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Jordan Algebras: Definitions and examples

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Abstract

A Jordan algebra is a nonassociative, commutative algebra that satisfies a weaker form of associativity known as the Jordan identity. We go through some basic properties and look at of some the most important classes of Jordan algebras: full, Hermitian, and quadratic factors. We formalize the notion of composition algebras which appear naturally as coordinates of certain Jordan matrix algebras. We state Macdonald's theorem and explore some of its important consequences, and give a brief exposition of Peirce decompositions used in studying the structure of Jordan algebras. Finally we sketch the development of the structure theory for Jordan algebras since their inception in 1933.

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Introduction

Jordan algebras were first introduced in 1933 by Pascual Jordan in order to find a suitable model of quantum mechanical *observables*. An observable in physics is any physical quantity that can be measured. In classical mechanics, the mathematical model for an observable is a real-valued function defined on the set of all possible states of a system. In the Copenhagen interpretation of quantum mechanics, observables are modelled by certain linear operators on a Hilbert space. The eigenvalues of these operators are always real numbers, and the collection of eigenvalues correspond to the possible values that the observable can be measured as having.

In finite dimensions, observables can be represented by Hermitian matrices. A *Hermitian* or *self-adjoint* matrix is a complex square matrix that is equal to its own conjugate transpose. There are many ways to combine Hermitian matrices into new Hermitian matrices, and Jordan sought to formalize this "algebra of observables". Many of the basic matrix operations do not preserve the Hermitian property, however. For example, multiplying a Hermitian matrix by a complex scalar does not produce a Hermitian matrix unless the scalar is a real number, and the product of two Hermitian matrices is not Hermitian unless the matrices commute. Representing observables as Hermitian matrices is therefore susceptible to the criticism that many matrix operations are not "observable"; they are not intrinsic operations and do not correspond to anything physically meaningful.

Jordan wanted to find an axiomatic framework that captured the algebraic properties of observables that would not be reliant on some ambient, outer structure with operations that did not make sense for observables. He set out to study the algebraic properties of Hermitian matrices in order to see what other possible non-matrix systems would satisfy these axioms. This led him to define the notion of a *Jordan algebra*, or an *r-number system*, as it was initially called. After some investigation, Jordan thought that all the algebraic properties of Hermitian matrices could be derived from two basic properties (this later turned out to be false). The first was commutativity. A product in a Jordan algebra needed to be commutative since it is only when two Hermitian matrices commute that their matrix product is again Hermitian. The second is known as the **Jordan identity**:

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x).$$

We can think of this property as a form of weaker associativity, requiring merely that for any element y, multiplying with some x and its square $x^2 := x \cdot x$ can be done in any order.

The Hermitian matrices live inside the full matrix algebra of complex square matrices. In general one can from any associative algebra A constuct a Jordan algebra A^+ by replacing the product xy of A with a derived product called *quasi-multiplication* or the **Jordan product**:

$$x \bullet y = \frac{1}{2}(xy + yx).$$

A Jordan algebra that lives inside some governing associative algebra is called *special*, otherwise it is called *exceptional*. Jordan's quest was to find an exceptional setting for quantum mechanics.

In 1934, Jordan, John von Neumann and Eugene Wigner showed in [JNW34] that every finite-dimensional, *formally real* Jordan algebra could be written as a direct sum of *simple* ones, and that these simple building blocks came in five basic types: Four of these were different types of Hermitian matrix algebras $H_n(A)$, living inside full associative matrix algebras $M_n(A)$. The last type was of a different nature. This was $H_3(\mathbb{O})$, a 27-dimensional algebra of 3×3 Hermitian matrices with entries from the **Cayley algebra** (or **octonions**) $\mathbb{O}$, known now as an **Albert algebra**. The Albert algebra did not seem special since its entries came from the *nonassociative* algebra $\mathbb{O}$. When the coordinates of matrices are not associative, then matrix multiplication is not associative, so $H_3(\mathbb{O})$ did not appear to live inside any associative algebra. Shortly after, A. A. Albert was able to prove in [AAA34] that it was indeed exceptional.

The classification of finite-dimensional, formally real Jordan algebras came as a disappointment to physicists. There were only one exceptional algebra in this list, of dimension 27. This was of course insufficient to serve as a model of quantum mechanics, and moreover, it provided little information as to the possible existence of an infinite-dimensional exceptional one. Hope remained that one could find exceptional Jordan algebras of infinite dimensions.

In 1979-1983, mathematician Efim Zelmanov showed in a series of papers [Zel79a], [Zel79b], [Zel83] that even in the infinite-dimensional case, there were no simple exceptional Jordan algebras other than the Albert algebras. He also gave a classification similar to the finite-dimensional case, but for arbitrary simple Jordan algebras of any dimension. This was the conclusive end to the search for an exceptional setting for quantum mechanics.

While the initial objective had failed, it later became apparent that Jordan algebras had a rich and interesting theory in their own right, with many applications to various other areas of mathematics. The application of Jordan algebras to the theory of Lie algebras was historically the first example, and the connections between Lie algebras and Jordan algebras brought about continuous interest to Jordan theory. Another important connection was with differential geometry and symmetric spaces.

In 1994, Zelmanov was awarded the Fields Medal for his solution to the *Restricted Burnside Problem* (english translation in [Zel91],[Zel92]). Burnside's original problem asks whether periodic groups (every element has finite order) that are finitely generated are necessarily finite. The answer to Burnside's original problem was answered in the negative by Golod in 1964. However, there were several variants to the original problem, among them the *restricted* Burnside problem. It was known that this problem could be reduced to a problem regarding certain Lie p-rings (Lie algebras over $\mathbb{Z}_p$ for a prime p). Zelmanov found that there was a natural Jordan algebra structure in characteristic p that could be used to solve the problem about the Lie p-rings. Notably for characteristic $p = 2$, both Lie algebras and ordinary, "linear" Jordan algebras don't work very well since the Jordan product $xy + yx$ and the Lie bracket $xy - yx$ become indistinguishable. The theory of Jordan algebras had developed over fields of characteristic $\neq 2$ due to the nature of the Jordan product, but in 1967 Kevin McCrimmon gave a definition of a *quadratic* Jordan algebra, with a product which, loosely speaking, looked like $(x, y) \mapsto xyx$. This definition filled the gap of characteristic 2, and it was precisely the quadratic formulation that enabled Zelmanov to use Jordan algebras to solve the problem.

2

Preliminaries

2.1 Algebras

Definition 2.1.1 (Multilinear map)**.** Let $V_1, V_2, \ldots, V_n, W$ be vector spaces over the same field F. A map $f : V_1 \times V_2 \times \cdots \times V_n \to W$ is called **n-multilinear** if it is linear in each of its n arguments: for each i and any $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$, the map

$$v_i \mapsto f(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n)$$

is linear. In particular for $n = 2$ and $n = 3$ a multilinear map is called **bilinear** and **trilinear**, respectively.

Definition 2.1.2 (Algebra over a field)**.** An **algebra over a field** F, or simply an **algebra**, is a vector space A over F together with a bilinear product $p : A \times A \to A$. The product is usually abbreviated as concatenation $xy := p(x, y)$. Explicitly, we have for all $x, y, z \in A$, $\alpha, \beta \in F$

$$\begin{aligned}(x + y)z &= xz + yz, \\ z(x + y) &= zx + zy, \\ (\alpha x)(\beta y) &= (\alpha\beta)(xy).\end{aligned}$$

An algebra is called **associative** if it is associative with respect to p, and **commutative** if it is commutative with respect to p. An algebra is called **unital** if it has a two-sided identity element with respect to p, i.e. if there exists some element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

In some situations, associativity of the product is assumed in the definition of an algebra. Since we are interested in Jordan algebras whose product is not necessarily associative, *we shall never make the assumption that an algebra is associative unless explicitly mentioned*. An algebra that is not assumed to be associative is sometimes called a *nonassociative* algebra, meaning it may or may not be associative.

Definition 2.1.3 (Algebra homomorphism)**.** An **algebra homomorphism** is a linear map $\varphi : A \to B$ between algebras A, B that preserves multiplication: $\varphi(xy) = \varphi(x)\varphi(y)$. If the algebras are unital, we impose the additional requirement that $\varphi(1_A) = 1_B$.

Definition 2.1.4 (Subalgebras and ideals)**.** For any algebra A, a **subalgebra** S of A is a vector subspace that is closed under multiplication with itself: $SS \subset A$. For unital algebras, we must also require that $1 \in S$. A (two-sided) **ideal** I of A is a vector subspace closed under left and right multiplication by A: $AI \subset I$ and $IA \subset I$. A **proper** ideal is an ideal that is different from the **improper** ideals A and $\mathbf{0}$.

Definition 2.1.5 (Subalgebra generated by a set)**.** Let A be an algebra over a field F and let $S \subset A$. The **subalgebra of** A **generated by** S is the smallest subalgebra of A containing S, i.e. it is the intersection of all subalgebras of A containing S.

Definition 2.1.6 (Quotient algebra)**.** Any ideal I of an algebra A is the kernel of the canonical projection homomorphism $\pi : x \mapsto \overline{x}$ from A to the **quotient algebra** A/I consisting of all cosets $\overline{x} = x + I$ with the induced operations $\lambda\overline{x} = \overline{\lambda x}$, $\overline{x} + \overline{y} = \overline{x+y}$, and $\overline{x}\ \overline{y} = \overline{xy}$.

Definition 2.1.7 (Direct sum and product)**.** The **direct product** $\prod_{i\in I} A_i$ of a family of algebras $\{A_i\}_{i\in I}$ indexed by some set I is the Cartesian product of the underlying sets equipped with the usual componentwise operations of scalar multiplication, addition, and multiplication.

The **direct sum** $\bigoplus_{i\in I} A_i$ is the subalgebra of $\prod_{i\in I} A_i$ consisting of all tuples with all but finitely many entries equal to zero. An element of the direct sum can be represented as a finite sum $a_{i_1} + \dots a_{i_n}$ of elements $a_j \in A_j$. When necessary, the symbol $\boxplus$ is used to denote an algebra direct sum to distinguish it from a mere vector space direct sum.

For each $i \in I$ there are canonical projections π_i of both the direct product and direct sum onto the ith component A_i.

Definition 2.1.8 (Simple algebra)**.** An algebra A is called **simple** if it is nontrivial ($A \cdot A \neq \mathbf{0}$) and has no proper ideals. An algebra A is **semisimple** if it is a finite direct sum of simple algebras.

Definition 2.1.9 (Associator and commutator)**.** In any algebra A, there is a trilinear map $[-,-,-] : A \times A \times A \to A$ defined by $[x, y, z] = (xy)z - x(yz)$ called the **associator**. Similarly one defines the bilinear map $[x, y] = xy - yx$ called the **commutator**. Saying that an algebra is associative is thus to say that $[x, y, z] = 0$ for all x, y, z, and similarly an algebra is commutative iff $[x, y] = 0$ for all x, y.

We may think of the associator as in some sense measuring "how far" three elements are from associating, and the commutator as measuring how far two elements are from commutating.

Definition 2.1.10 (Nucleus and center)**.** The **nucleus** $\mathrm{Nuc}(A)$ of an algebra A is the part that associates with every element of the algebra:

$$\mathrm{Nuc}(A) := \{n \in A \mid [n, a, a'] = [a, n, a'] = [a, a', n] = 0 \text{ for all } a, a' \in A\}.$$

The **center** $\mathrm{Cent}(A)$ is the "scalar" part of the algebra, the elements that commute and associate with all elements of the algebra:

$$\mathrm{Cent}(A) := \{c \in \mathrm{Nuc}(A) \mid [c, a] = 0 \text{ for all } a \in A\}.$$

A unital algebra A can always be considered as an algebra over its center. When A is unital, it contains a copy $F1$ of the base field in its center (via the map $\lambda \mapsto \lambda \cdot 1$). An algebra A is called **central** if its center is precisely the scalar multiples $F1$, and we say that A is **central-simple** if it is simple and central. We collect some basic properties of the nucleus and center.

Proposition 2.1.1. Let A be an algebra over a field F.

(i). The nucleus $\mathrm{Nuc}(A)$ is an associative subalgebra of A, and the center $\mathrm{Cent}(A)$ is a commutative and associative subalgebra of A.

(ii). The nucleus and the center are invariant as sets under any automorphism of A.

(iii). If A is unital, then $\lambda 1 \in \mathrm{Cent}(A)$ for all $\lambda \in F$.

(iv). If A is unital, then $\mathrm{Cent}(A)$ is a commutative unital ring and A can be considered as an algebra over $\mathrm{Cent}(A)$.

(v). If A is unital and simple, then $\mathrm{Cent}(A)$ is a field.

We have the usual isomorphism theorems, or fundamental homomorphism theorems, for algebras:

Theorem 2.1.1 (Fundamental homomorphism theorems)**.**

(i) If $\varphi : A \to B$ is an algebra homomorphism, then $\ker \varphi$ is an ideal of A, the image $\varphi(A)$ is a subalgebra of B, and $A/\ker(\varphi)$ is isomorphic to $\varphi(A)$.

(ii) There is a bijective correspondence between the ideals (subalgebras, respectively) $\overline{C}$ of the quotient algebra A/B and those C of A which contain B. For any such ideals C, we have $(A/B)/\overline{C} \cong A/C$.

(iii) If B is an ideal of A and C a subalgebra of A, then $C/(C \cap B) \cong (C + B)/B$.

In Jordan matrix algebras, the entries are in a natural way elements of alternative algebras. The most important example of alternative algebras are the *octonion* algebras of dimension 8.

Definition 2.1.11 (Alternative algebra)**.** An **alternative algebra** A is an algebra that satisfies the following two conditions for all x, y:

$$x(xy) = (xx)y \tag{2.1.1}$$

$$(yx)x = y(xx). \tag{2.1.2}$$

The two identities (called the left and right **alternative laws**) become in terms of associators

$$[x, x, y] = [y, x, x] = 0, \quad x, y \in A. \tag{2.1.3}$$

Every associative algebra is alternative, and one may view alternativity as a weaker notion than associativity. The reason for the name "alternative" is that the associator in an alternative algebra A *alternates* in the sense that if π is any permutation of $1, 2, 3$, then

$$[x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}] = \operatorname{sign}(\pi)[x_1, x_2, x_3], \tag{2.1.4}$$

where $\operatorname{sign}(\pi)$ is the number 1 if π is even and -1 if π is odd. To see this, it is enough to show that

$$[x, y, z] = -[y, x, z] = [y, z, x]$$

for all $x, y, z \in A$, since a transposition and a cycle generate all the permutations. Using linearity of the associator, we have

$$\begin{aligned} [x + y, x + y, z] &= [x, x, z] + [x, y, z] + [y, x, z] + [y, y, z] \\ &= [x, y, z] + [y, x, z] = 0, \end{aligned}$$

showing $[x, y, z] = -[y, x, z]$. In the same way $[y, z, x] = -[y, x, z]$.

In particular $0 = [x, x, y] = -[x, y, x]$, so every alternative algebra satisfies the *flexible law*:

$$(xy)x = x(yx).$$

Thus we may write products xyx unambiguously, omitting brackets. From this we can derive the *Moufang identities*

$$\begin{aligned} (xax)y &= x(a(xy)), \\ y(xax) &= ((yx)a)x, \\ (xy)(ax) &= x(ya)x \end{aligned}$$

for all $x, y, a \in A$. For the first identity, we have

$$\begin{aligned} (xax)y - x(a(xy)) &= [xa, x, y] + [x, a, xy] \\ &= -[x, xa, y] - [x, xy, a] \\ &= -(x(xa))y + x((xa)y) - (x(xy))a + x((xy)a) \\ &= -(x^2a)y - (x^2y)a + x((xa)y + (xy)a) \\ &= -[x^2, a, y] - [x^2, y, a] - x^2(ay) - x^2(ya) + x((xa)y + (xy)a) \\ &= x(-x(ay) - x(ya) + (xa)y + (xy)a) \\ &= x([x, a, y] + [x, y, a]) \\ &= 0. \end{aligned}$$

The second identity can be proved by considering the *opposite algebra* A^{op}, defined the vector space A with product $x \cdot_{op} y := yx$. In A^{op}, the left alternative law becomes the right alternative law and vice versa, so A^{op} is alternative if A is. The first and second Moufang identities have the same reciprocal relationship as the alternative laws. The third identity follows from the first:

$$\begin{aligned}(xy)(ax) - x(ya)x &= [x, y, ax] + x(y(ax) - (ya)x) \\ &= -[x, ax, y] - x[y, a, x] \\ &= -(xax)y + x((ax)y - [y, a, x]) \\ &= -x(-[a, x, y] + [y, a, x]) \\ &= 0.\end{aligned}$$

The second Moufang identity is equivalent to the identity

$$[y, xa, x] = -[y, x, a]x, \tag{2.1.5}$$

since

$$[y, ax, x] = (y(xa))x - y(xax) = (y(xa))x - ((yx)a)x = -[y, x, a]x.$$

With $x = x + z$, (2.1.5) becomes

$$[y, xa, z] + [y, za, x] = -[y, x, a]z - [y, z, a]x. \tag{2.1.6}$$

A useful way to characterize an alternative algebra as a weakened form of an associative algebra is a result due to Artin.

Theorem 2.1.2 (Artin)**.** The subalgebra generated by any two elements of an alternative algebra is associative.

The proof is from Theorem III.3.1 in [Schaf66].

Proof. Let A be an alternative algebra. Let $p(x, y)$ denote an arbitrary nonassociative product of k elements z_i, $i = 1, \dots, k$ where each z_i is equal to either x or y. Denote the number k of $p(x, y)$ with $d(p(x, y))$. To show that a subalgebra generated by two elements x and y is associative, we must show that

$$[p, q, r] = 0$$

for all nonassociative products $p = p(x, y)$, $q = q(x, y)$ and $r = r(x, y)$. We proceed by induction on $n = d(p) + d(q) + d(r)$. For $n = 1, 2$ the statement clearly holds. Suppose $[p, q, r] = 0$ whenever $n < k$ for some integer k. Then in particular $d(p) < n$, thus writing p as a product of any three parts p_1, p_2, p_3 shows that any way of putting parentheses is equivalent; $[p_1, p_2, p_3] = 0$ for any choice of parts of p by the induction hypothesis. Extending the associative law to the generalized associative law, we find that all parentheses in p are unnecessary, and we may write $p = z_1 \dots z_{d(p)}$ unambiguously. Since p, q, r are in x, y, two of p, q, r must begin with the same element, say x. We may use the fact that the associator map is alternating for an alternative algebra, and assume without loss of generality that q and r begin with x for which only the sign of the associator may have changed. The situation can now be divided into four cases:

1. $d(q) = d(r) = 1$. Then $[p, q, r] = [p, x, x] = 0$ by the right alternative law.

2. If only one of q or r has degree > 1, say $d(r) = 1$ and $d(q) > 1$, then we may write $q = xq'$, so that $[p, q, r] = [p, xq', x] = -[p, x, q']x$ using (2.1.5). By the induction hypothesis $[p, x, q'] = 0$.

3. If both $d(q) > 1$ and $d(r) > 1$, write $q = xq'$ and $r = xr'$. Put $y = xr'$, $a = q'$ and $z = p$ in (2.1.6) to get

$$\begin{aligned}
[p, q, r] = [p, xq', xr'] &= -[xr', xq', p] \\
&= [xr', pq', x] + [xr', x, q']p + [xr', p, q']x \\
&= -[pq', xr', x] \\
&= [pq', x, r']x
\end{aligned}$$

using (2.1.5). Now $d(pq') + d(x) + d(r') = d(p) + d(q') + 1 + d(r') < d(p) + d(q) + d(r)$, so the above is zero by the induction hypothesis.

□

Another weak form of associativity is power associativity. An algebra is said to be **power associative** if the subalgebra generated by any single element is associative. This means that in any repeated product of x with itself, the order of the operations does not matter, e.g. $((xx)x)x = (xx)(xx)$. Equivalently, an algebra A is power associative if $x^n x^m = x^{n+m}$ for all nonnegative integers m, n. Every associative algebra is of course power associative, and every alternative algebra is power associative, so power associativity is a weaker notion than alternativity. Almost all interesting algebras are power associative, and Jordan algebras are no exception.

A **division algebra** is an non-trivial algebra D such that whenever $xy = 0$, we have either $x = 0$ or $y = 0$. If D is an associative algebra, then D is a division algebra if and only if it is unital and every non-zero element $x \in D$ has a multiplicative inverse $x^{-1} \in D$: $xx^{-1} = x^{-1}x = 1$. This is *not* true in general for nonassociative algebras.

There is a more general notion of an **algebra over a ring**, which is defined in precisely the same way except that the field F is now a commutative unital ring R, and A is an R-module rather than a vector space. In the theory of associative algebras, one may define an algebra over a commutative unital ring R as a ring A together with a ring homomorphism from R into the center of A. Since the definition of a ring requires that multiplication is associative, this is not applicable for general nonassociative algebras.

2.2 Categories

We shall state some of the results informally in the language of category theory. A **category** $\mathcal{C}$ consists of a collection of **objects** $\mathcal{O}$ and a set $\mathcal{M}(X, Y)$ of **morphisms** for each pair of objects X, Y, satisfying:

1. For objects X, Y and Z, we have a binary *composition* operation $\mathcal{M}(Y,Z) \times \mathcal{M}(X,Y) \to \mathcal{M}(X,Z)$ that is *associative*, and every object X has an *identity morphism* $1_X \in \mathcal{M}(X,X)$:

$$(h \circ g) \circ f = h \circ (g \circ f), \quad f \circ 1_X = 1_Y \circ f$$

for all $f \in \mathcal{M}(X,Y)$, $g \in \mathcal{M}(Y,Z)$ and $h \in \mathcal{M}(Z,W)$.

We think intuitively of objects as sets with some kind of additional structure, and morphisms as set-theoretic maps that preserve that structure. However, it is important to keep in mind that a category is an *abstract* notion and its objects and morphisms can be anything for which the axioms hold.

A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping that associates each object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$, and each morphism $f \in \mathcal{M}(X,Y)$ to a morphism $F(f) : F(X) \to F(Y)$, such that $F(1_X) = 1_{F(X)}$ for all X in $\mathcal{C}$, and

$$F(g \circ f) = F(g) \circ F(f)$$

for all $f \in \mathcal{M}(X,Y)$ and $g \in \mathcal{M}(Y,Z)$.

We think of a functor as a **construction** of $\mathcal{D}$-objects out of $\mathcal{C}$-objects in a way that uses only ingredients from the $\mathcal{C}$-object. Any morphism of the $\mathcal{C}$-object preserves these ingredients, and therefore induces a morphism of the constructed $\mathcal{D}$-object. Some examples are constructions of free algebraic objects from sets, such as free groups, free modules, and polynomial rings. These constructions are functors from the category of sets with morphisms ordinary set functions, to the category of some algebraic structure with morphisms the maps that preserve that structure: group homomorphisms, module homomorphisms, etc. In the other direction, we can for any algebraic object consisting of an underlying set with some additional structure consider functors that "forget" all structure and keeps only the underlying set; these are the *forgetful* functors from a category of some type of algebraic object to the category of sets that sends each object to its underlying set and each structure-preserving homomorphism to its underlying set function.

3

Overview

In section 4, we discuss the inception of Jordan algebras beginning in the 1930s when Pascual Jordan investigated the algebraic properties of Hermitian matrices. The theory of infinite-dimensional algebras was not developed at this time even in the associative case. Jordan hoped that by capturing the algebraic properties of finite-dimensional operators in the form of Hermitian matrices in an axiomatic system, he could then set out to classify the resulting structures and hopefully find a suitable family of finite-dimensional algebras parametrized by natural numbers n, so that letting n tend to infinity one could obtain a suitable setting for quantum mechanics in infinite dimensions.

We have a first look at three examples of Jordan algebras in section 5: full, Hermitian, and spin factors, before turning to the first classification result from the fundamental 1934 paper by Jordan, von Neumann and Wigner in section 6. In section 7, we go over some basic concepts such as multiplication operators and linearization. We give a linearized version of the Jordan identity, define some auxiliary products and show that Jordan algebras are always power associative. We present the definition of a quadratic Jordan algebra finalized by McCrimmon in 1967 using the auxiliary U-product.

In section 8, we formalize the notion of composition algebras. The exceptional Albert algebras consists of matrices with entries from 8-dimensional octonion algebras, a type of composition algebra obtained through the Cayley-Dickson construction. We prove a famous theorem of Hurwitz asserting that the only composition algebras are the base field of dimension 1, a quadratic extension of dimension 2, a quaternion algebra of dimension 4 and an octonion algebra of dimension 8. At the end of the section I prove that the Hermitian 2×2 matrix algebras appearing in Jordan, von Neumann and Wigner's classification are isomorphic to spin factors.

In the following three sections 9, 10, 11 we return to the examples of full, Hermitian and spin factor Jordan algebras of section 5 in slightly more detail. In section 9 we will see how a Jordan algebra can be constructed from an alternative algebra. In section 10 we show how full Jordan algebras arises as certain Hermitian Jordan algebras, and we define on the most important class of Hermitian Jordan algebras with finiteness conditions: the matrix algebras. In section 11 we show how we can construct a Jordan algebra from a vector space and a quadratic form, and how the spin factors from 5 arise as special cases of this construction.

In section 12 we define the notion of free Jordan algebras and state Macdonald's theorem. We explore some of the many consequences of Macdonald's theorem, in particular Cohn's reversible theorem and Cohn symmetry. We introduce the notion of i-special and i-exceptional Jordan algebras and discuss how the failure of Jordan's axiomatic framework to capture all of the algebraic properties of Hermitian matrices led to some unintended fortunate consequences, most notably the Albert algebras. I show how operator power-associativity for the U-operator follows from Macdonald's theorem.

Next we give a brief exposition in 13 of the method of Peirce decomposition in the Jordan case. In the classical Artin-Wedderburn theory of associative algebras, decomposition with respect to a family of orthogonal idempotents is a key tool used to break up an algebra into smaller pieces with multiplication behaving like multiplication of elementary matrices E_{ij} (1 at entry ij and the rest zeroes). Peirce decompositions play an important role also in the structure theory of Jordan algebras. We finish the section by showing how the entries in a Hermitian matrix algebra come naturally from an alternative algebra.

In section 14 we note some similarities and differences between Jordan algebras and Lie algebras. In particular we will see how the class of special Jordan algebras cannot be characterized by identities in the same way as with Lie algebras. We give some results as to when and how homomorphic images of special Jordan algebras are special, and we give an example of an i-special, or identity-special, but exceptional Jordan algebra. We also mention some surprising connections between the exceptional Albert algebras and the exceptional simple Lie algebras.

In the final section 15, we briefly sketch the development of the Jordan structure theory from its inception in the 1930s to Zelmanov's complete classification of simple Jordan algebras of arbitrary dimensions in the early 1980s.

4

Jordan's initial definition

Definition 4.0.1 (Hermitian matrix)**.** A **Hermitian matrix**, or **self-adjoint matrix**, is a complex square matrix X that is equal to its own conjugate transpose, i.e. $x_{ij} = \overline{x}_{ji}$ for all indices i and j, where $\overline{x}$ denotes the complex conjugate of x.

