}(\mathbb{R})/Z(H_{2n+1}(\mathbb{R}))$$

hence $Z_2 = H_{2n+1}(\mathbb{R})$ and it follows that $H_{2n+1}(\mathbb{R})$ is nilpotent with nilpotency class 2. $\square$

8 Representation Theory of the Heisenberg group

In this section we will work out the the unitary dual of the Heisenberg group $H_{2n+1}(\mathbb{R}^n)$. The Stone-von Neumann theorem will be proved and we can use it to classify elements of the unitary dual as either being equivalent to a Schrödinger representation or being characters of $\mathbb{R}^{2n}$. Knowing the unitary dual of the Heisenberg group will enable us to give a more concrete view of the C^*-algebra of the Heisenberg group as sitting in a particular type of continuous field of C^*-algebras.

8.1 Notation and Haar-measure

Recall that the Heisenberg group, $H_{2n+1}(\mathbb{R})$ is the collection of all real valued matrices on the form

$$\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & \mathbf{I}_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

where $\mathbf{a}^\top, \mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. To make the notation more compact and easier to read we drop the bold notation and write the matrix above simply as (a, b, c) and then the group law becomes

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + a \cdot b') \tag{1}$$

where $a \cdot b$ denotes the scalar product in $\mathbb{R}^n$.

Proposition 8.1.1. *The Haar measure of the Heisenberg group, $H_{2n+1}(\mathbb{R})$ when viewed as a subset of $\mathbb{R}^{2n+1}$ is the $2n+1$-dimensional Lebesgue measure, m_{2n+1}.*

Proof. The map that sends the matrix

$$\begin{pmatrix} 1 & \mathbf{a} & c \\ \mathbf{0} & \mathbf{I}_n & \mathbf{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

to $(\mathbf{a}^\top, \mathbf{b}, c) \in \mathbb{R}^{2n+1}$ is a homeomorphism and therefore establishes a bijection between the Borel sets of $H_{2n+1}(\mathbb{R})$ and the Borel sets of $\mathbb{R}^{2n+1}$. Since the Lebesgue measure is a Haar measure it follows that it defines a non-zero Radon measure on $H_{2n+1}(\mathbb{R})$. It remains to show translation-invariance. If E is a Borel set in $H_{2n+1}(\mathbb{R})$ and $(a, b, c) \in H_{2n+1}(\mathbb{R})$ then it follows that translation can be written as

$$(a, b, c)E = (a, b, c) + L_a(E)$$

where $L_a : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the linear transformation with matrix

$$\begin{pmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n & a^\top \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$$

in the standard basis. It is clear that $\det(L_a) = 1$ and using well known properties of the Lebesgue measure we have

$$m_{2n+1}((a,b,c)E) = m_{2n+1}((a,b,c) + L_a(E)) = m_{2n+1}(L_a(E)) = m_{2n+1}(E)$$

□

8.2 The Stone-von Neumann Theorem

Example 8.2.1 (The Schrödinger representation). *For any fixed non-zero $h \in \mathbb{R}$ we can define the function $\rho_h : H_{2n+1}(\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^n))$ by*

$$(\rho_h(a,b,c)f)(x) = e^{i(b\cdot x+hc)}f(x+ha)$$

Since $\rho_h(a,b,c)$ is a composition of a translation operator and a multiplication operator by a complex exponential, both being unitary operators on $L^2(\mathbb{R}^n)$ it follows that $\rho_h(a,b,c)$ is unitary. The calculation

$$\begin{aligned}
&(\rho_h(a,b,c)\rho_h(a',b',c')f)(x) \\
&= e^{i(b\cdot x+hc)}(\rho_h(a',b',c')f)(x+ha) \\
&= e^{i(b\cdot x+hc)}e^{i(b'\cdot(x+ha)+hc')}f(x+ha+ha') \\
&= e^{i((b+b')\cdot x+h(c+c'+a\cdot b'))}f(x+h(a+a')) \\
&= (\rho(a+a',b+b',c+c'+a\cdot b')f)(x) \\
&= (\rho((a,b,c)(a',b',c'))f)(x)
\end{aligned}$$

shows that ρ_h is a group homomorphism. The continuity of ρ_h follows from the fact that $b \mapsto M_{e^{ib\cdot x}}$ and $a \mapsto L_{-ha}$ are continuous mappings $\mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ and multiple uses of the triangle inequality. This is shown in a more general situation in the Stone-von Neumann theorem. It follows that $\rho_h : H_{2n+1}(\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^n))$ is a unitary representation for any non-zero $h \in \mathbb{R}$. This representation is called the Schrödinger representation. It is clear that ρ_h and $\rho_{h'}$ for distinct choices of h and h' are non-equivalent since they in particular are non-equivalent when they are restricted to the center of the Heisenberg group.

Proposition 8.2.2. *The Schrödinger representation $(\rho_h, L^2(\mathbb{R}^n))$ of $H_{2n+1}(\mathbb{R})$ for any $h \neq 0$ is irreducible.*

Proof. Assume that K is closed non-trivial subspace of $L^2(\mathbb{R}^n)$ such that

$$\rho(H_{2n+1}(\mathbb{R}))K \subseteq K.$$

Then it follows that there exists a non-zero $g \in L^2(\mathbb{R}^n)$ such that

$$\langle \rho_h(x)f, g\rangle = 0$$

for all $f \in K$ and all $x \in H_{2n+1}(\mathbb{R})$. Pick a non-zero $f \in K$ and define $F : H_{2n+1}(\mathbb{R}) \to \mathbb{C}$ by

$$F(x) = \langle \rho_h(x)f, g\rangle.$$

By assumption, $F = 0$ hence

$$
\begin{aligned}
0 = \|F\|_2^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} |\langle \rho(a,b,c)f, g\rangle|^2 dbdadc \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} | \int_{\mathbb{R}^n} e^{i(b\cdot y + hc)} f(y+ha)\overline{g(y)}dy|^2 dbdadc \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} |(\widehat{f_{-ha}\overline{g}})(b)|^2 dbdadc.
\end{aligned}
$$

From Plancherel's Theorem it follows that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} |(\widehat{f_{-ha}\overline{g}})(b)|^2 dbdadc \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} |(f(y+ha)\overline{g}(y)|^2 dydadc \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^\ltimes} \int_{\mathbb{R}^\ltimes} |(f(ha)\overline{g(y)}|^2 dadydc \\
&= \frac{1}{|h|^n} \int_{\mathbb{R}} (\int_{\mathbb{R}^\ltimes} |f(a)|^2 da)(\int_{\mathbb{R}^\ltimes} |g(y)|^2 dy)dc \\
&= \frac{1}{|h|^n} \int_{\mathbb{R}} \|f\|_2^2 \|g\|_2^2 dc.
\end{aligned}
$$

The last integral can only be zero if $\|f\|_2 = 0$ or $\|g\|_2 = 0$ which contradicts our choice of f and g. $\square$

Define ρ'_h and ρ''_h on $\mathbb{R}^n$ by

$$
\begin{aligned}
\rho'_h(x) &= \rho_h(x,0,0) \\
\rho''_h(x) &= \rho_h(0,x,0).
\end{aligned}
$$

Ít is then clear that both ρ'_h and ρ''_h are unitary representations of $\mathbb{R}^n$. Since

$$(x,0,0)(0,y,0) = (0,0,x\cdot y)(0,y,0)(x,0,0)$$

for any $x, y \in \mathbb{R}^n$ it follows that

$$\rho'_h(x)\rho''_h(y) = e^{ihx\cdot y}\rho''_h(y)\rho'_h(x).$$

where

$$\rho(0,0,c) = e^{ihc}.$$

It is also clear that ρ'_h and ρ''_h determine ρ_h since

$$\rho_h(a,b,c) = e^{ihc}\rho''_h(b)\rho'_h(a) = \rho'_h(a)\rho''_h(b)$$

The Stone-von Neumann Theorem shows when this relationship between pairs of representations of $\mathbb{R}^n$ can be extended to a representation of the Heisenberg *group and vice versa in the general case. It also characterises any unitary representation of the Heisenberg group that satisfies*

$$\pi(0,0,c) = e^{ihc} id_H$$

for some non-zero h as being unitarily equivalent to a direct sum of Schrödinger representations.

