SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Ordinals and the Burali-Forti paradox

av

Hao Ju

2023 - K15

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Hao Ju

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Peter LeFanu Lumsdaine

2023

Abstract

The aim of this thesis is to introduce ordinal numbers, demonstrate the Burali–Forti paradox, the generalization of natural numbers to ordinal numbers and show any well-ordered set is isomorphic to a unique ordinal number. This thesis is heavily influenced by Karel Hrbacek and Thomas Jech 1999 “Introduction to Set Theory”. For some of the proofs I also give credits to Burak Kaya 2017–2018 “Lecture notes of Math 320 Set theory”.

In Chapter 1 we write the connections between chapters.

In Chapter 2 we introduce some axioms of set theory and show some of their consequences.

In Chapter 3 we introduce some important relations and orderings.

In Chapter 4 we construct natural numbers based on induction principle and axiom of Infinity. The properties of natural numbers are highlighted such as the set of all natural numbers is a totally ordered set, and furthermore it is a well-ordered set.

In Chapter 5 we generalize natural numbers to ordinal numbers, where the natural numbers are exactly the finite ordinal numbers, and the set of all natural numbers $\mathbb{N}$ is the first infinite ordinal number. Since there exists a set $\mathbb{N}$ which contains all natural numbers, a question is asked if there exists a set of all ordinal numbers. The Burali–Forti paradox demonstrates the existence of “the set of all ordinal numbers” leads to a contradiction. This contradiction is constructed by showing that there exists a successor ordinal of the supremum of the set of all ordinals and this successor ordinal is not contained in the set of all ordinals. As further analysis we show that every well-ordered set is isomorphic to one unique ordinal number, and therefore ordinal numbers can be used to represent any well-ordered set.

In Chapter 6 we write the conclusion of this thesis.

Acknowledgements

I would like to thank my supervisor Dr Peter LeFanu Lumsdaine for all his help with this thesis. From the first seeds to the finished work, I have felt great support from his comments and suggestions on how to improve my thesis. I have been free to develop my own ideas, but also to get help when needed.

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1 Introduction

In set theory, one familiar ordering relation is *partial order* relation, which defines a notion of comparison. A set with partial order relation is called a partially ordered set (also called poset). Two elements x and y in partially ordered set may have four mutually exclusive relationships to each other: either $x < y$, or $x = y$, or $x > y$, or x and y are incomparable.[1] If x and y do not have incomparable relation, in other words, x and y have only 3 mutually exclusive relationships to each other: either $x < y$, or $x = y$, or $x > y$, then this partial order relation is called *total order* relation. Like the definition of partially ordered set, a set with total order relation is called totally ordered set. Furthermore, total order relation is called *well-founded* relation if every nonempty subset of a totally ordered set has a minimal element. Likewise, a set with well-founded relation is called well-ordered set.

The most familiar example of well-ordered sets are natural numbers. We can generalize natural numbers to ordinal numbers, where the natural numbers are exactly the finite ordinal numbers, and the set of all natural numbers $\mathbb{N}$ is the first infinite ordinal number. Since there exists a set $\mathbb{N}$ containing all natural numbers, we may ask if there is a set containing all ordinal numbers. The Burali–Forti paradox demonstrates the existence of "the set of all ordinal numbers" leads to a contradiction. Actually, ordinal numbers can represent any well-ordered sets. This is because every well-ordered set is isomorphic to a unique ordinal number.

The aim of this thesis is to illustrate the relations between axiom of set theory, partial order relation, well-ordered relation, natural numbers, and ordinal numbers. In the end we will demonstrate the contradiction of the Burali–Forti paradox and show that a unique ordinal number can represent any well-ordered set.

[1] https://en.wikipedia.org/wiki/Partially_ordered_set , accessed 2021-11-19.

2 Sets

2.1 References

In this chapter the definition and examples of Axiom of Set theory are heavily influenced by the chapter 1 of Karel Hrbacek and Thomas Jech 1999 “Introduction to Set Theory”. For the symbol expression in the definition of Axiom of Set theory I give credits to Burak Kaya 2017 –2018 “Lecture notes of Math 320 Set theory”.

2.2 Some historical remarks

If one examines the history of mathematics, one may see that towards the end of 19th century, some mathematicians started to investigate the nature of mathematical objects. For example, Dedekind gave a construction for the real numbers, Peano axiomatized the natural numbers, Cantor established a rigorous way to deal with the notion of infinity. These works may be considered as first step to understand what mathematical objects are.

In the early 20th century, arose what is known as the foundational crisis of mathematics. Mathematicians searched for a proper foundation of mathematics which is free of contradictions and is sufficient to carry out all traditional mathematical reasoning. With the work of Dedekind and Cantor, the idea that mathematics established on set theory became more and more common. This eventually led to the development of the Zermelo-Fraenkel set theory with the axiom of Choice.

Today, some mathematicians consider ZFC as the foundation of mathematics, by which one can formalize virtually all known mathematical reasoning. In this thesis, we aim to investigate by the axioms of Zermelo-Fraenkel set theory but without the axiom of Choice.

2.3 Some axioms of ZFC and their consequences

We begin to set up our axiomatic system and try to make clear the intuitive meaning of each axiom.

The first step is to make sure that some sets exist. To be specific, we claim the existence of an empty set.

The Axiom of Existence *There exists a set with no elements.*

In other words, there exists a set X such that for all element y, $y \notin X$. A set with no element will be referred to as an empty set $\emptyset$.

$$\exists X\, \forall y\ \ y \notin X$$

The next axiom allows us to compare sets by their elements.

The Axiom of Extensionality *If every element of X is an element of Y and every element of Y is an element of X, then X=Y.*

$$\forall X\ \forall Y\big(X = Y \leftrightarrow \forall z(z \in X \leftrightarrow z \in Y)\big)$$

With The Axiom of Extensionality, we can show that $\emptyset$ is unique. Assume for contradiction that there is another empty set $\emptyset'$ than $\emptyset$; since there is no element of $\emptyset$ is the element of $\emptyset'$; and there is no element of $\emptyset'$ is the element of $\emptyset$; we have $\emptyset = \emptyset'$. Therefore $\emptyset$ is unique.

□

The next axiom allows us to construct a new set for the elements of existing sets.

The Axiom of Pairing *For any sets X and Y, there exists a set Z such that $W \in Z$ if and only if $W = X$ or $W = Y$.*

$$\forall X\ \forall Y\ \exists Z\ \forall W\big(W \in Z \leftrightarrow (W = X \vee W = Y)\big)$$

In other words, for any sets X and Y, the collection $\{X, Y\}$ is indeed a set, which has exactly X and Y as its elements.

Here we introduce unordered pair. Ordered pair will be introduced in Definition 3.1.

Definition 2.1 **(**Unordered) pair of X and Y is the set having exactly X and Y as its elements which is denoted by $\{X, Y\}$. If $X = Y$, we write $\{X\}$ instead of $\{X, X\}$.

Example 2.2 *The Axiom of Pairing*

a) By pairing $\emptyset$ with itself, since $\{\emptyset, \emptyset\} = \{\emptyset\}$, we have the set $\{\emptyset\}$ exists.

b) By pairing the set $\{\emptyset\}$ with $\emptyset$, we can construct the set $\{\emptyset, \{\emptyset\}\}$.

c) Let X and Y be sets. By pairing X with itself, the set $\{X\}$ existes; by pairing X with Y, the set $\{X, Y\}$ exists; by pairing $\{X\}$ and $\{X, Y\}$, the set $\{\{X\}, \{X, Y\}\}$ exists.

The next axiom allows us to collect all the elements from the elements of sets and then construct a new set by the elements collected.

The Axiom of Union *For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.*

The set U is called the union of S and denoted by $\cup\, S$.

Example 2.3 *The Axiom of Union*

a) Let $S = \{\emptyset, \{\emptyset\}\}$; $x \in \cup S$ if and only if $x \in A$ for some $A \in S$, i.e., if and only if $x \in \emptyset$ or $x \in \{\emptyset\}$. Therefore, $x \in \cup S$ if and only if $x = \emptyset$, thus $\cup S = \{\emptyset\}$.

b) Let M and N be sets; $x \in \cup \{M, N\}$ if and only if $x \in M$ or $x \in N$. The set $\cup \{M, N\}$ is called the union of M and N and is denoted as $M \cup N$.

The next Axiom allows us to choose certain elements from a set by certain properties, and then construct a new set by the chosen elements.

The Axiom Schema of Comprehension *Let **P**(x) be a property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and **P**(x) holds.*

This is a schema of axioms, i.e., for each property ***P***, we have one axiom. For example,

If ***P***(x) is "$x = x$" then the axiom says:

For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and "$x = x$".

If ***P***(x) is "$x \notin x$" then the axiom says:

For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and "$x \notin x$".

Lemma 2.4 For every A, there is *only one* set B such that $x \in B$ if and only if $x \in A$ and ***P****(x)*.