The first step in analysing the algebraic properties of Hermitian matrices was to decide what the basic "observable" operations were. Mathematically, this means finding ways of combining Hermitian matrices to get another Hermitian matrix. While the sum of two Hermitian matrices is always Hermitian, the ordinary matrix product is in general not Hermitian—this happens only if the two matrices commute. The set of Hermitian $n \times n$ matrices does not form a vector space over the complex numbers $\mathbb{C}$, since the set is not closed under multiplication by complex scalars. For example, the identity matrix I_n is equal to its own conjugate transpose, but $i \cdot I_n$ is not. The Hermitian matrices *are* closed under multiplication by real scalars, however, and forms a vector space over $\mathbb{R}$.

There many ways to combine Hermitian matrices to obtain new ones, but after some experimentation, Jordan found that they could all be expressed in terms of *quasi-multiplication*, which later became known as the **Jordan product**:

$$A \bullet B = \frac{1}{2}(AB + BA).$$

Here AB is ordinary matrix multiplication. Given two Hermitian matrices, the Jordan product is also Hermitian, as we shall see. The Jordan product is bilinear, making the $\mathbb{R}$-vector space of Hermitian matrices into an $\mathbb{R}$-algebra with this product.

Jordan investigated what laws or axioms the algebra of Hermitian matrices obeyed. He thought that the key property of the Jordan product besides it being commutative, was a weaker form of associativity:

$$(A \bullet A) \bullet (B \bullet A) = ((A \bullet A) \bullet B) \bullet A.$$

Today, this is known as the **Jordan identity**.

Another important property of the algebra of Hermitian matrices under the Jordan product is *positive-definiteness* or *formal reality*.

Definition 4.0.2 (Symmetric bilinear form)**.** Let V be a vector space over a field F. In general, a multilinear map $f : V_1 \times \cdots \times V_n \to W$ is called a **n-multilinear form** if the codomain $W = F$ is the field of scalars. An 1-multilinear form is called a **linear form** (or *linear functional*, *one-form*, *co-vector*), and a 2-multilinear form is called a **bilinear form**. A bilinear form $b : V_1 \times V_2 \to F$ is **symmetric** if $V_1 = V_2 =: V$ and $b(u, v) = b(v, u)$ for all $u, v \in V$.

A symmetric bilinear form $b : V \times V \to F$ on a real or complex vector space V is called **positive definite** if $b(x, x) > 0$ for all nonzero x. Let $A = (a_{ij})$ be a Hermitian $n \times n$ matrix, i.e. $a_{ij} = \overline{a}_{ji}$ for all $i, j = 1, \dots, n$. Looking at the diagonal entries of the square $A^2 = (a'_{ij})$ (under ordinary matrix multiplication), we have

$$a'_{ii} = \sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{j=1}^{n} a_{ij}\overline{a}_{ij} = \sum_{j=1}^{n} |a_{ij}|^2.$$

If b is the *trace bilinear form* $b(A, B) := \operatorname{tr}(AB)$, then

$$b(A, A) = \operatorname{tr}(A^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2,$$

so that $b(A, A) > 0$ unless all a_{ij} are zero, so b is positive definite. This property had algebraic consequences for the Jordan product. Namely, a sum of squares $A^2 + B^2 + \dots$ is never equal to zero unless all $A, B, \dots = 0$. Indeed, if $A^2 + B^2 + \cdots = 0$, then $\operatorname{tr}(A^2) + \operatorname{tr}(B^2) + \cdots = 0$, and every term is nonnegative, forcing each term to be zero, which happens precisely when $A, B, \dots = 0$.

More generally, for any F-algebra A on which there exist a symmetric positive definite bilinear form $b : A \times A \to F$, we have

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \iff x_1 = x_2 = \cdots = x_n = 0.$$

Jordan was familiar with this "formal reality" property from the Artin-Schreier theory of formally real fields.

After some experimentation, it seemed to Jordan that all other laws satisfied by the Jordan product were consequences of commutativity, the Jordan identity, and positive-definiteness (formal reality), although he did not make formal reality as part of the axioms. He made the following definition.

Definition 4.0.3 (Jordan's initial definition)**.** A **Jordan algebra** $J = (V, p)$ consists of a vector space V over $\mathbb{R}$ equipped with a bilinear product $p : V \times V \to V$, abbreviated $p(x, y) = x \bullet y$, satisfying

$$x \bullet y = y \bullet x \qquad \text{(Commutativity)}$$
$$(x \bullet x) \bullet (y \bullet x) = (x \bullet x) \bullet y) \bullet x \qquad \text{(Jordan identity)}$$

A Jordan algebra is called **Euclidean** or **formally real** if

$$x_1^2 + x_2^2 + \cdots = 0 \implies x_1 = x_2 = \cdots = 0 \quad (x^2 := x \bullet x).$$

Jordan originally called these *r-number algebras*. The term "Jordan algebra" was first used by A. A. Albert in 1946.

We know now that Jordan had overlooked some algebraic properties of Hermitian matrices, and his initial definition had captured something slightly more general than what was intended. The *quadratic product* xyx and the *inverse* x^{-1} were two natural operations on Hermitian matrices that were not considered. These can be defined using the Jordan product, but it was not noticed for another 30 years. The quadratic product and the inverse were later used by McCrimmon to provide an alternative axiomatic formulation for Jordan algebras (see Definition 7.3.2).

Something else that was overlooked were the *n-tad products*, defined as

$$\{x_1, \ldots, x_n\} = x_1 x_2 \ldots x_n + x_n x_{n-1} \ldots x_1.$$

If $X_1, \ldots, X_n$ are Hermitian matrices, then so is $\{X_1, \ldots, X_n\}$ for all n. The 2-tad is just twice the Jordan product $2x \bullet y$. For $n = 3$, the *triad* $\{x_1, x_2, x_3\}$ can, somewhat surprisingly, also be expressed in terms of the Jordan product as we will see in section 7. However, for $n \geq 4$, the n-tads cannot be written as Jordan products. The n-tad operations for $n \geq 4$ were thus not included in the theory for Jordan algebras. In particular the exclusion of the *tetrad* for $n = 4$ lead to some interesting unintended Jordan algebras. These were the *spin factors* and the exceptional *Albert algebras*, both of which are not closed under the tetrad.

Jordan also missed some laws, or identities, that could not be built up from the Jordan bullet product. The first and smallest of these is Glennie's Identity G_8 (Definition 12.2.3), an expression in three variables of degree 8 that was discovered in 1963.

5

Examples

Let $M_n(\mathbb{C})$ be the $\mathbb{R}$-algebra of complex $n\times n$ matrices. The set $H_n(\mathbb{C})$ of Hermitian matrices is a subspace of $M_n(\mathbb{C})$, and we may equip it with the Jordan product, making it into a Jordan algebra.

In general, given any associative algebra A, we can construct a Jordan algebra A^+ by taking the vector space part of A and equipping it with the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$ derived from the original product in A. The Jordan product is clearly commutative with the square $x \bullet x = x^2$ coinciding with the associative square, and it satisfies the Jordan identity:

$$\begin{aligned}
((x \bullet x) \bullet y) \bullet x &= \frac{1}{4}\left[(x^2y + yx^2)x + x(x^2y + yx^2)\right] \\
&= \frac{1}{4}\left[x^2yx + yx^3 + x^3y + xyx^2\right] \\
&= \frac{1}{4}\left[x^2(yx + xy) + (yx + xy)x^2\right] \\
&= (x \bullet x) \bullet (y \bullet x).
\end{aligned}$$

The construction of A^+ is an example of a *special* Jordan algebra.

Definition 5.0.1 (Special and exceptional)**.** A Jordan algebra J is called **special** if it is isomorphic to a subalgebra of A^+ for some associative algebra A. A Jordan algebra that is not special is called **exceptional**.

We shall frequently identify special Jordan algebras J as a subsets of A^+.

The algebra A^+ is not associative in general. For example, take $A = M_2(\mathbb{R})$ and let E_{ij} denote the **elementary matrix** with entry 1 at ij and 0 elsewhere, and let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then in the ordinary matrix product of A we have $E_{11}E_{22} = E_{22}E_{11} = 0$, and a straightforward computation shows that

$$(E_{11} \bullet E_{22}) \bullet X = 0, \quad E_{11} \bullet (E_{22} \bullet X) = \frac{1}{4}X.$$

5.1 Full

The first example of Jordan algebras are the full algebras A^+ we have just seen. These are obtained by taking an associative algebra and "symmetrizing" it, replacing its product with the Jordan product.

Definition 5.1.1 (Full Jordan algebra)**.** Given a (not necessarily associative) algebra A over $\mathbb{R}$ with product xy, we define A^+ to be the algebra obtained from the vector space A equipped with the Jordan product

$$x \bullet y := \frac{1}{2}(xy + yx).$$

We saw that when A is associative, A^+ is a Jordan algebra. Later, we shall see that we can weaken the assumption on A from being associative to being alternative and still obtain a Jordan algebra through this construction (Theorem 9.0.2).

The full Jordan algebras are rarely formally real. For example, if we take $A = M_n(R)$ to be the algebra of $n \times n$ matrices with entries from an arbitrary unital ring R, then A^+ is never formally real for $n \geq 2$. Indeed, consider any elementary matrix E_{ij}. Squaring E_{ij}, one finds that $E_{ij}^2 = 0$ whenever $i \neq j$. However, if we restrict ourselves to the set of *symmetric* real matrices or the *Hermitian* complex matrices, then A^+ will be formally real.

5.2 Hermitian

The Jordan algebra of complex Hermitian matrices is an example of a more general class of Jordan algebras. These are algebras obtained from an alternative algebra by means of an *involution*.

Definition 5.2.1 (Algebra involution)**.** Let A be any algebra, not necessarily commutative nor associative. An **involution** $\star : A \to A$ on A is an *anti-automorphism* on A of period 2. That is, it is a linear map from A to itself that *reverses* the order of products:

$$(xy)^\star = y^\star x^\star \quad \forall x, y \in A,$$

and for which $(x^\star)^\star = x$ for all $x \in A$.

The notion of involutions generalizes complex conjugation in $\mathbb{C}$. Conjugation in $\mathbb{C}$ is an involution, but due to commutativity conjugation in $\mathbb{C}$ is just an ordinary automorphism of period 2.

Definition 5.2.2. A **$\star$-algebra** ("star-algebra") or **algebra with involution** $(A, \star)$ is any algebra A together with a choice of involution. A **$\star$-algebra homomorphism** $\varphi : (A, \star) \to (A', \star')$ is an algebra homomorphism that respects the involution: $\varphi(x^\star) = \varphi(x)^{\star'}$, or $\varphi \circ \star = \star' \circ \varphi$. Similarly, a **$\star$-ideal** of $(A, \star)$ is the same as an algebra ideal I of A except that it must also be invariant under the involution: $x \in I \implies x^\star \in I$.

The archetypical example of a $\star$-algebra is the algebra of complex numbers with conjugation. Other important involutions are the standard trace involutions on a quaternion or octonion algebra in section 8.

Definition 5.2.3 (Hermitian elements)**.** Let $(A, \star)$ be a $\star$-algebra. An element $x \in A$ fixed by the involution, $x^\star = x$, is called **Hermitian**, or **self-adjoint**, and the set of all Hermitian elements is denoted with $H(A, \star)$. If $(A, \star)$ is a composition algebra (Definition 8.0.2) equipped with its standard trace involution, we may simply write $H(A)$.

There are two general ways to construct a Hermitian element out of an arbitrary $x \in A$, the **norm** $n(x) := xx^\star$ and the **trace** $t(x) := x + x^\star$. If the base field has characteristic different from 2, then every Hermitian element x is a trace $\frac{1}{2}t(x)$, and when A is unital, every trace $t(x)$ is a norm $n(x+1) - n(x) - n(1)$.

Remark 5.2.1. The term "norm" has no connection to the concept of norms in metric spaces. Rather, the trace and the norm are polynomial functions analogous to the trace and the determinant of a matrix. There is a general notion of norm in finite-dimensional power associative algebras A over a field F. If $x_1, \ldots, x_n$ is a basis for A, then the element $x = t_1 x_1 + \ldots t_n x_n$ in the polynomial ring $F[t_1, \ldots, t_n]$ is a *generic* element of A in the sense that every element of A arises as x through the specialization $t_i \mapsto \lambda_i$ of the indeterminates to scalars in F. The generic element x satisfies a *generic minimum polynomial* that looks like the characteristic polynomial for matrices:

$$x^k - a_1(x)x^{k-1} + \cdots + (-1)^k a_k(x)1 = 0,$$

where the $a_i(x)$ can be shown to be polynomial functions $a_i \in F[t_1, \ldots, t_n]$ that are homogeneous of degree i in the indeterminates $t_1, \ldots, t_n$. The functions $a_1(x) =: t(x)$ and $a_k(x) =: n(x)$ are called the *generic trace* and *generic norm* of A, and the number k is the *degree* of the algebra. See e.g. [Jac89] for a detailed treatment.

When $A = M_n(\mathbb{C})$ is the algebra of complex square matrices and $\star$ is the conjugate transpose involution, then $H(A, \star)$ is the set of ordinary Hermitian complex matrices. If $A = M_n(\mathbb{R})$ is the algebra of real matrices, then $H(A, \star)$ is the set of *symmetric* matrices.

When the $\star$-algebra A is associative, it is easy to verify that the Hermitian elements $H(A, \star)$ form a Jordan algebra under the Jordan product.

Theorem 5.2.1. Let $(A, \star)$ be a $\star$-algebra. If A is associative, then $H(A, \star)$ is a Jordan subalgebra of A^+ under the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$.

We prove this in section 10 when we cover Hermitian Jordan algebras in more detail. Note that while $H(A, \star)$ is closed under the Jordan product, it is not in general closed under the possibly noncommutative product xy of A, since $(xy)^\star = y^\star x^\star = yx$ for $x, y \in H(A, \star)$. Hence $H(A, \star)$ is a Jordan subalgebra of A^+, but not a subalgebra of A.

Given a unital associative $\star$-algebra A with involution denoted with $x \mapsto \overline{x}$, we can define an involution on the matrix algebra $A := M_n(A)$ by $X \mapsto \overline{X}^T$, i.e. by sending a matrix $X = (x_{ij})$ to its conjugate transpose $\overline{X}^T = (\overline{x}_{ji})$. This construction gives us the most important class of Hermitian Jordan algebras.

Theorem 5.2.2. Let $(A, -)$ be a unital associative $\star$-algebra over a field F with involution denoted with $x \mapsto \overline{x}$. The conjugate transpose mapping $X \mapsto \overline{X}^T$ is an involution on the algebra $M_n(A)$ of all $n \times n$ matrices with entries from A under the usual matrix product XY. The set $H_n(A, -) := H(M_n(A), -^T)$ of all "Hermitian" matrices ($\overline{X}^T = X$) under this involution is closed under the Jordan product $X \bullet Y = \frac{1}{2}(XY + YX)$ and forms a Jordan algebra. If A is a composition algebra equipped with its standard trace involution, we may simply write $H_n(A)$.

Such a Jordan matrix algebra is formally real if the $\star$-algebra A is formally real in the sense that

$$\sum_j x_j \overline{x}_j = 0 \implies \text{all } x_j = 0 \quad x_j \in A.$$

In particular, taking $A = \mathbb{R}$ to be the algebra of real numbers with *trivial* involution: $\overline{r} = r$ for all $r \in \mathbb{R}$, then $H_n(A, -)$ is a formally real Jordan algebra of symmetric real matrices under the Jordan product. If we take $A = \mathbb{C}$ to be the algebra of complex numbers over itself with involution defined as complex conjugation, we get a formally real Jordan algebra of Hermitian matrices.

Similar to how we may think of complex numbers $a + bi$ as pairs (a, b) of real numbers, Hamilton's **quaternions** $\mathbb{H}$ can be thought of as pairs of complex numbers. The quaternions is an associative, noncommutative $\mathbb{R}$-algebra on which there is a natural involution, defined using the involution on $\mathbb{C}$. Thus $H_n(\mathbb{H}, -)$ is a Jordan algebra. When we construct the **Cayley algebra**, or the **octonions** $\mathbb{O}$ as pairs of quaternions by the same recipe as for constructing $\mathbb{C}$ and $\mathbb{H}$, we will find that $\mathbb{O}$ is neither commutative nor associative. Hence $H_n(\mathbb{O})$ is no longer guaranteed to be a Jordan algebra. Specifically, $H_n(\mathbb{O})$ is not a Jordan algebra for $n \geq 4$, but for $n = 3$ it is (cf. Theorem 13.2.3). For $n = 3$ we obtain an Albert algebra $H_3(\mathbb{O})$ which is an exceptional Jordan algebra.

5.3 Spin factors

The first Jordan algebras that were not Hermitian matrix algebras were discovered by Max Zorn. These are called the *spin factors*. Let $\langle -, - \rangle$ denote the usual inner product (dot product) on the vector space $\mathbb{R}^n$, and set $\mathcal{J}_{spin}(n) := \mathbb{R}^n \oplus \mathbb{R}$. Define a product on the space $\mathcal{J}_{spin}(n)$ by

$$(\mathbf{x}, \alpha) \bullet (\mathbf{y}, \beta) := (\beta\mathbf{x} + \alpha\mathbf{y}, \langle \mathbf{x}, \mathbf{y} \rangle + \alpha\beta).$$

The unit element on $\mathcal{J}_{spin}(n)$ is $(\mathbf{0}, 1)$. The product is commutative since the inner product is symmetric:

$$
\begin{aligned}
(\mathbf{x},\alpha)\bullet(\mathbf{y},\beta) &= (\beta\mathbf{x}+\alpha\mathbf{y},\langle\mathbf{x},\mathbf{y}\rangle+\alpha\beta)\\
&= (\alpha\mathbf{y}+\beta\mathbf{x},\langle\mathbf{y},\mathbf{x}\rangle+\beta\alpha)\\
&= (\mathbf{y},\beta)\bullet(\mathbf{x},\alpha).
\end{aligned}
$$

To see that the product satisfies the Jordan identity, note that

$$
(\mathbf{x},a)^2 = (2a\mathbf{x},\langle\mathbf{x},\mathbf{x}\rangle+a^2) = (\langle\mathbf{x},\mathbf{x}\rangle-a^2)(\mathbf{0},1)+2a(\mathbf{x},a).
$$

This shows $(\mathbf{x},a)^2$ is a linear combination of $(\mathbf{x},a)$ and the unit element. If A is any commutative unital algebra over $\mathbb{R}$, $x,y\in A$ and $x^2=\alpha 1+\beta x$ for some $\alpha,\beta\in\mathbb{R}$, then

$$
(x^2y)x = x(x^2y) = x((\alpha 1+\beta x)y) = \alpha(xy)+\beta x(xy),
$$

and

$$
x^2(yx) = x^2(xy) = (\alpha 1+\beta x)(xy) = \alpha(xy)+\beta x(xy),
$$

so that $(x^2y)x = x^2(yx)$ and the Jordan identity holds. In general, any unital commutative algebra for which every element satisfies a degree 2 equation (i.e., the algebra is of degree 2, cf. remark 5.2.1) is automatically a Jordan algebra (Lemma 11.1). The Jordan algebra $\mathcal{J}_{spin}(n)$ is also formally real. This follows from the fact that an inner product is positive-definite, and that

$$
\begin{aligned}
\sum_j(\mathbf{x}_j,a_j)^2 &= \sum_j(2a_j\mathbf{x}_j,\langle\mathbf{x}_j,\mathbf{x}_j\rangle+a_j^2)\\
&= 2\left(\sum_j a_j\mathbf{x}_j,\ 0\right)+\left(\sum_j\langle\mathbf{x}_j,\mathbf{x}_j\rangle+a_j^2\right)(\mathbf{0},1),
\end{aligned}
$$

where the coefficient of $(\mathbf{0},1)$ is zero if and only if all $\mathbf{x}_j=\mathbf{0}$ by positive-definiteness of the inner product and all $a_j=0$.

The reason the Jordan algebras $\mathcal{J}_{spin}(n)$ is called spin factors comes from something called the *spin group* in physics. The spin group is the universal covering space of the special orthogonal group $\mathrm{SO}(n)$. The special orthogonal group is sometimes called the *rotation group* due to the fact that its elements act on $\mathbb{R}^2$ as rotations around a point (or around a line in $\mathbb{R}^3$).

It turns out that even though the spin factors $\mathcal{J}_{spin}(n)$ are not constructed as Hermitian matrix algebras, there is nevertheless an embedding of $\mathcal{J}_{spin}(n)$ into a large Jordan algebra of real hermitian (symmetric) $2^n\times 2^n$ matrices, so the spin factors $\mathcal{J}_{spin}(n)$ are special Jordan algebras.

6

Classification of finite-dimensional formally real Jordan algebras

After Jordan had decided on the basic axioms of his then-called r-number systems, he sought to classify them. An algebra suitable for quantum mechanics would have to be infinite-dimensional, but at this point in time the theory of infinite-dimensional algebras was not well developed even in the associative case. Obtaining a full classification of infinite-dimensional Jordan algebras seemed out of reach. Instead, Jordan thought that if, in the finite-dimensional case, one could find a family of exceptional Jordan algebras J_n parametrized by natural numbers n, one could let n tend to infinity and hopefully find a suitable infinite-dimensional exceptional Jordan algebra.

Jordan called in the help of physicist Eugene Wigner and mathematician John von Neumann, and in their joint paper published in 1934 they were able to show that every finite-dimensional, formally real Jordan algebra were built out of Hermitian matrix algebras and the spin factors $\mathcal{J}_{spin}(n)$.

Theorem 6.0.1 (Jordan, von Neumann, Wigner)**.** Every finite-dimensional formally real Jordan algebra is a direct sum of simple ideals, of which there are five types. Four types are Hermitian matrix algebras corresponding to the four real composition division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ of dimensions 1,2,4,8, and the last type are spin factors. Every finite-dimensional formally real simple Jordan algebra is isomorphic to one of the following:

1. $H_n(\mathbb{R})$;
2. $H_n(\mathbb{C})$;
3. $H_n(\mathbb{H})$;
4. $H_3(\mathbb{O})$;
5. $\mathcal{J}_{spin}(n)$.

The division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ all carry a positive definite quadratic form

$$q\left(\sum \lambda_i x_i\right) = \sum \lambda_i^2$$

relative to suitable bases x_i. The form q *admits composition* in the sense that $q(xy) = q(x)q(y)$ for all x, y. A famous theorem of Hurwitz (Theorem 8.1.3) states that the only possible composition algebras over any field are the field itself (dimension 1), a quadratic extension (dimension 2), a quaternion algebra (dimension 4) and an octonion algebra (dimension 8). We shall formalize the notion of composition algebras in Section 8.

When $n = 1$, the Jordan algebras $H_1(A)$ for $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are just the sets of Hermitian elements under the Jordan product. The elements of $\mathbb{C}$ that are equal to their own conjugate is just the real numbers, and the same is true for $\mathbb{H}$ and $\mathbb{O}$. In general, an involution on an algebra A over F is called *scalar* if the set of Hermitian elements are contained in the scalar multiples of the unit: $H(A, \star) \subset F1$. Thus the algebras $H_1(A)$ are all isomorphic to the Jordan algebra $\mathbb{R}^+$.

For $n = 2$, the algebra $H_2(\mathbb{R})$ of *symmetric* real matrices is isomorphic to the spin factor $\mathcal{J}_{spin}(2) = \mathbb{R}^2 \oplus \mathbb{R}$, and the algebra $H_2(\mathbb{C})$ of complex Hermitian matrices is isomorphic to $\mathcal{J}_{spin}(3) = \mathbb{R}^3 \oplus \mathbb{R}$. Similarly we have $H_2(\mathbb{H}) \cong \mathcal{J}_{spin}(5)$ and $H_2(\mathbb{O}) \cong \mathcal{J}_{spin}(9)$ (Proposition 8.1.2).

From the example of Hermitian Jordan algebras, we have seen that the first three algebras of the classification are special, living inside the full associative matrix algebras $M_n(A)$ for the associative coordinate algebras $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We have also noted that $\mathcal{J}_{spin}(n)$ can be embedded in an algebra of large Hermitian matrices, so that it too is special.

However, the fourth item $H_3(\mathbb{O})$ on the list did not seem to be special since it had entries from the nonassociative Cayley algebra. Jordan, Wigner and von Neumann were not able to prove that this was the case, but soon after, Abraham Adrian Albert was able to show that $H_3(\mathbb{O})$ was indeed exceptional. An element of $H_3(\mathbb{O})$ looks like

$$\begin{pmatrix} \lambda_1 & x & y \\ \overline{x} & \lambda_2 & z \\ \overline{z} & \overline{y} & \lambda_3 \end{pmatrix}$$

where $\lambda_1, \lambda_2, \lambda_3$ are Hermitian elements in $\mathbb{R}1$, and $x, y, z \in \mathbb{O}$. Since the dimension of $\mathbb{O}$ is 8, we see that there are $8 + 8 + 8 + 1 + 1 + 1 = 27$ degrees of freedom for such a matrix, and the dimension of $H_3(\mathbb{O})$ is 27.

Due to Albert's proof of its exceptionality and his later constructions of Jordan division algebras that are *forms* of $H_3(\mathbb{O})$, these 27-dimensional exceptional algebras are now known as *Albert algebras*.

Definition 6.0.1 (Form of an algebra)**.** If K/F is a field extension and A an algebra over F, we say that A is a **form** of an algebra B over K if A becomes isomorphic to B when we extend its scalars to K:

$$K \otimes_F A \cong B.$$

Here, the extension $K \otimes_F A$ of A is defined as the tensor product as vector spaces with the induced multiplication

$$(a \otimes x) \cdot (b \otimes y) := (ab) \otimes (xy).$$

7

Jordan algebras: basics

In this section, we give the basic definition of a Jordan algebra over an arbitrary field of characteristic different from 2, and define some auxiliary operations. We show that a Jordan algebra is always power associative, i.e. that any subalgebra of a Jordan algebra generated by a single element is associative, and give the definition of a quadratic Jordan algebra.

We assume that all fields have characteristic different from 2 in order to make sense of the examples we have seen with the Jordan bullet product $x \bullet y$.

Definition 7.0.1 (Jordan algebra)**.** Let F be a field of characteristic different from 2. A **Jordan algebra** J over F is an algebra with product $x \bullet y$ satisfying

$$[x, y] = 0 \qquad \text{(Commutative law)}$$
$$[x \bullet x, y, x] = 0 \qquad \text{(Jordan identity)}$$

for all $x, y \in J$.

With F as in the definition, an associative algebra A over F is a Jordan algebra if and only if it is commutative. If A is not commutative, we can equip it with the Jordan product to obtain a Jordan algebra A^+, thereby "symmetrizing" it as in the full example (Section 5.1). When A is associative and has an involution $\star$, the Hermitian elements $H(A, \star)$ form a Jordan subalgebra over F of A^+ under the Jordan product.

Usually, but not always, we shall denote the product of a Jordan algebra with a bullet $\bullet$, and when we refer to the Jordan product $x \bullet y := \frac{1}{2}(xy + yx)$ defined in terms of the product xy of some associative algebra we shall make this clear from context.

Definition 7.0.2 (Jordan algebra homomorphism)**.** A **homomorphism** of Jordan algebras is the same as an algebra homomorphism, i.e. an F-vector space homomorphism (a linear map) φ that preserves multiplication: $\varphi(x \bullet y) = \varphi(x) \bullet' \varphi(y)$.

