SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Eulerian number

av

Subash K C

2023 - K13

Eulerian number

Subash K C

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Per Alexandersson

2023

Abstract

This thesis is focused on exploring the mathematical concepts of Eulerian numbers, Eulerian polynomials, and Grassmannian permutations. The main objective of this research is to provide an introductory understanding of Eulerian polynomials and Grassmannian permutations, with a particular emphasis on proving the polynomial recursion. The thesis also aims to prove an important property of Eulerian polynomials, namely that they are real-rooted.

Contents

1 Introduction

The basic concept of discrete mathematics has been studied by mathematicians from many different cultures and time periods. In ancient India, the concept of permutations was studied as part of Combinatorics. Indian mathematicians developed a system for counting the number of permutations of n objects, which is known as the "factorial" function[1]. For example, the number of permutations of 4 objects is $4 \cdot 3 \cdot 2 \cdot 1 = 24$. The concept of permutation dates back to ancient Greek mathematics, where it was used in the study of symmetry and geometric transformations. In the 17th century, Blaise Pascal, Pierre de Fermat, and Abraham de Moivre continued this line of research in discrete mathematics. Pascal is well known for his contributions to the field through his work on Pascal's triangle. De Moivre's contributions to combinatorics were more focused on probability, where he developed the theory of probabilities and introduced the concept of normal distribution. In the 18th century, the Swiss mathematician Leonhard Euler made significant contributions to the field of combinatorics through his work on graph theory, partitions, and Latin squares[14].

In the 20th century, permutations found further applications in areas such as cryptography, coding theory, and computer science. The study of permutations continues to play a central role in modern mathematics and science, and new applications of permutations are still being discovered. Permutations can be used to solve many real-world problems, such as counting the number of possible ways to arrange a group of people in a line or assigning tasks to a team of workers. In its simplest form, a permutation is a way to arrange a set of n distinct objects in a specific order. For example, if we have a set of three distinct objects A, B, and C, then we can arrange them in six different ways: ABC, ACB, BAC, BCA, CAB, and CBA.

Eulerian numbers are integers that count the number of permutations of n elements that have k descents, where a descent is defined as a position in the permutation where the value is greater than the next value. Eulerian polynomials are polynomials that generalize Eulerian numbers. Eulerian numbers and polynomials are essential concepts in combinatorics, with a history dating back to the 18th century. These mathematical objects are named after the renowned mathematician Leonhard Euler, who made significant contributions to their development[5].

2 Permutations

A permutation is a bijective function that maps a set of elements to itself.

Definition 2.1. *A permutation of a set A is a bijective function $\phi : A \to A$ that maps each element of the set A to a unique element of A, in such a way that each element of A is included in the mapping exactly once.*

For example, Let $A = \{1, 2, 3, 4, 5\}$ be a set, then $\phi(1) = 2, \phi(2) = 3, \phi(3) = 1, \phi(4) = 5$ and $\phi(5) = 4$ is one of its permutation. We can write this in standard notation as

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}.$$

Another way to write it is as $\phi = (1\ 2\ 3)(4\ 5)$. This notation is called cycle notation because it emphasizes the cycles of the permutation. Each cycle represents a group of elements that are swapped by the permutation.

In general, the set of all permutations of a set of size n is known as the symmetric group on n elements, denoted by S_n[3].

Lemma 2.1 (Permutation Formula)**.** *The number of permutations of a set of size n is $n!$. The number of elements in S_n is $n!$.*

Proof of Lemma 2.1. Base case: When $n = 1$. In this case, there is only one permutation of the set, which is the set itself.
Now assume that the formula holds for $n = k$, where k is a positive integer.
We need to show that the formula holds for $k + 1$ distinct elements. To prove the formula for $n = k + 1$, consider a set S of size $k + 1$ distinct elements. We can choose any one of these elements to be the first element in the sequence, giving us $k + 1$ choices. Then we are left with a set of k elements to arrange, which can be done in $k!$ ways by our inductive hypothesis. Therefore, the total number of permutations of S_n is $(k + 1) \cdot k! = (k + 1)!$.
By mathematical induction, the formula $n!$ for the total number of permutations of a set of size n is true for all positive integers n. $\square$

3 Generating functions

A generating function is a formal power series of the form

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

where the coefficients $a_0, a_1, a_2, a_3, ...$ are typical elements of some algebraic structure, such as the integers or polynomials. The coefficients of the power series correspond to the terms in the sequence, and the variable x is a formal variable that is used to keep track of the position of each term in the sequence. Generating functions are used to study the properties of sequences and to solve combinatorial and algebraic problems. Generating functions are used to study many different types of sequences, including sequences of integers, polynomials, partitions, permutations, and more, and they have applications in combinatorics, number theory, probability, statistics, physics, and engineering[17].

