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C*-simplicity of discrete groups and étale groupoids

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Abstract

In this thesis, we look at a dynamical characterisation of C*-simplicity of discrete groups as presented in [1], and groupoid C*-algebras defined for étale groupoids as considered in [2]. Background information and a more detailed outline of the thesis is provided in the introduction. The introduction is followed by a chapter providing necessary preliminary information on C*-algebras, where for example the Gelfand representation and the Gelfand-Naimark-Segal construction are covered. Following the article [1], we discuss in the next chapter a dynamic characterisation of C*-simplicity of discrete groups. Here, we also show that a discrete group G is C*-simple if and only if the reduced crossed product $C(\partial_F G) \rtimes_r G$ is simple, where $\partial_F G$ denotes the Furstenberg boundary associated to G. In the final chapter, we present a definition of groupoid C*-algebras that appears in [2]. We also present a definition of the reduced groupoid C*-algebra, although in a different form from what appears in [2], that applies to étale groupoids with totally disconnected locally compact Hausdorff space of units, and show that the reduced groupoid C*-algebra with this definition is a groupoid C*-algebra.

Contents

Chapter 1

Introduction

The thesis is divided into three main chapters. In the first chapter, we largely follow the book "C*-algebras and operator theory" by Gerald Murphy [3], and present necessary prerequisites about C*-algebras that are needed for the following chapters. Topics covered include the Gelfand representation and the Gelfand-Naimark-Segal (GNS) construction. In the subsequent chapter we begin our study of C*-simplicity of discrete groups. As we shall see, one can associate to each discrete group a C*-algebra, which is called the reduced group C*-algebra. A discrete group is called C*-simple if the reduced group C*-algebra is simple. Our goal is to present some results relating to C*-simplicity found in the paper [1] written by Breuillard–Kalantar–Kennedy–Ozawa, where the authors solved several open problems on C*-simplicity. In particular, we will in Section 3.3 look at a dynamical characterisation of C*-simplicity. In the final chapter, we look at ways to extend the definitions of group C*-algebras and the reduced group C*-algebra to groupoids, following [2]. Groupoids can be viewed as more general groups. The main difference is that not all elements are composable in a groupoid, meaning for example that a groupoid can contain several units.

Von Neumann algebras are special cases of what we now call C*-algebras that were introduced by John von Neumann and studied by von Neumann and Francis Murray starting in 1930 [4]. The more general C*-algebras were then considered by Gelfand and Naimark in 1943 [5]. Specifically, a C*-algebra is a Banach algebra A together with an involution $a \mapsto a^*$ such that the so called C*-identity $\|a^*a\| = \|a\|^2$ holds. An example of a C*-algebra is the algebra of continuous functions on a compact topological space, with complex conjugation as the involution and its norm being the supremum norm. Another example is the algebra of bounded operators on some Hilbert space equipped with the map taking each operator to its adjoint and the operator norm. A fundamental result is that any C*-algebra can be represented as bounded operators on some Hilbert space, which can be shown using the GNS construction [5, 6]. The theory of C*-algebras has for example seen use in physics where bounded operators on Hilbert spaces are of particular interest.

For us, the main class of C*-algebras that are of interest are those that arise as the reduced group C*-algebra of a group, which is a C*-algebra one can naturally associate to any given discrete group. More specifically, we aim to understand under what conditions the reduced group C*-algebra is simple, meaning that it contains no nontrivial closed ideals. As an example of a reduced group C*-algebra, we mention that the reduced group C*-algebra of the discrete group $\mathbf{Z}$ is the C*-algebra of continuous functions on the circle S^1, which is shown using methods from Fourier analysis (see [7, Example F.4.7]). C*-simplicity of the free group on two generators was shown by Powers in an article published in 1975 [8]. What motivated Powers to approach this problem was a question whether every C*-algebra is generated by its projections. Powers' method, which is based on combinatorial arguments, was refined and continued to see use, for example in [9], and was the primary method for studying C*-simplicity. Kalantar and Kennedy in [10] and Breuillard, Kalantar, Kennedy, and Ozawa in [1] applied new methods, using properties of the group action on a certain topological space called the Furstenberg boundary, and was able to resolve a number of open questions on C*-simplicity presented in [1]. In Chapter 3, we will look more closely at the methods and constructions that the authors of [1] use. Two key constructions are those of the Furstenberg boundary and the Hamana boundary that are both compact topological spaces one can associate to any given discrete group, and on which the group acts. In fact, it can be shown that the two boundaries are isomorphic [10, Theorem 3.11], and so properties of one boundary can be inferred from properties of the other. The Furstenberg boundary was introduced by Furstenberg [11], and it was first used in studying simple Lie groups.

As a generalisation of reduced group C*-algebras, we are also interested in reduced *groupoid* C*-algebras. A group can be viewed as a small category consisting of just one object in which all morphisms are invertible. Similarly, a groupoid is a also a small category in which all morphisms are invertible, but where we allow for more than one object. This makes groups special cases of groupoids. A key difference between groupoids and groups is that any two groupoid elements cannot necessarily be multiplied. Unlike a group, a groupoid can contain several units. The set of units in a groupoid is called the unit space. All groups considered in connection to C*-simplicity in this thesis are equipped with the discrete topology. Instead of the discrete topology, we consider groupoids equipped with a topology making them étale. An étale groupoid carries a topology such that the groupoid locally looks like its unit space considered as a subspace of the groupoid. If the topology on a group is étale, then the group is a discrete group because the unit space consists of just one element. In [2], the authors present a new notion of groupoid C*-algebras for étale groupoids. The definition of the reduced groupoid C*-algebra of an étale groupoid that we use is equivalent to the one used in [2], although the definition that we use is formulated slightly differently, and only works with the added condition that the unit space is totally disconnected. A benefit of this approach is that it is technically less heavy. The main result of the final chapter is the fact that reduced groupoid C*-algebra is a groupoid C*-algebra.

Chapter 2

Preliminaries

In this chapter, we present the necessary prerequisites on C*-algebras that are required for the two following chapters. We largely follow chapters 1-3 from G. Murphy's book [3]. Some proofs are omitted in order to keep focus on the parts that are most relevant for our purposes. Two important topics in this chapter include the Gelfand representation and the Gelfand-Naimark-Segal construction.

As a convention, vector spaces and algebras throughout are assumed to be over the complex field **C** unless otherwise stated.

2.1 C*-algebras

We begin by recalling some basic definitions.

Definition 2.1.1. An *algebra* is a vector space A equipped with a bilinear map,

$$A \times A \to A, \quad (a, b) \mapsto ab,$$

for which it holds that $a(bc) = (ab)c$, for all $a, b, c \in A$. A *subalgebra* of an algebra A is a vector subspace of A that is closed under multiplication.

Definition 2.1.2. A *homomorphism* between algebras A and B is a linear map $\varphi : A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

Definition 2.1.3. A *normed algebra* is an algebra A together with a norm $\|\cdot\|$ on A that is submultiplicative in the sense that

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

A normed algebra A is a *unital normed algebra* if there exists a multiplicative unit $1 \in A$ such that $\|1\| = 1$.

If the norm on a normed algebra A is complete, then A is a *Banach algebra*. A *unital Banach algebra* is a Banach algebra A that is also a unital normed algebra.

Example 2.1.4. Let X be a topological space, and denote by $C_b(X)$ the space of bounded continuous functions $X \to \mathbf{C}$. With the supremum norm $\|\cdot\|_\infty$, the space $C_b(X)$ is a unital Banach algebra. In order to show completeness, consider a Cauchy sequence (f_n) in $C_b(X)$. Because $\mathbf{C}$ is complete, there is a function $f : X \to \mathbf{C}$ such that

$$\lim f_n(x) = f(x), \quad (x \in X).$$

That is, (f_n) converges pointwise to f. Given $\varepsilon > 0$, there is an $N \geq 0$ such that $\|f_n - f_m\|_\infty < \varepsilon/2$ if $n, m \geq N$. Fix $x \in X$. Then

$$|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| < |f(x) - f_m(x)| + \varepsilon/2,$$

if $n, m \geq N$. Now take $m \geq N$ so large that $|f(x) - f_m(x)| < \varepsilon/2$, so

$$|f(x) - f_n(x)| < \varepsilon.$$

Because the choice of N does not depend on x, we get

$$n \geq N \implies \|f_n - f\|_\infty < \varepsilon,$$

showing that $\lim_{n\to\infty} \sup_{x\in X} |f(x) - f_n(x)| = 0$. That is, (f_n) converges uniformly to f. It is clear from this that f is bounded. Continuity of f is equivalent to there for each $\varepsilon > 0$ and $x \in X$ existing an open neighbourhood U of x such that $y \in U \implies |f(x) - f(y)| < \varepsilon$. Let $x \in X$ and $\varepsilon > 0$ be given. Choose $n \in \mathbf{N}$ such that $\sup_{x\in X} |f(x) - f_n(x)| < \varepsilon/3$. By continuity of f_n there is an open neighbourhood U of x such that $y \in U \implies |f_n(x) - f_n(y)| < \varepsilon/3$. Then $y \in U$ implies

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.$$

Example 2.1.5. Let X be a locally compact Hausdorff space, and denote by $C_0(X)$ the space of continuous functions $X \to \mathbf{C}$ that vanish at infinity. That a continuous function $f : X \to \mathbf{C}$ vanishes at infinity means that the set

$$\{x \in X : |f(x)| \geq \varepsilon\}$$

is compact for all $\varepsilon > 0$. Note that a continuous function that vanishes at infinity is necessarily bounded because the image of a compact set under a continuous function is bounded in $\mathbf{C}$. Together with the supremum norm $\|\cdot\|_\infty$, the space $C_0(X)$ is a Banach algebra. We know already from the previous example that any Cauchy sequence (f_n) in $C_0(X)$ must converge to a bounded continuous function f, so what remains to be proven in order to show completeness of $C_0(X)$ is that f vanishes at infinity. Take n such that $\|f - f_n\| \leq \varepsilon/2$. Then for all $x \in X$,

$$\varepsilon \leq |f(x)| \implies \varepsilon/2 \leq |f_n(x)|,$$

so $\{x \in X : |f(x)| \geq \varepsilon\}$ is a subset of the compact set $\{x \in X : |f_n(x)| \geq \varepsilon/2\}$. Since $\{x \in X : |f(x)| \geq \varepsilon\}$ is closed by continuity of the map $x \mapsto |f(x)|$, we can conclude that $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

Definition 2.1.6. The *unitisation* of an algebra A is the vector space $\tilde{A} = A \oplus \mathbf{C}$ together with the multiplication map given by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \mu\lambda), \quad ((a, \lambda), (b, \mu) \in A \oplus \mathbf{C}).$$

This makes $\tilde{A}$ a unital algebra with unit $(0, 1)$.

The subalgebra of $\tilde{A}$ consisting of elements of the form $(a, 0)$, $a \in A$, is naturally identified with A. Furthermore, it is straightforward to check that A is an ideal of $\tilde{A}$ in the sense that A is a vector subspace of A and $ax, xa \in A$ for all $a \in A$ and $x \in \tilde{A}$. In the case that A is a normed algebra, the unitisation $\tilde{A}$ with norm given by

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

is a unital normed algebra. With this norm, A is closed in $\tilde{A}$. If A is a Banach algebra, then the completeness of A and $\mathbf{C}$ imply that $\tilde{A}$ is a Banach algebra.

Any homomorphism $\varphi : A \to B$ between an algebra A and a unital algebra B extends uniquely to a unital homomorphism $\tilde{\varphi} : \tilde{A} \to B$, which is given by $\tilde{\varphi}(a, \lambda) = \varphi(a) + \lambda$.

If A is a unital algebra we denote by $\mathrm{Inv}(A)$ the set of invertible elements in A.

Definition 2.1.7. Let A be an algebra and suppose a is an element of A. The *spectrum* of a is the set

$$\sigma(a) = \begin{cases} \{\lambda \in \mathbf{C} : \lambda - a \notin \mathrm{Inv}(A)\}, & A \text{ unital}, \\ \{\lambda \in \mathbf{C} : \lambda - a \notin \mathrm{Inv}(\tilde{A})\}, & A \text{ non-unital}. \end{cases}$$

For clarity, we sometimes write $\sigma_A(a)$ to emphasise that we are considering the spectrum of a as an element of A.

Example 2.1.8. Let X be a compact Hausdorff space. Consider the unital Banach algebra $C(X)$. A function $f \in C(X)$ has an inverse precisely when 0 is not in the range of f. It follows that $\sigma(f)$ is just the range of f.

The following result is very useful, and is presented here without proof.

Proposition 2.1.9. ([3, Lemma 1.2.4]) *Let A be an algebra and suppose a is an element of A. The spectrum $\sigma(a)$ is closed in $\mathbf{C}$ and bounded by $\|a\|$.*

By the above proposition, the *spectral radius* $r(a) = \sup_{x \in \sigma(a)} |x|$ of an element a satisfies $r(a) \leq \|a\|$. In fact, even more can be said. It can be shown (see [3, Theorem 1.27]) that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ for all elements of A.

Example 2.1.10. Let X be a locally compact Hausdorff space that is not compact. Since X is not compact, the algebra $C_0(X)$ is not unital. For $f \in C_0(X)$, we try to determine $\sigma(f)$. The element $(f, 0) - (0, \lambda)$ in $(C_0(X))^{\sim}$ is invertible if and only if there exists $g \in C_0(X)$ and $\mu \in \mathbf{C}$ such that

$$(0, 1) = (f, -\lambda)(g, \mu) = (fg + \mu f - \lambda g, \mu\lambda).$$

This equation implies directly that $\lambda \neq 0$ (and so $0 \in \sigma(f)$), and $\mu = -1/\lambda$. It further implies that for all $x \in X$,

$$g(x)(f(x) - \lambda) = \frac{1}{\lambda} f(x).$$

From this we get that $\lambda \in \sigma(f)$ if λ is in the range of f. Conversely, if $\lambda \neq 0$ is not in the range of f, then the function defined by

$$g(x) = \frac{1}{\lambda} \frac{f(x)}{f(x) - \lambda}$$

is continuous on X. We check that $g \in C_0(X)$. Let $\varepsilon > 0$. If $|g(x)| \geq \varepsilon$, then $|f(x)| \geq |\lambda|^2 \varepsilon / (1 + |\lambda|\varepsilon)$. By continuity of g, the set

$$S = \{x \in X : |g(x)| \geq \varepsilon\}$$

is closed in X, and since $f \in C_0(X)$ we see that S is a subset of a compact set. Thus, S is compact, so $g \in C_0(X)$. We have now shown that $\sigma(f) = f(X) \cup \{0\}$. Note that this is consistent with the fact that the spectrum is compact. For f extends to a continuous function on the one point compactification of X, and the image of this extension is just $f(X) \cup \{0\}$, which is compact as it is the continuous image of a compact set.

Remark. *With notation as in the above example, define a map* $\varphi : (C_0(X))^\sim \to C(\tilde{X})$ *via* $\varphi(f, \lambda) = \tilde{f} + \lambda$, *where* $\tilde{f}$ *is the unique continuous extension of* f *to* $\tilde{X}$. *If* $\varphi(f, \lambda) = 0$, *then* $\lambda = 0$ *because* $\tilde{f}(\infty) = 0$, *which in turn implies that* $f = 0$. *This shows that* φ *is injective. If* $g \in C(\tilde{X})$, *then the restriction of* $g - g(\infty)$ *to* X *is in* $C_0(X)$, *implying that* φ *is surjective. Straightforward computations show that* $\varphi((f_1, \lambda_1)(f_2, \lambda_2)) = \varphi(f_1, \lambda_1)\varphi(f_2, \lambda_2)$. *We conclude that* φ *is an isomorphism between the algebras* $(C_0(X))^\sim$ *and* $C(\tilde{X})$.

However, $\|(f, \lambda)\| = \sup_{x \in X} |f(x)| + |\lambda|$, *while* $\|\tilde{f} + \lambda\| = \sup_{x \in X} |f(x) + \lambda|$, *meaning that* φ *is not necessarily isometric.*

Definition 2.1.11. Let A be an abelian Banach algebra. A non-zero homomorphism $A \to \mathbf{C}$ is called a *character* on A.

If A is unital and τ is a character on A, then it holds that $\tau(1) = 1$ since $\tau(1)^2 = \tau(1)$ and $\tau(1) \neq 0$. Take $a \in A$. Then $\tau(\tau(a)1 - a) = 0$, so $\tau(a)1 - a$ cannot be invertible. This implies that $\tau(a) \in \sigma(a)$. Thus, we can apply Proposition 2.1.9 and use the fact that $\tau(1) = 1$ to see that $\|\tau\| = 1$. If A is non-unital, then the unital extension $\tilde{\tau} : \tilde{A} \to \mathbf{C}$ of any character τ on A is a character on $\tilde{A}$. Thus, as in the unital case, we have $\tau(a) \in \sigma(a)$ and $\|\tau\| \leq 1$.

Definition 2.1.12. Let A be an abelian Banach algebra. The *spectrum* of A is the space of characters on A viewed as a subspace of A^* with the weak*-topology (we know from the preceding discussion that all characters are bounded). The spectrum of A is denoted $\Omega(A)$.

Proposition 2.1.13. ([3, Theorem 1.3.3]) *For an abelian unital Banach algebra A, the spectrum is nonempty, and for each maximal ideal I of A there exists a unique character τ such that* $\ker\tau = I$. *Conversely, the ideal* $\ker\tau$ *is maximal for all $\tau \in \Omega(A)$.*

Proof. We only give a proof of uniqueness here. Assume that τ, τ' are characters with $\ker\tau = \ker\tau'$. Take $a \in A$. Since $\tau(a-\tau(a)) = 0$, we have $\tau'(a-\tau(a)) = 0$. We know that $\tau'(1) = 1$, so $\tau'(a) = \tau(a)$. □

Proposition 2.1.14. *Let A be an abelian Banach algebra. If A is unital, then the spectrum of A is compact. In general it holds that $\Omega(A) \cup \{0\}$ is compact and that the spectrum of A is locally compact.*

Proof. Suppose first A is an abelian Banach algebra, either unital or non-unital. We have already seen that $\Omega(A)$ is a subspace of the closed unit ball in A^*. Let (τ_λ) be a *net* in $\Omega(A)$ converging to $\tau \in A^*$ (see Appendix A for properties of nets). Convergence in the weak*-topology is the same as pointwise convergence. Then for all $a, b \in A$ we can use the continuity of characters to show that

$$\tau(ab) = \lim_\lambda \tau_\lambda(ab) = \lim_\lambda \tau_\lambda(a)\tau_\lambda(b) = \lim_\lambda \tau_\lambda(a) \lim_\lambda \tau_\lambda(b) = \tau(a)\tau(b),$$

which tells us that τ a homomorphism.

If A is unital, we can from $\tau(1) = \lim_\lambda \tau_\lambda(1) = 1$ deduce that τ is nonzero, hence $\tau \in \Omega(A)$. We conclude that $\Omega(A)$ is closed in A^* in the case that A is unital. If A is non-unital, then a limit of characters is either a character or 0, so $\Omega(A) \cup \{0\}$ is closed. By the Banach-Alaoglu theorem, $\Omega(A)$ must be compact if A is unital, and in the non-unital case we get that $\Omega(A) \cup \{0\}$ is compact.

Take a point $\tau \in \Omega(A)$. By the Hausdorff property there are disjoint open subsets U, V of $\Omega(A) \cup \{0\}$ such that $\tau \in U$ and $0 \in V$. Then $0 \notin \bar{U}$, so the closure of U in $\Omega(A)$ is the same as the closure in $\Omega(A) \cup \{0\}$. Because closed subsets of compact spaces are compact, this shows that U is a precompact open neighbourhood of τ in $\Omega(A)$. Thus, we conclude that $\Omega(A)$ is locally compact. □

Let A be an abelian Banach algebra. Each element $a \in A$ defines a function $\hat{a} : \Omega(A) \to \mathbf{C}$ via $\hat{a}(\tau) = \tau(a)$. By definition of the weak*-topology, each function $\hat{a}$ is continuous. Consider the set

$$S = \{\tau \in \Omega(A) : |\hat{a}(\tau)| \geq \varepsilon\}.$$

Let (τ_λ) be a net in S converging to an element τ in $\Omega(A) \cup \{0\}$. We have that $\lim \tau_\lambda(a) = \tau(a)$, so $|\tau(a)| \geq \varepsilon$, and we see that $\tau \neq 0$. It then follows that $|\hat{a}(\tau)| \geq \varepsilon$, and we conclude that $\tau \in S$. We have now shown that S is closed. Since S is closed and lies in the compact space $\Omega(A) \cup \{0\}$, we conclude that S is compact. Thus, $\hat{a} \in C_0(\Omega(A))$.

Proposition 2.1.15. ([3, Theorem 1.3.4]) *Let A be an abelian Banach algebra and suppose a is an element of A. If A is unital, then*

$$\sigma(a) = \hat{a}(\Omega(A)).$$

If A is non-unital, then

$$\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}.$$

Proof. Assume first that A is unital. If $\lambda \in \sigma(A)$, then $a - \lambda$ is not invertible, so the ideal $I = (a - \lambda)$ is proper. Thus, I is contained in some maximal ideal, which by Proposition 2.1.13 is equal to $\ker \tau$ for some $\tau \in \Omega(A)$. From $a - \lambda \in I \subset \ker \tau$, we get $\hat{a}(\tau) = \tau(a) = \lambda\tau(1) = \lambda$. This shows that $\sigma(a) \subset \hat{a}(\Omega(A))$.

Conversely, note that for all $\tau \in \Omega(A)$, it holds that $\tau(a - \tau(a)) = 0$. Because τ is a homomorphism, $a - \tau(a)$ cannot be invertible. This shows that $\tau(a) \in \sigma(A)$, and we can conclude that $\hat{a}(\Omega(A)) \subset \sigma(a)$.

Now that we are done with the unital case, consider the case when A is non-unital. Any character τ' on the unitisation $\tilde{A}$ that is nonzero on A restricts to a character in $\Omega(A)$, so $\tau' = \tilde{\tau}$ for some $\tau \in \Omega(A)$ because the unital extension is unique. The only character on $\tilde{A}$ that vanishes on A is the character τ_0 defined by $\tau_0(a, \lambda) = \lambda$. The uniqueness follows from the fact that all characters on $\tilde{A}$ are unital. Thus, $\Omega(\tilde{A}) = (\Omega(A))^{\sim} \cup \{\tau_0\}$. From the unital case, we get $\sigma(a) = \hat{a}(\Omega(\tilde{A})) = \hat{a}((\Omega(A))^{\sim} \cup \{\tau_0\}) = \hat{a}(\Omega(A)) \cup \{0\}$. □

Definition 2.1.16. Let A be an abelian Banach algebra. The homomorphism $\varphi : A \to C_0(\Omega(A))$ defined by $\varphi(a) = \hat{a}$ is called the *Gelfand representation* of A.

Proposition 2.1.17. *Let A be an abelian Banach algebra. The Gelfand representation $\varphi : A \to C_0(\Omega(A))$ of A satisfies $\|\varphi\| \leq 1$. If A is unital, then φ is unital.*

Proof. Using Proposition 2.1.15, we deduce that the supremum norm of $\hat{a}$ is just the spectral radius $r(a)$. Since $r(a) \leq \|a\|$ it follows that φ is bounded and $\|\varphi\| \leq 1$. If A is unital, then $\tau(1) = 1$ for all $\tau \in \Omega(A)$. From this it follows that $\varphi(1)$ is the constant function equal to 1 on the compact space $\Omega(A)$, which is the unit in $C_0(\Omega(A)) = C(\Omega(A))$. □

We shall see in later sections that if we demand A to be an abelian C*-algebra, then the Gelfand representation becomes an isomorphism.

Definition 2.1.18. Let A be an algebra. A conjugate linear map $a \mapsto a^*$ on A is said to be an *involution* if $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. We usually call a^* the adjoint of a. An algebra together with an involution is a **-algebra*. A subset S of a *-algebra is *self-adjoint* if an element of A is in S if and only if the adjoint of the element is in S.

Definition 2.1.19. A homomorphism $\varphi : A \to B$ between *-algebras is called a **-homomorphism* if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Definition 2.1.20. A *Banach *-algebra* A is a Banach algebra together with an involution $a \mapsto a^*$ such that $\|a\| = \|a^*\|$ for all $a \in A$. A Banach *-algebra A is called a *C*-algebra* if $\|a\|^2 = \|a^*a\|$ for all $a \in A$.

Note that every closed self-adjoint subalgebra of a C*-algebra is a C*-algebra.

Proposition 2.1.21. *Let $\pi : A \to B$ be a *-homomorphism between C*-algebras. The map π is injective if and only if it is* faithful *in the sense that $\pi(a^*a) = 0 \implies a = 0$ for all $a \in A$.*

Proof. Assume that π is injective. If $0 = \pi(a^*a) = \pi(a)^*\pi(a)$, then $0 = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2$, so $\pi(a) = 0$. By injectivity, $a = 0$.

To prove the other direction, assume that π is faithful. If $\pi(a) = 0$, then $\pi(a^*a) = \pi(a)^*\pi(a) = 0$, so $a = 0$. □

Example 2.1.22. Let X be a locally compact Hausdorff space. Complex conjugation defines an involution on $C_0(X)$. For $f \in C_0(X)$, we check that

$$\|f^*f\| = \sup_{x \in X} |f(x)|^2 = \|f\|^2.$$

This shows that $C_0(X)$ is a C*-algebra.