We shall use the abbreviation x^2 for $x \bullet x$. Since a Jordan algebra is nonassociative, however, when the exponent is > 2, x^n is a priori not defined unless we specify an order of multiplication. When the product is the Jordan product, $x \bullet x = \frac{1}{2}(xx + xx) = xx$ coincides with the ordinary associative square.

The axioms defining a Jordan algebra, and many other properties, can be expressed concisely using *multiplication operators*.

Definition 7.0.3. Let A be any algebra over a field or commutative ring F and $a \in A$ any element. The (left) **multiplication operator** $L_a : A \to A$ is the map defined by

$$L_a(b) = ab.$$

Similarly we define $R_a(b) = ba$.

Remark 7.0.1. Let F be a field, and suppose A and B are two vector spaces over F. Let $\mathrm{Hom}_F(A, B)$ be the set of all F-linear maps (vector space homomorphisms) from A to B. The set $\mathrm{Hom}_F(A, B)$ itself inherits a vector space structure by defining addition in the usual pointwise fashion, and scalar multiplication in the obvious way: $(\lambda f)(x) := \lambda f(x)$. If $A = B$, we write $\mathrm{End}_F(A)$ for the vector space $\mathrm{Hom}_F(A, A)$. In this situation, the composition $f \circ g$ of two linear maps $f, g \in \mathrm{End}_F(A)$ is defined, and is again a linear map from A to itself. Additionally, composition of maps obeys the associative law. Hence $\mathrm{End}_F(A)$ carries a unital associative F-algebra structure under composition of maps, with multiplicative identity given by the identity map.

The preceding also holds more generally when $F = R$ is a commutative ring and A is an R-module. The set $\mathrm{End}_R(A) = \mathrm{Hom}_R(A, A)$ then carries a unital associative R-algebra structure.

The definition of an algebra A over a field or over a commutative unital ring R requires that the product is bilinear. This can be expressed in terms of multiplication operators as saying that the map $L : A \to \mathrm{End}_R(A)$ defined by $a \mapsto L_a$ is linear (an R-module homomorphism) from the R-module A into the R-module $\mathrm{End}_R(A)$, and likewise for $a \mapsto R_a$. Indeed, saying that L is linear means that for all $a, b \in A$,

$$L_{a+b}(x) = L_a(x) + L_b(x), \quad \text{and} \quad L_{\lambda a}(x) = \lambda L_a(x)$$

for all $x \in A$, which is equivalent to the element identities

$$(a + b)x = ax + bx, \quad \text{and} \quad (\lambda a)x = \lambda(ax)$$

for all $a, b, x \in A$. The other algebra axioms follow from linearity of $a \mapsto R_a$. The maps L and R given by $a \mapsto L_a$ and $a \mapsto R_a$ are known as the *left* and *right regular representations* of the algebra A. As long as A has a multiplicative unit, the left and right regular representations are injective since $L_a = L_b$ implies $ax = bx$ for all $x \in A$ so $a = b$ for $x = 1$ (and similarly for R).

Saying that A is commutative is saying that $L_a = R_a$ for all $a \in A$, i.e. that the left and right regular representations coincide. Expressing the associative law $(ab)c = a(bc)$ in terms of operator identities can be done in three ways, depending on which element one views as the variable:

$$L_{ab} = L_a L_b, \quad R_c R_b = R_{bc}, \quad R_c L_a = L_a R_c.$$

Here, concatenation ST of operators S, T is the product in $\mathrm{End}(A)$, i.e. function composition.

The Jordan identity can be expressed as $L_{x^2}L_x = L_xL_{x^2}$, or $[L_{x^2}, L_x] = 0$ using the commutator. Hence a Jordan algebra is an algebra for which $L_x = R_x$ and $[L_{x^2}, L_x] = 0$ for all x.

7.1 Linearizing the Jordan identity

There is a useful technique, particularly in nonassociative algebras, called *linearization* or *polarization*. For any homogeneous (all terms have the same degree) polynomial of degree n in one variable $p(x)$, we can define a polynomial $p'(x_1, \ldots, x_n)$ that is *multilinear* and coincides with $p(x)$ on the "diagonal": $p(x) = p'(x, \ldots, x)$. For example, the linearizations of x^2 and x^3 are

$$\frac{1}{2}(x_1x_2 + x_2x_1),$$
$$\frac{1}{6}(x_1x_2x_3 + x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2 + x_3x_2x_1).$$

As long as we are in a field of characteristic 0 one can do this freely, but one must take care in general, for linearizing requires division with $n!$. The process can be described more formally as follows.

Replace x with a formal linear combination $x_1 + \lambda x_2$ for some *indeterminate* scalar λ; expanding $p(x_1 + \lambda x_2)$, we have

$$p(x_1 + \lambda x_2) = p(x_1) + \lambda p_1(x_1, x_2) + \lambda^2 p_2(x_1, x_2) + \cdots + \lambda^n p(x_2)$$

where $p_i(x_1, x_2)$ is homogeneous with degree $n - i$ in x_1 and i in x_2. Intuitively, we obtain $p_i(x_1, x_2)$ from replacing i x_1's in $p(x_1)$ by x_2's in all possible ways. The coefficient $p_1(x_1, x_2)$ of λ is now linear in x_2 and of degree $n-1$ in x_1. We repeat this process until we obtain a full linearization (apart from a factor of $n!$). Equivalently, by immediately replacing x with $\lambda_1 x_1 + \ldots \lambda_n x_n$, we obtain the full linearization of $p(x)$ as the coefficient of $\lambda_1 \ldots \lambda_n$ in $p(\lambda_1 x_1 + \cdots + \lambda_n x_n)$.

Proposition 7.1.1. Any Jordan algebra J over a field F, $\mathrm{char}(F) \neq 2$, satisfies the *linearized Jordan identities*:

$$[x^2, y, z] + 2[x \bullet z, y, x] = 0 \tag{JL1}$$
$$[x \bullet z, y, w] + [z \bullet w, y, x] + [w \bullet x, y, z] = 0 \tag{JL2}$$

Proof. The associator $[x^2, y, x]$ in the Jordan identity $[x^2, y, x] = 0$ is homogeneous of degree 3. First, replace x with $x + \lambda z$ for some scalar λ. The constant term is $[x^2, y, x]$ and the λ^3-term is $\lambda^3[z^2, y, z]$. Subtract these terms to obtain

$$\begin{aligned}&[(x + \lambda z)^2, y, x + \lambda z] - [x^2, y, x] - \lambda^3[z^2, y, z]\\ &= \lambda\big([x^2, y, z] + 2[x \bullet z, y, x]\big) + \lambda^2\big([z^2, y, x] + 2[z \bullet x, y, z]\big).\end{aligned}$$

Now the identity $[x^2, y, x] = 0$ holds for $x = x$, $x = z$ and $x = x + \lambda z$ individually, so the above must equal zero. With

$$f(x, y, z) := [x^2, y, z] + 2[x \bullet z, y, x],$$

we may rewrite the above as

$$\lambda f(x, y, z) + \lambda^2 f(z, y, x) = 0.$$

This equation must hold for any $\lambda \in F$, so in particular taking $\lambda = 1$ and subtracting from it the equation for $\lambda = -1$ we find $2f(x, y, z) = 0$. Since $\mathrm{char}(F) \neq 2$, $f(x, y, z) = 0$, showing (JL1).

The expression $f(x, y, z)$ is of degree 2 in x and of degree 1 in y and z, so linearizing again with $x = x + \lambda w$ and subtracting the constant and λ^3 terms, we obtain

$$2\lambda\big([x \bullet w, y, z] + [x \bullet z, y, w] + [w \bullet z, y, x]\big),$$

which is equal to 0 by applying the original Jordan identity to x, w and $x + \lambda w$. Taking $\lambda = \frac{1}{2}$, we obtain the second identity. $\square$

7.2 Power associativity

Lemma 7.1. From the Jordan identity, we obtain the operator identities

$$L_{x^2 \bullet y} = -2L_x L_y L_x + L_{x^2} L_y + 2L_{x \bullet y} L_x \tag{7.2.1}$$
$$L_{(x \bullet z) \bullet y} = -L_x L_y L_z - L_z L_y L_x + L_{z \bullet x} L_y + L_{x \bullet y} L_z + L_{z \bullet y} L_x \tag{7.2.2}$$

Proof. The identity (JL1) in Proposition 7.1.1 considered with z as the variable, becomes in terms of operators

$$\begin{aligned} 0 &= (L_{x^2 \bullet y} - L_{x^2} L_y) + 2(R_x R_y L_x - R_{y \bullet x} L_x) \\ &= (L_{x^2 \bullet y} - L_{x^2} L_y) + 2(L_x L_y L_x - L_{x \bullet y} L_x), \end{aligned}$$

using commutativity. Thus (7.2.1) holds. Now we linearize (7.2.1) with $x \mapsto x + \frac{1}{2}z$, so that the left hand side becomes

$$L_{(x + \frac{1}{2}z)^2 \bullet y} = L_{x^2 \bullet y} + L_{(x \bullet z) \bullet y} + \frac{1}{4} L_{z^2 \bullet y}.$$

Using (7.2.1) for the first and third of the terms on the right and subtracting this from the above, one obtains (7.2.2):

$$L_{(x \bullet z) \bullet y} = -L_x L_y L_z - L_z L_y L_x + L_{z \bullet x} L_y + L_{x \bullet y} L_z + L_{z \bullet y} L_x.$$

$\square$

Proposition 7.2.1. Let J be a unital Jordan algebra, and define powers of $x \in J$ inductively by $x^0 = 1$, $x^1 = x$ and $x^{n+1} = x \bullet x^n$. Then

(i) All multiplication operators L_{x^n} are polynomials in the commuting operators L_x and L_{x^2}, and therefore they all commute: $[L_{x^n}, L_{x^m}] = 0$ for all m, n.

(ii) J is power associative: For any x, the subalgebra of J generated by x is associative; we have
$$x^m \bullet x^n = x^{m+n}.$$

Proof. To show (i), we use induction on n. For $n = 0, 1, 2$ the assertion is trivial, so suppose it holds for all powers smaller than $n + 2$. Now put $y \mapsto x$, $z \mapsto x^n$ in Lemma 7.1(ii) to obtain

$$\begin{aligned} L_{x^{n+2}} &= L_{(x \bullet x^n) \bullet x} \\ &= -L_x L_x L_{x^n} - L_{x^n} L_x L_x + L_{x^n \bullet x} L_x + L_{x^2} L_{x^n} + L_{x^n \bullet x} L_x \\ &= L_{x^{n+1}} L_x + (L_{x^2} - L_x L_x) L_{x^n} + (L_{x^{n+1}} - L_{x^n} L_x) L_{x^n}. \end{aligned}$$

By induction, $L_{x^n}, L_{x^{n+1}}$ are polynomials in L_x, L_{x^2}, hence so is $L_{x^{n+2}}$.

For (ii), we use induction on $n + m$. For $n + m = 0, 1, 2$ the result holds, so assume it holds for exponents smaller than $n + m$ with $m \geq 2$. Then

$$x^n \bullet x^m = (L_{x^n} L_x)(x^{m-1}) = (L_x L_{x^n})(x^{m-1}) = L_x(x^{n+m-1}) = x^{n+m},$$

where the second equality follows from commutativity using (i), and the third by the induction hypothesis. $\square$

7.3 Auxiliary products and the quadratic definition

Definition 7.3.1 (Auxiliary products)**.** If J is any Jordan algebra with product denoted with $x \cdot y$ for brevity, we define the following auxiliary products (squares, brace products, U-products, triple products and V-products):

$$\begin{aligned} x^2 &:= x \cdot x \\ \{x, y\} &:= 2(x \cdot y) \\ U_x(y) &:= 2(x \cdot (x \cdot y)) - x^2 \cdot y = (2L_x^2 - L_{x^2})(y) \\ \{x, y, z\} &:= U_{x,z}(y) := (U_{x+z} - U_x - U_z)(y) = 2\Big(x \cdot (z \cdot y) + z \cdot (x \cdot y) - (x \cdot z) \cdot y\Big) \\ V_x(y) &:= \{x, y\} \\ V_{x,y}(z) &:= \{x, y, z\} = U_{x,z}(y). \end{aligned}$$

The brace product $\{x, y\}$ has the same notation as the n-tad $\{x, y\}$ for $n = 2$ since when the product is the Jordan bullet product they coincide.

The same is true for the *Jordan triple product* $\{x, y, z\}$. The expression for the Jordan triple product arises from the ordinary triple product, defined in any associative algebra:
$$xyz + zyx.$$

Of course, the triple product does not make sense for general nonassociative algebras. However, in an associative algebra it is possible to express the triple product in terms of the Jordan bullet product $x \bullet y = \frac{1}{2}(xy + yx)$:

$$xyz + zyx = 2\Big((x \bullet y) \bullet z + (y \bullet z) \bullet x - (x \bullet z) \bullet y\Big).$$

We use this expression to *define* a triple product $\{x, y, z\}$ in any Jordan algebra.

Note that $\{x, y, x\} = 2U_x(y)$, and $V_x = V_{x,1} = U_{x,1} = V_{1,x} = 2L_x$. We think of the U-operator as *simultaneous left and right multiplication*, or *outer multiplication*, and the V operators as *left plus right multiplications*. This is precisely what these operators correspond to in special Jordan algebras: If J is a subalgebra of A^+ for some associative algebra A, then in terms of the associative product xy in A,

$$\begin{aligned} U_x(y) &= xyx \\ V_x(y) &= xy + yx \\ V_{x,y}(z) &= U_{x,z}(y) = \{x, y, z\} = xyz + zyx. \end{aligned}$$

Also note that the U-operator is quadratic, in the sense that $U_{\lambda x} = \lambda^2 U_x$, and its linearization

$$\begin{aligned} U_{x,y} = U_{x+y} - U_x - U_y &= 2(L_x L_y + L_y L_x) - (L_{xy} + L_{yx}) \\ &= 2(L_x L_y + L_y L_x - L_{x \bullet y}) \end{aligned}$$

is bilinear in x, y.

The n-tads and brace products use the same notation as for finite sets, which could potentially cause some confusion particularly in section 12, but it should always be clear what is meant from context.

One advantage with the auxiliary products, in particular the U-product, is that they are not limited to when the base field has characteristic different from 2, unlike the Jordan bullet product $x \bullet y = \frac{1}{2}(xy + yx)$. If one could find a definition of Jordan algebras that made sense for arbitrary fields or rings of scalars, it would not only fill the gap of characteristic 2, but also make possible the study of Jordan *rings*, where the ring of scalars is taken to be $\mathbb{Z}$ (a $\mathbb{Z}$-algebra is a ring similar to how a $\mathbb{Z}$-module is an abelian group).

In 1958, I.G. Macdonald first proved the **fundamental formula** for arbitrary Jordan algebras:

$$U_{U_x(y)} = U_x U_y U_x. \tag{7.3.1}$$

The U-operator simplified many notions that were more cumbersome to express with L- or R-operators. The search began for an axiomatic definition of Jordan algebras in terms of the quadratic U-operator such that it would coincide with the ordinary definition when the field or ring of scalars contained $\frac{1}{2}$.

In 1967, Kevin McCrimmon finalized a quadratic definition of Jordan algebras.

Definition 7.3.2 (Quadratic Jordan algebra)**.** Let R be a commutative unital ring. A **unital quadratic Jordan algebra** is an R-module J equipped with a product $U_x(y)$ that is linear in y and quadratic in x (i.e. the map $x \mapsto U_x$ is a quadratic mapping from J to $\mathrm{End}_F(J)$), together with a choice of unit element 1, such that the following three operator axioms hold *strictly*:

1. $U_1 = 1_J$
2. $V_{x,y}U_x = U_x V_{y,x}$
3. $U_{U_x(y)} = U_x U_y U_x$

That the axioms hold strictly means that these axioms hold not only for all $x, y \in J$, but they also continue to hold in all scalar extensions of J.

8

Composition algebras

Recall from Definition 5.2.2 that a $\star$-algebra is an algebra A over a field F together with an involution, i.e. a map $\star : A \to A$ that is an anti-isomorphism of order 2: $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$. In any $\star$-algebra we can build Hermitian elements from traces $t(x) := x + \overline{x}$ and norms $n(x) := x\overline{x}$.

First, we begin by formalizing the notion of a *composition algebra*. We begin with some basic terminology of quadratic and bilinear forms.

Definition 8.0.1 (Quadratic form)**.** Let F be a field and V a vector space over F. A **quadratic form** q is a map $V \to F$ such that $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in F$ and all $v \in V$, and such that the associated map $\hat{q} : V \times V \to F$ defined as

$$\hat{q}(u, v) := q(u + v) - q(u) - q(v)$$

is bilinear.

We call $\hat{q}(u, v)$ the **bilinear form associated with** q. Note that $\hat{q}$ is symmetric. From the definition, we have

$$\hat{q}(v, v) = q(2v) - 2q(v) = 4q(v) - 2q(v) = 2q(v),$$

so as long as $\mathrm{char}(F) \neq 2$, we can recover the quadratic form from the associated bilinear form, and the theory of quadratic and bilinear forms over fields of characteristic $\neq 2$ is essentially the same.

In general, a symmetric bilinear form $b(x,y)$ is called **nondegenerate** if for any $x \in V$,

$$b(x,y) = 0 \text{ for all } y \in V \implies x = 0.$$

A quadratic form q is nondegenerate if its associated bilinear form $\hat{q}$ is nondegenerate. In this case

$$b(x,z) = b(y,z) \text{ for all } z \implies x = y \tag{8.0.1}$$

since $b(x,z) - b(y,z) = b(x-y,z) = 0$ for all z.

A vector $v \in V$ is called **isotropic** if $q(v) = 0$, and **anisotropic** if $q(v) \neq 0$. A form is isotropic if it has nonzero isotropic vectors, and anisotropic if it has none, i.e. if $q(v) = 0 \iff v = 0$. If the quadratic form q is anisotropic, then it is necessarily nondegenerate since for any $v \neq 0$ we have $0 \neq q(v) = \frac{1}{2}\hat{q}(v,v)$.

Definition 8.0.2 (Composition algebra)**.** A **composition algebra** A is a unital, not necessarily associative algebra over a field F together with a nondegenerate quadratic form q, satisfying the *composition law*:

$$q(ab) = q(a)q(b) \quad \text{for all } a, b \in A.$$

The quadratic form q is referred to as the **norm** on A.

The composition law together with nondegeneracy has significant algebraic consequences. From $q(x) = q(1x) = q(1)q(x)$ we immediately conclude $q(1_A) = 1_F$. We also have

$$\begin{aligned} \hat{q}(xy, xz) &= q(x)\hat{q}(y,z), \\ \hat{q}(yx, zx) &= \hat{q}(y,z)q(x). \end{aligned}$$

If we take the latter of these and replace x with $x_1 + x_2$, then by linearity

$$\hat{q}(yx_1 + yx_2, zx_1 + zx_2) = \hat{q}(yx_1, zx_1) + \hat{q}(yx_1, zx_2) + \hat{q}(yx_2, zx_1) + \hat{q}(yx_2, zx_2). \tag{8.0.2}$$

On the other hand, the left hand side is

$$\begin{aligned} \hat{q}(yx_1 + yx_2, zx_1 + zx_2) &= q((y+z)(x_1+x_2)) - q(y(x_1+x_2)) - q(z(x_1+x_2)) \\ &= q(x_1+x_2)(q(y+z) - q(y) - q(z)) \\ &= q(x_1+x_2)\hat{q}(y,z). \end{aligned}$$

Subtracting the two terms $\hat{q}(yx_1, zx_1) = \hat{q}(y,z)q(x_1)$ and $\hat{q}(yx_2, zx_2) = \hat{q}(y,z)q(x_2)$ from (8.0.2), the left hand side becomes

$$q(x_1+x_2)\hat{q}(y,z) - \hat{q}(y,z)q(x_1) - \hat{q}(y,z)q(x_2) = \hat{q}(x_1,x_2)\hat{q}(y,z),$$

hence

$$\hat{q}(yx_1, zx_2) + \hat{q}(yx_2, zx_1) = \hat{q}(y,z)\hat{q}(x_1,x_2). \tag{8.0.3}$$

We have linearized the equation $\hat{q}(yx, zx) = \hat{q}(y,z)q(x)$ with respect to x, which we may write $x \to x_1, x_2$ for short. Taking $y = x$, $z = y$, $x_1 = 1$ and $x_2 = x$ in (8.0.3) gives a special case

$$\hat{q}(x, yx) + \hat{q}(x^2, y) = \hat{q}(x,y)\hat{q}(1,x). \tag{8.0.4}$$

Proposition 8.0.1. Let A be a composition algebra with norm q. Then

$$x^2 - \hat{q}(x, 1)x + q(x)1 = 0 \tag{8.0.5}$$

for all $a \in A$.

Proof. We have

$$\begin{aligned}\hat{q}(x^2 - \hat{q}(x, 1) + q(x)1, y) &= \hat{q}(x^2, y) - \hat{q}(x, 1)\hat{q}(x, y) + q(x)\hat{q}(1, y)\\ &= \hat{q}(x^2, y) - \hat{q}(x, 1)\hat{q}(x, y) + \hat{q}(x, yx)\\ &= 0,\end{aligned}$$

where the last equality follows from (8.0.4). Since y was arbitrary and $\hat{q}$ is nondegenerate, the claim is proved. □

In other words, every element of a composition algebra satisfies a generic minimum polynomial of degree 2 (cf. remark 5.2.1).

If we linearize $x \to x, y$ in (8.0.5), we obtain

$$xy + yx - \hat{q}(x, 1)y - \hat{q}(y, 1)x + \hat{q}(x, y)1 = 0. \tag{8.0.6}$$

Corollary 8.0.0.1. The norm q on a composition algebra A is uniquely determined by the algebra structure of A.

Proof. For $x = \lambda 1 \in F1$, we have $q(x) = q(\lambda 1) = \lambda^2$, and when $x \notin F1$, the equation (8.0.5) is the unique polynomial of minimum degree for x. □

Definition 8.0.3 (Involution and trace)**.** Let A be any algebra and $q : A \to F$ a quadratic form. If $q(c) = 1$ for some $c \in A$, we say that q has a **base point** c. For such a q we define the **trace** linear form

$$\begin{aligned}T : V &\longrightarrow F\\ x &\longmapsto \hat{q}(x, c)\end{aligned}$$

and the **standard trace involution** $x \mapsto \overline{x}$ on V by $\overline{x} := T(x)c - x$.

Note that the trace and the standard trace involution are linear maps by definition. Any composition algebra has $q(1) = 1$, so the multiplicative unit is always a base point in a composition algebra.

Proposition 8.0.2. For any algebra A with quadratic form q with base point c, the standard trace involution preserves base points, norms and traces and is of period 2:

$$\overline{c} = c \tag{8.0.7}$$

$$T(\overline{x}) = T(x) \tag{8.0.8}$$

$$q(\overline{x}) = q(x) \tag{8.0.9}$$

$$\overline{\overline{x}} = x. \tag{8.0.10}$$

Proof. First, note that $T(c) := \hat{q}(c,c) = 4q(c) - 2q(c) = 2$, so $\bar{c} := T(c)c - c = 2c - c = c$, showing (8.0.7). Also

$$T(\bar{x}) = T(T(x)c - x) = T(x)T(c) - T(x) = T(x),$$

showing (8.0.8). Now

$$\begin{aligned} q(\bar{x}) = q(T(x)c + (-x)) &= \hat{q}(T(x)c, -x) + q(T(x)c) + q(-x) \\ &= T(x)T(-x) + T(x)^2 q(c) + (-1)^2 q(x) \\ &= -T(x)^2 + T(x)^2 + q(x) \\ &= q(x), \end{aligned}$$

showing (8.0.9). Lastly,

$$\bar{\bar{x}} = \overline{(T(x)c - x)} = T(x)\bar{c} - \bar{x} = T(x)c - (T(x)c - x) = x.$$

$\square$

Proposition 8.0.3. When A is a composition algebra, the standard trace involution is an algebra involution, and we have

$$x\bar{x} = \bar{x}x = q(x)1 \tag{8.0.11}$$

$$\overline{(xy)} = \bar{y}\,\bar{x} \tag{8.0.12}$$

$$T(x)1 = x + \bar{x} \tag{8.0.13}$$

Proof. (8.0.11) follows from Proposition 8.0.1:

$$x\bar{x} = x(T(x)1 - x) = T(x)x - x^2 = q(x)1,$$

and similarly $\bar{x}x = q(x)1$. For (8.0.12), we have

$$\begin{aligned} \bar{y}\,\bar{x} &= (T(y)1 - y)(T(x)1 - x) \\ &= T(x)T(y)1 - T(x)y - T(y)x + yx \\ &= \hat{q}(x,1)\hat{q}(y,1)1 - xy - \hat{q}(x,y)1 && \text{using (8.0.6)} \\ &= \hat{q}(xy,1)1 - xy && \text{by (8.0.3)} \\ &= \overline{(xy)}. \end{aligned}$$

Lastly, $x + \bar{x} = x + T(x)1 - x = T(x)1$ is immediate from the definition. $\square$

Corollary 8.0.0.2. Any composition algebra is simultaneously a $\star$-algebra under the standard trace involution. Moreover, when $\mathrm{char}(F) \neq 2$, the involution is *scalar*, i.e. every Hermitian element is a scalar multiple of 1: $H(A) \subset F1 \subset A$.

Proof. Every Hermitian element $x = \bar{x}$ in A is a trace $\frac{1}{2}t(x) = \frac{1}{2}(x + \bar{x})$, which by (8.0.13) lie in $F1$. $\square$

As long as A is unital, all traces are built from norms:

$$t(x) = n(x+1) - n(x) - n(1).$$

Hence for unital $\star$-algebras over a field of characteristic not 2, it is enough for all norms $x\overline{x} = \overline{x}x$ to lie in $F1$ to conclude that all Hermitian elements do too.

Proposition 8.0.4. Let A be a composition algebra with norm q.

(i). A satisfies the *left* and *right adjoint formulas*:

$$\hat{q}(xy, z) = \hat{q}(y, \overline{x}z), \quad \text{and} \quad \hat{q}(xy, z) = \hat{q}(x, z\overline{y}). \tag{8.0.14}$$

(ii). The *left* and *right Kirmse identities* hold:

$$\overline{x}(xy) = x(\overline{x}y) = q(x)y, \quad (y\overline{x})x = (yx)\overline{x} = q(x)y. \tag{8.0.15}$$

(iii). A is always alternative.