The generating function can be used to derive various identities and closed formulas such as the Fibonacci numbers closed-form expression. The generating function for Fibonacci numbers is

$$F(x) = x/(1 - x - x^2)$$

where n:th number F_n is the coefficient of the expansion of $F(x)$.

3.1 Ordinary generating function

Definition 3.1. *The ordinary generating function $A(x)$ of the sequence (a_n) is the power series defined by*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the summation ranges over all non-negative integers n. Note that we don't care about convergence.

The coefficients of the power series $A(x)$ are the terms of the sequence (a_n), i.e., a_n is the coefficient of x^n in the power series expansion of $A(x)$.

A very simple example of the sequence $a_n = 1, 2, 3, 4, \ldots$, which is the sequence of positive integers starting from 1, which has generating function, $1 + 2x + 3x^2 + 4x^3 + \ldots$, and now we can see that

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}(1 + x^2 + x^3 + x^4 + \cdots)$$

which the well known geometric series $\frac{1}{1-x} = (1 + x^2 + x^3 + x^4 + \cdots)$. Hence

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}\frac{1}{(1-x)} = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(1+n)x^n.$$

Now let us consider the Fibonacci numbers and find out an exact formula for the function $F(x) = \sum_{n\geq 0} F_n x^n$, which generates the Fibonacci numbers. The recurrence relation of Fibonacci numbers is

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \text{ with } F_1 = 0, F_2 = 1.$$

We have the function $F(x) = \sum_{n\geq 0} F_n x^n$ and if we separate the initial terms and use the recurrence relation we get,

$$F(x) = \sum_{n\geq 0} F_n x^n = x + \sum_{n\geq 2} F_n x^n = x + \sum_{n\geq 2}(F_{n-1} + F_{n-2})x^n.$$

Now if we rewrite the two sums in terms of generating function we get,

$$\begin{aligned}\sum_{n\geq 2}(F_{n-1} + F_{n-2})x^n &= \sum_{n\geq 2} F_{n-1}x^n + \sum_{n\geq 2} F_{n-2}x^n \\ &= x\sum_{n\geq 2} F_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2}x^{n-2} \\ &= x\sum_{n\geq 0} F_n x^n + x^2 \sum_{n\geq 0} F_n x^n \\ &= xF(x) + x^2F(x).\end{aligned}$$

We have that

$$F(x) = x + xF(x) + x^2F(x) \implies F(x) = \frac{x}{(1-x-x^2)}.$$

Once we get the closed formula for generating function, we need to express this as power series. The roots of the polynomial $(1-x-x^2)$ are $-r_1$ and $-r_2$, where $r_1 = \frac{1-\sqrt{5}}{2}$ and $r_2 = \frac{1+\sqrt{5}}{2}$. So the polynomial factors are $(1-x-x^2) = -(x+r_1)(x+r_2)$.
Now by using partial fraction decomposition, we obtain

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{r_1}{x+r_1} - \frac{r_2}{x+r_2}\right).$$

Since $r_2 = -\frac{1}{r_1}$, we can rewrite this as,

$$\begin{aligned}F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-r_2x} - \frac{1}{1-r_1x}\right) \\ &= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty} r_2^n x^n - \sum_{n=0}^{\infty} r_1^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(r_2^n - r_1^n)x^n.\end{aligned}$$

The definition of $F(x) = \sum_{n\geq 0} F_n x^n$ gives the closed formula for n-th Fibonacci numbers as,

$$F_n = \frac{1}{\sqrt{5}}\left(r_2^n - r_1^n\right).$$

3.2 Exponential generating function

An exponential generating function is a generating function used in combinatorics to count certain types of objects. While an ordinary generating function is used to count the number of objects of a certain size, an exponential generating function counts objects based on their structure or composition. For example, an ordinary generating function can be used to count the number of ways to choose k items from a set of n items. In contrast, an exponential generating function can be used to count the number of permutations of n objects based on their cycle structure, or the number of partitions of a set of n objects based on the number of parts and their sizes[13][17].