Theorem 8.2.3 (The Stone-von Neumann Theorem)**.**
a) If (π', H) and (π'', H) are unitary representations of $\mathbb{R}^n$ which satisfy

$$\pi'(x)\pi''(y) = e^{ihx\cdot y}\pi''(y)\pi'(x)$$

for some real non-zero h then there exists a unitary map $U : H \to \bigoplus_{i\in I} L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} U\pi'(x)U^{-1} &= \bigoplus_{i\in I} \rho'_h(x) \\ U\pi''(x)U^{-1} &= \bigoplus_{i\in I} \rho''_h(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$.
b) If (π, H) is a unitary representation of $H_{2n+1}(\mathbb{R})$ such that

$$\pi(0,0,c) = e^{ihc} id_H$$

for some non-zero real h then π is unitarily equivalent to a direct sum of copies of ρ_h.

Proof. We begin by showing that the statements a) and b) are equivalent. Suppose that (π, H) is a unitary representation of $H_{2n+1}(\mathbb{R})$ such that

$$\pi(0,0,c) = e^{ihc}\mathrm{id}_H$$

for some non-zero h. Define unitary representations of $\mathbb{R}^n$ satisfying

$$\begin{aligned} \pi'(x) &= \pi(x,0,0) \\ \pi''(x) &= \pi(0,x,0). \end{aligned}$$

It follows that the condition of a) is satisfied since

$$\begin{aligned} &\pi'(x)\pi''(y) = \pi(x,0,0)\pi(0,y,0) = \pi(x,y,x\cdot y) \\ &= \pi((0,0,x\cdot y)(0,y,0)(0,0,x)) \\ &= e^{ihx\cdot y}\pi''(y)\pi'(x). \end{aligned}$$

Hence there exists a unitary map $U : H \to \bigoplus_{i\in I} L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} U\pi'(x)U^{-1} &= \bigoplus_{i\in I} \rho'_h(x) \\ U\pi''(y)U^{-1} &= \bigoplus_{i\in I} \rho''_h(y) \end{aligned}$$

for any $x, y \in \mathbb{R}^n$. On any copy of $L^2(\mathbb{R}^n)$ we then have

$$
\begin{aligned}
&U\pi(x,y,c)U^{-1} = U\pi((0,0,c)(0,y,0)(0,0,x))U^{-1} \\
&= e^{ihc}U\pi''(y)\pi'(x)U^{-1} = e^{ihc}U\pi''(y)U^{-1}U\pi'(x)U^{-1} \\
&= e^{ihc}\rho_h''(y)\rho_h'(x) = \rho_h((0,0,c)(0,y,0)(0,0,x)) \\
&= \rho(x,y,c)
\end{aligned}
$$

and it follows that π is unitarily equivalent to a direct sum of copies of ρ_h. This shows that a) implies b). If instead (π', H) and (π'', H) are unitary representations of $\mathbb{R}^n$ which satisfy

$$\pi'(x)\pi''(y) = e^{ihx\cdot y}\pi''(y)\pi'(x)$$

for some real non-zero h then we can define $\pi : H_{2n+1}(\mathbb{R}) \to \mathcal{U}(H)$ by

$$\pi(x,y,c) = e^{ihc}\pi''(y)\pi'(x).$$

The operators $\pi(x,y,c)$ are clearly unitary. The map $\pi : H_{2n+1}(\mathbb{R}) \to \mathcal{U}(H)$ is also continuous since multiple applications of the triangle inequality and the fact that the operators are isomtetries yields the estimate

$$\|\pi(x,y,c)\xi - \xi\| \leq |e^{ihc} - 1|\|\xi\| + \|\pi''(y)\pi(x)\xi - \pi(x)\xi\| + \|\pi'(x)\xi - \xi\|$$

The first and third term clearly tend to 0 as $(a,b,c) \to 0$ for any $\xi \in H$. For the middle term we have

$$
\begin{aligned}
&\|\pi''(y)\pi(x)\xi - \pi(x)\xi\| = \|\pi''(y)\pi(x)\xi - \pi''(y)\xi\| + \|\pi''(y)\xi - \pi(x)\xi\| \\
&\leq 2\|\pi(x)\xi - \xi\| + \|\pi''(y)\xi - \xi\|.
\end{aligned}
$$

This shows that $\pi : H_{2n+1}(\mathbb{R}) \to \mathcal{U}(H)$ is continuous. It follows that (π, H) is a unitary representation since

$$
\begin{aligned}
&= \pi((x,y,c)(x',y',c')) = \pi(x+x', y+y', c+c'+x\cdot y') \\
&= e^{ih(c+c'+x\cdot y')}\pi''(y+y')\pi'(x+x') \\
&= e^{ih(c+c'+x\cdot y')}\pi''(y)\pi''(y')\pi'(x)\pi'(x') \\
&= e^{ihc}e^{ihc'}e^{ihx\cdot y'}\pi''(y)e^{-ihx\cdot y'}\pi'(x)\pi''(y')\pi'(x') \\
&= e^{ihc}\pi''(y)\pi'(x)e^{ihc'}\pi''(y')\pi'(x') \\
&= \pi(x,y,c)\pi(x',y',c').
\end{aligned}
$$

The representation also satisfies $\pi(0,0,c) = e^{ihc}\mathrm{id}_H$ hence *b*) gives us that π is unitarily equivalent to a direct sum of copies of ρ_h. Let $U : H \to \bigoplus_{i\in I} L^2(\mathbb{R}^n)$ be a unitary map giving this equivalence. Since $U\pi(x,y,c)U^{-1} = \rho_h(x,y,c)$ on any copy of $L^2(\mathbb{R}^n)$ it is clear that *a*) holds by simply inserting $y = 0, c = 0$ and $x = 0, c = 0$ in the equation. We have thus showed that *a*) and *b*) are equivalent statements.