Proof. (i). The left adjoint formula follows from linearity and symmetry of $\hat{q}$ and taking $y = x$, $z = 1$, $x_1 = y$, $x_2 = z$ in (8.0.3):

$$\begin{aligned}
\hat{q}(y, \overline{x}z) &= \hat{q}(y, (\hat{q}(x,1)1 - x)z) = \hat{q}(x,1)\hat{q}(y,z) - \hat{q}(y, xz) \\
&= \hat{q}(xy, z) + \hat{q}(xz, y) - \hat{q}(y, xz) \\
&= \hat{q}(xy, z).
\end{aligned}$$

The right adjoint formula the left one and Proposition 8.0.2:

$$\hat{q}(xy, z) = \hat{q}(y, \overline{x}z) = \hat{q}(\overline{y}, \overline{(\overline{x}z)}) = \hat{q}(\overline{y}, \overline{z}x) = \hat{q}(z\overline{y}, x) = \hat{q}(x, z\overline{y}).$$

(ii). To prove the left Kirmse identity, we have for any $z \in A$

$$\begin{aligned}
\hat{q}(x(\overline{x}y), z) &= \hat{q}(\overline{x}y, \overline{x}z) && \text{by the left adjoint formula} \\
&= q(x)\hat{q}(y, z) && \text{since } q(\overline{x}) = q(x) \\
&= \hat{q}(q(x)y, z).
\end{aligned}$$

Since $\hat{q}(x(\overline{x}y), z) = \hat{q}(q(x)y, z)$ for all $z \in A$ we have $x(\overline{x}y) = q(x)y$ by (8.0.1). If we replace x by $\overline{x}$, we obtain $\overline{x}(xy) = q(\overline{x})y = q(x)y$. The right identity is obtained from the left by taking $y = \overline{y}$ and conjugating:

$$q(x)y = \overline{q(x)\overline{y}} = \overline{\overline{x}(x\overline{y})} = \overline{(x\overline{y})}x = (y\overline{x})x.$$

Finally $(yx)\overline{x} = q(\overline{x})y = q(x)y$ by taking $x = \overline{x}$ in the above.

(iii). Recall that an alternative algebra is one for which $x(xy) = (xx)y$ and $(yx)x = y(xx)$ for all x, y. In terms of operators these identities can be written $L_x^2 = L_{x^2}$ and $R_x^2 = R_{x^2}$. Either Kirmse identity (8.0.15) applied to $y = 1$ gives

$x\overline{x} = \overline{x}x = q(x)1$, which is equivalent to the degree 2 equation in Proposition 8.0.1 since

$$0 = x^2 - T(x)x + q(x)1 = (x - T(x)1)x + q(x)1 = -\overline{x}x + q(x)1.$$

In terms of operators, the left Kirmse identity is then equivalent to left alternative law. With I and 0 denoting the identity operator and the zero operator on A respectively, we have

$$\begin{aligned} 0 = L_{\overline{x}}L_x - q(x)I_A = L_{T(x)1-x}L_x - q(x)L_1 &= L_{T(x)x-x^2} - q(x)L_1 \\ &= L_{T(x)x-q(x)1} - L_x^2 \\ &= L_{x^2} - L_x^2, \end{aligned}$$

and similarly for right alternativity.

□

8.1 The Cayley-Dickson construction and Hurwitz's theorem

Definition 8.1.1 (Cayley-Dickson construction)**.** Let A be a unital $\star$-algebra over a field F, and let $\lambda \in F$ be a nonzero (invertible) element. Define $\mathcal{CD} := \mathcal{CD}(A, \lambda) := A \oplus A$ as the vector space direct sum of two copies of A, and define a product on $\mathcal{CD}(A, \lambda)$ by

$$(a, b) \cdot (c, d) := (ac + \lambda d^\star b, da + bc^\star),$$

and an involution by

$$(a, b)^\star := (a^\star, -b).$$

We call the resulting algebra $\mathcal{CD}(A, \lambda)$ the **Cayley-Dickson algebra** obtained from the unital $\star$-algebra A and the invertible scalar λ by the **Cayley-Dickson construction** or **doubling process**.

Let $u = (0_A, 1_A) = (0, 1)$. Elements in $\mathcal{CD}$ of the form $(a, 0)$ are identified with $a \in A$ by $a \mapsto (a, 0)$. Elements of the form $(0, b)$ can be written as $(b, 0) \cdot (0, 1)$, or bu. In this way we may express every $(a, b) \in \mathcal{CD}$ as a sum $a + bu$, and $\mathcal{CD} = A \oplus Au$. The product and involution on $\mathcal{CD}$ in this formulation looks like:

$$(a \oplus bu) \cdot (c \oplus du) = (ac + \lambda d^\star b) \oplus (da + bc^\star)u,$$

and

$$(a \oplus bu)^\star = a^\star - bu.$$

For instance when A is the $\star$-algebra $\mathbb{R}$ of real numbers over itself with involution defined as the identity map, then $\mathcal{CD}(\mathbb{R}, -1)$ is isomorphic to the complex numbers $\mathbb{C}$ (and u is denoted with the imaginary unit i).

Proposition 8.1.1. The algebra $\mathcal{CD}(A, x)$ is a unital $\star$-algebra, and its product can be broken up into the following *product rules*: for all $a = (a, 0)$ and $b = (b, 0)$,

$$
\begin{aligned}
au &= ua^\star, && \text{(CD1)}\\
ab &= ab \text{ (i.e. } (a,0)(b,0) = (ab,0)), && \text{(CD2)}\\
a(bu) &= (ba)u, && \text{(CD3)}\\
(au)b &= (ab^\star)u, && \text{(CD4)}\\
(au)(bu) &= \lambda b^\star a. && \text{(CD5)}
\end{aligned}
$$

Proof. A simple computation shows $1_{\mathcal{CD}} = (1, 0)$ is a multiplicative identity in $\mathcal{CD}$. It is clear that the involution $\star$ on $\mathcal{CD}$ is of period 2, and its linearity follows from the linearity of the involution on A. To show that it is an anti-homomorphism, we compute:

$$
\begin{aligned}
\Big((a,b)\cdot(c,d)\Big)^\star &= ((ac + xd^\star b)^\star, -(da + bc^\star))\\
&= (c^\star a^\star + xb^\star d, -bc^\star - d(a^\star)^\star)\\
&= (c^\star, -d)\cdot(a^\star, -b) = (c,d)^\star \cdot (a,b)^\star.
\end{aligned}
$$

For the first product rule, $(a, 0)\cdot(0, 1) = (0, a)$ and $(0, 1)\cdot(a^\star, 0) = (0, (a^\star)^\star) = (0, a)$. The second is immediate from the definition of the product in $\mathcal{CD}$.

For the third, $bu = (0, b)$ and $(a, 0)\cdot(0, b) = (0, ba)$ whereas $(b, 0)\cdot(a, 0) = (ba, 0)$, and $(ba, 0)\cdot(0, 1) = (0, ba)$.

Fourth, $au = (0, a)$ and $ab^\star = (ab^\star, 0)$, thus $(0, a)\cdot(b, 0) = (0, ab^\star) = (ab^\star, 0)u$.

Lastly, $(au)(bu) = (0, a)\cdot(0, b) = (\lambda b^\star a, 0) = \lambda b^\star a$. □

The following theorem is Theorem II.2.5.2 in [Cri04].

Theorem 8.1.1 (Inheritance theorem)**.** Let A be a unital $\star$-algebra over a field F with involution written $a \mapsto \overline{a}$, and $\lambda \in F$ a nonzero element. Then the Cayley-Dickson algebra $\mathcal{CD}(A, \lambda)$ satisfies the following properties.

(i) The involution on $\mathcal{CD}(A, \lambda)$ is always nontrivial (i.e. never the identity map). Moreover, the involution on $\mathcal{CD}(A, \lambda)$ is a scalar involution if and only if the involution on A is scalar: If the involution on A is such that $a\overline{a} = q(a)1 \in F1 \subset A$ and $a + a^\star = t(a)1 \in F1 \subset A$ for some quadratic norm form q and linear trace t (and hence all Hermitian elements lie in $F1$), then the involution $\mathcal{CD}(A, \lambda)$ is scalar with new norm and trace

$$Q(a \oplus bu) = q(a) - \lambda q(b), \quad T(a \oplus bu) = t(a).$$

(ii) $\mathcal{CD}(A, \lambda)$ is commutative if and only if A is commutative with trivial involution.

(iii) $\mathcal{CD}(A, \lambda)$ is associative if and only if A is both commutative and associative.

(iv) $\mathcal{CD}(A,\lambda)$ is alternative if and only if A is associative with *central* involution, i.e. all Hermitian elements $H(A,\star)$ of A lie in the center of A: $H(A,\star)\subset \mathrm{Cent}(A)$.

Proof. (i). It is clear that the involution is nontrivial (at the very least, as long as $0\neq 1$ and $\mathrm{char}(F)\neq 2$). If the original involution on A is scalar with $a+\overline{a}=t(a)1$, then $\overline{a}=t(a)1-a$ commutes with a and $a\overline{a}=\overline{a}a=q(a)1$. The new trace in $\mathcal{CD}$ is then

$$(a\oplus bu)+\overline{(a\oplus bu)}=(a\oplus bu)+(\overline{a}\oplus -bu)=(a+\overline{a})\oplus 0=t(a)1,$$

and the new norm is

$$(a\oplus bu)\overline{(a\oplus bu)}=(a\oplus bu)(\overline{a}\oplus -bu)=(a\overline{a}-\lambda\overline{b}b)\oplus(-ba+b\overline{\overline{a}})u=(q(a)-\lambda q(b))1.$$

(ii). Note that A is a subalgebra of $\mathcal{CD}(A,\lambda)$ since it is a subspace closed under multiplication by Proposition 8.1.1(2), so it is certainly necessary that A is commutative. It is also necessary that the involution is trivial, since the commutator $[a,u]=au-ua=(a-\overline{a})u$ must vanish in $\mathcal{CD}(A,\lambda)$. These two conditions are also sufficient for making the product

$$(a_1\oplus b_1u)\cdot(a_2\oplus b_2u)=(a_1a_2+\lambda b_1b_2)\oplus(a_1b_2+b_1a_2)u$$

symmetric in the indices 1 and 2.

(iii). For $\mathcal{CD}(A,\lambda)$ to be associative it is necessary that the subalgebra A is associative. Commutativity of A is also necessary for the associator $[a,b,u]=(ab)u-a(bu)=(ab-ba)u$ (using (CD3)) to vanish. To show that these conditions are sufficient, we show that any associator in $\mathcal{CD}(A,\lambda)$ vanishes under these conditions:

$$\begin{aligned}&[a_1\oplus b_1u,a_2\oplus b_2u,a_3\oplus b_3u]\\&\quad=\Big((a_1a_2+\lambda\overline{b_2}b_1)a_3+\lambda\overline{b_3}(b_1\overline{a_2}+b_2a_1)-a_1(a_2a_3+\lambda\overline{b_3}b_2)-\lambda\overline{(b_2\overline{a_3}+b_3a_2)}b_1\Big)\\&\quad+\Big((b_1\overline{a_2}+b_2a_1)\overline{a_3}+b_3(a_1a_2+\lambda\overline{b_2}b_1)-(b_2\overline{a_3}+b_3a_2)a_1-b_1\overline{(a_2a_3+\lambda\overline{b_3}b_2)}\Big)u.\end{aligned}$$

Under commutativity and associativity, this expression becomes

$$\begin{aligned}&\Big(a_1a_2a_3+\lambda b_1\overline{b_2}a_3+\lambda a_1b_2\overline{b_3}+\lambda b_1\overline{a_2}\overline{b_3}-a_1a_2a_3-\lambda a_1b_2\overline{b_3}-\lambda b_1\overline{a_2}\overline{b_3}-\lambda b_1\overline{b_2}a_3\Big)\\&+\Big(b_1\overline{a_2}\overline{a_3}+a_1b_2\overline{a_3}+a_1a_2b_3+\lambda b_1\overline{b_2}b_3-a_1b_2\overline{a_3}-a_1a_2b_3-b_1\overline{a_2}\overline{a_3}-\lambda b_1\overline{b_2}b_3\Big)u\\&=0.\end{aligned}$$

(iv). If $\mathcal{CD}(A,\lambda)$ is alternative, then its associator alternates (2.1.4). To see that associativity of A is necessary, we compute

$$\begin{aligned}[c,au,\overline{b}]+[au,c,\overline{b}]&=\Big((ac)b-(ab)c+(a\overline{c})b-a(\overline{c\overline{b}})\Big)u\\&=\Big((at(c)1)b-[a,b,c]-a(bt(c)1)\Big)u=-[a,b,c]u.\end{aligned}$$

The left hand side vanishes in an alternative algebra, in which case all associators in A vanish. A central involution is also necessary, which can be seen by computing

$$[au, au, bu] = \Big(\lambda b(\overline{a}a) - a\overline{(\lambda\overline{b}a)}\Big)u = \lambda\Big(b(\overline{a}a) - (a\overline{a})b\Big)u.$$

Letting $b = 1$, we see that the above is zero if and only if $a\overline{a} = \overline{a}a$ (for $b = 1$), and that this element commutes with all b. Thus all norms lie in $\mathrm{Cent}(A)$ for an associative A, and hence also all traces due to $\mathrm{char}(F) \neq 2$. Since all Hermitian elements are traces, this shows $H(A, \star) \subset \mathrm{Cent}(A)$. A straightforward computation shows that associativity of A implies that $\mathcal{CD}(A, \lambda)$ is alternative, see e.g. [Schaf66] Ch. III.4. $\square$

There is a natural way in which Cayley-Dickson doubling process takes place inside a composition algebra, and one can repeatedly perform the doubling until the entire composition algebra is exhausted. We will see that all composition algebras have finite dimension. A priori we do not know this, so we shall focus for the moment on finite-dimensional subalgebras of composition algebras.

Recall that for two vectors $v, w \in V$ are *orthogonal* with respect to a symmetric bilinear form $b : V \to V$ if $b(v, w) = 0$, and if $W \subset V$ is a subspace, the *orthogonal complement* $W^\perp$ of W is the set of vectors of V that are orthogonal to all vectors in W.

We shall make use of the following linear algebra fact. Let W be a vector space, possibly infinite-dimensional, equipped with a symmetric bilinear form b, and let $V \subset W$ be a subspace. Every vector $v \in V$ determines a linear map from V into its dual space $V^* := \mathrm{Hom}(V, F)$ by $v \mapsto (v' \mapsto b(v', v))$ or $v \mapsto b(\,\cdot\,, v)$ for short. When b is nondegenerate on V, this map is injective, so V is isomorphic to a subspace of V^*. If in addition V is finite-dimensional, $v \mapsto b(\,\cdot\,, v)$ is an isomorphism $V \to V^*$, because V and V^* have the same finite dimension. Likewise every $w \in W$ produces an element $b(\,\cdot\,, w)$ of V^*. For every such w there must be some $v \in V$ for which $b(\,\cdot\,, w) = b(\,\cdot\,, v)$ by the isomorphism $V \to V^*$, so $b(\,\cdot\,, w - v) = 0$ and $w - v \in V^\perp$. Hence every element $w \in W$ decomposes into a sum $v + (w - v)$ where $v \in V$, $w - v \in V^\perp$.

The following result (which I assume is due to Nathan Jacobson) is Theorem II.2.6.1 in [Cri04].

Theorem 8.1.2 (Jacobson necessity theorem)**.** Suppose A is a proper ($A \neq 0, C$) finite-dimensional (unital) subalgebra of a composition algebra C over some field F. Assume that q is nondegenerate on A. Then there are elements $u \in A^\perp$ for which $q(u) = -\lambda \neq 0$, and for any such element, $A + Au$ is a subalgebra of C that is isomorphic and isometric to the Cayley-Dickson algebra $\mathcal{CD}(A, \lambda)$.

Proof. Since q is nondegenerate on A and $\hat{q}(A, A^\perp) = \mathbf{0}$, we have $A \cap A^\perp = \mathbf{0}$. Since A is finite-dimensional, we get a decomposition $C = A \oplus A^\perp$. By assumption A is proper, so $A^\perp$ is nonzero. Since q is nondegenerate on C, $\mathbf{0} \neq \hat{q}(A^\perp, C) = \hat{q}(A^\perp, A + A^\perp) = \hat{q}(A^\perp, A^\perp)$, so $\hat{q}$ is not identically zero on $A^\perp$. Hence there must exist some $u \in A^{\perp}$ with nonzero norm; define $\lambda := -q(u)$. We shall show that the algebra $A+Au$ has the structure of the Cayley-Dickson algebra $\mathcal{CD}(A,u) = A \oplus Au$.

First, we show that $A + Au = A \oplus Au$, i.e. that $A \cap Au = \mathbf{0}$. This follows since $A \cap A^{\perp} = \mathbf{0}$ and $Au \subset A^{\perp}$, the latter because

$$q(au, A) = q(u, \overline{a}A) \subset q(u, A) = \mathbf{0},$$

using the left adjoint formula (8.0.14) and the fact that A is a subalgebra and orthogonal to u.

The fact that $Au \subset A^{\perp}$ also implies that

$$q(a+bu) = q(a) + \hat{q}(a, bu) + q(b)q(u) = q(a) - \lambda q(b),$$

since $\hat{q}(a, bu) = 0$ and $q(u) = -\lambda$. Also $T(au) = q(1, \underline{au}) \subset q(A, Au) = 0$ (using that A is unital), so the involution on $A \oplus Au$ becomes $\overline{a+bu} = T(a+bu)1 - (a+bu) = T(a)1 - a - bu = \overline{a} - bu$. Thus the norm and the involution are the same on $A+Au$ and $\mathcal{CD}(A, \lambda)$:

$$q(a+au) = q(a) - \lambda q(b), \quad \overline{a+bu} = \overline{a} - bu. \tag{8.1.1}$$

It remains to show that the product on $A + Au$ and on $\mathcal{CD}(A, \lambda)$ coincide. We will do this by showing that all the product rules in Proposition 8.1.1 hold. Notice first that $u^2 = \lambda 1$, since $u^2 = -u\overline{u}$ by the involution above, and $-u\overline{u} = -q(u)1$ by either Kirmse identity taking $y = 1$, and $-q(u)1 = \lambda 1$ by definition of u.

Now (CD1) follows directly from the involution: $u\overline{a} = -\overline{u}\,\overline{a} = -\overline{(au)} = au$. The second is trivial. For (CD3), we linearize the right Kirmse identitiy (8.0.15) $(yx)\overline{x} = q(x)y$ with $x \to x_1, x_2$ to get

$$(yx_1)\overline{x}_2 + (yx_2)\overline{x}_1 = \hat{q}(x_1, x_2)y. \tag{8.1.2}$$

Then

$$\begin{aligned}
a(bu) - (ba)u &= -a\overline{(bu)} + (ba)\overline{u} && \text{by the involution (8.1.1)}\\
&= -(1a)\overline{bu} - (1(bu))\overline{a} + (bu)\overline{a} + (ba)\overline{u} && \\
&= -1\hat{q}(a, bu) + b\hat{q}(a, u) && \text{using (8.1.2)}\\
&= 0 && \text{since } Au \subset A^{\perp}.
\end{aligned}$$

For (CD4), we have

$$\begin{aligned}
(a\overline{b})u - (au)b &= (a\overline{b})u + (a\overline{u})b && \text{by (8.1.1)}\\
&= a\hat{q}(b, u) && \text{by (8.1.2)}\\
&= 0 && \text{since } Au \subset A^{\perp}.
\end{aligned}$$

Finally, we use the linearized left Kirmse identity

$$\overline{x}_1(x_2 y) + \overline{x}_2(x_1 y) = \hat{q}(x_1, x_2)y \tag{8.1.3}$$

to show (CD5):

$$\begin{aligned}
\lambda\bar{b}a - (au)(bu) &= u^2(\bar{b}a) + \overline{(au)}(bu) && \text{by (8.1.1) and } u^2 = \lambda 1 \\
&= \bar{b}(au^2) + \overline{(au)}(bu) && \text{since } u^2 \in \mathrm{Cent}(A + Au) \\
&= \bar{b}((au)u) + \overline{(au)}(bu) && \text{by right alternative law (2.1.2)} \\
&= q(b, au)u && \text{by (8.1.3)} \\
&= 0 && \text{since } Au \subset A^{\perp}.
\end{aligned}$$

Hence all the rules for multiplying in the Cayley-Dickson algebra hold, and $A + Au \cong \mathcal{CD}(A, \lambda)$. ∎

We can now combine the inheritance theorem 8.1.1 with the above to prove Hurwitz's theorem.

Theorem 8.1.3 (Hurwitz's Theorem)**.** Any composition algebra A over a field F of characteristic $\neq 2$ has finite dimension 1, 2, 4 or 8 and is one of the following.

1. The base field $A_0 = F1$ of dimension 1 that is commutative, associative and has trivial involution.

2. A *binarion* algebra $A_1 = \mathcal{CD}(A_0, \lambda_1)$ of dimension 2 that is commutative, associative with nontrivial involution.

3. A *quaternion* algebra $A_2 = \mathcal{CD}(A_1, \lambda_2)$ of dimension 4 that is associative and noncommutative.

4. An *octonion* algebra $A_3 = \mathcal{CD}(A_2, \lambda_3)$ of dimension 8 that is noncommutative and nonassociative but alternative.

Proof. The subalgebra $A_0 = F1$ of A has dimension 1 and is unital, commutative and associative with trivial involution: $\overline{\lambda 1} = \lambda\bar{1} = \lambda 1$. The norm form is nondegenerate because $\mathrm{char}(F) \neq 2$: $\hat{q}(\alpha 1, \beta 1) = \alpha\beta T(1) = 2\alpha\beta$. If $A_0 = A$, we are done.

If $A_0 \subsetneq A$, then by Theorem 8.1.2 we can pick i orthogonal to A_0 with $q(i) = -\lambda_1 \neq 0$ and obtain a subalgebra $A_1 \cong \mathcal{CD}(A_0, \lambda_1)$ of dimension 2. By the inheritance theorem 8.1.1, A_1 is commutative and associative with nontrivial involution.

If $A_1 \subsetneq A$, then by Jacobson necessity we can pick j orthogonal to A_1 with $N(j) = -\lambda_2 \neq 0$ and obtain a 4-dimensional subalgebra $A_2 \cong \mathcal{CD}(A_1, \lambda_2)$ that is associative but not commutative.

If $A_2 \subsetneq A$, we proceed in the same way to obtain an 8-dimensional subalgebra $A_3 \cong \mathcal{CD}(A_2, \lambda_3)$ that is neither commutative nor associative, but is alternative.

Now suppose that A_3 is still not the whole algebra A. Then we repeat the process to obtain a 16-dimensional subalgebra $\mathcal{CD}(A_3, \lambda_4)$. But this is no longer alternative since A_3 is not associative, so it cannot be a composition algebra by 8.0.4(iii). Hence $A_3 = C$, and the process must stop. ∎

Since Hurwitz's theorem makes no assumption of finite-dimensionality, this shows that finite-dimensionality is an intrinsic property of composition algebras.

If we take $A_0 = \mathbb{R}$ to be the composition algebra of real numbers (over itself) with trivial involution and norm $q(r) = r^2$, with $\lambda_1 = \lambda_2 = \lambda_3 = -1$ in the Cayley-Dickson doubling, then A_1 is the complex numbers $\mathbb{C}$, A_2 is the Hamilton quaternion algebra $\mathbb{H}$ and A_3 is the Cayley algebra of octonions $\mathbb{O}$, as in Theorem 6.0.1. As algebras over $\mathbb{R}$, these are the only possible composition algebras by Hurwitz' theorem. They are also *division* algebras: $xy = 0$ if and only if $x = 0$ or $y = 0$. Indeed, when the norm q is anisotropic, every nonzero x has an inverse: $x\overline{x} = \overline{x}x = q(x)1$ hence $x^{-1} = \frac{\overline{x}}{q(x)}$. For associative algebras, the existence of multiplicative inverses for all nonzero elements is equivalent to being a division algebra. It is clear that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are division algebras. However, for nonassociative algebras this is *not* true in general. Indeed, if we apply the Cayley-Dickson doubling to the octonions following the same recipe, we get a 16-dimensional algebra called the *sedenions*. The elements of this algebra all have multiplicative inverses as before, but the sedenions have zero divisors and is not a division algebra.

Note that the norm $q(r) = r^2$ on $\mathbb{R}$ is positive definite, i.e. $q(r) > 0$ for all nonzero $r \in \mathbb{R}$. The new norm in $\mathbb{C} := \mathcal{CD}(\mathbb{R}, -1)$ becomes $q((a, b)) = q(a) + q(b) = a^2 + b^2$ which is positive definite if and only if the norm on $\mathbb{R}$ is. Thus the norms in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are all positive definite. In particular the nonassociative algebra $\mathbb{O}$ is a division algebra, for if $xy = 0$, then $q(x)q(y) = 0$ so either $q(x)$ or $q(y)$ is zero, which by positive definiteness implies either $x = 0$ or $y = 0$. This fails for the sedenions because the norm no longer admits composition.

We finish this section by showing that the Hermitian 2×2 matrix algebras $H_2(A)$ in the classification theorem 6.0.1 are isomorphic to spin factors (Example 5.3).

Let $\{e_0, e_1, \ldots, e_7\}$ be a basis for $\mathbb{O}$ and $x = x_0e_0 + x_1e_1 + \ldots x_7e_7$ an arbitrary element, where we identify the multiplicative unit e_0 with the real number 1. Then

$$q(x) = \sum x_i^2,$$

and

$$\begin{aligned} T(x) := \hat{q}(x, 1) &:= q(x + 1) - q(x) - q(1) \\ &= (x_0 + 1)^2 + \sum_{i=1}^{7} x_i^2 - \sum_{i=0}^{7} x_i^2 - 1 \\ &= 2x_0. \end{aligned}$$

The involution looks like

$$\overline{x} := T(x)1 - x = 2x_0e_0 - x = x_0e_0 - x_1e_1 - \cdots - x_7e_7.$$

The bilinear form is just twice the ordinary dot product:

$$\hat{q}(x, y) = q(x + y) - q(x) - q(y) = \sum((x_i + y_i)^2 - x_i^2 - y_i^2) = 2\sum x_iy_i.$$

Similar statements hold also for $\mathbb{R}, \mathbb{C}, \mathbb{H}$ for suitable bases.

Proposition 8.1.2. The Hermitian matrix algebras $H_2(A)$ for $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are isomorphic to the spin factor $\mathcal{J}_{spin}(n)$ for $n = \dim A + 1$.