Definition 3.2. *The exponential generating function of a sequence b_n is defined as*

$$F(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

where the summation ranges over all non-negative integers n.

Exponential generating functions are useful for counting permutations and other ordered structures. For example, the exponential generating function for all permutations of n objects is given by

$$F(x) = \sum_{n} n! \frac{x^n}{n!}.$$

This can be derived using the identity $F(x) = 1 + xF(x)$ where $F(x)$ stands for the exponential generating function counting permutations. This can be seen as the first term is a trivial "empty set" and the term $xF(x)$ counts the permutation of non-empty sets multiplication principle. Solving for $F(x)$, we get

$$F(x) = \frac{1}{1-x}.$$

So,

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} x^n.$$

Comparing both sides gives $a_n = n!$ which counts the number of permutation of set $[n]$ which is in agreement with Lemma 2.1.

4 Descents and ascents

Definition 4.1. *Let π be a permutation on set $[n] = \{1, 2, ..., n\}$. We say π has a* descent *at $i \in \{1, ..., n-1\}$ if $\pi_i > \pi_{i+1}$ and we say π has an* ascent *at $i \in \{1, ..., n-1\}$ if $\pi_i < \pi_{i+1}$.*

For example the permutation the permutation $\pi = 9\ 2\ 5\ 4\ 7\ 6\ 8\ 1\ 3$ has 4 descents and 4 ascents. The descents occur at positions $i = 1$, $i = 3$, $i = 5$, and $i = 7$. The ascents occur at positions $i = 2$, $i = 4$, $i = 6$, and $i = 8$.

4.1 Eulerian number

The Eulerian number is a family of integers that are named after the Swiss mathematician Leonhard Euler. They arise in combinatorial problems that involve permutations and partitions of sets. The Eulerian number $A_{n,k}$ counts the number of permutations of the set $1, 2, ..., n$ that have exactly k descents or ascents[5][9] .

Definition 4.2. *For $n \in \mathbb{N}$, the number of permutations in the π that have exactly k descents is denoted by $A_{n,k}$, and such numbers are known as Eulerian number.*

Example 4.1. *Let $n = 3$ and $k = 1$. Then $A_{n,k} = 4$, there are four permutations of $n = 3$ with exactly $k = 1$ descent which are $132, 213, 312, 231$. As we can see that 213 and 312 have descents at position $i = 1$ and 132 and 231 at position $i = 2$.*

We see that $A_{n,0} = 1$ and $A_{n,n-1} = 1$ since the only permutation of π with no descents is the identity permutation, and the only permutation with $n-1$ descents is $(n, n-1, \cdots, 1)$. For $k \geq n$, $A_{n,k} = 0$, since the maximum amount of descents a permutation of set n can have is $(n-1)$. As usual, we defined $A_{0,0} = 1$, since $A_{0,0}$ represents the number of permutations of an empty set with no descents or ascents, and this value is considered to be 1 by convention. We can also see that $A_{n,n} = 0$ since there can be only $(n-1)$ descent in the permutation of set $[n]$.

Theorem 4.1 (Analytic formula)**.** *For all integers n and k such that, $k < n$ we have*

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

Proof of theorem 4.1. We begin by considering the case when $k = 0$. In this case, we are interested in the number of permutations of $[n]$ with no descents. The only permutation satisfying this condition is the identity permutation. Thus, $A_{n,0} = 1$. To verify the formula, we substitute $k = 0$ into the equation

$$A_{n,0} = \sum_{j=0}^{0} (-1)^j \binom{n+1}{j} (1+k-j)^n = (-1)^0 \binom{n+1}{0} 1^n.$$

Next, we consider the case when $k \geq 1$. For this case, we will construct an n, k-arrangement of $[n]$ with k black squares forming $k+1$ segments such that each segment is ascending runs. Segments are allowed to be empty also. For example, an n, k-arrangements of $n = 5$ with $k = 3$ black squares is

$$1 \blacksquare 3 \blacksquare 25 \blacksquare 4$$

We observe that for an n, k-arrangement, the number of segments is one greater than the number of black squares. We claim that the number of n, k-arrangements is equal to $(k+1)^n$. To see why, we can construct an n, k-arrangement in the following way. First, we distribute k black squares to create $k+1$ empty segments. We then distribute each number i from $[n]$ among the $k+1$ segments, ensuring that the numbers within each segment are arranged in ascending order. Each number has $k+1$ choices of segments where it can be placed. Thus, by applying the product principle, the total number of n, k-arrangements is $(k+1)^n$.