Example 2.1.23. Consider the Banach algebra $\mathcal{B}(H)$ where H is a Hilbert space. For each $u \in \mathcal{B}(H)$ there is a unique operator $u^* \in \mathcal{B}(H)$ such that $\langle u(x), y\rangle = \langle x, u^*(y)\rangle$ for all $x, y \in H$. The map $u \mapsto u^*$ is an involution on $\mathcal{B}(H)$. Note that

$$\|u^*(y)\| = \sup_{\|x\|=1} |\langle x, u^*(y)\rangle| = \sup_{\|x\|=1} |\langle u(x), y\rangle| \leq \|u\|\|y\|,$$

so $\|u^*\| \leq \|u\|$. Similarly, $\|u\| \leq \|u^*\|$, meaning that $\|u^*\| = \|u\|$. Further,

$$\begin{aligned}
\|u\|^2 &= \sup_{\|x\|=1} \|u(x)\|^2 \\
&= \sup_{\|x\|=1} \langle u(x), u(x)\rangle \\
&= \sup_{\|x\|=1} \langle u^*u(x), x\rangle \\
&\leq \|u^*u\| \\
&\leq \|u^*\|\|u\| \\
&= \|u\|^2,
\end{aligned}$$

showing that $\|u\|^2 = \|u^*u\|$. Thus, $\mathcal{B}(H)$ is a C*-algebra.

A closed self-adjoint subalgebra of $\mathcal{B}(H)$ is also a C*-algebra, and is called a *concrete C*-algebra*.

Example 2.1.24. Let X be a locally compact Hausdorff space, and let μ be a σ-finite Borel measure with full support on X. We show that the Banach *-algebra $C_b(X)$ can be identified with a concrete C*-algebra. Let $m : C_b(X) \to B(L^2(X, \mu))$ be the map $f \mapsto m_f$, where $m_f(\xi)(x) = f(x)\xi(x)$ for all $\xi \in L^2(X, \mu)$ and $x \in X$. It is clear that $m_{\lambda_1 f_1 + \lambda_2 f_2} = \lambda_1 m_{f_1} + \lambda_2 m_{f_2}$ and $m_{f_1 f_2} = m_{f_1} m_{f_2}$ for all $f_1, \in f_2 \in C_b(X)$ and $\lambda_1\lambda_2 \in \mathbf{C}$. Then m is a *-homomorphism because

$$\langle m_{\bar{f}}(\xi_1), \xi_1\rangle = \int_X \bar{f}\xi_1\bar{\xi}_2 d\mu = \int_X \xi_1\bar{f}\bar{\xi}_2 d\mu = \langle \xi_1, m_f\xi_2\rangle$$

shows that $m_{f^*} = (m_f)^*$. Furthermore, m is isometric, which we now show. By definition $\|m_f\| = \sup_{\|\xi\|\leq 1} \|m_f\xi\|$. It follows from

$$\|m_f\xi\|^2 = \int_X |f|^2|\xi|^2 d\mu \leq \|f\|^2 \int_X |\xi|^2 d\mu = \|f\|^2\|\xi\|^2$$

that $\|m_f\| \leq \|f\|$. Take $\varepsilon > 0$ small and define $A = \{x \in X : |f| > \|f\| - \varepsilon\}$. By continuity A is open, and by definition of the norm A cannot be empty. Since μ has full support, A cannot have measure 0. In case the measure of A is ∞ we instead use A to denote a subset of $\{x \in X : |f| > \|f\| - \varepsilon\}$ of finite nonzero measure. This exists since μ is σ-finite. Introduce $\xi = (\mu(A))^{-1/2}\chi_A \in L^2(x, \mu)$. Then

$$\|m_f\xi\|^2 = \int_X |f|^2|\xi|^2 d\mu = (\mu(A))^{-1} \int_A |f|^2 d\mu \geq (\|f\| - \varepsilon)^2,$$

so $\|m_f\xi\| \geq \|f\| - \varepsilon$. Taking $\varepsilon \to 0+$ shows that $\|m_f\| \geq \|f\|$. Thus, $\|m_f\| = \|f\|$, meaning that m is isometric. That m is isometric implies that m is injective. It also implies together with $C_b(X)$ being complete that $m(C_b(X))$ is complete and thus closed in $B(L^2(X, \mu))$. Via the isometric $*$-homomorphism m we can then identify $C_b(X)$ with the concrete C*-algebra $m(C_b(X))$.

Definition 2.1.25. An *ideal* in a C*-algebra A is a vectors subspace $I \subset A$ such that for any $b \in I$ and $a \in A$, both ab and ba are in A. A C*-algebra A is said to be *simple* if the only closed ideals of A are 0 and A.

Given a closed ideal I in a C*-algebra A, we can multiply elements in the quotient A/I in the natural way: $(a + I)(b + I) = ab + I$. Further, one defines a norm on A/I by setting $\|a + I\| = \inf_{b\in I} \|a + b\|$. As can be shown (see [3, Theorem 3.1.4] for example) the quotient A/I becomes a C*-algebra with this norm.

We saw in a previous section that any algebra A is included in a unital algebra $\tilde{A}$. If A is a C*-algebra, it can be shown (see [3, Theorem 2.1.6]) that there is a unique C*-norm on $\tilde{A}$ extending the norm of A. Further, any *-homomorphism from A to some unital *-algebra can be uniquely extended to a unital *-homomorphism defined on $\tilde{A}$.

The following result will be used in order to prove that *-homomorphisms are continuous.

Proposition 2.1.26. ([3, Theorem 2.1.1]) *If $a \in A$ is an element of a C*-algebra A such that $a^* = a$, then $r(a) = \|a\|$.*

Proof. From $\|a\|^2 = \|a^*a\| = \|a^2\|$ it follows that $\|a\|^{2^n} = \|a^{2^n}\|$. Since $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$, we also have $r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{2^{-n}} = \|a\|$. □

The norm of any element a in a C*-algebra A can be expressed in terms of the norm of a self-adjoint element. That is, $\|a\| = \|a^*a\|^{1/2}$. However, by the above proposition the norm of self-adjoint elements is determined completely be the algebraic properties of A. As a corollary then, there cannot be more than one norm on a *-algebra making it a C*-algebra.

Proposition 2.1.27. *Any unital *-homomorphism $\varphi : A \to B$ between unital C*-algebras is bounded.*

Proof. Note that the image of any invertible element under φ is invertible. Thus, if $\lambda \in \sigma(\varphi(a))$, then $\varphi(a-\lambda)$ is not invertible, implying that $a-\lambda$ is not invertible. It follows that $\sigma(\varphi(a)) \subset \sigma(a)$. In particular, $r(\varphi(a^*a)) \leq r(a^*a)$, so

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

We see from the above equation that $\|\varphi\| \leq 1$. $\square$

By considering the unitisations of domain or codomain, one sees that in general any *-homomorphism between C*-algebras is bounded.

Proposition 2.1.28. *Let A be a C*-algebra. For all $\tau \in \Omega(A)$ and $a \in A$, it holds that $\tau(a^*) = \tau(a)^-$. This shows that characters on C*-algebras are always *-homomorphisms.*

Proof. Take $a \in A$. Define self-adjoint elements $b = (a^* + a)/2$ and $c = i(a^* - a)/2$, so $a = b + ic$. It can be shown ([3, Theorem 2.1.8]) that the spectrum of any self-adjoint element in a C*-algebra is contained in the real numbers. Thus, Proposition 2.1.15 implies that $\tau(b)$ and $\tau(c)$ are real. Then

$$\tau(a^*) = \tau(b - ic) = \tau(b) - i\tau(c) = (\tau(b) + i\tau(c))^- = \tau(a)^-.$$

$\square$

We are now almost ready to prove the Gelfand theorem, stating that every nonzero abelian C*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. But first, we shall need the Stone-Weierstrass theorem for locally compact Hausdorff spaces. Recall the Stone-Weierstrass theorem for compact Hausdorff spaces:

Theorem 2.1.29. *Let X be a compact Hausdorff space. If A is a self-adjoint subalgebra of $C(X)$ separating the points of X and vanishing at no point of X, then A is dense in $C(X)$.*

The Stone-Weierstrass theorem for locally compact Hausdorff spaces follows readily from the compact case.

Theorem 2.1.30. *Let X be a locally compact Hausdorff space. If A is a self-adjoint subalgebra of $C_0(X)$ separating the points of X and vanishing at no point of X, then A is dense in $C_0(X)$.*

Proof. Suppose $A \subset C_0(X)$ is a self-adjoint subalgebra separating the points of X and vanishing at no point of X. Let $\tilde{X} = X \cup \{\infty\}$ be the one point compactification of X. If $f \in C_0(X)$, then f extends uniquely to a continuous function $\tilde{f} \in C(\tilde{X})$ with $\tilde{f}(\infty) = 0$. Let $\tilde{A} = \{\tilde{f} \in C(\tilde{X}) : f \in A\}$, which is self-adjoint because A is self-adjoint. Introduce $B = \tilde{A} \oplus \mathbf{C} \subset C(\tilde{X})$. Since

$$(\tilde{f} + r)(\tilde{g} + s) = \tilde{f}\tilde{g} + r\tilde{g} + s\tilde{f} + rs = (fg)^{\sim} + r\tilde{g} + s\tilde{f} + rs \in B,$$

for all $f, g \in A$ and $r, s \in \mathbf{C}$, we have that B is a self-adjoint subalgebra of $C(\tilde{X})$. The corresponding properties of A imply that B vanishes at no point of $\tilde{X}$ and separates the points of $\tilde{X}$. By the Stone-Weierstrass theorem, B is dense in $C(\tilde{X})$.

Let f be an arbitrary element of $C_0(X)$. For $\varepsilon > 0$, there exists an element $g \in A$ and $r \in \mathbf{C}$ such that $\|\tilde{f} - \tilde{g} - r\| < \varepsilon$. In particular $|r| < \varepsilon$. Then

$$\|f - g\| = \|\tilde{f} - \tilde{g}\| \leq \|\tilde{f} - \tilde{g} - r\| + |r| < 2\varepsilon.$$

This shows that A is dense in $C_0(X)$. □

Theorem 2.1.31. (Gelfand Theorem) *Let A be a nonzero abelian C*-algebra. The Gelfand representation $\varphi : A \to C_0(\Omega(A))$ is an isometric *-isomorphism.*

Proof. The Gelfand representation preserves adjoints since

$$\varphi(a^*)\tau = \tau(a^*) = \tau(a)^- = (\varphi(a)\tau)^- = \varphi(a)^*\tau$$

for all $a \in A$ and $\tau \in \Omega(A)$. Further, $\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(a^*a)$. Using that a^*a is self-adjoint, we find that $r(a^*a) = \|a^*a\| = \|a\|^2$. Thus, φ is an isometric *-homomorphism. Since φ is isometric, the image of A under φ is closed in $C_0(\Omega(A))$. We see that $\varphi(A)$ vanishes at no point of $\Omega(A)$ since characters are non-zero by definition. That $\varphi(A)$ separates the points of $\Omega(A)$ is immediate from the definition of $\varphi(a) = \hat{a}$. By the Stone-Weierstrass theorem for locally compact Hausdorff spaces $\varphi(A) = \overline{\varphi(A)} = C_0(\Omega(A))$. □

Proposition 2.1.32. ([3, Theorem 3.1.5]) *Any injective *-homomorphism $\varphi : A \to B$ between C*-algebras is isometric.*

Proof. Suppose first that A and B are unital and abelian, and that φ is unital. Introduce the map $\varphi' : \Omega(B) \to \Omega(A)$ such that $\varphi'(\tau) = \tau \circ \varphi$. That φ' actually maps characters on B to characters on A is a consequence of φ being a homomorphism and unital, where being unital ensures that $\tau \circ \varphi$ is nonzero. Recall that we always consider the spectrum of a C*-algebra as a topological space equipped with the weak *-topology. It is straightforward to check that φ' is continuous by showing that if the net (τ_λ) converges to τ in $\Omega(B)$, then the net $(\varphi'(\tau_\lambda))$ converges to $\varphi'(\tau)$ in $\Omega(A)$. Thus, $\varphi'(\Omega(B))$ is compact since $\Omega(B)$ is compact.

Suppose $\varphi'(\Omega(B)) \neq \Omega(A)$. Using Urysohn's lemma, we find a nonzero $f \in C(\Omega(A))$ that vanishes on $\varphi'(\Omega(B))$. By the Gelfand theorem, there exists $a \in A$ such that $f = \hat{a}$. It follows that for all $\tau \in \Omega(B)$, we have $0 = \hat{a}(\varphi'(\tau)) = \tau(\varphi(a))$. Because the Gelfand representation is isometric, $\|a\| = \sup_{\tau \in \Omega(A)} \tau(a)$ and $\|\varphi(a)\| = \sup_{\tau \in \Omega(B)} \tau(\varphi(a))$. In particular, $\varphi(a) = 0$. By injectivity, $a = 0$ so $f = \hat{a} = 0$, which is a contradiction. We conclude that $\varphi'(\Omega(B)) = \Omega(A)$. Now

$$\|\varphi(a)\| = \sup_{\tau \in \Omega(B)} \tau(\varphi(a)) = \sup_{\tau \in \Omega(B)} \varphi'(\tau)(a) = \sup_{\tau \in \Omega(A)} \tau(a) = \|a\|$$

shows that φ is isometric.

We return to the general case, where we do not put any restrictions on the C*-algebras A and B. We want to show that $\|\varphi(a)\|^2 = \|\varphi(a^*a)\|$ is equal to $\|a\|^2 = \|a^*a\|$ for all $a \in A$. Consider the unital *-homomorphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$. If $0 = \tilde{\varphi}(a, \lambda) = (\varphi(a), \lambda)$, then $\lambda = 0$ and $a = 0$ because φ is injective. This shows that $\tilde{\varphi}$ is injective. Consider the C*-algebra A' generated by a^*a and 1 in $\tilde{A}$. Since a^*a is self-adjoint, A' is abelian. In $\tilde{B}$, we have that $B' = \overline{\varphi(A')}$ is an abelian unital C*-algebra. Applying the first part of the proof to the restriction of $\tilde{\varphi}$ to A' and B' shows that $\|\varphi(a^*a)\| = \|a\|^2$ as we wanted. □

2.2 The GNS construction

The main goal of this section is to present the Gelfand-Naimark-Segal construction and the universal representation, which represents any C*-algebra as bounded operators on some Hilbert space. For this purpose, we need to know about positive elements of C*-algebras.

Definition 2.2.1. Let a be a self-adjoint element of a C*-algebra A. We say that a is *positive* if $\sigma(a)$ is a subset of the nonnegative real numbers. The set of positive elements in A is denoted by A^+. A map $\varphi : A \to B$ between C*-algebras is said to be *positive* if $\varphi(A^+) \subset B^+$.

The notion of positivity of elements in a C*-algebra A defines a binary relation $\leq$ on the self-adjoint elements A_{sa}. That is, for elements $a, b \in A_{sa}$ we say that $a \leq b$ if $b - a$ is positive.

Example 2.2.2. Let X be a locally compact Hausdorff space. In Examples 2.1.8 and 2.1.10, we found that the spectrum of an element $f \in C_0(X)$ is either $f(X)$ if X is compact, or $f(X) \cup \{0\}$ if X is not compact. The self-adjoint elements are just the real valued functions in $C_0(X)$, so the positive elements are the nonnegative functions.

Next is an important result that will sometimes be used implicitly without mention. For a proof, see [3, Section 2.2].

Proposition 2.2.3. *Let A be C*-algebra and B a C*-subalgebra of A. Then $B^+ = A^+ \cap B$.*

Proposition 2.2.4. *For every C*-algebra A it holds that $A^+ = \{a^*a : a \in A\}$.*

Proof. We first give a proof when $A = C_0(X)$ for some locally compact Hausdorff space X. It is clear that $f^*f = |f|^2$ is a nonnegative function for all $f \in C_0(X)$. Conversely, if $f \in C_0(X)$ is a nonnegative function, then $f = \sqrt{f}\sqrt{f}$.

Let A be an arbitrary C*-algebra and let $b \in A^+$ be a positive element. Consider the C*-algebra B generated by b. Because b is self-adjoint, B is abelian. By Proposition 2.2.3, b is positive in B. Via the Gelfand representation, B is isomorphic to a C*-algebra of the form $C_0(X)$ for some locally compact Hausdorff space X. In particular, an element of B is positive if and only if its image under the Gelfand representation is positive. By the first part of the proof, there therefore exists $a \in B$ such that $b = a^*a$. Conversely, since a^*a is self-adjoint for all $a \in A$, the Gelfand representation can be applied in a similar way to the C*-algebra generated by a^*a to show that a^*a is positive. □

Corollary. *Any *-homomorphism $\varphi : A \to B$ between C*-algebras is positive.*

Proof. This follows from the previous proposition and the fact that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ for all $a \in A$. □

Consider the C*-algebra $C(X)$, where X is a compact Hausdorff space. Let t be a real number and f a self-adjoint (same as real valued in this case) element in $C(X)$. If $\|f - t\| \leq t$, then

$$|f(x) - t| \leq t \implies t - f(x) \leq t \implies 0 \leq f(x)$$

for all $x \in X$, so f is positive. Conversely, if f is positive and satisfies $\|f\| \leq t$, then $f(x) - t \leq f(x) \leq t$ and $t - f(x) \leq t$, so $\|f - t\| \leq t$. Via the Gelfand representation, we can translate these relations to unital C*-algebras.

Proposition 2.2.5. *Let A be a unital C*-algebra. If a is a self-adjoint element of A and $\|a-t\| \leq t$ holds for some real number t, then a is positive. Conversely, if a is positive and $\|a\| \leq t$ for some real number t, then $\|a - t\| \leq t$.*

Proof. Let $a \in A$ be self-adjoint and t be some real number. Consider the unital and abelian C*-algebra B generated by 1 and a in A. Let $\varphi : B \to C_0(\Omega(B))$ be the Gelfand representation of B. If $\|a - t\| \leq t$, then $\|\varphi(a) - t\| \leq t$, so $\varphi(a)$ and hence a are positive.

For the second part, assume now that a is positive and $\|a\| \leq t$. By the Gelfand theorem, $\varphi(a)$ is positive and $\|\varphi(a)\| \leq t$, so $\|\varphi(a) - t\| \leq t$. This implies that $\|a - t\| \leq t$. □

Let a be a self-adjoint element of a unital C*-algebra A. Then

$$\|\|a\| - a - \|a\|\| = \|a\|.$$

By using Proposition 2.2.5, we see that $a \leq \|a\|$.

Proposition 2.2.6. *If a, b are positive elements of a C*-algebra A, then $a + b$ is positive.*

Proof. Since A is a C*-subalgebra of the unital C*-algebra $\tilde{A}$, it suffices to show that $a + b$ is positive in $\tilde{A}$. By Proposition 2.2.5 we have that

$$\|a + b - \|a\| - \|b\|\| \leq \|a - \|a\|\| + \|b - \|b\|\| \leq \|a\| + \|b\|.$$

By another application of Proposition 2.2.5, this shows that $a+b$ is positive. □

Corollary. *Let A be a C*-algebra. The binary relation on A_{sa} given by $a \leq b$ if and only if $b - a \in A^+$ is a preorder on A_{sa}.*

Proof. That $a \leq a$ for all $a \in A$ follows from $\sigma(0) = \{0\}$. Thus we have reflexivity.

To show transitivity, suppose $a \leq b$ and $b \leq c$. Since $c - a = c - b + b - a$ is positive as a sum of positive elements, we conclude that $a \leq c$. □

Proposition 2.2.7. *For a C*-algebra A, the set of positive elements A^+ is closed in A.*

Proof. Because $\|a\| = \|a^*\|$ for every $a \in A$, the set of self-adjoint elements A_{sa} is closed in A.

Assume first that A is unital. Let (a_n) be a sequence of positive elements converging to $a \in A$. Because (a_n) converges, there is a $t \in \mathbf{R}$ such that $\|a_n\| \leq t$. By Proposition 2.2.5 we have $\|a_n - t\| \leq t$, and it follows that $\|a - t\| \leq t$. Since a is self-adjoint, we can apply Proposition 2.2.5 to conclude that a is positive.

If A is non-unital, we consider the unitisation $\tilde{A}$ and use that $A^+ = A \cap \tilde{A}^+$, allowing us to apply the above argument to show that A^+ is closed in A. □

Proposition 2.2.8. *A contractive unital map $\varphi : A \to B$ between unital C*-algebras that preserves adjoints is positive.*

Proof. If $a \in A$ is positive, then $\|a - \|a\|\| \leq \|a\|$. Thus,

$$\|\varphi(a) - \|a\|\| = \|\varphi(a - \|a\|)\| \leq \|a - \|a\|\| \leq \|a\|.$$

Since $\varphi(a)$ is self-adjoint, we conclude that $\varphi(a)$ is positive. □

Example 2.2.9. Let H be a Hilbert space and $H_0 \subset H$ a Hilbert subspace. Denote by $\iota : H_0 \hookrightarrow H$ the inclusion and by $p : H \to H_0$ the orthogonal projection onto H_0. We define a map $\varphi : \mathcal{B}(H) \to B(H_0)$ via $\varphi(u) = pu\iota$. If $x, y \in H_0$, then

$$\begin{aligned}
\langle \varphi(u)x, y \rangle &= \langle pu\iota x, y \rangle \\
&= \langle pux, y \rangle \\
&= \langle ux, py \rangle \\
&= \langle ux, y \rangle \\
&= \langle x, u^* y \rangle \\
&= \langle px, u^* \iota y \rangle \\
&= \langle x, pu^* \iota y \rangle.
\end{aligned}$$

This shows that $\varphi(u^*) = \varphi(u)^*$. Since $\|\iota\| = \|p\| = 1$, we have $\|\varphi(u)\| \leq \|u\|$, so φ is contractive. Further, φ is unital since $p\iota$ is the identity on H_0. As a unital contractive map that preserves adjoints, φ is positive. This tells us that if $u \in \mathcal{B}(H)$ is positive and $u(H_0) \subset H_0$, then the restriction of u to H_0 is positive as an element of $B(H_0)$.

Proposition 2.2.10. *For every positive element a of a C*-algebra A there exists a positive square root $b \in A^+$ such that $a = b^2$.*

Proof. Let B be the C*-algebra generated by a and let $\varphi : B \to C_0(\Omega(B))$ be the Gelfand representation. Since $\varphi(a)$ is positive, the positive element $f = \sqrt{\varphi(a)}$ is the unique positive square root for $\varphi(a)$. A positive square root for a is then $b = \varphi^{-1}(\sqrt{\varphi(a)})$. □

Proposition 2.2.11. *Any self-adjoint element a of a C*-algebra A can be written as a difference of two positive elements.*

Proof. Because a is self-adjoint, $a^2 = aa = a^*a$ is positive. Let b be a positive square root of a^2, let B be the C*-algebra generated by b, and let $\varphi : B \to C_0(\Omega(B))$ be the Gelfand representation. Since $\varphi(b)$ is the positive square root of $\varphi(a^2)$, it holds that $-\varphi(b) \leq \varphi(a) \leq \varphi(b)$, so $-b \leq a \leq b$. In particular, $a_+ = (b + a)/2$ and $a_- = (b - a)/2$ are positive. Since $a = a_+ - a_-$, the proof is finished. Note that $a_+a_- = 0$ since $b^2 - a^2 = a^2 - a^2 = 0$. □

Example 2.2.12. If A is a unital C*-algebra, then there is another way to write a self-adjoint element $a \in A$ as a difference of two positive elements. We have that $a = (a + \|a\|)/2 - (\|a\| - a)/2$. Since $-a \leq \|a\|$ and $a \leq \|a\|$, the elements $(a + \|a\|)/2$ and $(\|a\| - a)/2$ are indeed positive.

Proposition 2.2.13. *If $a, b \in A$ are self-adjoint elements of a C*-algebra A such that $-b \leq a \leq b$, then $\|a\| \leq \|b\|$.*

Proof. We work in the unitisation $\tilde{A}$. As we have seen earlier, $b \leq \|b\|$, and multiplying this relation with -1 gives $-\|b\| \leq -b$. Thus, it holds that $-\|b\| \leq a \leq \|b\|$. Let B the C*-algebra generated by a and 1 in $\tilde{A}$ and let $\varphi : B \to C(\Omega(B))$ be the Gelfand representation. The relation $-\|b\| \leq a \leq \|b\|$ translates via the Gelfand representation to the relation $-\|b\| \leq \varphi(a) \leq \|b\|$, which implies $\|a\| = \|\varphi(a)\| \leq \|b\|$. □

Proposition 2.2.14. *Let $\varphi : A \to B$ be a positive map between C*-algebras. It holds that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. That is, positive maps between C*-algebras preserve adjoints.*

Proof. Take any $a \in A$. We introduce self-adjoint elements $b = (a + a^*)/2$ and $c = i(a^* - a)/2$ such that $a = b + ic$. Then $a = b_+ - b_- + ic_+ - ic_-$. Since φ is positive, $\varphi(b_\pm)$ and $\varphi(c_\pm)$ are all self-adjoint elements of B. Thus,

$$\begin{aligned}
\varphi(a^*) &= \varphi(b_+ - b_- - ic_+ + ic_-) \\
&= \varphi(b_+) - \varphi(b_-) - i\varphi(c_+) + i\varphi(c_-) \\
&= (\varphi(b_+) - \varphi(b_-) + i\varphi(c_+) - i\varphi(c_-))^* \\
&= \varphi(a)^*.
\end{aligned}$$

□

Proposition 2.2.15. *Let $\varphi : A \to B$ be a positive map between C*-algebras, with A unital. The map φ is bounded.*

Proof. Let a be a self-adjoint element of A. Since φ is positive, it holds that

$$0 \leq \varphi(\|a\| - a) = \|a\|\varphi(1) - \varphi(a).$$

Replacing a with $-a$, we also get $-\|a\|\varphi(1) \leq \varphi(a)$. Thus, $\|\varphi(a)\| \leq \|a\|\|\varphi(1)\|$.