Proof. An arbitrary element of $H_2(A)$ looks like

$$X = \begin{pmatrix} \alpha & x \\ \overline{x} & \beta \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}1,\ x \in A.$$

We may rewrite this as

$$X = \begin{pmatrix} a+b & x \\ \overline{x} & a-b \end{pmatrix}$$

for $a = \frac{1}{2}(\alpha + \beta)$, $b = \frac{1}{2}(\alpha - \beta)$. We claim that $H_2(A) \cong (A \oplus \mathbb{R}) \oplus \mathbb{R}$ via the linear map φ sending $X \mapsto ((x, b), a)$. Indeed, if

$$Y = \begin{pmatrix} c+d & y \\ \overline{y} & c-d \end{pmatrix},$$

then the product in $H_2(A)$ becomes

$$X \bullet Y = \frac{1}{2}(XY + YX) = \frac{1}{2}\begin{pmatrix} 2(a+b)(c+d) + x\overline{y} + y\overline{x} & 2ay + 2cx \\ 2c\overline{x} + 2a\overline{y} & 2(a-b)(c-d) + \overline{x}y + \overline{y}x \end{pmatrix}.$$

Now

$$x\overline{y} + y\overline{x} = x\overline{y} + \overline{(x\overline{y})} = T(x\overline{y})1 = \hat{q}(x\overline{y}, 1)1 = \hat{q}(x, y)1$$

by the right adjoint formula (8.0.14). Similarly $\overline{x}y + \overline{y}x = T(\overline{x}y)1 = \hat{q}(\overline{x}y, 1) = \hat{q}(x, y)1$ by the left adjoint formula. Hence

$$X \bullet Y = \frac{1}{2}\begin{pmatrix} 2(ac+bd) + \hat{q}(x,y) + 2(ad+bc) & 2ay + 2cx \\ 2c\overline{x} + 2a\overline{y} & 2(ac+bd) + \hat{q}(x,y) - 2(ad+bc) \end{pmatrix}.$$

Thus

$$\varphi(X \bullet Y) = ((ay + cx, ad + bc), \frac{1}{2}\hat{q}(x, y) + ac + bd).$$

On the other hand, in the spin factor $(A \oplus \mathbb{R}) \oplus \mathbb{R}$, the product becomes

$$((x, b), a) \bullet ((y, d), c) = (c(x, b) + a(y, d), ac + (x, b) \cdot (y, d)).$$

The dot product in $A \oplus \mathbb{R}$, which is isomorphic to $\mathbb{R}^k \oplus \mathbb{R}$ for $k = \dim A$ as vector spaces, is just $(x, b) \cdot (y, d) = \frac{1}{2}\hat{q}(x, y) + bd$. Thus $\varphi(X \bullet Y) = \varphi(X) \bullet \varphi(Y)$, so φ is an algebra homomorphism. It has an obvious inverse $((x, b), a) \mapsto X$, so it is an isomorphism. The dimension of $A \oplus \mathbb{R}$ over $\mathbb{R}$ is $\dim A + 1$, completing the proof. $\square$

8.2 Split composition algebras

One can show that in each of the dimensions 2, 4 and 8 there are up to isomorphism exactly one composition algebra with isotropic norm, i.e. any two composition algebras with isotropic norm of the same dimension are isomorphic (see e.g. [SV00] Theorem 1.8.1). This makes it particularly easy to describe them.

Definition 8.2.1 (Split composition algebras)**.** The **split composition algebras** over a field F are the $\star$-algebras of dimension 1, 2, 4, 8 isomorphic to the following models.

Split unarions $\mathcal{U}(F) = F$, the scalars F with trivial involution and norm $q(x) = x^2$.

Split binarions $\mathcal{B}(F) = F \boxplus F$ (a direct algebra sum) of scalars with the *exchange involution* $(x, y) \mapsto (y, x)$ and norm $q((x, y)) = xy$.

Split quaternions $\mathcal{Q}(F)$, the algebra $M_2(F)$ of 2×2-matrices with *symplectic involution*

$$\overline{x} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with norm $q(x) = \det(x)$.

Split octonions $\mathcal{O}(F) = \mathcal{Q}(F) \oplus \mathcal{Q}(F)\ell$ with standard involution $x \oplus y\ell = \overline{x} - y\ell$ and norm $q(x \oplus y\ell) = \det(x) - \det(y)$.

As algebras over the real numbers $\mathbb{R}$, the split composition algebras are in a sense completely opposite to the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are obtained from the Cayley-Dickson process by choosing $\lambda_i = 1$ instead of $\lambda_i = -1$.

9

Full algebras

We now return to the examples in section 5 in some more detail, starting with the full algebras.

Recall from Definition 5.1.1 that for any algebra A, we define A^+ to be the vector space A equipped with the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$ derived from the product xy in A.

Theorem 9.0.1 (Full properties)**.** Let A be any algebra with product xy.

(i). If A is associative, then A^+ is a Jordan algebra.

(ii). If A is associative, then the auxiliary Jordan products are

$$x^2 = xx, \quad \{x, y\} = xy + yx, \quad U_x(y) = xyx, \quad \{x, y, z\} = xyz + zyx.$$

(iii). Any homomorphism or anti-homomorphism of associative algebras $A \to B$ is simultaneously a homomorphism of Jordan algebras $A^+ \to B^+$. Any associative subalgebra (ideal) of A is also a Jordan subalgebra (ideal) of A^+.

(iv). A^+ is unital if and only if A is unital, in which case their units coincide.

(v). A^+ is unital and simple as a Jordan algebra if and only if A is unital simple as an associative algebra.

Proof.

(i). The Jordan product is clearly commutative, and satisfies the Jordan identity due to associativity of A:

$$\begin{aligned}((x \bullet x) \bullet y) \bullet x &= \frac{1}{4}\left[(x^2y + yx^2)x + x(x^2y + yx^2)\right] \\ &= \frac{1}{4}\left[x^2yx + yx^3 + x^3y + xyx^2\right] \\ &= \frac{1}{4}\left[x^2(yx + xy) + (yx + xy)x^2\right] \\ &= (x \bullet x) \bullet (y \bullet x).\end{aligned}$$

(ii). The first two are trivial, and the fourth follows from the third by linearization. The third one follows from

$$2U_x(y) = \{x, \{x, y\}\} - \{x^2, y\} = x(xy + yx) + (xy + yx)x - (xxy + yxx) = 2xyx.$$

(iii). It is clear that any map that preserves or reverses the ordinary product xy in A also preserve the symmetric Jordan product. Any subspace that is closed under the associative product xy is certainly also closed under the derived Jordan product.

(iv). If 1_A is the associative unit of A, then clearly $x \bullet 1_A = 1_A \bullet x = x$ for all x. For the converse direction, note that a Jordan unit 1 is in particular an idempotent ($1^2 = 1$) with respect to the associative product. On the one hand $U_1(y) = y$ by definition of U and the unit element, but $U_1(y) = 1y1$ by (ii), so $y = 1y1$. Then $1y = 1(1y1) = (11)y1 = 1y1 = y$ and similarly $y1 = y$, so 1 is an associative unit.

(v). If A^+ is simple, then it is necessary that A is simple, since any proper associative ideal is automatically a proper Jordan ideal. Suppose conversely that A is simple. If I is a proper ideal of A^+, then $B = \{aba : a \in A, b \in I\}$ generates an associative ideal of A for any nonzero $b \in I$. Since A is unital, B is not the zero ideal. But any $aba \in B$ can be written in terms of the Jordan bullet product as $U_a(b)$; hence $0 \neq B \subset I$ and B is a proper associative ideal of A, contradicting the simplicity assumption. □

The requirement that A is associative for A^+ to be a Jordan algebra can be weakened somewhat. The proof follows § 1.5 Theorem 3 in [Jac68].

Theorem 9.0.2. Suppose A is a unital algebra over a field F ($\mathrm{char}(F) \neq 2$) for which the left alternative law (2.1.1) holds. Then A^+ is a special Jordan algebra.

Proof. In operator form, the left alternative law becomes

$$L_{x^2} = L_x^2.$$

Then

$$L_{(x+y)^2} = L_{x+y}^2 = (L_x + L_y)^2 = L_x^2 + L_x L_y + L_y L_x + L_y^2,$$

and

$$L_{(x^2+xy+yx+y^2)} = L_{x^2} + L_x L_y + L_y L_x + L_{y^2}.$$

Since $L_x^2 = L_{x^2}$, we get $L_{xy+yx} = L_x L_y + L_y L_x$, and hence that

$$L_{x \bullet y} = L_x \bullet L_y$$

in terms of the Jordan product. By assumption A has an identity 1, so the left regular representation $x \mapsto L_x$ is injective. Moreover, the map $x \mapsto L_x$ is, by the above, a Jordan homomorphism of A^+ into $\mathrm{End}_F(A)^+$. Hence A^+ is isomorphic to a subalgebra $\mathrm{End}_F(A)^+$ for the associative algebra $\mathrm{End}_F(A)$, so A^+ is special. □

We can describe the construction of A^+ by saying that we have a **plus functor** from the category of associative F-algebras to the category of Jordan F-algebras that sends A to A^+ and is the identity on the morphisms.

The full algebra A^+ is in some sense too close to being associative, and the more important roles in the theory of Jordan algebras are played by certain subalgebras of A^+: the Hermitian algebras and the quadratic factors.

10

Hermitian Jordan algebras

The archetypical example of Jordan algebras is the class of Hermitian Jordan algebras. These are subalgebras selected from an algebra A^+, A associative, by means of an involution. Recall that a $\star$-homomorphism $\varphi : (A, \star) \to (A', \star')$ is a homomorphism of $\star$-algebras, i.e. an algebra homomorphism for which $\varphi(x^\star) = \varphi(x)^{\star'}$, and the $\star$-ideals are the kernels of $\star$-homomorphisms; they are ideals invariant under the involution.

Theorem 10.0.1 (Hermitian properties)**.** Let $(A, \star)$ be an associative $\star$-algebra.

1. The set of hermitian elements $H(A, \star)$ forms a Jordan subalgebra of A^+ which is unital if A is. Any $\star$-homomorphism or anti-homomorphism $(A, \star) \to (A', \star')$ restricts to a Jordan homomorphism $H(A, \star) \to H(A', \star')$.

2. Let B be a subalgebra and I and ideal of A. Then $H(B, \star)$ is a Jordan subalgebra and $H(I, \star)$ a Jordan ideal of $H(A, \star)$.

3. If $(A, \star)$ is $\star$-simple, then $H(A, \star)$ is a simple Jordan algebra.

Proof. The anti-automorphism $\star$ is in particular a Jordan automorphism of A^+ by Theorem 9.0.1(iii), so the set of fixed points $H(A, \star)$ forms a Jordan subalgebra, since the set of fixed elements under any Jordan automorphism forms a Jordan subalgebra.

Any $\star$-homomorphism or anti-homomorphism is a homomorphism $A^+ \to (A')^+$ by 9.0.1(iii) that takes Hermitian elements to Hermitian elements: $x \in H(A, \star) \implies \varphi(x)^{\star'} = \varphi(x^\star) = \varphi(x) \in H(A', \star')$.

For any $C \subset A$, $H(C, \star) = C \cap H(A, \star)$ so the second assertion follows since intersections of subalgebras (ideals) are again subalgebras (ideals).

For the simplicity assertion we refer to [Cri04] II.3.2.2.

$\square$

For any algebra A, we can define the **opposite algebra** A^{op} by reversing the order of the product: $x \cdot_{op} y := y \cdot x$.

Proposition 10.0.1 (Exchange involution embedding)**.** Every algebra A is a subalgebra of a $\star$-algebra, namely, its **exchange algebra** $\mathrm{Ex}(A) := (A \boxplus A^{op}, ex)$ with **exchange involution** $ex(a, b) := (b, a)$.

The symbol $\boxplus$ refers to the algebra direct sum, the cartesian product under the usual componentwise operations. The map $a \mapsto (a, 0)$ is the embedding mentioned. The Hermitian elements are given by $H(\mathrm{Ex}(A), ex) = \{(a, a) : a \in A\}$.

Proposition 10.0.2. For any algebra A,

$$A^+ \cong H(\mathrm{Ex}(A), ex).$$

Proof. The map $a \mapsto (a, a)$ is linear and a bijection $A^+ \to H(\mathrm{Ex}(A), ex)$ that preserves the bullet product:

$$(a, a) \bullet (b, b) = \frac{1}{2}(ab + ba, ba + ab) = (a \bullet b, b \bullet a) = (a \bullet b, a \bullet b).$$

$\square$

Initially, the Hermitian Jordan algebra $H(A, \star)$ arises as a certain subalgebra of the full algebra A^+, but the above shows that at the same time A^+ arises as a certain Hermitian algebra.

The most important Hermitian Jordan algebras are the Hermitian $n \times n$ matrix algebras.

Hermitian matrix algebras. Let $(D, -)$ be a (unital) $\star$-algebra over some field F with involution denoted $d \mapsto \overline{d}$ and let $X^\star := \overline{X}^T$ be the standard conjugate transpose involution on the algebra $M_n(D)$ of all $n \times n$ matrices with entries in D under the ordinary matrix product XY. The **Hermitian matrix algebra** $H_n(D, -)$ is the algebra given by the vector space $H(M_n(D), \star)$ over F of Hermitian matrices equipped with the Jordan bullet product $X \bullet Y = \frac{1}{2}(XY + YX)$ or the brace product $\{X, Y\} := XY + YX$.

With E_{ij} the elementary matrices with ij-entry 1 and all other entries zero, the algebra $H_n(D, -)$ is spanned by the basic Hermitian elements written in **Jacobson box notation** (see [Jac68] Ch. 3.2 or [Cri04] II.3.2.4) as

$$\begin{aligned}
&\delta[ii] := \delta E_{ii} \quad \text{for Hermitian } \delta \in H(D, -),\\
&d[ij] := dE_{ij} + \overline{d}E_{ji} \quad \text{for any } d \in D,\\
&d[ij] = \overline{d}[ji].
\end{aligned}$$

For the ordinary matrix square $X^2 = XX$ and the brace product $\{X, Y\} = XY + YX$ and for distinct indices i, j, k, we have the basic *product rules*

$$\begin{aligned}
\delta[ii]^2 &= \delta^2[ii], & \{\delta[ii], \gamma[ii]\} &= (\delta\gamma + \gamma\delta)[ii],\\
d[ij]^2 &= d\overline{d}[ii] + \overline{d}d[jj], & \{d[ij], b[ij]\} &= (d\overline{b} + b\overline{d})[ii] + (\overline{d}b + \overline{b}d)[jj],\\
\{\delta[ii], d[ij]\} &= \delta d[ij], & \{d[ij], \delta[jj]\} &= d\delta[ij],\\
\{d[ij], b[jk]\} &= db[ik],
\end{aligned}$$

as well as an *orthogonality rule*

$$\{d[ij], b[k\ell]\} = 0 \quad \text{if} \quad \{i, j\} \cap \{k, \ell\} = \emptyset.$$

When $n \geq 4$, in order for $H_n(A, \star)$ to be a Jordan algebra the relations the elements of A must satisfy forces A to be associative, but for $n = 3$ it suffices for the entries to be from an alternative algebra (Theorem 13.2.3). The algebra $H_n(\mathbb{O})$ is Jordan also for $n < 3$; but is isomorphic to the base field $\mathbb{R}$ and a spin factor $\mathcal{J}_{spin}(9)$ for $n = 1$ and $n = 2$ respectively. This explains why $H_n(\mathbb{O})$ appears only for $n = 3$ in the classification theorem 6.0.1.

From a category theoretic perspective, we have a **Hermitian functor** from the category of associative $\star$-algebras with $\star$-homomorphisms to the category of Jordan algebras, sending $(A, \star)$ to $H(A, \star)$ and $\star$-morphisms φ to their restrictions $\varphi|_{H(A,\star)}$.

11

Quadratic factors

Another important class of Jordan algebras are the *quadratic factors*. These are Jordan algebras constructed from quadratic forms with a base point.

Lemma 11.1. Suppose A is a unital and commutative algebra such that every $x \in A$ satisfies a degree 2 equation:

$$x^2 - \alpha x + \beta 1 = 0$$

for some $\alpha, \beta \in F$ depending on x. Then A is a Jordan algebra.

Proof. We only need to show that the Jordan identity holds. The degree 2 equation implies that we have $[x^2, y, x] = [\alpha x - \beta 1, y, x] = \alpha[x, y, x] - \beta[1, y, x]$, but $[1, y, x] = 0$ since $1 \in \mathrm{Nuc}(A)$ and $[x, y, x] = 0$ since $(xy)x = x(xy) = x(yx)$ holds in any commutative algebra. □

11.1 Quadratic factors

Theorem 11.1.1 (Quadratic factor construction)**.** Suppose J is a vector space equipped with a quadratic form $q : J \to F$ with a base point $q(c) = 1$ for some $c \in J$. Define a product on J with

$$x \bullet y := \frac{1}{2}\Big(T(x)y + T(y)x - \hat{q}(x, y)c\Big). \tag{11.1.1}$$

(i) J under this product is a unital Jordan algebra with multiplicative unit c, and every $x \in J$ satisfies the degree 2 identity:

$$x^2 - T(x)x + q(x)c = 0. \tag{11.1.2}$$

(ii) The standard trace involution is an algebra involution, i.e. $\overline{x \bullet y} = \overline{x} \bullet \overline{y}$, and the auxiliary U-product is given in terms of the trace involution by

$$U_x(y) = \hat{q}(x, \overline{y})x - q(x)\overline{y},$$

and the form q *permits Jordan composition* with U:

$$q(U_x(y)) = q(x)q(y)q(x).$$

(iii) If q is nondegenerate, then J is a simple Jordan algebra unless J is 2-dimensional and q is isotropic.

Proof. The product is symmetric in x, y as $\hat{q}$ is symmetric, so commutativity is clear. Letting $y = c$ in (11.1.1) gives

$$x \bullet c = c \bullet x = \frac{1}{2}\Big(T(x)c + 2x - T(x)c\Big) = x$$

since $T(c) = \hat{q}(c, c) = 2$ and $\hat{q}(x, c) = T(x)$. Hence $1 := c$ is a multiplicative unit.

The degree 2 identity follows from (11.1.1) by setting $y = x$, since $\hat{q}(x, x) = 2q(x)$. Thus J is a Jordan algebra by Lemma 11.1.

Since the standard trace involution is linear and preserves traces and norms and fixes the base point c (Proposition 8.0.2), it also preserves the product $x \bullet y$ that is built out of these components.

To obtain the expression for the U-operator, we have

$$\begin{aligned} U_x(y) &:= 2x \bullet (x \bullet y) - x^2 \bullet y \\ &= x \bullet \Big(T(x)y + T(y)x + \hat{q}(x, y)c\Big) - \Big(T(x)x - q(x)c\Big) \bullet y \\ &= \Big(T(x)x \bullet y + T(y)x^2 - \hat{q}(x, y)x\Big) - \Big(T(x)x \bullet y - q(x)y\Big) \\ &= T(y)\Big(T(x)x - q(x)c\Big) - \hat{q}(x, y)x + q(x)y \\ &= \Big(T(x)T(y) - \hat{q}(x, y)\Big)x - q(x)\Big(T(y)c - y\Big) \\ &= \hat{q}(x, \overline{y})x - q(x)\overline{y}, \end{aligned}$$

where the last equality follows from $\overline{y} = T(y)c - y$ and $\hat{q}(x, c) = T(x)$. We now use this expression for the U-operator to show that q permits Jordan composition:

$$\begin{aligned} q(U_x(y)) &= q(\hat{q}(x, \overline{y})x - q(x)\overline{y}) \\ &= \hat{q}(x, \overline{y})^2 q(x) - \hat{q}(x, \overline{y})q(x)\hat{q}(x, \overline{y}) + q(x)^2 q(\overline{y}) \\ &= q(x)^2 q(y). \end{aligned}$$

To prove the simplicity claim, note that if I is any proper ideal of J, then I cannot contain any nonzero scalar multiple of the unit c (since then $c \in I$ because F is a field), so I must be totally isotropic: for any $b \in I$, by (11.1.2) we have $q(b)1 = T(b)b - b^2 \in I$, so we must have $q(b) = 0$. Moreover, no nonzero $b \in I$ can have zero trace, because if $T(b) = \hat{q}(1, b) = 0$, then by nondegeneracy of q there must be some $x \in J$ for which $\hat{q}(x, b) \neq 0$. Scaling x if necessary so that $\hat{q}(x, b) = 1$ and using that $\{x, y\} := x \bullet y + y \bullet x = T(x)y + T(y)x - q(x, y)c$, we would have

$$c = \hat{q}(x, b)c = T(x)b + T(b)x - \{x, b\} = T(x)b - \{x, b\}.$$

The right hand side is in I, so $c \in I$ contradicting the fact that I was proper. Since T is linear, it must be injective into the base field F when restricted to I, which means I can only be one-dimensional. If we scale b so that $T(b) = 1$, then for any $y \in J$

$$y - \hat{q}(y, b)c = T(b)y - \hat{q}(y, b)c = \{y, b\} - T(y)b,$$

where the right hand side is in I. Hence all $y \in J$ lie in $F1 + I = F1 + Fb$, so J is 2-dimensional and q is isotropic.

□

We call the Jordan algebra of Theorem 11.1.1 a **quadratic factor** and denote it with $\mathcal{J}(q, c)$.

From a category-theoretic perspective, we can form the category of *quadratic forms with base point* by taking the objects to be pairs (q, c) of quadratic forms with base point c, and as morphisms the F-linear *isometries* $\varphi : (q, c) \to (q', c')$ which preserve the base point: $q'(\varphi(x)) = q(x)$ and $\varphi(c) = c'$. Such a map φ automatically preserves traces:

$$T'(\varphi(x)) = \hat{q}'(\varphi(x), c') = \hat{q}'(\varphi(x), \varphi(c)) = \hat{q}(x, c) = T(x)$$

and hence also preserves the product $x \bullet y$ in J, so it is a Jordan algebra homomorphism. Thus we have a **quadratic functor** from the category of F-quadratic forms with base point to unital Jordan algebras over F, with $(q, c) \mapsto \mathcal{J}(q, c)$ on objects and the identity $\varphi \to \varphi$ on morphisms.

11.2 Spin factors

Theorem 11.2.1 (Spin factor construction)**.** Let V be an F-vector space and σ a symmetric bilinear form. Let $J := F1 \oplus V$ and define a product on J by

$$(\alpha 1 \oplus v) \bullet (\beta 1 \oplus w) := (\alpha\beta + \sigma(v, w))1 \oplus (\beta v + \alpha w).$$

Then J is a unital Jordan algebra, which we denote with $\mathcal{J}_{spin}(V, \sigma)$, called a *spin factor*.

Proof. The spin factor $\mathcal{J}_{spin}(V, \sigma)$ is equal to the quadratic factor $\mathcal{J}(q, c)$ for

$$c = 1 \oplus 0, \quad q(\alpha 1 \oplus v) = \alpha^2 - \sigma(v, v), \quad T(\alpha 1 \oplus v) = 2\alpha.$$

The associated bilinear form becomes

$$\begin{aligned}\hat{q}(\alpha 1 \oplus v, \beta 1 \oplus w) &= q((\alpha + \beta)1 \oplus (v + w)) - q(\alpha 1 \oplus v) - q(\beta 1 \oplus w)\\ &= 2\alpha\beta - 2\sigma(v, w)\end{aligned}$$

using that σ is symmetric. Thus the product in $\mathcal{J}(q, c)$ becomes

$$\begin{aligned}(\alpha 1 \oplus v) \bullet (\beta 1 \oplus w) &= \frac{1}{2}\Big(2\alpha(\beta 1 \oplus w) + 2\beta(\alpha 1 \oplus v) - (2\alpha\beta - 2\sigma(v, w)) \bullet (1 \oplus 0)\Big)\\ &= (\alpha\beta + \sigma(v, w))1 \oplus (\alpha w + \beta v).\end{aligned}$$

□

Since we are only dealing with the cases where $\mathrm{char}(F) \neq 2$, we can reverse this process: every quadratic factor $\mathcal{J}(q,c)$ is naturally isomorphic to the spin factor $\mathcal{J}_{spin}(V,\sigma)$ for $\sigma(v,w) = -\frac{1}{2}\hat{q}(v,w)$, the negative of the restriction of $\frac{1}{2}\hat{q}$ to $V := c^{\perp} = \{x \in J : T(x) = \hat{q}(x,c) = 0\}$ since we have the natural decomposition $J = Fc \oplus V$.

In category theoretical terms, we can form a category of symmetric F-bilinear forms whose morphisms are F-linear isometries $\varphi : \sigma \to \sigma'$ (i.e. $\sigma'(\varphi(v),\varphi(w)) = \sigma(v,w)$). The spin factor construction is then a **spin functor** from the category of symmetric F-bilinear forms to unital Jordan algebras over F given by $\sigma \mapsto \mathcal{J}_{spin}(V,\sigma)$, where the Jordan homomorphism corresponding to an F-linear isometry is defined by extending φ unitally to $J = F1 \oplus V$: $\mathcal{J}_{spin}(\varphi)(\alpha 1 \oplus v) := \alpha 1 \oplus \varphi(v)$.

Remark 11.2.1. All nondegenerate quadratic forms q on vector spaces of dimension $n+1$ over an algebraically closed field Ω for $\mathrm{char}(\Omega) \neq 2$ are *equivalent*, i.e. there exists a bijective Ω-linear isometry between them. Such a quadratic form can be represented, relative to a suitable basis, as the dot product on Ω^{n+1}: $q(v) = \sigma(v,v) = v^T v$ for column vectors v. This means that every resulting Jordan algebra $\mathcal{J}(q,c)$ is isomorphic to $\mathcal{J}_{spin}(\Omega^n,\cdot)$ where $\cdot$ is the dot product.

When $V = \mathbb{R}^n$ and σ the ordinary dot product, then $\mathcal{J}_{spin}(V,\sigma) = \mathcal{J}_{spin}(n)$ as in the spin factor example 5.3.

12

Free algebras. Macdonald's theorem.

Macdonald's theorem is a general result in Jordan algebras which says that any identity (i.e. equality between two expressions) in three variables that is of degree at most 1 in one of the variables and holds in every special Jordan algebra will hold in *any* Jordan algebra.

In order to state the theorem, we must make precise the notions of *free algebras*, in particular free *nonassociative* algebras. Free algebras, like other free objects (such as free groups, free modules, polynomial rings, etc.) can be thought of as a generalisation of the notion of bases in vector spaces. Similar to how a linear map between vector spaces is entirely determined by its values on the basis elements of its domain, a homomorphism from a free algebraic object into some other object of the same kind is determined by its values on the generating set of the free object.

An associative algebra A can be thought of as a vector space with a *monoid* structure, i.e. A is simultaneously a vector space and a set equipped with an associative binary multiplication operation $p : A \times A \to A$, such that the vector space structure interacts nicely with the monoid structure (multiplication distributes over addition, etc.). Similarly we may regard a nonassociative algebra as a vector space and a set with a nonassociative binary operation.

12.1 Free Jordan algebras

Definition 12.1.1 (Monoid)**.** A (unital) **monoid** is a set X equipped with an associative binary operation $X \times X \to X$, written xy, with a distinguished element $1 \in X$ for which $1x = x1 = x$ for all $x \in X$. A map $\varphi : M \to M'$ between two monoids is called a **monoid homomorphism** if $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(1_M) = 1_{M'}$.

Definition 12.1.2 (Magma)**.** If the binary operation in the above definition is not assumed to be associative, the structure is instead called a (unital) **magma**. A magma homomorphism is defined in the same way as for monoids; it is a map between (unital) magmas for which $\varphi(xy) = \varphi(x)\varphi(y)$ (and $\varphi(1) = 1$).

Let us first examine the case for associative algebras. If A is a (unital) associative algebra over a field F, then A is a monoid with respect to the multiplication in A. Now suppose M is a monoid and F a field. Consider the vector space A over F with M as a basis (i.e. the free F-module A with basis M). We can define a product on A by

$$\left(\sum a_i m_i\right) \cdot \left(\sum b_j n_j\right) := \sum a_i b_j (m_i n_j)$$

where $a_i, b_j \in F$ and $m_i, n_j \in M$. This product makes A into an associative unital algebra over F. Indeed, associativity follows directly from the fact that the binary operation of M is associative, and the multiplicative identity of A is identified with 1_M in the canonical inclusion $M \hookrightarrow A$. If we take M to be a magma instead, this construction yields a nonassociative algebra.