Proof. If B' is another set such that $x \in B'$ if and only if $x \in A$ and ***P***(x) holds, then $x \in B$ if and only if $x \in B'$. By the Axiom of Extensionality, we have $B = B'$.

□

Definition 2.5 $\{x \in A |\ \boldsymbol{P}(x)\}$ is the set of all $x \in A$ with the property ***P****(x)*.

Before the next axiom we introduce a definition.

Definition 2.6 A is a subset of B if and only if every element of A *is* the element of B. In other words, A is a subset of B if and only if $x \in A$ *implies* $x \in B$.

We write $A \subseteq B$ to denote A is a subset of B.

Example 2.7 *Subset*

a) If $A \in S$, *then* $A \subseteq \cup S$.

b) $\{x \in A | \boldsymbol{P}(x)\} \subseteq A$.

The next Axiom allows us to construct a new set by using all the subsets of a set as elements.

The Axiom of Power Set *For any set S there exists a set* $\mathcal{P}$ *such that* $X \in \mathcal{P}$ *if and only if* $X \subseteq S$.

We call the set of all subsets of S the *power set* of S and denote it as $\mathcal{P}(S)$.

Example 2.8 *Power Set*

The elements of $\mathcal{P}(\{a, b\})$ are $\emptyset, \{a\}, \{b\}, and$ $\{a, b\}$.

All these axioms above are not yet all the axioms in ZFC set theory, we will introduce two more axioms *the Axiom of Infinity* in chapter 4 Natural Numbers and *the Axiom Schema of Replacement* in chapter 5 Ordinal Numbers.

3 Relations and Orderings

3.1 References

In this chapter the definition and examples of ordered pairs, relations, and orderings are heavily influenced by the chapter 2 of Karel Hrbacek and Thomas Jech 1999 “Introduction to Set Theory”. Proofs are influenced by both “Introduction to Set Theory” and Burak Kaya 2017–2018 “Lecture notes of Math 320 Set theory”.

3.2 Ordered Pairs

If x and y are sets, by the Axiom of Pairing we can construct the set $\{x, y\}$; By the Axiom of Extensionality, we have $\{x, y\} = \{y, x\}$; In other words the order of x and y does not make any difference. Here we construct a pair where the order of x and y makes difference.

By the Axiom of Pairing the set $\{\{x\}, \{x, y\}\}$ exists.

Definition 3.1 The set $\{\{x\}, \{x, y\}\}$ is called *Ordered Pair* (x, y), where x is the first coordinate and y is the second coordinate.

Since the coordinate of x and y makes difference, we cannot simply compare 2 pairs by the Axiom of Extensity. Here we introduce a theorem to compare them.

Theorem 3.2 $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$.

Proof: If $x = x'$ and $y = y'$, then $(x, y) = \{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} = (x', y')$.

If $(x, y) = (x', y')$, then by the definition of ordered pair, we have $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$.

There are 2 cases either x $\neq y$ or $x = y$. We will show for both cases $x = x'$ and $y = y'$ hold.

If $x \neq y$, then $\{x\} = \{x'\}$ and $\{x, y\} = \{x', y'\}$, which leads to $x = x'$ and $y = y'$.

If $x = y$, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ implies $\{\{x\}\} = \{\{x'\}, \{x', y'\}\}$, which leads to $x = x'$ and $x = y'$.

□

3.3 Relations

The ordered pair (x, y) can be seen as a subset of power set $\mathcal{P}(\{x, y\})$ which satisfies certain property. Here we define a direct term to describe it.

Definition 3.3 A set R is a *binary relation* if all the elements of *R* are ordered pairs, i.e., if for any $z \in R$ then there exist *x* and *y* such that $z = (x, y)$.

It is customary to write xRy instead of $(x, y) \in R$. We say that *x* is in relation *R* with *y* if xRy holds. Several types of relations are of special interest which will be used in natural numbers and ordinal numbers.

Here we introduce some frequent used relations.

Definition 3.4 The *membership relation* on *A* is defined by

$$\epsilon_A = \{\, (a, b) | a \in A, b \ \in A, \text{and } a \in b\}$$

Definition 3.5 Let *A* and *B* be sets. The set of all ordered pairs whose first coordinate comes from *A* and second coordinate comes from *B* is called the *cartesian product* of *A* and *B* and denoted by $A \times B$. In other words,

$$A \times B = \{\, (a, b) |\ a \ \in A \text{ and } b \ \in B\}$$

Thus $A \times B$ is a relation in which every element of *A* is related to every element of *B*.

Cartesian product of *A* and *B* exists because by the Axiom of Union and the Axiom of Power Set, the set $\mathcal{P}(\mathcal{P}(A \cup B))$ exists. By the Axiom Schema of Comprehension $A \times B$ exist, namely the following set exists.

$$\mathrm{A} \times B = \{\, X \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists\, a \in A,\ b \in B,\ X = (a, b)\}$$

Definition 3.6 Let *A* be a set and *R* be a relation on *A*, such that $R \subseteq A \times A$

a) *R* is called *reflexive* on *A* if for all $a \in A$, *aRa* holds.

b) *R* is called *symmetric* on *A* if for all $a, b \in A$, *aRb* implies *bRa*.

c) *R* is called *transitive* on *A* if for all $a, b, c \in A$, *aRb* and *bRc* imply *aRc*.

d) *R* is called an *equivalence relation* on *A* if it is reflexive, symmetric, and transitive on *A*.

3.4 Orderings

Orderings are another frequently used relations.

Definition 3.7 A binary relation R in A *is antisymmetric* if for all $a, b \in A, aRb$ and bRa *imply* $a = b$.

Definition 3.8 A binary relation R in A which is reflexive, antisymmetric, and transitive is called a *(partial) ordering* in A. The pair*(A, R)* is called an *ordered set.*

Example 3.9 *(partial) ordering*

a) $\leq$ is an ordering on the set of all (natural, rational, real) numbers.
b) Define the relation $\subseteq_A$ in A as follows: $x \subseteq_A y$ if and only if $x \subseteq y$ and $x, y \in A$. Then $\subseteq_A$ is an ordering of the set A.

Here we introduce a relation which will be used to define strict ordering as compared to the symmetric relation in partial ordering.

Definition 3.10 A relation S in A is *asymmetric* if aSb implies that bSa does not hold (for any $a, b \in A$). That is, aSb and bSa can never both be true.

Definition 3.11 A relation S in A is a *strict ordering* if it is asymmetric and transitive.

Example 3.12 *strict ordering*

Let R be an ordering of A; then the relation S defined in A by

$$aSb \text{ if and only if } aRb \text{ and } a \neq b$$

is a strict ordering of A.

If we cannot compare all the elements in a set, then we cannot put them in order. The following definition formalizes the capability of comparing the elements.

Definition 3.13 Let $a, b \in A$, and let $\leq$ be an ordering of A. We say that a and b are *comparable* in the ordering $\leq$ if $a \leq b$ or $b \leq a$. We say that a and b are *incomparable* if they are not comparable (i.e., if neither $a \leq b$ nor $b \leq a$ holds). Both definitions can be stated equivalently in terms of the corresponding strict ordering $<$; for example, a and b are incomparable by $<$ if $a \neq b$ and neither $a < b$ nor $b < a$ hold.

Example 3.14 *comparable and incomparable*

a) Any two real numbers are comparable by the ordering $\leq$.

b) 2 and 3 are incomparable by the ordering division.

With the definition of comparable we can develop the ordering to following:

Definition 3.15 An ordering $\leq$ (or <) of A is called *linear* or *total* if any two elements of A are comparable. The pair $(A, \leq)$ is then called a *linearly(total) ordered set*.

Example 3.16 The ordering $\leq$ of positive integers is total.

Here we define some properties of ordered set.

Definition 3.17 Let $\leq$ be an ordering of A, and let $B \subseteq A$

a) $b \in B$ is the *least* element of B in the ordering $\leq$ if $b \leq x$ holds for every $x \in B$.

b) $b \in B$ is the *minimal* element of B in the ordering $\leq$ if there does not exist $x \in B$ such that $x \leq b$ and $x \neq b$.

c) $b \in B$ is the *greatest* element of B in the ordering $\leq$ if for every $x \in B$ $x \leq b$ holds.

d) $b \in B$ is the *maximal* element of B in the ordering $\leq$ if there does not exist $x \in B$ such that $b \leq x$ and $x \neq b$.

It follows from the transitivity of $\leq$ that least and greatest elements, if they exist, are unique. Being least is clearly stronger than being minimal, i.e., least element are also minimal elements. However, the converse is not always true. i.e., in totally ordered set, minimal element is least element because all the elements are comparable. But if not in totally ordered set, since not all elements are comparable, minimal element may not be least element.

4 Natural Numbers

4.1 References

In this chapter the definition and examples are heavily influenced by the chapter 3 of Karel Hrbacek and Thomas Jech 1999 “Introduction to Set Theory”. Proofs are influenced by both “Introduction to Set Theory” and Burak Kaya 2017–2018 “Lecture notes of Math 320 Set theory”.