Now let $a \in A$ be an arbitrary element, not necessarily self-adjoint. Put $b = (a + a^*)/2$ and $c = i(a^* - a)/2$. Then $a = b + ic$. Since b and c are self-adjoint and have norm bounded above by $\|a\|$, it holds that

$$\|\varphi(a)\| \leq \|\varphi(b)\| + \|\varphi(c)\| \leq 2\|\varphi(1)\|\|a\|.$$

□

Definition 2.2.16. A *state* τ on a C*-algebra A is a positive linear functional $\tau : A \to \mathbf{C}$ such that $\|\tau\| = 1$. The *state space* $\mathcal{S}(A)$ is the set of states on A equipped with the weak *-topology.

We present without proofs three results about positive linear functionals on C*-algebras that we will have much use for in the sequel.

Proposition 2.2.17. *Let A be a C*-algebra.*

(i) ([3, Theorem 3.3.2]) *For all $a \in A$ and positive linear functionals $\tau : A \to \mathbf{C}$, it holds that $\tau(a^*) = \tau(a)^-$. This actually follows from the fact that positive maps preserve adjoints, but see* [3] *for a different proof.*

(ii) ([3, Theorem 3.3.6]) *If A is nonzero, then for any normal element $a \in A$, meaning that $a^*a = aa^*$, there is a state $\tau \in \mathcal{S}(A)$ such that $\|a\| = \tau(a)$. As a corollary, there exists at least one state on any nonzero C*-algebra.*

(iii) ([3, Theorem 3.3.7]) *Fix a positive linear functional $\tau : A \to \mathbf{C}$. An element $a \in A$ satisfies $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for every $b \in A$. Further, $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$ for all $a, b \in A$.*

Let τ be a positive linear functional on a C*-algebra A. Then $\sigma(a, b) := \tau(b^*a)$ defines a positive sesquilinear form on A. The fact that σ is positive follows from $\sigma(a, a) = \tau(a^*a) \geq 0$. From the Cauchy-Schwarz inequality for positive sesquilnear forms, we get $\|\tau(b^*a)\|^2 \leq \tau(a^*a)\tau(b^*b)$.

Consider a C*-algebra A. For a state τ on A, we introduce the set $N_\tau = \{a \in A : \tau(a^*a) = 0\}$. As the kernel of the continuous map $a \mapsto \tau(a^*a)$, the set N_τ is closed in A. Suppose $a, b \in N_\tau$. Then

$$|\tau((a + b)^*(a + b))| \leq |\tau(b^*a)| + |\tau(a^*b)| \leq 2\tau(a^*a)^{1/2}\tau(b^*b)^{1/2} = 0$$

by Cauchy-Schwarz, showing that N_τ is closed under addition. That N_τ is closed under scalar multiplication is clear, so N_τ is vector subspace of A. Further, if $a \in N_\tau$, then for all $b, c \in A$ we have $\tau(cba) = 0$ by Proposition 2.2.17 (iii), meaning that $ba \in N_\tau$ by the same proposition. This shows that N_τ is a left ideal in A.

Consider the vector space A/N_τ equipped with the inner product

$$\langle a + N_\tau, b + N_\tau \rangle = \tau(b^*a).$$

This is well defined, for if $c, d \in A$ are such that $c - a \in N_\tau$ and $d - b \in N_\tau$, then

$$\tau(d^*c) = \tau((d-b)^*c) + \tau(b^*c) = \tau(c^*(d-b))^- + \tau(b^*(c-a)) + \tau(b^*a) = \tau(b^*a),$$

where we used Proposition 2.2.17 (i) and (iii). It follows from Proposition 2.2.17 (i) that $\langle \cdot, \cdot \rangle$ is conjugate symmetric, and since $\tau(a^*a) \geq 0$ because τ is positive, we get together with the definition of N_τ that $\langle \cdot, \cdot \rangle$ is positive definite. Thus, $\langle \cdot, \cdot \rangle$ is indeed an inner product. The Hilbert space completion of N_τ is denoted H_τ.

We will now define a *-homomorphism $\pi_\tau : A \to \mathcal{B}(H_\tau)$. We start by defining $\pi_\tau(a)$ on the dense subset A/N_τ via

$$\pi_\tau(a)(b + N_\tau) = ab + N_\tau,$$

which is well defined because N_τ is a left ideal. Since

$$\|\pi_\tau(a)(b + N_\tau)\|^2 = \tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b) = \|a\|^2\|b + N_\tau\|^2,$$

by Proposition 2.2.17 (iii), $\pi(a)$ is bounded and $\|\pi(a)\| \leq \|a\|$. Thus, $\pi(a)$ can be continuously extended to H_τ. It can be seen directly from the definition that π_τ is a homomorphism. To show that it preserves adjoints, take $a, b, c \in A$ and calculate

$$\begin{aligned}
\langle b + N_\tau, \pi_\tau(a^*)(c + N_\tau) \rangle &= \langle b + N_\tau, a^*c + N_\tau \rangle \\
&= \tau(c^*ab) \\
&= \langle ab + N_\tau, c + N_\tau \rangle \\
&= \langle \pi_\tau(a)(b + N_\tau), c + N_\tau \rangle.
\end{aligned}$$

In general, a *representation* of a *-algebra is a *-homomorphism into the C*-algebra of bounded operators on some Hilbert space. The representation $\pi_\tau : A \to \mathcal{B}(H_\tau)$ is known as the *Gelfand-Naimark-Segal representation*, or GNS representation for short, associated to the state τ. Suppose A is nonzero, so $\mathcal{S}(A) \neq \emptyset$ by Proposition 2.2.17 (ii). If we introduce the Hilbert space direct sum $H = \bigoplus_{\tau \in \mathcal{S}(A)} H_\tau$, we get a representation $\pi : A \to \mathcal{B}(H)$ from taking the direct sum of the representations $\pi_\tau : A \to \mathcal{B}(H_\tau)$, $\tau \in \mathcal{S}(A)$. The representation $\pi : A \to \mathcal{B}(H)$ is called the *universal representation* of A. Note that the universal representation is unital if A is unital since each representation π_τ is unital in that case.

Theorem 2.2.18. ([3, Theorem 3.4.1]) *The universal representation $\pi : A \to \mathcal{B}(H)$ of a non-zero C*-algebra A is faithful.*

Proof. Assume that $\pi(a) = 0$ for some $a \in A$. The element a^*a is self-adjoint and thus normal, so by Proposition 2.2.17 (ii) there is a state $\tau \in \mathcal{S}(A)$ such that $\|a\|^2 = \|a^*a\| = \tau(a^*a)$. Since a^*a is positive, there exists a positive element $b \in A^+$ such that $b^2 = a^*a$, and a positive element $c \in A^+$ such that $c^2 = b$. Note that $(\pi(c))^4 = \pi(c^4) = \pi(a^*a) = 0$, so for all $\xi \in H$ we have

$$\|\pi_\tau(c^2)\xi\|^2 = \langle \pi_\tau(c)^2\xi, \pi_\tau(c)^2\xi\rangle = \langle \pi_\tau(c)^4\xi, \xi\rangle = 0,$$

implying that $\pi_\tau(c^2) = 0$. Similarly,

$$\|\pi_\tau(c)\xi\|^2 = \langle \pi_\tau(c)\xi, \pi_\tau(c)\xi\rangle = \langle \pi_\tau(c)^2\xi, \xi\rangle = 0,$$

so $\pi_\tau(c) = 0$. Putting these relations together, we get

$$\|a\|^2 = \tau(a^*a) = \tau(c^4) = \|c^2 + N_\tau\|^2 = \|\pi_\tau(c)(c + N_\tau)\|^2 = 0.$$

□

The above theorem is incredibly useful. Since injective *-homomorphisms between C*-algebras are isometric, it shows that any C*-algebra can be identified with a C*-subalgebra of the C*-algebra of bounded operators on some Hilbert space.

We now continue with investigating representations of *-algebras.

Definition 2.2.19. A representation $\pi : A \to \mathcal{B}(H)$ of a *-algebra is called *nondegenerate* if for any nonzero $\xi \in H$ there exists $a \in A$ such that $\varphi(a)\xi \neq 0$.

Proposition 2.2.20. *A representation* $\pi : A \to \mathcal{B}(H)$ *of a *-algebra* A *is nondegenerate if and only if* $\overline{\text{span}}(\pi(A)H) = H$.

Proof. Assume that the representation is nondegenerate. If $\xi \in \overline{\text{span}}(\pi(A)H)^\perp$, then for all $a \in A$

$$0 = \langle \xi, \pi(a^*a)\xi\rangle = \|\pi(a)\xi\|^2.$$

Since the representation is nondegenerate, this implies $\xi = 0$. Thus, we conclude that $\overline{\text{span}}(\pi(A)H) = H$.

Conversely, assume that $\overline{\text{span}}(\pi(A)H) = H$. If $\pi(a)\xi = 0$ for all $a \in A$, then

$$\langle \eta, \pi(a^*)\xi\rangle = \langle \pi(a)\eta, \xi\rangle = 0$$

for all $\eta \in H$. This shows that $\xi \in \overline{\text{span}}(\pi(A)H)^\perp$, so $\xi = 0$. Thus, we conclude that the representation is nondegenerate. □

Definition 2.2.21. A representation $\pi : A \to \mathcal{B}(H)$ of a *-algebra is called *cyclic* if there exists a so called *cyclic vector* $\xi \in H$ such that $\overline{\varphi(A)\xi} = H$. We use a triple (π, H, ξ) to say that $\pi : A \to \mathcal{B}(H)$ is a cyclic representation with cyclic vector ξ.

Note that cyclic representations are nondegenerate.

Proposition 2.2.22. *Let A be a *-algebra, and let (π, H, ξ) and (π', H', ξ') be cyclic representations of A such that $\langle \pi(a)\xi, \xi\rangle = \langle \pi'(a)\xi', \xi'\rangle$ for all $a \in A$. There is a unitary map $u : H \to H'$ such that $\pi'(a) = u\pi(a)u^*$ for all $a \in A$.*

Proof. We want to define $u(\pi(a)\xi) = \pi'(a)\xi'$ for all $a \in A$. To see that u is well defined, note that

$$\|\pi(a)\xi\|^2 = \langle \pi(a^*a)\xi, \xi\rangle = \langle \pi'(a^*a)\xi', \xi'\rangle = \|\pi'(a)\xi'\|^2,$$

so if $\pi(a)\xi = \pi(b)\xi$, then $\pi'(a)\xi' = \pi'(b)\xi'$ by linearity. The above equation also shows that u is isometric. In particular, u extends to an isometric map from $H = \overline{\varphi(A)\xi}$. As H is complete and $u : H \to H'$ is isometric, the image $u(H)$ is closed in H'. Since the image $u(H)$ contains the dense subset $\pi(A)\xi'$ of H', we see that u is surjective. As a surjective isometric map, u is unitary. □

Finally, we present the following result without proof, which states that the GNS representation corresponding to each state has a natural cyclic vector associated to it.

Proposition 2.2.23. ([3, Theorem 5.1.1]) *Let A be a C^*-algebra. If τ is a state on A, then the GNS representation of A associated with τ is cyclic. Furthermore, there exists a unique unit cyclic vector $\xi_\tau \in H_\tau$ such that $\pi_\tau(a)\xi_\tau = a + N_\tau$ and $\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau\rangle$ for all $a \in A$.*

Chapter 3

C*-simplicity of discrete groups

This chapter first covers some theory on operator systems and compact G-spaces. We shall then see in the final section of the chapter how these concepts can be used to study simplicity of the reduced group C*-algebra, which is a C*-algebra one can associate to any any discrete group. One of the main results we present and prove in this chapter is Theorem 3.3.8 that can be used to characterise simplicity of the reduced group C*-algebra in terms of the group action on the Furstenberg boundary. We define the reduced crossed product of G-C*-algebras, which is also a G-C*-algebra. The two final results of the chapter, Theorems 3.3.10 and 3.3.11, show that simplicity of the reduced group C*-algebra is equivalent to simplicity of a certain reduced crossed product.

3.1 Operator systems

In some situations, for example in the proof of 3.3.8, we encounter objects that are not quite C*-algebras, but share some properties. These are called operator systems, and in this section we present some basic properties of operator systems that will be of use in subsequent sections.

Consider a *-algebra A. The space $M_n(A)$ of $n \times n$ matrices with entries in A is a *-algebra with matrix multiplication and the involution $(a_{ij}) \mapsto (a_{ji})^*$. If $\varphi : A \to B$ is a *-homomorphism between *-algebras, then one checks that the *amplification*

$$\varphi_n : M_n(A) \to M_n(B), \quad (\varphi_n(a))_{ij} = \varphi_n(a_{ij})$$

is also a *-homomorphism.

Just as $M_n(\mathbf{C})$ can be identified with $B(\mathbf{C}^n)$, we can for any Hilbert space H identify $M_n(\mathcal{B}(H))$ with $B(H^{(n)})$, where $H^{(n)}$ denotes the Hilbert space direct sum $H^{(n)} = \bigoplus_{i=1}^n H$. To do this, we associate to each matrix $u \in M_n(\mathcal{B}(H))$ the operator $\varphi(u) \in B(H^{(n)})$ mapping $\xi \in H^{(n)}$ to $\nu \in H^{(n)}$ where $\nu_i = \sum_j u_{ij}(\xi_j)$. We want to show that $\varphi(u)$ is bounded. For $u' \in \mathcal{B}(H)$ let $\delta_{ij}(u')$ denote the matrix in $M_n(\mathcal{B}(H))$ with u' as the entry in place ij and all other entries 0. For $\xi' \in H$, we will also use $\delta_i(\xi') \in H^{(n)}$ to denote the element with ξ' as the i:th coordinate, and 0 as all other coordinates. That is, δ_i is the inclusion of the i:th copy of H into $H^{(n)}$. We have that $u = \sum_{ij} \delta_{ij}(u_{ij})$ and $\xi = \sum_i \delta_i(\xi_i)$, so

$$\|\varphi(u)\xi\| = \|\sum_{ij} \varphi(\delta_{ij}(u_{ij}))\xi\| \leq \sum_{ij} \|\delta_i(u_{ij}\xi_j)\| \leq \sum_{ij} \|u_{ij}\|\|\xi\|,$$

implying that $\|\varphi(u)\| \leq \sum_{ij} \|u_{ij}\|$. As is the case for the corresponding map when considering matrices over $\mathbf{C}$, the map φ is an injective *-homomorphism. If we let $p_i : H^{(n)} \to H$ denote the orthogonal projection onto the i:th copy of H, then any operator $\tilde{u} \in B(H^{(n)})$ is the image under φ of the matrix u with entries $u_{ij} = p_i \tilde{u} \delta_j$. As a composition of bounded maps we indeed have $u_{ij} \in \mathcal{B}(H)$, so φ is surjective and thus a *-isomorphism. From $u_{ij} = p_i \varphi(u) \delta_j$, we get $\|u_{ij}\| \leq \|\varphi(u)\|$. We put this together with our previous bound for $\|\varphi(u)\|$:

$$\|u_{ij}\| \leq \|\varphi(u)\| \leq \sum_{ij} \|u_{ij}\|. \tag{3.1}$$

Because $M_n(\mathcal{B}(H))$ and $B(H^{(n)})$ are isomorphic as *-algebras, the map $u \mapsto \|\varphi(u)\|$ defines a C*-norm on $M_n(\mathcal{B}(H))$, making $M_n(\mathcal{B}(H))$ a C*-algebra.

Let A be a C*-algebra. We will use the universal representation $\pi : A \to \mathcal{B}(H)$ to turn $M_n(A)$ into a C*-algebra. Let $\pi_n : M_n(A) \to M_n(\mathcal{B}(H))$ denote the amplification of π. Injectivity of π implies injectivity of π_n, so the map $a \mapsto \|\pi_n(a)\|$ defines a norm on $M_n(A)$. This norm has the property that $\|a^*a\| = \|a\|^2$ for all $a \in M_n(A)$ since $M_n(\mathcal{B}(H))$ is a C*-algebra. What we still need in order to conclude that $M_n(A)$ is a C*-algebra under the norm $\|a\| := \|\pi_n(a)\|$ is completeness. To show completeness, suppose (a_k) is a Cauchy sequence in $M_n(A)$. By Equation (3.1) each matrix entry of the sequence $(\pi_n(a_k))$ is also part of a Cauchy sequence, which is convergent by completeness of $\mathcal{B}(H)$. From completeness of A, it follows that $\pi(A)$ is closed in $\mathcal{B}(H)$, so again by Equation (3.1), the sequence $(\pi_n(a_k))$ converges to $\pi_n(a)$ for some $a \in M_n(A)$. Thus, (a_k) converges to a showing that $M_n(A)$ is complete, hence $M_n(A)$ is a C*-algebra.

Definition 3.1.1. Let H be some Hilbert space. A self-adjoint subspace V of $\mathcal{B}(H)$ containing the unit of $\mathcal{B}(H)$ is called an *operator system*.

Because the universal representation of a unital C*-algebra is unital, every unital C*-algebra can be naturally identified with an operator system.

An operator system $V \subset \mathcal{B}(H)$ inherits an order structure from $\mathcal{B}(H)$, so we may speak of positive maps between operator systems. Using the same proof as for C*-algebras, it is shown that every linear positive map between operator systems is bounded. The space $M_n(V)$ is a subspace of the C*-algebra $M_n(\mathcal{B}(H))$, so it also makes sense to speak about positive elements of $M_n(V)$. If V, W are operator systems, then a linear map $\varphi_n : V \to W$ is called *completely positive* if all amplifications φ_n, $n \geq 1$, are positive.

Proposition 3.1.2. *Any positive map $\varphi : V \to A$ between an operator system V and a commutative unital C^*-algebra A is completely positive.*

Proof. By the Gelfand theorem, we may assume $A = C(X)$ where X is a compact Hausdorff space. We also know that $f \in C(X)$ is positive if and only if $f(x) \geq 0$ for all $x \in X$.

Suppose $v \in M_n(V)$ is positive and let H be a Hilbert space such that $V \subset \mathcal{B}(H)$. Then $v = u^*u$ for some $u \in M_n(\mathcal{B}(H))$. For any $z \in \mathbf{C}^n$, we have

$$\sum_{ij} \bar{z}_i v_{ij} z_j = \sum_{ijk} (u_{ki} z_i)^* u_{kj} z_j = \sum_k \left(\sum_i u_{ki} z_i \right)^* \left(\sum_i u_{ki} z_i \right) \geq 0. \tag{3.2}$$

Let $\mathrm{ev}_{x,n} : M_n(C(X)) \to M_n(\mathbf{C})$ be the amplification of the evaluation map at $x \in X$. By considering the spectrum of matrices in $M_n(C(X))$ one checks that a matrix $F \in M_n(C(X))$ is positive if and only if $\mathrm{ev}_{x,n}(F)$ is positive in $M_n(\mathbf{C})$ for all $x \in X$. If $v \in V$ is positive, then by Equation (3.2) we have

$$\sum_{ij} \bar{z}_i (\varphi_n(v))_{ij}(x) z_j = \varphi(\sum_{ij} \bar{z}_i v_{ij} z_j)(x) \geq 0$$

for all $x \in X$ and $z \in \mathbf{C}^n$. Thus, we conclude that $\varphi_n(v) \geq 0$, so φ is completely positive. □

Proposition 3.1.3. *If $\varphi : A \to W$ is a positive unital map between a C^*-algebra A and an operator system W such that the amplification φ_2 is also positive, then $\varphi(a^*)\varphi(a) \leq \varphi(a^*a)$ for all $a \in A$.*

Proof. Introduce the matrix

$$B = \begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \in M_2(A)^+.$$

We have that

$$\varphi_2(B) = \begin{pmatrix} 1 & \varphi(a) \\ \varphi(a^*) & \varphi(a^*a) \end{pmatrix} \in M_2(W)^+.$$

Let H be the Hilbert space such that $W \subset \mathcal{B}(H)$. For all $\xi \in H$, it then holds by positivity of $\varphi_2(B)$ that

$$0 \leq \langle \begin{pmatrix} -\varphi(a)\xi \\ \xi \end{pmatrix}, \begin{pmatrix} 1 & \varphi(a) \\ \varphi(a^*) & \varphi(a^*a) \end{pmatrix} \begin{pmatrix} -\varphi(a)\xi \\ \xi \end{pmatrix} \rangle = \langle \xi, (\varphi(a^*a) - \varphi(a^*)\varphi(a))\xi \rangle.$$

Thus, $\varphi(a^*a) \geq \varphi(a^*)\varphi(a)$. □

Corollary. *If $\varphi : A \to W$ is a positive unital map between a C^*-algebra A and an operator system W such that the amplification φ_2 is also positive, then φ is contractive.*

Proof. Take $a \in A$. It holds that $\|\varphi(a^*a)\| \leq \|a^*a\|$ since $0 \leq a^*a \leq \|a^*a\|$ and φ is positive and unital. Further, $0 \leq \varphi(a)^*\varphi(a) = \varphi(a^*)\varphi(a) \leq \varphi(a^*a)$, so $\|\varphi(a)^*\varphi(a)\| \leq \|\varphi(a^*a)\|$. Thus,

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| \leq \|\varphi(a^*a)\| \leq \|a^*a\| = \|a\|^2.$$

This shows that $\|\varphi\| \leq 1$. □

Let G be a group and V an operator system. A group homomorphism $\alpha : G \to \mathrm{End}(V)$ such that α_g is a unital completely positive map for each $g \in G$ is called a G-action on V. We write $G \curvearrowright V$ to indicate that G acts on V, and say that V is a *G-operator system*. By a G-unital completely positive map between G-operator systems we mean a G-equivariant unital completely positive map. We sometimes write gx instead of $\alpha_g(x)$ for the action of g on $x \in V$.

Similarly, a C*-algebra A is said to be a G-C*-algebra if G acts on A via *-isomorphisms. Note that any G-C*-algebra is also a G-operator system because *-homomorphisms are completely positive.

Definition 3.1.4. Let V be a G-operator system. We say that V is *injective* if every G-unital completely positive map $X \to V$ can be extended to a G-unital completely positive map $Y \to V$ for any inclusion of G-operator systems $X \subset Y$.

Example 3.1.5. Let G be a discrete group. As an example of an injective operator system, consider the abelian C*-algebra $\ell^\infty(G)$ on which G acts via $(gf)(x) = f(g^{-1}x)$ for all $g, x \in G$ and $f \in \ell^\infty(G)$. An element of $\ell^\infty(G)$ is positive if and only if its range is nonnegative. Thus, G acts via positive maps. In fact, by Proposition 3.1.2, G acts via completely positive maps. It is clear that G acts via unital maps, so $\ell^\infty(G)$ is a G-operator system.

Let $V \subset W$ be an inclusion of G-operator systems and let $\varphi : V \to \ell^\infty(G)$ be a G-unital completely positive map. Let $\tau_e : V \to \mathbf{C}$ be the map given by the composition $\tau_e = \mathrm{ev}_e \circ \varphi$. For $g \in G$, let $\tau_g : V \to \mathbf{C}$ be the map such that $\tau_g(x) = \tau_e(g^{-1}x)$. Note that τ_g is positive for each $g \in G$ because φ is positive and because G acts via positive maps. Because φ is G-equivariant, we have $\tau_g = \mathrm{ev}_g \circ \varphi$.

We show that $\|\tau_e\| = 1$. As a positive map, $\|\tau_e\| < \infty$. Take $\varepsilon > 0$ and $x \in V$ such that $\|x\| \leq 1$ and $\|\tau_e\| - \varepsilon < |\tau_e(x)|$. If $\theta \in [0, 2\pi)$ is such that $\tau_e(x) = |\tau_e(x)|e^{i\theta}$, then $\|\tau_e\| - \varepsilon < \tau_e(xe^{-i\theta})$. Introduce the self-adjoint element $y = (xe^{-i\theta} + x^*e^{i\theta})/2$, which also satisfies $\|y\| \leq 1$. As a positive map, τ_e preserves adjoints and so $\tau_e(y) = \tau_e(x)$. From $y \leq \|y\|$ it follows that $\tau_e(y) \leq \|y\|\tau_e(1) = \|y\|$. It follows that $\|\tau_e\| - \varepsilon < \|y\| \leq 1$. Taking $\varepsilon \to 0+$, we get $\|\tau_e\| \leq 1$. Since τ_e is unital it follows that $\|\tau_e\| = 1$.