Note that some black squares are removable and, the remaining segments still maintain the ascending runs. For example,

$$\blacksquare\, 1 \,\blacksquare\, 3 \,\blacksquare\, 25 \,\blacksquare\,\blacksquare 4 \,\blacksquare$$

The arrangement has 6 black squares, but we can remove the black square at the front, in the end, between 1 and 3 and one black square between 25 and 4 while preserving the ascending runs within the remaining segments. A black square is removable if it is between two numbers that are in increasing order, if it is alone at the very beginning, if it is alone at the very end, or if there is no number in between the black squares.
We claim that the number of n, k-arrangements with no removable black squares is equal to the Eulerian numbers. Since there is a bijection between these arrangements and the permutations with k descents. Therefore, we have the following relationship

$$A_{n,k} = \text{number of } n, k\text{-arrangements with no removable square.}$$

Next, we define positions in the arrangement as the spaces between consecutive entries of a permutation, as well as the space before the first entry and after the last entry. For example, permutation 12345 has 6 positions

$$p\ 1\ p\ 2\ p\ 3\ p\ 4\ p\ 5\ p \qquad p \text{ represent the position in this permutation.}$$

For an n-permutation, we have a total of $n+1$ positions. Let $S \subseteq [n+1]$, and let A_S represent the number of arrangements in which there is a black square in each position belonging to S. Let $|S| = i$. Then we claim $A_S = (k+1-i)^n$. In order to see this first take any $n, k-$arrangement that contains $(k-i)$ black squares. There are $(k-i+1)^n$ such arrangements. Now, we will insert i extra black squares by placing one in each position that belongs to S. If a position already contains a black square, we simply put the new black square immediately to the right of it. This procedure generates an $n, k-$arrangement that falls within the set A_S. Conversely, every $n, k-$arrangement belonging to A_S can be obtained precisely once using this method. Specifically, if a is an element of A_S, we can obtain the original $n, k-$arrangement by removing one black square from each of the i positions that belong to S. This results in a unique $n, k-$arrangement with $k-i-1$ black squares, which corresponds to a. Thus, we have successfully constructed n, k-arrangements with removable black squares in the set S. It now follows that there are $(k-i+1)^n$ many such arrangements. By the Principle of Inclusion and Exclusion, we get

$$A_{n,k} = \text{number of } n, k\text{-arrangements} - \sum_{\substack{S\subseteq\{0,1,\ldots,n\},\\ 1\le|S|\le k}} (-1)^{|S|+1} A_S$$

$$\begin{aligned}
A_{n,k} &= (k+1)^n - \sum_{\substack{S\subseteq\{0,1,\ldots,n\},\\ 1\le|S|\le k}} ((-1)^{|S|+1}(k+1-|S|)^n \\
&= (k+1)^n - \sum_{\substack{S\subseteq\{0,1,\ldots,n\},\\ 1\le|S|\le k}} (-1)^{i+1}(k+1-i)^n \\
&= (k+1)^n - \sum_{i=1}^{k}(-1)^i\binom{n+1}{i}(k+1-i)^n \\
&= \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n.
\end{aligned}$$

This completes the proof. [4] $\square$

Lemma 4.1 (Recursive Formula for Eulerian Numbers)**.** *The Eulerian numbers satisfy the following recursive formula*

$$A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1} \qquad (A_{n,0} = 1 \ (n \geq 0), A_{n,k} = 0 \ (k \geq n)). \tag{1}$$

Proof of Lemma 4.1. We will use a bijective argument to prove this lemma. Note that the Eulerian numbers A_n, k are defined as the number of permutations of $[n]$ that have exactly k descents.
Now let us consider a permutation σ counted by A_n, k and a permutation σ' obtained by removing the entry n. We want to remove the element n from σ and insert it back into the permutation in such a way that we end up with a permutation of $[n-1]$ with either k or $k-1$ descents. There are two cases to consider:

- Removing n from σ does not change the number of descents k. In this case, the resulting permutation is a permutation of $[n-1]$ with k many descents, so there are $A_{n-1,k}$ many of them. To reconstruct the permutation σ we have to insert n at the end of σ', or we have to insert n between two entries that form one of the k descents of σ'. This means we have $k+1$ choices for the position of n. Therefore, the contribution from this case is $(k+1)A_{n-1,k}$.