4.2 Introduction to Natural Numbers

We know natural numbers intuitively: 0, 1, 2, 3, ...15, etc. And we can easily give examples of sets with zero, one, two, or three elements:

Example 4.1 *natural numbers*

a) empty set $\emptyset$ has 0 element.

b) $\{\alpha\}$ for any α, has one element.

c) $\{\alpha, \beta\}$ where $\alpha \neq \beta$, has two elements, etc.

Here we introduce set theory model of natural numbers. The idea is to define a natural number as the set of all smaller natural numbers:

a) $0 = \emptyset$

b) $1 = \{0\} = \{\emptyset\}$

c) $2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$

…

We notice that each natural number is connected to the next in some way. Here we introduce successor set to formalize it. Successor set has applications in both natural numbers and ordinal numbers.

Definition 4.2 The *successor* of a set x is the set $S(x) = x \cup \{x\}$.

By Axiom of Pairing, $\{x\}$ is a set; By Axiom of Union, $S(x) = x \cup \{x\}$ is a set.

From definition above we know that for every element $y \in S(x)$, either $y \subseteq x$ or $y = x$. Intuitively, there is no set between set x and $S(x)$ because $S(x)$ simply adds one element x into the set x.

Theorem 4.3 *Let* $x \subsetneq S(x)$ then there is no set z such that $x \subsetneq z \subsetneq S(x)$.

Proof. Assume for the contradiction that there was a set z such that $x \subsetneq z \subsetneq S(x)$; then there must exist an element y such that $y \notin x\ but\ y \in z, and\ y \in S(x)$; By **Definition 4.2**, $S(x)$ has exactly all the elements of x and x itself as elements; Therefore, y can only be the case that $y = x$ which means that $z = S(x)$ instead of $z \subsetneq S(x)$. Therefore, there is no such z that $x \subsetneq z \subsetneq S(x)$.

□

With the definition of successor, we can simplify natural numbers as:

0 = $\emptyset$

1 = $S(0)$

2 = $S(1)$

We can develop further the construction of natural numbers as follows:

a) 0 is a natural number.

b) If n is a natural number, then its successor $S(n)$ is also a natural number.

c) All natural numbers are obtained by application of (a) and (b)

To define the set of all nature numbers we consider a set containing 0 and closed under successor. Here we prepare a definition:

Definition 4.4 A set *I* is called *inductive* if

a) $0 \in I$.

b) If $n \in I$, then $S(n) \in I$

The precise meaning of c) is that the set of all natural numbers should be the smallest inductive set. In other words, it is in every inductive set.

Definition 4.5 The set of all *natural numbers* is the set

$$\mathbb{N} = \{x \mid x \in I \text{ for every inductive set } I\}$$

The elements of ℕ are called *natural numbers*. By the definition of ℕ the set *x* is a natural number if and only if it belongs to every inductive set.

But there is a question that whether there exists such set ℕ or not. Because previous axioms have a general form:

"For every set *X*, there exists a set *Y* such that…,"

where if set *X* is finite then set *Y* is also finite. Since the only set having existence by the Axiom is the empty set $\emptyset$, therefore all the other sets whose existences are proved by the Axioms are also finite. To complete the definition of ℕ we need an axiom for infinity.

The Axiom of Infinity *An inductive set exists.*

By the Axiom of Infinity, we know that there exists an inductive set I. It follows from the Axiom Schema of Comprehension that the collection below is a set.

$$\{x \in I: \text{for all } J(\text{J is inductive} \rightarrow x \in J)\}$$

This set is called the *set of all natural numbers* denoted by $\mathbb{N}$.

4.3 Properties of Natural Numbers

After defining $\mathbb{N}$ it is interesting to study the properties of it. First we prepare a fundamental tool.

The Induction Principle Let $\mathbf{P}(x)$ be a property (possibly with parameters). Assuming that:

a) $\mathbf{P}(0)$ holds.

b) For all $n \in \mathbb{N}$, $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$.

Then $\mathbf{P}(n)$ holds for all natural numbers $n \in \mathbb{N}$.

Proof. *Let I be the set* $\{n \in \mathbb{N} | \boldsymbol{P}(n)\}$. By the Axiom Schema of Comprehension, I is a subset of $\mathbb{N}$. Since I is inductive, by

Definition 4.*5* $\mathbb{N}$ is subset of any inductive set. Therefore $I = \mathbb{N}$ which implies for $I = \{n \in \mathbb{N} | \boldsymbol{P}(n)\}$ all the property $\mathbf{P}(n)$ must hold for all $n \in \mathbb{N}$.

□

Definition 4.6

a) The relation < on $\mathbb{N}$ is defined by $m < n$ if and only if $m \in n$.

b) The relation = on $\mathbb{N}$ is defined by $m = n$ if and only if $m = n$.

c) The relation $\leq$ on $\mathbb{N}$ is defined by either < holds or = holds.

With these definitions we introduce some lemmas and theorems.

Lemma 4.7

i. $0 \leq n$ for all $n \in \mathbb{N}$.

ii. For all $k, n \in \mathbb{N}, k < n + 1$ if and only if $k < n$ or $k = n$.

Proof.

i. Let $\mathbf{P}(x)$ be the property "$0 \leq n$" and we establish the assumptions for the Induction Principle.

a. $\mathbf{P}(0)$ holds. $\mathbf{P}(0)$ is the statement "$0 \leq x$" , which is true since $0 \leq 0$.

b. $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$. Assume $\mathbf{P}(n)$ holds, i.e., $0 \leq n$, then by **Definition 4.6** either $0 = n$ or $0 \in$ n holds; Since $0 = n$ implies $0 \in \{n\}$ and $0 <$ n implies $0 \in n$, we have $0 \in n \cup \{n\}$. Since $n \cup \{n\} = S(n) = n + 1$, we have $0 < n + 1$, so $\mathbf{P}(n+1)$ holds.

Having shown (a) and (b) by the Induction Principle we can conclude that $\mathbf{P}(n)$ holds for all $n \in \mathbb{N}$, so $0 \leq n$ holds for all $n \in \mathbb{N}$.

ii. For all $k, n \in \mathbb{N}, k < n + 1$, we have $k \in n \cup \{n\}$ which implies either $k = n$ or $k \in n$. This means either $k = n$ or $k < n$ holds.

Conversely, if $k < n$ or $k = n$ holds, then we have either $k \in n$ or $k = n$, which implies $k \in n \cup \{n\}$ and $k < n + 1$.

□

The set $\mathbb{N}$ have some properties as below.

Theorem 4.8 *(*$\mathbb{N}, <$) is a *totally ordered* set.

Proof.

i. The relation < is *transitive* on $\mathbb{N}$.

We will show that for all $k, m, n \in \mathbb{N}$, $k < m$ and $m < n$ imply $k < n$. Let $\mathbf{P}(n)$ be the property that "for all $k, m, n \in \mathbb{N}$, if $k < m$ and $m < n,$ then $k < n$".

a) $\mathbf{P}(0)$ holds. $\mathbf{P}(0)$ asserts that for all $k, m \in \mathbb{N}$, if $k < m$ and $m < 0$ then $k < 0$. Since 0 in $\mathbb{N}$ is defined as empty set and there is no element in empty set, we have there is no $m \in \mathbb{N}$ such that $m < 0$. Therefore $\mathbf{P}(0)$ holds.

b) Assume $\mathbf{P}(n)$ holds, that is for all $k, m \in \mathbb{N}$, if $k < m$ and $m < n$ then $k < n$, we need to show that $\mathbf{P}(n+1)$ holds, that is $k < m$ and $m < n + 1$ imply $k < n + 1$. If $m < n + 1$, by **Lemma 4.7** we have $m < n$ or $m = n$. In the case of $m < n$, we have $k < n$ by the assumption of $\mathbf{P}(n)$; In the case of $m = n$, we have $k < n$ by replacing m with n in $k < m$. Since in both cases $k < n$ hold which implies $k \in n \subset n \cup \{n\}$, we have $k < n + 1$. Therefore $\mathbf{P}(n+1)$ holds.

Since the Induction Principle asserts the validity of $\mathbf{P}(n)$ for all $n \in \mathbb{N}$, we have the transitivity of $(\mathbb{N}, <)$.

ii. The relation < is *asymmetric* on $\mathbb{N}$.

Assume for the contradiction that both $k < n$ and $n < k$ hold; since $<$ is transitive on $\mathbb{N}$, $n < k < n$ implies $n < n$. Let $\mathbf{P}(n)$ be the property that "for all $n \in \mathbb{N}$, $n < n$ does not hold.

a) $\mathbf{P}(0)$ holds. Since $\emptyset \in \emptyset$ does not hold, we have $0 < 0$ does not hold.

b) Assume **P**(n) holds, we will **P**(n+1) holds. Assume for the contradiction that $n < n$ does not hold but $n + 1 < n + 1$ holds; By **Lemma 4.7**, we have either $n + 1 < n$ or $n + 1 = n$; For both cases, since $n < n + 1$ holds, by the transitivity of <, we have $n < n$, which contradicts to **P**(n).