Let H be a Hilbert space such that $V \subset W \subset \mathcal{B}(H)$. Using the Hahn-Banach theorem, we can extend extend τ_e to linear functional $\bar{\tau}_e : \mathcal{B}(H) \to \mathbf{C}$ such that $\|\bar{\tau}_e\| = \|\tau_e\| = 1$. According to [3, Corollary 3.3.4] a bounded unital linear functional on a unital C*-algebra is positive if and only if it has norm equal to 1. As a consequence of this result, $\bar{\tau}_e$ is positive. The positive map

$\bar{\tau}_g : W \to \mathbf{C}$ given by $\bar{\tau}_g(x) = \bar{\tau}_e(g^{-1}x)$ extend τ_g. We define $\bar{\varphi} : W \to \ell^\infty(G)$ as the map such that $\mathrm{ev}_g(\bar{\varphi}(x)) = \bar{\tau}_g(x)$ for all $g \in G$ and $x \in W$. The map $\bar{\varphi}$ is positive because each map $\bar{\tau}_g$ is positive, and by Proposition 3.1.2 $\bar{\varphi}$ is completely positive. Since $\tau_g = \mathrm{ev}_g \circ \varphi$, we have that $\bar{\varphi}$ extends φ, hence $\bar{\varphi}$ is unital. Finally, we have

$$(\tilde{\varphi}(gx))(h) = \tau_h(gx) = \tau_{g^{-1}h}(x) = (\tilde{\varphi}(x))(g^{-1}h) = (g\tilde{\varphi}(x))(h)$$

showing that $\bar{\varphi}$ is G-equivariant. Thus, $\bar{\varphi}$ is a G-unital completely positive map extending φ.

Definition 3.1.6. An inclusion $V \subset W$ of G-operator systems is called *essential* if every G-unital completely positive map $W \to X$ between G-operator systems that is completely isometric on V is completely isometric on W.

Definition 3.1.7. An inclusion $V \subset W$ of G-operator systems is called *rigid* if every G-unital completely positive map $W \to W$ that is equal to the identity on V is the identity map on W.

Let V be a G-operator system. An *injective envelope* of V is an essential inclusion $V \subset W$ of G-operator systems, with W injective. By [12, Theorem 4.1], an injective envelope of V always exists. We will now show that it is unique.

Proposition 3.1.8. *If V is a G-operator system and $V \subset W_1$ and $V \subset W_2$ are essential inclusions with W_1 and W_2 injective, then there exist G-unital completely positive maps $\varphi_1 : W_2 \to W_1$ and $\varphi_2 : W_1 \to W_2$ such that $\varphi_1 \circ \varphi_2 = \mathrm{id}_{W_1}$ and $\varphi_2 \circ \varphi_1 = \mathrm{id}_{W_2}$.*

Proof. By [12, Lemma 3.7], if the G-operator system W is injective, the inclusion $V \subset W$ is essential if and only if it is rigid. Let $V \subset W_1$ and $V \subset W_2$ be essential inclusions of G-operator systems, where W_1 and W_2 are injective. Then the inclusions are rigid. By injectivity of W_1, the inclusion map $V \hookrightarrow W_1$ can be extended to G-unital completely positive map $\varphi_1 : W_2 \to W_1$. Similarly, we get a G-unital completely positive map $\varphi_2 : W_1 \to W_2$. By rigidity of W_1, the map $\varphi_2 \circ \varphi_1$ is the identity map on W_1. Similarly, the map $\varphi_1 \circ \varphi_2$ is the identity map on W_2. Thus, W_1 and W_2 are isomorphic. □

Now that we know that the injective envelope of a G-operator system V is unique, we will use $\mathrm{I}_G(V)$ to denote the injective envelope of V.

3.2 The Furstenberg and Hamana boundaries

In this section we introduce G-boundaries. We are going to look at the Furstenberg boundary and relate properties of the Furstenberg boundary as a G-boundary to properties of the G-operator system of continuous functions on the Furstenberg boundary. Specifically, we will see that the Furstenberg boundary is isomorphic to the Hamana boundary, which is also introduced in this section. These properties are to be used for studying C*-simplicity in the final section of the chapter.

Definition 3.2.1. Let G be a discrete group and let X be a compact Hausdorff space. We say that X is a *compact G-space* if G acts on X via homeomorphisms. A compact G-space X is said to be *minimal* if $\overline{Gx} = X$ for every $x \in X$.

Proposition 3.2.2. *A compact G-space X is minimal if and only if there are no proper nonempty closed subsets $A \subset X$ such that $GA \subset A$.*

Proof. Assume that there are no proper nonempty closed subsets $A \subset X$ such that $GA \subset A$. Take $x \in X$ and consider the closed subset $\overline{Gx}$. Since $g(Gx) = Gx$ for all $g \in G$ and because G acts by homeomorphisms, we have $g(\overline{Gx}) = \overline{Gx}$. Hence,

$$G(\overline{Gx}) = \bigcup_{g \in G} g(\overline{Gx}) = \overline{Gx}.$$

By the assumption made at the start, we must have $\overline{Gx} = X$, so X is minimal.

Conversely, assume that there is a nonempty closed proper subset $A \subset X$ such that $GA \subset A$. Take an element $x \in A$. Then $Gx \subset A$, so $\overline{Gx} \subset A \neq X$, showing that X is not minimal. □

Let G be a discrete group and suppose X is a compact G-space. Consider the C*-algebra $C(X)$. We define a G-action on $C(X)$ by setting $(gf)(x) = f(g^{-1}x)$ for all $f \in C(X)$, $g \in G$ and $x \in X$. This way, G acts on $C(X)$ via *-isomorphisms, making $C(X)$ a G-C*-algebra. Now consider the state space $\mathcal{S}(C(X)) \subset C(X)^*$ as usual equipped with the weak *-topology. Note that $\mathcal{S}(C(X))$ is compact by the Banach-Alaoglu theorem. We define a G-action on $\mathcal{S}(C(X))$ by setting $(g\tau)(f) = \tau(g^{-1}f)$ for all $f \in C(X)$, $g \in G$ and $\tau \in \mathcal{S}(C(X))$. To see that the map $\tau \to g\tau$ is continuous, let (τ_λ) be a net converging to τ in $\mathcal{S}(C(X))$. Then

$$(g\tau)(f) = \tau(g^{-1}f) = \lim \tau_\lambda(g^{-1}f) = \lim (g\tau_\lambda)(f)$$

for all $f \in C(X)$. Thus, $\mathcal{S}(C(X))$ is indeed a compact G-space.

Let $\mathrm{ev} : X \to \mathcal{S}(C(X))$ be the map taking $x \in X$ to the evaluation map ev_x on $C(X)$. By Urysohn's lemma, the map ev is injective. To show that it is continuous, let (x_λ) be a net converging to x in X. For $f \in C(X)$, we have

$$\mathrm{ev}_x(f) = f(x) = f(\lim x_\lambda) = \lim f(x_\lambda) = \lim \mathrm{ev}_{x_\lambda}(f),$$

which by the definition of the weak*-topology shows that $\mathrm{ev} : X \to \mathcal{S}(C(X))$ is continuous. Because, X is compact and $\mathcal{S}(C(X))$ is Hausdorff, the map ev is a homeomorphism onto its image. Note also that

$$\mathrm{ev}_{gx}(f) = f(gx) = (g^{-1}f)(x) = \mathrm{ev}_x(g^{-1}f) = (g\mathrm{ev}_x)(f).$$

Thus, the G-space X can be identified with its image $\mathrm{ev}(X)$ in the G-space $\mathcal{S}(C(X))$.

By a probability measure on X we mean a regular Borel measure μ on X that is positive and satisfies $\mu(X) = 1$. The set of probability measures is denoted

$\mathcal{P}(X)$. To each $\mu \in \mathcal{P}(X)$ we associate a state on $C(X)$ defined by $f \mapsto \int_X f d\mu$. By the Riesz representation theorem, this gives a bijective correspondence between probability measures on X and states on $C(X)$. Using this bijective correspondence, $\mathcal{P}(X)$ inherits the weak *-topology from $\mathcal{S}(C(X))$. If $g \in G$ and $\mu \in \mathcal{P}(X)$, then $g\mu$ is the probability measure defined by $(g\mu)(U) = \mu(g^{-1}U)$. Suppose $\tau \in \mathcal{S}(C(X))$ and $\mu \in P(X)$ are such that $\tau(f) = \int_X f d\mu$, then $(g\tau)(f) = \int_X f d(g\mu)$, so we may consider $\mathcal{S}(C(X))$ and $\mathcal{P}(X)$ the same as G-spaces. Furthermore, X which we identify with $\mathrm{ev}(X)$ in $\mathcal{S}(C(X))$, corresponds to the Dirac measures in $\mathcal{P}(X)$. In some situations it will be easier for us to work with $\mathcal{P}(X)$ instead of $\mathcal{S}(C(X))$.

Proposition 3.2.3. *If X is a compact Hausdorff space, then the Dirac measures are the extreme points of $\mathcal{P}(X)$. In other words, the set of extreme points in $\mathcal{P}(X)$ is X.*

Proof. Let δ_x be given, and assume that $\frac{1}{2}\mu + \frac{1}{2}\nu = \delta_x$ for some $\mu, \nu \in \mathcal{P}(X)$. Applying both sides of the equation to $\{x\}$, we get $\mu(\{x\}) = \nu(\{x\}) = 1$, which implies that $\mu = \nu = \delta_x$. This shows that the Dirac measures are extreme.

Conversely, let $\mu \in \mathcal{P}(X)$ be extreme. Assume that there is a Borel set $U \subset X$ such that U and U^c have non-zero measure, and introduce the probability measures μ_1 and μ_2 on X defined by

$$\mu_1(A) = \mu(A \cap U)/\mu(U)$$

and

$$\mu_2(A) = \mu(A \cap U^c)/\mu(U^c)$$

for all Borel sets A. Then

$$\mu = \mu(U)\mu_1 + \mu(U^c)\mu_2.$$

Since μ is an extreme point, $\mu = \mu_1 = \mu_2$. This gives a contradiction since $\mu_1(U) \neq \mu_2(U)$. We conclude that for any Borel set U, either $\mu(U) = 1$ or $\mu(U) = 0$.

We use a contradiction argument to show that X has a point for which all neighbourhoods have measure 1. Suppose every point $x \in X$ has an open neighbourhood U_x of measure 0. By compactness, there are points $x_1, \ldots, x_n \in X$ such that $X = U_{x_1} \cup \cdots \cup U_{x_n}$, which contradicts $\mu(X) = 1$. Therefore, there indeed exists a point x such that $\mu(U) = 1$ for all open sets $U \subset X$ containing x. By regularity of the measure, $\mu(\{x\}) = 1$. If a Borel set $U \subset X$ does not contain x, then $\mu(U^c) = 1$, so $\mu(U) = 0$. This shows that $\mu = \delta_x$. $\square$

Definition 3.2.4. Let G be a discrete group and let X be a compact G-space. We say that X is *strongly proximal* if $\overline{G\mu} \cap X \neq \emptyset$ for all $\mu \in \mathcal{P}(X)$.

Proposition 3.2.5. *A compact G-space is strongly proximal if and only if for all $\mu, \nu \in \mathcal{P}(X)$, there exists a net (g_λ) in G such that* $\lim g_\lambda \mu = \lim g_\lambda \nu$.

Proof. Assume that for each $\mu, \nu \in \mathcal{P}(X)$, there exists a net (g_λ) in G such that $\lim g_\lambda \mu = \lim g_\lambda \nu$. Let $\mu \in \mathcal{P}(X)$ be given. Take $\delta_x \in X$, then there there exists a net (g_λ) in G such that $\lim g_\lambda \mu = \lim \delta_{g_\lambda x}$. Since X is closed, there is some $\delta_y \in X$ such that $\lim g_\lambda \mu = \lim \delta_{g_\lambda x} = \delta_y$, showing that $\overline{G\mu} \cap X \neq \emptyset$.

Conversely, assume that X is strongly proximal and let $\mu, \nu \in \mathcal{P}(X)$ be given. Then there is a net (g_λ) in G such that

$$\lim g_\lambda (\frac{1}{2}\mu + \frac{1}{2}\nu) = \delta_x$$

for some $\delta_x \in \mathcal{P}(X)$. Passing to convergent subnets,

$$\lim g_\lambda \frac{1}{2}\mu + \lim g_\lambda \frac{1}{2}\nu = \delta_x.$$

Since the Dirac measures are extreme points, it follows that $\lim g_\lambda \mu = \lim g_\lambda \nu = \delta_x$. $\square$

Definition 3.2.6. A compact G-space X is called a G-*boundary* if it is minimal and strongly proximal. If X is a G-boundary and if for every other G-boundary Y there is a G-equivariant continuous map $X \to Y$, then X is called a *Furstenberg boundary.*

The existence of a Furstenberg boundary $\partial_F G$ associated to a discrete group G is shown in [13, p.32].

Proposition 3.2.7. *Let $\varphi : Y \to \mathcal{P}(X)$ be a G-equivariant continuous map from a compact G-space to the space of probability measures on a strongly proximal G-space X. Then $\varphi(Y) \cap X \neq \emptyset$. Further, if Y is minimal, then $\varphi(Y) \subset X$ and there are no other G-equivariant continuous maps $Y \to \mathcal{P}(X)$. If X is minimal, then $X \subset \varphi(Y)$.*

Proof. Take a point $y \in Y$. We have that $\varphi(Gy) = G\varphi(y)$, and so $\overline{\varphi(Gy)} \cap X \neq \emptyset$. Because Y is compact and $\mathcal{P}(X)$ is Hausdorff, $\varphi(Y)$ is closed, so $\overline{\varphi(Gy)} \subset \varphi(Y)$. We conclude that $\varphi(Y)$ intersects X.

Suppose now that Y is minimal. We know now that there exists some $y \in Y$ such that $\varphi(y) = \delta_x$ for some $x \in X$. Since Y is minimal, we have $\overline{Gy} = Y$. As a continuous map from a compact space to a Hausdorff space, φ is closed. Thus, $\varphi(\overline{Gy}) = \overline{\varphi(Gy)} = \overline{G\delta_x}$. Since X is a compact G-space, $\varphi(Y) = \overline{G\delta_x} \subset X$. If $\varphi' : Y \to \mathcal{P}(X)$ is a G-equivariant continuous map, then so is the map $\Phi : Y \to \mathcal{P}(X)$ defined by $\Phi(y) = (\delta_{\varphi(y)} + \delta_{\varphi'(y)})/2$. The image of Φ must also be contained X. Noting that the points of X are extreme in $\mathcal{P}(X)$, we see that $\delta_{\varphi(y)} = \delta_{\varphi'(y)}$ for all $y \in Y$, so $\varphi = \varphi'$.

If X is minimal, then $X = \overline{G\delta_x} = \overline{\varphi(Gy)} \subset \overline{\varphi(Y)} = \varphi(Y)$. $\square$

Proposition 3.2.8. *If X and Y are Furstenberg boundaries, then they are isomorphic.*

Proof. From the definition of a Furstenberg boundary, we get G-equivariant continuous maps $\varphi_1 : X \to Y$ and $\varphi_2 : Y \to X$. Applying the uniqueness result in Proposition 3.2.7, we conclude that $\varphi_2 \circ \varphi_1 = \mathrm{id}_X$ and $\varphi_1 \circ \varphi_2 = \mathrm{id}_Y$, showing that X and Y are isomorphic. □

Proposition 3.2.9. *The inclusion $\mathbf{C} \subset C(\partial_F G)$ of G-operator systems is rigid.*

Proof. Let $\varphi : C(\partial_F G) \to C(\partial_F G)$ be a G-unital completely positive map. By Proposition 3.2.12, the map $\varphi' : \partial_F G \to \mathcal{S}(C(\partial_F G))$, $\varphi'(x) = \mathrm{ev}_x \circ \varphi$ is continuous and G-equivariant. Since $\mathcal{S}(C(\partial_F G)) = \mathcal{P}(\partial_F G)$, we can use Proposition 3.2.7 to see that φ' is the identity. This means that $f(x) = \varphi(f)(x)$ for all $x \in \partial_F G$ and $f \in C(\partial_F G)$, so φ is the identity. □

Suppose X is a compact G-space. We say that $G \curvearrowright \mathcal{P}(X)$ is *irreducible* if the only closed convex G-invariant subsets of $\mathcal{P}(X)$ are the empty set and $\mathcal{P}(X)$.

Let V be a G-operator system. Just as we made $\mathcal{S}(C(X))$ a compact G-space for a compact G-space X, we make $\mathcal{S}(V)$ a compact G-space by defining $(g\tau)(x) = \tau(g^{-1}x)$ for all $g \in G$, $\tau \in \mathcal{S}(V)$, and $x \in V$. We say that $G \curvearrowright \mathcal{S}(V)$ is *irreducible* if the only closed convex G-invariant subsets of $\mathcal{S}(V)$ are the empty set and $\mathcal{S}(V)$. Note that if we view $C(X)$ as G-operator system, our definitions of irreducibility of $\mathcal{P}(X) \simeq \mathcal{S}(C(X))$ agree, so there is no issue with ambiguity.

We state without proof the following two useful results relating boundaries and irreducible G-actions, and essentiality of inclusions and irreducible G-actions, respectively.

Proposition 3.2.10. ([14, Proposition 23]) *A compact G-space X is a boundary if and only if $G \curvearrowright \mathcal{P}(X)$ is irreducible.*

Proposition 3.2.11. ([14, Proposition 16]) *Given that $G \curvearrowright \mathcal{S}(V)$ is irreducible, the inclusion $V \subset W$ of G-operator systems is essential if and only if $G \curvearrowright \mathcal{S}(W)$ is irreducible.*

Consider the G-operator system $\mathbf{C}$ on which G acts trivially. We know that the injective envelope $\mathrm{I}_G(\mathbf{C})$ is an injective G-operator system such that the inclusion $\mathbf{C} \subset \mathrm{I}_G(\mathbf{C})$ of G-operator systems is essential. By a result due to Choi and Effros [15, Theorem 3.1] one can define on any injective G-operator system a product, called the Choi-Effros product, making it a C*-algebra. If the operators system in question is included into an abelian C*-algebra, then the operator system with the Choi-Effros product is an abelian G-C*-algebra. Henceforth, we consider the injective envelope $\mathrm{I}_G(\mathbf{C})$ as a C*-algebra with the Choi-Effros product. In the construction of the injective envelope, the injective envelope is included in an abelian C*-algebra, so $\mathrm{I}_G(\mathbf{C})$ is an abelian C*-algebra [12]. In particular, the spectrum $\Omega(\mathrm{I}_G(\mathbf{C}))$ is a compact G-space, where the G-action on $\Omega(\mathrm{I}_G(\mathbf{C}))$ is inherited from $\mathcal{S}(\mathrm{I}_G(\mathbf{C}))$. Note that the inclusion $\Omega(\mathrm{I}_G(\mathbf{C})) \subset \mathcal{S}(\mathrm{I}_G(\mathbf{C}))$ holds since characters are unital *-homomorphisms into $\mathbf{C}$. The compact G-space $\Omega(\mathrm{I}_G(\mathbf{C}))$ is called the Hamana boundary and is denoted $\partial_H G$. By the Gelfand theorem, we have that $C(\Omega(\mathrm{I}_G(\mathbf{C})))$ is isomorphic as a C*-algebra to $\mathrm{I}_G(\mathbf{C})$. If $a \in \mathrm{I}_G(\mathbf{C})$ and $\tau \in \partial_H G$, then

$$(ga)^\wedge(\tau) = \tau(ga) = \hat{a}(g^{-1}\tau) = g\hat{a}.$$

showing that the Gelfand representation is G-equivariant. Thus, $C(\partial_H G)$ is isomorphic as a G-C*-algebra to $\mathrm{I}_G(\mathbf{C})$. In particular, $C(\partial_H G)$ is a G-injective operator system and the inclusion $\mathbf{C} \subset C(\partial_H G)$ is essential.

We are almost ready to show the main result of this chapter: that the Hamana and Furstenberg boundaries are isomorphic. The following proposition is the final ingredient needed for the proof.

Proposition 3.2.12. ([14, Proposition 7]) *For a G-operator system V and a compact G-space X, we associate to each map $\varphi : V \to C(X)$ a map $x \mapsto ev_x \circ \varphi$. This association gives a bijection between G-unital completely positive maps $V \to C(X)$ and G-equivariant maps $X \to \mathcal{S}(V)$.*

Theorem 3.2.13. *The Furstenberg boundary is isomorphic to the Hamana boundary as compact G-spaces.*

Proof. By the definition of the injective envelope, the inclusion $\mathbf{C} \subset \mathrm{I}_G(\mathbf{C})$ is essential. Since $\mathrm{I}_G(\mathbf{C}) = C(\Omega(\mathrm{I}_G(\mathbf{C}))) = C(\partial_H G)$, the inclusion $\mathbf{C} \subset C(\partial_H G)$ is essential. The only state on $\mathbf{C}$ is the identity, so in particular the action $G \curvearrowright \mathcal{S}(\mathbf{C})$ is irreducible. By Proposition 3.2.11, the action $G \curvearrowright \mathcal{S}(C(\partial_H G))$ is irreducible. Since $\mathcal{S}(C(\partial_H G)) = \mathcal{P}(\partial_H G)$ it follows from Proposition 3.2.10 that $\partial_H G$ is a G-boundary. From the definition of the Furstenberg boundary we get a continuous G-equivariant map $\varphi : \partial_F G \to \partial_H G$. It is straightforward to check that the map $\varphi' : C(\partial_H G) \to C(\partial_F G)$, $\varphi'(f) = f \circ \varphi$ is a unital *-homomorphism.

Now we will find an inverse to φ'. Since $C(\partial_H G)$ is an injective G-operator system, the inclusion map $\mathbf{C} \hookrightarrow C(\partial_H G)$ can be extended to a G-unital completely positive map $\psi : C(\partial_F G) \to C(\partial_H G)$. By Proposition 3.2.12, the map $\psi' : \partial_H G \to \mathcal{S}(C(\partial_F G))$ defined by $\psi'(x) = \mathrm{ev}_x \circ \psi$ is continuous and G-equivariant. Using that $\partial_H G$ is a boundary and that $\mathcal{S}(C(\partial_F G)) = \mathcal{P}(\partial_F G)$, Proposition 3.2.7 tells us that $\psi'(\partial_H G) \subset \partial_F G$. In particular, there exists for each $x \in \partial_H G$ a point $y \in \partial_F G$ such that $\psi'(x) = \mathrm{ev}_y$. This implies that ψ is a *-homomorphism. To see this, note that $\psi(f_1 f_2)(x) = \psi'(f_1 f_2) = f_1(y) f_2(y)$ and $\psi(f_1^*)(x) = f_1^*(y) = \psi(f_1)^*(x)$ for $f_1, f_2 \in C(\partial_F G)$. The inclusion $\mathbf{C} \subset C(\partial_H G)$ is rigid because $\mathbf{C} \subset C(\partial_H G)$ is essential and $C(\partial_H G)$ injective. We proved rigidity of $\mathbf{C} \subset \partial_F G$ in Proposition 3.2.9. Because ψ and φ' and are unital *-homomorphism, rigidity implies that ψ and φ' are inverses to each other, showing that ψ is a G-equivariant *-isomorphism. Then the map $\mathcal{S}(C(\partial_H G)) \to \mathcal{S}(C(\partial_F G)), \tau \mapsto \tau \circ \psi$ is a G-equivariant affine homeomorphism. Since an affine isomorphism gives a one-to-one correspondence between extreme points, the map restricts to a G-equivariant homeomorphism between $C(\partial_H G)$ and $C(\partial_F G)$, which are the extreme points in $\mathcal{S}(C(\partial_H G))$ and $\mathcal{S}(C(\partial_F G))$, respectively. ∎

3.3 The reduced group C*-algebra

At this point we have gathered the prerequisites on C*-algebras and operator systems needed to begin our study of C*-simplicity of discrete groups. For example, we prove a part of Theorem 3.1 in [1] giving a dynamic characterisation of C*-simplicity. We begin the section with constructing the reduced group C*-algebra.

Let G be a discrete group. Consider the vector space $\mathbf{C}G$, that per definition has a basis consisting of the elements of G. We denote by u_g the basis vector in $\mathbf{C}G$ associated to $g \in G$. By extending linearly, we get a multiplication operation on $\mathbf{C}G$ defined by $u_g u_h = u_{gh}$ for all $g, h \in G$. By extending conjugate linearly, we get an involution on $\mathbf{C}G$ defined by $u_g^* = u_{g^{-1}}$ for all $g \in G$. A unit for $\mathbf{C}G$ is u_e. Thus, we have a unital *-algebra naturally associated to G. We will now see how to construct a unital C*-algebra containing $\mathbf{C}G$ that is naturally associated to G.