We prove now prove the Stone-von Neumann Theorem by proving a). Assume that (π', H) and (π'', H) are unitary representations of $\mathbb{R}^n$ which satisfy

$$\pi'(x)\pi''(y) = e^{ihx\cdot y}\pi''(y)\pi'(x) \tag{1}$$

for some real non-zero h. If we replace $\pi''(y)$ with the representation $\pi''(\frac{y}{h})$ it follows that (1) is satisfied with $h = 1$ hence we can assume that $h = 1$. By Theorem 6.6.5 there is a unique projection valued measure $P : \mathcal{B}(\widehat{\mathbb{R}}^n) \to \mathscr{B}(H)$ such that

$$\tilde{\pi}''(f) = \int_{\widehat{\mathbb{R}}^n} \hat{f}(\gamma^{-1})dP(\gamma)$$

for all $f \in L^1(\mathbb{R}^n)$. It is well known that $\widehat{\mathbb{R}}^n \cong \mathbb{R}^n$ and each element of $\widehat{\mathbb{R}}^n$ is of the form $\gamma(t) = e^{it\cdot y}$ for any $t \in \mathbb{R}$. It follows that translating P that

$$\tilde{\pi}''(f) = \int_{\mathbb{R}^n} \hat{f}(-y)dP(y).$$

Fix an x in G and consider the unitary representations σ and τ given by $\sigma(y) = \pi'(x)\pi''(y)\pi'(x)^{-1}$ and $\tau(y) = e^{ix\cdot y}\pi''(y)$. The calculation

$$\begin{aligned}
&\langle \int_{\mathbb{R}^n} f(y)\sigma(y)dy\xi, \eta\rangle = \int_{\mathbb{R}^n} \langle f(y)\sigma(y)\xi, \eta\rangle dy \\
&= \int_{\mathbb{R}^n} \langle f(y)\pi'(x)\pi''(y)\pi'(x)^{-1}\xi, \eta\rangle dy \\
&= \int_{\mathbb{R}^n} f(y)\langle \pi''(y)\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta\rangle dy \\
&\langle \int_{\mathbb{R}^n} f(y)\pi''(y)dy\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta\rangle \\
&\langle \pi'(x)\tilde{\pi}''(f)\pi'(x)^{-1}\xi, \eta\rangle
\end{aligned}$$

shows that

$$\tilde{\sigma}(f) = \pi(x)'\tilde{\pi}''(f)\pi(x)'^{-1}.$$

A similar straightforward calculation shows that

$$\tilde{\tau}(f) = \tilde{\pi}''(gf)$$

where $g(y) = e^{ix\cdot y}$. Let E^σ be the projection valued measure corresponding to σ and pick an open set V in $\mathbb{R}^n$. There exists a sequence (f_n) of functions in

$L^1(\mathbb{R}^n)$ such that $\hat{f}_n(-y) \to \chi_V(y)$ monotonously for all $y \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
&\langle E^\sigma(V)\xi, \eta\rangle = \int_{\mathbb{R}} \chi_V dE^\sigma_{\xi,\eta} = \lim_{n\to+\infty} \int_{\mathbb{R}} \hat{f}_n(-y) dE^\sigma_{\xi,\eta}(y) \\
&= \lim_{n\to+\infty} \langle \tilde{\sigma}(f_n)\xi, \eta\rangle = \lim_{n\to+\infty} \langle \tilde{\pi}''(f_n)\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta\rangle \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^n} \hat{f}_n(-y) dP_{\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta}(y) \\
&= \int_{\mathbb{R}^n} \chi_V dP_{\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta} = \langle P(V)\pi'(x)^{-1}\xi, \pi'(x)^{-1}\eta\rangle \\
&= \langle \pi'(x)P(V)\pi'(x)^{-1}\xi, \eta\rangle.
\end{aligned}$$

From which it follows that

$$E^\sigma(\omega) = \pi'(x)P(\omega)\pi'(x)^{-1}$$

for any Borel set ω. If E^τ is as above and V and (f_n) are as above then we also have

$$\begin{aligned}
&\langle E^\tau(V)\xi, \eta\rangle = \int_{\mathbb{R}} \chi_V dE^\tau_{\xi,\eta} = \lim_{n\to+\infty} \int_{\mathbb{R}} \hat{f}_n(-y) dE^\tau_{\xi,\eta}(y) \\
&= \langle \tilde{\tau}(f_n)\xi, \eta\rangle = \langle \tilde{\pi}''(gf_n)\xi, \eta\rangle = \lim_{n\to+\infty} \int_{\mathbb{R}^n} \widehat{(gf)}(-y) dP_{\xi,\eta} \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^n} \widehat{f}_n(-(y+x)) dP_{\xi,\eta}(y) = \int_{\mathbb{R}^n} \chi_{V+x} dP_{\xi,\eta} \\
&= \langle P(V+x)\xi, \eta\rangle
\end{aligned}$$

hence

$$E^\tau(\omega) = P(\omega + x)$$

for any Borel set ω. But by assumption $\sigma = \tau$ hence

$$\pi'(x)P(\omega)\pi'(x)^{-1} = P(\omega + x). \tag{2}$$

By Theorem 6.6.6 there correponds to P a continuous *-representation $\tilde{m} : B(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to \mathscr{B}(H)$ such that

$$\tilde{m}(\chi_\omega) = P(\omega)$$

for any Borel set ω. Assume that $f \in B(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is a function such that $\|f\|_\infty = 0$. Then there exists a sequence of simple functions (s_n) that converges to f in the norm $\|\cdot\|_\infty$. It follows that $\|s_n\|_\infty = 0$ hence $\tilde{m}(f) = \lim_{n\to\infty} \tilde{m}(s_n) = 0$. This shows that $\tilde{m}$ factors through $L^\infty(P)$. Denote this map by $m : L^\infty(P) \to \mathscr{B}(H)$. From (2) it follows that

$$\pi'(x)m(\chi_\omega)\pi'(x)^{-1} = m(\chi_{\omega+x})$$

and since the simple functions are dense in $L^\infty(P)$ it follows that

$$\pi'(x)m(f)\pi'(x)^{-1} = m(L_{-x}f)$$

for any $f \in L^\infty(P)$. By a special case of the imprivitivity Theorem, see 6.31 in [5] and the definitions of 6.4 implies that there exists a unitary isomorphism $U : H \to \bigoplus_{i\in I} L^2(\mathbb{R}^n)$ for some index set I such that

$$U\pi'(x)U^{-1} = \bigoplus_{i\in I} L^2(\mathbb{R}^n)\rho_1'(x)$$
$$Um(f)U^{-1} = \bigoplus_{i\in I} L^2(\mathbb{R}^n)M_f$$

where $M_f : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the multiplication operator. This shows the first part of a) regarding π'. For π'' we have

$$\begin{aligned} U\pi''(y)U^{-1} &= U\int_{\mathbb{R}^n} e^{ix\cdot y}dP(y)U^{-1} = Um(y\mapsto e^{ix\cdot y})U^{-1} \\ &= M_{y\mapsto e^{ix\cdot y}} = \rho_1''(y) \end{aligned}$$

which concludes the proof. □

The Stone-Von Neumann Theorem lets us characterize the irreducible representations of the Heisenberg group.

Corollary 8.2.3.1. *The irreducible unitary representations of $H_{2n+1}(\mathbb{R}^n)$ are either unitarily equivalent to a Schrödinger representation $(\rho_h, L^2(\mathbb{R}^n))$ for some $h \neq 0$ or are being equivalent to a representation $\phi_{\alpha,\eta} : H_{2n+1}(\mathbb{R}^n) \to \mathbb{C}$*

$$\phi_{\alpha,\beta}(x, y, c) = e^{i(\alpha\cdot x+\beta\cdot y)}.$$

that factors through $\mathbb{R}^{2n}$.