Since the Induction Principle asserts the validity of **P**(n) for all $n \in \mathbb{N}$, we have the asymmetric of $(\mathbb{N}, <)$.

iii. < is a *total ordering* of $\mathbb{N}$.

We will show that for all $m, n \in \mathbb{N}$, $m\ and\ n$ are comparable, that is either $m < n$ or $m = n$ or $n < m$. Let **P**(n) be the property that for all $m, n \in \mathbb{N}$, if either $m < n$ or $m = n$ or $n < m$ holds.

a) **P**(0) holds. For all $m \in \mathbb{N}$, since if $m \leq\ 0$ then $m = 0$, we have $0 < n + 1$ holds.

b) Assume **P**(n) holds, we will show **P**(n+1) that either $m < n + 1$ or $m = n + 1$ or $n + 1 < m$ holds.

If $m < n$, since $n < n + 1$, by transitivity we have $m < n < n + 1$ implies $m < n + 1$.

If $m = n$, since $n < n + 1$, by replacing n with m, we have $m < n + 1$.

If $n < m$, we will show that $either$ $n + 1 < m$ or $n + 1 = m$ by another induction. Let **Q**(m) be the property that "for all $m, n \in \mathbb{N}$, $n < m$ implies $n + 1 < m$ or $n + 1 = m$.

1) **Q**(0) holds. Since for all $m, n \in \mathbb{N}$, $n < 0 + 1$ implies $n = 0$; we have $n = m = 0$; therefore $n + 1 = m + 1$ holds.

2) Assume **Q**(m) holds, we will show **Q**($m + 1$) that $n < m + 1$ implies $n + 1 < m + 1$ or $n + 1 = m + 1$ holds. Since $n < m + 1$ holds if and only if either $n = m$ or $n < m$, we have two cases to show. In the case of $n = m$, since $n + 1 = n + 1$, by replacing n with m on the right side of equation, we have so $n + 1 = m + 1$; In the case of $n < m$, since by assumption $n < m$ implies $n + 1 < m$ or $n + 1 = m$, by transitive property we have $n + 1 < m + 1$ or $n + 1 = m + 1$.

Since the Induction Principle asserts the validity of **Q**(m) for all $m \in \mathbb{N}$, we have **Q**(m) holds.

Since the Induction Principle asserts the validity of **P**(n) for all $n \in \mathbb{N}$, we have < is a *total ordering* of $\mathbb{N}$.

□

We define a development of total ordering as below:

Definition 4.9 A total ordering $<$ of a set A is a *well-ordering* if every nonempty subset of A has a least element. This ordered set$(A, <)$ is called a *well-ordered set*.

When showing $<$ is a *total ordering* of $\mathbb{N}$, we used nested induction. In the loop of showing property $\mathbf{P}(n)$ we used another induction with property $\mathbf{Q}(m)$ in which n is treated as a fixed value. It seems complicated, we will show a more convenient version of Induction Principle.

The Induction Principle, Second Version Let $\mathbf{P}(n)$ be a property (possibly with parameters). Assume for all $n \in \mathbb{N}$,

If $\mathbf{P}(k)$ holds for all $k < n$, then $\mathbf{P}(n)$ holds.

Then $\mathbf{P}$ holds for all $n \in \mathbb{N}$.

Proof. We will show this second version by the first version of the Induction Principle. Let $\mathbf{Q}(n)$ be the property that if $\mathbf{P}(k)$ holds for all $k < n$, then $\mathbf{P}(n)$ holds.

Firstly, we show $\mathbf{Q}(0)$ holds. Since there is no $k < 0$, we have $\mathbf{P}(k)$ holds for all $k < 0$ is false, which implies $\mathbf{Q}(0)$ holds.

Secondly, we show that for all $n \in \mathbb{N}$, $\mathbf{Q}(n)$ implies $\mathbf{Q}(n+1)$. $\mathbf{Q}(n)$ states that if $\mathbf{P}(k)$ holds for all $k < n$, then $\mathbf{P}(n)$ holds; $\mathbf{Q}(n+1)$ states that if $\mathbf{P}(k)$ holds for all $k < n+1$, then $\mathbf{P}(n+1)$ holds. Since the equivalent logic of $\mathbf{Q}(n+1)$ is that if $\mathbf{P}(n+1)$ does not hold, then there exists $k < n+1$ such that $\mathbf{P}(k)$ does not hold; We assume for the contradiction that $\mathbf{P}(n+1)$ does not hold; then there exists either $k < n$ or $k = n$ such that $\mathbf{P}(k)$ does not hold. But by $\mathbf{Q}(n)$ we have $\mathbf{P}(k)$ holds for all $k < n$ and consequently $\mathbf{P}(n)$ holds. Therefore, $\mathbf{P}(n+1)$ holds.

By the Induction Principle, we have $\mathbf{Q}(n)$ holds for all for $n \in \mathbb{N}$. Therefore, we can conclude that $\mathbf{P}(n)$ holds for all $n \in \mathbb{N}$.

□

Theorem 4.10 $(\mathbb{N}, <)$ is a *well-ordered* set.

Proof. Lex X be a nonempty subset of $\mathbb{N}$, Since **Theorem 4.8** shows that $(\mathbb{N}, <)$ is totally ordered set, we just need to show that X has a least element. We assume for the contradiction that X does not have least element and we will show $X = \emptyset$ by the Second Version of Induction Principle. Let $\mathbf{P}(n)$ be a property that if for all $k < n$, $k \in \mathbb{N} - X$, then $n \in \mathbb{N} - X$. Assume for the contradiction that $n \notin \mathbb{N} - X$ then $n \in X$; and since for all $k < n$, $k \in \mathbb{N} - X$, there does not exist $m \in X$ such that $m < n$. Since we have shown that $(\mathbb{N}, <)$ is total ordering, it must be that for all $m \in X$, either $n < m$ or $n = m$, that is n is a least element in X. But this contradicts to X does not have a least element. Therefore $\mathbf{P}(n)$ holds.

By the second version of the Induction Principle, we conclude that $n \in \mathbb{N} - X$ holds for all $n \in \mathbb{N}$, where $\mathbf{P}(n)$ is the property that if $k \in \mathbb{N} - X$ for all $k < n$, then $n \in \mathbb{N} - X$ holds. Therefore $X = \emptyset$, that is there does not exist such nonempty subset X. Therefore, $(\mathbb{N}, <)$ is strictly well-ordered set.

$\square$

4.4 Finite and Countable sets

With natural numbers we can formalize the way of counting set.

Definition 4.11 Sets A and B are *equipotent* (have the *same cardinality*) if there is a one-to-one function f with domain A and range B. We denote this as $|A| = |B|$.

Definition 4.12 A set S is *finite* if it is equipotent to some natural number $n \in \mathbb{N}$. We define $|S| = n$ and say that S has n element. A set is *infinite* if it is not finite.

By our definition, we can count set by cardinality, we see that cardinal number of finite sets are the natural numbers. Obviously, natural numbers are themselves finite set, and $|n| = n$ for all $n \in \mathbb{N}$.

5 Ordinal Numbers

5.1 References

In this chapter the definition and examples are heavily influenced by the chapter 6 of Karel Hrbacek and Thomas Jech 1999 "Introduction to Set Theory". Proofs are influenced by both "Introduction to Set Theory" and Burak Kaya 2017–2018 "Lecture notes of Math 320 Set theory".

5.2 Introduction to Ordinal Numbers

In set theory, an ordinal number, also called ordinal, is the generalization of the natural number. We will show that natural numbers are exactly the finite ordinals, the set of all natural numbers $\mathbb{N}$ is the least infinite ordinal, and how to represent well-ordered set by ordinals.

Before defining ordinal number we prepare some definitions.

Definition 5.1 A set T is *transitive* if for every element $v \in T$, v is also a subset of T, that is $v \subseteq T$.

This definition is equivalent to say that a set T is *transitive* if for every element v, u, $u \in v \in T$ implies $u \in T$. This is because by the definition of subset, $v \subseteq T$ holds if and only if $u \in v \in T$ implies $u \in T$.

Now we introduce the definition of ordinal number.

Definition 5.2 The set α is an *ordinal number* if

a) α is transitive, and

b) $(\alpha, \epsilon_\alpha)$ is a strictly well-ordered set.

We need to notice the difference between relation < is transitive for the set W and the set α is transitive. The relation < is transitive if for all $\alpha, \beta, \gamma \in W$, $\alpha < \beta$ and $\beta < \gamma$ implies $\alpha < \gamma$. The set α is transitive if $\beta \in \alpha$ and $\gamma \in \beta$ implies $\gamma \in \alpha$ or equivalently $\beta \in \alpha$ if and only if $\beta \subseteq \alpha$. Note that it is restricted to membership relation. The definition above shows that natural number is in relation to ordinal number in some way. We formalize this relation by next theorem.