Equipped with the counting measure, G is a measure space. For $x \in G$, consider the unitary operator

$$\lambda_x : \ell^2(G) \to \ell^2(G), \quad \lambda_x(f)(g) = f(x^{-1}g).$$

We extend the map $x \mapsto \lambda_x$ linearly to $\mathbf{C}G$. This map is unital because λ_e is the identity on $\ell^2(G)$. Note that

$$\lambda_{xy} f(g) = f(y^{-1}x^{-1}g) = \lambda_x \lambda_y f(g),$$

so λ is a homomorphism. Simple calculations involving the inner product on $\ell^2(G)$ show that $\lambda_x^* = \lambda_x^{-1} = \lambda_{x^{-1}}$ for all $x \in G$. Because $u_x^* = u_{x^{-1}}$, it follows that λ is a *-homomorphism. For $g \in G$, let $\delta_g \in \ell^2(G)$ be the function such that $\delta_g(g) = 1$ and $\delta_g(h) = 0$ if $h \neq g$. Given a linear combination $u = \sum c_i u_{x_i}$ of the vectors $u_{x_1}, \ldots, u_{x_n}$ in $\mathbf{C}G$ where $i \neq j \implies x_i \neq x_j$, such that $\lambda(u) = 0$, it holds that

$$0 = \lambda(u)\delta_{x_i^{-1}}(e) = c_i$$

for all $1 \leq i \leq n$, which implies that $u = 0$. This shows that λ is an injective *-homomorphism, so we may identify $\mathbf{C}G$ with its image in the C*-algebra $B(\ell^2(G))$. The norm closure of $\lambda(\mathbf{C}G)$ in $B(\ell^2(G))$ is called the reduced group C*-algebra, and is denoted $\mathrm{C}_r^*(G)$. For $u \in \mathrm{C}_r^*(G)$ and $g \in G$, we define a G-action on $\mathrm{C}_r^*(G)$ via $gu = \lambda_g u \lambda_g^*$. One checks for example that $gu^* = (\lambda_g u \lambda_g^*)^*$, and further that G acts via *-isomorphisms. Thus, $\mathrm{C}_r^*(G)$ provides us with a unital G-C*-algebra naturally associated with G.

Example 3.3.1. Consider the discrete group $\mathbf{Z}$. By definition, $\mathbf{CZ}$ acts on $\ell^2(\mathbf{Z})$ via the left regular representation: $\lambda_x(f)(t) = f(t-x)$ for $f \in \ell^2(\mathbf{Z})$ and $x, t \in \mathbf{Z}$. Via the Fourier transform $\mathcal{F} : \ell^2(\mathbf{Z}) \to L^2(S^1)$ satisfying $\mathcal{F}(\delta_n) = e^{2\pi i n t}$ for $n \in \mathbf{Z}$, the Hilbert space $\ell^2(\mathbf{Z})$ is isomorphic as a Hilbert space to $L^2(S^1)$. We denote the corresponding action of $\mathbf{CZ}$ on $L^2(S^1)$ by $\bar{\lambda}_x : L^2(S^1) \to L^2(S^1)$. We calculate that

$$\bar{\lambda}_x(e^{2\pi i n t}) = \mathcal{F}\lambda_x\mathcal{F}^{-1}(e^{2\pi i n t}) = \mathcal{F}(\delta_{n+x}) = e^{2\pi i x t}e^{2\pi i n t}.$$

Writing z for the coordinate function on S^1, it follows that $\bar{\lambda}(\mathbf{C}\mathbf{Z})$ is equal to the set of multiplication operators $m_f \in B(L^2(S^1))$, where f is a polynomial function in z and z^* on S^1.

Let $m : C(S^1) \to B(L^2(S^1))$ be the map taking each $f \in C(S^1)$ to the corresponding multiplication operator in $B(L^2(S^1))$. As is shown in Example 2.1.24, the map m isometrically embeds $C(S^1)$ in $B(L^2(S^1))$. Thus, we can identify $\bar{\lambda}(\mathbf{C}\mathbf{Z})$ with the polynomial functions in z and z^* on S^1, and $\mathrm{C}^*_r(\mathbf{Z})$ is the closure of $\mathbf{C}[z, z^*]$ in $C(S^1)$. By the Stone-Weierstrass theorem, $\mathbf{C}[z, z^*]$ is dense in $C(S^1)$. Therefore, $\mathrm{C}^*_r(\mathbf{Z}) = C(S^1)$.

Definition 3.3.2. Let G be a discrete group. The *canonical tracial state* is the linear map $\tau : \mathrm{C}^*_r(G) \to \mathbf{C}$ defined by $x \mapsto \langle x\delta_e, \delta_e\rangle$.

Let G be a discrete group and consider the canonical tracial state τ on $\mathrm{C}^*_r(G)$. From Cauchy-Schwarz we get the bound

$$|\langle x\delta_e, \delta_e\rangle| \leq \|x\delta_e\|\|\delta_e\| \leq \|x\|.$$

Together with $\tau(1) = 1$, this shows that $\|\tau\| = 1$. Further,

$$\tau(x^*x) = \langle x\delta_e, x\delta_e\rangle \geq 0,$$

so τ is positive. Thus, τ is indeed a state on $\mathrm{C}^*_r(G)$. If $\tau(x^*x) = 0$, then $x\delta_e = 0$. By definition of $\mathrm{C}^*_r(G)$, the element x is in the closed linear span of $\{\lambda_g : g \in G\}$. For $h \in G$, introduce the operator

$$\rho_h : \ell^2(G) \to \ell^2(G), \quad f(g) \mapsto f(gh).$$

Because $\rho_h(\delta_g) = \delta_{gh^{-1}}$, we have that ρ_h commutes with λ_g for all $g, h \in G$, so ρ_h commutes with x. Thus, $0 = \rho_h x\delta_e = x\delta_{h^{-1}}$ for all $h \in G$. This shows that $x = 0$, meaning that τ is faithful. Further, τ is *tracial* in the sense that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathrm{C}^*_r(G)$ since

$$\tau(\lambda_g\lambda_h) = \langle \lambda_g\lambda_h\delta_e, \delta_e\rangle = \begin{cases} 1, & gh = e, \\ 0, & gh \neq e, \end{cases}$$

is equal to $\tau(\lambda_h\lambda_g)$ for all $g, h \in G$.

Let A be a G-C*-algebra, assumed for the moment to be concrete. That is, $A \subset \mathcal{B}(H)$ for some Hilbert space H. Of course, using the universal representation any C*-algebra can be identified with a concrete C*-algebra, meaning this is not really a restriction. Our goal now is to define the *reduced crossed product* of A and the discrete group G. The reduced crossed product of A and G is a G-C*-algebra containing both A and $\mathrm{C}^*_r(G)$, which we will use to characterise simplicity of $\mathrm{C}^*_r(G)$.

By definition, the algebraic tensor product $H \otimes_{alg} \ell^2(G)$ is dense in the Hilbert space tensor product $H \otimes \ell^2(G)$. Let $\sum \xi_i \otimes f_i$ be an arbitrary element in $H \otimes_{alg} \ell^2(G)$. By approximating each f_i with functions in $C_c(G)$ it follows that $H \otimes_{alg} C_c(G)$ is dense in $H \otimes_{alg} \ell^2(G)$ and thus in $H \otimes \ell^2(G)$. If $f \in C_c(G)$, then f is a linear combination of the functions δ_g, $g \in G$. Therefore, any element in $H \otimes_{alg} C_c(G)$ can be written as a finite sum

$$\sum \xi_g \otimes \delta_g,$$

and the elements of this form are dense in $H \otimes \ell^2(G)$. Note that every element $H \otimes_{alg} C_c(G)$ can be written uniquely in this form by linear independence. For $a \in A$, consider the linear map

$$\Delta(a) : H \otimes_{alg} C_c(G) \to H \otimes_{alg} C_c(G)$$

such that $\Delta(a)(\xi \otimes \delta_g) = (g^{-1}a)\xi \otimes \delta_g$. Since

$$\begin{aligned}
\|\Delta(a) \sum \xi_g \otimes \delta_g\|^2 &= \| \sum (g^{-1}a)\xi_g \otimes \delta_g\|^2 \\
&= \sum \|(g^{-1}a)\xi_g\|^2 \\
&\leq \sum \|g^{-1}a\|^2 \|\xi_g\|^2 \\
&= \|a\|^2 \sum \|\xi_g\|^2 \\
&= \|a\|^2 \| \sum \xi_g \otimes \delta_g\|^2,
\end{aligned}$$

we see that $\Delta(a)$ is bounded on $H \otimes_{alg} C_c(G)$. Because $H \otimes_{alg} C_c(G)$ is dense in $H \otimes \ell^2(G)$ and because $H \otimes \ell^2(G)$ is complete, we can extend $\Delta(a)$ to a bounded map $H \otimes \ell^2(G) \to H \otimes \ell^2(G)$. We now show that $\Delta : A \to B(H \otimes \ell^2(G))$ is a *-homomorphism. We have

$$\begin{aligned}
\Delta(ab)(\xi \otimes \delta_g) &= (g^{-1}(ab))\xi \otimes \delta_g \\
&= (g^{-1}a)(g^{-1}b)\xi \otimes \delta_g \\
&= \Delta(a)((g^{-1}b)\xi \otimes \delta_g) \\
&= \Delta(a)\Delta(b)(\xi \otimes \delta_g),
\end{aligned}$$

and linearity is shown in a similar way. Further,

$$\begin{aligned}
\langle \Delta(a)(\xi \otimes \delta_g), \xi' \otimes \delta_{g'} \rangle &= \langle (g^{-1}a)\xi \otimes \delta_g, \xi' \otimes \delta_{g'} \rangle \\
&= \langle (g^{-1}a)\xi, \xi' \rangle \langle \delta_g, \delta_{g'} \rangle \\
&= \langle \xi, (g^{-1}a)^*\xi' \rangle \langle \delta_g, \delta_{g'} \rangle \\
&= \langle \xi, (g^{-1}a^*)\xi' \rangle \langle \delta_g, \delta_{g'} \rangle \\
&= \langle \xi \otimes \delta_g, \Delta(a^*)(\xi' \otimes \delta_{g'}) \rangle,
\end{aligned}$$

so it follows from continuity that Δ is a *-homomorphism. If $\Delta(a) = 0$ for some $a \in A$, then

$$0 = \Delta(a)(\xi \otimes \delta_e) = a\xi \otimes \delta_e \implies 0 = a\xi,$$

for all $\xi \in H$, implying that $a = 0$. This shows that Δ is injective. As an injective *-homomorphism, Δ is isometric. Also observe that Δ is unital if A is unital. Thus, we may identify A with its image $\Delta(A)$ in $B(H \otimes \ell^2(G))$.

Via the inclusion $i : B(\ell^2(G)) \to B(H \otimes \ell^2(G))$, $u \mapsto \mathrm{id}_H \otimes u$, the C*-algebra $B(\ell^2(G))$ is contained in $B(H \otimes \ell^2(G))$. In particular, the left regular representation $\lambda : \mathbf{C}G \to B(H \otimes \ell^2(G))$ is a unital injective *-homomorphism and $B(H \otimes \ell^2(G))$ contains $\mathrm{C}^*_r(G)$. We calculate

$$\begin{aligned}\lambda_x \Delta(a) \lambda_x^* (\xi \otimes \delta_g) &= \lambda_x \Delta(a)(\xi \otimes \delta_{x^{-1}g}) \\ &= \lambda_x((g^{-1}xa)\xi \otimes \delta_{x^{-1}g}) \\ &= (g^{-1}xa)\xi \otimes \delta_g \\ &= \Delta(xa)(\xi \otimes \delta_g),\end{aligned}$$

which shows together with a continuity argument that $\lambda_x \Delta(a) \lambda_x^* = \Delta(xa)$. The *-algebra generated by $\Delta(A) \simeq A$ and $\lambda(\mathbf{C}G) \simeq \mathbf{C}G$ in $B(H \otimes \ell^2(G))$ is denoted $A \rtimes_a G$ and called the *algebraic crossed product* of A and G. The algebraic crossed product makes sense even for non-concrete C*-algebras A, for then we just replace A with the concrete C*-algebra associated to it via the universal representation. Since $\mathbf{C}G$ contains the unit of $B(H \otimes \ell^2(G))$, the algebraic crossed product is unital. Using that $\lambda_x \Delta(a) \lambda_x^* = \Delta(xa)$, one shows that

$$A \rtimes_a G = \mathrm{span}\{\Delta(a)\lambda_x : x \in G, a \in A\}.$$

Suppose $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in G$, with the x_i's distinct, are such that $0 = \sum \Delta(a_i)\lambda_{x_i}$. Then

$$0 = \sum \Delta(a_i)\lambda_{x_i}(\xi \otimes \delta_e) = \sum (x_i^{-1}a_i)\xi \otimes \delta_{x_i}$$

for all $x \in \xi$, so by linear independence $x_i^{-1}a_i = 0$ for each value of i. This implies $a_1 = \cdots = a_n = 0$, showing that each element of $A \rtimes_a G$ can be written uniquely on the form $\sum \Delta(a_i)\lambda_{x_i}$ with distinct x_i's. Furthermore, the algebraic crossed product has the following useful property.

Proposition 3.3.3. *Let G be a discrete group, let A be a unital G-C*-algebra, and let B a unital *-algebra. We denote by $\mathcal{U}(B)$ the group of elements $u \in B$ such that $u^* = u^{-1}$. Given a unital *-homomorphism $\varphi_A : A \to B$ and a group homomorphism $\varphi_G : G \to \mathcal{U}(B)$ such that $\varphi_G(g)\varphi_A(a)\varphi_G(g)^* = \varphi_A(ga)$ for all $g \in G$ and $a \in A$, there exists a unique unital *-homomorphism $\varphi : A \rtimes_a G \to B$ such that $\varphi|_A = \varphi_A$ and $\varphi|_G = \varphi_G$.*

Proof. Suppose $\varphi : A \rtimes_a G \to B$ is a unital *-homomorphism such that $\varphi|_A = \varphi_A$ and $\varphi|_G = \varphi_G$. Then $\varphi(\Delta(a)\lambda_x) = \varphi_A(a)\varphi_G(x)$. Since $A \rtimes_a G$ is spanned by elements of the form $\Delta(a)\lambda_x$, we see that φ indeed is determined uniquely.

Conversely, we now define $\varphi : A \rtimes_a G \to B$ via $\varphi(\Delta(a)\lambda_x) = \varphi_A(a)\varphi_G(x)$ and extend linearly, and then show that φ is the desired unital *-homomorphism. Note that this is well defined because each element in $A \rtimes_a G$ can be written uniquely as a sum $\sum \Delta(a_x)\lambda_x$. It is clear that $\varphi|_A = \varphi_A$ and $\varphi|_G = \varphi_G$, and that φ is unital. We have that

$$\begin{aligned}\varphi((\Delta(a)\lambda_x)^*) &= \varphi(\lambda_x^*\Delta(a^*)) \\ &= \varphi(\Delta(x^{-1}a^*)\lambda_x^*) \\ &= \varphi(\Delta((x^{-1}a)^*)\lambda_x^*) \\ &= \varphi_A(x^{-1}a)^*\varphi_G(x)^* \\ &= (\varphi_G(x)\varphi_A(x^{-1}a))^* \\ &= (\varphi_A(xx^{-1}a)\varphi_G(x))^*,\end{aligned}$$

so by linearity φ preserves the adjoint. To show that φ is multiplicative, one only needs to show that $\varphi(\Delta(a_1)\lambda_{x_1}\Delta(a_2)\lambda_{x_2}) = \varphi(\Delta(a_1)\lambda_{x_1})\varphi(\Delta(a_2)\lambda_{x_2})$, which can be done by first using $\Delta(a_1)\lambda_{x_1}\Delta(a_2)\lambda_{x_2} = \Delta(a_1a_2)\lambda_{x_1^{-1}x_2}$, and then proceeding to make the straightforward albeit a bit long winded calculations. Thus, the existence of a unital *-homomorphism $\varphi : A \rtimes_a G \to B$ such that $\varphi|_A = \varphi_A$ and $\varphi|_G = \varphi_G$ is established. □

Definition 3.3.4. Let G be a discrete group and let A be a unital G-C*-algebra. The *reduced crossed product* $A \rtimes_r G$ of A and G is the closure of $A \rtimes_a G$ in $B(H \otimes \ell^2(G))$.

We will now look at some properties of the reduced crossed product of a group G and G-C*-algebra A. Since $A \rtimes_a G$ contains A, we have an inclusion of C*-algebras $A \subset A \rtimes_r G$. Further, since $A \rtimes_a G$ contains $\mathbf{C}G$, we have another inclusion of C*-algebras $\mathrm{C}_r^*(G) \subset A \rtimes_r G$. For $u \in A \rtimes_r G$, we define $gu = \lambda_g u \lambda_g^*$, making $A \rtimes_r G$ a G-C*-algebra. It is immediately clear from the definition of the G-action on $\mathrm{C}_r^*(G)$ that $\mathrm{C}_r^*(G) \subset A \rtimes_r G$ is an inclusion of G-C*-algebras, and from $\lambda_x\Delta(a)\lambda_x^* = \Delta(xa)$ it follows that $A \subset A \rtimes_r G$ is an inclusion of G-C*-algebras.

Suppose A is a unital G-C*-algebra such that $A \subset \mathcal{B}(H)$ for some Hilbert space H. By definition $A \rtimes_r G \subset B(H \otimes \ell^2(G))$. We make the identification $H = H \otimes \mathbf{C}\delta_e$. Let $p_e : H \otimes \ell^2(G) \to H$ be the orthogonal projection onto H. We shall define a map $E : A \rtimes_r G \to A$, called the *canonical conditional expectation*. Take $b \in A \rtimes_r G$. Then we set

$$E(b)\xi = p_e b\xi \otimes \delta_e, \quad \xi \in H,$$

defining $E(b)$ as an operator on H. Since,

$$\|E(b)\| = \sup_{\|\xi\|=1} \|E(b)\xi\| = \sup_{\|\xi\|=1} \|p_e b\xi \otimes \delta_e\| \leq \|p_e b\| \leq \|b\|,$$

we see the $E(b) \in \mathcal{B}(H)$, and so E maps $A \rtimes_r G$ continuously into $\mathcal{B}(H)$. Suppose b is of the form $b = \Delta(a)\lambda_g$. Then

$$b\xi \otimes \delta_e = \Delta(a)\xi \otimes \delta_g = (g^{-1}a)\xi \otimes \delta_g.$$

Since δ_g is orthogonal to δ_e if $g \neq e$, we get

$$E(b)\xi = \begin{cases} a\xi, & g = e, \\ 0, & g \neq e. \end{cases}$$

By linearity,

$$E(\sum \Delta(a_g)\lambda_g) = a_e.$$

By continuity of E, this shows that E maps into A. Similarly, any other continuous map $A \rtimes_r G \to A$ satisfying the above equation must be equal to E. Note that

$$E(\lambda_x)\xi = \begin{cases} \xi, & x = e, \\ 0, & x \neq e, \end{cases}$$

so $E|_{C^*_r(G)} = \tau$ where $\tau : C^*_r(G) \to \mathbf{C}$ is the canonical tracial state.

Proposition 3.3.5. *For any unital G-C*-algebra A, the canonical conditional expectation $E : A \rtimes_r G \to A$ is faithful.*

Proof. Let $\alpha : G \to \mathrm{Aut}(A)$ be the G-action on A. Using the universal representation $\pi : A \to \mathcal{B}(H)$, one defines a G-action $\bar{\alpha} : G \to \mathrm{Aut}(\pi(A))$, $\bar{\alpha}_g(\pi(a)) = \pi(\alpha_g(a))$ for all $a \in A$. We will now define a G-action $\hat{\alpha}$ by unitary maps on the Hilbert space direct sum $H = \bigoplus_{\tau \in \mathcal{S}(A)} H_\tau$. Recall that H_τ is the Hilbert space completion of A/N_τ where $N_\tau = \{a \in A : \tau(a^*a) = 0\}$ and A/B_τ is equipped with the inner product $\langle a, b\rangle = \tau(b^*a)$. Introducing the maps,

$$\Lambda_\tau : A \to A/N_\tau, \quad a \mapsto a + N_\tau,$$

we start by defining

$$\hat{\alpha}_g(\Lambda_\tau(a)) = \Lambda_{g\tau}(\alpha_g(a)),$$

where $g\tau(x) = \tau(\alpha_{g^{-1}}(x))$ for all $x \in A$. If $b - a \in N_\tau$, then

$$g\tau(\alpha_g((b-a)^*(b-a)) = \tau((b-a)^*(b-a)) = 0,$$

implying that $\alpha_g(b-a) \in N_{g\tau}$, so $\hat{\alpha}_g$ is well defined. Further,

$$\begin{aligned} \|\hat{\alpha}_g(\Lambda_\tau(a))\|^2 &= \|\Lambda_{g\tau}(\alpha_g(a))\|^2 \\ &= \|\alpha_g(a) + N_{g\tau}\|^2 \\ &= g\tau(\alpha_g(a^*a)) \\ &= \tau(a^*a) \\ &= \|\Lambda_\tau(a)\|^2, \end{aligned}$$

showing that $\hat{\alpha}_g : A/N_\tau \to A/N_{g\tau}$ is isometric. Thus, we can extend to an isometric map $\hat{\alpha}_g : H_\tau \to H_{g\tau}$. Let H' be a the vector space direct sum of the Hilbert spaces H_τ as τ runs through $\mathcal{S}(A)$. We consider H' as a subspace of H, which is the Hilbert space completion of H'. We extend linearly to a map

$\hat{\alpha}_g : H' \to H'$. Suppose $\xi = \sum \xi_\tau \in H'$ where $\xi_\tau \in H_\tau$. If $\tau \neq \tau'$, then $g\tau \neq g\tau'$, and in particular $\hat{\alpha}_g(\xi_\tau)$ is orthogonal to $\hat{\alpha}_g(\xi_{\tau'})$. Then

$$\|\hat{\alpha}_g(\xi)\|^2 = \|\sum \hat{\alpha}_g(\xi_\tau)\|^2 = \sum \|\hat{\alpha}_g(\xi_\tau)\|^2 = \sum \|\xi_\tau\|^2 = \|\xi\|^2.$$

Thus, $\hat{\alpha}_g : H' \to H'$ is isometric and we can once again extend by continuity to an isometric map $\hat{\alpha}_g : H \to H$. Since α_g has an inverse $\alpha_{g^{-1}}$, we see that α_g is a unitary operator on H.

so we can extend isometrically our definition of $\hat{\alpha}_g$ to each Hilbert space H_τ, and then to H since $\hat{\alpha}_g(H_\tau)$ and $\hat{\alpha}_g(H_{\tau'})$ are orthogonal if $\tau \neq \tau'$. Since

$$\begin{aligned}
\hat{\alpha}_{gh}(\Lambda_\tau(a)) &= \Lambda_{gh\tau}(\alpha_g(\alpha_h(a))) \\
&= \hat{\alpha}_g(\Lambda_{h\tau}(\alpha_h(a))) \\
&= \hat{\alpha}_g(\hat{\alpha}_h(\Lambda_\tau(a))),
\end{aligned}$$

$\hat{\alpha}$ does indeed define a G-action on H by unitary operators.

For all $\tau \in \mathcal{S}(A)$, $a, b \in A$, and $g \in G$, we have

$$\begin{aligned}
\hat{\alpha}_g(\pi(a)(\Lambda_\tau(b))) &= \hat{\alpha}_g(\pi(a)(b + N_\tau)) \\
&= \hat{\alpha}_g(ab + N_\tau) \\
&= \hat{\alpha}_g(\Lambda_\tau(ab)) \\
&= \Lambda_{g\tau}(\alpha_g(ab))
\end{aligned}$$

and

$$\begin{aligned}
\bar{\alpha}_g(\pi(a))(\hat{\alpha}_g(\Lambda_\tau(b))) &= \bar{\alpha}_g(\pi(a))(\Lambda_{g\tau}(\alpha_g(b))) \\
&= \pi(\alpha_g(a))(\alpha_g(b) + N_{g\tau}) \\
&= \alpha_g(a)\alpha_g(b) + N_{g\tau} \\
&= \alpha_g(ab) + N_{g\tau} \\
&= \Lambda_{g\tau}(\alpha_g(ab)),
\end{aligned}$$

so

$$\hat{\alpha}_g(\pi(a)(\xi)) = \bar{\alpha}_g(\pi(a))(\hat{\alpha}_g(\xi)) \tag{3.3}$$

for all $\xi \in H$.

Define $\nu_g = \hat{\alpha}_g \otimes \rho_g \in B(H \otimes \ell^2(G))$. For all $a \in A$, $g, x \in G$, and $\xi \in H$, we have

$$\nu_g \sigma(a)\xi \otimes \delta_x = \nu_g(x^{-1}a)\xi \otimes \delta_x = \hat{\alpha}_g((x^{-1}a)\xi) \otimes \delta_{xg^{-1}}$$

and

$$\sigma(a)\nu_g \xi \otimes \delta_x = \sigma(a)\hat{\alpha}_g(\xi) \otimes \delta_{xg^{-1}} = (gx^{-1}a)\hat{\alpha}_g(\xi) \otimes \delta_{xg^{-1}},$$

so by Eq. (3.3), $\nu_g\sigma(a)\xi \otimes \delta_x = \sigma(a)\nu_g\xi \otimes \delta_x$. Since ρ_g and λ_h commute for all $g, h \in G$, we conclude that ν_g commutes with all elements of the form $\sigma(a)\lambda_x$, and thus with all elements in $A \rtimes_r G$.