- Removing n from σ decreases the number of descents from k to $k-1$. In this case, the resulting permutation is a permutation of $[n-1]$ with $k-1$ many descents, so there are $A_{n-1,k-1}$ many of them. To reconstruct the permutation σ we have to insert n at the beginning of σ' or at any of the $n-k-1$ places that are not descents in σ'. This means that we have $n-k$ choices to insert n in σ'. Hence, we conclude that the contribution from this possibility is $(n-k)A_{n-1,k-1}$.

Adding the two cases, we get the recurrence relation $A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1}$.
This completes the proof. [4] $\square$

Proposition 4.1. *Given a positive integer n and $0 \leq k \leq n-1$, we have $A_{n,k} = A_{n,n-k-1}$.*

Proof of Proposition. 4.1 We can simply take the reverse of any permutation with k descents to obtain a permutation with $n-k-1$ descents. Therefore, $A_{n,k} = A_{n,n-k-1}$, as claimed. $\square$

4.2 Eulerian polynomials

The Eulerian polynomials are a sequence of polynomials which coefficients are Eulerian numbers[9].

Definition 4.3. *Given a non-negative integer n, the n-th Eulerian polynomial $A_n(x)$ is defined as*

$$A_n(x) = \sum_{k=0}^{n} A_{n,k} x^k$$

where $A_{n,k}$ is an Eulerian number.

The first few terms of eulerian polynomials are,

$$\begin{aligned}
A_0(x) &= 1 \\
A_1(x) &= 1 \\
A_2(x) &= 1 + x \\
A_3(x) &= 1 + 4x + x^2 \\
A_4(x) &= 1 + 11x + 11x^2 + x^3 \\
A_5(x) &= 1 + 26x + 66x^2 + 26x^3 + x^4 \\
A_6(x) &= 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5 \\
A_7(x) &= 1 + 120x + 1191x^2 + 2416x^3 + 1191x^4 + 120x^5 + x^6.
\end{aligned}$$

Lemma 4.2. *The following identity holds for all integers n and k:*

$$A_n(x) = \sum_{k\geq 0} \sum_{j=0}^{k} \binom{n+1}{j} (-1)^j (1 + k - j)^n x^k.$$

Proof of Lemma 4.2. The Eulerian number $A_{n,k}$ satisfies the analytic formula as shown in 4.1 and Eulerian numbers are coefficients of Eulerian polynomials. So the claim follows. □

Proposition 4.2. *For $n \geq 0$, we have the identity*

$$\sum_{k\geq 0} (k+1)^n x^k = \frac{1}{(1-x)^{n+1}} A_n(x).$$

Proof of proposition. Let us consider the Lemma 4.2. We have

$$\sum_{k\geq 0} A_{n,k} x^k = \sum_{k\geq 0} \sum_{j=0}^{k} \binom{n+1}{j} (-1)^j (1 + k - j)^n x^k.$$

If we now rearrange the terms in this expression,

$$\sum_{k\geq 0} A_{n,k} x^k = \sum_{j\geq 0} \binom{n+1}{j} (-1)^j x^j \sum_{k\geq 0} (1 + k)^n x^k.$$

The binomial theorem tells us that $(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k$. Using the binomial theorem, we can write $\sum_{j\geq 0} \binom{n+1}{j} (-1)^j x^j = (1-x)^{n+1}$. Finally, by rearranging the terms and using the binomial theorem we arrive at

$$\sum_{k\geq 0} A_{n,k} x^k = (1-x)^{n+1} \sum_{k\geq 0} (1+k)^n x^k,$$

which then implies the formula. $\square$

Proposition 4.3. *The Eulerian polynomials satisfy the following,*

$$A_n(x) = x(1-x)A'_{n-1}(x) + (1+(n-1)x)A_{n-1}(x), \qquad (n \geq 1) \tag{2}$$

where $A'_{n-1}(x)$ denotes the derivative of $A_{n-1}(x)$ with respect to x and with intial conditions $A_1(x) = 1$.