Theorem 5.3 Every natural number is an ordinal.

Proof. Let n be a natural number. Since **Theorem 4.8** (i) shows that membership relation $\in$ is transitive on $\mathbb{N}$, we have n is transitive. By **Theorem 4.10** we know that $(\mathbb{N}, <)$ is a well-ordered set, therefore every nonempty subset of $\mathbb{N}$ is well-ordered. Since $n \in \mathbb{N}$ is also a subset of $\mathbb{N}$, we have every natural number is well-ordered by $\in$. By **Definition 5.2** we have that every natural number is an ordinal.

□

In Theorem 5.13 we will show that natural numbers are exact the finite ordinal numbers and $\mathbb{N}$ is the first infinite ordinal number. Here we prepare a definition for it in advance.

Definition 5.4 $\omega = \mathbb{N}$

5.3 Properties of Ordinal Numbers

We will show that ordinal numbers continues the procedure of generating larger number into transfinite.

Lemma 5.5 If α is an ordinal number, so is $S(\alpha)$.

Proof. Firstly, we show that $S(\alpha)$ is a transitive set. Let α be an ordinal, we will show for all $\beta \in S(\alpha)$, $\beta \subseteq S(\alpha)$ holds. Let $\beta \in S(\alpha)$, by definition $S(\alpha) = \alpha \cup \{\alpha\}$, we have either $\beta \in \alpha$ or $\beta \in \{\alpha\}$. In the case of $\beta \in \alpha$, since α is an ordinal, it has transitive property, therefore $\beta \subseteq \alpha$. In the case of $\beta \in \{\alpha\}$, since $\{\alpha\}$ has exact one element, we have $\beta = \alpha$. By replacing α with β in $\alpha \subseteq S(\alpha)$, we have $\beta \subseteq S(\alpha)$. Since in both cases $\beta \subseteq S(\alpha)$ holds, we have $S(\alpha)$ is transitive set.

Secondly, we show that every nonempty subset of $S(\alpha)$ has a least element. Let E be a nonempty subset of $S(\alpha)$, then either $\alpha \in E$ or $\alpha \notin E$. From Theorem 4.3 we know that there does not exist any set E such that $\alpha \subsetneq E \subsetneq S(\alpha)$. Therefore, if $\alpha \notin E$, then $E \subseteq \alpha$, hence E has a least element by the property of ordinal α. If $\alpha \in E$, since α is the greatest element in $S(\alpha)$, so if $E - \{\alpha\} = \emptyset$, then α is the least element in E, and if $E - \{\alpha\} \neq \emptyset$, then E has the same least element as $E - \{\alpha\} \subseteq \alpha$.

Thirdly, we will show that $S(\alpha)$ is strictly total ordered set. Since α is an ordinal, it is strictly total ordered set. By the definition of $S(\alpha) = \alpha \cup \{\alpha\}$, we have all the elements in α keep same order in $S(\alpha)$ and α is the greatest element in $S(\alpha)$. Therefore $S(\alpha)$ is strictly total ordered set.

Since $S(\alpha)$ is strictly total ordered set and every nonempty subset has a least element, we have $S(\alpha)$ is strictly well-ordered by $\in_{S(\alpha)}$. Having shown the transitivity of $S(\alpha)$, we can conclude that $S(\alpha)$ is an ordinal number.

□

From **Lemma 5.5** we know the successor of an ordinal is still an ordinal, but there are also other ordinals which are not the successor of an ordinal. Here we formalize the definition for them.

Definition 5.6 An ordinal β is said to be a *successor ordinal* if $\beta = S(\alpha)$ for some ordinal α.

An ordinal β is said to be a *limit ordinal* if $\beta \neq \emptyset$ and β is not a successor ordinal.

We denote the successor of α by $\alpha + 1$:

$$\alpha + 1 = S(\alpha) = \alpha \cup \{\alpha\}$$

Next we prepare some lemmas for **Theorem 5.10**.

Lemma 5.7 If α is an ordinal number, then $\alpha \notin \alpha$.

Proof. Let set α be an ordinal. Assume for contradiction that $\alpha \in \alpha$, then the set α has an element $x = \alpha$ such that $x \in \alpha$. By replacing set α with element x we have $x \in x$. By the asymmetric property of ordinal number α, $x \in x$ does not hold. Therefore if α is an ordinal number, then $\alpha \notin \alpha$.

□

Lemma 5.8 Every element of an ordinal number is an ordinal number.

Proof. Let α be an ordinal, x be an element of α. To show x is an ordinal, we need to show that *x* is transitive and strictly well-ordered set by membership relation.

Firstly, we show *x* is transitive set. Let $u \in v \in x$, we wish to show $u \in x$. By the transitivity of ordinal α, $u \in v \in x \in \alpha$ implies $u \in v \in \alpha$. By applying the transitivity of α again, we have $u \in \alpha$. Therefore, we have $x \in \alpha$, $v \in \alpha$, $u \in \alpha$ are all elements in α. Since α is transitive, we have $u \in v \in x$ implies $u \in x$.

Secondly, we show *x* is strictly well-ordered set. There are 2 properties need to show. One is every nonempty subset $v \subseteq x$ has a least element, the other is x is strictly total ordered set. Since α is transitive, we have $x \in \alpha$ if and only if $x \subseteq \alpha$. Since $v \subseteq x \subseteq \alpha$ implies $v \subseteq \alpha$, and every nonempty subset of α has a least element, we have v has a least element. Since α is strictly total ordered set, the subset $x \subseteq \alpha$ is also strictly total ordered set.

Since *x* is both transitive and well-ordered by membership relation, we have *x* is an ordinal.

□

Lemma 5.9 If α *and* β are ordinal numbers such that α is proper subset of β, then $\alpha \in \beta$.

Proof. If α and β are ordinal numbers such that α is proper subset of β, then $\beta - \alpha$ is nonempty subset of β and has a least element denoted by γ. We aim to show that actually $\alpha = \gamma$, and by replacing γ with α in $\gamma \in \beta$, we have $\alpha \in \beta$.

To show $\alpha = \gamma$ we need to show both $\gamma \subseteq \alpha$ and $\alpha \subseteq \gamma$ hold.

Firstly, to show $\gamma \subseteq \alpha$ we need to show for any element $\delta \in \gamma$, $\delta \in \alpha$ holds. Clearly the least element $\gamma \in \beta - \alpha$ is an element of β. For any element $\delta \in \gamma$, by the transitivity of ordinal β,

$\delta \in \gamma \in \beta$ implies $\delta \in \beta$. Since γ is the least element in $\beta - \alpha$, there does not exist any element of γ in $\beta - \alpha$, otherwise γ would not be the least element in $\beta - \alpha$. Therefore, for any element $\delta \in \gamma$, we have $\delta \notin \beta - \alpha$. Since $\delta \in \beta$ and $\delta \notin \beta - \alpha$, we have $\delta \in \alpha$. That is for any element $\delta \in \gamma$, $\delta \in \alpha$ holds. Therefore, $\gamma \subseteq \alpha$.

Secondly, to show $\alpha \subseteq \gamma$ we need to show for any element $\delta \in \alpha$, $\delta \in \gamma$ holds. Since α is the proper subset of β, we have $\delta \in \alpha$ implies $\delta \in \beta$. By Lemma 5.8 we know that as the elements of ordinal β, δ and γ are also ordinals. Since ordinal β has strictly total ordering property, it makes that for the elements $\delta \in \beta$ and $\gamma \in \beta$, only 3 cases can mutually exclusively hold: $\gamma \in \delta$, $\gamma = \delta$ or $\delta \in \gamma$. For the case $\gamma \in \delta$, since $\delta \in \alpha$, we have $\gamma \in \delta \in \alpha$. By the transitivity of α, we have $\gamma \in \alpha$. But choosing $\gamma \in \beta - \alpha$ implies $\gamma \notin \alpha$, which contradicts to $\gamma \in \alpha$. Therefore case $\gamma \in \delta$ does not hold. For case $\gamma = \delta$, since $\delta \in \alpha$, by replacing δ with γ, we have $\gamma \in \alpha$. Because choosing $\gamma \in \beta - \alpha$ implies $\gamma \notin \alpha$, which contradicts to $\gamma \in \alpha$. Therefore case $\gamma = \delta$ does not hold. In the end only the case $\delta \in \gamma$ is left, since one of these three cases $\gamma \in \delta$ or $\gamma = \delta$ or $\delta \in \gamma$ must hold, we have $\delta \in \gamma$ holds. Since for any element $\delta \in \alpha$, $\delta \in \gamma$ holds, we have $\alpha \subseteq \gamma$.