Assume that $E(b^*b) = 0$ for some $b \in A \rtimes_r G$. Then for all $\xi \in H$,

$$\begin{aligned} 0 &= \langle E(b^*b)\xi, \xi \otimes \delta_e \rangle \\ &= \langle p_e b^* b\xi \otimes \delta_e, \xi \otimes \delta_e \rangle \\ &= \langle b^* b\xi \otimes \delta_e, p_e \xi \otimes \delta_e \rangle \\ &= \langle b\xi \otimes \delta_e, b\xi \otimes \delta_e \rangle \\ &= \|b\xi \otimes \delta_e\|^2. \end{aligned}$$

That is, $b\xi \otimes \delta_e = 0$. Since ν_g commutes with b for all $g \in G$, it follows that $0 = b\nu_g(\xi \otimes \delta_e) = b\hat{\alpha}_g(\xi) \otimes \delta_{g^{-1}}$. Using that $\hat{\alpha}_g$ is surjective for every $g \in G$, we conclude that $b = 0$. This shows that E is faithful. □

Lemma 3.3.6. *If $\varphi : A \to \mathcal{B}(H)$ is a unital positive map, where A is a C*-algebra and H is a Hilbert space, then the image $\mathrm{Im}(\varphi)$ is an operator system.*

Proof. We need to check that $\mathrm{Im}(\phi)$ is unital and self-adjoint. That $\mathrm{Im}(\phi)$ is unital is clear from ϕ being unital. Let a be a self-adjoint element of A. We have that

$$\phi(a) = \phi(a_+ - a_-) = \phi(a_+) - \phi(a_-)$$

is self-adjoint since positive elements are self-adjoint by definition. If a is an arbitrary element of A, it can be written as $a = b + ic$, where b, c are self-adjoint. Applying ϕ, we get $\phi(a) = \phi(b) + i\phi(c)$. Observe that $\phi(b), \phi(c)$ are self-adjoint, so

$$\phi(a)^* = \phi(b) - i\phi(c) = \phi(b - ic) = \phi(a^*),$$

showing that $\mathrm{Im}(\phi)$ is self-adjoint. □

The *multiplicative domain* of a unital completely positive map $\phi : A \to B$ between operator systems is the set

$$D_\phi = \{a \in A : \phi(ab) = \phi(ba) \text{ and } \phi(ba) = \phi(b)\phi(a) \text{ for all } b \in A\}.$$

The following proposition nicely characterises the multiplicative domain.

Proposition 3.3.7. ([16, Proposition 1.5.7]). *If $\phi : A \to B$ is a unital completely positive map between operator systems, then*

$$D_\phi = \{a \in A : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}$$

and D_ϕ is a C-subalgebra of A.*

A discrete group G is a said to be *C*-simple* if the reduced group C*-algebra $C^*_r(G)$ is simple as a C*-algebra.

Theorem 3.3.8. *Let G be a discrete group. If G acts freely on $\partial_F G$, then the C*-algebra $C^*_r(G)$ is simple.*

Proof. This proof follows the proof given in [1, Theorem 3.1]

Let I be a non-trivial ideal of $C^*_r(G)$. Then the quotient $C^*_r(G)/I$ is a C*-algebra, which we can identify with a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. The quotient map $\pi : C^*_r(G) \to \mathcal{B}(H)$ is a unital *-homomorphism onto $C^*_r(G)/I$. If we can show that π is injective, then I must be the zero ideal, and simplicity of $C^*_r(G)$ follows. The map π defines a G-action on $\mathcal{B}(H)$ via $gu = \pi(\lambda_g)u\pi(\lambda_g)^*$ for all $g \in G$ and $u \in \mathcal{B}(H)$, which makes $\mathcal{B}(H)$ a G-C*-algebra.

We have seen that $C^*_r(G)$ is C*-subalgebra of the reduced crossed product $C(\partial_F G) \rtimes_r G$. Because π is a *-homomorphism, it is completely positive. By Arveson's extension theorem (see [17, Theorem 1.2.3]) π can be extended to a unital completely positive map $\phi : C(\partial_F G) \rtimes_r G \to \mathcal{B}(H)$. As ϕ restricts to a *-homomorphism on $C^*_r(G)$, we have $C^*_r(G) \subset D_\phi$ by Proposition 3.3.7. Therefore,

$$\phi(gu) = \phi(\lambda_g u \lambda_g^*) = \pi(\lambda_g)\phi(u)\pi(\lambda_g)^* = g\phi(u).$$

for all $b \in C(\partial_F G) \rtimes_r G$, showing that ϕ is G-equivariant.

Since ϕ extends π, if ϕ is faithful, then π is also faithful and thus injective as π is a *-homomorphism. Recall that the inclusion $\mathbf{C} \subset C(\partial_F G)$ is essential because $C(\partial_F G) = C(\partial_H G)$. Because ϕ is unital, the restriction $\phi|_{\mathbf{C}}$ is an isometry, and so it follows that the restriction $\phi|_{C(\partial_F G)}$ is an isometry. Define $A = \mathrm{Im}(\phi|_{C(\partial_F G)})$ and $B = \mathrm{Im}(\phi)$. Since $\partial_F G$ is compact, $C(\partial_F G)$ contains the unit of $C(\partial_F G) \rtimes_r G$, making $\phi|_{C(\partial_F G)}$ unital. It follows from $\phi|_{C(\partial_F G)}$ and ϕ being unital and positive that A and B are operator systems. Further, $A \subset B$ is an inclusion of G-operator systems. Using the injectivity of the operator system $C(\partial_F G)$, the inverse $(\phi|_{C(\partial_F G)})^{-1} : A \to C(\partial_F G)$ extends to G-equivariant unital completely positive map $\tau : B \to C(\partial_F G)$. The composition $\psi = \tau \circ \phi : C(\partial_F G) \rtimes_r G \to C(\partial_F G)$ is a G-equivariant unital completely positive map. By rigidity of the inclusion $\mathbf{C} \subset C(\partial_F G)$, the map ψ restricts to the identity on $C(\partial_F G)$. This implies that $C(\partial_F G) \subset D_\psi$.

For $s \in G$ and $f \in C(\partial_F G)$, it holds that

$$\psi(\lambda_s)f = \psi(\lambda_s)\psi(f) = \psi(\lambda_s f) = \psi((sf)\lambda_s) = (sf)\psi(\lambda_s).$$

We now assume that the G-action on $\partial_F G$ is free, so we have $s^{-1}x \neq x$ for all $s \in G \setminus \{e\}$ and $x \in \partial_F G$. Given $s \in G \setminus \{e\}$ and $x \in \partial_F G$, Urysohn's lemma tells us that there is $f \in C(\partial_F G)$ such that $f(x) \neq 0$ while

$$0 = f(s^{-1}x) = (sf)(x).$$

From

$$\psi(\lambda_s)(x)f(x) = (sf)(x)\psi(\lambda_s)(x) = 0$$

it follows that $\psi(\lambda_s)(x) = 0$. Since x was arbitrary we conclude $\psi(\lambda_s) = 0$. We know that $C(\partial_F G) \subset D_\psi$. Combined with $\psi(\lambda_s) = 0$ for $s \in G \setminus \{e\}$, we get

$$\psi(\sum a_x \lambda_x) = \psi(a_e)\psi(\lambda_e) = a_e.$$

Because ψ is continuous, this shows that $\psi = E$. Since we know that the canonical conditional expectation E is faithful, the proof is finished. □

In fact, much more can be said in regards to the above theorem. The authors of [1] show that a discrete group G is C*-simple if and only if the G-action on $\partial_F G$ is free. They also show that a discrete group G is C*-simple if and only if the G-action on $\partial_F G$ is topologically free. Using the universal property of the Furstenberg boundary, they also show that this in turn is equivalent to there existing some G-boundary on which the G-action is topologically free. That the G-action is topologically free means that for each $g \in G \setminus \{e\}$, the set of fixed points of g has empty interior.

We continue our study of C*-simplicity and investigate how it relates to simplicity of the reduced crossed product $C(\partial_F G) \rtimes_r G$.

Proposition 3.3.9. *Let* G *be a discrete group. The* G*-C*-algebra* $C(\partial_F G) \rtimes_r G$ *is included in the* G*-C*-algebra* $\mathrm{I}_G(\mathrm{C}^*_r(G))$.

Proof. Using injectivity of $\mathrm{I}_G(\mathrm{C}^*_r(G))$ and $C(\partial_F G)$ together with the inclusions $\mathrm{C}^*_r(G) \subset C(\partial_F G) \rtimes_r G$ and $\mathrm{C}^*_r(G) \subset \mathrm{I}_G(\mathrm{C}^*_r(G))$, we obtain G-unital completely positive maps ϕ_1, ϕ_2 such that the below diagram commutes, where τ denotes the canonical tracial state.

$$\begin{array}{ccc} C(\partial_F G) \rtimes_r G & \xrightarrow{\phi_1} & \mathrm{I}_G(\mathrm{C}^*_r(G)) \\ \uparrow & \nearrow & \downarrow{\scriptstyle \phi_2} \\ \mathrm{C}^*_r(G) & \xrightarrow{\tau} & C(\partial_F G) \end{array}$$

Denote by ϕ the map

$$\phi = (\phi_2 \circ \phi_1)|_{C(\partial_F G)} : C(\partial_F G) \to C(\partial_F G).$$

Because the maps under consideration are all unital, ϕ is equal to the identity on $\mathbf{C}$. Now we use rigidity of the inclusion $\mathbf{C} \subset C(\partial_F G)$ to see that ϕ is the identity map. In particular, $C(\partial_F G)$ is in the multiplicative domain of $\phi_2 \circ \phi_1$. From the diagram, we also see that $(\phi_2 \circ \phi_1)|_{\mathrm{C}^*_r(G))} = \tau$. Thus, for any finite sum $\sum a_x \lambda_x$ in $C(\partial_F G) \rtimes_r G$, we have

$$\phi_2(\phi_1(\sum a_x \lambda_x)) = a_e,$$

and by continuity we conclude that $\phi_2 \circ \phi_1 = E$, where E is the canonical conditional expectation. In particular, ϕ_1 is faithful. Since faithful *-homomorphisms are injective, it is sufficient in order to finish the proof to show that ϕ_1 is a *-homomorphism. By construction, ϕ_1 restricts to the identity on $\mathrm{C}^*_r(G)$, so $\mathrm{C}^*_r(G)$ is in the multiplicative domain of ϕ_1. Now we show that $C(\partial_F G)$ is also in the multiplicative domain of ϕ_1. We will do this using the Cauchy-Schwarz inequality in Proposition 3.1.3. Suppose $\phi_1(a^*a) > \phi_1(a)^*\phi_1(a)$ for some $a \in C(\partial_F G)$. The fact that ϕ_2 is a G-unital completely positive map implies that

$$(\phi_2 \circ \phi_1)(a^*a) > \phi_2(\phi_1(a)^*\phi_1(a)) \geq (\phi_2 \circ \phi_1)(a)^*(\phi_2 \circ \phi_1)(a).$$

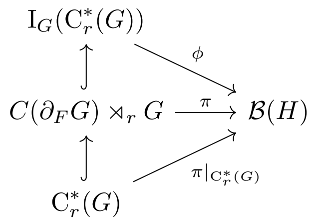
However, this contradicts the fact that $C(\partial_F G)$ is in the multiplicative domain of $\phi_2 \circ \phi_1$. Thus, we must have $\phi_1(a^*a) = \phi_1(a)^*\phi_1(a)$, showing that $C(\partial_F G)$ is contained in the multiplicative domain of ϕ_1. Since both $\mathrm{C}^*_r(G)$ and $C(\partial_F G)$ are in the multiplicative domain of ϕ_1, ϕ_1 must be multiplicative. As a positive map that is multiplicative, ϕ_1 is a *-homomorphism because positive maps preserve adjoints. □

Theorem 3.3.10. *Let G be a discrete group. If $\mathrm{C}^*_r(G)$ is simple, then $C(\partial_F G) \rtimes_r G$ is simple.*

Proof. By the previous proposition we have the following inclusions of G-C*-algebras

$$\mathrm{C}^*_r(G) \subset C(\partial_F G) \rtimes_r G \subset \mathrm{I}_G(\mathrm{C}^*_r(G)).$$

Let $\pi : C(\partial_F G) \rtimes_r G \to \mathcal{B}(H)$ be a unital *-homomorphism, where H is some Hilbert space. We want to show that π is injective. As in the proof of Theorem 3.3.8, we consider $\mathcal{B}(H)$ as a G-C*algebra and obtain from Arveson's extension theorem a G-unital completely positive map $\phi : \mathrm{I}_G(\mathrm{C}^*_r(G)) \to \mathcal{B}(H)$ such that the diagram below commutes.



Since $\mathrm{C}^*_r(G)$ is simple, the unital map $\pi|_{\mathrm{C}^*_r(G)}$ is injective. As an injective *-homomorphism of C*-algebras, $\pi|_{\mathrm{C}^*_r(G)}$ and all its amplifications are isometric. By essentiality of the inclusion $\mathrm{C}^*_r(G) \subset \mathrm{I}_G(\mathrm{C}^*_r(G))$, we conclude that ϕ is completely isometric. Therefore, π is injective as it is a restriction of a injective map. □

Theorem 3.3.11. *Let G be a discrete group. If $C(\partial_F G) \rtimes_r G$ is simple, then $\mathrm{C}^*_r(G)$ is simple.*

Proof. To any $x \in \partial_F G$, we can associate the stabilizer subgroup G_x consisting of all group elements g fixing x. As G_x is a subset of G, the Hilbert space $\ell^2(G_x)$ is a subspace of $\ell^2(G)$ and $H \otimes \ell^2(G_x)$ is a subspace of $H \otimes \ell^2(G)$. Let $p : H \otimes \ell^2(G) \to H \otimes \ell^2(G_x)$ be the orthogonal projection onto $H \otimes \ell^2(G_x)$. We define a map

$$\phi_1 : B(H \otimes \ell^2(G)) \to B(H \otimes \ell^2(G_x)), \quad \phi_1(b) = pb,$$

where we interpret b in the composition pb as the restriction of b to $H \otimes \ell^2(G_x)$. Comparing with Example 2.2.9, we see that ϕ_1 is a positive map. Since $\|p\| = 1$, the map ϕ_1 satisfies $\|\phi_1\| \leq 1$. For $f \in C(\partial_F G)$, $\xi \in H$, $g \in G$, and $h \in G_x$, we have

$$pf\lambda_g \xi \otimes \delta_h = p(((gh)^{-1}f)\xi \otimes \delta_{gh}) = \begin{cases} f\lambda_g \xi \otimes \delta_h, & g \in G_x \\ 0, & g \notin G_x. \end{cases} \tag{3.4}$$

This determines ϕ_1 on $C(\partial_F G) \rtimes_a G$, and we also see that $\phi_1(C(\partial_F G) \rtimes_a G) \subset C(\partial_F G) \rtimes_r G_x$. By continuity, ϕ_1 must map $C(\partial_F G) \rtimes_r G$ into the complete space $C(\partial_F G) \rtimes_r G_x$, so ϕ_1 restricts to a map

$$\phi_1 : C(\partial_F G) \rtimes_r G \to C(\partial_F G) \rtimes_r G_x.$$

If $f \in C(\partial_F G)$, then we can define an operator $f \in B((\mathbf{C} \oplus H) \otimes \ell^2(G_x))$ extending the definition $f((r, \xi) \otimes \delta_h) = (f(x)r, (h^{-1}f)\xi) \otimes \delta_h$ continuously to all of $(\mathbf{C} \oplus H) \otimes \ell^2(G_x)$. Similarly, for $\lambda_g \in \mathrm{C}^*_r(G)$, we get an operator $\lambda_g \in B((\mathbf{C} \oplus H) \otimes \ell^2(G_x))$ extending the definition $\lambda_g((r, \xi) \otimes \delta_h) = (r, \xi) \otimes \delta_{gh}$. Thus, we can view $C(\partial_F G) \rtimes_r G_x$ as a C*-subalgebra of $B((\mathbf{C} \oplus H) \otimes \ell^2(G_x))$.

Let $q : (\mathbf{C} \oplus H) \otimes \ell^2(G_x) \to (\mathbf{C} \oplus 0) \otimes \ell^2(G_x) = \ell^2(G_x)$ be the orthogonal projection onto $\ell^2(G_x)$. We define a map

$$\phi_2 : C(\partial_F G) \rtimes_r G_x \to B(\ell^2(G_x)), \quad \phi_2(b) = qb,$$

where we interpret b in the composition pb as the restriction of q to $(\mathbf{C} \oplus 0) \otimes \ell^2(G_x) = \ell^2(G_x)$. Because $\|q\| = 1$, we have $\|\phi_2\| \leq 1$. For all $f \in C(\partial_F G)$ and $g, h \in G_x$ we have

$$f\lambda_g((1, 0) \otimes \delta_h) = (f(x), 0) \otimes \delta_{gh} = (1, 0) \otimes (f(x)\delta_{gh}), \tag{3.5}$$

so $\phi_2(f\lambda_g) = f(x)\lambda_g \in \mathrm{C}^*_r(G)$. Thus, ϕ_2 maps $C(\partial_F G) \rtimes_a G_x$ into $\mathrm{C}^*_r(G)$, and by continuity of ϕ_2 it follows that ϕ_2 maps $C(\partial_F G) \rtimes_r G$ into $\mathrm{C}^*_r(G)$. Since both $(\mathbf{C} \oplus 0) \otimes \ell^2(G_x) = \ell^2(G_x)$ and $(0 \oplus H) \otimes \ell^2(G_x) = (\ell^2(G_x))^\perp$ are invariant under $f\lambda_g$ for all $f \in C(\partial_F G)$ and $g \in G_x$ and thus under all operators in $C(\partial_F G) \rtimes_r G$, we have that q commutes with $C(\partial_F G) \rtimes_r G_x$. If $b_1, b_2 \in C(\partial_F G) \rtimes_r G_x$, then

$$\phi_2(b_1 b_2) = qb_1 b_2 = q^2 b_1 b_2 = qb_1 qb_2 = \phi_2(b_1)\phi_2(b_2).$$

Since $qb^* = (bq)^* = (qb)^*$ for all $b \in C(\partial_F G) \rtimes_r G$ we conclude that ϕ_2 is a *-homomorphism.

Introduce the map $E_x = \phi_2 \circ \phi_1 : C(\partial_F G) \rtimes_r G \to \mathrm{C}^*_r(G)$. By Equations 3.4 and 3.5 we have

$$E_x(f\lambda_g) = \begin{cases} f(x)\lambda_g, & g \in G_x, \\ 0, & g \notin G_x, \end{cases}$$

for all $f \in C(\partial_F G)$ and $g \in G$. The map E_x is unital, and also positive as a composition of positive maps. That ϕ_2 is positive follows from it being a *-homomorphism, but could also be shown in the same way as is done for ϕ_1. Since $\|\phi_1\| \leq 1$ and $\|\phi_2\| \leq 1$ we have $\|E_x\| \leq 1$. By [1, Proposition 2.7], the group G_x is *amenable*, so there exists by [7, Corollary G.3.9] a *-homomorphism $\varepsilon_x : \mathrm{C}^*_r(G_x) \to \mathbf{C}$ mapping λ_g to 1 for all $g \in G_x$. We will not need any more properties of amenable groups, but [7, Appendix G] contains some information for the interested reader. As a unital *-homomorphism ε_x satisfies $\|\varepsilon_x\| = 1$. We introduce the map $\tau_x = \varepsilon_x \circ E_x$ which satisfies

$$\tau_x(f\lambda_g) = \begin{cases} f(x), & g \in G_x, \\ 0, & g \notin G_x. \end{cases}$$

From $\tau_x(1) = 1$ and $\|\tau_x\| \leq \|\varepsilon_x\|\|E_x\| \leq 1$ it follows that $\|\tau_x\| = 1$. As a linear functional with $1 = \tau_x(1) = \|\tau_x\|$, it follows from [3, Corollary 3.3.4] that τ_x is a state.

By [1, Theorem 3.1], if $\mathrm{C}^*_r(G)$ is not simple, then the G-action on the Furstenberg boundary is not topologically free. For contradiction, we assume that $\mathrm{C}^*_r(G)$ is not simple, so the G-action on the Furstenberg boundary is not topologically free, and set out to find a non-injective unital *-homomorphism $\pi : C(\partial_F G) \rtimes_r G \to \mathcal{B}(\ell^2(G/G_x))$ for some $x \in \partial_F G$, which will show that $C(\partial_F G) \rtimes_r G$ is not simple. Here, G/G_x denotes the set of cosets $\{gG_x : g \in G\}$ equipped with the discrete topology.

If the G-action on $\partial_F G$ is not topologically free, then there exists $g \in G \setminus \{e\}$ and an open nonempty subset $U \subset \partial_F G$ such that $gx = x$ for all $x \in U$. We pick an x in U and define a linear map

$$\pi_{C(\partial_F G)} : C(\partial_F G) \to \mathcal{B}(\ell^2(G/G_x)),$$

which extends the definition $\pi_{C(\partial_F G)}(f)\delta_{gG_x} = f(gx)\delta_{gG_x}$ for all $g \in G$ and $f \in C(\partial_F G)$. Note that this is well defined because if $h^{-1}g \in G_x$, then $hx = gx$, and that $\pi_{C(\partial_F G)}(f)$ is bounded on $C_c(G/G_x)$ as a consequence of f being bounded, which allows the definition to be extended to $\mathcal{B}(\ell^2(G/G_x))$. It then follows almost directly from the definition that $\pi_{C(\partial_F G)}$ is a *-homomorphism. Let $\lambda' : G \to \mathcal{B}(\ell^2(G/G_x))$ be the map such that $\lambda'_g(\delta_{hG_x}) = \delta_{ghG_x}$. As with the left regular representation, we have that λ'_g is unitary and that λ' is a group homomorphism into the group of unitary operators on $\ell^2(G/G_x)$. We check that

$$\lambda'_g \pi_{C(\partial_F G)}(f)(\lambda'_g)^* \delta_{hG_x} = f(g^{-1}hx)\delta_{hG_x} = (gf)(hx)\delta_{hG_x} = \pi_{C(\partial_F G)}(gf)\delta_{hG_x},$$

so by continuity $\lambda'_g \pi_{C(\partial_F G)}(f)(\lambda'_g)^* = \pi_{C(\partial_F G)}(gf)$. By the universal property of the algebraic crossed product, there then exists a unital *-homomorphism $\pi : C(\partial_F G) \rtimes_a G \to \mathcal{B}(\ell^2(G/G_x))$ such that $\pi|_{C(\partial_F G)} = \pi_{C(\partial_F G)}$ and $\pi|_G = \lambda'$.

We have that

$$\langle \pi(f\lambda_g)\delta_{G_x}, \delta_{G_x} \rangle = \langle \pi_{C(\partial_F G)}(f)\lambda'_g \delta_{G_x}, \delta_{G_x} \rangle = \begin{cases} f(x), & g \in G_x, \\ 0, & g \notin G_x, \end{cases}$$

so $\tau_x(fu_g) = \langle \pi(f\lambda_g)\delta_{G_x}, \delta_{G_x} \rangle$. It follows by linearity that

$$\tau_x(b) = \langle \pi(b)\delta_{G_x}, \delta_{G_x} \rangle$$

for all $b \in C(\partial_F G) \rtimes_a G$. Now let $(\pi_{\tau_x}, H_{\tau_x}, \xi_{\tau_x})$ be the GNS representation of $C(\partial_F G) \rtimes_r G$ associated to the state τ_x. By restricting $(\pi_{\tau_x}, H_{\tau_x}, \xi_{\tau_x})$, we get a representation of $C(\partial_F G) \rtimes_a G$. Note that the restriction of the GNS representation is also cyclic since $C(\partial_F G) \rtimes_a G$ is dense in $C(\partial_F G) \rtimes_r G$. The vector δ_{G_x} is cyclic with respect to the representation $\pi : C(\partial_F G) \rtimes_a G \to \mathcal{B}(\ell^2(G/G_x))$ since compactly supported functions are dense in $\ell^2(G/G_x)$, so it follows from Propositions 2.2.22 and 2.2.23 that there exists a unitary map $u : H_{\tau_x} \to B(\ell^2(G/G_x))$ such that $\pi(b) = u\pi_{\tau_x}(b)u^*$ for all $b \in C(\partial_F G) \rtimes_a G$. We define a new *-homomorphism $b \mapsto u\pi_{\tau_x}(b)u^* : C(\partial_F G) \rtimes_r G \to B(\ell^2(G/G_x))$, and see that this extends π. Henceforth, π denotes this extension to $C(\partial_F G) \rtimes_r G$.

Let $g \in G \setminus \{e\}$ be such that $gy = y$ for all $y \in U$. We have

$$\begin{aligned}\pi(\lambda_g f)\delta_{hG_x} &= \pi(\lambda_g f)\delta_{hG_x} \\ &= \lambda'_g \pi_{C(\partial_F G)}(f)\delta_{hG_x} \\ &= \lambda'_g f(hx)\delta_{hG_x} \\ &= f(hx)\delta_{ghG_x}\end{aligned}$$

for all $h \in G$ and $f \in C(\partial_F G)$. Meanwhile, $\pi(f)\delta_{hG_x} = f(hx)\delta_{hG_x}$. Now let $f \in C(\partial_F G)$ be a nonzero function supported in U. Such a function exists by Urysohn's lemma. If $h \in G_x$, then

$$\pi(\lambda_g f)\delta_{hG_x} = f(hx)\delta_{ghG_x} = f(hx)\delta_{G_x} = \pi(f)\delta_{hG_x},$$

since $g \in G_x$. If $h \notin G_x$, then $gh \notin G_x$, so $hx \notin U$, and

$$\pi(\lambda_g f)\delta_{hG_x} = \pi(f)\delta_{hG_x} = f(hx)\delta_{hG_x} = 0.$$

This shows that $\pi(\lambda_g f) = \pi(f)$.