Proof. We can prove this by using the identity from Proposition 4.2. We have the expression for $A_n(x)$. Using this we can find $A_{n-1}(x)$ as $(1-x)^n \sum (1+k)^{n-1} x^k$. By taking the derivative of $A_{n-1}(x)$, we can express it as

$$A'_{n-1}(x) = n(1-x)^{n-1} \sum (1+k)^{n-1} x^k + (1-x)^n \sum k(1+k)^{n-1} x^{k-1}.$$

Substituting the expression we obtained and performing simple calculations of right-hand side (RHS) yields

$$\begin{aligned} RHS &= (1-x)^n \sum_{k\geq 0} (1+k)^{n-1} x^k \Big(1 + (n-1)x - nx + (1-x)k\Big) \\ &= (1-x)^n \sum_{k\geq 0} (1+k)^{n-1} x^k (1+k)(1-x) \\ &= (1-x)^{n+1} \sum_{k\geq 0} (1+k)^n x^k \\ &= A_n(x). \end{aligned}$$

The recurrence holds as claimed.

$\square$

4.3 Grassmannian permutation

The study of Grassmannian permutations is essential to Schubert calculus, as the Schubert polynomials depend on the relative order of the elements of a given permutation. Schubert calculus is a field of algebraic geometry that involves the study of geometric objects called Schubert varieties and their intersection theory. To compute the number of intersection points of Schubert varieties, the Schubert calculus rule, a combinatorial formula involving Schubert polynomials, is used[6][10][16].

Definition 4.4. *Let π be a permutation on the set $[n] = 1, 2, 3, \ldots, n$. We say that π is a Grassmannian permutation if and only if the permutation has at most one descent.*

For example, the permutation 312 has only one descent at position $i = 2$ and is considered to be a Grassmannian permutation of the set $\{1, 2, 3\}$. On the other hand, the permutation 321 has two descents at positions $i = 1$ and $i = 2$, and is not a Grassmannian permutation.

Proposition 4.4. *The explicit formula for the total number of Grassmannian permutations on set $[n]$ is given by $2^n - n$.*

Proof. We can consider the analytic formula for Eulerian numbers 4.1,

$$A(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (1 + k - j)^n,$$

which calculates the number of permutations on set $[n]$ that have exactly k descents. Putting $k = 1$ gives,

$$\begin{aligned}
A(n,1) &= \sum_{j=0}^{1} (-1)^j \binom{n+1}{j} (1 + k - j)^n \\
&= (-1)^0 \binom{n+1}{0} (2^n) + (-1)^1 \binom{n+1}{1} (1^n) \\
&= 2^n - (n+1)1^n \\
&= 2^n - (n+1).
\end{aligned}$$

□

Hence, there are $2^n - (n + 1)$ number of permutations of S_n that have exactly one descent. The only permutation with zero descent is the identity permutation itself. Therefore the total number of Grassmannian permutations of $\pi = A(n, 1) + 1 = 2^n - n$.

4.3.1 Bijective proof of Proposition 4.4

We want to show that the number of Grassmannian permutations is given by $2^n - n$. Consider the set $[n] = \{1, 2, 3, \cdots, n\}$.
For every subset A of $[n]$, we generate Grassmannian permutations by arranging the elements of A in increasing order, followed by the elements of the complement of A (i.e., the elements not in A), also listed in increasing order. This construction guarantees that the resulting permutation has at most one descent.

Now, let us analyze the number of permutations obtained through this construction, which is equivalent to the total number of subsets of $[n]$. Since we are listing the elements of A in ascending order, followed by the elements of the complement of A, there are 2^n possible subsets of $[n]$. However, we need to consider that this construction yields $n+1$ instances of identical permutations. Thus, we subtract n from the 2^n. Consequently, the number of Grassmannian permutations of $[n]$ is $2^n - n$.[6]

5 Real-rooted

In this section, we will study the property of roots of Eulerian polynomials. The real roots of a polynomial are important because they give information about the behavior of the polynomial. We will show that the Eulerian polynomials have only real roots[5][18].