Since both $\gamma \subseteq \alpha$ and $\alpha \subseteq \gamma$ hold, we have $\gamma = \alpha$. And since both $\gamma = \alpha$ and $\gamma \in \beta$ hold, by replacing γ with α, we have $\alpha \in \beta$.

□

We have prepared the relations of elements in *one* ordinal as lemmas above, here we will show some relations between ordinals.

Theorem 5.10 Let $\alpha, \beta, and\ \gamma$ be ordinal numbers

a) If $\alpha \in \beta$ and $\beta \in \gamma$, then $\alpha \in \gamma$.

b) $\alpha \in \beta$ and $\beta \in \alpha$ cannot both hold.

c) Either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$ holds.

d) Every nonempty set of ordinal numbers has a $\in$ least element. Consequently, every set of ordinal numbers is well-ordered by $\in$.

□

Actually, if we imagine there is an ordinal containing $\alpha, \beta, and\ \gamma$, then a) can be interpreted as *transitive* property of this ordinal; b) can be interpreted as *asymmetric* property; c) can be interpreted as total ordering property; d) can be interpreted as well-ordering property. But does such imagined ordinal exist or not? Yes, the successor of $\alpha \cup \beta \cup \gamma$ denoted by $S(\alpha \cup \beta \cup \gamma)$ is such ordinal. We will formalize this hypothesis by Lemma 5.11. Before that we will show theorem 5.10 with the prepared lemmas.

Theorem 5.10 a): If $\alpha \in \beta$ and $\beta \in \gamma$, then $\alpha \in \gamma$.

Proof. If $\alpha \in \beta$ and $\beta \in \gamma$, we have $\alpha \in \beta \in \gamma$. By the transitivity of ordinal number γ, we have $\alpha \in \gamma$.

□

Theorem 5.10 b): $\alpha \in \beta$ and $\beta \in \alpha$ cannot both hold.

Proof. Assume for contradiction that $\alpha \in \beta$ and $\beta \in \alpha$ both hold, then we have $\alpha \in \beta \in \alpha$. By the transitivity property of ordinal α, we have $\alpha \in \alpha$. But this contradicts to **Lemma 5.7** that $\alpha \notin \alpha$.

□

Theorem 5.10 c): Either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$ holds.

Proof. Firstly, we show that as the intersection of α and β, $\alpha \cap \beta$ is transitive set. There are 2 cases of $\alpha \cap \beta$, either $\alpha \cap \beta = \emptyset$ or $\alpha \cap \beta \neq \emptyset$. For the case $\alpha \cap \beta = \emptyset$ since there is no element in $\alpha \cap \beta$, we have $\alpha \cap \beta$ is transitive set. For the case $\alpha \cap \beta \neq \emptyset$ choosing any $\gamma \in \alpha \cap \beta$, we have $\gamma \in \alpha$ and $\gamma \in \beta$. Since α and β are ordinals, α and β have transitive property, we have $\gamma \in \alpha$ and $\gamma \in \beta$ hold if and only if $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$ hold, therefore $\gamma \subseteq \alpha \cap \beta$. Since for any $\gamma \in \alpha \cap \beta$ holds, $\gamma \subseteq \alpha \cap \beta$, $\alpha \cap \beta$ is transitive set.

Secondly, we show that $\alpha \cap \beta$ is strictly well-ordered. For the case $\alpha \cap \beta = \emptyset$ since there is no element in $\alpha \cap \beta$, we have $\alpha \cap \beta$ is strictly well-ordered set. For the case $\alpha \cap \beta \neq \emptyset$, since both α and β are strictly total ordered, we have $\alpha \cap \beta$ is strictly total ordered. Since every nonempty subset of α or β has a least element, we have then $\alpha \cap \beta$ as a subset of α and β has a least element.

Now since $\alpha \cap \beta$ satisfies ordinal **Definition 5.2**, we have $\alpha \cap \beta$ is ordinal.

Thirdly, we will show either $\alpha \cap \beta = \beta$ holds or $\alpha \cap \beta = \alpha$ holds, and then by applying Lemma 5.9 we conclude the proof. For $\alpha \cap \beta$ there are 4 cases: $\alpha \cap \beta \subsetneq \alpha$, $\alpha \cap \beta \subsetneq \beta$, $\alpha \cap \beta = \alpha$, or $\alpha \cap \beta = \beta$. Therefore, the combinations for these 4 cases are:

1. Both $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta = \beta$ hold.
2. Both $\alpha \cap \beta \subsetneq \beta$ and $\alpha \cap \beta = \alpha$ hold.
3. Both $\alpha \cap \beta = \alpha$ and $\alpha \cap \beta = \beta$ hold.
4. Both $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta \subsetneq \beta$ hold.

For case 1 and 2, by the Lemma **5.9** we have $\alpha \in \beta$ or $\beta \in \alpha$ holds respectively. For case 3, clearly $\alpha = \beta$ holds. But case 4 cannot hold. Assume for contradiction that both $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta \subsetneq \beta$ hold, by applying Lemma 5.9 we have $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$ hold, that is $\alpha \cap \beta \in \alpha \cap \beta$, but this contradicts to Lemma 5.7 that $\alpha \cap \beta \notin \alpha \cap \beta$. From above we conclude that either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$ holds.

□

Theorem 5.10 d): Every nonempty set of ordinal numbers has a least element by $\in$. Consequently, every set of ordinal numbers is well-ordered by $\in$.

Proof. Let X be a nonempty set with ordinal elements. Choose any element $\alpha \in X$. Then the intersection of α and X has 2 cases, either $\alpha \cap X = \emptyset$ or $\alpha \cap X \neq \emptyset$. We will show in both cases X has least element by ordering $\in$, and consequently, every set of ordinal numbers is well-ordered by $\in$.

Firstly, we show that for the case $\alpha \cap X = \emptyset$, X has a least element by ordering $\in$. For the case $\alpha \cap X = \emptyset$, there does not exist any element $\beta \in X$ such that $\beta \in \alpha$. By **Theorem 5.10** c), we know that for ordinals β and α, either $\beta \in \alpha$ or $\beta = \alpha$ or $\alpha \in \beta$ holds. After excluding the case $\beta \in \alpha$ we have cases $\beta = \alpha$ and $\alpha \in \beta$ left. Since $\beta = \alpha$ implies β and α are exactly same element, we have in both cases α is the least element in X.

Secondly, we show that for the case $\alpha \cap X \neq \emptyset$, X has a least element by ordering $\in$. Since $\alpha \cap X$ is a nonempty subset of ordinal α, there is a least element $\beta \in \alpha \cap X$ by ordering $\in_{\alpha \cap X}$. We claim that β is the least element in X. Because if β were not the least element in X, then there must exist an ordinal $\gamma \in X$ such that $\gamma \in \beta$. By the transitivity of α, $\gamma \in \beta \in \alpha$ implies $\gamma \in \alpha$, hence β is not the least element in $\alpha \cap X$ which is a contradiction.

Now we have shown that X has a least element by ordering $\in$. By Theorem 5.10 c) the set X with ordinal numbers is totally ordered, therefore set X is well-ordered by $\in$.

□

5.4 The Burali–Forti paradox

The Burali–Forti paradox demonstrates the existence of "the set of all ordinal numbers" leads to a contradiction. It is named after Cesare Burali–Forti, who published a paper in 1897 proving a theorem which, unknown to him, contradicted a previously proved result by Cantor.

Before showing the Burali–Forti paradox we prepare a lemma.

Lemma 5.11 For every set of ordinal numbers X, there is an ordinal γ such that $\gamma \notin X$.

Proof. Let X be a set of ordinals. We will begin with by showing $\cup X$ is an ordinal, and then showing that the successor of $\cup X$ denoted by $S(\cup X)$ is also an ordinal but *not* the element of X.

To show $\cup X$ is an ordinal we need to show $\cup X$ is transitive and strictly well-ordered set. Firstly, we show $\cup X$ is transitive by showing that $\gamma \in \alpha \in \cup X$ implies $\gamma \in \cup X$. Since $\cup X$ has exact all the elements of these ordinals in set X, we have α is an element of certain ordinal β in X, that is $\gamma \in \alpha \in \beta \in X$. By the transitivity of β, we have $\gamma \in \beta \subseteq \cup X$, by which we can conclude that $\cup X$ is transitive set.

Secondly, we show that $\cup X$ is strictly well-ordered set. We claim that any element $\alpha \in \cup X$ is an ordinal. This is because α is an element of certain $\beta \in X$, and by Lemma 5.8 every element of an ordinal is an ordinal, since α is element of ordinal β, we have α is an ordinal. By

Theorem 5.10 d) that every nonempty set of ordinal is strictly well-ordered, we have $\cup X$ strictly well-ordered.

Since $\cup X$ is both transitive set and strictly well-ordered by $\in$, by Definition 5.2 we have $\cup X$ is an ordinal.