Definition 12.1.3 (Free monoid)**.** Let X be a nonempty set. A *word* on X is a finite sequence $x_1 \ldots x_k$ of elements of X (i.e. a word is a map $f : \{1, \ldots, k\} \to X$). Denote with

$$M(X) := \{x_1 \ldots x_k \mid k \in \mathbb{N},\ x_i \in X\}$$

the set of all finite sequences of elements of X, where we include the *empty word*, the empty sequence of zero length, which we denote with 1. We think of the elements of X as *letters*. The *length* of a word $x_1 \ldots x_k$ is the number k, and the length of the empty word 1 is 0. Define a binary operation by *concatenation*:

$$(x_1 \ldots x_k) \cdot (y_1 \ldots y_\ell) = x_1 \ldots x_k y_1 \ldots y_\ell.$$

It is clear that this operation is associative, with 1 acting as the identity element. The set $M(X)$ with this binary operation is called the **free monoid on** X.

Free monoids have the following universal property. Suppose M is any monoid, X any set, and $\varphi : X \to M$ an arbitrary set function. Then there is a unique monoid homomorphism $\tilde{\varphi} : M(X) \to M$ extending φ, i.e. $\tilde{\varphi}|_X = \varphi$. Put differently, every set function φ factors uniquely through the inclusion $\iota : X \to M(X)$ (sending $x \in X$ to the word $x \in M(X)$) via a homomorphism $\tilde{\varphi} : M(X) \to M$ of monoids: $\varphi = \tilde{\varphi} \circ \iota$.

Indeed, every element $m \in M(X)$ can be expressed as the concatenation of one-letter words (words of length 1), so if $m = x_1 \dots x_k$, then a monoid homomorphism $\tilde{\varphi}$ with the desired properties must satisfy $\tilde{\varphi}(m) = \tilde{\varphi}(x_1) \dots \tilde{\varphi}(x_k) = \varphi(x_1) \dots \varphi(x_k)$. We therefore define

$$\tilde{\varphi}(m) := \varphi(x_1) \dots \varphi(x_k), \quad m = x_1 \dots x_k \in M(X)$$

with $\tilde{\varphi}(1) := 1_M$, we find that $\tilde{\varphi}$ is the desired monoid homomorphism.

Definition 12.1.4 (Free unital magma)**.** Let X be a nonempty set. Define *nonassociative words* by induction on *degree*. The only word of degree 0 is the word 1. Each $x \in X$ is a nonassociative word of degree 1. If all nonassociative words of degree up to $k-1$ are defined, then the words of degree k are precisely all (mn) (an object consisting of m followed by n, surrounded by parentheses) for words m, n of degrees $d_m, d_n > 0$ such that $d_m + d_n = k$. Let $N(X)$ be the set of all nonassociative words, and define a multiplication on $N(X)$ by $(a, b) \mapsto (ab)$ if $a, b \neq 1$, and $(1, a) \mapsto a$ and $(a, 1) \mapsto a$. This defines a unital magma $N(X)$.

For instance if $a = x_4((x_1x_2)x_3)$ and $b = (x_5x_6)$, then their product is

$$a \cdot b = (x_4((x_1x_2)x_3)(x_5x_6)).$$

We have the following universal property. Let X be a set and M a unital magma. For every map of sets $\varphi : X \to M$, there exists a unique unital magma homomorphism $\tilde{\varphi} : M(X) \to M$ extending φ. The unique homomorphism is constructed as follows. We define $\tilde{\varphi}$ inductively on degree n. For $n = 0$ we set $\tilde{\varphi}(1) = 1_M$, and for $n = 1$ we define $\tilde{\varphi}(x) := \varphi(x)$. If $\tilde{\varphi}$ is defined for all nonassociative words up to degree n, then we define $\tilde{\varphi}((xy)) := \tilde{\varphi}(x)\tilde{\varphi}(y)$.

Definition 12.1.5 (Free associative algebra generated by a set)**.** Let X be a set and $M(X)$ the free monoid on X, and let F be a field. The **free (unital) associative algebra** $A(X)$ on X is the free F-module with basis $M(X)$ together with multiplication defined by

$$\left(\sum_{m_i \in M(X)} a_i m_i\right) \cdot \left(\sum_{n_j \in M(X)} b_j n_j\right) := \sum_{m_i, n_j \in M(X)} a_i b_j m_i n_j \quad a_i, b_j \in F,$$

i.e. the product of basis elements is the product of $M(X)$, and the product of two elements of $A(X)$ is determined by the requirement that the product in an algebra is bilinear.

Another way to think of free associative algebras generated by a set X is that they are noncommutative analogues of polynomial rings. If we think of $X = \{x_1, x_2, \dots\}$ as indeterminates, then the product in $A(X)$ is identical to the product of polynomials in the polynomial ring $F[x_1, x_2, \dots]$, except that the indeterminates no longer commute. We identify X as a subset of $A(X)$ by the map $\iota : X \hookrightarrow M(X) \hookrightarrow A(X)$ sending $x \in X$ to the associative polynomial $1x$. The free associative algebra $A(X)$ has the universal property that any set function $\varphi : X \to A$ into a unital associative algebra A factors uniquely through ι via a unital algebra homomorphism $\tilde{\varphi} : A(X) \to A$. In other words, for any such φ there exists a unique associative homomorphism $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \iota = \varphi$. The map $\tilde{\varphi}$ is the map whose values on basis elements $m = x_1 \dots x_k \in M(X)$ is defined by $\varphi(x_1)\varphi(x_2)\dots\varphi(x_k)$ as in the universal property of free monoids. We extend $\tilde{\varphi}$ linearly to arbitrary elements of $A(X)$.

Definition 12.1.6 (Free nonassociative algebra generated by a set)**.** In the previous definition, we may take the (unital) magma $N(X)$ rather than $M(X)$ and define the **free (unital) nonassociative algebra** $A'(X)$ on X as the free F-module with basis $N(X)$, with product defined in the same way:

$$\left(\sum_{m_i \in N(X)} a_i m_i\right) \cdot \left(\sum_{n_j \in N(X)} b_j n_j\right) := \sum_{m_i, n_j \in N(X)} a_i b_j (m_i n_j),$$

where $(m_i n_j)$ denotes the product in $N(X)$.

The universal property of $A'(X)$ says that any set function $\varphi : X \to A$ into a nonassociative unital algebra A factors uniquely through the natural inclusion $\iota : X \hookrightarrow N(X) \hookrightarrow A'(X)$ via a unital algebra homomorphism $\tilde{\varphi} : A'(X) \to A$. The map $\tilde{\varphi}$ is defined on basis elements as in the universal property of $N(X)$, and extended linearly to $A'(X)$.

We may now define the notion of a *free special* Jordan algebra. Recall that a Jordan algebra is special if it is isomorphic to a subalgebra of A^+ for some associative algebra A. Suppose for simplicity that A is a unital associative algebra and that J is a subalgebra of A^+ containing the multiplicative unit of A, so the unit of A is also the unit of J (Theorem 9.0.1(iv)).

The free unital associative algebra $A(X)$ can be equipped with an involution, called the **reversal involution** ρ, defined by reversing the order of products of basis elements: $\rho(x_1 x_2 \dots x_k) = x_k \dots x_2 x_1$. This map is the identity on the generators X, and is clearly of period 2. Also, $\rho(x_1 \dots x_k y_1 \dots y_\ell) = y_\ell \dots y_1 x_k \dots x_1 = \rho(y_1 \dots y_\ell)\rho(x_1 \dots x_k)$, so ρ is indeed an involution on $A(X)$.

Definition 12.1.7 (Free special unital Jordan algebra)**.** Let $(A(X), \rho)$ be the $\star$-algebra $A(X)$ equipped with the reversal involution ρ. The set $H(A(X), \rho)$ of Hermitian elements of $A(X)$ fixed by ρ becomes a special Jordan algebra (a subalgebra of $A(X)^+$) under the Jordan product that contains X and 1. The Jordan subalgebra of $H(A(X), \rho)$ generated by X and 1 is called the **free (unital) special Jordan algebra** $FSJ(X)$.

Alternatively we may define $FSJ(X)$ as the subalgebra of $A(X)^+$ generated by 1 and X. In this case it is clear that $FSJ(X) \subset H(A(X), \rho)$ since the Hermitian elements contain 1 and X.

Proposition 12.1.1 (Universal property of free special Jordan algebras)**.** Let J be an arbitrary unital special Jordan algebra, where we identify J as a subalgebra $J \subset A^+$ for some associative algebra A. Let $X = \{x_1, \ldots, x_k\}$. Then for any $y_1, \ldots, y_k \in J$, there exists a unique homomorphism of $FSJ(X)$ into J sending $1 \mapsto 1$ and $x_j \mapsto y_j$ for $j = 1, \ldots, k$.

Proof. Let $\varphi : X \to A$ be the set function such that $\varphi(x_j) = y_j$ for $j = 1, \ldots, k$. By the universal property of free unital associative algebras, this extends uniquely to a homomorphism $\tilde{\varphi} : A(X) \to A$ such that $\tilde{\varphi}(1) = 1$ and $\tilde{\varphi}(x_j) = y_j$, $j = 1, \ldots, k$. The map $\tilde{\varphi}$ is simultaneously a homomorphism $A(X)^+$ to A^+ of Jordan algebras (Theorem 9.0.1). If we restrict this map to $FSJ(X)$, then since the y_j are in J, we get a map from $FSJ(X)$ into J for which $1 \mapsto 1$ and $x_j \mapsto y_j$, $j = 1, \ldots, k$. Since the elements $1, x_1, \ldots, x_k$ generate $FSJ(X)$, this resulting homomorphism is unique. $\square$

Remark 12.1.1. If X and Y are sets of the same cardinality, then $FSJ(X)$ is isomorphic to $FSJ(Y)$. If $f : X \to Y$ is a bijection, then by the universal property the set functions $f : X \to Y \subset FSJ(Y)$ and $f^{-1} : Y \to X \subset FSJ(X)$ induce homomorphisms $\varphi : FSJ(X) \to FSJ(Y)$ and $\psi : FSJ(Y) \to FSJ(X)$. Now the identity set function $f \circ f^{-1} : X \to X \subset FSJ(X)$ induces the identity homomorphism $FSJ(X) \to FSJ(X)$, so by uniqueness $\varphi \circ \psi$ must be the identity. Similarly one finds $\psi \circ \varphi$ to be the identity on $FSJ(Y)$, and $FSJ(X) \cong FSJ(Y)$. We may unambiguously write FSJ^k for the special Jordan algebra on a set of k elements. The same is true also for other free objects constructed from sets, such as free groups, free modules, and so on.

The idea for constructing the free unital Jordan algebra is to take the free nonassociative algebra and quotient out the ideal generated by all elements of the form $(ab) - (ba)$ and $((((aa)b)a) - ((aa)(ba)))$, corresponding to the commutative law and the Jordan identity.

Definition 12.1.8. The **free (unital) Jordan algebra** $FJ(X)$ on a set X is the quotient algebra $A'(X)/I$ where I is the ideal generated by all elements $((ab) - (ba))$ and $(((aa)b)a) - ((aa)(ba))$ for $a, b \in A'(X)$.

The algebra $FJ(X) = A'(X)/I$ has a unit $1_{A'} + I$ and satisfies the commutative law and the Jordan identity by this construction, so is a unital Jordan algebra. The algebra $FJ(X)$ has the universal property that any set function $\varphi : X \to J$ of a set X into a unital Jordan algebra has a unique extension into a homomorphism $\tilde{\varphi} : FJ(X) \to J$. When $X = \{x_1, \ldots, x_k\}$ we may write FJ^k to denote the unique (up to isomorphism) free unital Jordan algebra on k elements (in view of the above remark applying analogously in this case).

If $X \subset Y$, then $FJ(X)$ can be identified with the Jordan subalgebra of $FJ(Y)$ generated by X. The homomorphism of $FJ(X)$ onto this subalgebra induced by $X \hookrightarrow Y \hookrightarrow FJ(Y)$ is surjective. It is also injective because it has a left inverse $FJ(Y) \to FJ(X)$ induced by $Y \to X \cup \{0\}$ where $x \in X \mapsto x$ and $y \in Y \setminus X \mapsto 0$. In particular we shall make use of this in the case of two and three generators by identifying $FJ(\{x, y\})$ with the elements of $FJ(\{x, y, z\})$ of degree 0 in z.

Remark 12.1.2. The constructions of free objects can be described in category theoretic terms as saying we have a functor F from the category of sets to the category of monoids, associative algebras, special Jordan algebras, etc. that sends a set X to the free object $F(X)$ constructed from that set, and a set function $\varphi : X \to Y$ to the homomorphism $F(f) := \tilde{\varphi} : F(X) \to F(Y)$ induced by the universal property by $X \to Y \hookrightarrow F(Y)$.

For any associative unital algebra A and any set function $\varphi : X \to A$, the induced map $\tilde{\varphi} : A(X) \to A$ can be thought of as an **evaluation map** in the following sense. If $X = \{x_1, \ldots, x_n\}$, then $\tilde{\varphi}$ takes an arbitrary *generic associative polynomial* $p(x_1, \ldots, x_n) \in A(X)$ to $p(a_1, \ldots, a_n) \in A$, obtained by substituting the elements $a_i = \varphi(x_i)$ for the variables x_i. In a similar way we think of elements of $A'(X)$ as generic *nonassociative* polynomials, and elements of FSJ^n as generic *special Jordan* polynomials, and so on.

Definition 12.1.9. A **Jordan polynomial** $f(x_1, \ldots, x_n)$ in n variables is an element of $FJ(\{x_1, \ldots, x_n\})$.

As is the case with any polynomial, f determines a map $(a_1, \ldots, a_n) \mapsto f(a_1 \ldots, a_n)$ from $J^n \to J$ for any Jordan algebra J by evaluating the variables. This is just the universal property: the map φ sending $x_i \mapsto a_i$ induces a homomorphism $\tilde{\varphi} : FJ(\{x_1, \ldots, x_n\}) \to J$ and $f(a_1, \ldots, a_n) = \tilde{\varphi}(f(x_1, \ldots, x_n))$. We say that f **vanishes** on J if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in J$. This process of evaluating polynomials is sometimes called *specializing* the indeterminates x_i to the values a_i; note however that a Jordan polynomial can be evaluated in any Jordan algebra, not just special Jordan algebras.

12.2 Macdonald's theorem

We may now state Macdonald's theorem (as in [Jac68], Ch. 1.9).

Theorem 12.2.1 (Macdonald's theorem)**.** Let FJ^3 be the free unital Jordan algebra generated by three elements x, y, z, similarly let FSJ^3 be the free special unital Jordan algebra generated by u, v, w. Let $\tilde{\varphi} : FJ^3 \to FSJ^3$ be the unique homomorphism sending $1 \mapsto 1$, $x \mapsto u$, $y \mapsto v$ and $z \mapsto w$, and let J' be the set of elements of FJ^3 that have degree at most 1 in z. Then $\ker \tilde{\varphi} \cap J' = 0$.

Proof. See [Jac68] Ch. 1.9 or [Cri04] IV.B. □

Macdonald's theorem says that the homomorphism $\tilde{\varphi}$ induced by the universal property is injective on the elements of degree at most 1 in z. Note that a Jordan polynomial $f(x, y, z)$ vanishes on all Jordan algebras if and only if it vanishes in FJ^3 by the universal property: $f(x, y, z) = 0$ in FJ^3 if and only if $f(a, b, c) = 0$ for all evaluations $x, y, z \mapsto a, b, c$ of elements in any Jordan algebra J. The same is true also for FSJ^3. Hence the theorem says that any Jordan polynomial $f(x, y, z) \in FJ^3$ of degree at most 1 in z that vanishes in all special Jordan algebras necessarily vanishes in *all* Jordan algebras. In particular, if f vanishes in all A^+ for associative algebras A, then it necessarily vanishes in all Jordan algebras, i.e. is an *identity* (i.e. an equality of expressions $f = 0$) for Jordan algebras. The condition that one of the variables has degree at most 1 is necessary, as we shall see. Macdonald's theorem has a number of useful consequences.

If $M \in \mathrm{End}_F(A)$ is an operator in an algebra A, then the operator identity $M = 0$ is equivalent to the element identity $M(z) = 0$ for all z. This is always linear in the variable z. If $f(x, y, z)$ is a Jordan polynomial with all terms having z-degree 1, then $f(x, y, z) = M_{x,y}(z)$ defines a multiplication operator in x and y, and $f(x, y, z)$ vanishes on an algebra J if and only if $M_{x,y}$ does. Macdonald's theorem thus implies that

> *any multiplication operator in two variables that vanishes on all special Jordan algebras vanishes on all Jordan algebras.*

In particular, we have the following.

Corollary 12.2.1.1. The Jordan U- and V-operators satisfy

$$U_{x^n} = U_x^n, \quad V_{x^n,x^m} = V_{x^{m+n}}.$$

More generally, for any polynomials f, g in the subalgebra $F[x]$ generated by x,

$$U_{f(x)}U_{g(x)} = U_{(fg)(x)}, \quad V_{f(x),g(x)} = V_{(fg)(x)}.$$

Proof. When $J \subset A^+$ is special, the U-operator amounts to simultaneous left and right multiplication with respect to the associative product in A (Theorem 9.0.1). Since any Jordan algebra is power associative (Proposition 7.2.1), the subalgebra $F[x]$ is associative (and commutative) and consists of polynomials in the variable x. The product of two such polynomials $f(x), g(x)$ is the ordinary product $(fg)(x)$ of polynomials, and

$$(U_{f(x)}U_{g(x)})(z) - U_{(fg)(x)}(z) = f(x)(g(x)zg(x))f(x) - (fg)(x)z(fg)(x) = 0$$

vanishes on all special Jordan algebras, hence on all Jordan algebras by Macdonald's theorem. Similarly, using the definition of the V-product (Definition 7.3.1) and Theorem 9.0.1,

$$\begin{aligned} V_{f(x),g(x)}(z) = \{f(x), g(x), z\} &= f(x)g(x)z + zg(x)f(x) \\ &= (fg)(x)z + z(fg)(x) \\ &= \{(fg)(x), z\} = V_{(fg)(x)}(z). \end{aligned}$$

Taking $f(x) = g(x) = x$, we obtain $U_x^2 = U_{x^2}$ and $V_{x,x} = V_{x^2}$, and the case for general n follows inductively. □

Another important corollary is that any free Jordan algebra generated by two elements is special.

Corollary 12.2.1.2 (Shirshov's theorem)**.** A free Jordan algebra generated by two elements $X = \{x, y\}$ is special: the homomorphism $\tilde{\varphi} : FJ(X) \to FSJ(X)$ induced by the map $\varphi : X \to FSJ(X)$ sending $x \mapsto x$, $y \mapsto y$ is an isomorphism.

Remark 12.2.1. Macdonald's original version of the theorem in 1958 only concerned Jordan polynomials $f(x, y, z)$ of z-degree *equal* to 1. The assertion for polynomials of degree 0 in z is precisely Shirshov's theorem. The version of Macdonald given above is the union of the original 1958 theorem and Shirshov's earlier 1956 theorem.

Proof. The homomorphism $\tilde{\varphi}$ determined by φ is surjective since its image contains the generators x, y. We show that it is also injective. As noted before, we can identify $FJ(\{x, y\})$ and $FSJ(\{x, y\})$ with the subalgebras $B \subset FJ(\{x, y, z\})$ and $B' \subset FSJ(\{x, y, z\})$ generated by $\{x, y\}$. The map $\tilde{\varphi}$ is then equal to the restriction of the universal surjective homomorphism $\psi : FJ(\{x, y, z\}) \to FSJ(\{x, y, z\})$ to B, with $\psi(B) = B'$. By Macdonald's theorem, the map ψ is injective on Jordan polynomials $f(x, y, z)$ of degree at most 1 in z, so in particular ψ is injective on B. Hence φ is injective. □

In Definition 12.1.7, we defined the free special unital Jordan algebra $FSJ(X)$ as the subalgebra of the Hermitian elements $H(A(X), \rho)$ containing X. By definition $FSJ(X) \subset H(A(X), \rho)$, and a natural question to ask is how close these special Jordan algebras are from being equal. Recall from Section 4 that the n-tad products are the maps defined by

$$\{x_1, \ldots, x_n\} = x_1 x_2 \ldots x_n - x_n x_{n-1} \ldots x_1.$$

In particular when $n = 4$ we have the *tetrad*: $\{x_1, x_2, x_3, x_4\} = x_1x_2x_3x_4 - x_4x_3x_2x_1$. For $n \leq 3$, the n-tads are all expressible as Jordan bullet products. The tetrad however is a reversible element that can *not* be written as a Jordan product. To show this, it suffices to give an example of a special Jordan algebra $J \subset A^+$ containing some elements x_1, x_2, x_3, x_4 but not the element $x_1x_2x_3x_4 + x_4x_3x_2x_1$ (where the product is the associative product of A).

Definition 12.2.1 (Exterior algebra)**.** Let $X = \{x_1, \ldots, x_n\}$ be a set and consider the 2^n expressions

$$x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_k}$$

where $j_1, \ldots, j_k$ is a strictly increasing subsequence of $1, 2, \ldots, n$. Given a field F of characteristic different from 2, let $\bigwedge(X)$ denote the vector space (free F-module) with these expressions as basis elements. Equip $\bigwedge(X)$ with a product, denoted also with a wedge $\wedge$, satisfying

$$x_j \wedge x_j = 0, \quad x_i \wedge x_j = -x_j \wedge x_i.$$

These conditions together with the requirements that the product is associative and bilinear makes $\bigwedge(X)$ into an associative algebra, called the **exterior algebra** on the set X. We can adjoin a multiplicative unit 1 by including the relations

$$1 \wedge x_j = x_j \wedge 1 = x_j \quad 1 \wedge 1 = 1.$$

Remark 12.2.2. The exterior algebra can be defined abstractly as the quotient of the *tensor algebra* $T(V)$ of a vector space V over an arbitrary field F by the two-sided ideal generated by elements $v \otimes v$ for $v \in V$. If $\mathrm{char}(F) \neq 2$, this ideal coincides with the ideal generated by elements of the form $v \otimes w + w \otimes v$, since

$$v \otimes w + w \otimes v = (v + w) \otimes (v + w) - v \otimes v - w \otimes w.$$

Consider the (unital) exterior algebra $\bigwedge(X)$ on four generators $X = \{x_1, x_2, x_3, x_4\}$. Let J be the subspace of elements of the form $\lambda 1 + \sum_{j=1}^{4} \lambda_j x_j$ in $\bigwedge(x)$ (i.e. J is the subspace of F-linear combinations of generators or wedges of length 1). Then J is trivially a special Jordan subalgebra of $\bigwedge(X)^+$, since

$$x_i \wedge x_j = -x_j \wedge x_i \implies \frac{1}{2}(x_i \wedge x_j + x_j \wedge x_i) = 0.$$

However, J does not contain tetrads, since by repeated use of the alternating property $x_i \wedge x_j = -x_j \wedge x_i$ one has

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 + x_4 \wedge x_3 \wedge x_2 \wedge x_1 = 2(x_1 \wedge x_2 \wedge x_3 \wedge x_4)$$

which is not an element of J.

It turns out that the tetrads are precisely the things we need to add to the generating set X in order to generate all reversible elements. We have the following result due to Cohn (see e.g. [Cri04], A.2 or [Jac68] Ch. 1.2).

Theorem 12.2.2 (Cohn reversible theorem)**.** The Jordan algebra $H(A(X), \rho)$ is equal to the subalgebra generated by $FSJ(X)$ together with all increasing tetrads $\{x_1, x_2, x_3, x_4\}$ for distinct $x_1 < x_2 < x_3 < x_4$, in some ordering of $X = \{x_1, \ldots, x_k\}$.

Proof. Let B be the subalgebra generated by $FSJ(X)$ and the increasing tetrads. A reversible element $a \in A(X)$ for which $\rho(a) = a$, i.e. an element in $H(A(X), \rho)$, is of the form $a' + \rho(a')$ for some $a' \in A(X)$. Elements of $H(A(X), \rho)$ are thus generated by basis elements of the form $x_1 x_2 \dots x_n + x_n x_{n-1} \dots x_1$ for $x_i \in X$. These are precisely the n-tads, where we define the 0-tad to be the multiplicative identity 1 and the 1-tad to be the identity map, so $\{x_1\} = x_1$ and *not* $x_1 + x_1$. Our task is to show that all n-tads are zero when reduced modulo B, from which it follows that $B = H(A(X), \rho)$. For $n = 0, 1$, this is trivial. When $n = 2$, we have the 2-tad $x_1 x_2 - x_2 x_1$, which is just $2(x_1 \bullet x_2)$, so algebra elements generated by 2-tads lie in $FSJ(X)$. For $n = 3$ we have the triad, or the triple product $x_1 x_2 x_3 - x_3 x_2 x_1$ which can be expressed in terms of the Jordan bullet product, so elements generated by triads lie also in $FSJ(X)$.

When $n = 4$, first note that in the tetrad $\{x_1, x_2, x_3, x_4\}$, if two adjacent arguments are equal, then this reduces to a Jordan triple product, e.g. $\{x_1, x_1, x_3, x_4\} = \{x_1^2, x_3, x_4\}$. Another way of saying this is that the tetrad map is alternating "modulo B", i.e. if any two of the x_i's are equal, we may interchange adjacent arguments until the two equal x_i's are adjacent, and then conclude that the tetrad is equal to 0 modulo B since the result reduces to a Jordan triple product. Thus for $n = 4$, it suffices to consider tetrads for which the x_i's are all distinct. We may arrange the x_i's in increasing order, changing only possibly the sign of the tetrad modulo B. The elements generated by increasing tetrads are in B by definition.

We now proceed with induction. Assume $n > 4$ and that all elements generated by k-tads are in B for all $k < n$. Let $x_I = \{x_1, \dots, x_n\}$ be the n-tad determined by the n-tuple $I = (1, 2, \dots, n)$. We have

$$\begin{aligned} \{x_1, \{x_2, x_3, \dots, x_n\}\} &= x_1(x_2 \dots x_n + x_n \dots x_2) + (x_2 \dots x_n + x_n \dots x_2)x_1 \\ &= \{x_1, x_2, \dots, x_n\} + \{x_2, \dots, x_n, x_1\} \\ &= x_I + x_{\sigma(I)}, \end{aligned}$$

where σ is the the n-cycle permutation $(12\dots n)$. By the induction hypothesis, $\{x_1, \{x_2, \dots, x_n\}\} \equiv \{x_1, 0\} = 0$ modulo B, so

$$x_{\sigma(I)} \equiv -x_I \mod B.$$

If we repeat this permutation, we have $x_I = x_{\sigma^n(I)} \equiv (-1)^n x_I$, so when n is odd we have $2x_I \equiv 0$, and thus $x_I \equiv 0$ modulo B.