5.1 Interlacing

Definition 5.1. *Let f and g be polynomials with real coefficients and with real roots f_i and g_i respectively. We say that f interlaces g if $deg(f) + 1 = deg(g) = n$ and the roots of f and g are real and arranged in increasing order: $g_1 \leq f_1 \leq \cdots \leq f_{n-1} \leq g_n$.*

Moreover, we say that f alternates left of g if $deg(f) = deg(g) = n$ and $f_1 \leq g_1 \leq \cdots \leq f_n \leq g_n$.We say that f interleaves g if either f interlaces g or f alternates left of g. We write this as $f \ll g$.

5.2 Eulerian polynomials are real-rooted

Proposition 5.1. *The Eulerian polynomial $A_n(x)$ has only real roots.*

Proof. 5.1 Lets consider the Eulerian polynomials

$$A_n(x) = \sum_{k=0}^{n} A_{n,k} x^k.$$

For the first few terms of the Eulerian polynomials, it is easy to check that the roots are indeed real. We will prove this proposition by induction. Base case: For $n = 0, 1$, the Eulerian polynomial $A_0(x)$ and $A_1(x)$ is simply 1, which has no roots. Thus, the proposition holds for the base case.

Inductive Step: Assume that the proposition holds for all $n \leq k-1$, where $k \geq 0$. We will prove that it holds for $n = k$. Now consider the Eulerian polynomials recurrence 4.3 for $n \geq 2$. We have

$$\begin{aligned} A_n(x) &= x(1-x)A'_{n-1}(x) + (1+(n-1)x)A_{n-1}(x) \\ &= xA'_{n-1}(x) + A_n(x) - x^2A'_{n-1}(x) - xA_{n-1}(x) + nxA_{n-1}(x) \end{aligned}$$

Let us introduce $\tilde{A}_n(x) = A_n(x)x$ so that $\tilde{A}'_n(x) = A_n(x) + xA'_n(x)$ which gives the new recurrence

$$\tilde{A}_n(x) = x(1-x)\tilde{A}'_{n-1}(x) + nx\tilde{A}_{n-1}(x).$$

We conclude that $nx\tilde{A}_{n-1}(x) \ll x(1-x)\tilde{A}'_{n-1}(x)$ even if the leading coefficient is negative. Note that $\tilde{A}'_{n-1}(x) \ll \tilde{A}_{n-1}(x)$. However the sum of the $x(1-x)\tilde{A}'_{n-1}(x) + nx\tilde{A}_{n-1}(x)$ has positive leading coefficient due to $n \geq 2$. Now if we draw a graphic illustration of how this looks,

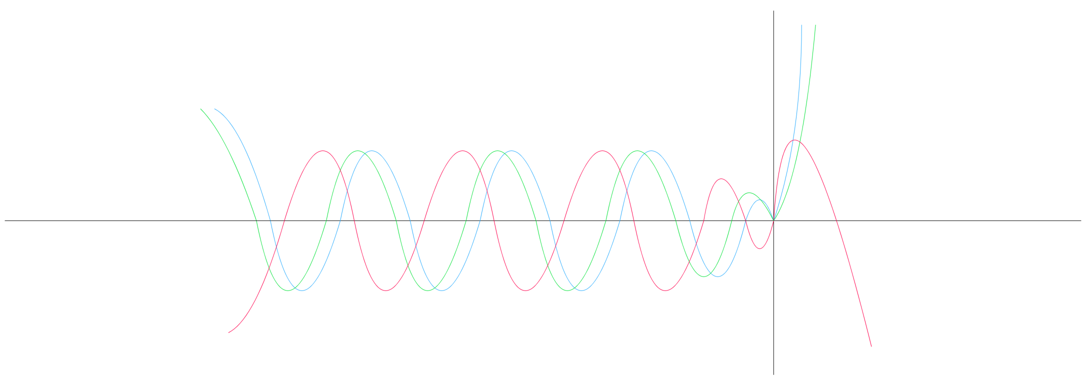

Figure 1: Graphic illustration of recurrence relation

In Figure 1 $\tilde{A}_n(x) =$ blue color, $x(1-x)\tilde{A}'_{n-1}(x) =$ red color and $x\tilde{A}_{n-1}(x) =$ green color. Let $b_1, b_2, \ldots, b_n$ be the roots of $\tilde{A}_n(x)$ in in decreasing order and let $c_1, c_2, \ldots, c_m$ be the roots of $\tilde{A}_{n-1}(x)$ in decreasing order. We claim that

$$b_m \leq c_n \leq b_{m-1} \leq c_{n-1} \leq b_{m-2} \leq \cdots \leq c_1 \leq b_1.$$