Thirdly, we show that $S(\cup \mathrm{X}) \notin \mathrm{X}$. Since $\cup X$ is an ordinal, by Lemma 5.5, $S(\cup \mathrm{X})$ is also an ordinal. Assume for the contradiction that $S(\cup \mathrm{X}) \in X$, since $\cup \mathrm{X}$ contains exact all the elements of elements of X, we have $S(\cup \mathrm{X}) \subseteq \cup \mathrm{X}$. There are 2 cases for $S(\cup \mathrm{X})$, either $S(\cup \mathrm{X})$ is proper subset of $\cup \mathrm{X}$ or $S(\cup \mathrm{X}) = \cup \mathrm{X}$. For the case of $S(\cup \mathrm{X}) \subsetneq \cup \mathrm{X}$, since both $\cup \mathrm{X}$ and $S(\cup \mathrm{X})$ are ordinals, by Lemma 5.9 we have $S(\cup \mathrm{X}) \in \cup \mathrm{X}$. But since $\cup \mathrm{X} \in S(\cup \mathrm{X})$ also holds by the definition of $S(\cup \mathrm{X})$, we have $\cup \mathrm{X} \in S(\cup X) \in \cup \mathrm{X}$. By the transitivity of ordinal $\cup \mathrm{X}$, we have $\cup \mathrm{X} \in \cup \mathrm{X}$ which contradicts to lemma 5.7 that $\cup X \notin \cup X$. Therefore $S(\cup \mathrm{X}) \subsetneq \cup \mathrm{X}$ does not hold. For the case of $S(\cup X) = \cup X$, since $S(\cup X)$ contains the element $\cup \mathrm{X}$, we have $\cup \mathrm{X} \in S(\cup \mathrm{X}) = \cup \mathrm{X}$, in other words $\cup \mathrm{X} \in \cup \mathrm{X}$ which also contradicts Lemma 5.7 that $\cup \mathrm{X} \notin \cup \mathrm{X}$. Therefore $S(\cup \mathrm{X}) = \cup \mathrm{X}$ does not hold. Since neither $S(\cup \mathrm{X}) = \cup \mathrm{X}$ nor $S(\cup \mathrm{X}) \subsetneq \cup \mathrm{X}$ holds, we have $S(\cup \mathrm{X}) \notin X$. In other words, $S(\cup \mathrm{X})$ is the ordinal γ such that $\gamma \notin X$.

□

The Burali–Forti paradox. It demonstrates the existence of "the set of all ordinal numbers" leads to a contradiction.

Proof We show it directly by Lemma 5.11. Assume for contradiction that there exists a set of all ordinal numbers denoted by X, then by Lemma 5.11, there exists an ordinal $S(\cup \mathrm{X})$ such that $S(\cup \mathrm{X}) \notin X$. Therefore, there does not exist any set X which contains *all* the ordinals.

□

5.5 Supremum of ordinal numbers

The ordinal number $\cup X$ used in the proof of Lemma 5.11 and *the Burali–Forti paradox* plays such important role that it is worthwhile to analyze.

Theorem 5.12 For any set X of ordinals, $\cup X$ is the *least ordinal* such that for all $\alpha \in X$ either $\alpha \in \cup X$ or $\alpha = \cup X$.

Proof. First, we will show that $\cup X$ is an ordinal such that for all $\alpha \in X$ either $\alpha \in \cup X$ or $\alpha = \cup X$ holds, and then we will show $\cup X$ is the *least ordinal* for this.

In Theorem 5.11 we have shown that $\cup X$ is an ordinal containing all the elements of ordinals in set X. Let α be an ordinal in set X, we have $\alpha \subseteq \cup X$. There are 2 cases, either α is proper subset of $\cup X$ or $\alpha = \cup X$. For the case of $\alpha \subsetneq \cup X$, by Lemma 5.9 we have $\alpha \in \cup X$. Therefore $\cup X$ is an ordinal such that for all $\alpha \in X$ either $\alpha \in \cup X$ or $\alpha = \cup X$.

Assume for the contradiction that $\cup X$ is not the least ordinal, then there exists an ordinal $Y \in \cup X$ such that for all $\alpha \in X$ either $\alpha \in Y$ or $\alpha = Y$. Since $\cup X$ contains exactly all the elements of ordinals in X, we have $\cup X$ is either proper subset of Y, or $\cup X = Y$. For the case of $\cup X \subsetneq Y$, by Lemma 5.9 we have $\cup X \in Y$. But by Lemma 5.7, $\cup X \in Y$ implies $Y \notin \cup X$ which contradicts to our assumption that $Y \in \cup X$. For the case $\cup X = Y$ we consider $\cup X$ and Y are exactly same ordinal. Since in both cases $Y \in \cup X$ does not hold, we have $\cup X$ is the least ordinal such that for all $\alpha \in X$ either $\alpha \in \cup X$ or $\alpha = \cup X$.

□

$\cup X$ is called the *supremum* of X denoted by *sup X*. As shown above $\cup X$ may or may not be contained in set X.

5.6 Generalize natural numbers to ordinal numbers

In Theorem 5.3 we have shown that natural numbers are ordinal numbers. The following theorem restates the fact that ordinals are the generalization of the natural number.

Theorem 5.13 The natural numbers are exactly the finite ordinal numbers.

Proof. By Theorem 5.3 we know that every natural number is an ordinal, and by Definition 4.12 we have every natural number is a finite set. Therefore, we only need to show that all ordinals which are not natural numbers are infinite sets. If α is an ordinal such that $\alpha \notin \omega$, where ω is defined by Definition 5.4, then by Theorem 5.10 b) it must be the case that either $\omega \in \alpha$ or $\omega = \alpha$, since α is transitive, we have either $\omega \subsetneq \alpha$ or $\omega = \alpha$. Both $\omega \subsetneq \alpha$ and $\omega = \alpha$ imply that α has an infinite subset and hence is infinite.

□

Theorem 5.13 implies that ω is the first infinite ordinal number. We have the following picture.

$$0\ \ 1\ \ 2\ \ 3\ \ \ldots \omega$$

By applying the successor operation, we can construct the following ordinals.

$$0\ \ 1\ \ 2\ \ 3 \ldots \omega\ \ S(\omega)\ \ \ S\big(S(\omega)\big) \ldots$$

It is easily seen that the collection

$$\{0\ \ 1\ \ 2\ \ 3 \ldots \omega\ \ S(\omega)\ \ \ S\big(S(\omega)\big) \ldots\}$$

if is a set, then it must be an ordinal. It turns out that without further axiom we cannot prove this collection is a set. Here we introduce an axiom allowing us to construct this ordinal.

The Axiom Schema of Replacement Let ***P***(x, y) be a property such that for every x there is a unique y for which ***P***(x, y) holds.

> For every set A, there is a set B such that, for every $x \in A$, there is $y \in B$ for which ***P***(x, y) holds.

Let $\boldsymbol{F}$ be the operation defined by the property $\boldsymbol{P}$, that is let $\boldsymbol{F}(x)$ denote the unique y for which ***P***(x, y) holds. The corresponding Axiom Schema of Replacement can be stated as follows:

> For every set A there is a set B such that for all $x \in A, \boldsymbol{F}(x) \in B$

This operation $\boldsymbol{F}$ is called class function. Consider $\boldsymbol{F}(0) = \omega$, $\boldsymbol{F}(1) = S(\omega)$, $\boldsymbol{F}(2) = S(S(\omega))$ …By The Axiom Schema of Replacement, the set $\boldsymbol{F}(\omega)$ exists. By the Axiom of Union, following set exists.

$$\cup\, \boldsymbol{F}(x) = \{0\ \ 1\ \ 2\ \ 3 \ldots \omega\ \ S(\omega)\ \ S\big(S(\omega)\big) \ldots\}$$

Since $\{0\ \ 1\ \ 2\ \ 3 \ldots \omega\ \ S(\omega)\ \ S\big(S(\omega)\big) \ldots\}$ is a set, we see that $\{0\ \ 1\ \ 2\ \ 3 \ldots \omega\ \ S(\omega)\ \ S\big(S(\omega)\big) \ldots\}$ is an ordinal.

5.7 Represent well-ordered sets by ordinal numbers

Having introduced that ordinal numbers are defined in such a way that each is well-ordered by membership $\in$ relation, we may wonder if well-ordered sets can be represented by ordinal numbers. Theorem 5.20 will show that every well-ordered set is isomorphic to a unique ordinal number. Before that we prepare some definitions and lemmas.

Definition 5.14 *Let* $(L, <)$ be a totally ordered set. A set $S \subsetneq L$ is called an *initial segment* of L if S is proper subset of L and if for every $a \in S$, all $x < a$ are also elements of S.

Example 5.15 Both the set of all negative reals and the set of all nonpositive reals are initial segments of the set of all real numbers.

Lemma 5.16 If $(W, <)$ is a well-ordered set and if P is an initial segment of $(W, <)$, then there exists $x \in W$ such that $P = \{y \in W | y < x\}$.