Proof. Let (π, H) be an irreducible unitary representation of $H_{2n+1}(\mathbb{R}^n)$ and consider the map $\pi_0 : \mathbb{R} \to \mathcal{U}(H)$ given by $\pi_0(c) = \pi(0, 0, c)$. The map π_0 is a representation of $\mathbb{R}$ and since $\mathbb{R}$ is abelian it follows that there exists an $h \in \mathbb{R}$ such that $\pi_0(c) = e^{ihc}\mathrm{id}_H$. If h is not 0 the Stone-Von Neumann Theorem allows us to conclude that (π, H) must be equivalent to a direct sum of the Schrödinger representation $(\rho_h, L^2(\mathbb{R}^n))$ and since π is irreducible it follows that (π, H) is equivalent to $(\rho_h, L^2(\mathbb{R}^n))$. If $h = 0$ then the restriction of π to the center is trivial:

$$\pi|_Z = \mathrm{id}_H$$

and it follows that π factors through $H_{2n+1}(\mathbb{R})/Z \cong \mathbb{R}^{2n}$. The representations of $\mathbb{R}^{2n}$ are all of the form $\phi(x, y) = e^{i(\alpha\cdot x+\beta\cdot y)}$ for any $\alpha, \beta \in \mathbb{R}^n$. It follows that

$$\pi(x, y, c) = e^{i(\alpha\cdot x+\beta\cdot y)}$$

for some $\alpha, \beta \in \mathbb{R}^n$. □

8.3 The Unitary Dual of the Heisenberg Group

The unitary dual $\widehat{H_{2n+1}}(\mathbb{R}^n)$ is by the corollary above given (as a set) as a union of a line without the origin and a plane through the origin, $\mathbb{R}^\times \cup \mathbb{R}^{2n}$. Restricting our attention to the plane we know that functions of positive type on the plane $\mathbb{R}^{2n} \subseteq \mathbb{R}^\times \cup \mathbb{R}^{2n}$ correspond up to scalar multiple to representations of $\mathbb{R}^{2n}$. That is, elements of $\widehat{\mathbb{R}}^{2n}$ and it is well known in Fourier analysis that $\widehat{\mathbb{R}}^{2n}$ is homeomorphic to $\mathbb{R}^{2n}$. In other words the topology of the plane part of $\mathbb{R}^\times \cup \mathbb{R}^{2n}$ coincides with the Euclidean topology on $\mathbb{R}^{2n}$. If we instead restrict to $\mathbb{R}^\times$ and recall that the map $h \mapsto L_h f \in L^2(\mathbb{R}^n)$ for any fixed $f \in L^2(\mathbb{R}^n)$ is uniformly continuous (see 1.1.5 in [9]). Then we can show that for any $\xi \in L^2(\mathbb{R}^n)$ and $h' \in \mathbb{R}^\times$ that the expression

$$|\langle \rho_h(a,b,c)\xi, \xi\rangle - \langle \rho_{h'}(a,b,c)\xi, \xi\rangle|$$

can be made arbitrarily small on compact subsets of $H_{2n+1}(\mathbb{R})$ as h gets closer to h'. This shows that the topology on the unitary dual when restricted to $\mathbb{R}^\times$ coincides with the usual topology on $\mathbb{R}^\times$. However, not everything is as well-behaved as one might expect. It is shown in 2.2.2 in [11] that for any character $e^{i\alpha\cdot x + i\beta\cdot y}$ that there exists a family of functions $(f_h)_{h\in\mathbb{R}^\times}$ in $L^2(\mathbb{R}^n)$ such that

$$\lim_{h\to 0} \langle \rho_h(a,b,c) f_h, f_h\rangle = e^{i\alpha\cdot x + i\beta\cdot y}$$

for any $a, b, c \in H_{2n+1}(\mathbb{R}^{2n})$ and this convergence is uniform on compact subsets of $H_{2n+1}(\mathbb{R}^{2n})$. It follows that ρ_h converges to any character on $\mathbb{R}^{2n}$ in $\mathbb{R}^\times \cup \mathbb{R}^{2n}$ as h goes to 0 which shows that $\mathbb{R}^\times \cup \mathbb{R}^{2n}$ is not Hausdorff.

8.4 A Bundle description for the Group C*-Algebra

In this section we will obtain a description for $C^*(H_{2n+1}(\mathbb{R}))$ as a C^*-bundle. This part follows closely the work of J. Ludwig, J and L. Turowska in [11] although the presentation is slightly different. The reason for this is that the bundle is presenteted here as a continuous field of C^*-algebras while it is worked on more concretely in [11] through the Fourier transform and operators fields. The benefit of this more concrete description is that a better understanding of the fibre at 0 is obtained but it also requires more work. The definition of continuous fields of C^*-algebras is given in 10.3 in [6]. The idea behind the definition is to have a C^*-algebra that sits inside a vector bundle over a topological space. The fibers of the bundle will be C^*-algebras and the C^*-algebra that sits in this bundle will be a collection of sections.

Througout this section we set $H = H_{2n+1}(\mathbb{R})$ for a fixed choice of n. Let $\rho_h : H \to \mathcal{U}(L^2(\mathbb{R}^n))$ denote the Schrödinger representation

$$(\rho_h(a,b,c)f)(x) = e^{i(b\cdot x + hc)} f(x + ha)$$

where h is a real non-zero number. We will abuse notation and also denote the integrated representation $\tilde{\rho}_h : C^*(H) \to \mathscr{B}(L^2(\mathbb{R}^n))$ by ρ_h. Define for each real

h

$$E(h) = \begin{cases} \rho_h(C^*(H)), & \text{if } h \in \mathbb{R}^\times \\ C^*(\mathbb{R}^{2n}), & \text{if } h = 0 \end{cases}$$

Denote by $\prod_{h\in\mathbb{R}} E(h)$ the set of all functions x such that $x(h) \in E(h)$ for each $h \in \mathbb{R}$. Let $\Gamma \subseteq \prod_{h\in\mathbb{R}} E(h)$ be the set of all $x \in \prod_{h\in\mathbb{R}} E(h)$ such that the restriction $x|_{\mathbb{R}^\times} : \mathbb{R}^\times \to \mathscr{B}(L^2(\mathbb{R}^n))$ is norm continuous and $\|x(\cdot)\| : \mathbb{R} \to \mathbb{R}$ is continuous at 0. It is straightforward to show that $\mathscr{C} = (\Gamma, E(h)_{h\in\mathbb{R}})$ is a continuous field of C^*-algebras over $\mathbb{R}$. Let $\Theta \subseteq \Gamma$ be the set of all $x \in \Gamma$ such that the function $\|x(\cdot)\|$ vanishes at infinity. Then Θ is a C^*-algebra when it is equipped with the norm

$$\|x\| = \sup_{h\in\mathbb{R}} \|x(h)\|.$$

Now consider $I : L^1(H) \to L^1(\mathbb{R}^{2n})$ given by

$$I(f)(x, y) = \int_{\mathbb{R}} f(x, y, c)dc$$

It is clear that I is well-defined and it is not too hard to prove that I is a surjective *-homomorphism. Composing I with the canonical injection $\iota_{R^n} : L^1(\mathbb{R}^{2n}) \to C^*(\mathbb{R}^{2n})$ gives us a *-homomorphism $L^1(H) \to C^*(\mathbb{R}^{2n})$. By the characteristic property of $C^*(H)$ we get an extension $I : C^*(H) \to C^*(\mathbb{R}^{2n})$ such that $I \circ \iota_H = \iota_{\mathbb{R}^{2n}} \circ I$. In order to prove that I maps onto $C^*(\mathbb{R}^{2n})$ we first note that $I(C^*(H))$ is closed in $C^*(\mathbb{R}^{2n})$ because $I(C^*(H))$ is a C^*-algebra and therefore it is complete. It then follows that

$$\begin{aligned} C^*(\mathbb{R}^{2n}) &= \overline{\iota_{\mathbb{R}^{2n}}(L^1(\mathbb{R}^{2n}))} = \overline{\iota_{\mathbb{R}^{2n}}(I(L^1(H)))} && (3) \\ &= \overline{I(\iota_H(L^1(H)))} \subseteq \overline{I(C^*(H))} && (4) \\ &= I(C^*(H)) && (5) \end{aligned}$$

and we see that I is surjective. Define the *-homomorphism $\Phi : C^*(H) \to \prod_{h\in\mathbb{R}} E(h)$ by

$$\Phi(x)(h) = \begin{cases} \rho_h(x), & \text{if } h \in \mathbb{R}^\times \\ I(x), & \text{if } h = 0 \end{cases}$$