What is left in order to show that π is not injective is to show that $\lambda_g f \neq f$. This is clear however because for all $\xi \in H$, where H is such that $C(\partial_F G) \subset \mathcal{B}(H)$, we have

$$\lambda_g f\xi \otimes \delta_e = f(\xi) \otimes \delta_g$$

and

$$f\xi \otimes \delta_e = f(\xi) \otimes \delta_e,$$

so if $\xi \in H$ is such that $f(\xi) \neq 0$, then

$$\lambda_g f\xi \otimes \delta_e \neq f\xi \otimes \delta_e,$$

so $\lambda_g f \neq f$. □

Combining Theorems 3.3.10 and 3.3.11, we see that C*-simplicity of the discrete group G is equivalent to simplicity of $C(\partial_F G) \rtimes_r G$.

Chapter 4

The reduced groupoid C*-algebra

So far, we have studied simplicity of the reduced group C*-algebra of a discrete group G. In this chapter we look at a generalisation of the reduced group C*-algebra and of group C*-algebras presented in [2], to allow for the more general *groupoids* instead of groups. The definition of the reduced groupoid C*-algebra that we use is slightly different from what appears in [2], and only applies to étale groupoids with totally disconnected locally compact Hausdorff space of units. The trade-off is that our approach is technically less heavy. Our main result in this chapter is that the reduced groupoid C*-algebra with our definition is a groupoid C*-algebra.

We begin with the definition of a groupoid and then state some basic facts. See [18] for more information on groupoids.

Definition 4.0.1. A *groupoid* $\mathcal{G}$ is a set together with a set of composable pairs $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ and two maps $(x,y) \mapsto xy : \mathcal{G}^2 \to \mathcal{G}$ and $x \mapsto x^{-1} : \mathcal{G} \to \mathcal{G}$ having the following properties:

(i) $(x^{-1})^{-1} = x$

(ii) if $(x,y) \in \mathcal{G}^2$ and $(y,z) \in \mathcal{G}^2$, then $(xy,z) \in \mathcal{G}^2$ and $(x,yz) \in \mathcal{G}^2$, and $(xy)z = x(yz)$

(iii) $(x^{-1},x) \in \mathcal{G}^2$ and $(x,x^{-1}) \in \mathcal{G}^2$

(iv) if $(x,y) \in \mathcal{G}^2$, then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

Lemma 4.0.2. *Let $\mathcal{G}$ be a groupoid. If $(xy,z) \in \mathcal{G}^2$, then $(y,z) \in \mathcal{G}^2$. Similarly, if $(x,yz) \in \mathcal{G}^2$, then $(x,y) \in \mathcal{G}^2$.*

Proof. Since $(x^{-1},x) \in \mathcal{G}^2$ and $(x,y) \in \mathcal{G}^2$, we have $(x^{-1},xy) \in \mathcal{G}^2$, so $(y,z) = (x^{-1}(xy),z) \in \mathcal{G}^2$. The second part is shown in much the same way. □

The above lemma together with property (ii) in Definition 4.0.1 imply that we can write a product $(xy)z$ as xyz and have no issues with ambiguity. To illustrate this, we note that if $(x,y) \in \mathcal{G}^2$, then $(xy)^{-1} = y^{-1}x^{-1}$ since

$$y^{-1}x^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = (xy)^{-1}.$$

Lemma 4.0.3. *Let $\mathcal{G}$ be a groupoid. The pair $(x,y) \in \mathcal{G} \times \mathcal{G}$ is composable if and only if $yy^{-1} = x^{-1}x$.*

Proof. Assume that $(x,y) \in \mathcal{G}^2$. Then $x^{-1}x = x^{-1}xyy^{-1} = yy^{-1}$.

Conversely, if $x^{-1}x = yy^{-1}$, then $(x,y) \in \mathcal{G}^2$ because yy^{-1} is composable with y from the right. □

The maps r and s on $\mathcal{G}$ defined by $r(x) = xx^{-1}$ and, $s(x) = x^{-1}x$ are called the *range* and *source* maps, respectively. By the above lemma, the pair (x,y) is composable if and only if the range of y is equal to the source of x.

The *unit space* of a groupoid $\mathcal{G}$ is the set $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G})$. The elements of $\mathcal{G}^{(0)}$ are called *units* of $\mathcal{G}$. If $u \in \mathcal{G}$, we define $\mathcal{G}^u = r^{-1}(u)$ and $\mathcal{G}_u = s^{-1}(u)$. The intersection $\mathcal{G}_u^u = \mathcal{G}^u \cap \mathcal{G}_u$ is called the *isotropy group* at u.

Lemma 4.0.4. *Suppose $\mathcal{G}$ is a groupoid and u is a unit in $\mathcal{G}$, then the isotropy group at u is a group under the product map of $\mathcal{G}$.*

Proof. By definition, the source and range of any two element in $\mathcal{G}_u^u$ coincide, so all elements are composable. Since $r(x) = s(x^{-1})$, we have that $x^{-1} \in \mathcal{G}_u^u$ for all $x \in \mathcal{G}_u^u$. The identity element is u since $ux = xu = x$ and $x^{-1}x = xx^{-1} = u$ for all $x \in \mathcal{G}_u^u$. ☐

Example 4.0.5. Suppose the group G acts on the set X. The underlying set of the *action groupoid* $\mathcal{G}$ is $G \times S$. The composable elements are

$$\mathcal{G}^2 = \{((g, x), (h, y)) \in \mathcal{G} \times \mathcal{G} : y = gx\}$$

and multiplication is given by $(g, x)(h, gx) = (gh, x)$. The inverse of (g, x) is (g^{-1}, gx). Further, $r(g, x) = (e, x)$ and $s(g, x) = (e, gx)$. In particular, $s(g, x) = r(h, y) \iff y = gx$, which is consistent with our finding that two elements are composable if and only if the source and range agree. We also see that the unit space $\mathcal{G}^{(0)} = \{e\} \times X$ can be identified with X.

Definition 4.0.6. A *topological groupoid* is a groupoid $\mathcal{G}$ with a topology such that the product and inversion maps are continuous, and the range and source maps are open as maps into the subspace $\mathcal{G}^{(0)}$. If the range and source maps are local homeomorphisms, then $\mathcal{G}$ is an *étale groupoid.*

Lemma 4.0.7. *If $\mathcal{G}$ is a topological groupoid, then the range and source maps are continuous.*

Proof. Because inversion is continuous, the map $x \mapsto (x, x^{-1})$ is continuous into $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$. Since, x and x^{-1} are composable and multiplication is continuous, the range and source maps $x \mapsto xx^{-1}$ and $x \mapsto x^{-1}x$ are continuous as compositions of continuous maps. ☐

A subset $U \subset \mathcal{G}$ of a groupoid is a *bisection* if the restrictions of the source and range maps to U are injective. If $\mathcal{G}$ is a topological groupoid, then by the above lemma the restrictions of the range and source maps to any open bisection are homeomorphisms between U and the open subsets $r(U)$ and $s(U)$, respectively, of the subspace $\mathcal{G}^{(0)}$.

Lemma 4.0.8. *Let $\mathcal{G}$ be a groupoid. A subset $U \subset \mathcal{G}$ is a bisection if and only if $UU^{-1} \subset \mathcal{G}^{(0)}$ and $U^{-1}U \subset \mathcal{G}^{(0)}$.*

Proof. Suppose U is a bisection. If $(x, y^{-1}) \in (U \times U^{-1}) \cap \mathcal{G}^2$, then $s(x) = r(y^{-1}) = s(y)$, which implies that $x = y$. Thus, $UU^{-1} = \bigcup_{x \in U} xx^{-1} \subset \mathcal{G}^{(0)}$. Similarly, it follows from injectivity of r that $U^{-1}U = \bigcup_{x \in U} x^{-1}x \subset \mathcal{G}^{(0)}$.

Conversely, suppose $UU^{-1} \subset \mathcal{G}^{(0)}$ and $U^{-1}U \subset \mathcal{G}^{(0)}$. If $x, y \in U$ are such that $s(x) = s(y) = r(y^{-1})$, then $xy^{-1} \in \mathcal{G}^{(0)}$. Then $y = xy^{-1}y = x$. Injectivity of r is shown in a similar way. ☐

Let $\mathcal{G}$ be a topological groupoid. We denote by $\Gamma(\mathcal{G})$ the set of open bisections of $\mathcal{G}$. We now show that $\Gamma(\mathcal{G})$ is an inverse semigroup under the multiplication map $(U, V) \mapsto UV$.

Proposition 4.0.9. *Let $\mathcal{G}$ be a topological groupoid. The open bisections $\Gamma(\mathcal{G})$ form an inverse semigroup.*

Proof. As we have seen, if U is a bisection, then $U^{-1}U$ and UU^{-1} are contained in the unit space. In particular, $UU^{-1}U = U$ and $U^{-1}UU^{-1} = U^{-1}$, showing that U^{-1} serves as the inverse of $U \in \Gamma(\mathcal{G})$. Note that U^{-1} is indeed an open bisection if U is an open bisection because the inversion map on $\mathcal{G}$ is a homeomorphism.

What we still need to show is that if U, V are open bisections of $\mathcal{G}$, then UV is an open bisection of $\mathcal{G}$. First we note that UV is a bisection since $UV(UV)^{-1} = UVV^{-1}U^{-1} \subset UU^{-1} \subset \mathcal{G}^{(0)}$ since $VV^{-1} \subset \mathcal{G}^{(0)}$ and $UU^{-1} \subset \mathcal{G}^{(0)}$, and similarly $(UV)^{-1}UV \subset \mathcal{G}^{(0)}$.

Now we want to show that UV is open. Introduce the open bisections $\tilde{U} = s^{-1}(r(V)) \cap U$ and $\tilde{V} = r^{-1}(s(U)) \cap V$. Note that $\tilde{U} = s^{-1}(r(\tilde{V})) \cap U$ and $\tilde{V} = r^{-1}(s(\tilde{U})) \cap V$. By construction, $UV = \tilde{U}\tilde{V}$, every element of $\tilde{U}$ is composable with precisely one element of $\tilde{V}$, and every element of $\tilde{V}$ is composable with precisely one element of $\tilde{U}$.

Let $\tau : \mathcal{G}^{r(\tilde{V})} \to \tilde{U}$ be the map mapping each $y \in \mathcal{G}^{r(\tilde{V})}$ to the unique $x \in \tilde{U}$ such that $(x, y) \in \mathcal{G}^2$. Denote by $\pi_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ the projection map onto the second factor. If $X \subset \tilde{U}$, then $\tau^{-1}(X) = \pi_2(\mathcal{G}^2 \cap (X \times \mathcal{G}^{r(\tilde{V})}))$, so if X is open, then $\tau^{-1}(X)$ is open. Thus, τ is continuous. Let $\tau' : \mathcal{G}^{r(\tilde{U})} \to \tilde{U}$ be the map mapping each $x \in \mathcal{G}^{r(\tilde{U})}$ to the unique $y \in \tilde{U}$ such that $r(y) = r(x)$. If $X \subset \tilde{U}$, then $(\tau')^{-1}(X) = r^{-1}(X)$, so τ' is continuous.

Take $y \in \mathcal{G}^{r(\tilde{V})}$. Then $\tau'(\tau(y)y) \in \tilde{U}$ and $r(\tau'(\tau(y)y)) = r(\tau(y)y) = r(\tau(y))$. Since also $\tau(y) \in \tilde{U}$, it follows from injectivity that $\tau'(\tau(y)y) = \tau(y)$. Take $x \in \mathcal{G}^{r(\tilde{U})}$. Then $\tau(\tau'(x)^{-1}x) \in \tilde{U}$ and $s(\tau(\tau'(x)^{-1}x)) = r(\tau'(x)^{-1}x) = r(\tau'(x)^{-1}) = s(\tau'(x))$. Since also $\tau'(x) \in \tilde{U}$, it follows from injectivity that $\tau(\tau'(x)^{-1}x) = \tau'(x)$.

Let $\varphi : \mathcal{G}^{r(\tilde{V})} \to \mathcal{G}^{r(\tilde{U})}$ be the map such that $\varphi(y) = \tau(y)y$. The map is continuous since it is a composition of continuous maps. Let $\varphi' : \mathcal{G}^{r(\tilde{U})} \to \mathcal{G}^{r(\tilde{V})}$ be the map such that $\varphi(x) = \tau'(x)^{-1}x$, which is also continuous a composition of continuous maps. We calculate that

$$\varphi'(\varphi(y)) = \tau'(\tau(y)y)^{-1}\tau(y)y = \tau(y)^{-1}\tau(y)y = y$$

and

$$\varphi(\varphi'(x)) = \tau(\tau(x)^{-1}x)\tau'(x)^{-1}x = \tau'(x)\tau'(x)^{-1}x = x.$$

This shows that φ is a homeomorphism. In particular, $UV = \varphi(\tilde{V})$ is open. $\square$

Proposition 4.0.10. *Let $\mathcal{G}$ be a Hausdorff étale groupoid. The unit space $\mathcal{G}^{(0)}$ is a clopen set in $\mathcal{G}$.*

Proof. Take an arbitrary element u in $\mathcal{G}^{(0)}$. Let γ be an open bisection containing u. Because the range and source maps by definition are one-to-one on bisections and because $s(g) = r(g^{-1})$ for all $g \in \mathcal{G}$, we have $\gamma\gamma^* = \{xx^{-1} : x \in \gamma\}$, so the open bisection $\gamma\gamma^*$ is a subset of $\mathcal{G}^{(0)}$. We also have $u \in \gamma\gamma^*$ since $u = uu^{-1}$. Since u was arbitrary, this shows that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.

Suppose (u_λ) is a net in $\mathcal{G}^{(0)}$ converging to g in $\mathcal{G}$. Viewing the range map as a continuous map into $\mathcal{G}$, we get that $(u_\lambda) = (r(u_\lambda))$ is a net converging to $r(g)$ in $\mathcal{G}$. Since the limits of nets in Hausdorff spaces are unique, we conclude that $g = r(g)$, so g is a unit. This shows that $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$. □

Observe that in the above proof the Hausdorff property was not needed in order to show that the unit space is open.

In the sequel, we will want to study étale groupoids with some requirements on the topology of the unit space. One of those being the following.

Definition 4.0.11. A topological space X is *totally disconnected* if all connected components are one point sets.

Example 4.0.12. Let C be a subset of the space of rational numbers that contains at least two points, say p, q. If x is an irrational number such that $p < x < q$, then the sets $C \cap \{r : r < x\}$ and $C \cap \{r : r > x\}$ is a separation of C. This shows that the rational numbers are totally disconnected.

Let X be a topological space. We introduce an equivalence relation on X by saying that $x \sim y$ if there are no pairs (U, V) of disjoint nonempty open sets such that $U \cup V = X$ and $x \in U$ and $y \in V$. That $\sim$ is reflexive and symmetric is clear. To show that the relation is transitive, suppose $x \sim y$ and $y \sim z$. If (U, V) are disjoint nonempty open sets such that $U \cup V = X$, then either $y \in U$ or $y \in V$. Thus, if $x \in U$ and $z \in V$, we either get a contradiction to $x \sim y$ or $y \sim z$, so we cannot have $x \in U$ and $z \in V$. This shows that $x \sim z$. The equivalence classes corresponding to $\sim$ are called *quasi components*. If x and y are in the same connected component, then it follows immediately from definition that $x \sim y$. We will next consider the converse.

Proposition 4.0.13. *Let X be a compact Hausdorff topological space. If x, y are points in X such that $x \sim y$, then x and y are in the same connected component.*

Proof. For clarity, we will write the equivalence relation with a subscript $\sim_X$ indicating what space the equivalence relation originates from. Suppose $x \sim_X y$. Let S be the set of all closed subsets C of X containing x and y such that $x \sim_C y$, and order S by inclusion. Since $X \in S$, the set S is non empty. Let $T \subset S$ be a nonempty totally ordered subset. Consider the set $C_0 = \bigcap_{C \in T} C$. Clearly, C_0 is closed and $x, y \in C_0$. Assume there are disjoint nonempty closed sets $U, V \subset C_0$ whose union is C_0 and such that $x \in U$ and $y \in V$. Because X is compact Hausdorff, there are disjoint open sets U', V' of X such that $U \subset U'$ and $V \subset V'$. Let T' be the collection of all sets of the form $C \setminus (U' \cup V')$ with $C \in T$. Because each C is closed, all sets in T' are compact. Further, T' is totally ordered by inclusion because T is. Note also that each set $C \setminus (U' \cup V')$ is nonempty since $x \sim_C y$. Thus, as the intersection of a totally ordered collection of nonempty compact sets, $C_0 \setminus (U' \cup V') = \bigcap_{C \in T} C \setminus (U' \cup V')$ is nonempty. This, gives a contradiction to $C_0 \subset U' \cup V'$, so we conclude that $x \sim_{C_0} y$. In particular, $C_0 \in T$, showing that T has a lower bound. By Zorn's lemma, S has a minimal element m. Suppose A, B is a separation of m. We can without loss of generality assume that $x, y \in A$. If U, V are disjoint closed subsets of A such that $x \in U$ and $y \in V$ and $U \cup V = A$, then $U, V \cup B$ are disjoint closed subsets of m such that $x \in U$ and $y \in V \cup B$ and $U \cup (V \cup B) = m$. This contradicts $x \sim_m y$, so $x \sim_A y$. However, this contradicts minimality of m, so there cannot exist a separation of m. This shows that m is connected, meaning that x and y are in the same connected component. □

Proposition 4.0.14. *A locally compact Hausdorff space X is totally disconnected if and only if it has a basis of open compact sets.*

Proof. Suppose X is totally disconnected. Take $x \in X$ and an open neighbourhood U of x. We can find a precompact neighbourhood V of x such that $\bar{V} \subset U$. Any connected component of $\bar{V}$ is connected in X, so $\bar{V}$ is totally disconnected. If we can show that totally disconnected compact Hausdorff spaces have bases of open compact sets, then we are done, for then there is a compact open neighbourhood of x contained V.

Assume now that X is compact. Then $x \sim_X y$ if and only if $x = y$. In particular, for every $y \in U^c$, there is a clopen set C_y such that $y \in C_y$ and $x \in C_y^c$. By compactness, there is finite number of elements $y_1, \dots, y_n$ such that $U^c \subset C_{y_1} \cup \dots \cup C_{y_n}$. Then $x \in C_{y_1}^c \cap \dots \cap C_{y_n}^c \subset U$. As a finite intersection of clopen sets, $C_{y_1}^c \cap \dots \cap C_{y_n}^c$ is clopen, and thus compact since X is compact. This shows that X has a basis of open compact sets.

Conversely, suppose that X has a basis of compact open sets. Take $x, y \in X$. If x and y are not equal, we can find an open neighbourhood U of x not containing y. Because X has a basis of clopen sets, we can find a clopen neighbourhood W of x contained in U. Thus, W is a clopen set containing x and not containing y, meaning that x and y are not in the same component. This shows that the components of X are one point sets. □

Proposition 4.0.15. *If $\mathcal{G}$ is an étale groupoid with totally disconnected locally compact Hausdorff space of units, then $\mathcal{G}$ has a basis of compact open bisections.*

Proof. Take $x \in \mathcal{G}$ and let U be an open neighbourhood of x. Since $\mathcal{G}$ is étale, we may without loss of generality assume that U is an open bisection containing x. The range map $r : U \to r(U) \subset \mathcal{G}^{(0)}$ is a homeomorphism onto an open subset of $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is a totally disconnected locally compact Hausdorff space, there exists an open compact set V that is contained in $r(U)$ and contains $r(x)$. Then $r^{-1}(V)$ is a compact open bisection containing x that is contained in U. □

We denote by $B(\mathcal{G})$ the set of open compact bisections of a groupoid $\mathcal{G}$.

Proposition 4.0.16. *Let $\mathcal{G}$ be a topological groupoid with Hausdorff space of units. The open compact bisections $B(\mathcal{G})$ is an inverse subsemigroup of $\Gamma(\mathcal{G})$.*

Proof. We need to show that $B(\mathcal{G})$ is closed under multiplication and inversion. That is, if $U, V \in B(\mathcal{G})$, we need to show that U^{-1} and UV are compact.

Since the inverse map on $\mathcal{G}$ is a homeomorphism, the set U^{-1} must be compact if U is compact.

Let Δ denote the diagonal in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. By the Hausdorff property, Δ is closed. Introduce the map $\varphi : \mathcal{G} \times \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ such that $\varphi(x, y) = (s(x), r(y))$. Then $\mathcal{G}^2 = \varphi^{-1}(\Delta)$. By continuity, $\mathcal{G}^2$ is closed. The product UV is the image under the multiplication map of $\mathcal{G}^2 \cap (U \times V)$. If U and V are compact, then $\mathcal{G}^2 \cap (U \times V)$ is compact. It follows from continuity of the multiplication map that UV is compact. □

Henceforth, we will sometimes denote elements of $\Gamma(\mathcal{G})$ with γ to emphasise the semigroup properties of $\Gamma(\mathcal{G})$. In this context, we also write γ^* instead of γ^{-1}.

Let $\mathcal{G}$ be an étale groupoid with totally disconnected locally compact space of units. We denote by $B(\mathcal{G})$ the inverse semigroup of compact open bisections. We associate to $B(\mathcal{G})$ the vector space $\mathbf{C}[B(\mathcal{G})]'$ that has as its basis the elements of $B(\mathcal{G})$. We denote by u_γ the vector corresponding to $\gamma \in B(\mathcal{G})$ in $\mathbf{C}[B(\mathcal{G})]'$. Extending linearly the multiplication on $B(\mathcal{G})$ turns $\mathbf{C}[B(\mathcal{G})]'$ into an algebra, and extending the map $\gamma \mapsto \gamma^*$ conjugate linearly turns $\mathbf{C}[B(\mathcal{G})]'$ into *-algebra. For $\gamma_1, \gamma_2 \in B(\mathcal{G})$, we say that $\gamma_1 \perp \gamma_2$ if $r(\gamma_1) \cap r(\gamma_2) = \emptyset$ and $s(\gamma_1) \cap s(\gamma_2) = \emptyset$. Note that $\gamma_1 \cup \gamma_2 \in B(\mathcal{G})$ if $\gamma_1 \perp \gamma_2$. Introduce the set

$$S = \{u_{\gamma_1 \cup \gamma_2} - u_{\gamma_1} - u_{\gamma_2} : \gamma_1 \perp \gamma_2\} \subset \mathbf{C}[B(\mathcal{G})]'.$$

Let $\tilde{S}$ be the *-algebra generated by S in $\mathbf{C}[B(\mathcal{G})]'$. We define $\mathbf{C}[B(\mathcal{G})]$ to be the *-algebra given by the quotient $\mathbf{C}[B(\mathcal{G})]'/\tilde{S}$.

We now set out to define the reduced groupoid C*-algebra of an étale groupoid $\mathcal{G}$ with totally disconnected locally compact space of units. For $x \in \mathcal{G}^{(0)}$, recall that $\mathcal{G}_x = s^{-1}(x)$. Take any $g \in \mathcal{G}_x$. There is an open bisection U containing g. Since the source map is one-to-one on U, we have $\mathcal{G}_x \cap U = \{g\}$. Thus, $\mathcal{G}_x$ is a discrete space. For $u_\gamma \in \mathbf{C}[B(\mathcal{G})]'$, we define an operator $\lambda_x(u_\gamma)$ on $C_c(\mathcal{G}_x)$ via

$$\lambda_x(u_\gamma)\delta_g = \begin{cases} \delta_{\gamma g}, & r(g) \in s(\gamma), \\ 0, & \text{otherwise.} \end{cases}$$

We use a slight abuse of notation in the above definition. If $s(\gamma) \cap r(g) \neq \emptyset$, then there exists a unique element $h \in \gamma$ such that $s(h) = r(g)$, and by γg we really mean hg. To see that $\lambda_x(u_\gamma)$ is bounded in the supremum norm, note that if $g_1, g_2 \in \mathcal{G}_x$ are such that $r(g_1) \neq r(g_2)$, then $\gamma g_1 \neq \gamma g_2$ as the range of these elements are non equal as a consequence of the range map being one-to-one on γ, and if $r(g_1) = r(g_2)$ and $\gamma g_1 = \gamma g_2$, then $hg_1 = hg_2$ for some $h \in \gamma$, so $g_1 = g_2$. This shows that if $g_1 \neq g_2$, then $\gamma g_1 \neq \gamma g_2$. From this it follows that $\lambda_x(u_\gamma)$ is bounded. By extending $\lambda_x(u_\gamma)$ continuously, we may view $\lambda_x(u_\gamma)$ as a bounded operator on $\ell^2(\mathcal{G}_x)$. We may then extend λ_x to a linear map $C[B(\mathcal{G})]' \to B(\ell^2(\mathcal{G}_x))$. Straightforward calculations show that $\lambda_x : \mathbf{C}[B(\mathcal{G})]' \to B(\ell^2(\mathcal{G}_x))$ is a *-homomorphism.