Proof. By the Axiom Schema of Comprehension set P exists. Since P is a proper subset of W, the nonempty complement set $W - P$ has a least element by the well-ordering $<$ denoted by x. As well-ordered set W is strictly total ordered, so is P and $W - P$. Therefore for any elements $y, z \in W$ such that $y < x$ and $z < x$, y and z are elements in P, and y, z have the same ordering as they are in W.

□

We call the set $P = \{y \in W | y < x\}$ as the *initial segment of W given by* x denoted by $pred(x)$. Clearly if x is the least element of W, $pred(x) = \emptyset$.

For well-ordered sets, there exists isomorphism for their initial segments. Before clarifying this theorem, we prepare a definition.

Definition 5.17 An *isomorphism* between two ordered sets$(P, <)$ and $(Q, <)$ is a one-to-one function h with domain P and range Q such that for all $p_1, p_2 \in P$ we have

$$p_1 < p_2 \text{ if and only if } h(p_1) < h(p_2).$$

If an isomorphism exists between $(P, <)$ and $(Q, <)$, then $(P, <)$ and $(Q, <)$ are *isomorphic*.

Here we prepare some lemmas about isomorphism.

Lemma 5.18 If h is isomorphism between $(P, <)$ and $(Q, <)$, then the inverse one-to-one function h^{-1} is an isomorphism between $(Q, <)$ and $(P, <)$.

Proof. Since h is isomorphism between $(P, <)$ and $(Q, <)$, by Definition 5.17 we have for all $p_1, p_2 \in P$, $p_1 < p_2$ $if\ and\ only\ if$ $h(p_1) < h(p_2)$. Since h is one-to-one function, its inverse function h^{-1} has the property that $h^{-1} \circ h(p_1) < h^{-1} \circ h(p_2)$ if and only if $h(p_1) < h(p_2)$, that is $p_1 < p_2$ $if\ and\ only\ if$ $h(p_1) < h(p_2)$ between $(Q, <)$ and $(P, <)$.

□

The following lemma states that isomorphism has the transitivity property.

Lemma 5.19 If f is isomorphism between $\left(P, <_p\right)$ and $\left(Q, <_Q\right)$, and if g is isomorphism between $\left(Q, <_Q\right)$ and $(W, <_W)$, then the composition function $g \circ f$ is isomorphism between $\left(P, <_p\right)$ and $(W, <_W)$.

Proof. Since f and g are isomorphic, by Definition 5.17 we have for all $p_1, p_2 \in P$

$$p_1 <_P p_2 \text{ } if \text{ } and \text{ } only \text{ } if \text{ } f(p_1) <_Q f(p_2) \text{ },$$

and for all $q_1, q_2 \in Q$,

$$q_1 <_Q q_2 \text{ } if \text{ } and \text{ } only \text{ } if \text{ } g(q_1) <_W g(q_2).$$

Hence we have for all $p_1, p_2 \in P$, if $p_1 <_P p_2$, then $f(p_1) <_Q f(p_2)$, then we have

$$g \circ f(p_1) <_W g \circ f(p_2)$$

Conversely, by Lemma 5.18, for all $w_1, w_2 \in W$, if $w_1 < w_2$, then

$$g^{-1}(w_1) <_Q g^{-1}(w_2), \text{ then}$$

$$f^{-1} \circ g^{-1}(w_1) <_P f^{-1} \circ g^{-1}(w_2), \text{ that is } p_1 <_P p_2 \text{ }.$$

Therefore, the composition function $g \circ f$ is isomorphism between $\left(P, <_p\right)$ and $(W, <_W)$.

□

Here we generalize the Induction Principle for natural numbers to the Transfinite Induction Principle for ordinals.

The Transfinite Induction Principle Let $(W, <)$ be a well-ordered set, and $P \subseteq W$

be a subset of W. Assume that:

> For all $x \in W$, if for all $y \in W$, we have for all $y < x$, $y \in P$ holds, then $x \in P$ holds.

Then $x \in P$ holds for all $x \in W$.

In other words, $\forall x \in W \ ((\forall y \in W, y < x \rightarrow y \in P) \rightarrow x \in P) \rightarrow (x \in P)$.

Proof. We will show $W - P = \emptyset$ by contradiction. Assume $W - P$ is not empty, then it has a least element denoted by x. By $x \in W - P$ we have $x \notin P$. We notice that the equivalent logic of $\forall x \in W \ ((\forall y \in W, y < x \rightarrow y \in P) \rightarrow x \in P) \rightarrow (x \in P)$ is that $\forall x \in W \left((x \notin P) \rightarrow \exists y \in W, y < x, y \notin P\right)$, that is $x \notin P$ implies there exists a y such that $y < x$ and $y \notin P$. Therefore $y \in W - P$. But $y < x$ and $y \in W - P$ contradicts to the assumption that x is the least element in $W - P$. Hence $W - P = \emptyset$ which shows that $x \in P$ holds for all $x \in W$.

□

Theorem 5.20 Every well-ordered set is *isomorphic* to a unique ordinal number.

Proof. Let W be a well-ordered set. By transfinite recursion we define a function $f: W \to Ord$ as $f(x) = \{f(y) | y \in pred(x)\}$ for $x, y \in W$. We will show that f gives an isomorphism $(W, <) \cong (\{f(x) | x \in W\}, \in)$ and $f(x)$ is an ordinal by following steps:

1. $y < x$ implies $f(y) \in f(x)$.
2. Function f is injective.
3. $f(y) \in f(x)$ implies $y < x$.
4. For all $x \in W, f(x)$ is transitive.

Firstly, we show that $y < x$ implies $f(y) \in f(x)$. By the definition of initial segment $pred(x) = \{y \in W | y < x\}$, we have $y < x$ implies $y \in pred(x)$. By $f(y) \in \{f(y) | y \in pred(x)\} = f(x)$, we have $f(y) \in f(x)$.

Secondly, we show that for all $x \in W$ and for all $y < x$, $f(y) \neq f(x)$ holds by transfinite induction, and so function f is injective. We assume the induction hypothesis is for all $z < y < x$, $f(z) \neq f(y)$ holds, and we will show that for all $y < x$, $f(y) \neq f(x)$ holds. Assume for the contradiction that $y < x$ but $f(x) = f(y) = \{f(z) | z \in pred(y)\}$ holds. So there exists $z \in pred(y)$ such that $f(y) = f(z)$. Since $z \in pred(y)$ implies $z < y$, we have $f(y) = f(z)$ contradicts to our induction hypothesis. Therefore, by the transfinite induction, for all $x \in W$ and for all $y < x$, $f(y) \neq f(x)$ holds, in other words f is injective.

Thirly, we show that $f(y) \in f(x)$ implies $y < x$. By $f(y) \in f(x) = \{f(z) | z \in pred(x)\}$, there exists $z < x$ such that $f(y) = f(z)$. By injectivity, we have $y = z < x$, that is $y < x$.

Having shown that f is injective and for all $x, y \in W$, $y < x$ if and only if $f(y) \in f(x)$, we have function f gives an isomorphism $(W, <) \cong (\{f(x) | x \in W\}, \in)$.

Fourthly, we show that $f(x)$ is transitive. Suppose $f(z) \in f(y)$ and $f(y) \in f(x)$, since f gives isomorphism, we have $z < y$ and $y < x$ respectively; And since W is transitive, we have $z < x$; Let f gives isomorphism again, we have $f(z) \in f(x)$. Therefore, $f(x)$ is transitive.

Since $f(x)$ is transitive and f gives an isomorphism $(W, <) \cong (\{f(x) | x \in W\}, \in)$, we can conclude that $f(x)$ is an ordinal and well-ordered set W is isomorphic to a unique ordinal number by function f.

□

6 Conclusion

This thesis begins with the introduction of Some axioms of ZFC, such as the Axiom of Existence, the Axiom of Extensionality, the Axiom of Pairing, and the Axiom of Power Set with related examples. And then relations and orderings are introduced. From the familiar partial order relation, we generalize to well-ordered relation. Since natural number is a familiar example, we construct the natural numbers based on Induction Principle and the Axiom of Infinity. The properties of natural numbers are also highlighted such as the set of all natural numbers is totally ordered set, and moreover it is a well-ordered set.

After this we generalize natural numbers to ordinal numbers, where the natural numbers are exactly the finite ordinal numbers, and the set of all natural numbers $\mathbb{N}$ is the first infinite ordinal number. Since there exists a set $\mathbb{N}$ containing all natural numbers, a question is asked if there is a set of all ordinal numbers. The Burali–Forti paradox demonstrates the existence of "the set of all ordinal numbers" leads to a contradiction. This contradiction is constructed by showing there exists the successor of the supremum of the set of all ordinals is not contained in the set of all ordinals. As further analysis we show that every well-ordered set is isomorphic to a unique ordinal number, and therefore ordinal numbers can be used to represent any well-ordered set. The related concrete explanations are written in Chapter 5.

7 References

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