If $\Phi(a) = 0$ then $\rho_h(a) = 0$ for all $h \in \mathbb{R}^\times$ and $I(a) = 0$. Let $(\pi, \mathcal{H})$ be an irreducible representation of $C^*(H)$. If π is equivalent to a Schrödinger representation then $\pi(a) = 0$. Otherwise π corresponds to an irreducible unitary representation $(\pi, \mathcal{H})$ of H such that $\pi|_{Z(H)} = \mathrm{id}_{\mathcal{H}}$. It follows that π factors through $\mathbb{R}^{2n}$ hence $\pi(a, b, c) = e^{i(-\alpha\cdot a - \beta\cdot b)} id_{\mathcal{H}}$ for some α, β in $\mathbb{R}^n$. A straighforward calculation then shows that for any $f \in L^1(G)$ and $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned} \langle \pi(f)\xi, \eta\rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(-\alpha\cdot a - \beta\cdot b)} \int_{\mathbb{R}} f(a, b, c)\langle \xi, \eta\rangle dcdadb \\ &= \widehat{I(f)}(\alpha, \beta)\langle \xi, \eta\rangle = \langle \widehat{I(f)}(\alpha, \beta)\xi, \eta\rangle \end{aligned}$$

It follows that $\pi(f) = \widehat{I(f)}(\alpha, \beta)$. Which implies that $\pi(a) = 0$. We have showed that if $\Phi(a) = 0$ then $\pi(a) = 0$ for any irreducible representation $(\pi, \mathcal{H})$. By Theorem 5.1.2 in [12] there exists an irreducible representation $(\pi_a, \mathcal{H}_a)$ such that $\|\pi_a(a)\| = \|a\|$ which implies that $a = 0$. This shows that Φ is injective.

We will show that Φ maps onto Θ and to do that we first need to examine what it does on any non-zero fiber. For $h \in \mathbb{R}^\times$ and $f \in L^1(H)$ we have

$$\begin{aligned}
\langle \rho_h(f)\xi, \eta\rangle &= \int_H \langle f(x)\rho_h(x)\xi, \eta\rangle dx \\
&= \int_H \int_{\mathbb{R}^n} f(x)(\rho_h(x)\xi)(y)\overline{\eta(y)}dydx \\
&= \int_{\mathbb{R}^n} (\int_H f(x)(\rho_h(x)\xi)(y)dx)\overline{\eta(y)}dy
\end{aligned}$$

and it follows that if we define $K_f^h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$(K_f^h\xi)(f) = \int_H f(x)(\rho_h(x)\xi)(y)dx$$

then it follows that

$$\langle \rho_h(f)\xi, \eta\rangle = \langle K_f^h\xi, \eta\rangle.$$

hence

$$\rho_h(f) = K_f^h$$

for all $h \in \mathbb{R}^\times$ and $f \in L^1(H)$. Using the definition of $\rho_h(f)$ we obtain

$$\begin{aligned}
(K_f^h\xi)(y) &= \int_H f(x)(\rho_h(x)\xi)(y)dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(a, b, c)e^{i(b\cdot y + hc)}\xi(y + ha)dadbdc \\
&= \frac{1}{|h|^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\frac{a-y}{h}, b, c)e^{i(b\cdot y + hc)}\xi(a)dadbdc \\
&= \int_{\mathbb{R}^n} (\frac{1}{|h|^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(\frac{a-y}{h}, b, c)e^{i(b\cdot y + hc)}dbdc)\xi(a)da
\end{aligned}$$

.

It follows that K_f^h is an integral operator with kernel

$$\begin{aligned}
k_f^h(a, y) &= \frac{1}{|h|^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(\frac{a-y}{h}, b, c)e^{i(b\cdot y + hc)}dbdc \\
&= \frac{1}{|h|^n} \widehat{f^{2,3}}(\frac{a-y}{h}, -y, -h)
\end{aligned}$$

Where $\hat{f}^{2,3}$ is the Fourier transform of $f \in L^1(H)$ with respect to the second and third argument when viewed as a function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Theorem 1.6 in [13] shows that $K_f^h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is well defined bounded linear operator. If we identify H with $\mathbb{R}^{2n+1}$ then we can let $\mathcal{S}(H)$ be the Schwartz functions on H. Since the Schwartz functions are dense in L^1 it follows that $\mathcal{S}(H)$ are dense in $C^*(H)$. Pick an element x in $C^*(H)$ and a sequence of Schwartz functions converging to a. By continuity of $\Phi(\cdot)(h)$ for any non-zero h it follows that $\Phi(f_n)(h)$ converges to $\Phi(a)(h)$ in $\mathscr{B}(L^2(\mathbb{R}^n))$. It is easy to show that k_f is in $L^2(\mathbb{R}^{2n})$ when $\hat{f}$ is a Schwartz function and Theorem 4.7 in [13] therefore shows that $\Phi(f_n)(h)$ is a compact operator. Denote the set of all compact operators from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ by $\mathcal{K}$. Theorem 4.18 in [7] shows that $\mathcal{K}$ is a norm closed subspace of $\mathscr{B}(L^2(\mathbb{R}^n))$ and it follows that $\Phi(a)(h)$ is a compact operator. We have thus showed that $\Phi(\cdot)(h)$ maps $C^*(H)$ into $\mathcal{K}$ when $h \neq 0$. To show that this map is surjective we recall that the finite rank operators in $\mathscr{B}(L^2(\mathbb{R}^n))$ are dense in $\mathcal{K}$, see Theorem 4.4 [13]. Since linear functionals on Hilbert spaces are evaluations of inner products it follows that any finite rank operator, $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ can be written on the form

$$F(\xi) = \sum_{i=1}^{n} \langle \xi, u_i \rangle h_i$$

for some positive integer n and $u_i, h_i \in L^2(\mathbb{R}^n)$ for each i and $\{h_i\}$ is an orthogonal set. By expanding the inner product it follows that F is an integral operator with kernel f_F given by

$$f_F(x, y) = \sum_{i=1}^{n} h_i(y)\overline{u_i(x)}$$

Since the Schwartz functions are dense in $L^2(\mathbb{R}^n)$ it is enough to prove, by linearity that for any u and h in $S(\mathbb{R}^n)$ there exists an $f \in L^1(H)$ such that

$$\frac{1}{|h|^n}\widehat{f^{2,3}}(\frac{x-y}{h}, -y, -h) = h(y)\overline{u(x)} \qquad \text{a.e.}$$