In order to justify why $\tilde{A}_{n-1}(x)$ interlaces $\tilde{A}_n(x)$, we can consider the term $\tilde{A}_{n-1}(x)$ in the recurrence relation which has only real roots by assumption. Then the term $\tilde{A}'_{n-1}(x)$ also has real roots since $\tilde{A}'_{n-1}(x) \ll \tilde{A}_{n-1}(x)$. Thus the polynomial $\tilde{A}_{n-1}(x)$ and $\tilde{A}'_{n-1}(x)$ changes signs at the roots just like in Figure 1.

Now, consider an interval (a, b) between two roots. We want to show that the $\tilde{A}_n(x)$ has a real root at the interval. In the interval (a, b), both $\tilde{A}_{n-1}(x)$ and $\tilde{A}'_{n-1}(x)$ maintain their signs (either positive or negative) due to the ordering of the real roots. Therefore, the polynomial $\tilde{A}_n(x)$ in the interval (a, b) maintains the same sign as the sum of these two terms, which is positive if they have the same sign and negative if they have opposite signs. By the Intermediate Value Theorem, since $\tilde{A}_{n-1}$ is negative and $\tilde{A}_n(x)$ is positive (or vice versa), there exists at least one root of $\tilde{A}_n(x)$ in the interval (a, b). Since this is true for any real root x of $\tilde{A}_n(x)$. This accounts for $(n-1)$ roots of $\tilde{A}_n(x)$. Since we have accounted for all but one root, the remaining last root must be real since complex roots of polynomials with real coefficients come in conjugate pair we conclude that $\tilde{A}_n(x)$ has only real roots.
This completes the proof. [4]

$\square$

References

[1] Discrete Mathematics: Past, Present, and Future — ias.edu. https://www.ias.edu/math/csdm/dm.

[2] Per Alexandersson. The symmetric functions catalog. https://www.symmetricfunctions.com.

[3] Norman L. Biggs. *Discrete Mathematics*. Oxford University Press, 2002.

[4] Miklós Bóna. *Combinatorics of Permutations*. Chapman and Hall/CRC, April 2022.

[5] Dominique Foata. Eulerian polynomials: From euler's time to the present. In *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, pages 253–273. Springer New York, 2010.

[6] Jair A. Gil and Jeffrey A. Tomasko. Restricted grassmannian permutations. https://arxiv.org/pdf/2112.03338.pdf, 2021.

[7] Ralph P. Grimaldi. *Discrete and Combinatorial Mathematics*. Pearson Education, 5 edition, 2004.

[8] K. Menon and A. Singh. Grassmannian permutations avoiding identity. https://arxiv.org/pdf/2212.13794.pdf, 2022.

[9] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. Cambridge University Press, 1997.

[10] Wikipedia contributors. Schubert calculus — Wikipedia, the free encyclopedia, 2022.

[11] Wikipedia contributors. Discrete mathematics — Wikipedia, the free encyclopedia, 2023.

[12] Wikipedia contributors. Eulerian number — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Eulerian_number&oldid=1152441140, 2023. [Online; accessed 2-May-2023].

[13] Wikipedia contributors. Generating function — Wikipedia, the free encyclopedia, 2023.

[14] Wikipedia contributors. Leonhard euler — Wikipedia, the free encyclopedia, 2023.

[15] Wikipedia contributors. Permutation — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Permutation&oldid=1150837092, 2023. [Online; accessed 2-May-2023].

[16] Wikipedia contributors. Schubert polynomial — Wikipedia, the free encyclopedia, 2023.

[17] Herbert S. Wilf. *Generatingfunctionology*. Academic Press, Inc., 2 edition, 1994.

[18] Tian Xiong, Henry Tsao, and James I. Hall. General eulerian numbers and eulerian polynomials. https://arxiv.org/pdf/1207.0430.pdf, 2018.

[19] Yan Zhuang. Eulerian polynomials and descent statistics. https://arxiv.org/pdf/1610.07218.pdf, 2017.