Suppose that $\gamma_1, \gamma_2 \in B(\mathcal{G})$ are such that $\gamma_1 \perp \gamma_2$. Then

$$\lambda_x(u_{\gamma_1 \cup \gamma_2})\delta_g = \begin{cases} \delta_{(\gamma_1 \cup \gamma_2)g}, & r(g) \in s(\gamma_1 \cup \gamma_2), \\ 0, & \text{otherwise.} \end{cases}$$

Since $s(\gamma_1 \cup \gamma_2)$ is a disjoint union of $s(\gamma_1)$ and $s(\gamma_2)$, we see that $\lambda_x(u_{\gamma_1 \cup \gamma_2}) = \lambda_x(u_{\gamma_1}) + \lambda_x(u_{\gamma_2})$. Thus, λ_x factors into a representation $\lambda_x : \mathbf{C}[B(\mathcal{G})] \to B(\ell^2(\mathcal{G}_x))$. We define $\lambda : \mathbf{C}[B(\mathcal{G})] \to B(\bigoplus_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x))$ as the Hilbert space direct sum of the representations λ_x as x runs through $\mathcal{G}^{(0)}$. The *reduced groupoid C*-algebra* denoted $\mathrm{C}^*_r(\mathcal{G})$ is the closure of $\lambda(\mathbf{C}[B(\mathcal{G})])$ in $B(\bigoplus_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x))$.

Let $\mathcal{G}$ be an étale groupoid. For $\gamma \in \Gamma(\mathcal{G})$, we denote by ψ_γ the homeomorphism $\psi_\gamma = r|_\gamma \circ (s|_\gamma)^{-1} : s(\gamma) \to r(\gamma)$. Observe that $\psi_{\gamma^*} = \psi_\gamma^{-1}$. If $x \in s(\gamma)$, then $x = y^{-1}y$ for some $y \in \gamma$, so $\psi_\gamma(x) = yy^{-1} = x^{-1}$, meaning that ψ_γ is a restriction of the inversion map.

Just as the reduced group C*-algebra is a group C*-algebra, we want to define groupoid C*-algebras in a way making the reduced groupoid C*-algebra a groupoid C*-algebra. As we shall see, the following definition proposed in [2] will do just that.

Definition 4.0.17. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. A unital $\mathcal{G}$-C*-algebra is a unital C*-algebra A together with the following data.

(i) An injective unital *-homomorphism $\iota : C_0(\mathcal{G}^{(0)}) \to A$.

(ii) For each $\gamma \in \Gamma(\mathcal{G})$ a *-isomorphism $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$, where in general for open $U \subset \mathcal{G}^{(0)}$ we put $A_U = \overline{C_0(U)AC_0(U)}$ (using the inclusion ι this makes A_U well defined because $C_0(U)$ can be included in $C_0(\mathcal{G}^{(0)})$ as U is open in the unit space). For all $\gamma \in \Gamma(\mathcal{G})$ and all $f \in C_0(s(\gamma))$ we require that $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma(\iota(f))$. We also require that for all $\gamma_1, \gamma_2 \in \Gamma(\mathcal{G})$ the following diagram commutes.

$$\begin{array}{ccc} A_{\psi_{\gamma_2^*}(s(\gamma_1))} & \xrightarrow{\alpha_{\gamma_1\gamma_2}} & A_{\psi_{\gamma_1}(r(\gamma_2))} \\ \Big\downarrow \alpha_{\gamma_2} & \nearrow \alpha_{\gamma_1} & \\ A_{s(\gamma_1) \cap r(\gamma_2)} & & \end{array}$$

Example 4.0.18. Suppose G is a discrete group and that A is a unital G-C*-algebra. Consider now G as a groupoid. We show that A is a unital groupoid C*-algebra. The space of units is just the unit e in G and the open bisections are the one point sets in G. Since $C_0(e) = \mathbf{C}$ and $\mathbf{C}$ is included in A because A is unital, we have an inclusion $\iota : C_0(\mathcal{G}^{(0)}) \to A$. For $\gamma \in G$, we take α_γ to be the action of G as a group on A. By definition, α_γ is a unital *-isomorphism on A.

Note that $A_{r(\gamma)} = A_{s(\gamma)} = A$. Because α_γ is unital and ψ_γ is the identity map on $\{\gamma\}$, the identity $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma(\iota(f))$ holds for all $f \in C_0(s(\gamma)) = C_0(\gamma)$. The fact that $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$ on A completes our of proof of A being a unital groupoid C*-algebra.

We investigate how we can make the reduced groupoid C*-algebra a groupoid C*-algebra.

Proposition 4.0.19. *Let $\mathcal{G}$ be an étale groupoid with totally disconnected locally compact Hausdorff space of units. There is a unital *-homomorphism $\iota : C_0(\mathcal{G}^{(0)}) \to \mathrm{C}^*_r(\mathcal{G})$ of C*-algebras.*

Proof. We first mention that any compact open set in $\mathcal{G}^{(0)}$ is a compact open bisection in $B(\mathcal{G})$ because the range and source maps restrict to the identity on the unit space and because the unit space is open in $\mathcal{G}$.

Consider the vector space $V' = \operatorname{span}\{u_\gamma : \gamma \in B(\mathcal{G}), \gamma \subset \mathcal{G}^{(0)}\} \subset \mathbf{C}[B(\mathcal{G})]'$. Because products and inverses of units are also units, V' is a self-adjoint subalgebra of $\mathbf{C}[B(\mathcal{G})]'$. Denote by $C_c^{lc}(\mathcal{G}^{(0)})$ the self-adjoint subalgebra of $C_0(\mathcal{G}^{(0)})$ consisting of locally constant functions with compact support. Let $\varphi : V' \to C_c^{lc}(\mathcal{G}^{(0)})$ be the linear map defined by $\varphi(u_\gamma) = \chi_\gamma$. That $\chi_\gamma \in C_c^{lc}(\mathcal{G}^{(0)})$ is a consequence of γ being compact and open. Since $\gamma^* = \gamma$ as γ consists of units, φ preserves adjoints. If $\gamma_1, \gamma_2 \in B(\mathcal{G})$ are contained in the unit space, then it holds that $\gamma_1\gamma_2 = \gamma_1 \cap \gamma_2$, so $\varphi(u_{\gamma_1\gamma_2}) = \varphi(u_{\gamma_1})\varphi(u_{\gamma_2})$ because $\chi_{\gamma_1}\chi_{\gamma_2} = \chi_{\gamma_1\cap\gamma_2}$. This shows that φ is a *-homomorphism.

For $\gamma_1, \gamma_2 \in B(\mathcal{G})$ contained in the unit space we have that $\gamma_1 \perp \gamma_2$ if and only if $\gamma_1 \cap \gamma_2 = \emptyset$. If $\gamma_1 \cap \gamma_2 = \emptyset$, then $\varphi(u_{\gamma_1\cup\gamma_2}) = \chi_{\gamma_1} + \chi_{\gamma_1}$, so φ factors through $\mathbf{C}[B(\mathcal{G})]$ to a *-homomorphism $\varphi : V \to C_c^{lc}(\mathcal{G}^{(0)})$, where V denotes the image of V' under the quotient map $\mathbf{C}[B(\mathcal{G})]' \to \mathbf{C}[B(\mathcal{G})]$.

Suppose $\varphi(\sum r_i u_{\gamma_i}) = 0$ where $r_i \in \mathbf{C}$ and γ_i are compact open bisections contained in the unit space. We have that $\gamma_1 \cap \gamma_2$, $\gamma_1 \setminus \gamma_2$, and $\gamma_2 \setminus \gamma_1$ are all compact open bisections, and so by the rule $u_{\gamma_1\cup\gamma_2} = u_{\gamma_1} + u_{\gamma_2}$ we may without loss of generality assume that all γ_i in the sum $\sum r_i u_{\gamma_i}$ are disjoint. In this case, $0 = \varphi(\sum r_i u_{\gamma_i}) = \sum r_i \chi_{\gamma_i}$ implies that all coefficients $r_i = 0$, so $\sum r_i u_{\gamma_i} = 0$. This shows that φ is injective.

Now we show surjectivity. Suppose $f \in C_c^{lc}(\mathcal{G}^{(0)})$. Because f is locally constant, $f^{-1}(z)$ is open for each $z \in \mathbf{C}$. By covering $\operatorname{supp}(f)$ with the open sets $f^{-1}(z)$, $z \in \mathbf{C}$, and using that f has compact support, it is seen that the range of f is finite. Let $z_1, \dots, z_n$ be the nonzero elements in the range of f. By continuity and f having compact support, $f^{-1}(z_i)$ is compact, and f being locally constant means that $f^{-1}(z_i)$ is open. Thus, we get $f = \varphi(\sum z_i u_{f^{-1}(z_i))})$. From this we also get that the inverse of φ is the map $\phi : C_c^{lc}(\mathcal{G}^{(0)}) \to V$, such that $\phi(f) = \sum z_i u_{f^{-1}(z_i))}$. We conclude that φ and in particular ϕ are *-isomorphisms.

Our next goal is to show that $\lambda \circ \phi : C_c^{lc}(\mathcal{G}^{(0)}) \to \mathrm{C}^*_r(\mathcal{G}) \subset B(\bigoplus_{x\in\mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x))$ is bounded. Take x in the unit space and consider $\lambda_x(\phi(f))\delta_g$ where $g \in \mathcal{G}_x$. We saw earlier that if $z_1, \dots, z_n$ are the nonzero elements in the range of f, then

$\phi(f) = \sum z_i u_{f^{-1}(z_i)}$. using the definition of λ_x, we get

$$\lambda_x(\phi(f))\delta_g = f(r(g))\delta_g.$$

It follows that $\|\lambda_x(\phi(f))\| \leq \|f\|_\infty$. Further, $\lambda_x(\phi(f))\delta_x = f(x)\delta_x$, so $|f(x)| \leq \|\lambda_x(\phi(f))\|$ and $\|f\|_\infty \leq \sup_{x\in\mathcal{G}^{(0)}} \|\lambda_x(\phi(f))\|$. Hence $\|f\|_\infty = \sup_{x\in\mathcal{G}^{(0)}} \|\lambda_x(\phi(f))\|$. In particular, $\|\lambda(\phi(f))\| \leq \|f\|_\infty$ showing that $\lambda \circ \phi$ is bounded. From $|f(x)| \leq \|\lambda_x(\phi(f))\| \leq |\lambda(\phi(f))\|$, we get $\|f\|_\infty \leq \|\lambda(\phi(f))\|$. Thus, $\lambda \circ \phi$ is isometric on $C_c^{lc}(\mathcal{G}^{(0)})$. Using the Stone-Weierstrass theorem for locally compact Hausdorff spaces one can show that $C_c^{lc}(\mathcal{G}^{(0)})$ is dense in $C_0(\mathcal{G}^{(0)})$. Extending $\lambda \circ \phi$, we get an isometric *-homomorphism $\iota : C_0(\mathcal{G}^{(0)}) \to \mathrm{C}_r^*(\mathcal{G})$. Isometric maps are injective and ι is unital as a composition of unital map, so ι is our desired map. □

Proposition 4.0.20. *Let $\mathcal{G}$ be an étale groupoid with totally disconnected locally compact Hausdorff space of units. We saw in the above proposition that there exists an inclusion $\iota : C_0(\mathcal{G}^{(0)}) \to \mathrm{C}_r^*(\mathcal{G})$ of unital C*-algebras. For all compact open bisections γ contained in the unit space $\mathcal{G}^{(0)}$, it holds that*

$$A_\gamma = \overline{C_0(\gamma)\mathrm{C}_r^*(\mathcal{G})C_0(\gamma)} = \chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma.$$

Proof. Because γ is compact we have that $C_0(\gamma) = C(\gamma)$. Using the inclusion $\iota : C_0(\mathcal{G}^{(0)}) \to C_r^*(\mathcal{G})$, we consider $C(\gamma)$ is a subset of $C_r^*(\mathcal{G})$.

Because γ is clopen, we have $\chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma \subset C(\gamma)\mathrm{C}_r^*(\mathcal{G})C(\gamma)$, giving the first inclusion.

We now show that $\chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma$ is closed in $\mathrm{C}_r^*(\mathcal{G})$. Suppose $(\chi_\gamma a_n \chi_\gamma)_{n\in\mathbf{N}}$ is a sequence converging to $a \in \mathrm{C}_r^*(\mathcal{G})$. By continuity,

$$a = \lim_{n\to\infty} \chi_\gamma a_\lambda \chi_\gamma = \chi_\gamma (\lim_{n\to\infty} \chi_\gamma a_\lambda \chi_\gamma)\chi_\gamma = \chi_\gamma a \chi_\gamma \in \chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma,$$

showing that $\mathrm{C}_r^*(\mathcal{G})$ is closed. Let fug be an element of $C(\gamma)\mathrm{C}_r^*(\mathcal{G})C(\gamma)$, where $f, g \in C(\gamma)$ and $u \in \mathrm{C}_r^*(\mathcal{G})$. Since $\chi_\gamma f = f$ and $g\chi_\gamma = g$, it holds that $fug = \chi_\gamma fug\chi_\gamma \in \chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma$. Thus, $C(\gamma)\mathrm{C}_r^*(\mathcal{G})C(\gamma) \subset \chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma$, and because $\chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma$ is closed we get $\overline{C(\gamma)\mathrm{C}_r^*(\mathcal{G})C(\gamma)} \subset \chi_\gamma \mathrm{C}_r^*(\mathcal{G})\chi_\gamma$. □

Proposition 4.0.21. *Let $\mathcal{G}$ be an étale groupoid with totally disconnected locally compact Hausdorff space of units. For $\gamma \in B(\mathcal{G})$, we define a map $\alpha_\gamma : \mathrm{C}_r^*(\mathcal{G})_{s(\gamma)} \to \mathrm{C}_r^*(\mathcal{G})_{r(\gamma)}$ via $\alpha_\gamma(a) = u_\gamma a u_{\gamma^*}$. These maps are *-isomorphisms and for all $\gamma_1, \gamma_2 \in B(\mathcal{G})$ the diagram below commutes.*

$$\begin{array}{ccc} \mathrm{C}_r^*(\mathcal{G})_{\psi_{\gamma_2^*}(s(\gamma_1))} & \xrightarrow{\alpha_{\gamma_1\gamma_2}} & \mathrm{C}_r^*(\mathcal{G})_{\psi_{\gamma_1}(r(\gamma_2))} \\ \downarrow \alpha_{\gamma_2} & \nearrow \alpha_{\gamma_1} & \\ \mathrm{C}_r^*(\mathcal{G})_{s(\gamma_1)\cap r(\gamma_2)} & & \end{array}$$

Proof. By the previous proposition, we have that $C_r^*(\mathcal{G})_{s(\gamma)} = \mu_{\gamma^*\gamma} C_r^*(\mathcal{G})_{s(\gamma)}\mu_{\gamma^*\gamma}$. If $a \in C_r^*(\mathcal{G})$, then $\mu_\gamma \mu_{\gamma^*\gamma} a \mu_{\gamma^*\gamma}\mu_{\gamma^*} \in C_r^*(\mathcal{G})_{r(\gamma)}$, so the domain and codomain of

α_γ are correct. Since $u_\gamma a_1 a_2 u_{\gamma^*} = u_\gamma a_1 u_{\gamma^*} u_\gamma a_2 u_{\gamma^*}$ and $(u_\gamma a u_{\gamma^*})^* = u_\gamma a^* u_{\gamma^*}$, the map α_γ is a *-homomorphism. The inverse of α_γ is α_{γ^*}, showing that α_γ is a *-isomorphism.

That the diagram commutes is just a consequence of the identity $(\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*$. □

Proposition 4.0.22. *Let $\mathcal{G}$ be an étale groupoid with totally disconnected locally compact Hausdorff space of units. It holds that $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma(\iota(f))$ for all $f \in C_0(s(\gamma))$ and $\gamma \in B(\mathcal{G})$.*

Proof. Take any $\gamma' \in B(\mathcal{G})$ such that $\gamma' \subset s(\gamma)$. We first check the identity for $f = \chi_{\gamma'}$. Take $x \in r(\gamma)$. We have that

$$f(\psi_{\gamma^*}(x)) = \begin{cases} 1, & x \in \psi_\gamma(\gamma'), \\ 0, & \text{otherwise.} \end{cases}$$

Because γ' is contained in the space of units, $\gamma' = (\gamma')^{-1}$, which in turn is equal to $\psi_\gamma(\gamma')$ so $f \circ \psi_{\gamma^*} = \chi_{\gamma'}$ and $\iota(f \circ \psi_{\gamma^*}) = u_{\gamma'}$

Meanwhile, $\chi_{\gamma'} = \chi_{s(\gamma)}\chi_{\gamma'}\chi_{s(\gamma)} \in \mathrm{C}^*_r(\mathcal{G})_{s(\gamma)}$, so $\alpha_\gamma(\iota(f))$ is indeed well defined. We get $\alpha_\gamma(f) = u_{\gamma\gamma^*\gamma\gamma'\gamma^*\gamma\gamma^*} = u_{\gamma\gamma'\gamma^*}$. If $x \in \gamma'$, then $x = y^{-1}y$ for some $y \in \gamma$. It follows that $\gamma\gamma'\gamma^* = \{yy^{-1}yy^{-1} : y \in \gamma \text{ and } y^{-1}y \in \gamma'\} = (\gamma')^{-1} = \gamma'$, so $\alpha_\gamma(\iota(f)) = u_{\gamma'}$.

We see now that $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma(\iota(f))$ for all $f \in C(s(\gamma))$ of the form $f = \chi_{\gamma'}$ where $\gamma' \subset s(\gamma)$ is clopen in $s(\gamma)$. By linearity, $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma(\iota(f))$ holds for all $f \in C^{lc}(s(\gamma))$, and by Stone-Weierstrass and continuity the identity holds for all $f \in C(s(\gamma))$. □

Comparing Propositions 4.0.19, 4.0.21, and 4.0.22 with Definition 4.0.17, we see that we have almost shown that the reduced groupoid C*-algebra carries a structure making it a groupoid C*-algebra. What is missing is that we have only defined the maps α_γ for $\gamma \in B(\mathcal{G})$, and not for every $\gamma \in \Gamma(\mathcal{G})$. However, it is shown in [2, Proposition 4.6] that the subsemigroup $B(\mathcal{G}) \subset \Gamma(\mathcal{G})$ is sufficiently large for there to be a well-behaved correspondence between groupoid C*-algebras according to Definition 4.0.17, and the groupoid C*-algebras where one restricts attention to the subsemigroup $B(\mathcal{G}) \subset \Gamma(\mathcal{G})$.

Appendix A

A.1 Limits of nets

A directed set Λ consists of a nonempty set together with a preorder $\leq$ such that every two element subset of Λ has an upper bound. A net in a topological space X is a function from a directed set Λ to X. We denote a net by $(x_\lambda)_{\lambda\in\Lambda}$, where x_λ is the function value at λ. Note that sequences are special cases of nets.

We say that the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in X$ if for every open neighbourhood U of x there exists a λ_0 such that $\lambda_0 \leq \lambda \implies x_\lambda \in U$.

Proposition A.1.1. *Let X be a Hausdorff space. If $(x_\lambda)_{\lambda\in\Lambda}$ is a net converging to x_1 and x_2, then $x_1 = x_2$.*

Proof. Suppose $x_1 \neq x_2$. Let U and V be disjoint open sets such that $x_1 \in U$ and $x_2 \in V$. Let λ_1 be such that $\lambda \geq \lambda_1 \implies x_\lambda \in U$ and let λ_2 be such that $\lambda \geq \lambda_2 \implies x_\lambda \in V$. If λ_0 is an upper bound for λ_1 and λ_2, then x_{λ_0} is in both U and V. This contradiction shows that $x_1 = x_2$. □

Proposition A.1.2. *Let X be a topological space. If $A \subset X$ and $x \in X$, then $x \in \bar{A}$ if and only if x the limit of some net in A.*

Proof. We start with the forward implication. If $x \in X$, we can just put $\lambda = x$, and $\Lambda = \{x\}$, and define $x_\lambda = x$. Since $\bar{A}$ is the union of A and the limit points of A, what is left to check is that limit points of A are limits of nets in A. Assume that x is a limit point of A. Define Λ to be the set of neighbourhoods of x in X. Inclusion defines a preorder on Λ via $V \leq U \iff U \subset V$. The set Λ together with $\leq$ defines a directed set since two neighbourhoods U, V of x has upper bound $U \cap V$ in Λ. Because each set $\lambda \in \Lambda$ has nonempty intersection with A, we associate with λ an element $x_\lambda \in \lambda \cap A$. This defines a net $(x_\lambda)_{\lambda\in\Lambda}$ in A. Now we check that x is the limit of this net. Let U be an open neighbourhood of x. Set $\lambda_0 = U$, so $\lambda_0 \in \Lambda$. If $\lambda_0 \leq \lambda$, then λ is a subset of U, and therefore $x_\lambda \in U$.

The definition of a net implies directly that a limit of net in A is either in A or a limit point of A. □

Proposition A.1.3. *Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if and only if* $\lim x_\lambda = x \implies \lim f(x_\lambda) = f(x)$ *for all nets* $(x_\lambda)_{\lambda\in\Lambda}$ *in* X.

Proof. Assume first that $f : X \to Y$ is continuous and that $(x_\lambda)_{\lambda\in\Lambda}$ is a net with limit x. Let V be an open neighbourhood of $f(x)$. Then $f^{-1}(V)$ is an open neighbourhood of x, so there exists a $\lambda_0 \in \Lambda$ with $\lambda_0 \leq \lambda \implies x_\lambda \in f^{-1}(V) \implies f(x_\lambda) \in V$.

For the other direction, let $f : X \to Y$ be a map such that $\lim x_\lambda = x \implies \lim f(x_\lambda) = f(x)$ for all nets $(x_\lambda)_{\lambda\in\Lambda}$ in X. Let V be an open set in Y. Assume for contradiction that $f^{-1}(V)$ is not open. Then there is a point $x \in f^{-1}(V)$ such that all open neighbourhoods of x intersect $(f^{-1}(V))^c$. Define Λ to be the set of neighbourhoods of x in X. Inclusion defines a preorder on Λ via $V \leq U \iff U \subset V$. The set Λ together with $\leq$ defines a directed set since two neighbourhoods U, V of x has upper bound $U \cap V$ in Λ. For each $\lambda \in \Lambda$ we associate an element $x_\lambda \in (f^{-1}(V))^c \cap \lambda$. This defines a net $(x_\lambda)_{\lambda\in\Lambda}$ in X with limit x. By assumption, $\lim f(x_\lambda) = f(x) \in V$, so there is a λ with $f(x_\lambda) \in V$. This contradicts that $x_\lambda \notin f^{-1}(V)$. □

Let A be a Banach algebra and let $(x_\lambda)_{\lambda\in\Lambda}$ be an indexed family of elements in A. We denote by Λ' the collection of finite subsets of Λ. We get a preorder on Λ' via $F \leq G \iff F \subset G$. The union $F \cup G$ gives an upper bound for the pair $F, G \in \Lambda'$. By defining $x_F = \sum_{\lambda\in F} x_\lambda$ we get a net $(x_F)_{F\in\Lambda'}$ in A. If the limit of this net exists we define $\sum_{\lambda\in\Lambda} x_\lambda = \lim_{F\in\Lambda'} x_F$.

Consider now the case that all x_λ are real numbers, that $(x_\lambda)_{\lambda\in\Lambda}$ is increasing in the sense that $\lambda \leq \lambda' \implies x_\lambda \leq x_{\lambda'}$ and that $\lim_{\lambda\in\Lambda} x_\lambda = x$ exists. It follows from the definition of the limit and a contradiction argument that $x \leq \sup_{\lambda\in\Lambda} x_\lambda$. Conversely, we show that x is an upper bound for $(x_\lambda)_{\lambda\in\Lambda}$. Assume for contradiction that there is a λ' such that $x < x_{\lambda'}$. There is a $\lambda_0 \in \Lambda$ with $\lambda_0 \leq \lambda \implies x_\lambda < x_{\lambda'}$. Let λ_1 be an upper bound for λ_0 and λ'. Then $x_{\lambda_1} < x_{\lambda'}$ while also $x_{\lambda'} \leq x_{\lambda_1}$ because the net is increasing. This is a contradiction, and we conclude that $\lim_{\lambda\in\Lambda} x_\lambda = \sup_{\lambda\in\Lambda} x_\lambda$.

Assume now that $\sup_{\lambda\in\Lambda} x_\lambda = x < \infty$. For every $\varepsilon > 0$ there exists a λ_0 such that $x - \varepsilon < x_{\lambda_0} \leq x$. Because the net is increasing, we have

$$\lambda_0 \leq \lambda \implies x - \varepsilon < x_\lambda \leq x.$$

Thus, $\lim_{\lambda\in\Lambda} x_\lambda$ exists and is equal to x. motivated by this, we define $\lim_{\lambda\in\Lambda} x_\lambda = \infty$ if the limit does not exist, so

$$\lim_{\lambda\in\Lambda} x_\lambda = \sup_{\lambda\in\Lambda} x_\lambda$$

always holds true. If $(x_\lambda)_{\lambda\in\Lambda}$ is an indexed family of nonnegative real numbers, then $(x_F)_{F\in\Lambda'}$ is increasing. Thus,

$$\sum_{\lambda\in\Lambda} x_\lambda = \sup_{F\in\Lambda'} \sum_{\lambda\in F} x_\lambda.$$

In particular, if the sum $\sum_{\lambda\in\Lambda} x_\lambda$ is finite, then $x_\lambda \neq 0$ for at most a countable number of λ. For otherwise, there would be an $n \in N$ such that $x_\lambda \geq 1/n$ for an infinite number of λ, implying that $\sup_{F\in\Lambda'} \sum_{\lambda\in F} x_\lambda = \infty$.

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