Denote by $\mathcal{F} : \mathcal{S}(\mathbb{R}^{2n+1}) \to \mathcal{S}(\mathbb{R}^{2n+1})$ the partial Fourier transform $\mathcal{F}f = \hat{f}^{2,3}$ and define $T : \mathcal{S}(\mathbb{R}^{2n+1}) \to \mathcal{S}(\mathbb{R}^{2n+1})$ by $(Tf)(x, y, h) = f(hx - y, -y, h)$ it is easy to show that these maps are well defined. Since the (ordinary) Fourier transform maps maps Schwartz functions onto Schwartz functions and the partial Fourier transform fixes the first argument it follows that $\mathcal{F}$ has an inverse $\mathcal{F}^{-1}$. Let $\tilde{T} = \mathcal{F}^{-1}T\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ then $T\mathcal{F} = \mathcal{F}\tilde{T}$. If $f(x, b, c) = |h|^n\overline{u(x)}h_0(b)g(c)e^{-ich}$ where $h_0 \in \mathcal{S}(\mathbb{R}^n)$ is the unique element such that $\widehat{h_0} = h$ and $g \in S(\mathbb{R}^n)$ is positive with $\|g\|_1 = 1$. Then it follows that

$\tilde{T}f \in S(\mathbb{R}^n)$ and

$$\frac{1}{|h|^n}\mathcal{F}(\tilde{T}f)(\frac{x-y}{h},-y,-h) = \frac{1}{|h|^n}T\mathcal{F}f(\frac{x-y}{h},-y,-h) = \frac{1}{|h|^n}\mathcal{F}f(x,y,-h) =$$
$$= \frac{1}{|h|^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(x,b,c)e^{-iy\cdot b}e^{ihc}dbdc = \overline{u(x)}\int_{\mathbb{R}^n} h_0(b)e^{-iy\cdot b}db$$
$$= \overline{u(x)}h(y)$$

We have thus showed that the image $\Phi(S(H))(h)$ contains the finite rank operators of $\mathcal{K}$. Since the Schwartz functions are dense in $C^*(H)$ and the finite rank operators are dense in the compact operators it follows that $\Phi(\cdot)(h)$ is surjective for each $h \in \mathbb{R}^\times$.

The next step is to calculate the image of the map $\Phi : C^*(H) \to \prod_{h\in\mathbb{R}} E(h)$ where we have showed that $E(h) = \mathcal{K}$, the compact operatators on $L^2(\mathbb{R}^n)$ when $h \neq 0$. In particular we want to show that this image is Θ. If we pick a Schwartz function $f \in \mathcal{S}(\mathbb{R}^{2n+1})$ then the kernel function k_f^h is in $\mathcal{S}(\mathbb{R}^{2n})$. For any $\xi \in L^2(\mathbb{R}^n)$ we then have

$$\int_{\mathbb{R}^n} |(K_f^h\xi)(x)|^2dx \leq \int_{\mathbb{R}^n}(\int_{\mathbb{R}^n}|k_f^h(y,x)||\xi(y)|dy)^2dx$$
$$\leq \int_{\mathbb{R}^n}(\int_{\mathbb{R}^n}|k_f^h(y,x)|^2dy^{1/2}\int_{\mathbb{R}^n}|\xi(y)|^2dy^{1/2})^2dx$$
$$= \|\xi\|_2^2\|k_f^h\|_2^2$$

which shows that K_f^h is a bounded linear operator with norm bounded by $\|k_f^h\|_2$. Pick a compactly supported Schwartz function $f \in \mathcal{S}(H)$ and a non-zero h_0, then we have

$$\|K_f^h - K_f^{h_0}\|^2 \leq \|k_f^h - k_f^{h_0}\|$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\frac{1}{|h|^n}\hat{f}^{2,3}(\frac{a-y}{h},-y,-h) - \frac{1}{|h_0|^n}\hat{f}^{2,3}(\frac{a-y}{h_0},-y,-h_0)|^2dady$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\frac{|h_0|^{n/2}}{|h|^{n/2}}\hat{f}^{2,3}(h_0a,-y,-h) - \frac{|h|^{n/2}}{|h_0|^{n/2}}\hat{f}^{2,3}(ha,-y,-h_0)|^2dady$$

Using Plancherel's Theorem we can rewrite the last expression with respect to partial Fourier transform with respect to the last variable, denoted by $\hat{f}^3$ yielding

$$\|K_f^h - K_f^{h_0}\|^2$$
$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\frac{|h_0|^{n/2}}{|h|^{n/2}}\hat{f}^3(h_0a,-b,-h) - \frac{|h|^{n/2}}{|h_0|^{n/2}}\hat{f}^3(ha,-b,-h_0)|^2dadb$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\int_{\mathbb{R}}(\frac{|h_0|^{n/2}}{|h|^{n/2}}f(h_0a,-b,c)e^{ihc} - \frac{|h|^{n/2}}{|h_0|^{n/2}}f(ha,-b,c)e^{ih_0c})dc|^2dadb.$$

Define $F_h : H \to \mathbb{C}$ by

$$F_h(a,b,c) = \frac{|h_0|^{n/2}}{|h|^{n/2}} f(h_0 a, -b, c) e^{ihc} - \frac{|h|^{n/2}}{|h_0|^{n/2}} f(ha, -b, c) e^{ih_0 c}$$

it then follows that

$$\|K_f^h - K_f^{h_0}\| \leq \|I(F_h)\|_2$$

It is straightforward to show that there exists a constant $C > 0$ such that

$$\|I(g)\|_2 \leq C\|g\|_{2,0}$$

for any $g \in \mathcal{S}(H)$ where $\|\cdot\|_{2,0}$ is the Schwartz norm

$$\|g\|_{2,0} = \sup_{a,b,c} |g(a,b,c)|(1 + |a|^2 + |b|^2 + c^2)^2.$$

Since F_h is compactly supported there exists an $R > 0$ such that for all h close enough to h_0 we have

$$\begin{aligned}
\|F_h\|_{2,0} &\leq (1 + (2n+1)R^2)^2 \sup_{a,b,c} |\frac{|h_0|^{n/2}}{|h|^{n/2}} f(h_0 a, -b, c) e^{ihc} - \frac{|h|^{n/2}}{|h_0|^{n/2}} f(ha, -b, c) e^{ih_0 c}| \\
&\leq (1 + (2n+1)R^2)^2 (\frac{|h_0|^{n/2}}{|h|^{n/2}} - \frac{|h|^{n/2}}{|h_0|^{n/2}}) \|f\|_\infty \\
&+ (1 + (2n+1)R^2)^2 \frac{|h|^{n/2}}{|h_0|^{n/2}} \sup_{a,b,c} |f(h_0 a, -b, c) - f(ha, -b, c) e^{i(h_0 - h)c}|.
\end{aligned}$$