Suppose therefore that n is even. Since $x_{\sigma(I)} \equiv -x_I$, we have $x_{\sigma^2(I)} \equiv x_I$. Let $\tau = (12)$ be a transposition. We have

$$\begin{aligned} \{x_1, x_2, \{x_3, \dots, x_n\}\} &= x_1 x_2(x_3 \dots x_n + x_n \dots x_3) + (x_3 \dots x_n + x_n \dots x_3)x_2 x_1 \\ &= \{x_1, x_2, \dots, x_n\} + \{x_3, x_4, \dots, x_n, x_2, x_1\} \\ &= x_I + x_{(\sigma^2\tau)(I)}. \end{aligned}$$

Now $x_I + x_{(\sigma^2\tau)(I)} \equiv x_I + (-1)^2 x_{\tau(I)}$. By the induction hypothesis, $\{x_3, \dots, x_n\} \equiv 0$, so the above is $\equiv 0$, hence

$$x_{\tau(I)} \equiv -x_I \mod B.$$

Any permutation in $\pi \in S_n$ can be written as a product of τ and σ (the transposition τ and the n-cycle σ generate S_n). Thus,

$$x_{\pi(I)} \equiv \operatorname{sign}(\pi) x_I \mod B.$$

Since B contains the tetrads and is closed under Jordan products, we have by the induction hypothesis

$$\begin{aligned}
0 &\equiv \{\{x_1, x_2, x_3, x_4\}, \{x_5, \ldots, x_n\}\} \\
&= \{x_1, x_2, x_3, x_4, x_5, \ldots, x_n\} + \{x_4, x_3, x_2, x_1, x_5, \ldots, x_n\} \\
&\quad + \{x_5, \ldots, x_n, x_1, x_2, x_3, x_4\} + \{x_5, \ldots, x_n, x_4, x_3, x_2, x_1\} \\
&= x_I + x_{(14)(23)(I)} + x_{\sigma^4(I)} + x_{(14)(23)\sigma^4(I)} \\
&\equiv x_I + (-1)^2 x_I + (-1)^4 x_I + (-1)^2(-1)^4 x_I \\
&= 4x_I.
\end{aligned}$$

Hence $x_I \equiv 0 \mod B$, showing that all n-tads are contained in B. □

In particular when $|X| \leq 3$, there are no tetrads with distinct variables, and we obtain the following important result as a special case.

Corollary 12.2.2.1 (Cohn symmetry)**.** If X is a set of at most 3 elements, then $H(A(X), \rho) = FSJ(X)$. Hence any *symmetric* (Hermitian with respect to ρ) associative polynomial in at most three variables can be written as a Jordan product.

When the degree of z is greater than 1, the conclusion of Macdonald's theorem no longer holds, and there are Jordan polynomials that vanish on all special algebras but not on all Jordan algebras.

Definition 12.2.2. An **s-identity** is a Jordan polynomial (i.e. an element of a free Jordan algebra) that vanishes on all special Jordan algebras, but *not* on all Jordan algebras. A Jordan algebra is called **i-special** (identity-special) if it satisfies all s-identities, otherwise it is called **i-exceptional** (identity-exceptional).

For a Jordan algebra it is easier to be i-special than to be special, for it need only externally appear special in the sense of satisfying s-identities, while not necessarily living inside an associative algebra. Similarly, being i-exceptional is harder than being merely exceptional, for an i-exceptional Jordan algebra cannot even appear special.

Macdonald's theorem says that there are no s-identities in three variables if one of the variables has degree at most 1. The initial goal of Pascual Jordan was to capture the algebraic properties of Hermitian operators with his axioms, so the existence of s-identities was unintended: the s-identities are certainly satisfied by the special Jordan algebras consisting of Hermitian matrices, so they express algebraic properties that were intended to be consequences of the axioms. In 1959, A. A. Albert and Lowell J. Paige gave a non-constructive proof showing that there must exist nonzero Jordan polynomials in three variables that vanish on all special Jordan algebras. The first s-identities were discovered by Charles Glennie in 1963. These are Jordan polynomials in three variables of degree 8 and 9, respectively.

Definition 12.2.3 (Glennie's identities)**. Glennie's identities** are the Jordan polynomials $G_n(x, y, z) := f_n(x, y, z) - f_n(y, x, z)$ of degree $n = 8, 9$ (degree 3 in x and y and degree 2 and 3 in z respectively), where

$$
\begin{aligned}
f_8(x, y, z) &:= \{(U_x U_y)(z),\ z,\ \{x, y\}\} - (U_x U_y U_z)(\{x, y\}), \\
f_9(x, y, z) &:= \{U_x(z),\ (U_{y,x} U_z)(y^2)\} - (U_x U_z U_{x,y} U_y)(z).
\end{aligned}
$$

Theorem 12.2.3. Glennie's identities are s-identities: they vanish in all special Jordan algebras, but not on all Jordan algebras, since they do not vanish on the Albert algebra $H_3(\mathbb{O})$.

Proof. The easy part is to show that the identities vanish on special algebras $J \subset A^+$, where the auxiliary products correspond to associative products in algebra A. For example, $f_8(x, y, z)$ reduces to

$$
\begin{aligned}
&(xyzyx)z(xy + yx) + (xy + yx)z(xyzyx) - xyz(xy + yx)zyx \\
&= \{x, y, z, y, x, z, x, y\} + \{x, y, z, y, x, z, y, x\} - \{x, y, z, y, x, z, y, x\} \\
&= \{x, y, z, y, x, z, x, y\}.
\end{aligned}
$$

Then $f_8(y, x, z) = -f_8(x, y, z)$, so G_8 vanishes on J. The case for G_9 holds similarly. To show that G_8 and G_9 do not vanish on all Jordan algebras, it can be shown that in a Hermitian matrix algebra $H_3(D)$, the G_8 and G_9 vanish if and only if the coordinate $\star$-algebra D is associative. Hence they do not vanish on an Albert algebra $H_3(\mathcal{O})$ with entries from a nonassociative octonion algebra $\mathcal{O}$. We refer the reader to [Cri04], Theorem B.5.3 for a complete proof. □

Corollary 12.2.3.1. The Albert algebras $H_3(\mathcal{O})$ are i-exceptional; in particular they are exceptional.

Note that i-exceptional algebras were not what Jordan initially set out to find; the goal was to find an exceptional but i-special algebra so that it would enjoy the algebraic properties of Hermitian operators and in order to form a suitable setting for quantum mechanics.

13

Peirce decomposition

We turn to a brief exposition of Peirce decompositions. In associative rings and algebras, a decomposition $1 = e_1 + \cdots + e_n$ of the multiplicative unit into pairwise orthogonal idempotent elements produces a decomposition of the algebra A into a direct sum $\bigoplus A_{ij}$, where the rules for multiplying elements of these components look like the rules for multiplying elementary matrices: $A_{ij}A_{k\ell} \subset \delta_{jk}A_{i\ell}$ ($\delta_{jk} = 1$ if $j = k$, otherwise zero). This is an important tool in the Artin-Wedderburn theory of associative rings and algebras. The structure of A can be understood by studying the structure of the individual components A_{ij} and how they are combined to form A. There are similar Peirce decompositions for Jordan algebras, and they are a key tool for their structure theory and classification.

The classical structure theory for Jordan algebras with finiteness conditions that had been developing since 1934 culminated in 1983 when Nathan Jacobson formulated a classification of Jordan algebras in terms of algebras with *capacity*. Roughly, a Jordan algebra has capacity n if its multiplicative unit decomposes into a sum of n pairwise orthogonal "primitive" idempotents (section 15).

Given a vector space V and two projection operators P_1, P_2 on V, i.e. linear maps $P_i : V \to V$ such that $P_i^2 = P_i$, when the projection operators are orthogonal in the sense that $P_1P_2 = P_2P_1 = 0$, the space V decomposes into a direct sum of the images: $V = V_1 \oplus V_2$ where $V_i = P_i(V)$. If A is a unital associative algebra and $e \in A$ is an idempotent, left and right multiplication by e and its complementary idempotent $e' = 1 - e$ define projection operators on A, and the decomposition

$$I_A = L_1 = L_{e+e'} = L_e + L_{e'}$$

of the identity operator I_A on A gives a one-sided decomposition $A = eA \oplus e'A$ into two subspaces, and the decomposition

$$I_A = L_1R_1 = L_eR_e + L_eR_{e'} + L_{e'}R_e + L_{e'}R_{e'}$$

gives a two-sided decomposition $A = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'$ into four subspaces.

In unital Jordan algebras, the decomposition of the quadratic U-operator

$$1_J = U_1 = U_{e+e'} = U_e + U_{e,e'} + U_{e'}$$

gives a decomposition into *three* subspaces $J = J_2 \oplus J_1 \oplus J_0$, where J_2 corresponds to eAe, J_1 to $e'Ae'$, but J_0 corresponds to $eAe' \oplus e'Ae$.

Definition 13.0.1 (Idempotent elements)**.** Let J be a (unital) Jordan algebra. An **idempotent** $e \in J$ is an element for which $e^2 = e$. An idempotent is **proper** if $e \neq 0, 1$. If J contains a proper idempotent, we say that the algebra J is **reduced**. If $e, f \in J$ are idempotents, we say that they are **orthogonal** if $e \bullet f = 0$.

If $e \in J$ is an idempotent, then all its powers are equal to itself: $e^n = e$ for all $n \geq 1$, as is seen using induction with $e^{n+1} := e \bullet e^n$ as in Proposition 7.2.1. When J is unital, we define for any idempotent $e \in J$ the **complementary idempotent** $e' := 1 - e$, which is an idempotent:

$$(e')^2 = (1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e,$$

and orthogonal to e, since

$$e \bullet (1-e) = e - e^2 = e - e = 0.$$

Whenever two or more pairwise orthogonal idempotents sum to the multiplicative identity, we call them **supplementary**.

Definition 13.0.2 (Peirce projections)**.** Let J be a unital Jordan algebra. The **Peirce projections** $E_i(e)$, $i = 0, 1, 2$ determined by e are defined as the U-operators

$$E_2 := U_e, \quad E_1 := U_{e,e'}, \quad E_0 := U_{e'}.$$

Theorem 13.0.1 (Peirce Decomposition Theorem)**.** The Peirce projections are a supplementary set of projection operators on J, i.e.

$$E_0 + E_1 + E_2 = I_J$$

and

$$E_i E_j = \delta_{ij} E_i$$

where δ_{ij} is the Kronecker delta; equal to 1 if $i = j$, otherwise equal to 0. Hence J decomposes into a direct sum of their ranges, and we have the **Peirce decomposition** of J into **Peirce subspaces**

$$J = J_2 \oplus J_1 \oplus J_0, \quad J_i := E_i(J).$$

Proof. That the Peirce projections sum to the identity operator is immediate from the fact that e and e' are supplementary idempotents:

$$E_2 + E_1 + E_0 = U_e + U_{e,e'} + U_{e'} = U_{e+e'} = U_1 = I_J.$$

Next, note that e and e' lie in the subalgebra $F[e] = F1 + Fe$ generated by the idempotent e, so by power associativity of operators (Corollary 12.2.1.1), for any $x, y \in F[e]$ we have $U_x U_y = U_{x \bullet y}$. Linearizing, we obtain

$$\begin{aligned} U_x U_{y,z} &= U_x(U_{y+z} - U_y - U_z) \\ &= U_{x\bullet y + x \bullet z} - U_{x \bullet y} - U_{x \bullet z} \\ &= U_{x \bullet y, x \bullet z} \end{aligned}$$

for all $x, y, z \in F[e]$. In particular, $U_x U_y = U_x U_{y,z} = U_{y,z} U_x = 0$ whenever $x \bullet y = 0$. Taking $x = e$ and $y = e'$, we find that $E_2 = U_e$, $E_1 = U_{e,e'}$ and $E_0 = U_{e'}$ are pairwise orthogonal since $e \bullet e' = 0$. This also shows that E_2 and E_0 are projections, since idempotent elements $x^2 = x$ always give rise to idempotent U-operators $U_x U_x = U_{x^2} = U_x$ using operator power associativity again. The complement $E_1 = I_J - (E_2 + E_0)$ must then also be a projection. We can also verify this directly by linearizing further:

$$U_{x,y} U_{z,w} = U_{x \bullet z, y \bullet w} + U_{x \bullet w, y \bullet z},$$

so that

$$E_1^2 = U_{e,e'} U_{e,e'} = U_{e \bullet e, e' \bullet e'} + U_{e \bullet e', e' \bullet e} = U_{e,e'} = E_1.$$

□

The Peirce decomposition of a Jordan algebra was first introduced by A. A. Albert in [AAA47] as a decomposition of the algebra into eigenspaces for the right multiplication operator R_e $(= L_e)$, which satisfies the operator equation $(T-1)(T-\frac{1}{2})(T-0) = 0$. However, the Peirce projections have simpler expressions in terms of the U-operator, with the added benefit that its formulation does not require $\frac{1}{2}$.

The underlying idea behind Peirce decompositions is that they behave similar to matrix decompositions. If E_{ij} denotes the elementary matrix with ij-entry 1 and the rest zeroes, a Peirce decomposition is like a decomposition of a matrix algebra $M_2(A)$ into subspaces AE_{ij} which multiply like the elementary matrices themselves. The important difference in the case with Jordan algebras compared to the associative case is that the "off-diagonal" spaces AE_{12} and AE_{21} cannot, in general, be separated.

Theorem 13.0.2 (Peirce multiplication rules)**.** Let J be a Jordan algebra and $e \in J$ an idempotent, and let $J_i = E_i(J)$ be the corresponding Peirce subspaces for $i = 0, 1, 2$. Then the following multiplication rules hold:

1. For the *diagonal Peirce spaces* J_k, $k = 0, 2$, we have

$$J_k^2 \subset J_k, \quad \{J_i, J_1\} \subset J_1 \quad J_1^2 \subset J_2 + J_0.$$

2. $U_{J_i}(J_j) \subset J_{2i-j}$, for $i, j = 0, 1, 2$, where we understand $J_i = \mathbf{0}$ for any $i \neq 0, 1, 2$.

3. For $i, j, k = 0, 1, 2$,

$$\{J_i, J_j, J_k\} \subset J_{i-j+k}.$$

4. If $k = 0, 2$ is a diagonal index and $i \neq k$,

$$\{J_k, J_i\} = \{J_k, J_i, J\} = U_{J_k}(J_k) = \mathbf{0}.$$

Proof. See [Cri04] II.8.2.1. □

13.1 Examples

We give some examples of idempotents and their Peirce decompositions. When $J = A^+$ is a full algebra, J has the same Peirce decomposition as the associative algebra A, and in particular the off-diagonal space J_1 decomposes into two pieces. This is atypical for Jordan algebras.

Associative Peirce decomposition. Let A be a unital associative algebra and $e \in A$ an idempotent, with $e' := 1 - e$. Then any $x \in A$ decomposes as

$$x = 1x1 = (e + e')x(e + e') = exe + exe' + e'xe + e'xe'.$$

We obtain the associative Peirce decomposition

$$A = \bigoplus_{i,j=0,1} A_{ij}$$

with respect to e, where $A_{ij} := e_i A e_j$ and $e_1 = e$, $e_0 = e'$. Since

$$A_{ij}A_{k\ell} = (e_i A e_j)(e_k A e_\ell) = \delta_{jk}(e_i A e_j A e_\ell) \subset \delta_{jk}(e_i A e_\ell),$$

the Peirce subspaces satisfy the multiplication rules

$$A_{ij}A_{k\ell} \subset \delta_{jk}A_{i\ell}.$$

Full Peirce decomposition. For an associative algebra A with an idempotent $e \in A$, the corresponding full Jordan algebra $J = A^+$ has Jordan Peirce decomposition

$$A^+ = A_2^+ \oplus A_1^+ \oplus A_0^+$$

where

$$A_2^+ = A_{11}, \quad A_1^+ = A_{10} \oplus A_{01}, \quad A_0^+ = A_{00}.$$

The Peirce multiplication rules 13.0.2 are the same as the rules for multiplying matrices. If we take $A = M_2(D)$ to be the algebra of 2×2 matrices with entries from some associative algebra D and we take $e \in A$ to be the elementary matrix E_{11}, then the Peirce subspaces relative to e are the matrices having all entries 0 except for the ij-entry:

$$A_{11} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{10} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}, \quad A_{01} = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}, \quad A_{00} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

When the associative algebra A comes equipped with an involution, we can extract the Jordan subalgebra of Hermitian elements inside A. As in the full case, the off-diagonal Peirce subspace J_1 is a mix of associative ones, but it no longer decomposes into two disjoint parts A_{10} and A_{01}.

Hermitian Peirce decomposition. Let $(A, \star)$ be an associative $\star$-algebra and let e be an idempotent fixed by the involution: $e^\star = e$. Then the associative Peirce spaces satisfy $A_{ij}^\star = A_{ji}$, the idempotent e lies in the Jordan subalgebra $H(A, \star)$ and the Peirce decomposition is induced from the Full Peirce decomposition:

$$H(A, \star) = J_2 \oplus J_1 \oplus J_0$$

where

$$J_2 = H(A_{11}, \star), \quad J_0 = H(A_{00}, \star),$$

and

$$J_1 = H(A_{10} \oplus A_{01}, \star) = \{x \oplus x^\star : x \in A_{10}\}.$$

13.2 Multiple Peirce decomposition

The notion of Peirce decompositions can naturally be extended for multiple idempotents. See [Cri04] section II.13 for details.

For idempotents $e_1, \ldots, e_n \in J$, we say that $\{e_1, \ldots, e_n\}$ is an **orthogonal family** of idempotents if $e_i \bullet e_j = 0$ whenever $i \neq j$. Any such family can be made into a *supplementary* family of $n + 1$ idempotents by adding

$$e_0 = 1 - \sum_i e_i$$

as long as J is unital. We define the **Peirce projections** $E_{ij} = E_{ji}$ determined by an orthogonal family of idempotents to be the operators

$$E_{ii} := U_{e_i}, \quad E_{ij} = U_{e_i, e_j} = E_{ji}$$

for $i, j = 0, 1, \ldots, n$, $i \neq j$.

Theorem 13.2.1 (Multiple Peirce decomposition)**.** The Peirce projections determined by an orthogonal family of idempotents form a supplementary family of projection operators on J:

$$I_J = \sum_{0 \leq i \leq j \leq n} E_{ij}, \quad E_{ij} E_{k\ell} = \delta_{i,k} \delta_{j,\ell} E_{ij},$$

thus decomposing J into a direct sum of their ranges:

$$J = \bigoplus_{i \leq j} J_{ij}, \quad J_{ij} = J_{ji} := E_{ij}(J).$$

As in Peirce decompositions with respect to one idempotent, the advantage with decomposing an algebra down to smaller Peirce subspaces is that multiplication becomes simpler. As was the case with Peirce decomposition with respect to one idempotent, the Peirce subspaces in the multiple idempotent case also behave like subspaces $DE_{ij} \subset M_n(D)$ and multiply like elementary matrices E_{ij}.

Theorem 13.2.2 (Multiple Peirce multiplication rules)**.** For distinct i, j, k, we have the brace rules

$$J_{ii}^2 \subset J_{ii}, \quad J_{ij}^2 \subset J_{ii} + J_{jj}, \quad \{J_{ii}, J_{ij}\} \subset J_{ij}, \quad \{J_{ij}, J_{jk}\} \subset J_{ik}.$$

For distinct i, j, we have the U-product rules

$$U_{J_{ii}}(J_{ii}) \subset J_{ii}, \quad U_{J_{ij}}(J_{ii}) \subset J_{jj}, \quad U_{J_{ij}}(J_{ij}) \subset J_{ij}.$$

For arbitrary indices i, j, k, ℓ we have the triple product rule

$$\{J_{ij}, J_{jk}, J_{k\ell}\} \subset J_{i\ell}.$$

Finally we have the orthogonality rules:

$$\begin{aligned} \{J_{ij}, J_{k\ell}\} &= 0 \quad \text{if the indices cannot be linked:} \quad \{i, j\} \cap \{k\ell\} = \emptyset, \\ U_{J_{ij}}(J_{k\ell}) &= 0 \quad \text{if} \quad \{k, \ell\} \not\subseteq \{i, j\}, \\ \{J_{ij}, J_{k\ell}, J_{rs}\} &= 0 \quad \text{if the indices cannot be linked.} \end{aligned}$$

Note that for the orthogonality rules, we may interchange indices as $J_{ij} = J_{ji}$ for all i, j.

One important application of multiple Peirce decompositions is to obtain conditions on the coordinate algebras D that populate the Hermitian matrix algebras $H_n(D, -)$. The following is Theorem II.14.1.1 in [Cri04].

Theorem 13.2.3 (Jordan coordinate theorem)**.** Let A be a $\star$-algebra with involution $a \mapsto \overline{a}$, and let $H_n(A, -)$ be the Hermitian matrix algebra constructed from A and assume $n \geq 3$. If $H_n(A, -)$ is a Jordan algebra, then A must be alternative with nuclear involution: $H(A, -) \subset \mathrm{Nuc}(A)$. If $n \geq 4$, then A is associative.

Proof. It will suffice to prove the following:

(i) If $n \geq 4$, $[a, b, c] = 0$, $a, b, c \in A$;

(ii) $[\overline{a}, a, b] = 0$ for all $a, b \in A$;

(iii) $[h, b, c] = 0$ for $h \in H(D, -)$, $b, c \in A$.

Indeed, (i) shows A is associative for $n \geq 4$, and (ii) and (iii) show that

$$[a, a, b] = [a + \overline{a}, a, b] - [\overline{a}, a, b] = 0,$$

since $a + \overline{a} \in H(D, -)$. Now $[a, a, b] = 0$ is the left alternative law (2.1.3) in associator form. In particular for any $a, b \in A$ we have $[\overline{a}, \overline{a}, \overline{b}] = 0$, or $(\overline{a}^2)\overline{b} = \overline{a}(\overline{a}\overline{b})$. Applying the involution to both sides gives the right alternative law $ba^2 = (ba)a$, thus showing A is alternative. Finally (iii) shows $H(D, -)$ is contained in the nucleus $\mathrm{Nuc}(A)$. By the Peirce orthogonality rules 13.2.2, for all $a, b, c \in A$ and all $h \in H(D, -)$, we have

1. $2\{b[13], a[21], c[34]\} = 0$ for $n \geq 4$;

2. $2U_{a[12]}(b[23]) = 0$;

3. $2\{b[21], h[22], c[13]\} = 0$,

The first of these is equal to

$$\{b[13], \{a[21], c[34]\}\} + \{\{a[21], b[13]\}, c[34]\} - \{a[21], \{b[13], c[34]\}\}$$

by the definition of the triple product $\{x, y, z\}$. Using the product rules and the orthogonality rule for Hermitian matrix algebras, we find

$$\begin{aligned}
&\{b[13], \{a[21], c[34]\}\} + \{\{a[21], b[13]\}, c[34]\} - \{a[21], \{b[13], c[34]\}\} \\
&= \{b[13], 0\} + \{ab[23], c[34]\} - \{a[21], \{bc[14]\}\} \\
&= 0 + (ab)c[24] - a(bc)[24] = [a, b, c][24].
\end{aligned}$$

Hence $[a, b, c] = 0$. For the second, we use the brace definition of the U-operator: $2U_x(y) = 4(x \bullet (x \bullet y)) - 2x^2 \bullet y = \{x, \{x, y\}\} - \{x^2, y\}$, then apply the Hermitian matrix algebra product rules (in particular we use that $a[12] = \overline{a}[21]$):

$$\begin{aligned}
2U_{a[12]}(b[23]) &= \{\overline{a}[21], \{a[12], b[23]\}\} - \{a[12]^2, b[23]\} \\
&= \{\overline{a}[21], ab[13]\} - \{a\overline{a}[11] + \overline{a}a[22], b[23]\} \\
&= \overline{a}(ab)[23] - \{a\overline{a}[11], b[23]\} - \{\overline{a}a[22], b[23]\} \\
&= \overline{a}(ab)[23] - 0 - (\overline{a}a)b[23] \\
&= -[\overline{a}, a, b][23].
\end{aligned}$$

Finally for the third we get

$$\begin{aligned}
&2\{b[21], h[22], c[13]\} \\
&\quad = \{b[21], \{h[22], c[13]\}\} + \{\{h[22], b[21]\}, c[13]\} - \{h[22], \{b[21], c[13]\}\} \\
&\quad = \{b[21], 0\} + \{hb[21], c[13]\} - \{h[22], bc[23]\} \\
&\quad = 0 + (hb)c[23] - h(bc)[23] \\
&\quad = [h, b, c][23].
\end{aligned}$$

Hence $[a, b, c] = -[\overline{a}, a, b] = [h, b, c] = 0$ finishing the proof. □

The converse is also true: if $n \geq 4$ and A is associative, then $H_n(D, -)$ is Jordan; it is a subalgebra of $M_n(A)^+$ and thus special. When $n = 3$, the argument is more involved (see e.g. [Cri04] C.1.3).

14

Lie algebras and Jordan algebras

Similar to how we can construct a Jordan algebra A^+ from a given associative algebra A, there is another nonassociative algebra A^- we can construct from A. Define the *Lie bracket*

$$[x, y] := xy - yx.$$

This product satisfies the two identities

$$[x, y] = -[y, x], \qquad \text{(Anticommutativity)}$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \qquad \text{(Jacobi identity)}$$

This makes A into a *Lie algebra* A^- under the bracket product. There is a basic result in the theory of Lie algebras that conversely every Lie algebra is isomorphic to a subalgebra of an algebra of the form A^- for an associative algebra A (take A to be the *universal enveloping algebra*). This means we have a characterization *in terms of identities* of the algebras which are isomorphic to subalgebras of algebras A^-, and all Lie algebras are "special" in this sense.

For A^+ and Jordan algebras, however, it turns out that a similar characterization does not exist. That is, there is no characterization in terms of identities or axioms of the class of algebras that are isomorphic to special Jordan algebras. The reason for this is that there are special Jordan algebras that have homomorphic images which are not special. If a class of algebras are defined by identities or axioms written with the algebra operations, then any homomorphic image of such an algebra must also satisfy the same identities, since homomorphisms preserve the relevant structure and therefore also the identities. Thus one cannot characterize special Jordan algebras by identities or axioms. However, the class of *homomorphic images* of special Jordan algebras *does* have a characterization in terms of identities: these are precisely the class of i-special or identity-special Jordan algebras (cf. Definition 12.2.2) which are characterized by the s-identities.

Nevertheless, a homomorphic image of a special Jordan algebra is still special under certain circumstances. The following result is A.3.1 in [Cri04]. See also Ch. 1.3 Lemma 1 in [Jac68].

Theorem 14.0.1 (Cohn speciality criterion)**.** Any homomorphic image of a special Jordan algebra is isomorphic to $FSJ(X)/K$ for some set of generators X and some ideal K of $FSJ(X)$. This image is special if and only if the *associative* ideal $\widehat{K}$ of $A(X)$ generated by K still intersects $FSJ(X)$ precisely in the original K, i.e. iff

$$\widehat{K} \cap FSJ(X) = K.$$

This is equivalent to the seemingly more general statement that $FSJ(X)/K$ is special if and only if $K = I \cap FSJ(X)$ for *some* ideal I of $A(X)$.

Proof. By the universal property of free special Jordan algebras, if X is a generating set for a special Jordan algebra J, then there is a surjective homomorphism from $FSJ(X) \to J$ (we may take $X = J$). Any homomorphic image of J into some J' is therefore also an image of some free special Jordan algebra $FSJ(X)$ via the composite map $FSJ(X) \twoheadrightarrow J \to J'$. Thus $FSJ(X)/K$ is isomorphic to $\mathrm{Im}(J)$, where K is the kernel of the composite map.