The first term above clearly goes to 0 as $h \to h_0$. For the second term we recall that if $r \neq 0$ and if $T_r(g)(x) = g(rx)$ is the dilation operator on $C_c(\mathbb{R}^n)$ then the map $r \mapsto T_r(g)$ is continuous on $\mathbb{R}^\times$ for any fixed $g \in C_c(\mathbb{R}^n)$. This implies that the second term above goes to 0 as $h \to h_0$. It follows that $\Phi(f)|_{\mathbb{R}^\times}$ is a continuous map $\mathbb{R}^\times \to \mathcal{K}$. A similar calculation using Plancherel's Theorem shows that

$$\|\Phi(f)(h)\| \leq \|k_f^h\|_2 \leq \frac{1}{|h|^{n/2}} \|I(f)\|_2$$

and it follows that $\|\Phi(f)(\cdot)\|$ vanishes at infinity.
To show that $\Phi(f)$ is an element of Θ it remains to prove that $\|\Phi(f)(\cdot)\|$ is continuous at 0 but this is not however easy since we haven't got a good description of what happens at the 0-fiber. It is true that $\|\Phi(f)(\cdot)\|$ is continuous at 0 and this follows from Theorem 2.12 in [11]. This shows that $\Phi(C^*(H)) \subseteq \Theta$ However Φ is not onto Θ and instead $C^*(H) \cong \Phi(C^*(H))$ sits inside Θ

Let $C_0(\mathbb{R}^\times, \mathcal{K})$ be the set of functions continuous functions $f : \mathbb{R}^\times \to \mathcal{K}$ that vanishes at infinity which means that $\|f(\cdot)\|$ is an element of $C_0(\mathbb{R}^\times)$. The proof that $C_0(X)$ is a C^*-algebra can be used to prove that $C_0(\mathbb{R}^\times, \mathcal{K})$ is a

C^*-algebra. It is not too hard to show, using convex combinations and uniform continuity that $C_0(\mathbb{R}^\times, \mathcal{FR})$ where $\mathcal{FR}$ is the finite rank operators of $\mathcal{K}$ is dense in $C_0(\mathbb{R}^\times, \mathcal{K})$. One can show that this implies that the restrictions $x|_{\mathbb{R}^\times}$ of elements x in the image $\Phi(C^*(H))$ contains $C_0(\mathbb{R}^\times, \mathcal{K})$ since Φ maps onto each fibre $\mathcal{K}$. We summarize our results below

Theorem 8.4.1. *Let $I : C^*(H) \to C^*(\mathbb{R}^{2n})$ be integration over the center and ρ_h, $h \in \mathbb{R}^\times$ be the Schrödinger representations of $H = H_{2n+1}(\mathbb{R})$. Define for each non-zero h and $x \in C^*(H)$*

$$\Phi(x)(h) = \rho_h(x)$$

and set $\Phi(x)(0) = I(x)$. Then each image $\Phi(\cdot)(h)$ is given by the compact operators, $\mathcal{K}$ on $L^2(\mathbb{R}^n)$ if $h \neq 0$ and $\Phi(\cdot)(0) = C^*(\mathbb{R}^{2n})$. The map $\Phi(x) = (\Phi(x)(h))_{h\in\mathbb{R}}$ defines an isometric *-isomorphism between $C^*(H)$ and a C^*-algebra defined by a continuous field of C^*-algebras over $\mathbb{R}$ where the fibers over non-zero h are $\mathcal{K}$ and the fiber over 0 is $C^*(\mathbb{R}^{2n})$. Furthermore the restrictions $x|_{\mathbb{R}^\times}$ of elements x in the image $\Phi(C^*(H))$ contains $C_0(\mathbb{R}^\times, \mathcal{K})$.

We conclude this section by examining the kernel of the integration over the center map $I : C^*(H) \to C^*(\mathbb{R}^{2n})$. By our theorem above we know that I factors as $I = \Phi \circ e_0$ where $e_0 : \Phi(C^*(H)) \to C^*(\mathbb{R}^{2n})$ is evaluation on the 0-fiber $x \mapsto x(0)$. If $e_0(x) = 0$ for some $x \in \Phi(C^*(H))$ Then $x(0) = 0$. Continuity of the vector field at 0 then shows that $\lim_{h\to 0} \|x(h)\| = 0$. This means that the restriction $x|_{\mathbb{R}^\times}$ is an element of $C_0(\mathbb{R}^\times, \mathcal{K})$. Define for $x \in \ker e_0$ the map $\psi(x) = x|_{\mathbb{R}^\times}$. The map ψ is then well-defined as a map $\ker e_0 \to C_0(\mathbb{R}^\times, \mathcal{K})$. By continuity there is only one way to extend a restriction $x|_{\mathbb{R}^\times}$ to a continuous vector field on $\mathbb{R}$, namely by setting $x(0) = 0$ and it follows that ψ is injective. Furthermore, since the restrictions $x|_{\mathbb{R}^\times}$ of elements x in the image $\Phi(C^*(H))$ contains $C_0(\mathbb{R}^\times, \mathcal{K})$ it follows that e_0 maps onto $C_0(\mathbb{R}^\times, \mathcal{K})$. It is easy to show that the map ψ is a *-homomorphism which shows that $\ker e_0$ is isometrically *-isomorphic to $C_0(\mathbb{R}^\times, \mathcal{K})$. Since $I(x) = 0$ if and only if $e_0(x) = 0$ it follows that the kernel of I is isometrically *-isomorphic to $C_0(\mathbb{R}^\times, \mathcal{K})$. It follows that we have get the following short-exact sequence for $C^*(H)$.

$$0 \to C_0(\mathbb{R}^\times, \mathcal{K}) \to C^*(H) \to C^*(\mathbb{R}^{2n}) \to 0.$$

Recall that a short exact sequence means that the image of any map equals the kernel of the map to the right of it and vice versa.

9 Rigidity

In this section a brief discussion on rigidity will be given and how induction arguments can be used for connected, simply connected, nilpotent Lie groups. We will also examine how the Heisenberg gives potential approaches for rigidity results for 2-step nilpotent groups. We begin with a definition of a strong form of rigidity. Some properties of nilpotent Lie groups will be assumed and these facts should be in any book on nilpotent Lie groups.

Definition 9.0.1 (C*-superrigidity)**.** *A locally compact group G is called C^*-superrigid if for any locally compact group H such that $C^*_{red}(H) \cong C^*_{red}(G)$ then H is isomorphic to G as topological groups.*

Not all groups are C^*-superrigid since for example $C^*(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{C}^4 \cong C^*(\mathbb{Z}_4)$ but clearly $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$. However, superrigidity for abelian groups has been known for a long time and the proof essentially follows from an article by Scheinberg [14] proving that if G and H are two connected locally compact abelian groups such that G is homeomorphic to H then G is isomorphic to H as topological groups.

Theorem 9.0.2. *Any locally compact, torsion-free abelian group is C^*-superrigid.*

Proof. Let G be a locally compact, simply connected, connected abelian group and H be any locally compact group such that $C^*_{\text{red}}(H)$ is isomorphic to $C^*_{\text{red}}(G)$. Since $C^*_{\text{red}}(G) \cong C_0(\widehat{H})$ it follows that $C^*_{\text{red}}(H)$ is abelian. It is not too hard to show that $C^*_{\text{red}}(H)$ is abelian if and only if H is abelian since H is abelian if and only if $C_c(G)$ is abelian and $C_c(G)$ is abelian if and only if $C^*_{\text{red}}(G)$ is abelian. It follows that H is an abelian group. Note that G is connected if and only if $\widehat{G}$ is torsion free, indeed if G is connected and assume that $\gamma \in \widehat{G}$ satisfies $\gamma^n = 1$ for some n. Then the image $\gamma(G)$ is a subset of the set of all points in $\mathbb{T}$ that have order less than n. This set is finite. Since the image of connected space by continuous function is conncted it follows that this image only has 1 element and it follows that $\gamma = 1$. which shows that $\widehat{G}$ is torsion-free. If $\widehat{G}$ is connected then Pontryagin's duality theorem (see 1.7 in [9]) shows that $G \cong \widehat{\widehat{G}}$ is torsion free. It follows that $\widehat{G}$ is connected since G is assumed to be torsion-free. The isomorphism $C^*_{\text{red}}(G) \cong C_0(\widehat{G})$ induces a homeomorphism $\widehat{G} \approx \widehat{H}$ and since $\widehat{G}$ is connected it follows by the theorem of Scheinberg that $\widehat{G} \cong \widehat{H}$ as topological groups. From the Pontryagin duality theorem it follows that $G \cong \widehat{\widehat{G}} \cong \widehat{\widehat{H}} \cong H$. □

The following weaker but natural rigidity properties are the natural next steps to consider from the abelian case

Definition 9.0.3. *Let $\mathscr{C}_\infty$ be the collection of all connected, simply connected, nilpotent Lie groups. Let $\mathscr{C}_n$ be the collection of all the groups in $\mathscr{C}_\infty$ of nilpotency class less than or equal to n.*

Definition 9.0.4. *We say that a group in G in some $\mathscr{C}_n$, $1 \leq n \leq \infty$ is rigid with respect to $\mathscr{C}_n$ if $C^*(G) \cong C^*(H)$ for some H in $\mathscr{C}_n$ implies that $G \cong H$.*

Remark 9.1. *It can be proved that a connected nilpotent Lie group is torsion-free if and only if it is connected simply connected.*

Remark 9.2. *The assumptions on $\mathscr{C}_1$ implies that any group in $\mathscr{C}_1$ is of the form $\mathbb{R}^n$ for some integer n.*

The reason for considering these classes for rigidity is that not much is known about them (except for the abelian case) and that they are closed under taking centers and quotients of centers. That is, since G is a connected, simply connected, nilpotent, Lie group of nilpotency degree n then it follows that its center $Z(G)$ is a connected simply connected Lie group of nilpotency degree less than or equal to 1. Nilpotency is required for this step to hold. Since $Z(G)$ is also closed in G properties of Lie groups then implies that $G/Z(G)$ is connected simply connected Lie group of nilpotency degree less than n. In short we have the following proposition

Proposition 9.0.5. *If G is in $\mathscr{C}_n$, $1 \leq n \leq \infty$ then $Z(G)$ and $G/(Z(G))$ are elements of $\mathscr{C}_n$.*

This proposition gives the opportunity to use induction arguments on the nilpotency degree n. The base case is the following statement: if G is an element of $\mathscr{C}_1$ and H is element of $\mathscr{C}_1$ such that $C^*(G) \cong C^*(H)$ then $G \cong H$. This is a much weaker statement than superrigidity for torsion-free abelian groups which we have already proved. The idea for the induction case is then the following we assume that G and H are elements of $\mathscr{C}_n$ such that $C^*(G) \cong C^*(H)$ then we want to show that

$$C^*(Z(G)) \cong C^*(Z(H))$$
$$C^*(G/Z(G)) \cong C^*(H/Z(H))$$

Since then the induction hypothesis would allow us to conclude that $Z(G) \cong Z(H)$ and $G/Z(G) \cong H/Z(H)$. Which will give the following diagram

$$\begin{array}{ccccccccc} 0 & \to & Z(G) & \to & G & \to & G/Z(G) & \to & 0 \\ & & \uparrow & & & & \uparrow & & \\ 0 & \to & Z(H) & \to & H & \to & H/Z(H) & \to & 0 \end{array}$$

of exact sequences where vertical arrows are isomorphisms. This is an extension problem for groups and group cohomological arguments such as the ones found in [16] could potentially be used to get further into the analysis.

9.1 2-step Nilpotence

For the 2-step nilpotent case we let G and H be two groups in $\mathscr{C}_2$ such that $\Phi : C^*(G) \cong C^*(H)$ is an isomorphism. By comparing what we got for our

Heisenberg groups H_{2n+1} which are all 2-step nilpotent we can look at reasonable properties to expect from general 2-step nilpotent groups that should aid the analysis to arrive both at the conclusion that the centers are isomorphic and the quotients are isomorphic. For the center we look at the unitary dual of the heisenberg group. For our Heisenberg groups H_{2n+1} we know that their unitary duals are given by $\mathbb{R}^\times \cup \mathbb{R}^{2n}$ and the subset $\mathbb{R}^\times$ is a dense open subset such that the elements on $\mathbb{R}^\times$ are uniquely determined by the value of their central charaters. A reasonable assumption to examine would therefore be the one given below.

9.1.1 (Assumption)**.** *For any 2-step nilpotent group G there exists a dense open set $U \subseteq \widehat{G}$ such that if $[\pi] \in U$ and $[\pi'] \in \widehat{G}$ such that $res\pi = res\pi'$ then $[\pi] = [\pi']$.*

Recall that for any isomorphism $C^*(G) \to C^*(H)$ we get an induced homeomorphism $f : \widehat{G} \to \widehat{H}$. We also have quotient maps by taking the central character $\mathrm{res} : \widehat{G} \to \widehat{Z(G)}$ and $\mathrm{res} : \widehat{H} \to \widehat{Z(H)}$. If we can use the central characters to induce a homeomorphism $g : \widehat{Z(G)} \to \widehat{Z(H)}$ then we are done by the theorem of Scheinberg. With the assumption above we get by standard properties of quotient maps that there exists a dense open set $\tilde{U} = \mathrm{res}(U) \subseteq \widehat{Z(G)}$ and a function g_0 such that $g_0 \circ \mathrm{res} = \mathrm{res} \circ f$ on U where U is a dense open subset of G.

We know turn our attention to the quotient. For the Heisenberg group we recall that integration over the center $I : C^*(H_{2n+1}) \to C^*(\mathbb{R}^{2n})$ is surjective and $H_{2n+1}/Z(H_{2n+1}) \cong \mathbb{R}^{2n}$. We therefore have a surjective map $I : C^*(H_{2n+1}) \to C^*(H_{2n+1}/Z(H_{2n+1}))$. We can for any locally compact group G define integration over the center first as a map on $L^1(G) \to L^1(G/Z(G))$ by the formula

$$I(f)(xZ) = \int_{Z(G)} f(zx) d\mu(z)$$

where μ is the Haar measure. This map is a surjective *-homomorphism and using the same arguments as for the Heisenberg group this map can be extended to a surjective *-homomorphism $I : C^*(G) \to C^*(G/Z(G))$. For our Heisenberg groups $H_{2n+1}(\mathbb{R})$ we know that the kernel of this map is isomorphic to $C_0(\widehat{Z(G)} - \{1\}, \mathcal{K})$ where $\mathcal{K}$ is the set of compact operators on $L^2(\mathbb{R}^n)$. If the work of J. Ludwig and L. Turowska can be generalized to show that the kernel of integration over the center is $C_0(\widehat{Z(G)} - \{1\}, \mathcal{K})$ for general 2-step nilpotent groups then we can conclude, given that the centers are homeomorphic that

$$\begin{aligned} C^*(G/Z(G)) &\cong C^*(G)/C_0(\widehat{Z(G)} - \{1\}, \mathcal{K}) \\ &\cong C^*(H)/C_0(\widehat{Z(H)} - \{1\}, \mathcal{K}) \cong C^*(H/Z(H)) \end{aligned}$$

which would show that $G/Z(G) \cong H/Z(H)$ by the induction hypothesis.

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