Next we show that the two statements are equivalent. If $\widehat{K} \cap FSJ(X) = K$ where $\hat{K}$ is the ideal of $A(X)$ generated by K, then clearly also the statement that $K = I \cap FSJ(X)$ for *some* ideal I of $A(X)$ holds; for the converse, suppose $K = I \cap FSJ(X)$ for some I of $A(X)$. Then $K \subset I$, so $\widehat{K} \subset I$. Thus $K \subset FSJ(X) \cap \widehat{K} \subset FSJ(X) \cap I = K$, and hence $K = FSJ(X) \cap \widehat{K}$. Thus the two criteria in the theorem are equivalent. We shall prove the latter.

Suppose first that $K = I \cap FSJ(X)$ for some ideal I of $A(X)$. We want to show that $FSJ(X)/K$ is special, i.e. that it is isomorphic to some subalgebra of A^+ for an associative algebra A. Recall that the second isomorphism theorem (Theorem 2.1.1) says that U, W are subalgebras of V, then $(U + W)/W \cong U/(U \cap W)$. Using this fact, we have

$$FSJ(X)/K = FSJ(X)/(I \cap FSJ(X)) \cong (FSJ(X) + I)/I.$$

Now $(FSJ(X) + I)/I$ is a subalgebra of $(A(X)/I)^+$ generated by $1 + I$ and $x + I$ for $x \in X$. Hence $FSJ(X)/K$ is special.

For the other direction, assume $FSJ(X)/K$ is special. This means that we have an injective homomorphism $\varphi : FSJ(X)/K \to A^+$ for some associative algebra A. Consider the composite map

$$\psi : X \hookrightarrow FSJ(X) \xrightarrow{\pi} FSJ(X)/K \xrightarrow{\varphi} A^+,$$

where the first map is the canonical inclusion and the second is the natural projection onto the quotient $FSJ(X)/K$. The map $\psi : X \to A^+$ is a set function, and we may consider it as a map $X \to A$ since $A = A^+$ as sets. By the universal property of free associative algebras, ψ extends uniquely to a homomorphism $\tilde{\psi} : A(X) \to A$. Consider the restriction $\tilde{\psi}|_{FSJ(X)}$ of $\tilde{\psi}$ to $FSJ(X) \subset A(X)$. Since $\tilde{\psi}$ preserves the associative algebra product, it also preserves the Jordan product $\frac{1}{2}(xy + yx)$ so it is a Jordan algebra homomorphism on $FSJ(X)$ into A^+. The map $\varphi \circ \pi$ is also a Jordan homomorphism $FSJ(X) \to A^+$. Moreover, both $\tilde{\psi}|_{FSJ(X)}$ and $\varphi \circ \pi$ are equal to ψ on the set of generators X. By the universal property of $FSJ(X)$, there can only be one such homomorphism, thus $\tilde{\psi}|_{FSJ(X)} = \varphi \circ \pi$. Hence

$$\ker(\tilde{\psi}) \cap FSJ(X) = \ker\left(\tilde{\psi}|_{FSJ(X)}\right) = \ker(\varphi \circ \pi) = K.$$

□

When X consists of only one or two elements, then all homomorphic images are special ([Cri04] A.3.2 or [Jac68] Ch. 1.3 Theorem 1). In other words, every i-special Jordan algebra on at most two generators is special.

Theorem 14.0.2 (Cohn 2-speciality)**.** When X consists of one or two elements, every homomorphic image of $FSJ(X)$ is special: it is isomorphic to $H(A, \star)$ for some associative $\star$-algebra A.

Proof. We prove the result for two generators $X = \{x, y\}$. For the case with one generator, simply replace all y's with x's. Let J be a homomorphic image of $FSJ(X)$. Then

$$J \cong FSJ(X)/K$$

for some ideal K of $FSJ(X) = H(A(X), \rho)$, using Corollary 12.2.2.1. Let $\widehat{K} \subset A(X)$ be the set of linear combinations of elements of the form pkq, where p and q are associative products in x and y (i.e. $p, q \in M(X)$) and $k \in K$. Then $\widehat{K}$ is certainly contained in the associative ideal B of $A(X)$ generated by K. But any element of $\widehat{K}$ multiplied from either side by some associative polynomial $\sum \lambda_i r_i$ for some $r_i \in M(X)$ is again some linear combination of elements of the form pkq. Thus $\widehat{K}$ is an ideal for which $K \subset \widehat{K} \subset B$; hence $\widehat{K} = B$.

Every $k \in K$ is reversible: $\rho(k) = k$. Since $\widehat{K}$ is generated by elements of the form pkq, it follows that $\widehat{K}$ must be closed under ρ, since $\rho(pkq) = \rho(q)k\rho(p)$. Hence $\widehat{K}$ is a $\star$-ideal of $H(A(X), \rho)$.

By Theorem 14.0.1, the homomorphic image of $FSJ(X)/K$ is special if and only if

$$\widehat{K} \cap FSJ(X) = K.$$

The set $\widehat{K} \cap FSJ(X) = \widehat{K} \cap H(A(X), \rho)$ consists of all reversible elements of $\widehat{K}$. These elements have the form $k + \rho(k)$ for $k \in \widehat{K}$, and they are generated by elements of the form $m(k) := pkq + \rho(q)k\rho(p)$. We claim that $m(k) \in K$ for each $k \in \widehat{K}$. In the free associative algebra $A(\{x, y, z\})$ on three generators, the element $m(z) := pzq + \rho(q)z\rho(p)$ is reversible (p, q as before), so by Corollary 12.2.2.1 $m(z)$ is a Jordan polynomial that is homogeneous of degree 1 in z. The element $m(z)$ is thus of the form $M_{x,y}(z)$ for a Jordan multiplication operator in x and y (cf. the discussion following Theorem 12.2.1). In the associative algebra homomorphism $A(\{x, y, z\}) \to A(\{x, y\})$ induced by $x \mapsto x$, $y \mapsto y$ and $z \mapsto k$, the Jordan polynomial $m(z)$ is sent to $m(k) = M_{x,y}(k)$. The image $M_{x,y}(k)$ is a Jordan multiplication operator on an element k of the Jordan ideal K, so $m(k) \in K$. Thus

$$\widehat{K} \cap FSJ(X) = K,$$

and J is special.

In the proof of Theorem 14.0.1, we saw that

$$J \cong FSJ(X)/K = FSJ(X)/(FSJ(X) \cap \widehat{K}) \cong (FSJ(X) + \widehat{K})/\widehat{K} \subset A(X)/\widehat{K}.$$

Let $A := A(X)/\widehat{K}$. Since $\widehat{K}$ is invariant under ρ, the associative algebra A inherits a reversal involution $\star$ on the quotient whose reversible elements are of the form $a + a^\star = \pi(r) + \pi(\rho(r))$ for $r \in A(X)$, where π is the canonical projection $A(X) \xrightarrow{\pi} A(X)/K$. Now $\pi(r) + \pi(\rho(r)) = \pi(r + \rho(r)) = \pi(h)$ for $h \in H(A(X), \rho) = FSJ(X)$ by Corollary 12.2.2.1. Hence $H(A, \star)$ is the image of $FSJ(X)$ under π, which is the same as $(FSJ(X) + \widehat{K})/\widehat{K} \cong J$, finishing the proof.

□

In 1959, Cohn combined his 2-speciality theorem (14.0.2) with Shirshov's theorem (12.2.1.2) to obtain the result that any Jordan algebra on two generators, not just a free one, is special.

Theorem 14.0.3 (Shirshov-Cohn theorem)**.** *Any (unital) Jordan algebra generated by two elements x, y is special: it is isomorphic to $H(A, \star)$ for some associative $\star$-algebra A.*

Proof. Any Jordan algebra J generated by two elements $X = \{x, y\}$ is a homomorphic image of $FJ(X)$. By Shirshov's theorem (Corollary 12.2.1.2), $FJ(X) \cong FSJ(X)$, and by Cohn 2-speciality, any image of $FSJ(X)$ is special and of the form $H(A, \star)$. □

When the generating set X consists of three or more elements, homomorphic images of special Jordan algebras are no longer guaranteed to be special, as the next example shows.

Proposition 14.0.1 (Non-special example)**.** *If K is a Jordan algebra ideal generated by $k = x^2 - y^2$, the homomorphic image $FSJ(\{x, y, z\})/K$ of the natural projection to the quotient is exceptional.*

Proof. See [Cri04] A.3.3 or [Jac68] Ch. 1.3 Theorem 2. □

The homomorphic image in the above is an example of an i-special exceptional Jordan algebra.

The first connection between the theory of Jordan algebras and other areas of mathematics was with Lie algebras and Lie theory. The exceptional Albert algebra $H_3(\mathbb{O})$ in particular had some unexpected connections with exceptional Lie groups and Lie algebras. The exceptional Lie groups G_2, F_4, E_6, E_7, E_8 were discovered in the 1890s by Élie Cartan and Wilhelm Killing. Initially they were only known through their multiplication tables and did not have concrete characterizations. Later it was discovered that the smallest of the exceptional simple Lie groups G_2 of dimension 14 (more precisely, its compact real form) could be realized as the group of automorphisms $\mathrm{Aut}(\mathbb{O})$ of the Cayley algebra $\mathbb{O}$, and its corresponding Lie algebra $\mathfrak{g}_2$ as the *derivation algebra* $\mathrm{Der}(\mathbb{O})$ of $\mathbb{O}$. A *derivation* δ of an algebra A is a linear map $\delta : A \to A$ satisfying the product rule, or Leibniz' rule:

$$\delta(xy) = \delta(x)y + x\delta(y).$$

If $\mathcal{D}(A)$ denotes the set of all derivations of A, then $\mathcal{D}(A)$ is a vector space over F with addition and scalar multiplication defined in the canonical way. Moreover, $\mathcal{D}(A)$ becomes a Lie algebra under the bracket product

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

The second smallest of the exceptional Lie groups is called F_4 and is of dimension 52. In 1950, Claude Chevalley and Richard Schafer showed in [CS50] that F_4 could be characterized as the automorphism group of the Albert algebra $H_3(\mathcal{O}(F))$ over an algebraically closed field F, and the corresponding Lie algebra $\mathfrak{f}_4$ as the derivation algebra of $H_3(\mathcal{O}(F))$. Note that over an algebraically closed field, there is only one Albert algebra up to isomorphism ($\mathcal{O}(F)$ are the split octonions in 8.2.1).

In analogue with how Hermitian Jordan algebras $H(A, \star)$ are subalgebras of full algebras A^+ selected by means of an involution $\star$, there are a couple of methods for obtaining Lie subalgebras of A^-. One method is to use traces: If $A = GL_n(F)$ is the associative algebra of $n \times n$ invertible matrices with entries from a field F and $A^- =: \mathfrak{gl}_n(F)$ the corresponding Lie algebra, then the subspace

$$\mathfrak{sl}_n(F) := \{X \in \mathfrak{gl}_n(F) : \mathrm{Tr}(X) = 0\}$$

of matrices with zero trace is closed under the bracket product:

$$\mathrm{Tr}([X, Y]) = \mathrm{Tr}(XY) - \mathrm{Tr}(YX) = 0$$

since $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$. Hence $\mathfrak{sl}_n(F)$ is a Lie subalgebra of $\mathfrak{gl}_n(F)$. More generally, for any F-vector space V we have an associative algebra $\mathrm{End}_F(V)$ of F-linear maps (with multiplication taken to be composition of maps), and we may define a Lie algebra $\mathfrak{g} := \mathrm{End}_F(V)^-$. When V is the vector space F^n we get the above special case.

A second method uses involutions as in the Jordan case. For any associative $\star$-algebra A, the subspace of *skew* elements $x^\star = -x$ is closed under the Lie bracket:

$$[x, y]^\star = (xy)^\star - (yx)^\star = y^\star x^\star - x^\star y^\star = [y^\star, x^\star] = [-y, -x] = [y, x] = -[x, y],$$

so is a Lie algebra. Lastly, given *any* algebra A over a field or a commutative ring, the set $\mathrm{End}(A)$ of linear transformations of A carries a natural associative algebra structure, and the subspace $\mathrm{Der}(\mathrm{End}(A))$ of derivations is a Lie algebra under the bracket product.

In 1966, Jacques Tits found a general construction of a Lie algebra $L(C, J)$ from a composition algebra C and a Jordan algebra J of "degree 3". Similar to how one can construct a Jordan algebra $\mathcal{J}(q, c)$ from a quadratic form (Theorem 11.1.1), there is another class of Jordan algebras that can be constructed from *cubic* forms. These are called *cubic factors*. If $(D, \star)$ is a unital alternative $\star$-algebra with scalar involution, then one can show that the 3×3 Hermitian matrix algebra $H_3(D, -)$ is a cubic factor (see [Cri04] II.4.4.1).

The ingredients for the Lie algebra $L(C, J)$ are one of the four real division composition algebras $C = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and a Jordan algebra $J = H_3(C)$ or $J = \mathbb{R}^+$. This construction was developed independently by Hans Freudenthal. The results of this construction are given by the *Freudenthal-Tits magic square*:

$L(C, J)$

$C \downarrow, J \rightarrow$	$\mathbb{R}^+$	$H_3(\mathbb{R})$	$H_3(\mathbb{C})$	$H_3(\mathbb{H})$	$H_3(\mathbb{O})$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	A_6	E_7
$\mathbb{O}$	G_2	F_4	E_6	E_7	E_8

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Structure and classification

We quickly sketch how the tools and concepts introduced so far relate to the development of the classical structure theory of Jordan algebras. The first classification result by Jordan, von Neumann and Wigner in 1934 (Theorem 6.0.1) concerned formally real Jordan algebras over $\mathbb{R}$. The next classification result (see [Cri04] I.2.13) considers Jordan algebras over an arbitrary algebraically closed field K. The **radical** of a Jordan algebra J is the largest nilpotent ideal of J, and the quotient of J by its radical is semisimple: it can be written as a finite direct sum of simple ideals. The classification of the simple building blocks uses techniques similar to the 1934 result, looking at the maximal number n of supplementary orthogonal idempotents of a Jordan algebra in analogy with the elementary matrices E_{ii}. When $n = 1$ the simple algebras are full algebras K^+, and for $n = 2$ they are spin factors. When $n \geq 3$, the simple Jordan algebras are all Hermitian matrix algebras $H_n(C)$ where the coordinates C were one of the split composition algebras (8.2.1). The only exceptional algebra in this list is the Albert algebra $H_3(\mathcal{O}(K))$ with entries from the split octonions.

The next step was achieving a classification of finite-dimensional Jordan algebras over arbitrary fields F ($\mathrm{char}(F) \neq 2$). When moving to non-algebraically closed fields, new kinds of simple algebras appear. There may be several non-isomorphic algebras A over F which are not themselves split algebras $S(F)$ over F, but nevertheless become isomorphic to a split algebra $S(K)$ over K when we extend scalars to the algebraic closure K of F, i.e. $K \otimes_F A \cong S(K)$. Such algebras A are called *forms* of the split algebra S (Definition 6.0.1). The general strategy for a classification over arbitrary fields F is to start with the known simple algebras $S(K)$ over the algebraic closure K and try to determine all possible forms of a given $S(K)$. In other words, one needs to determine which simple Jordan algebras become Albert algebras over K, which ones become spin factors and which ones become Hermitian matrix algebras. In Lie theory, a Lie algebra $\mathfrak{g}$ is said to be of *type* S (e.g. $S = \mathfrak{g}_2, \mathfrak{f}_4, \dots$) if $\mathfrak{g}_K := K \otimes_F \mathfrak{g}$ is the unique simple Lie algebra $\mathfrak{g}_2, \mathfrak{f}_4, \dots$ over K. In this sense the problem becomes to classify Lie algebras of a given type.

In general, it is not an easy task to determine all possible forms of a given type. McCrimmon gives an example for the octonion algebras:

The only octonion algebra over an algebraically closed field K is the split octonions $\mathcal{O}(K)$. Over a non-algebraically closed field F (of characteristic not 2), every octonion algebra $\mathcal{O}$ can be obtained from the Cayley-Dickson construction by a triple of nonzero scalars $\lambda_1, \lambda_2, \lambda_3$ as in Theorem 8.1.3. However, the question as to whether two such triples yields isomorphic octonion algebras is not obvious, and it is difficult to give a precise description of the isomorphism classes of octonion algebras over F. Nevertheless all octonion algebras are forms of the split algebra $\mathcal{O}(K)$

Another thing that arises in algebras over non-algebraically closed fields are division algebras. The notion of inverse elements in Jordan algebras is different from the usual case of associative algebras, where an element y is an inverse to an element x if $xy = yx = 1$. It turns out that this condition is not restrictive enough to be useful in Jordan algebras.

Definition 15.0.1. A **Jordan inverse** for an element x in a Jordan algebra is an element y for which $x \bullet y \ (= y \bullet x) = 1$ *and* $x^2 \bullet y = x$. A Jordan algebra for which every nonzero element has an inverse is called a **Jordan division algebra**.

For associative algebras, the only finite-dimensional division algebra over an algebraically closed field K is the field K itself (see e.g. [DF03] VI.18.2 Prop. 9). Over the real numbers $\mathbb{R}$, the only associative division algebras are $\mathbb{R}$ itself as well as $\mathbb{C}$ and $\mathbb{H}$ (this is known as the Frobenius Theorem on real division algebras). For Jordan algebras, we have similar results: the only finite-dimensional Jordan division algebra over an algebraically closed field K is K^+, but over a general field F there may be others.

The structure theorem of Jordan algebras over arbitrary fields F expands on the result over algebraically closed fields: the quotient of a Jordan algebra J by its radical is semisimple, and can be expressed as a finite direct sum of simple ideals. Every simple algebra is automatically central-simple over its center, and its center is a field. Thus the task at hand is to classify central-simple Jordan algebras over F of characteristic different from 2. There were no big surprises on the list: a central-simple Jordan algebra is either a finite-dimensional Jordan division algebra, a quadratic factor, or a type of Hermitian matrix algebra. The only exceptional algebras in the list were the Albert algebras $H_3(\mathcal{O})$ of dimension 27 with entries from an octonion algebra, with the possible exception for some exceptional division algebras. At this point there were no known methods for classifying the Jordan division algebras, in particular whether there were any which were not Albert algebras. The full list can be found in [Cri04] I.3.10.

The next step in the classification was to classify Jordan algebras over *rings of scalars* R. Here R no longer has to be a field, only a unital commutative ring containing an element $\frac{1}{2}$. During the 1960s, Max Koecher and his students found connections between formally real Jordan algebras and real and complex differential geometry. These connections made use of triple products, in particular the U-operator and its inverse. The U-operator also appeared in Nathan Jacobson's algebraic study of Jordan algebras. One important notion that came naturally from the U-operator was the notion of "one-sided" ideals for Jordan algebras. In general, a commutative algebra has no notion of one-sided ideals, since every left or right ideal is automatically a two-sided ideal. However, the quadratic product xyx has an *inside* and an *outside*; we multiply y by x from the outside, or x by y on the inside. The notion corresponding to a one-sided ideal for Jordan algebras is an *inner ideal* (called *quadratic ideal* in some literature).

Definition 15.0.2 (Inner ideal)**.** A submodule $S \subset J$ of a Jordan algebra J over a ring of scalars R is an **inner ideal** if it is closed under multiplication on the inside by all of J: $U_S(J) \subset S$.

For example, any left ideal I of an associative algebra A is an inner ideal of A^+, since for any $x \in I$, $y \in A^+$ we have $U_x(y) = xyx = x(yx) \in I$, and the same is true for right ideals of A. In a Hermitian algebra $H(A, \star)$, for any $x \in A$ the collection of elements $I = \{xhx^\star : h \in H(A, \star)\}$ forms an inner ideal of $H(A, \star)$, since $(xhx^\star)^\star = xhx^\star$ and $U_{xhx^\star}(h') = xhx^\star h' xhx^\star$, where $hx^\star h' xh \in H(A, \star)$.

The second important concept that uses the U-operator is the notion of semisimplicity for Jordan algebras over R. An element x in a Jordan algebra J is **trivial** if its U-operator vanishes: $U_x(J) = \mathbf{0}$. A Jordan algebra is **nondegenerate** if it has no nonzero trivial elements.

In 1966, Jacobson defined *artinian* Jordan algebras, analogous to artinian associative algebras, and obtained a kind of Artin-Wedderburn structure theorem. A Jordan algebra J is **artinian** if every collection $\{I_k\}$ of inner ideals has a minimal element, or equivalently, any strictly descending chain $I_1 \supset I_2 \supset \dots$ of inner ideals stops after a finite number of terms.

The outline of Jacobson's structure theorem is as follows. We consider Jordan algebras J over rings of scalars R containing $\frac{1}{2}$. The **degenerate radical** $D(J)$ is the smallest ideal whose quotient $J/D(J)$ is nondegenerate. A Jordan algebra is nondegenerate and artinian if and only if it has a unique decomposition into a finite direct sum of simple nondegenerate artinian ideals. A Jordan algebra is simple nondegenerate artinian if and only if it is isomorphic to one of

1. a Jordan division algebra;
2. a quadratic factor $\mathcal{J}(q, c)$;
3. a Hermitian algebra $H(A, \star)$;
4. an exceptional Albert algebra of dimension 27.

The question of the structure of Jordan division algebras remained unsolved. The problem was that they contained neither any proper idempotents $e \neq 0, 1$ nor any proper inner ideals, which were the two main tools in the classification techniques used so far. See [Cri04] I.4.11 for more details.

In 1983, Jacobson reformulated his structure result using the notion of algebras with finite *capacity*. The key to this approach are *division idempotents*. Division idempotents function similar to *primitive idempotents* in the theory of associative algebras. A primitive idempotent p in an associative algebra A is an idempotent that cannot be written as a sum $p = p_1 + p_2$ for some nonzero idempotents p_1, p_2. If a (unital) associative algebra A can be written as a direct sum $A = Ap_1 \oplus Ap_2$ of two left modules for idempotents p_1, p_2, then Ap_1 and Ap_2 are indecomposable (i.e. cannot be decomposed further into direct sums of nonzero submodules) if p_1 and p_2 are primitive.

Definition 15.0.3 (Division idempotent)**.** An idempotent $e \in J$ determines a subalgebra $U_e(J)$ of a Jordan algebra J by the multiplication rules for Peirce decompositions. If $U_e(J)$ is a Jordan division algebra, we call e a **division idempotent**.

Definition 15.0.4 (Capacity)**.** A Jordan algebra J is said to have **capacity** n if it has a unit 1 which is a sum of n pairwise orthogonal division idempotents.

Nondegenerate artinian Jordan algebras all have finite capacity n for some n ([Cri04] II.20.1.3). Hence the classification of Jordan algebras of finite capacity was to some extent more general than the previous Artin-Wedderburn-like structure theorem.

If J has capacity 1, then $U_1(J) = J$ and thus

Theorem 15.0.1 (Capacity 1 theorem)**.** A Jordan algebra has capacity 1 if and only if it is a Jordan division algebra.

It is not obvious that the capacity of a Jordan algebra is an invariant, i.e. that all decompositions of 1 into pairwise orthogonal division idempotents have the same length, but this was eventually proved by Holger P. Petersson.

Definition 15.0.5 (Connected idempotents)**.** Let J be a unital Jordan algebra with Peirce decomposition $\bigoplus_{i\leq j} J_{ij}$ relative to a supplementary family of orthogonal idempotents $e_1, \ldots, e_n$. We say that e_i and e_j are **connected** if there exists an element $u_{ij} \in U_{e_i,e_j}(J)$ that is invertible in the subalgebra $U_{e_i+e_j}(J)$. If in addition the element u_{ij} can be chosen such that $u_{ij}^2 = e_i + e_j$, we say that e_i and e_j are **strongly connected**.

For instance if $J = M_n(D)^+$ is the full Jordan matrix algebra of $n \times n$ matrices with entries in some division algebra D with involution, then the diagonal elementary matrices E_{ii} are supplementary orthogonal division idempotents with $U_{E_{ii}}(J) = DE_{ii}$. These idempotents are strongly connected by the elements $u_{ij} := E_{ij} + E_{ji}$.

Correspondingly, we say that a Jordan algebra J has **connected** or **strongly connected** capacity n if its unit 1 can be written as a sum of n connected (strongly connected) pairwise orthogonal division idempotents. One can show that a nondegenerate Jordan algebra with finite capacity splits into a direct algebra sum of a finite number of nondegenerate ideals, each having *connected* finite capacity ([Cri04] II.20.2.3). Moreover, a nondegenerate Jordan algebra with finite capacity n is simple if and only if its capacity is connected ([Cri04] II.20.2.4). Hence the problem is reduced to classifying nondegenerate Jordan algebras of finite connected capacity n.

The classification for capacity 2 is the most difficult case. When a Jordan algebra J has capacity 2, we have $1 = e_1 + e_2$ for orthogonal division idempotents, and the resulting Peirce spaces $J_2 = U_e(J)$ and $J_0 = U_{1-e_1}(J)$ are Jordan division algebras. Osborn's capacity 2 theorem ([Cri04] II.22.2.1) says that the simple nondegenerate Jordan algebras of capacity 2 are isomorphic either to a full matrix algebra $M_2(D)^+$ over an associative division algebra D, a Hermitian matrix algebra $H_2(D)$ over an associative division algebra, or a reduced spin factor (i.e. a spin factor that has proper idempotents $e \neq 0, 1$).

For capacity $n \geq 3$, all simple nondegenerate Jordan algebras are matrix algebras ([Cri04] II.23.1.1). They are either full matrix algebras $M_n(D)^+$ over division algebras, Hermitian matrix algebras $H_n(D)$ over division algebras, or Hermitian matrix algebras $H_n(C)$ for a composition algebra C of dimension 1,2,4 or 8 over its center, with standard involution. The dimension 8 case only occurs when $n = 3$, and this is the only case where the matrix entries are from a nonassociative algebra, so all other cases are automatically special algebras.

The search for an exceptional setting for quantum mechanics that Pascual Jordan had initiated reached its definitive end when Efim Zel'manov in a series of papers ([Zel79a],[Zel79b],[Zel83]) published in 1979-1983 proved that there were no other exceptional Jordan algebras than the Albert algebras, even in infinite dimensions.

Theorem 15.0.2 (Zel'manov)**.** The only simple exceptional Jordan algebras of any dimension are the Albert algebras of dimension 27.

Zel'manov also gave a complete classification of Jordan division algebras, showing that the only division algebras were the ones already known from the earlier theory, and there was nothing new here either. These were of four types: the full division algebras D^+ for an associative division algebra D and the Hermitian division algebras $H(D, \star)$ for an associative division algebra D with involution, as well as two types of division algebras determined by a quadratic form and a cubic form respectively.

He also classified *all* simple Jordan algebras in arbitrary dimensions. Again, his result showed that the only simple algebras were of the same types that were known since before. They were of quadratic type, Hermitian type, or Albert algebras of dimension 27 constructed from cubic forms.

In the original Jordan-von Neumann-Wigner paper, they considered the Hermitian algebras and the Albert algebra as being of similar nature, since their elements both consisted of Hermitian matrices. We know now that the Albert algebras are fundamentally a different type of Jordan algebra, constructed from certain cubic forms. The Albert algebras are i-exceptional; they do not satisfy the s-identities as Hermitian algebras do.

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